The Global Angular Form, The Euler Class, and the Thom Class

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Abstract: In this paper, we will construct an explicit formula of Thom class for an arbitrary $\mathbb{R}^2$-bundle in local coordinates.
References:
Differential forms in Algebraic Topology - Bott and Tu (p 70-75)
Characteristic Classes - Milnor and Stasheff

Contents

1 Recall: Definition of the Thom Class

Let M be an m dimensional manifold and $\pi : E \to M$ be an $\mathbb{R}^n$ be an oriented vector bundle. The Thom class Φ of this vector bundle is defined as the image of 1 in $H^0(M)$ under the Thom isomorphism
$\mathscr{T} : H^k(M) \to H^{k+n}_{cv}(E)$.

Proposition 1. *The Thom class Φ of the vector bundle given above is the unique cohomology class on $H^n_{cv}(E)$ whose restriction to to each fiber F is the generator of $H^n_c(F)$.*

This means that $\Phi|_F$ is a bump form on the fiber with total integral 1.

2 The Global Angular Form, The Euler Class, and the Thom Class

*Orientation: A top form on a manifold M is called positive if it is in the orientation class of M.
The deformation retraction $\pi : \mathbb{R}^n - \{0\} \to S^{n-1}$ implies a group isomorphism

$$\pi^* : H^{n-1}(S^{n-1}) \to H^{n-1}(\mathbb{R}^n - \{0\}) \tag{2.1}$$

The image of the generator σ of $H^{n-1}(S^{n-1})$, $\psi = \pi^*\sigma$ under this isomorphism is called the angular form on $\mathbb{R}^n - \{0\}$. If $\rho(r)$ is a continuously increasing function of the radius r which is -1 near 0 and 0 near infinity, $d\rho(r) = \frac{\partial \rho}{\partial r} dr$ is a bump function on $\mathbb{R}$ with total integral 1. Therefore, $d\rho.\psi$ is a compactly supported form on $\mathbb{R}^n$ and $(d\rho.\psi)$ is the generator of $H^n_c(\mathbb{R}^n)$. Since ψ is an element of the cohomology class, $d\psi = 0$ and therefore, $(d\rho.\psi) = d(\rho.\psi) = \Phi$ is the Thom class. To make sense of a function of r, we have to endow the total space with a Riemannian structure.

$$< .|. >_E = \sum_\alpha \rho_\alpha < .|. >_\alpha \tag{2.2}$$

Now we will generalize this construction to general $\mathbb{R}^2$ bundle. Let E^0 be the set of all nonzero elements of the total space E of an $\mathbb{R}^2$ vector bundle over a manifold M. Locally, $E^0|_{U_\alpha}$ has a frame $\pi^* x_1, ...\pi^* x_m, r_\alpha, \theta_\alpha$ of $U_\alpha \times \mathbb{R}^2$. Note that we can always find a local frame when we go from one local subspace U_α to U_β such that $r_\alpha = r_\beta$ on the intersection $U_\alpha \cap U_\beta$. However, the angular coordinate θ_α will change with some angle depending on local trivialization of the vector bundle unless the vector bundle is trivial. Denote this change by an angle $\pi^*\phi_{\alpha\beta}$ where $0 \leq \phi_{\alpha\beta} \leq 2\pi$.

$$\theta_\beta = \theta_\alpha + \pi^*\phi_{\alpha\beta} \tag{2.3}$$

*** $\phi_{\alpha\beta}$ is defined on the manifold M instead of the total space E. Because it kind of depends on how curved the base space is.
Note that the $\phi_{\alpha\beta}$'s do not satisfy cocycle condition.

$$\phi_{\alpha\beta} + \phi_{\beta\gamma} - \phi_{\alpha\gamma} = 2\pi\epsilon_{\alpha\beta\gamma} \tag{2.4}$$

where $\epsilon_{\alpha\beta\gamma} \in \mathbb{Z}$. $d\phi_{\alpha\beta}$ clearly satisfies cocycle condition.
Define a 1-form on U_α

$$\xi_\alpha = \frac{1}{2\pi}\sum_\gamma \rho_\gamma d\phi_{\alpha\gamma} \tag{2.5}$$

on open subset U_α where $\{\rho_\alpha\}$ is a partition of unity subordinate to the open cover $\{U_\alpha\}$.
On $U_\alpha \cap U_\beta$

$$\begin{aligned}
\xi_\beta - \xi_\alpha &= \frac{1}{2\pi}\sum_\gamma \rho_\gamma d\phi_{\beta\gamma} - \frac{1}{2\pi}\sum_\gamma \rho_\gamma d\phi_{\alpha\gamma} && (2.6)\\
&= \frac{1}{2\pi}\sum_\gamma \rho_\gamma (d\phi_{\beta\gamma} - d\phi_{\alpha\gamma}) && (2.7)\\
&= \frac{1}{2\pi}\sum_\gamma \rho_\gamma d\phi_{\alpha\beta} = \frac{1}{2\pi} d\phi_{\alpha\beta} && (2.8)
\end{aligned}$$

