Chapter 11
The Poincaré–Bendixson Theorem

We have already seen how periodic solutions in planar dynamical systems play an important role in electrical circuit theory. In fact the periodic solution in Van der Pol's equation, coming from the simple circuit equation in the previous chapter, has features that go well beyond circuit theory. This periodic solution is a "limit cycle," a concept we make precise in this chapter.

The Poincaré–Bendixson theorem gives a criterion for the detection of limit cycles in the plane; this criterion could have been used to find the Van der Pol oscillator. On the other hand, this approach would have missed the uniqueness.

Poincaré–Bendixson is a basic tool for understanding planar dynamical systems but for differential equations in higher dimensions it has no generalization or counterpart. Thus after the first two rather basic sections, we restrict ourselves to planar dynamical systems. The first section gives some properties of the limiting behavior of orbits on the level of abstract topological dynamics while in the next section we analyze the flow near nonequilibrium points of a dynamical system.

Throughout this chapter we consider a dynamical system on an open set W in a vector space E, that is, the flow ϕ_t defined by a C^1 vector field $f: W \to E$.

§1. Limit Sets

We recall from Chapter 9, Section 3 that $y \in W$ is an *ω-limit point* of $x \in W$ if there is a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} \phi_{t_n}(x) = y$. The set of all ω-limit points of y is the *ω-limit set* $L_\omega(y)$. We define *α-limit points* and the *α-limit set* $L_\alpha(y)$ by replacing $t_n \to \infty$ with $t_n \to -\infty$ in the above definition. By a *limit set* we mean a set of the form $L_\omega(y)$ or $L_\alpha(y)$.

Here are some examples of limit sets. If $\bar{x}$ is an asymptotically stable equilibrium, it is the ω-limit set of every point in its basin (see Chapter 9, Section 2). Any

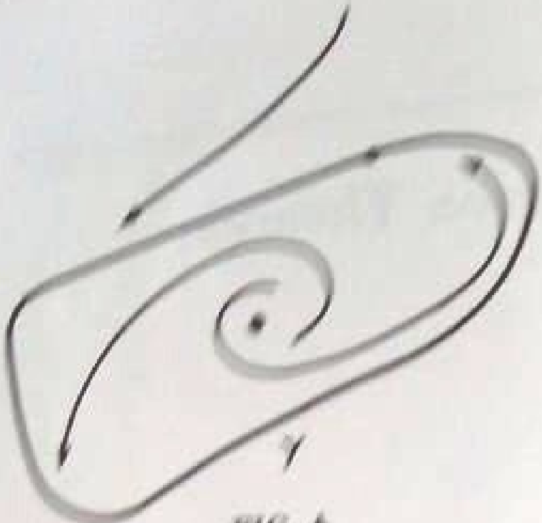

FIG. A

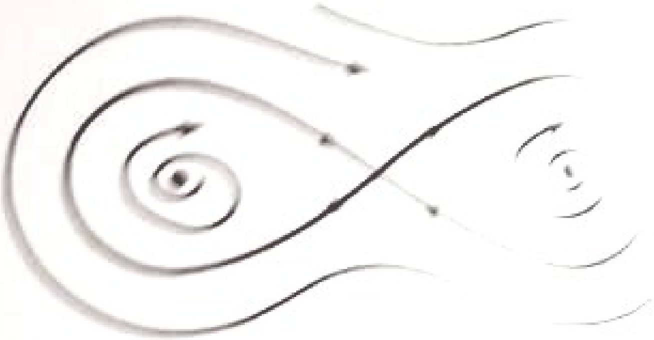

FIG. B

equilibrium is its own α-limit set and ω-limit set. A closed orbit is [illegible] ω-limit set of every point on it. In the Van der Pol oscillator there [illegible] orbit γ; it is the ω-limit of every point except the origin (Fig. A [illegible] α-limit set of every point inside γ. If y is outside γ, then $L_\alpha(y$ [illegible]

There are examples of limit sets that are neither closed orbits [illegible] example the figure 8 in the flow suggested by Fig. B. There are three [illegible] sources, and one saddle. The figure 8 is the ω-limit set of all points [illegible] right half of the 8 is the α-limit set of all points inside it except the [illegible] similarly for the left half.

In three dimensions there are extremely complicated examples [illegible] although they are not easy to describe. In the plane, however, limit [illegible] simple. In fact Fig. B is typical, in that one can show that a limit [illegible] closed orbit or equilibrium is made up of equilibria and trajectories [illegible] The Poincaré–Bendixson theorem says that if a compact limit set [illegible] contains no equilibria it is a closed orbit.

We recall from Chapter 9 that a limit set is closed in W, and is [illegible] the flow. We shall also need the following result:

Proposition (a) *If x and z are on the same trajectory, then $L_\omega(x$ [illegible] larly for α-limits.*

(b) *If D is a closed positively invariant set and $z \in D$, then $L_\omega(z$ [illegible] for negatively invariant sets and α-limits.*

(c) *A closed invariant set, in particular a limit set, contains the α-lim[illegible] sets of every point in it.*

Proof. (a) Suppose $y \in L_\omega(z)$, and $\phi_s(z) = z$. If $\phi_{t_n}(z) \to y$, then $\phi_{t_n - s}(z) \to y$. Hence $y \in L_\omega(z)$.

