# Picard's Existence and Uniqueness Theorem

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*These notes on the proof of Picard's Theorem follow the text* Fundamentals of Differential Equations and Boundary Value Problems*, 3rd edition, by Nagle, Saff, and Snider, Chapter 13, Sections 1 and 2. The intent is to make it easier to understand the proof by supplementing the presentation in the text with details that are not made explicit there. By no means is anything here claimed to be original work.*

One of the most important theorems in Ordinary Differential Equations is Picard's Existence and Uniqueness Theorem for first-order ordinary differential equations. Why is Picard's Theorem so important? One reason is it can be generalized to establish existence and uniqueness results for higher-order ordinary differential equations and for systems of differential equations. Another is that it is a good introduction to the broad class of existence and uniqueness theorems that are based on fixed points.

### Picard's Existence and Uniqueness Theorem

*Consider the Initial Value Problem (IVP)*

$$y' = f(x, y), \quad y(x_0) = y_0.$$

*Suppose* $f(x, y)$ *and* $\frac{\partial f}{\partial y}(x, y)$ *are continuous functions in some open rectangle* $R = \{(x, y) : a < x < b, c < y < d\}$ *that contains the point* $(x_0, y_0)$*. Then the IVP has a unique solution in some closed interval* $I = [x_0 - h, x_0 + h]$*, where* $h > 0$*. Moreover, the* Picard *iteration defined by*

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt$$

*produces a sequence of functions* $\{y_n(x)\}$ *that converges to this solution uniformly on* $I$*.*

**Example 1**: Consider the IVP

$$y' = 3y^{2/3}, \quad y(2) = 0$$

Then $f(x, y) = 3y^{2/3}$ and $\frac{\partial f}{\partial y} = 2y^{-1/3}$, so $f(x, y)$ is continuous when $y = 0$ but $\frac{\partial f}{\partial y}$ is not. Hence the hypothesis of Picard's Theorem does not hold. Neither does the conclusion; the IVP has two solutions, $y^{1/3} = x - 2$ and $y \equiv 0$.

There are many ways to prove the existence of a solution to an ordinary differential equation. The simplest way is to find one explicitly. This is a good approach for separable or exact equations, or linear equations with constant coefficients. But unfortunately there

are many equations that cannot be solved by elementary methods, so attempting to prove the existence of a solution with this approach is not at all practical. An alternative approach is to approximate a solution to an IVP by constructing a sequence of functions that converges to a solution. This is precisely the approach we will use for the proof of Picard's Theorem.

Before we discuss the idea behind successive approximations, let's first express a first-order IVP as an integral equation. For the IVP $y' = f(x, y)$, $y(x_0) = y_0$, suppose that $f$ is continuous on some appropriate rectangle and that there is a solution $y(x)$ that is continuous on some interval $I$. Then we may integrate both sides of the DE to obtain integral equation:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

Thus, under the assumptions of existence and continuity, the IVP is equivalent to the integral equation. This fact seems convenient and useful at first glance, but upon closer inspection we notice two problems:

- The integral equation is not well-defined unless we know that a solution exists.[1]
- The integral equation is very hard to solve, except for very elementary IVPs.

Suppose we define an operator $T$ that maps a function $y(x)$ to a function $T[y](x)$, given by

$$T[y](x) := y_0 + \int_{x_0}^{x} f(t, y(t)\, dt$$

Then the integral equation is simply $y = T[y]$, and any solution to the IVP must be a *fixed point* of $T$.[2]

To find fixed points, approximation methods are often useful. See Figure 1, below, for an illustration of the use of an approximation method to find a fixed point of a function. To find a fixed point of the transformation $T$ using Picard iteration, we will start with the function $y_0(x) \equiv y_0$ and then iterate as follows:

$$y_{n+1}(x) = y_n(x) + \int_{x_0}^{x} f(t, y_n(t))\, dt$$

to produce the sequence of functions $y_0(x)$, $y_1(x)$, $y_2(x)$, .... If this sequence converges, the limit function will be a fixed point of $T$.

[1] The term 'well-defined' means essentially that $(t, y(t))$ must be in the domain of $f(x, y)$. For example, the function $f(x) = \frac{1}{1-x}$ is well-defined for all $x \neq 1$. But if $x = 1$, then $f(x)$ is undefined.

