if Tr $A = 0$, and that in this case the origin is not a sink or a source. (*Hint*: An operator is area-preserving if and only if the determinant is ± 1.)

12. Describe in words the phase portraits of $x' = Ax$ for

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (d) $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

13. Suppose A is an $n \times n$ matrix with n distinct eigenvalues and the real part of every eigenvalue is less than some negative number α. Show that for every solution to $x' = Ax$, there exists $t_0 > 0$ such that

$$|x(t)| < e^{t\alpha} \quad \text{if} \quad t \geq t_0.$$

14. Let T be an invertible operator on $\mathbf{R}^n$, n odd. Then $x' = Tx$ has a nonperiodic solution.

15. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ have nonreal eigenvalues. Then $b \neq 0$. The nontrivial solutions curves to $x' = Ax$ are spirals or ellipses that are oriented clockwise if $b > 0$ and counterclockwise if $b < 0$. (*Hint*: Consider the sign of

$$\frac{d}{dt} \arctan(x_2(t)/x_1(t)).)$$

§5. A Nonhomogeneous Equation

We consider a nonhomogeneous nonautonomous linear differential equation

$$x' = Ax + B(t). \tag{1}$$

Here A is an operator on $\mathbf{R}^n$ and $B: \mathbf{R} \to \mathbf{R}^n$ is a continuous map. This equation is called *nonhomogeneous* because of the term $B(t)$ which prevents (1) from being strictly linear; the fact that the right side of (1) depends explicitly on t makes it *nonautonomous*. It is difficult to interpret solutions geometrically.

We look for a solution having the form

$$x(t) = e^{tA}f(t), \tag{2}$$

where $f: \mathbf{R} \to \mathbf{R}^n$ is some differentiable curve. (This method of solution is called "variation of constants," perhaps because if $B(t) \equiv 0$, $f(t)$ is a constant.) Every solution can in fact be written in this form since e^{tA} is invertible.

Differentiation of (2) using the Leibniz rule yields

$$x'(t) = Ae^{tA}f(t) + e^{tA}f'(t).$$

Since x is assumed to be a solution of (2),

$$Ax(t) + B(t) = Ax(t) + e^{tA}f'(t)$$

or

$$f'(t) = e^{-tA}B(t).$$

By integration

$$f(t) = \int_0^t e^{-As}B(s)\,ds + K,$$

so as a candidate for a solution of (1) we have

$$x(t) = e^{At}\left[\int_0^t e^{-As}B(s)\,ds + K\right], \qquad K \in \mathbf{R}^n. \tag{3}$$

Let us examine (3) to see that it indeed makes sense. The integrand in (3) and the previous equation is the vector-valued function $s \to e^{-As}B(s)$ mapping R into $\mathbf{R}^n$. In fact, for any continuous map g of the reals into a vector space $\mathbf{R}^n$, the integral can be defined as an element of $\mathbf{R}^n$. Given a basis of $\mathbf{R}^n$, this integral is a vector whose coordinates are the integrals of the coordinate functions of g.

The integral as a function of its upper limit t is a map from $\mathbf{R}$ into $\mathbf{R}^n$. For each t the operator acts on the integral to give an element of $\mathbf{R}^n$. So $t \to x(t)$ is a well-defined map from R into E.

To check that (3) is a solution of (1), we differentiate $x(t)$ in (3):

$$x'(t) = B(t) + Ae^{At}\left[\int_0^t e^{-As}B(s)\,ds + K\right]$$

$$= B(t) + Ax(t).$$

Thus (3) is indeed a solution of (1).

That every solution of (1) must be of the form (3) can be seen as follows. Let $y:\mathbf{R}^n \to E$ be a second solution of (1). Then

$$x' - y' = A(x - y)$$

so that from Section 1

$$x - y = e^{tA}K_0 \qquad \text{for some} \quad K_0 \text{ in } \mathbf{R}^n.$$

This implies that y is of the form (3) (with perhaps a different constant $K \in \mathbf{R}^n$).

