(1)

ΑΣΚΗΣΗ ΔΙΑΛΕΞΗ 11.

$$\begin{cases} \varepsilon y''' - y' + xy = 0 & (1) \\ y(0) = y'(0) = y(1) = 1 \end{cases}$$

Εξωτερική προσέγγιση (1ης τάξης)

$$y_{\varepsilon\xi}(x) = y_0(x) + \varepsilon y_1(x) + \dots$$

η εξίσωση γίνεται

$$\varepsilon y_0'''(x) + \varepsilon^2 y_1'''(x) - (y_0'(x) + \varepsilon y_1'(x)) + x(y_0(x) + y_1(x)\cdot\varepsilon) = 0$$

$$(-y_0'(x) + xy_0(x)) + \varepsilon(y_0'''(x) - y_1'(x) + xy_1(x)) + \varepsilon^2 y_1'''(x) + \dots = 0$$

ε^0:

$$-y_0'(x) + xy_0(x) = 0 \Leftrightarrow y_0'(x) - xy_0(x) = 0 \Leftrightarrow$$

$$e^{-\frac{x^2}{2}} y_0'(x) + (e^{-x^2/2})' y_0(x) = 0 \Leftrightarrow$$

$$(y_0(x)\cdot e^{-x^2/2})' = a_0 \Leftrightarrow$$

$$y_0(x) = a_0 e^{+x^2/2}$$

$$y(0) = 1 \Leftrightarrow \begin{cases} y_0(0) = 1 \\ y_1(0) = 0 \end{cases}$$

άρα $y_0(0) = 0 \Leftrightarrow a_0 = 1$

άρα $y_0(x) = e^{+\frac{x^2}{2}}$

ε^1

$$y_0'''(x) - y_1'(x) + xy_1(x) = 0 \qquad (2)$$

όπου $y_0(x) = e^{x^2/2}$, $y_0'(x) = x e^{x^2/2}$

$$y_0''(x) = e^{x^2/2} + x^2 e^{x^2/2}, \quad y_0'''(x) = xe^{x^2/2} + 2xe^{x^2/2} + x^3 e^{x^2/2}$$

$$y_0'''(x) = e^{x^2/2}(x^3 + 3x)$$

Για τη (2) γίνεται

$$e^{x^2/2}(x^3+3x) - y_1'(x) + x y_1(x) = 0$$

$$y_1'(x) - x y_1(x) = e^{x^2/2}(x^3+3x)$$

$$\left(y_1(x)\, e^{-x^2/2}\right)' = \left(\frac{x^4}{4} + \frac{3x^2}{2}\right)'$$

$$y_1(x)\, e^{-x^2/2} = \frac{x^4}{4} + \frac{3}{2}x^2 + B_1$$

$$y_1(x) = \frac{1}{4}\left(x^4 + 6x^2 + B_1\right) e^{x^2/2}$$

$$y_1(0) = 0 \to B_1 = 0$$

Άρα $y_1(x) = \frac{1}{4} x^2 (x^2+6)\, e^{x^2/2}$

Για $\boxed{y_{\varepsilon}(x) = e^{x^2/2}\left[1 + \varepsilon \frac{1}{4} x^2 (x^2+6)\right]}$ (3)

Β. Εσωτερική προσέγγιση στο $x=1$.

Θέτουμε $n = \frac{1-x}{\sqrt{\varepsilon}}$

και $Y(n) = Y_0(n) + \varepsilon^{1/2} Y_{1/2}(n) + \dots$

$Y(n) = Y\left(\frac{1-x}{\sqrt{\varepsilon}}\right) = y(x)$

$$\begin{cases} y'(x) = \dfrac{dY(n)}{dx} = \dfrac{\partial Y(n)}{\partial n}\dfrac{\partial n}{\partial x} = -\dfrac{1}{\sqrt{\varepsilon}}\dot{Y}(n) \\ y''(x) = \dfrac{1}{\varepsilon}\ddot{Y}(n) \\ y'''(x) = -\dfrac{1}{\varepsilon^{3/2}}\dddot{Y}(n) \end{cases}$$

Αντικαθιστούμε στην (1) και παίρνουμε

$$-\frac{1}{\varepsilon^{1/2}}\varepsilon\dddot{Y}(n) + \frac{1}{\varepsilon^{1/2}}\dot{Y}(n) + (1-n\sqrt{\varepsilon})Y(n)=0 \Rightarrow$$

$$-\varepsilon^{-\frac{1}{2}}\dddot{Y}(n) + \varepsilon^{-\frac{1}{2}}\dot{Y}(n) + (1-n\sqrt{\varepsilon})Y(n)=0$$

$$-\varepsilon^{-\frac{1}{2}}\left(\dddot{Y}_0(n) + \varepsilon^{\frac{1}{2}}\dddot{Y}_{1/2}(n)\right) + \varepsilon^{-\frac{1}{2}}\left(\dot{Y}_0(n) + \varepsilon^{\frac{1}{2}}\dot{Y}_{1/2}(n)\right) + (1-n\sqrt{\varepsilon})\left(Y_0(n) + \varepsilon^{1/2}Y_{1/2}(n)\right)=0$$

$$-\dddot{Y}_0(n) - \varepsilon^{1/2}\dddot{Y}_{1/2}(n) + \dot{Y}(n) + \varepsilon^{1/2}\dot{Y}_{1/2}(n) + \varepsilon^{1/2}(1-n\varepsilon^{1/2})\left(Y_0(n)\,\varepsilon^{1/2}Y_{1/2}(n)\right)=0$$

