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The Poincaré–Bendixson Theorem: from Poincaré to the XXIst century

Review Article

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Abstract: The Poincaré–Bendixson Theorem and the development of the theory are presented — from the papers of Poincaré and Bendixson to modern results.

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1. Introduction

The problem of existence of periodic orbits and critical points is fundamental in the analysis of behaviour of differential equations and in several applications. However, in many cases it is not easy to find such a solution. Two-dimensional systems play here an important role. One of reasons is that a one-dimensional equation of the second order may be reduced to a system of two equations of the first order. The Poincaré–Bendixson Theorem gives conditions which enable us to prove the existence of a periodic solution of the equation. Moreover, for two-dimensional systems in many cases the existence of a periodic orbit gives also the existence of a critical point. The existence of a particular kind of periodic orbits, i.e. limit cycles, is in many situations particularly interesting. Frequently, it follows from the Poincaré–Bendixson Theorem.

The Poincaré–Bendixson Theorem in its classical version says

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Theorem 1.1.

Consider a plane autonomous system $x' = f(x)$, where $x \in \mathbb{R}^2$, and assume that

($\star$) *the solutions of this system are given uniquely and defined for all $t \in \mathbb{R}$.*

Let the positive semitrajectory $\gamma^+(p)$ of a point $p \in \mathbb{R}^2$ be bounded and let the positive limit set $\omega(p)$ not contain any critical point. Then $\omega(p)$ is a periodic orbit. Moreover, either p is a periodic point or $\gamma^+(p)$ spirals towards a limit cycle of the system.

Let us present one of classical applications of this theorem. Consider the following example, due to [3].

Example 1.2.

We show that the equation $x'' - x'(1-3x^2-2(x')^2)+x = 0$ has a limit cycle. After substitution $y = x'$ and transformation to polar coordinates we get the system

$$r' = r\sin^2\theta(1-3r^2\cos^2\theta-2r^2\sin^2\theta), \qquad \theta' = -1+\frac{1}{2}\sin 2\theta(1-3r^2\cos^2\theta-2r^2\sin^2\theta).$$

We have $r' \geq 0$ for $r = 1/2$ and $r' \leq 0$ for $r = \sqrt{2}/2$, so positive solutions do not leave the annulus $\sqrt{2}/2 \leq r \leq 1/2$. On the other hand, $(0,0)$ is the unique critical point of the system. Thus, according to the Poincaré–Bendixson Theorem, the limit set of a point from the boundary of the annulus must be a periodic trajectory. It is easy to notice that it is a limit cycle. Moreover, the trajectory surrounds $(0,0)$, as it may be proved that inside a periodic trajectory in the plane there must be a critical point.

Several examples of applications of similar kind may be found in [78]. Some applications in economy are described in [76]. For some applications in biology, see for example [45]. Many other textbooks also discuss the Poincaré–Bendixson Theorem and its applications, for example [3, 22, 23, 34, 41, 57].

From the Poincaré–Bendixson Theorem there easily follow several corollaries concerning the behaviour of solutions and orbits in the two-dimensional systems. Below we present two of them (suitable definitions are given in the next section); their proofs can be found for example in [3, 78].

Theorem 1.3.

Consider an autonomous system $x' = f(x)$ on the plane or on the 2-dimensional sphere and assume ($\star$). Then a point p is contained in its limit set if and only if it is periodic or critical.

Theorem 1.4.

Consider an autonomous system $x' = f(x)$ on the plane or on the 2-dimensional sphere and assume ($\star$). Then any minimal set of the system is a single trajectory.

The theory started with the analysis of solutions of certain 2-dimensional differential equations more than one century ago. Now, there is a large number of results related to the original theorems. The research developed in many different directions. The behaviour of solutions was described more precisely, several new phenomena were observed, generalizations for broader class of spaces were presented. In some cases, they moved very far from the classical results, nevertheless, they have the Poincaré–Bendixson Theorem in its origin or have natural connections with the Poincaré–Bendixson Theorem. The theorem still has a remarkable influence on contemporary mathematical research.

In the paper, there are presented the history and the development of the theory — from the achievements of Poincaré and Bendixson up to some recent results.

2. Basic definitions

In this section there are presented definitions which will be of use in the further description of results.

Let X be a metric space. A *flow* (*dynamical system*) on X (which is called a *phase space*) is a triplet $(X, \mathbb{R}, \pi)$, where $\pi\colon \mathbb{R}\times X \to X$ is a continuous function such that $\pi(0,x) = x$ and $\pi(t,\pi(u,x)) = \pi(t+u,x)$ for any t,u,x. Consider an autonomous differential equation $x' = f(x)$ and assume that for any x the solution $x(t)$ through x is unique and exists for any $t \in (-\infty, +\infty)$. Then the function given by the solutions of the equation fulfils the conditions required from flows. Instead of formulation of the above conditions we will just say that the system gives a flow.

Definitions in the sequel will be given for flows, but they are applicable in an obvious way to systems of differential equations.

By *a trajectory* (*an orbit*) of a point x we mean the set $\gamma(x) = \{\pi(t,x) : t \in \mathbb{R}\}$. By *a positive semitrajectory* (*semiorbit*) we mean the set $\gamma^+(x) = \{\pi(t,x) : t \geq 0\}$. By *a negative semitrajectory* (*semiorbit*) we mean the set $\gamma^-(x) = \{\pi(t,x) : t \leq 0\}$.

A point x is said to be

- *critical* if $\pi(t,x) = x$ for every $t \geq 0$,
- *periodic* if there exists a $t > 0$ such that $\pi(t,x) = x$ and x is not critical,
- *regular* if it is neither periodic nor critical.

Note that for a system defined by differential equations a critical point x is a point where $f'(x) = 0$.

For a given point x we define *the positive limit set of* x (or *ω-limit set of* x) as $\omega(x) = \{y \in X : \pi(t_n,x) \to y$ for some $t_n \to \infty\}$ and *the negative limit set of* x (or *α-limit set of* x) as $\alpha(x) = \{y \in X : \pi(t_n,x) \to y$ for some $t_n \to -\infty\}$. We call these sets also the limit sets of the trajectory $\gamma(x)$.

A set A is *invariant* if $\pi(\mathbb{R}\times A) = A$. A set A is *minimal* if it is nonempty, closed, invariant and no proper subset of A has all these properties. A compact invariant set M is isolated if there exists a neighbourhood U of M such that M is the maximal invariant set contained in U. A critical point is called isolated if it is the only critical point in some its neighbourhood.

A set M is *a saddle set* if there exists a neighbourhood U of M such that every neighbourhood V of M contains a point x with $\gamma^+(x) \not\subset U$ and $\gamma^-(x) \not\subset U$.

An (ε, T, x)-chain, $\varepsilon > 0$, $T > 0$, is a collection of finite sets of points $\{x_0, \dots, x_{p+1}\}$ and numbers $\{t_0, \dots, t_p\}$ such that $x = x_0$, $x = x_{p+1}$, $t_j \geq T$ and $d(\pi(t_j, x_j), x_{j+1}) < \varepsilon$ for every $j = 0, 1, \dots, p$. A point $x \in X$ is called chain recurrent if for every $T > 0$ there is an (ε, T, x)-chain.