[2] A fixed point of an operator or a transformation is an element in the domain that the operator or transformation maps to itself. In terms of functions, a point $x = a \in dom_g$ is a fixed point of $g$ iff $g(a) = a$. Graphically, a fixed point is a point where the graph of $y = g(x)$ intersects the straight line $y = x$.

**Example 2**: Consider the IVP

$$y' = 2y, \quad y(0) = 1$$

This IVP is equivalent to $y = 1 + \int_0^x 2y\,dt$, so the Picard iterates are $y_0(x) \equiv 1$,

$$y_1(x) = 1 + \int_0^x 2y_0(t)\,dt = 1 + 2x$$

$$y_2(x) = 1 + \int_0^x 2(1 + 2t)\,dt = 1 + 2x + \frac{(2x)^2}{2!}$$

and so on. It can be shown by induction that the $n$th iterate is

$$y_n(x) = 1 + 2x + \frac{(2x)^2}{2!} + \ldots + \frac{(2x)^n}{n!} = \sum_{i=1}^{n} \frac{(2x)^i}{i!}$$

which is the $n$th partial sum of the Maclaurin series for $e^{2x}$. Thus, as $n \longrightarrow \infty$, $y_n(x) \longrightarrow e^{2x}$.

To carry out a rigorous test for convergence (which is especially necessary when we don't recognize the sequence of Picard iterates), we need some idea of distance between functions. The distance measure used in the proof of the Picard Theorem is based on the norm of a function, as given in the following definition.

**Definition** *Let $C[a,b]$ denote the set of all functions that are continuous on $[a,b]$. If $y \in C[a,b]$, then the* norm *of $y$ is $\|y\| := \max_{x\in[a,b]} |y(x)|$.*

The norm of a function $y(x)$ may be regarded as the distance between $y(x)$ and $y \equiv 0$, the function that is identically zero. [3] With this in mind we may define the distance between two functions, $y, z \in C[a,b]$ to be the norm of $y - z$, or

$$\|y - z\| = \max_{x\in[a,b]} |y(x) - z(x)|.$$

Using this measure of distance, we can define convergence of a sequence of functions to a limiting function.

**Definition** *A sequence $\{y_n(x)\}$ of functions in $C[a,b]$ converges uniformly to a function $y(x) \in C[a,b]$ iff $\lim_{n\to\infty} \|y_n - y\| = 0$.*

To illustrate uniform convergence, recall Example 2, where we found that $y_n(x) = \sum_{i=0}^{n} \frac{(2x)^i}{i!}$ and $y(x) = e^{2x}$ on the interval $[0,1]$. Then

$$\lim_{n\to\infty} \|y_n - y\| = \lim_{n\to\infty} \left[\max_{x\in[0,1]} \left|\sum_{i=0}^{n} \frac{(2x)^i}{i!} - e^{2x}\right|\right] = \lim_{n\to\infty} \left[\max_{x\in[0,1]} \left|\sum_{i=n+1}^{\infty} \frac{(2x)^i}{i!}\right|\right] = \lim_{n\to\infty} \sum_{i=n+1}^{\infty} \frac{2^i}{i!} = 0$$

[3] See Nagel, Saff, and Snider, page 837, for properties of this norm.

Therefore, $y_n \longrightarrow y$. One important detail to note in this example is that the uniform convergence of the sequence $\{y_n(x)\}$ to $y(x) = e^{2x}$ on $[a, b]$ occurs only when the interval is bounded on the right. In other words, $b$ must be finite. If we were to take $b$ to be infinite—for example, if the interval under consideration were the whole real line—then the sequence would not converge uniformly. The reason is that, if the value of $x$ is unbounded, then for any finite $n$, the norm of $\sum_{i=n+1}^{\infty} \frac{(2x)^i}{i!}$ is infinite.