(b) If $t_n \to \infty$ and $\phi_{t_n}(z) \to y \in L_\omega(z)$, then $t_n \geq 0$ for sufficiently large n so that $\phi_{t_n}(z) \in D$. Hence $y \in \bar{D} = D$.

(c) Follows from (b).

PROBLEMS

1. Show that a compact limit set is connected (that is, not the union of two disjoint non-empty closed sets.

2. Identify $\mathbf{R}^4$ with $\mathbf{C}^2$ having two complex coordinates (w, z), and consider the linear system

$$(*) \qquad w' = 2\pi i w,$$

$$z' = 2\pi\theta i z,$$

where θ is an *irrational* real number.

(a) Put $\alpha = e^{2\pi\theta i}$ and show that the set $\{\alpha^n \mid n = 1, 2, \ldots\}$ is dense in the unit circle $C = \{z \in \mathbf{C} \mid |z| = 1\}$.

(b) Let ϕ_t be the flow of $(*)$. Show that for n an integer,

$$\phi_n(w, z) = (w, \alpha^n z).$$

(c) Let (w_0, z_0) belong to the torus $C \times C \subset \mathbf{C}^2$. Use (a), (b) to show that

$$L_\omega(w_0, z_0) = L_\alpha(w_0, z_0) = C \times C.$$

(d) Find L_ω and L_α of an arbitrary point of $\mathbf{C}^2$.

3. Find a linear system on $\mathbf{R}^{2k} = \mathbf{C}^k$ such that if a belongs to the k-torus $C \times \cdots \times C \subset \mathbf{C}^k$, then

$$L_\omega(a) = L_\alpha(a) = C^k.$$

4. In Problem 2, suppose instead that θ is *rational*. Identify L_ω and L_α of every point.

5. Let X be a nonempty compact invariant set for a C^1 dynamical system. Suppose that X is *minimal*, that is, X contains no compact invariant nonempty proper subset. Prove the following:

(a) Every trajectory in X is dense in X;

(b) $L_\alpha(x) = L_\omega(x) = X$ for each $x \in X$;

(c) For any (relatively) open set $U \subset X$, there is a number $P > 0$ such that for any $x \in X$, $t_0 \in \mathbf{R}$, there exists t such that $\phi_t(x) \in U$ and $|t - t_0| < P$;

(d) For any x, y in X there are sequences $t_n \to \infty$, $s_n \to -\infty$ such that

$$|t_n - t_{n+1}| < 2P, \qquad |s_n - s_{n+1}| < 2P,$$

and

$$\phi_{t_n}(x) \to y, \qquad \phi_{s_n}(x) \to y.$$

6. Let X be a closed invariant set for a C^1 dynamical system on $\mathbf{R}^n$, such that $\phi_t(x)$ is defined for all $t \in \mathbf{R}$, $x \in X$. Suppose that $L_\alpha(x) = L_\omega(x) = X$ for all $x \in X$. Prove that X is compact.

§2. Local Sections and Flow Boxes

We consider again the flow ϕ_t of the C^1 vector field $f: W \to E$. Suppose th… 0 $\in E$ belongs to W.

A *local section* at 0 of f is an open set S containing 0 in a hyperplane $H \subset E$ … is transverse to f. By a *hyperplane* we mean a linear subspace whose dim… is one less than dim E. To say that $S \subset H$ is *transverse* to f means that … for all $x \in S$. In particular $f(x) \neq 0$ for $x \in S$.

Our first use of a local section at 0 will be to construct a "flow box" in a neig… hood of 0. A flow box gives a complete description of a flow in a neighbor… any nonequilibrium point of any flow, by means of special (nonlinear) coord… The description is simple: points move in parallel straight lines at constant …

We make this precise as follows. A *diffeomorphism* $\Psi: U \to V$ is a differ… map from one open set of a vector space to another with a differentiable … A *flow box* is a diffeomorphism

$$\mathbf{R} \times H \supset N \xrightarrow{\Psi} W$$

of a neighborhood N of $(0, 0)$ onto a neighborhood of 0 in W, which trans… the vector field $f: W \to E$ into the constant vector field $(1, 0)$ on $\mathbf{R} \times H$. The … of f is thereby converted to a simple flow on $\mathbf{R} \times H$:

$$\psi_s(t, y) = (t + s, y).$$

The map Ψ is defined by

$$\Psi(t, y) = \phi_t(y),$$

for (t, y) in a sufficiently small neighborhood of $(0, 0)$ in $\mathbf{R} \times H$. One ap… Chapter 15 to see that Ψ is a C^1 map. The derivative of Ψ at $(0, 0)$ is easily … to be the linear map which is the identity on $0 \times H$, and on $\mathbf{R} = \mathbf{R} \times 0$ it … 1 to $f(0)$. Since $f(0)$ is transverse to H, it follows that $D\Psi(0, 0)$ is an isom… Hence by the inverse function theorem Ψ maps an open neighborhood N of … diffeomorphically onto a neighborhood V of 0 in E. We take N of th…

FIG. A. The flow box.