A *semiflow* (*semi-dynamical system*) on X is a triplet $(X, \mathbb{R}_+, \pi)$ where $\pi\colon \mathbb{R}_+\times X \to X$ is a continuous function such that $\pi(0,x) = x$ and $\pi(t,\pi(u,x)) = \pi(t+u,x)$ for any t,u,x.

For a given semiflow, we define critical, periodic and regular points and positive semitrajectories in the same way as in the case of flows. We define *a negative solution through* x as a function $\sigma\colon (-\infty, 0] \to X$ such that $\sigma(0) = x$ and $\pi(t, \sigma(u)) = \sigma(t+u)$ for any t,u with $u \leq 0$, $t \geq 0$, $t+u \leq 0$. The image of a negative solution is called a *negative semitrajectory* (*semiorbit*) of x. We put $F(t,x) = \{y \in X : \pi(t,y) = x\}$, $F(\Delta, A) = \bigcup\{F(u,y) : u \in \Delta, y \in A\}$ for $A \subset X$, $\Delta \subset \mathbb{R}_+$. A set A is *positively invariant* if $\pi(\mathbb{R}_+\times A) = A$. A set A is *negatively invariant* if $F([0,\infty), A) = A$. We call the set A *weakly negatively invariant* if for any $x \in A$ there is a negative solution σ through x with $\sigma(-\infty, 0] \subset A$. A set A is *positively* (*weakly*) *minimal* if it is nonempty, closed, positively (weakly negatively) invariant and no proper subset of A has all these properties.

In the case of semiflows we define ω-limit sets in the same way as in the case of flows. However, α-limit sets must be defined in a slightly different way, as there may be many negative solutions through a given point x. Let σ be a negative solution through x. We define the α-limit set $\alpha_\sigma(x)$ as $\{y \in X : \sigma(t_n) \to y$ for some $t_n \to -\infty\}$. Note that for a given point x, different negative solutions may give different negative limit sets. Limit sets in semiflows are positively and weakly negatively invariant.

By a *Jordan arc* (a *Jordan curve*) we mean a homeomorphic image of a compact segment $[-1, 1]$ (a unit circle). We say that a Jordan curve J in the plane *surrounds* a point p if p is contained in a bounded component cut from the plane by J.

3. Poincaré and the XIXth century

Differential equations have been a very important area of mathematical research for many centuries. In the XIXth century this chapter of mathematics was of fundamental meaning. However, mathematicians used to concentrate on the methods of solving of different kinds of equations. It was Henri Poincaré (1854–1912) who was a pioneer of qualitative theory of differential equations. The main goal of this theory is to investigate properties of solutions of differential equations not necessarily knowing those solutions.

In the end of the XIXth century Poincaré published his memorable papers [58–61] in the *Journal de Mathématiques Pures et Appliquées*. There were published four papers (the first in 1881, the last in 1886) that gave the beginning to a lot of significant research in differential equations. Poincaré began the work on curves defined by the solutions of differential equations of particular type and investigated the global nature of those solutions. He did not work on integration of particular type of equations. Of Poincaré's interest there was a geometric picture of trajectories of a system given by a differential equation, without integrating this equation. Such a picture would help in understanding some physical phenomena of the system given by the investigated equation, especially the second order differential equations. Poincaré's research concerned asymptotic properties of solution curves and their topological properties. Poincaré used mainly analytical methods.

Poincaré also paid attention to singular points and their types. From the point of view of geometric picture, he characterised and classified different types of critical points that appear in the first order plane equations. The famous example of the differential equation on the two-dimensional torus, where each curve solution is dense and there is no critical point, was also considered by him.

Poincaré considered only the systems $x' = f(x)$ given by an analytic function f. However, in many cases he concentrated even on more narrow case, i.e. equations with polynomial right-hand side. More precisely, he worked on the first order differential equation $dx/X = dy/Y$, where X and Y are polynomials. Under this assumption he formulated the result which was the first version of the Poincaré–Bendixson Theorem. This was presented in the third paper [60], published in 1885.

Poincaré described also the qualitative behaviour of a flow in a neigbourhood of an isolated critical point for flows given by particular kind of differential equations, i.e. given by analytic functions. He showed that for the equations $x'(t) = Ax(t) + g(x)$, where A is linear, under some additional assumptions on g and A, the qualitative behaviour of the system coincides with the behaviour of the system $x'(t) = Ax(t)$ in some neighbourhood of 0. He investigated also the index of a critical point (this term will be described in the next section).

Let us mention also some aspects of terminology introduced by Poincaré. He introduced the concept of an orbit (or, in other words, a trajectory), i.e. a curve in the (x, x') plane parametrised by the time variable t. Such a curve was called by Poincaré *a characteristic* (in French: *caractéristique*). It can be obtained by eliminating the variable t from the given equations. Also the term "limit cycle" is due to Poincaré.

4. Bendixson and the period before World War II

The full story of the Poincaré–Bendixson Theorem starts in the very beginning of the XXth century. In 1901, Ivar Bendixson (1861–1935) published in *Acta Mathematica* his paper [11]. There he gave a more detailed description of bounded limit sets and critical points for planar differential equations. In the same volume there were published papers written by, among others, Emile Borel, Helge von Koch, Rudolf Lipschitz and Gösta Mittag-Leffler.

The Bendixson paper is nowadays famous mainly because of the Poincaré–Bendixson Theorem. However, in the paper the author obtained many other advanced results on planar systems. The paper consisted of 88 pages and up to now gives the inspiration to research. Besides the Poincaré–Bendixson Theorem and its several consequences, only some

Bendixson's results will be mentioned here.

In the very beginning of the paper Bendixson noted that his research is a continuation of Poincaré's results. Bendixson investigated planar systems with much weaker assumptions. He considered the system

$$\frac{dx}{dt} = X(x,y), \qquad \frac{dy}{dt} = Y(x,y)$$

and assumed that X, Y, $\partial X/\partial x$, $\partial X/\partial y$, $\partial Y/\partial x$, $\partial Y/\partial y$ are continuous in a certain domain contained in the plane. He investigated solutions of the system at a point (x_0, y_0) given uniquely and defined for all t.

Let us summarise briefly some of results from [11], under the assumptions given above (in modern formulation).

Theorem 4.1 ([11, Chapter I, Theorem II]).

If the positive semiorbit $\gamma^+(p)$ of a point $p \in \mathbb{R}^2$ is bounded and positive limit set $\omega(p)$ does not contain any critical point, then either $\gamma(p)$ is a periodic orbit or $\gamma^+(p)$ approximates a periodic orbit and this orbit is its limit set.

Bendixson gave also more detailed description of isolated critical points and periodic orbits.

Theorem 4.2.

If p is an isolated critical point, then at least one of the following conditions holds:

(4.2.1) *in any neighbourhood of p there exist infinitely many periodic orbits surrounding p,*

(4.2.2) *there exists a point $x \neq p$ such that $\omega(x) = \{p\}$ or $\alpha(x) = \{p\}$.*

Theorem 4.3.