We remark that if B in (1) is only defined on some interval, instead of on all of $\mathbf{R}$, then by the above methods, we obtain a solution $x(t)$ defined for t in that same interval.

We obtain further insight into (1) by rewriting the general solution (3) in the form

$$x(t) = u(t) + e^{At}K,$$

$$u(t) = e^{-at}\int_0^t e^{-As}B(s)\,ds.$$

Note that $u(t)$ is also a solution to (1), while $e^{At}K$ is a solution to the homogeneous equation

(4) $$y' = Ay$$

obtained from (1) by replacing $B(t)$ with 0. In fact, if $v(t)$ is any solution to (1) and $y(t)$ any solution to (4), then clearly $x = v + y$ is another solution to (1). Hence the general solution to (1) is obtained from a particular solution by adding to it the general solution of the corresponding homogeneous equation. In summary

Theorem *Let $u(t)$ be a particular solution of the nonhomogeneous linear differential equation*

(1) $$x' = Ax + B(t).$$

Then every solution of (1) has the form $u(t) + v(t)$ where $v(t)$ is a solution of the homogeneous equation

(4′) $$x' = Ax.$$

Conversely, the sum of a solution of (1) and a solution of (4′) is a solution of (1).

If the function $B(t)$ is at all complicated it will probably be impossible to replace the integral in (3) by a simple formula; sometimes, however, this can be done.

Example. Find the general solution to

(5) $$x_1' = -x_2,$$
$$x_2' = x_1 + t.$$

Here

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad B(t) = \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

Hence

$$e^{-As} = \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}$$

and the integral in (3) is

$$\int_0^t \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix} \begin{bmatrix} 0 \\ s \end{bmatrix} ds = \int_0^t \begin{bmatrix} s \sin s \\ s \cos s \end{bmatrix} ds$$
$$= \begin{bmatrix} \sin t - t \cos t \\ \cos t + t \sin t - 1 \end{bmatrix}.$$

To compute (3) we set

$$e^{At} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \qquad K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix};$$

hence the general solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} \sin t - t\cos t + K_1 \\ \cos t + t\sin t - 1 + K_2 \end{bmatrix}.$$

Performing the matrix multiplication and simplifying yields

$$x_1(t) = -t + K_1 \cos t + (1 - K_2)\sin t,$$

$$x_2(t) = 1 - (1 - K_2)\cos t + K_1 \sin t.$$

This is the solution whose value at $t = 0$ is

$$x_1(0) = K_1, \qquad x_2(0) = K_2.$$

PROBLEMS

1. Find all solutions to the following equations or systems:
   (a) $x' - 4x - \cos t = 0$; (b) $x' - 4x - t = 0$; (c) $x' = y$, $y' = 2 - x$;
   (d) $x' = y$, $y' = -4x + \sin 2t$; (e) $x' = x + y + z$, $y' = -2y + t$, $z' = 2z + \sin t$.
2. Suppose $T: \mathbf{R}^n \to \mathbf{R}^n$ is an invertible linear operator and $c \in E$ is a nonzero constant vector. Show there is a change of coordinates of the form
   $$x = Py + b, \qquad b \in \mathbf{R}^n,$$
   transforming the nonhomogeneous equation $x' = Tx + c$ into homogeneous form $y' = Sy$. Find P, b, and S. (*Hint*: Where is $x' = 0$?)
3. Solve Problem 1(c) using the change of coordinates of Problem 2.

§6. Higher Order Systems

Consider a linear differential equation with constant coefficients which involves a derivative higher than the first; for example,

(1)
$$s'' + as' + bs = 0.$$

By introducing new variables we are able to reduce (1) to a first order system of two equations. Let $x_1 = s$ and $x_2 = x_1' = s'$. Then (1) becomes equivalent to the