It follows that $d\xi_\alpha = d\xi_\beta$ on $U_\alpha \cap U_\beta$. It means that the local 2-forms which match on the intersection can bu glued together to a global 2-form e on M. This global 2-form $e \in H^2(M)$ is called the Euler class.
It is easy to check that the Euler class is independent of choice of 1 form ξ_α.
Now we can define the global angular form on E^0 which can be written in local coordinates as follows:

$$\psi|_{U_\alpha} = \frac{d\theta_\alpha}{2\pi} - \pi^*\xi_\alpha \tag{2.9}$$

The global angular form ψ is well-defined since

$$\frac{d\theta_\alpha}{2\pi} - \pi^*\xi_\alpha = \frac{1}{2\pi}d\theta_\beta - \frac{1}{2\pi}\pi^*\phi_{\alpha\beta} - \pi^*\xi_\beta + \frac{1}{2\pi}\pi^* d\phi_{\alpha\beta} \tag{2.10}$$

$$= \frac{d\theta_\beta}{2\pi} - \pi^*\xi_\beta \tag{2.11}$$

on the intersection $U_\alpha \cap U_\beta$. Note that the restriction of the global angular 1-form ψ to a fiber is the angular form $\frac{1}{2\pi}d\theta$ and the global angular form is not closed:

$$d\psi = -\pi^* d\xi_\alpha = -\pi^* e \tag{2.12}$$

Remark 1. *If the vector bundle is trivial, then the Euler class vanishes and the global angular form is closed.*

The Euler class can be computed explicitly in terms of transition functions of the total space E. Let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(2)$ be the transition functions. Since $SO(2)$ can be identified with $U(1)$, we can think of $g_{\alpha\beta}$ as the complex valued functions. Since an arbitrary element of $U(1)$ can be written as $e^{i\theta}$, we have

$$\theta_\alpha - \theta_\beta = -\pi^*\phi_{\alpha\beta} = \pi^*(\frac{1}{i}\log g_{\alpha\beta}) \tag{2.13}$$

$$\pi^* d\phi_{\alpha\beta} = -\pi^*(\frac{1}{i}d\log\ g_{\alpha\beta}) \tag{2.14}$$

Since π is surjective, π^* is injective and therefore

$$d\phi_{\alpha\beta} = -\frac{1}{i}d\log g_{\alpha\beta} \tag{2.15}$$

Thus, the local 1-forms we defined before become

$$\xi_\alpha = \frac{1}{2\pi}\sum_\gamma \rho_\gamma d\phi_{\gamma\alpha} = -\frac{1}{2\pi i}\sum_\gamma \rho_\gamma d\log(g_{\gamma\alpha}) \tag{2.16}$$

Therefore, the Euler class becomes

$$e(E) = d\xi_\alpha = -\frac{1}{2\pi i}\sum_\gamma d(\rho_\gamma d\log g_{\gamma\alpha}) \tag{2.17}$$

on $\pi^{-1}(U_\alpha)$.

Proposition 2. *If $f : N \to M$ is a C^∞ map between manifolds and E is an $\mathbb{R}^2$ oriented bundle on M, then*

$$e(f^{-1}E) = f^* e(E) \tag{2.18}$$

proof **:** Since the transition of the $f^{-1}(E)$ bundle is by definition $f^*g_{\alpha\beta}$ and pullback commutes with exterior derivative, it follows from the equation 2.15.

Now we can similarly define a cohomology class and show that it is in fact the Thom class:

$$\Phi = d(\rho(r).\psi) = d\rho(r).\psi - \rho(r)\pi^*(e) \tag{2.19}$$

The fact that ψ is defined only on E^0 does not matter as $d\rho(r)$ vanishes as r approaches 0.

Properties of Φ

- Φ has a compact support in each fiber.
- Φ is closed: $d\Phi = -d\rho.d\psi - d\rho.d\psi$
- restriction to each fiber has total integral 1
$$\int_0^\infty \int_0^{2\pi} d\rho(r)\frac{d\theta}{2\pi} = \rho(\infty) - \rho(0) = 1 \tag{2.20}$$
- the cohomology class of Φ is independent of choice of $\rho(r)$. Suppose $\rho'(r)$ is another function whose value is -1 near 0 and 0 near infinity. Then
$$\Phi - \Phi' = d((\rho - \rho').\psi) \tag{2.21}$$
Since $(\rho - \rho').\psi$ is a 1 form, it implies that the equivalence class of Φ equals to that of Φ'.

Therefore, Φ defines the Thom class.

Proposition 3. *The pullback of the Thom class to M by zero section is the Euler class.*

proof **:**

Let $s : M \to E$ be the zero section. Then

$$s^*\Phi = d(\rho(0)).s^*\psi - \rho(0)s^*\pi^*e = e \tag{2.22}$$

since $s^*\pi^* = id_M$.

Finally, we can write an explicit formula for the Thom class:

$$\Phi = d\rho.\psi - \rho d\psi \tag{2.23}$$

$$\psi = \frac{d\theta_\alpha}{2\pi} - \pi^* d\xi_\alpha \tag{2.24}$$

$$= \frac{d\theta_\alpha}{2\pi} + \frac{1}{2\pi i}\pi^* \sum_\gamma \rho_\gamma d\log g_{\alpha\beta} \tag{2.25}$$

Thus,

$$\Phi = d(\rho\frac{d\theta_\alpha}{2\pi}) + \frac{1}{2\pi i}d\rho\pi^* \sum_\gamma \rho_\gamma d\log g_{\alpha\beta} \tag{2.26}$$