Let $\gamma(p)$ be a periodic orbit. Then at least one of the following conditions holds:

(4.3.1) *in any neighbourhood of $\gamma(p)$ there exist infinitely many periodic orbits,*

(4.3.2) *there exists a point $x \notin \gamma(p)$ such that $\omega(x) = \gamma(p)$ or $\alpha(x) = \gamma(p)$.*

The proofs relied very strongly on the Jordan Curve Theorem. Also, another crucial fact was used there. In modern terms, we call it the local parallelizability in a small neighbourhood of a non-critical point. The phenomena will be discussed more precisely in the next section.

Let us mention also some other important results that appear in the paper [11]. They were either results used in the proofs on the way to main theorems, or consequences of main theorems.

Theorem 4.4.

If a limit set contains a periodic orbit, then it is equal to this orbit.

Theorem 4.5.

If a bounded limit set does not contain a critical point, then it is a periodic orbit.

Theorem 4.6.

If a limit set L is not a periodic orbit, then for any $x \in L$ the set $\omega(x)$ is a critical point and $\alpha(x)$ is a critical point.

Theorem 4.7.

In the bounded region given by a periodic orbit there is at least one critical point.

Theorem 4.8.

A bounded minimal set is either a periodic orbit or a critical point.

Bendixson's reasoning was rather purely geometrical whereas Poincaré used first of all analytic methods. In the paper [11] there were a few pictures; some of them are presented in Figure 1.

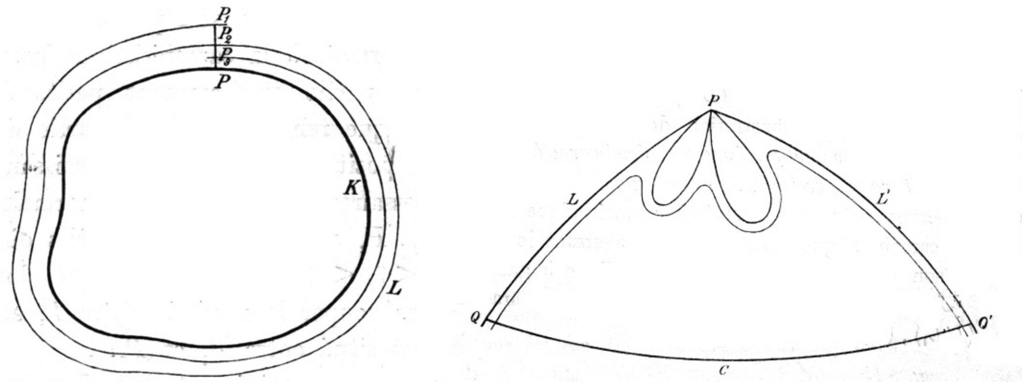

Figure 1.

Let us come back to Theorem 4.2. From this result Bendixson obtained the classification of isolated critical points. A critical point which fulfils the condition (4.2.1) is called a centre. There are two types of centres. Now they are frequently referred as Poincaré centres and Bendixson centres. By a Poincaré centre we mean an isolated critical point p such that there exists a neighbourhood U of p fulfilling the following properties:

(4.9.1) *U is invariant,*

(4.9.2) *all points in U except p are periodic,*

(4.9.3) *any periodic orbit contained in U surrounds p.*

For example, $(0,0)$ is a Poincaré centre in the system given by the equations (in polar coordinates)

$$r'(t) = 0, \qquad \theta'(t) = 1.$$

Poincaré centres were considered by Poincaré. However, the situation described below, presented by Bendixson, is impossible for planar system given by polynomials.

Assume now that (4.2.1) holds and for any periodic orbit $\gamma(q)$ surrounding p there exists a regular point v surrounded by $\gamma(q)$ (consequently, all points in $\gamma(v)$ are surrounded by $\gamma(q)$). A critical point satisfying this condition is called a Bendixson centre. For example, $(0,0)$ is a Bendixson centre in the system given by the equations (in polar coordinates)

$$r'(t) = \begin{cases} r^2 \sin \dfrac{\pi}{r} & \text{for} \quad r \neq 0, \\ 0 & \text{for} \quad r = 0, \end{cases} \qquad \theta'(t) = 1.$$

In this system, $(0,0)$ is an isolated critical point which is surrounded by infinitely many periodic orbits. Any circle of radius $r = 1/n$, $n = 1,2,3,\ldots$, is a periodic orbit. The orbits between the circle of radius $r = 1/n$ and the circle of radius $r = 1/(n+1)$ are spirals which spiral from one circle to the second one. For each of them, the α-limit set is the bigger circle and the ω-limit set is the smaller circle.

Examples of a Poincaré centre and a Bendixson centre are presented in Figure 2.

Another very interesting topic in [11] was a more detailed analysis of critical points fulfilling the condition (4.2.1). Bendixson got a crucial result concerning the index of a critical point. Let us describe very roughly this result. Assume that a planar vector field V is given and consider a Jordan curve C without critical points on it. The index of V along C

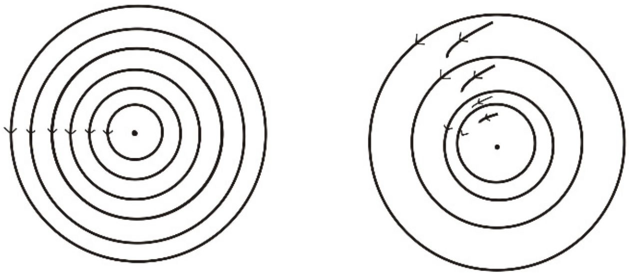

Figure 2.

is a number of rotations through angle 2π made by the vector of V while proceeding around C in the counterclockwise direction. If a critical point p of V is isolated, then for sufficiently small Jordan curves surrounding p indices along them are the same. Such an index is called the index of p.

Bendixson considered the division of a neighbourhood of an isolated critical point into sectors. This division was made by suitably defined separatrices. Those sectors were of two kinds: hyperbolic and elliptic. In a hyperbolic sector separatrices were asymptotes of trajectories. In an elliptic sector each trajectory was emerging or tending to the investigated critical point. Bendixson proved that the index is equal to $1 + (e - h)/2$, where e is a number of elliptic sectors and h is a number of hyperbolic sectors.

5. Between World War I and World War II; dynamical systems

1927 has made a great impact on the qualitative theory of differential equations. Then, George David Birkhoff (1884–1944) published his celebrated monograph [16]. It was a crucial step for the development of the theory. This monograph was the basis to a lot of research in the next twenty years and, in fact, it still has an influence on the theory. Birkhoff is commonly regarded as the founder of the theory of dynamical systems. In Birkhoff's book we can see the foundation of two important branches of the theory, i.e. topological theory and ergodic theory. It should be noted that the terminology: *α-limit points* and *ω-limit points*, was introduced by Birkhoff in this monograph. The definition of dynamical system presented in Chapter 2 was not yet formulated there, as Birkhoff considered only systems given by differential equations. The topological definition appeared soon after it.

The abstract definition of a dynamical system (a flow) was formulated independently in 1931 by Andrei Andreievich Markov (1903–1979) in [48] and in 1932 by Hassler Whitney (1907–1989) [81]. (By the way, A.A. Markov, mentioned above, was not Andrei Andreievich Markov known by Markov processes and Markov chains, but his son.) In 1933 Whitney in another memorable paper [82] introduced the concept now known as parallelizability. Roughly speaking, parallelizability of the system gives an opportunity of topological transformation of trajectories to get those trajectories as parallel lines.