Uniform convergence is particularly useful in that if a sequence of differentiable (and therefore continuous) functions is uniformly convergent, then the function to which it converges is also continuous.[4]

**Banach Fixed Point Theorem for Operators**

*Let $S$ denote the set of continuous functions on $[a, b]$ that lie within a fixed distance $\alpha > 0$ of a given function $y^t(x) \in C[a, b]$, i.e. $S = \{y \in C[a, b] : \|y - y^t\| \leq \alpha\}$. Let $G$ be an operator mapping $S$ into $S$ and suppose that $G$ is a* contraction *on $S$, that is*

$$\exists k \in R, 0 \leq k < 1 \text{ } s.\ t. \text{ } \|G[w] - G[z]\| \leq k\|w - z\| \forall w, z \in S.$$

*Then the operator $G$ has a unique fixed point solution in $S$. Moreover, the sequence of successive approximations defined by $y_{n+1} := G[y_n]$, $n = 0, 1, 2 \ldots$ converges uniformly to this fixed point, for any choice of starting function $y_0 \in S$.*

To prove the theorem we first show that the functions in the sequence $y_{n+1} = G[y_n]$ are well-defined; that is, every function in the sequence $\{y_n(x)\}$ is in the set $S$. Next we show that this sequence converges uniformly to a function $y_\infty \in S$. Finally, we show that this limit is a fixed point of $G$, that is, $y_\infty = G[y_\infty]$.

**Proof**:

Take any starting function $y_0 \in S$. Since $y_0 \in Dom_G$, then $y_1 = G[y_0]$ is defined. Since $G$ maps $S$ to itself, $y_1 \in S$. By induction, $y_n \in S$ and $G[y_n]$ is well-defined, for all $n \geq 0$.

Now rewrite $y_n = y_0 + (y_1 - y_0) + (y_2 - y_1) + \ldots + (y_n - y_{n-1})$, so that

$$y_n(x) = y_0(x) + \sum_{j=0}^{n-1} (y_{j+1}(x) - y_j(x)) \tag{1}$$

We now show that the sequence $\{y_n\}$ converges uniformly to an element in $S$. We do this by applying the *Weierstrass M-Test*, an extension of the Comparison Test.

[4]See the text *Introduction to Analysis* by James R. Kirkwood, pages 206-212, for the definitions and proofs of some properties of uniform convergence.

**Weierstrass M-Test** *Let $\{f_n\}$ be a sequence of functions defined on a set $E$. Suppose that for all $n \in N$, there exists $M_n \in R$ such that*

$$|f_n(x)| \leq M_n \forall x \in E$$

Then if $\sum M_n$ converges, $\sum f_n$ must converge uniformly on $E$.

What we need to do, then, is to find a bound $M$ on the terms (actually functions, of course) of the series (1).

**Claim**: $\|y_{j+1} - y_j\| \leq k^j \|y_1 - y_0\|$.

The claim is evidently true for $j = 0$. Suppose that it is true for $j = q$, where $q \in N, q \geq 0$. Then

$$\|G[y_{q+2}] - G[y_{q+1}]\| = \|G[G[y_{q+1}]] - G[G[y_q]]\| \leq k\|G[y_{q+1}] - G[y_q]\| \leq k^{q+1}\|y_1 - y_0\|,$$

proving the claim.

Returning to the series (1), it is clear from the claim that $\max\limits_{x\in[a,b]} |y_{j+1}(x) - y_j(x)| = \|y_{j+1} - y_j\| \leq k^j\|y_1 - y_0\|$. Let $M_j := k^j\|y_1 - y_0\|$. Because $\sum\limits_{j=1}^{\infty} M_j = \|y_0 - y_1\| \sum\limits_{j=1}^{\infty} k^j$ converges[5], the Weierstrass M-Test shows that $\{y_n\}$ converges uniformly to a continuous function, $y_\infty$. Moreover, $y_\infty \in S$ because the assumption that $\|y_\infty - y^t\| > \alpha$ implies that $\|y_n - y^t\| > \alpha$ for some $n$, contradicting the fact that $y_n \in S$.

Since $G$ is a contraction, we have that $\|G[y_\infty] - G[y_n]\| \leq k\|y_\infty - y_n\|$ for any $n$. But $\|y_\infty - y_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, so $\|y_\infty - y_{n+1}\| \longrightarrow 0$ as $n \longrightarrow \infty$. Of course, $G[y_n] = y_{n+1}$. Thus, $\lim\limits_{n\to\infty} \|G[y_\infty] - G[y_n]\| = \lim\limits_{n\longrightarrow\infty} \|G[y_\infty] - y_{n+1}\| \leq \lim\limits_{n\longrightarrow\infty} k\|G[y_\infty] - y_{n+1}\| = 0$. Finally, $G[y_\infty] - y_\infty = (G[y_\infty] - y_{n+1}) + (y_{n+1} - y_\infty)$, so that[6]

$$\|G[y_\infty] - y_\infty\| \leq \|G[y_\infty] - y_{n+1}\| + \|y_{n+1} - y_\infty\|.$$

Since both terms on the right side of the above equation approach zero as $n$ approaches $\infty$, it follows that $\|G[y_\infty] - y_\infty\| = 0$, or $G[y_\infty] = y_\infty$. Thus, $y_\infty$ is a fixed point of $G$.