ε^0: $-\dddot{Y}_0(n) + \dot{Y}(n) = 0$

$\varepsilon^{1/2}$: $-\dddot{Y}_{1/2}(n) + \dot{Y}_{1/2}(n) + Y_0(n)=0$

Επομένως $\dddot{Y}_0(n) - \dot{Y}_0(n) = 0 \Rightarrow$

$$\boxed{Y_0(n) = A_0 e^{n} + B_0 e^{-n} + C_0}$$

και επειδή θέλουμε η λύση να είναι φραγμένη $A_0=0$

άρα $Y_0(n) = B_0 e^{-n} + C_0$

$$-\dddot{Y}_{1/2}(n) + \dot{Y}_{1/2}(n) + Y_0(n)=0$$

$$\dddot{Y}_{1/2}(n) - \dot{Y}_{1/2}(n) = Y_0(n)$$

$$\dddot{Y}_{1/2}(n) - \dot{Y}_{1/2}(n) = B_0 e^{-n} + C_0$$

Η αντίστοιχη ομογενής λύση

$$Y_{1/2\,ομ}(n) = A_{1/2} e^{n} + B_{1/2} e^{-n} + C_{1/2}$$

και μια ειδική λύση της μη ομογενούς είναι η:

$$\boxed{Y_{1/2\,ε.λ.} = \frac{B_0}{2} n e^{-n} - C_0 n}$$

Άρα $x_{1/2}(n) = A_{1/2} e^{n} + B_{1/2} e^{-n} + c_{1/2} + \frac{B_0}{2} n e^{-n} - c_0 n$

επειδή θέλουμε φραγμένη λύση $A_{1/2} = 0$.

Έτσι $\boxed{x_0(n) = B_0 e^{-n} + c_0}$

$\boxed{x_{1/2}(n) = B_{1/2} e^{-n} + c_{1/2} + \frac{B_0}{2} n e^{-n} - c_0 n}$

$y(1) = 1 \rightarrow x_0(0) + x_{1/2}(0) \cdot \varepsilon^{1/2} = 1 \Rightarrow$

$x_0(0) = 1$ και $x_{1/2}(0) = 0$

$B_0 + c_0 = 1.$ $\qquad B_{1/2} + c_{1/2} = 0$

$\boxed{B_0 = 1 - c_0}$ $\qquad \boxed{B_{1/2} = -c_{1/2}.}$

άρα

$$\boxed{\begin{aligned} x_0(n) &= (1-c_0) e^{-n} + c_0 \\ x_{1/2}(n) &= -c_{1/2} e^{-n} + c_{1/2} + \frac{1-c_0}{2} n e^{-n} - c_0 n \end{aligned}}$$

γ. ΣΥΝΑΡΜΟΓΗ ΣΤΟ 1.

$\lim_{n\to+\infty} x_0(n) = y_{out}(1) \Rightarrow$

$c_0 = \sqrt[2]{e}$

$\lim_{n\to+\infty} x_{1/2}(n) = 0 \Rightarrow c_{1/2} - c_0 = 0 \Rightarrow c_{1/2} = \sqrt{e} = c_0$

Έτσι $\boxed{x_0(n) = (1-\sqrt{e}) e^{-n} + \sqrt{e}}$

$x_{1/2}(n) = -\sqrt{e}\, e^{-n} + \sqrt{e} + \frac{1-\sqrt{e}}{2} n e^{-n} - \sqrt{e}\, n$

$\boxed{x_{1/2}(n) = [-\sqrt{e} - \tfrac{1}{2}(\sqrt{e}-1) n] e^{-n} - \sqrt{e}\, n + \sqrt{e}}$

ΚΟΙΝΟ ΟΡΟ $\quad \lim_{\eta \to 0} Y(\eta) = \sqrt{e}$

Δ. ΟΜΟΙΟΜΟΡΦΗ ΠΡΟΣΕΓΓΙΣΗ

$y_{om} = y_{εξ} + y_{εσ}$ − κοιν. όρου

$$y_{om}(x) = e^{x^2/2}\left[1 + \varepsilon \frac{1}{4} x^2 (x^2+c)\right] + (1-\sqrt{e})\, e^{-\frac{1-x}{\sqrt{\varepsilon}}} + \cancel{\sqrt{e}}$$

$$- \sqrt{\varepsilon}\left\{\left[-\sqrt{e} - \frac{1}{2}(\sqrt{e}-1)\left(\frac{1-x}{\sqrt{\varepsilon}}\right)\right] e^{-\frac{1-x}{\sqrt{\varepsilon}}} - \sqrt{e}\,\frac{(1-x)}{\sqrt{\varepsilon}} + \sqrt{e}\right\} - \cancel{\sqrt{e}}$$

$$\boxed{y_{om}(x) = e^{\frac{x^2}{2}}\left[1 + \varepsilon \frac{1}{4} x^2 (x^2+c)\right] + (1-\sqrt{e})\, e^{-\frac{1-x}{\sqrt{\varepsilon}}} - \sqrt{\varepsilon}\left\{\left[-\sqrt{e} - \frac{1}{2}(\sqrt{e}-1)\left(\frac{1-x}{\sqrt{\varepsilon}}\right)\right] e^{-\frac{1-x}{\sqrt{\varepsilon}}} - \sqrt{e}\,\frac{(1-x)}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}\right\}}$$