By a *section* through x we mean a set S containing x such that for some $\lambda > 0$ the set $U = \pi((-\lambda, \lambda), S)$ is a neighbourhood of x and for every $y \in U$ there are a unique $z \in S$ and a unique $t \in (-\lambda, \lambda)$ with $\pi(t, z) = y$. By a *transversal* we mean a section that is simultaneously a Jordan arc or a Jordan curve.

For the proof of the Poincaré–Bendixson Theorem the notion of the transversality was of great importance. In the differential case, there is no problem with it. This follows easily as for any non-critical point p we may find a curve passing a given point which is not tangent to the solution curve through p. However, in the non-differential case the situation turns out to be much more complicated. Independently, Whitney in 1933 [82] and Bebutov in 1939 [9] defined sections for flows and proved the existence theorem which is now called the Whitney–Bebutov Theorem. The theorem in its modern form says that in a dynamical system for any non-critical point x there exists a section through x. The existence of a section through a given point shows that local parallelizability of the system is fulfilled and gives a very good tool for solving many problems as it is possible to describe very precisely the behaviour of a system in a neighbourhood of any non-critical point.

In the papers mentioned above, two authors not only presented different proofs of the theorem, but they even approached these problems from different sides. The results from those papers were of great influence to further research. In fact, Whitney considered not flows but special families of curves (more general case than families given by orbits of flows).

Sections were used in the next important result for planar flows of the Poincaré–Bendixson type. It was obtained by Harald Bohr and Werner Fenchel. By the way, Bohr was a younger brother of the famous physicists Niels Bohr and a very good football player, a member of the Danish national football team, a winner of a silver medal in 1908 Olympic Games. Bohr and Fenchel proved in [17]

Theorem 5.1.
Let $(\mathbb{R}^2, \mathbb{R}, \pi)$ be a flow and $p \in \mathbb{R}^2$ be a regular point. Then $p \notin \omega(p)$.

In fact, before the abstract dynamical systems were formally introduced, a valuable result of the Poincaré–Bendixson type for flows was obtained by Kneser. In 1924 he proved in [43] a theorem which now may be formulated in the following way:

Theorem 5.2.
Let $(\mathbb{K}, \mathbb{R}, \pi)$ be a flow without critical point on the Klein bottle. Then there exists a periodic point $x \in \mathbb{K}$.

Soon after it, an extremely valuable result for another surface in the differential case was obtained by Arnaud Denjoy (1884–1974) in [25]. Poincaré posed the question if for a flow on the torus $\mathbb{T}^2$, given by an analytic function f, the only possible minimal sets are points, periodic trajectories and the whole torus $\mathbb{T}^2$. Denjoy in 1932 solved this problem, showing the assertion is true even in a stronger version. He proved several results about the systems on the torus $\mathbb{T}^2$. We present here the theorem most connected to the Poincaré–Bendixson theory.

Theorem 5.3.
Assume that $x' = f(x)$, $x = (x_1, x_2)$ with a suitable identification, is an autonomous system on the torus $\mathbb{T}^2$, where f is of class C^2. Assume that this system gives a flow. Let M be a minimal set for this system. Then either M is a critical point, M is homeomorphic to the circle (i.e. is a periodic orbit), or $M = \mathbb{T}^2$.

As was also shown by Denjoy, the assumption that f is of class C^2 is essential. He presented an example of the system given by the function of class C^1 for which the assertion of the above theorem did not hold. His results were of great importance, gave a good description of the phenomena occurring in such systems and were a base for further investigations. Among others, he considered phenomena based on ergodicity and rotation numbers.

6. First generalization after World War II

Neither Poincaré nor Bendixson investigated limit sets with infinite number of critical points. However, the Poincaré–Bendixson Theorem can be generalised to such case.

Solntzev in 1945 [74] provided a generalization of Theorem 4.6. He split each compact limit set $\omega(p)$ into two parts, $\omega(p) = \omega_C(p) \cup \omega_N(p)$. Here $\omega_C(p)$ is the set of all critical points contained in $\omega(p)$, $\omega_N(p)$ the set of all non-critical points contained in $\omega(p)$. Any component of $\omega_C(p)$ was called *a singular component*. Solntzev proved the following theorem:

Theorem 6.1.
Consider an autonomous system

$$x' = f(x), \tag{1}$$

where $x \in \mathbb{R}^2$, and assume that this system gives a flow. Assume that the positive semitrajectory $\gamma^+(p)$ through a point $p \in \mathbb{R}^2$ is bounded. Then either

(6.1.1) *the positive limit set* $\omega(p)$ *is a periodic trajectory,* or

(6.1.2) *the set of non-critical trajectories contained in* $\omega(p)$ *is at most countable.*

If (6.1.2) *holds, then for any non-critical point* q *contained in* $\omega(p)$ *the set* $\alpha(q)$ *is contained in some singular component of* $\omega_C(p)$ *and the set* $\omega(q)$ *is contained in some singular component of* $\omega_C(p)$.

For another result of Solntzev (formulated here in a slightly different way than in the original paper), we need to introduce a special equivalence relation in a limit set. Define an equivalence class in $\omega(p)$ for a point p. We write $x \sim y$ if $x = y$ or x, y are critical points belonging to the same singular component of $\omega_C(p)$. Let $\Omega(p) = \omega(p)/\sim$, $\Omega_C(p) = \omega_C(p)/\sim$, $\Omega_N(p) = \omega_N(p)/\sim$. Of course, we may identify $\Omega_N(p)$ with $\omega_N(p)$. Now we have

Theorem 6.2.

Consider a system (1)*, where* $x \in \mathbb{R}^2$*, and assume that this system gives a flow. Assume that the positive semiorbit* $\gamma^+(p)$ *through a point* $p \in \mathbb{R}^2$ *is bounded. Then there exists a continuous surjective mapping* h *from a circle to* $\Omega(p)$ *such that* $h\restriction_{h^{-1}(\Omega_N(p))}$ *is a homeomorphism from a subset of a circle onto* $\Omega_N(p)$.

Roughly speaking, the theorem says that we can go along the whole limit set like along "cyclic paths" and meet any non-singular point precisely once.

The work of Solntzev was continued by Vinograd who in [79, 80] obtained similar result concerning unbounded limit sets. He proved that if a semitrajectory is unbounded and a limit set associated with this semitrajectory contains no critical points, then it consists of at most countable number of trajectories, each of which having empty α-limit set and ω-limit set and, moreover, each separating the plane. For limit sets, where there are no unbounded component consisting of critical points, Vinograd classified the components of this limit set into five types in terms of the behaviour of trajectories lying in them. He gave also a description of parametrizations of components similar to that presented by Solntzev.

The limit sets may be of different shape. For instance, a limit set may be in the shape of the circle and contain infinitely many regular trajectories (coming from one singular component to another one). On the other hand, it may be in the shape of a finite-leafed rose or infinite-leafed rose, with only one critical point common for all the leaves.

For example, consider points p_n, where (in polar coordinates (r, θ)) $p_n = (1, \pi/n)$, $n \geq 1$. Take as critical points all p_n, $(1, 0)$ and $(0, 0)$. The trajectories of points not contained in the unit circle spiral towards this circle, as well from inside as from outside. In the unit circle, the regular trajectories go from p_k to p_{k+1}.