$S \times (-\sigma, \sigma)$, where $S \subset H$ is a section at 0 and $\sigma > 0$. In this case we sometimes write $V_\sigma = \Psi(N)$ and call V_σ a *flow box* at (or about) 0 in E. See Fig. A. An important property of a flow box is that if $x \in V_\sigma$, then $\phi_t(x) \in S$ for a unique $t \in (-\sigma, \sigma)$.

From the definition of Ψ it follows that if $\Psi^{-1}(p) = (s, y)$, then $\Psi^{-1}(\phi_t(p)) = (s + t, y)$ for sufficiently small $|s|, |t|$.

We remark that a flow box can be defined about any nonequilibrium point x_0. The assumption that $x_0 = 0$ is no real restriction since if x_0 is any point, one can replace $f(x)$ by $f(x - x_0)$ to convert the point to 0.

If S is a local section, the trajectory through a point z_0 (perhaps far from S) may reach $0 \in S$ in a certain time t_0; see Fig. B. We show that in a certain local sense, t_0 is a continuous function of z_0. More precisely:

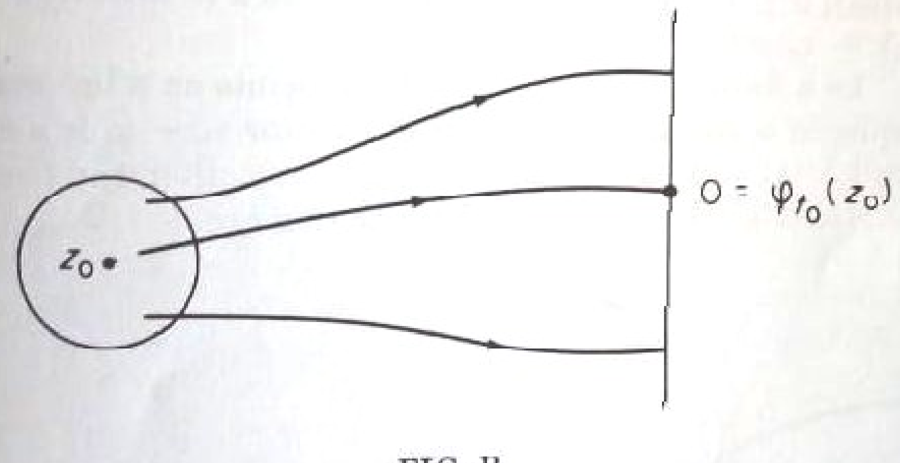

FIG. B

Proposition *Let S be a local section at 0 as above, and suppose $\phi_{t_0}(z_0) = 0$. There is an open set $U \subset W$ containing z_0 and a unique C^1 map $\tau: U \to \mathbf{R}$ such that $\tau(z_0) = t_0$ and*

$$\phi_{\tau(x)}(x) \in S$$

for all $x \in U$.

Proof. Let $h: E \to \mathbf{R}$ be a linear map whose kernel H is the hyperplane containing S. Then $h(f(0)) \neq 0$. The function

$$G(x, t) = h\phi_t(x)$$

is C^1, and

$$\frac{\partial G}{\partial t}(z_0, t_0) = h(f(0)) \neq 0.$$

By the implicit function theorem there is a unique C^1 map $x \to \tau(x) \in \mathbf{R}$ defined on a neighborhood U_1 of z_0 in W such that $\tau(z_0) = t_0$ and $G(x, \tau(x)) \equiv 0$. Hence $\phi_{\tau(x)}(x) \in H$; if $U \subset U_1$ is a sufficiently small neighborhood of z_0 then $\phi_{\tau(x)}(x) \in S$. This proves the proposition.

For later reference note that

$$D\tau(z_0) = -\left[\frac{\partial G}{\partial t}(z_0, t_0)\right]^{-1} \frac{\partial G}{\partial x}(z_0, t_0) = -\left[\frac{\partial G}{\partial t}(z_0, t_0)\right]^{-1} \circ h \circ D\phi_{t_0}(z_0).$$

§3. Monotone Sequences in Planar Dynamical Systems

We now restrict our discussion to planar dynamical systems.

Let $x_0, x_1, \ldots$ be a finite or infinite sequence of distinct points on the solution curve $C = \{\phi_t(x_0) \mid 0 \leq t \leq \alpha\}$. We say the sequence is *monotone along the trajectory* if $\phi_{t_n}(x_0) = x_n$ with $0 \leq t_1 < \cdots \leq \alpha$.

Let $y_0, y_1, \ldots$ be a finite or infinite sequence of points on a line segment I in $\mathbf{R}^2$. We say the sequence is *monotone along I* if the vector $y_n - y_0$ is a scalar multiple $\lambda_n(y_1 - y_0)$ with $1 < \lambda_2 < \lambda_3 < \ldots$ $n = 2, 3, \ldots$. Another way to say this is that y_n is between y_{n-1} and y_{n+1} in the natural order along I, $n = 1, 2, \ldots$.