Now suppose that $z \in S$ is any fixed point of $G$, i.e. that $z$ satisfies $G[z] = z$. Then $\|y_\infty - z\| = \|G[y_\infty] - G[z]\| \leq k\|y_\infty - z\| < \|y_\infty - z\|$, which is possible iff $\|y_\infty - z\| = 0$. In other words, $z = y_\infty$, so that $y_\infty$ is the unique fixed point of $G$. This completes the proof of the Banach Fixed Point Theorem for Operators.

Now we return to Picard's Theorem. Suppose that $u(x)$ is a solution to the IVP on $[x_0 - h, x_0 + h]$ and recall that $|f(x, y)| < M$ on a rectangle $R_1$, as shown in Figure 2, below.

[5] It is a convergent geometric series, because of the assumption that $0 \leq k < 1$.

[6] By the Triangle Inequality for Norms.

Since $u(x_0) = y_0$, the graph of $u(x)$ must lie in $R_1$ for all values of $x$ close enough to $x_0$. For such $x$, we have that $|f(x, u(x))| \leq M$. Then $|u'(x)| = |f(x, u(x))| \leq M$. So for $x$ close to $x_0$ the graph of $u = T[u]$ must lie within the shaded sector of Figure 2, where the parameters $\alpha_1$ and $h_2$ are defined geometrically.

The graph of $u(x)$ cannot escape from this rectangle on $[x_0 - h_2, x_0 + h_2]$ since, if it did, $|u'(x)| = |f(x, u(x))| > M$ at some point of the rectangle, which is clearly not possible. Thus, $|u(x) - y_0| \leq \alpha_1$, for all $x \in [x_0 - h, x_0 + h]$. So $u(x) \in S$.

**Proof of Picard's Theorem:**

To prove Picard's Theorem we apply the Banach Fixed Point Theorem for Operators to the operator $T$. The unique fixed point is the limit of the Picard Iterations given by

$$y_{n+1} = T[y_n], \quad y_0(x) \equiv y_0.$$

Recall that if $y$ is a fixed point of $T$, then $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$, which is equivalent to the IVP. If such a function, $y(x)$, exists, then it is the unique solution to the IVP.[7]

To apply the Banach Fixed Point Theorem for Operators, we must show that $T$ will map a suitable set $S$ to itself and that $T$ is a contraction. This may not be true for all real $x$. Also, our information pertains only to the particular intervals for $x$ and $y$ referred to the hypotheses of Picard's Theorem.

First we find an interval $I = [x_0 - h, x_0 + h]$ and $\alpha \in R$, $\alpha > 0$ such that $T$ maps $S = \{g \in C[I] : \|y - y_0\| \leq \alpha\}$ into itself and $T$ is a contraction. Here, $C[I] = C[x_0 - h, x_0 + h]$, and we adopt the norm $\|y\|_I = \max_{x \in I} |y|$. Choose $h_1$ and $\alpha_1$ such that

$$R_1 := \{(x, y) : |x - x_0| \leq h_1, |y - y_0| \leq \alpha_1\} \subseteq R$$

Because $f$ and $\frac{\partial f}{\partial y}$ are continuous on the compact set $R_1$, it follows that both $f$ and $\frac{\partial f}{\partial y}$ attain their supremum (and infimum) on $R_1$.[8]

It follows that there exist $M > 0$ and $L > 0$ such that

$$\forall (x, y) \in R_1, |f(x, y)| \leq \text{ and } \left|\frac{\partial f}{\partial y}\right| \leq L.$$

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[7] Note that if there is a fixed point $y(x)$, then $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$. By FTOC, $y(x)$ is differentiable and $y'(x) = \frac{d}{dx}\left(y_0 + \int_{x_0}^{x} f(t, y(t))\, dt\right) = f(x, y)$. Thus, $y(x)$ satisfies the DE and solves the IVP.