Another example may be described as follows. Consider the division of the plane given by half-lines: $\{x > 0,\ y = 0\}$; $\{x < 0,\ y = 0\}$; $\{x = 0,\ y > 0\}$ and $\{y = x/n,\ x > 0\}$ for $n \geq 1$. Denote the domains by D_n in successive way so that D_1 is the lower half-plane and for $k \geq 3$ each D_k is bounded by $y = x/(k-1)$, $y = x/(k-2)$. Now, in each D_k take a regular trajectory γ_k such that in both time directions it goes to $(0, 0)$ and its diameter is equal to $1/k$. For any n all points surrounded by γ_n are contained in trajectories with both limit sets equal to $(0, 0)$. Now we can construct the trajectory which spirals to the set $\Gamma = \{(0, 0)\} \cup \bigcup \{\gamma_n : n \geq 1\}$ and Γ is its ω-limit set.

Some of the possible phenomena are presented in Figure 3.

According to the above theorems we also have

Theorem 6.3.

Consider a system (1)*,* $x \in \mathbb{R}^2$*, and assume that this system gives a flow. Assume that the positive semitrajectory* $\gamma^+(p)$ *through a point* $p \in \mathbb{R}^2$ *is bounded and the limit set* $\omega(p)$ *contains precisely one critical point and infinitely many regular trajectories. Then the regular trajectories form a sequence of planar subsets with the diameters tending to* 0.

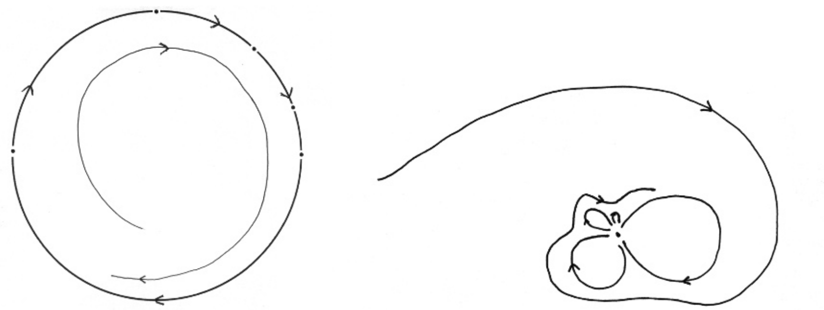

Figure 3.

Vinograd gave also another very interesting theorem concerning planar limit sets. He proved

Theorem 6.4.

Let B be a subset of $\mathbb{R}^2$. Then there exists a continuous function f such that the system (1) gives a planar flow with B being the ω-limit set for some point if and only if B is the boundary of some nonempty (and different from the whole plane) simply connected domain contained in $\mathbb{R}^2$.

It should be noted that Vinograd worked also in general flows. In particular, he proved in [79] that if in a flow on a locally compact first countable space the ω-limit set of some point is non-compact, then it does not have any compact component. From this theorem applied to flows on $\mathbb{R}^n$ it follows that the limit set associated with an unbounded semitrajectory does not have any bounded component. The results of Solntzev and Vinograd gave further precise characterization of limit sets.

In the next years, following Denjoy and Kneser, subsequent results for manifolds were published. In [30] a similar theorem for closed 2-dimensional orientable manifolds was stated. The author considered differential systems without critical points on manifolds which gave a flow. He claimed that a limit set containing a Jordan curve was equal to this curve provided the phase space was not a torus. However, the proof was not correct, as was pointed out by Peixoto [56]. The theorem of Denjoy was generalised in 1963 by Arthur J. Schwartz who proved [66]:

Theorem 6.5.

Assume that (1) is a system on a compact, connected 2-dimensional manifold X of class C^2, where f is of class C^2. Assume that this system gives a flow. Let M be a minimal set for this system. Then either M is a critical point, M is homeomorphic to the circle (i.e. is a periodic trajectory), or $M = X$ (i.e. is the whole manifold); in the last case the manifold X must be equal to the two-dimensional torus $\mathbb{T}^2$.

Another result of Schwartz was

Theorem 6.6.

Assume that (1) is a system on a compact, connected 2-dimensional orientable manifold X of class C^2, where f is of class C^2. Assume that this system gives a flow and that the manifold X is not a minimal set. Let p be such that $\omega(p)$ does not contain any critical point. Then $\omega(p)$ is homeomorphic to the circle.

For a non-orientable manifold and without the differentiability of the flow assumed, a similar theorem was obtained in 1969 by Markley [46]. This theorem may be regarded as a generalization of Theorem 5.2.

Theorem 6.7.

Let $(\mathbb{K}, \mathbb{R}, \pi)$ be a flow on the Klein bottle. Assume that $p \in \omega(p)$ or $p \in \alpha(p)$. Then p is either critical or periodic.

7. The theorem for dynamical systems: from Hájek to Gutiérrez

After World War II several generalizations of the Poincaré–Bendixson Theorem of various types had been obtained, also, several authors were proving many results for general flows. However, the classical theorem was still known only in the differential case.

One of main reasons was the problem with sections. The existence of sections was shown even in a very general case, however, not much was proved on the shape of sections even in the 2-dimensional case. This changed due to a result by Otomar Hájek.

One of basic results which could help in the generalization of the Poincaré–Bendixson Theorem for flows was local parallelizability of flows, guaranteed by the Whitney–Bebutov Theorem. However, the difficulty was moved to another point. In the case of flows given by differential equations, fulfilling suitable continuity assumptions, we could find transversals through any non-critical point without difficulty. This gave a good local description of solutions close to non-critical points and the proofs might use it. But without the assumption of differentiability, nothing was known about the topological shape of sections.

In fact, Whitney in [82] stated a remark on this topic. However, he did not write (and did not prove) that all sections in the 2-dimensional case must be Jordan arcs or curves. He only stated (and gave a nice explanation) that through any point one can construct a section which is an image of a closed interval.

The problem was solved by Hájek, who proved in 1965 [31] the following

Theorem 7.1.
Let X be a 2-dimensional manifold and let $(X, \mathbb{R}, \pi)$ be a flow. Then every section which is a locally connected continuum is either a Jordan arc or a Jordan curve.

This helped with the generalization of the Poincaré–Bendixson Theorem for flows in the 2-dimensional case (obtained by Hájek in [33]). Hájek gave a very precise description of limit sets in flows on the plane and dichotomic 2-dimensional manifolds, not necessarily compact. A 2-dimensional manifold X is called *dichotomic*, if for any Jordan curve J the set $X \setminus J$ has two components. Hájek proved, in particular,

Theorem 7.2.
Let $(X, \mathbb{R}, \pi)$ be a flow on a dichotomic manifold X. Assume that $\omega(p) \neq \emptyset$ for some $p \in X$. Then $\omega(p)$ consists of critical points and at most countable family $\{T_n : n \in A\}$, $A \subset \mathbb{N}$, of non-critical trajectories. Moreover, each compact subset of X without critical points has common points with at most finite number of T_n.