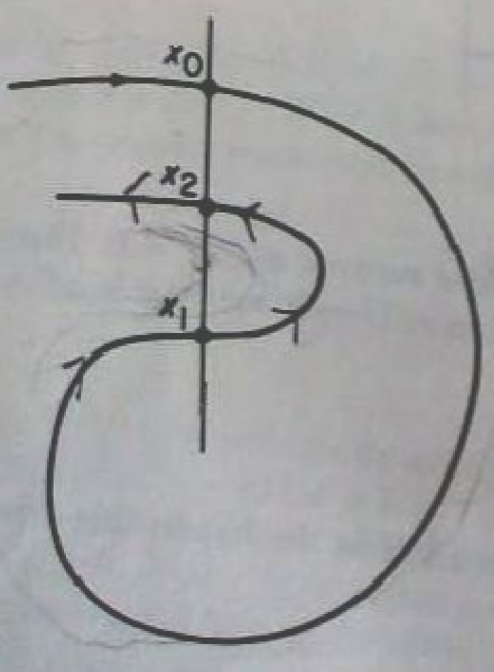

FIG. A

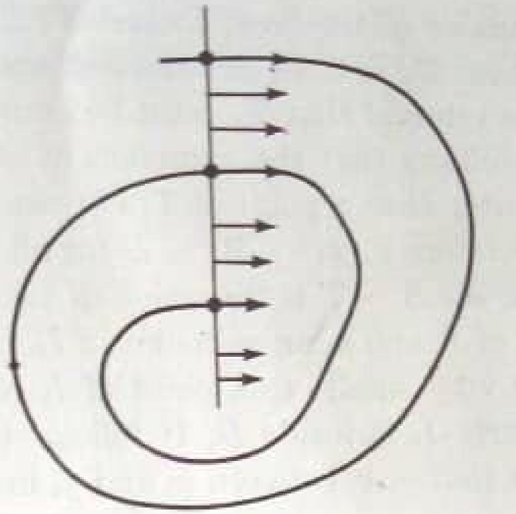

FIG. B

A sequence of points may be on the intersection of a solution curve and a segment I; they may be monotone along the solution curve but not along the segment, or vice versa; see Fig. A. However, *this is impossible if the segment is a local section.* Figure B shows an example; we suggest the reader experiment with paper and pencil!

Proposition 1 *Let S be a local section of a C^1 planar dynamical system and y_0, y_1, y_2, ... a sequence of distinct points of S that are on the same solution curve C. If the sequence is monotone along C, it is also monotone along S.*

Proof. It suffices to consider three points y_0, y_1, y_2. Let Σ be the simple closed curve made up of the part B of C between y_0 and y_1 and the segment $T \subset S$ between y_0 and y_1. Let D be the closed bounded region bounded by Σ. We suppose that the trajectory of y_1 *leaves* D at y_1 (Fig. C); if it enters, the argument is similar.

We assert that at any point of T the trajectory leaves D. For it either leaves or enters because, T being transverse to the flow, it crosses the boundary of D. The set of points in T whose trajectory leaves D is a nonempty open subset $T_- \subset T$, by

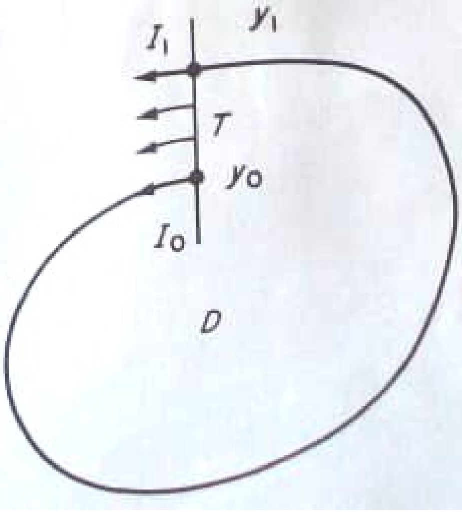

FIG. C

continuity of the flow; the set $T_+ \subset T$ where trajectories enter D is also open in T. Since T_- and T_+ are disjoint and $T = T_- \cup T_+$, it follows from connectedness of the interval that T_+ must be empty.

It follows that the complement of D is positively invariant. For no trajectory can enter D at a point of T; nor can it cross B, by uniqueness of solutions.

Therefore $\phi_t(y_1) \in \mathbf{R}^2 - D$ for all $t > 0$. In particular, $y_2 \in S - T$.

The set $S - T$ is the union of two half open intervals I_0 and I_1 with y_0 an endpoint of I_0 and y_1 an endpoint of I_1. One can draw an arc from a point $\phi_\epsilon(y_1)$ (with $\epsilon > 0$ very small) to a point of I_1, without crossing Σ. Therefore I_1 is outside D. Similarly I_0 is inside D. It follows that $y_2 \in I_1$ since it must be outside D. This shows that y_1 is between y_0 and y_2 in I, proving Proposition 1.

We come to an important property of limit points.