[8] A compact set is usually defined to be a set with the property that if the set is covered by the union of all members of a collection of open sets, then it is also covered by the union of a finite number of the open sets—any open cover has a finite subcover. An easier way to understand compactness is via the Heine-Borel Theorem and its converse which, together, state that a set in a Euclidean space is compact if and only if it is closed and bounded. For example, the set $[0, 1]$ is compact, since it is closed (i.e. contains its endpoints) and is clearly bounded. The set $(0, 1)$, on the other hand, is not compact.

Now let $g$ be a continuous function on $I_1 = [x_0 - h_1, x_0 + h_1]$ satisfying $|g(x) - y_0| \leq \alpha_1$ for all $x \in I$. Then $T[g](x) = y_0 + \int_{x_0}^{x} f(t, g(t))\, dt$ so that, for all $x \in I$,

$$|T[g](x) - y_0| = \left| \int_{x_0}^{x} f(t, g(t))\, dt \right| \leq \left| \int_{x_0}^{x} |f(t, g(t))|\, dt \right| \leq M \left| \int_{x_0}^{x} dt \right| = M|x - x_0|.$$

The situation is shown geometrically in Figure 2, below.

Choose $h$ such that $0 < h < \min\{h_1, \frac{\alpha_1}{M}, \frac{1}{L}\}$. Let $\alpha = \alpha_1$, $I = [x_0 - h, x_0 + h]$, and $S = \{g \in C(I) : \|g - y_0\|_I \leq \alpha\}$. Then $T$ maps $S$ into $S$. Moreover, $T[g](x)$ is clearly a continuous function on $I$ since it is differentiable (by the FTOC), and differentiability implies continuity.

For any $g \in S$, we have for any $x \in I$,

$$|T[g](x) - y_0| \leq M|x - x_0| \leq Mh < M\left(\frac{\alpha_1}{M}\right) = \alpha_1.$$

In other words, $\|T[g] - y_0\| \leq \alpha_1$, so $T[g] \in S$.

Now we show that $T$ is a contraction. Let $u, v \in S$. On $R_1$, $\left|\frac{\partial f}{\partial y}\right| \leq L$, so by the MVT there is a function $z(t)$ between $u(t)$ and $v(t)$ such that

$$\begin{aligned} |T[u](x) - T[v](x)| &= \left| \int_{x_0}^{x} \{f(t, u(t)) - f(t, v(t))\}\, dt \right| = \left| \int_{x_0}^{x} \frac{\partial f}{\partial y}(t, z(t))[u(t) - v(t)]\, dt \right| \\ &\leq L \left| \int_{x_0}^{x} |u(t) - v(t)|\, dt \right| \leq L\|u - v\|_I |x - x_0| \leq Lh\|u - v\|_I, \end{aligned}$$

for all $x \in I$. Thus $\|T[u](x) - T[v](x)\| \leq k\|u - v\|$, where $k = Lh < 1$, so $T$ is a contraction on $S$.

The Banach Fixed Point Theorem for Operators therefore implies that $T$ has a unique fixed point in $S$. It follows that the IVP $y' = f(x, y), y(x_0) = y_0$ has a unique solution in $S$. Moreover, this solution is the uniform limit of the Picard iterates.

Now we have found the unique solution to the IVP $y' = f(x, y), y(x_0) = y_0$ in $S$, there is one important point that remains to be resolved. We must show that any solution to the IVP on $I = [x_0 - h, x_0 + h]$ must lie in $S$.

Suppose that $u(x)$ is a solution to the IVP on $[x_0 - h, x_0 + h]$. Recall that $|f(x, y)| < M$ on the rectangle $R_1$. Since $u(x_0) = y_0$, the graph of $u(x)$ must lie in $R_1$ for $x$ close to $x_0$. For such an $x$, we have that $|f(x, u(x)| \leq M$, which implies that $|u'(x)| = |f(x, u(x))| \leq M$. Therefore, for $x$ close to $x_0$, the graph of $u = T[u]$ must lie within the shaded region of Figure 2. Moreover, the graph cannot escape from this region in $[x_0 - h, x_0 + h]$, since if it did, $|u'(x)| = |f(x, u(x)| > M$ at some point of the region, which is clearly impossible. Thus $|u(x) - y_0| \leq \alpha_1$ for all $x \in [x_0 - h, x_0 + h]$, which shows that $u(x) \in S$.