Theorem 7.3.
Let $(X, \mathbb{R}, \pi)$ be a flow on a dichotomic manifold X and let the closure of $\omega(p)$ be compact for some non-critical point $p \in X$. Then either p is a periodic point, or $\omega(p)$ is a periodic orbit and a limit cycle, or for every $x \in \omega(p)$ both $\alpha(x)$ and $\omega(x)$ are non-void compact connected sets containing only critical points.

Hájek also carried some results of Solntzev and Vinograd mentioned above to flows on dichotomic 2-manifolds.

Generally, the main points in Hájek's proof were similar to those used in the differentiable case. However, several parts had to be done in a different way because of the lack of the assumption of differentiation. Also, several other techniques were used. One of them was considering the inherent topology. For a given regular orbit, we may consider the Euclidean topology induced from the plane. On the other hand, we may define a topology on a regular trajectory taking as the base the images of open intervals through the solution. One of the main points of the proof was to show that these topologies are equal.

Hájek's results showed that the Poincaré–Bendixson Theorem is purely topological and does not depend on differentiability assumptions. As it turned out about twenty years later, it was related to much more general, deep property describing not only limit sets.

The problem if a flow on $\mathbb{R}^n$ must be given by a differential equation arose naturally after introducing the abstract definition of dynamical systems. In the one-dimensional case the answer is positive and the proof is not difficult. However, for higher dimension the problem turned out to be non-trivial. It was solved in 1974 by Chewning [19] who provided an example of a flow in $\mathbb{R}^4$ which is neither isomorphic nor equivalent to a differentiable system. Chewning cleverly used a topological example of a suitable set given by Bing. For $n = 2$ and $n = 3$ the problem was still open.

In 1986 Carlos Gutiérrez published a paper on smoothing continuous flows [29], fundamental for the theory of 2-dimensional systems. A corollary of the main theorem of this paper was that any continuous flow on a 2-dimensional compact manifold X of class C^∞ is topologically equivalent to a C^1 flow on X. Two continuous flows $(X, \mathbb{R}, \pi_1)$ and $(X, \mathbb{R}, \pi_2)$ are said to be *topologically equivalent*, if there exists a homeomorphism of X that takes trajectories of $(X, \mathbb{R}, \pi_1)$ to trajectories of $(X, \mathbb{R}, \pi_2)$, preserving the natural orientation of trajectories. Topological equivalence preserves topological properties of orbits, in particular, the properties investigated in the Poincaré–Bendixson Theorem. Note that earlier, in 1978, Neumann proved [54] that any continuous flow on a 2-dimensional compact orientable manifold X of class C^∞ is topologically equivalent to a C^∞ flow on X under the assumption that in the flow there are finite number of critical points and only periodic and critical points are contained in their limit sets. Together with the Poincaré–Bendixson Theorem this gave a topological equivalence of flows with finite number of critical points on the plane to C^∞ flows. Gutiérrez used Neumann's theorem in his proof.

Let us present main results of Gutiérrez's paper. His theorem generalised not only the result of [54], but first of all Theorem 6.5, which is the implication (7.4.1) ⇒ (7.4.3).

Theorem 7.4.

Let $(X, \mathbb{R}, \pi)$ be a flow on a compact 2-dimensional manifold X of class C^∞. Then there exists a C^1 flow on X which is topologically equivalent to $(X, \mathbb{R}, \pi)$. Furthermore, the following conditions are equivalent:

(7.4.1) $(X, \mathbb{R}, \pi)$ *is topologically equivalent to a C^2 flow on X,*

(7.4.2) $(X, \mathbb{R}, \pi)$ *is topologically equivalent to a C^∞ flow on X,*

(7.4.3) *if M is a minimal set in the flow $(X, \mathbb{R}, \pi)$, then either M is a critical point, or M is homeomorphic to the circle (i.e. is a periodic orbit), or $M = X$ (i.e the whole manifold); in the last case the manifold X must be equal to the two-dimensional torus $\mathbb{T}^2$.*

From this theorem it follows immediately that topological properties of differential planar autonomous systems hold for flows. In particular, the Poincaré–Bendixson Theorem for topological flows can be regarded as a consequence of the Poincaré–Bendixson Theorem in the differential case. Nevertheless, it should be pointed out that Gutiérrez used in his proof many advanced and complicated modern techniques and results. The other proofs of the Poincaré–Bendixson Theorem for flows, despite not easy, were really much more elementary.

The problem of smoothing the flow for $n = 3$ is, to the best of the author's knowledge, still open. Speaking about 3-dimensional flows in this aspect, it is natural to mention the Seifert conjecture. It states that every continuous flow without critical points given by differential equation on the 3-dimensional sphere has a closed orbit. In 1950 Seifert [69] asked if such a vector field exists, but did not formulate non-existence as a conjecture. In 1974 Schweitzer gave an example [67] of a C^1 flow without a closed orbit. In 1994 Kuperberg [44] constructed a C^∞ counterexample. Later this construction was used to obtain even more regular examples.

Note also that properties of Poincaré–Bendixson type in the 2-dimensional case for flows were considered not only for systems on surfaces, but also on subsets of the plane. In 1967 Seibert and Tulley [68] showed a similar theorem for flows defined on subsets of the plane. They generalised Theorem 5.1 and obtained

Theorem 7.5.

Let $X \subset \mathbb{R}^2$ and $(X, \mathbb{R}, \pi)$ be a flow. Assume that $p \in \mathbb{R}^2$ is a regular point. Then $p \notin \omega(p)$.

The reasoning in their proof did not use sections.

8. The Poincaré–Bendixson Theorem for semidynamical systems

In flows, we have the movement defined in both directions. However, one may consider only the movement defined in positive direction. This leads to the abstract definition of a semiflow, which was first formulated in 1965 by Hájek [32]. The theory of semiflows was developed soon later in the book [13].

The topological theory of flows grew from differential equations as a flow is a natural generalization of an autonomous equation with solutions given uniquely and defined for all $t \in \mathbb{R}$. Semiflows, with the movement defined only in one direction, appear now as a natural topic for further research of more general phenomena. It should be noted that from the point of view of differential equations the most interesting models of semiflows may be found in infinite dimensional systems. At the same time, from the topological point of view in many spaces there may appear several questions of interest. Also, it is intriguing which properties of a particular object are consequences of differentiable structure, which depend only on topological properties and on the possibility of movement in just one direction.

Semiflows have movement defined only "forward", but a natural question about negative continuations arises. For a given point x, we may introduce negative semitrajectories through x (there may be many such trajectories, but there may be none as well) and consider negative limit sets, depending not only on the point but also on a negative semitrajectory.

Thus a natural question arises about the Poincaré–Bendixson properties for 2-dimensional semiflows, not only for ω-limit sets but also for α_σ-limit sets, where σ is a negative solution through x.

In 1977 McCann [51] wrote an important paper about isomorphisms of semiflows. In particular, his results implied that investigating the topological properties of semiflows on 2-dimensional manifolds one can assume that any negative solution is defined on the interval $(-\infty, 0]$.

In the proof of the Poincaré–Bendixson Theorem for flows, transversals and sections played an important role. The local parallelizability of flows was fundamental for the local characterization of the neighbourhood of the system. However, for semiflows it is impossible to give such a good description, as here a trajectory can "glue" with other trajectories.