Proposition 2 *Let $y \in L_\omega(x) \cup L_\alpha(x)$. Then the trajectory of y crosses any local section at not more than one point.*

Proof. Suppose y_1 and y_2 are distinct points on the trajectory of y and S is a local section containing y_1 and y_2. Suppose $y \in L_\omega(x)$ (the argument for $L_\alpha(x)$ is similar). Then $y_k \in L_\omega(x)$, $k = 1, 2$. Let $V_{(k)}$ be flow boxes at y_k defined by some intervals $J_k \subset S$; we assume J_1 and J_2 disjoint (Fig. D). The trajectory of x enters $V_{(k)}$ infinitely often; hence it crosses J_k infinitely often. Hence there is a sequence

$$a_1, b_1, a_2, b_2, a_3, b_3, \ldots,$$

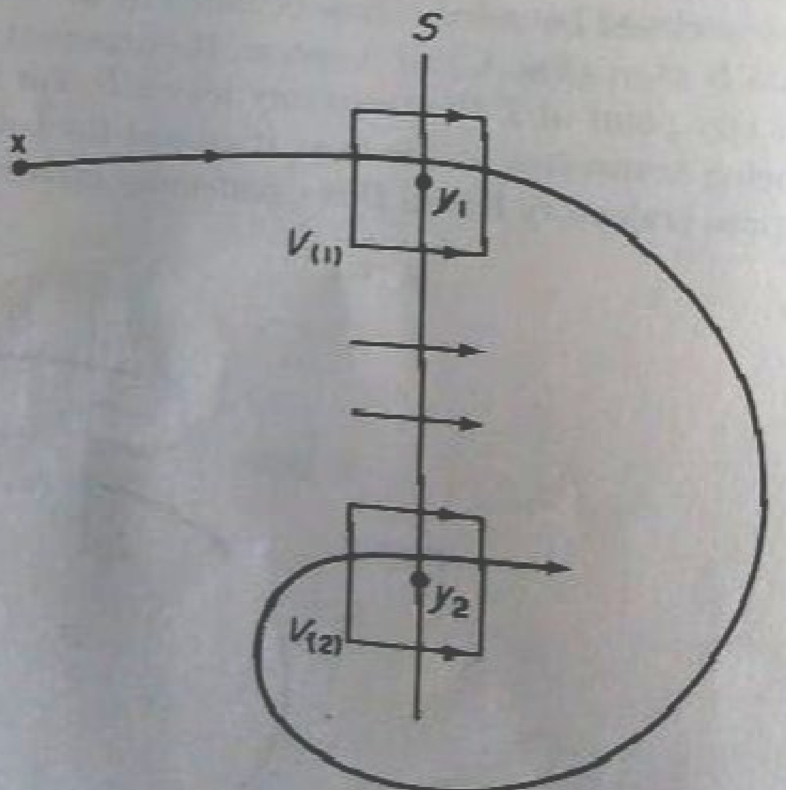

FIG. D

which is monotone along the trajectory of x, with $a_n \in J_1$, $b_n \in J_2$, $n = 1, 2, \ldots$. But such a sequence cannot be monotone along S since J_1 and J_2 are disjoint, contradicting Proposition 1.

PROBLEMS

1. Let $A \subset \mathbf{R}^2$ be the annulus

$$A = \{z \in \mathbf{R}^2 \mid 1 \leq |z| \leq 2\}.$$

Let f be a C^1 vector field on a neighborhood of A which points inward along the two boundary circles of A. Suppose also that every radial segment of A is local section (Fig. E). Prove there is a periodic trajectory in A.

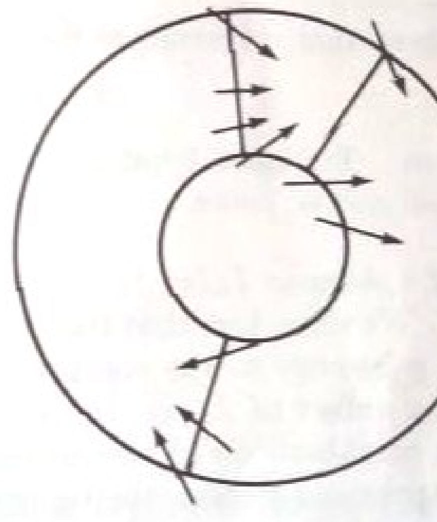

FIG. E

(*Hint:* Let S be a radial segment. Show that if $z \in S$ then $\phi_t(z) \in S$ for a smallest $t = t(z) > 0$. Consider the map $S \to S$ given by $z \mapsto \phi_{t(z)}(z)$.)

2. Show that a closed orbit of a planar C^1 dynamical system meets a local section in at most one point.

3. Let $W \subset \mathbf{R}^2$ be open and let $f\colon W \to \mathbf{R}^2$ be a C^1 vector field with no equilibria. Let $J \subset W$ be an open line segment whose end points are in the boundary of W. Suppose J is a *global section* in the sense that f is transverse to J, and for any $x \in W$ there exists $s < 0$ and $t > 0$ such that $\phi_s(x) \in J$ and $\phi_t(x) \in J$. Prove the following statements.
   (a) For any $x \in J$ let $\tau(x) \in \mathbf{R}$ be the smallest positive number such that $F(x) = \phi_{\tau(x)} \in J$; this map $F\colon J \to J$ is C^1 and has a C^1 inverse.
   (b) A point $x \in J$ lies on a closed orbit if and only if $F(x) = x$.
   (c) Every limit set is a closed orbit.

4. Let x be a *recurrent* point of a C^1 planar dynamical system, that is, there is a sequence $t_n \to \pm\infty$ such that

$$\phi_{t_n}(x) \to x.$$

(a) Prove that either x is an equilibrium or x lies on a closed orbit.
(b) Show by example that there can be a recurrent point for higher dimensional systems that is not an equilibrium and does not lie on a closed orbit.

§4. The Poincaré–Bendixson Theorem

By a *closed orbit* of a dynamical system we mean the image of a nontrivial periodic solution. Thus a trajectory γ is a closed orbit if γ is not an equilibrium and $\phi_p(x) = x$ for some $x \in \gamma$, $p \neq 0$. It follows that $\phi_{np}(y) = y$ for all $y \in \gamma$, $n = 0, \pm 1, \pm 2, \ldots$.

In this section we complete the proof of a celebrated result:

Theorem (Poincaré–Bendixson) *A nonempty compact limit set of a C^1 planar dynamical system, which contains no equilibrium point, is a closed orbit.*

Proof. Assume $L_\omega(x)$ is compact and $y \in L_\omega(x)$. (The case of α-limit sets is similar.) We show first that the trajectory of y is a closed orbit.

Since y belongs to the compact invariant set $L_\omega(x)$ we know that $L_\omega(y)$ is a nonempty subset of $L_\omega(x)$. Let $z \in L_\omega(y)$; let S be a local section at z, and V a flow box neighborhood of z about some open interval J, $z \in J \subset S$. By Proposition 2 of the previous section, the trajectory of y meets S at exactly one point. On the other hand, there is a sequence $t_n \to \infty$ such that $\phi_{t_n}(y) \to z$; hence infinitely many $\phi_{t_n}(y)$ belong to V. Therefore we can find $r, s \in \mathbf{R}$ such that $r > s$ and

$$\phi_r(y) \in S \cap V, \qquad \phi_s(y) \in S \cap V.$$

It follows that $\phi_r(y) = \phi_s(y)$; hence $\phi_{r-s}(y) = y$, $r - s > 0$. Since $L_\omega(x)$ contains no equilibrium, y belongs to closed orbit.

It remains to prove that if γ is a closed orbit in $L_\omega(x)$ then $\gamma = L_\omega(x)$. It is enough to show that

$$\lim_{t\to\infty} d(\phi_t(x), \gamma) = 0,$$

where $d(\phi_t(x), \gamma)$ is the distance from x to the compact set γ (that is, the distance from $\phi_t(x)$ to the nearest point of γ).

Let S be a local section at $z \in \gamma$, so small that $S \cap \gamma = z$. By looking at a flow box V_ϵ near z we see that there is a sequence $t_0 < t_1 < \cdots$ such that

$$\phi_{t_n}(x) \in S,$$

$$\phi_{t_n}(x) \to z,$$

$$\phi_t(x) \notin S \quad \text{for} \quad t_{n-1} < t < t_n, \quad n = 1, 2, \ldots .$$

Put $x_n = \phi_{t_n}(x)$. By Proposition 1, Section 3, $x_n \to z$ monotonically in S.

There exists an upper bound for the set of positive numbers $t_{n+1} - t_n$. For suppose $\phi_\lambda(z) = z$, $\lambda > 0$. Then for x_n sufficiently near z, $\phi_\lambda(x_n) \in V_\epsilon$ and hence

$$\phi_{\lambda+t}(x_n) \in S$$

for some $t \in [-\epsilon, \epsilon]$. Thus

$$t_{n+1} - t_n \leq \lambda + \epsilon.$$

Let $\beta > 0$. From Chapter 8, there exists $\delta > 0$ such that if $| x_n - u | < \delta$ and $|t| \leq \lambda + \epsilon$ then $| \phi_t(x_n) - \phi_t(u)| < \beta$.

Let n_0 be so large that $| x_n - z | < \delta$ for all $n \geq n_0$. Then

$$| \phi_t(x_n) - \phi_t(z) | < \beta$$

if $|t| \leq \lambda + \epsilon$ and $n \geq n_0$. Now let $t \geq t_{n_0}$. Let $n \geq n_0$ be such that

$$t_n \leq t \leq t_{n+1}.$$

Then

$$\begin{aligned} d(\phi_t(x), \gamma) &\leq | \phi_t(x) - \phi_{t-t_n}(z)| \\ &= | \phi_{t-t_n}(x_n) - \phi_{t-t_n}(z)| \\ &< \beta \end{aligned}$$

since $| t - t_n | \leq \lambda + \epsilon$. The proof of the Poincaré–Bendixson theorem is complete.

PROBLEMS

1. Consider a C^1 dynamical system in $\mathbf{R}^2$ having only a finite number of equilibria.
   (a) Show that every limit set is either a closed orbit or the union of equilibria and trajectories $\phi_t(x)$ such that $\lim_{t\to\infty} \phi_t(x)$ and $\lim_{t\to-\infty} \phi_t(x)$ are equilibria.
   (b) Show by example (draw a picture) that the number of distinct trajectories in $L_\omega(x)$ may be infinite.
2. Let γ be a closed orbit of a C^1 dynamical system on an open set in $\mathbf{R}^2$. Let λ be the period of γ. Let $\{\gamma_n\}$ be a sequence of closed orbits; suppose the period

Special case: Suppose $\omega(x_0)$ contains a unique equilibrium p.
Then either $\omega(x_0) = \{p\}$ or $\omega(x_0) = \{p\} \cup$ homoclinic.
Pf.
Let $q \in \omega(x_0)$ ... Then $p \in \omega(q)$. Suppose $\exists\, + \neq p$, in $\omega(q)$.


local sect.
isolating

of γ_n is λ_n. If there are points $x_n \in \gamma_n$ such that $x_n \to x \in \gamma$, prove that $\lambda_n \to$
(This result can be false for higher dimensional systems. It is true, however, that
if $\lambda_n \to \mu$, then μ is an integer multiple of λ.)