There were many attempts to generalise the notion of sections for semiflows. In 1992, the following definition was stated in [20]. A closed set S containing x is called a *section* through x if there are a $\lambda > 0$ and a closed set B such that $F(\lambda, B) = S$, $F([0, 2\lambda], B)$ is a neighbourhood (not necessarily open) of x and $F(\mu, B) \cap F(\nu, B) = \emptyset$ for $0 \leq \mu < \nu \leq 2\lambda$.

These sections give a good local description of a suitable neighbourhood of a non-critical point in general semiflows. In the case of flows, this definition gives a Whitney–Bebutov section. Also, the existence of sections in semiflows in the general case, i.e. without the assumption of local compactness of the phase space, was proved [20]. We have

Theorem 8.1.

Let a semiflow $(X, \mathbb{R}_+, \pi)$ on a metric space X be given. Then for any non-critical point x there exists a section through x. Moreover, if X is a manifold, then for any non-critical point x there exists a compact section through x.

According to this theorem and McCann's results, in a planar semiflow any non-critical point x can be contained in a suitable neighbourhood. This neighbourhood is a parallelizable "box" in which all the segments of trajectories go perfectly from one side to the opposite one in the time interval 2λ (all segments start on one side of the box and no other trajectory joins these segments).

This local characterization was an important step for the Poincaré–Bendixson Theorem for semiflows, which was proved in 1994 [21]. In that paper it was also shown that any compact section in a semiflow on a 2-manifold X is either a Jordan arc or a Jordan curve, which was important for the reasoning.

In the following theorems by a limit set L we mean either an ω-limit set $\omega(p)$ or an α_σ-limit set $\alpha_\sigma(p)$, where σ is a negative solution through a non-critical point p. In all cited theorems there is assumed that we have a semiflow on $\mathbb{R}^2$ or on a 2-dimensional sphere. However, some of them can be extended to some 2-dimensional manifolds.

Theorem 8.2.

If a limit set L is connected and does not contain critical points, then L is a single trajectory.

Theorem 8.3.

Let a semi-orbit (positive or negative) be bounded. Then either the limit set L associated with this orbit is a periodic orbit or any semi-orbit, contained in L, may contain in its limit set only critical points.

Theorem 8.4.

If $p \in \omega(p)$ (or $p \in \omega_\sigma(p)$) then p is either critical or periodic.

Theorem 8.5.

If a compact set A is either positively minimal or weakly minimal, then it is either a critical point or a periodic orbit.

Generally, the proofs were not a simple analogy to those for differential systems (or even to those for flows) as semiflows admit complicated situations which are impossible for flows. Moreover, all the earlier proofs of the Poincaré–Bendixson Theorem depended on the uniqueness of the negative semi-solutions and the possibility of unique continuous movement in both directions.

Note that the Gutiérrez theorem about the topological equivalence concerns only 2-dimensional flows, not semiflows. Because of the complicated structure of semiflows and the character of singular points in the finite dimensional case, one should not expect an analogous result for semiflows.

At the end of this section it should be pointed out that the Poincaré–Bendixson Theorem for semiflows shows that this theorem is not only purely topological, but in fact it depends only on the continuous movement defined for positive values of the time variable t. Roughly speaking, the essence of this theorem is a possibility of a continuous movement forward without being bothered about the backward direction.

9. Some other results

Many further results which may be regarded as some generalizations of the classical Poincaré–Bendixson Theorem were obtained after World War II. The Poincaré–Bendixson Theorem gave the beginning to other investigations and generalizations; all of them could now form a large collection. All those results could probably be a topic for a large mathematical monograph. Here, we mention rather briefly only some of those generalizations.

One of the advantages of the Poincaré–Bendixson Theorem was a precise description of planar systems in a neighbourhood of a periodic trajectory or a critical point, Theorem 4.2 and Theorem 4.3. This suggests a possible generalization and a question about the behaviour of the system in the neighbourhood of a compact invariant set, not necessarily for 2-dimensional systems. This was obtained (in the general case of flows) in 1960 by Ura and Kimura [77] and later developed by Saito in 1968 [64]. The Ura–Kimura Theorem is very important for several other applications.

Theorem 9.1.

Let $(X, \mathbb{R}, \pi)$ be a dynamical system on a locally compact space and $M \subset X$ be an isolated compact invariant set. Then there exists an $x \notin M$ such that either $\omega(x) \subset M$ or $\alpha(x) \subset M$.

Further generalization of this theorem, giving a more precise description, was obtained by Bhatia, see [12, 15].

Another generalization of the Poincaré–Bendixson Theorem was given in 1988 by Athanassopoulos and Strantzalos [7]. This was a generalization following the results of Schwartz and the assertion of the Ura–Kimura Theorem simultaneously. They proved that for a flow on a 2-dimensional manifold a compact minimal stable set is as in the assertion of the theorem of Schwartz. Also, they proved the assertion of the Poincaré–Bendixson Theorem for any compact minimal saddle set in such flows.

In 1996 Athanassopoulos proved in [5] that the assertion of the Poincaré–Bendixson Theorem for flows on the 2-dimensional sphere S^2 holds for actually larger class than compact limit sets. He proved the following theorems.

Theorem 9.2.

Let M be a 1-dimensional invariant chain recurrent continuum in a flow $(S^2, \mathbb{R}, \pi)$. If M contains a periodic orbit, then M is equal to this periodic orbit.

Theorem 9.3.

Let M be a 1-dimensional invariant chain recurrent continuum in a flow $(S^2, \mathbb{R}, \pi)$. If M contains no critical point, then M is a periodic orbit.

Some other results related in a very interesting way to chain recurrence, sections and the Poincaré–Bendixson type properties were proved by Hirsh and Pugh in 1988 [36]. Other properties, also grown from recurrence and having some connection to the Poincaré–Bendixson Theorem, were established in 1970 by Markley [47].

The results of Schwartz had an influence on many directions of research. In 1972 Sacker and Sell [63] gave the conditions under which there exists a nontrivial (i.e. not a singleton) minimal set in a flow on an orientable 2-manifold generated by a C^1 vector field. From this, they obtained some theorems on the existence of periodic orbits. This was continued by Neumann [53] who in 1978 proved a similar theorem on existence of periodic orbits in flows on orientable 2-manifolds (without the assumption on differentiation). One could wonder if those results would follow immediately from the differential case in view of later theorem of Gutiérrez (Theorem 7.4). However, several terms in the non-differentiable case were defined in [53] in a different way than in [63]. Recall that Gutiérrez in his proof (in 1986) used other results of Neumann, also published in 1978 [54]. A condition of yet another kind concerning probability measure, equivalent to the existence of a periodic orbit in a flow on the 2-dimensional torus was given by Athanassopoulos in [4]. The properties of the Poincaré–Bendixson type for flows on orientable 2-manifolds were also considered by Athanassopoulos, Petrescou and Strantzalos in 1997 [6]. They proved that only critical and periodic points are contained in their limit sets under the assumption of so-called D-stability.

The stability of periodic orbits and its connections with the Poincaré–Bendixson Theorem was also considered from another point of view, generally for strictly two-dimensional case. Some stability problems were investigated by Athanassopoulos in some of his papers mentioned above. Also, in 1979 Erle [26] proved a number of properties, particularly interesting from the point of view of mathematical models and applications.