§5. Applications of the Poincaré-Bendixson Theorem

We continue to suppose given a planar dynamical system.

Definition A *limit cycle* is a closed orbit γ such that $\gamma \subset L_\omega(x)$ or $\gamma \subset$
for some $x \notin \gamma$. In the first case γ is called an ω-limit cycle; in the second case
α-limit cycle.

In the proof of the Poincaré-Bendixson theorem it was shown that limit
enjoy a certain property not shared by other closed orbits: if γ is an ω-limit
there exists $x \notin \gamma$ such that

$$\lim_{t\to\infty} d(\phi_t(x), \gamma) = 0.$$

For an α-limit cycle replace ∞ by $-\infty$. Geometrically this means that some
jectory spirals toward γ as $t \to \infty$ (for ω-limit cycles) or as $t \to -\infty$
cycles). See Fig. A.

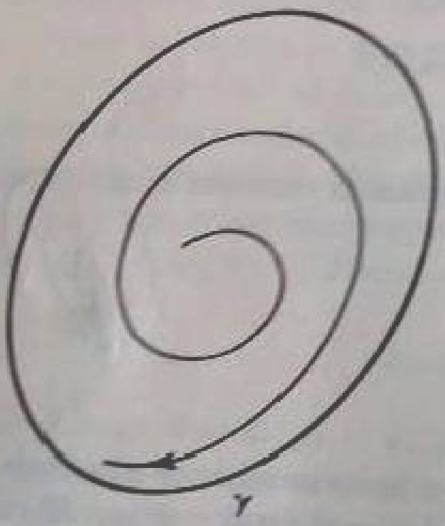

FIG. A. γ is an ω-limit cycle.

Limit cycles possess a kind of one-sided stability. Suppose γ is an ω-limit
and let $\phi_t(x)$ spiral toward γ as $t \to \infty$. Let S be a local section at $z \in \gamma$. The
will be an interval $T \subset S$ disjoint from γ bounded by $\phi_{t_0}(x)$, $\phi_{t_1}(x)$, with
and not meeting the trajectory of x for $t_0 < t < t_1$ (Fig. B). The region A bou
by γ, T and the curve

$$\{\phi_t(x) \mid t_0 \le t \le t_1\}$$

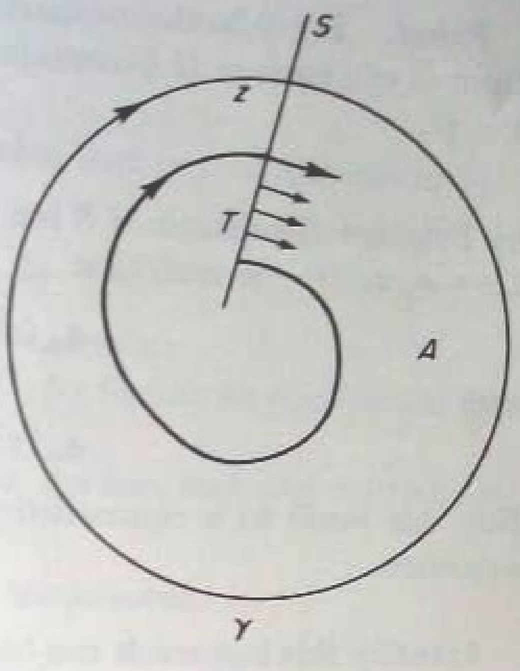

FIG. B

is positively invariant, as is the set $B = A - \gamma$. It is easy to see that $\phi_t(y)$ spirals toward γ for all $y \in B$. A useful consequence of this is

Proposition 1 *Let γ be an ω-limit cycle. If $\gamma = L_\omega(x)$, $x \notin \gamma$ then x has a neighborhood V such that $\gamma = L_\omega(y)$ for all $y \in V$. In other words, the set*

$$A = \{y \mid \gamma = L_\omega(y)\} - \gamma$$

is open.

Proof. For sufficiently large $t > 0$, $\phi_t(x)$ is in the interior of the set A described above. Hence $\phi_t(y) \in A$ for y sufficiently close to x. This implies the proposition.

A similar result holds for α-limit cycles.

Theorem 1 *A nonempty compact set K that is positively or negatively invariant contains either a limit cycle or an equilibrium.*

Proof. Suppose for example that K is positively invariant. If $x \in K$, then $L_\omega(x)$ is a nonempty subset of K; apply Poincaré–Bendixson.

The next result exploits the spiraling property of limit cycles.