The results for systems on non-orientable manifolds were also extended. In 2009 Demuner, Federson and Gutiérrez obtained [24] some results for vector fields on the Klein bottle $\mathbb{K}$. They introduced the notion of a *graph* in a flow which is, roughly speaking, the closed connected set with finite number of critical points and used it in the description of limit sets in Theorem 4.6. They proved that for certain kind of trajectories for flows on $\mathbb{K}$ with a finite number of critical points limit sets are either periodic orbits or graphs.

Solntzev and Vinograd presented a characterization of planar limit sets much more precise than in the classical Poincaré–Bendixson Theorem, see Section 6. Nevertheless, at several points those limit sets could be described more precisely, especially with respect of the relative orientation of trajectories in the limit set. This was investigated by Balibrea and Jiménez López in 1998 [8]. The authors gave, for the given collection of oriented curves and critical points, some necessary and sufficient conditions for the existence of a C^k map f such that this collection forms a limit set of a certain point in the system given by the equation $x' = f(x)$.

Later, Jiménez López and Soler López characterised limit sets for flows on the projective plane $\mathbb{P}^2$. In 2001 they proved [39] a theorem in some sense analogous to Theorem 6.4. They showed that any limit set in a flow on $\mathbb{P}^2$ is a boundary of a non-trivial region in $\mathbb{P}^2$ with connected complement. On the other hand, they proved that for each such a set there exists a C^∞ flow on $\mathbb{P}^2$ having this set as the limit set of one of its orbits. Results of similar character for closed surfaces were presented in 2006 in [40].

Another type of generalizations concerned the kind of systems. In 2004 Melin [52] showed that the assertion of the Poincaré–Bendixson Theorem holds under some assumptions for a planar autonomous equations with discontinuous right-hand side. In 2008 Bonotto and Federson proved [18] the Poincaré–Bendixson Theorem for planar impulsive semidynamical systems. Roughly speaking, impulsive systems are flows which admit some (suitably regular) "jumps"; in other words, they admit special discontinuities on trajectories. In [18] it was shown that under some additional assumption on impulses the assertion of the Poincaré–Bendixson Theorem is true.

Let us now turn to some particular kind of differential equations. In many cases it is possible to obtain the Poincaré–Bendixson-like results. We mention here one of particular importance. In 1956, Markus [49] presented a theorem of the Poincaré–Bendixson type for some kind of planar differential equations. He proved that under some assumptions on f and h, if some ω-limit set of the equation $x' = h(t, x)$ contains no critical points of the equation $x' = f(x)$, then it is the union of periodic trajectories of the second equation. Later on, in 1960 another version of this theorem was obtained by Opial [55]. These theorems had many interesting applications, cf. [75]. In 1992, Thieme [75] extended the Markus theorem to a more general case. Another analogue of the differentiable version of the Poincaré–Bendixson Theorem was proved by Filippov [28] in 1993. Using these results, Klebanov in 1997 [42] gave a precise description of orbits for some kind of planar equations and extended the results of Markus in a similar way to Thieme's theorems (the results of Klebanov and Thieme did not cover each other).

In 2006 Jiang and Liang [38] studied a model for n competing species with migration described by a system of autonomous ordinary differential equations. In particular, they showed that there a 4-dimensional system has 2-dimensional dynamics and as a consequence obtained the Poincaré–Bendixson Theorem for 2-species competitive systems with migration. In 2009 the Poincaré–Bendixson Theorem was used by Pokrovskii, Pokrovskiy and Zhezherun in [62] in proving the existence of a periodic solutions of a special kind, namely so-called periodic planar canards in perturbed systems. In 2011 Abate and Tovena [2] showed a type of Poincaré–Bendixson Theorem describing the recurrence properties and ω-limit sets of geodesics for a meromorphic connection on the complex projective one-dimensional space. In fact, their paper on holomorphic homogeneous vector fields can be regarded as a small monograph on the subject.

The classical Poincaré–Bendixson Theorem is strictly 2-dimensional. However, there are some generalizations of this theorem for higher dimensions or even infinite dimensional generalizations. In 1979 Hastings [35] proved a theorem for semiflows defined on 2-dimensional submanifolds of $\mathbb{R}^n$. Assuming that the semiorbit of a given point p is contained in a compact set A, he obtained some conditions of the style of Borsuk's shape theory for A. In the paper, there also were pointed out some differences between 2-dimensional cases and higher dimensional cases from the point of view of this theorem. Some ideas of the Poincaré–Bendixson Theorem were also transformed for some classes of n-dimensional equations by Smith in 1980 [71]. Continuing the research on the stability aspects of the Poincaré–Bendixson Theorem, in 1987 Smith [72] proved several results on orbital stability, extending the Poincaré–Bendixson Theorem in some way. The work of Sanchez of 2009 [65] was closely related with the research of Smith. Sanchez defined a class of systems that extend classical cooperative systems and proved that under some assumptions the limit set not containing critical point must be a periodic orbit.

For some type of scalar partial differential equations the results of this kind were obtained in 1989 by Fiedler and Mallet-Parret [27]. These results were followed in 1996, when the Poincaré–Bendixson type theorems for certain ordinary differential delay equations were proved by Mallet-Parret and Smith in [50]. They showed that in some autonomous delay monotone cyclic feedback systems, for a bounded trajectory of x either $\omega(x)$ is a periodic orbit or all α-limit sets and ω-limit sets contained in $\omega(x)$ consist of critical points. Some results for another type of infinite dimensional systems were obtained by Smith. In 1992, he proved that the assertion of the Poincaré–Bendixson Theorem holds (under several additional assumptions) for bounded positive semitrajectories in some autonomous retarded differential equations. This can be found in [73].

Let us mention also a generalization of Bendixson's work in another direction. Bendixson proved that the index of a critical point in a planar differential system is equal to $1 + (e - h)/2$, see Section 4. This was also a subject of further development. In particular, Izydorek, Rybicki and Szafraniec [37] gave in 1996 upper bounds for the numbers e and h in terms of indices of maps constructed explicitly from a vector field describing a flow. The reader is referred to [37] for details and more information about these aspects of the Poincaré–Bendixson theory.

The Poincaré–Bendixson Theorem has a lot of important applications, mainly for the problems connected with solving particular differential equations. However, some applications may be quite surprising. In 1966 Bhatia, Lazer and Leighton showed [14] that it is possible to prove the Brouwer Fixed Point Theorem (in the 2-dimensional case) as a corollary of the Poincaré–Bendixson Theorem.

Speaking about the Brouwer Fixed Point Theorem, let us mention that it can be used to prove the existence of a critical point inside a periodic trajectory in a planar system. In some sources this is proved with the use of advanced techniques and Bendixson's theory. However, it may be proved just as an application of the Brouwer Theorem in more general case of suitable invariant sets for flows, which can be found in [15]. This proof was simplified and adopted for semiflows in [21].

Finally, note that there are many different proofs of the Poincaré–Bendixson Theorem. Especially several proofs for flows differ noticeably from those mentioned earlier in the paper. A different approach to the problem and a proof without the use of sections was presented in [10]. In [1] there is another topological proof. Yet another proof with some original ideas was given in [70].

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