Proposition 2 *Let γ be a closed orbit and suppose that the domain W of the dynamical system includes the whole open region U enclosed by γ. Then U contains either an equilibrium or a limit cycle.*

Proof. Let D be the compact set $U \cup \gamma$. Then D is invariant since no trajectory from U can cross γ. If U contains no limit cycle and no equilibrium, then, for any $x \in U$,

$$L_\omega(x) = L_\alpha(x) = \gamma$$

by Poincaré–Bendixson. If S is a local section at a point $z \in \gamma$, there are sequences $t_n \to \infty$, $s_n \to -\infty$ such that

$$\phi_{t_n}(x) \in S, \qquad \phi_{t_n}(x) \to z,$$

and

$$\phi_{s_n}(x) \in S, \qquad \phi_{s_n}(x) \to z.$$

But this leads to a contradiction of the proposition in Section 3 on monotone sequences.

Actually this last result can be considerably sharpened:

Theorem 2 *Let γ be a closed orbit enclosing an open set U contained in the domain W of the dynamical system. Then U contains an equilibrium.*

Proof. Suppose U contains no equilibrium. If $x_n \to x$ in U and each x_n lies on a closed orbit, then x must lie on a closed orbit. For otherwise the trajectory of x would spiral toward a limit cycle, and by Proposition 1 so would the trajectory of some x_n.

Let $A \geq 0$ be the greatest lower bound of the areas of regions enclosed by closed orbits in U. Let $\{\gamma_n\}$ be a sequence of closed orbits enclosing regions of areas A_n such that $\lim_{n\to\infty} A_n = A$. Let $x_n \in \gamma_n$. Since $\gamma \cup U$ is compact we may assume $x_n \to x \in U$. Then if U contains no equilibrium, x lies on a closed orbit β of area $A(\beta)$. The usual section argument shows that as $n \to \infty$, γ_n gets arbitrarily close to β and hence the area $A_n - A(\beta)$, of the region between γ_n and β, goes to 0. Thus $A(\beta) = A$.

We have shown that if U contains no equilibrium, it contains a closed orbit β enclosing a region of minimal area. Then the region enclosed by β contains neither an equilibrium nor a closed orbit, contradicting Proposition 2.

The following result uses the spiraling properties of limit cycles in a subtle way.

Theorem 3 *Let H be a first integral of a planar C^1 dynamical system (that is, H is a real-valued function that is constant on trajectories). If H is not constant on any open set, then there are no limit cycles.*

Proof. Suppose there is a limit cycle γ; let $c \in \mathbf{R}$ be the constant value of H on γ. If $x(t)$ is a trajectory that spirals toward γ, then $H(x(t)) \equiv c$ by continuity of H. In Proposition 1 we found an open set whose trajectories spiral toward γ; thus H is constant on an open set.

PROBLEMS

1. The celebrated *Brouwer fixed point theorem* states that any continuous map f of the closed unit ball

$$D^n = \{x \in \mathbf{R}^n \mid |x| \le 1\}$$

into itself has a fixed point (that is, $f(x) = x$ for some x).

(a) Prove this for $n = 2$, assuming that f is C^1, by finding an equilibrium for the vector field $g(x) = f(x) - x$.

(b) Prove Brouwer's theorem for $n = 2$ using the fact that any continuous map is the uniform limit of C^1 maps.

2. Let f be a C^1 vector field on a neighborhood of the annulus

$$A = \{x \in \mathbf{R}^2 \mid 1 \le |x| \le 2\}.$$

Suppose that f has no zeros and that f is transverse to the boundary, pointing inward.

(a) Prove there is a closed orbit. (Notice that the hypothesis is weaker than in Problem 1, Section 3.)

(b) If there are exactly seven closed orbits, show that one of them has orbits spiraling toward it from both sides.

3. Let $f: \mathbf{R}^2 \to \mathbf{R}^2$ be a C^1 vector field with no zeros. Suppose the flow ϕ_t generated by f preserves area (that is, if S is any open set, the area of $\phi_t(S)$ is independent of t). Show that every trajectory is a closed set.

4. Let f be a C^1 vector field on a neighborhood of the annulus A of Problem 2. Suppose that for every boundary point x, $f(x)$ is a nonzero vector tangent to the boundary.

(a) Sketch the possible phase portraits in A under the further assumption that there are no equilibria and no closed orbits besides the boundary circles. Include the case where the boundary trajectories have opposite orientations.

(b) Suppose the boundary trajectories are oppositely oriented and that the flow preserves area. Show that A contains an equilibrium.

5. Let f and g be C^1 vector fields on $\mathbf{R}^2$ such that $\langle f(x), g(x) \rangle = 0$ for all x. If f has a closed orbit, prove that g has a zero.

6. Let f be a C^1 vector field on an open set $W \subset \mathbf{R}^2$ and $H: W \to \mathbf{R}$ a C^1 function such that

$$DH(x)f(x) = 0$$

for all x. Prove that:

(a) H is constant on solution curves of $x' = f(x)$;

(b) $DH(x) = 0$ if x belongs to a limit cycle;

(c) If x belongs to a compact invariant set on which DH is never 0, the... lies on a closed orbit.

Notes

P. Hartman's *Ordinary Differential Equations* [9], a good but advan... covers extensively the material in this chapter.

It should be noted that our discussion implicitly used the fact that a clos... in $\mathbf{R}^2$ which does not intersect itself must separate $\mathbf{R}^2$ into two connected re... bounded one and an unbounded one. This theorem, the Jordan curve theore... naïvely obvious, needs mathematical proof. One can be found in Newman's *T... of Plane Sets* [17].