$$\hat{\theta}_J = n\hat{\theta} - (n-1)\bar{\hat{\theta}}_{(\cdot)}, \quad \bar{\hat{\theta}}_{(\cdot)} = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_{(i)}$$

$$(x_1+x_2+x_3)(x_1+x_2+x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$$

$$s^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n} = \frac{1}{n}\left[\sum_{i=1}^{n}x_i^2 - n\bar{x}^2\right] = \frac{n\sum_{i=1}^{n}x_i^2 - \sum_{i=1}^{n}x_i\sum_{j=1}^{n}x_j}{n^2} =$$

$$= \frac{(n-1)\sum_{i=1}^{n}x_i^2 - \sum_{i=1}^{n}x_i\sum_{\substack{j=1\\j\neq i}}^{n}x_j}{n^2}$$

$$\hat{\theta}_{(i)} = \frac{(n-2)\sum_{\substack{j=1\\j\neq i}}^{n}x_j^2 - \sum_{\substack{j=1\\j\neq i}}^{n}\sum_{\substack{k=1\\k\neq i\\k\neq j}}^{n}x_jx_k}{(n-1)^2}$$

$$(n-1)\bar{\hat{\theta}}_{(\cdot)} = \frac{(n-1)}{n}\sum_{i=1}^{n}\hat{\theta}(i) =$$

$$= \frac{\cancel{(n-1)}}{n}\sum_{i=1}^{n}\left[\frac{(n-2)\sum_{\substack{j=1\\j\neq i}}^{n}x_j^2 - \sum_{\substack{j=1\\j\neq i}}^{n}\sum_{\substack{k=1\\j\neq k\neq i}}^{n}x_jx_k}{(n-1)^{\cancel{2}}}\right] =$$

$$= \frac{1}{n}\left[\frac{\cancel{(n-1)}(n-2)\sum_{i=1}^{n}x_i^2}{\cancel{(n-1)}} - \frac{(n-2)\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}x_ix_j}{(n-1)}\right] =$$

$$= \frac{1}{n}\left[(n-2)\sum_{i=1}^{n}x_i^2 - \frac{(n-2)}{(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}x_ix_j\right]$$

$$\hat{\theta}_J = n\hat{\theta} - (n-1)\bar{\hat{\theta}}_{(\cdot)} =$$

$$= \frac{(n-1)}{n}\sum_{i=1}^{n}x_i^2 - \frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}x_ix_j -$$

$$- \frac{1}{n}\left[(n-2)\sum_{i=1}^{n}x_i^2 - \frac{(n-2)}{(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}x_ix_j\right] =$$

$$= \frac{(n-1)^2\sum x_i^2 - (n-1)\sum\sum x_ix_j - (n-1)(n-2)\sum_{i=1}^{n}x_i^2 + (n-2)\sum\sum x_ix_j}{n(n-1)} =$$

$$= \frac{(n-1)\sum x_i^2(\cancel{n}-1-\cancel{n}+2) + [(n-2)-(n-1)]\sum\sum x_ix_j}{n(n-1)} =$$

$$= \frac{(n-1)\sum_{i=1}^{n}x_i^2 - \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}x_ix_j}{n(n-1)} = \quad (n-1)\sum_{i=1}^{n}x_i^2 - \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}x_ix_j = \sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$= \frac{1}{(n-1)}\cdot\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{(n-1)}$$

$$\hat{\theta}_J = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{(n-1)}$$

$$\hat{p}_{ML} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$R = \sum_{i=1}^{n} x_i$$

$\downarrow$ μετράει αριθμό επιτυχιών στις $n$ δοκιμές

$$\hat{p}_{ML} = \frac{R}{n}$$

$R$ διωνυμική

$$E(R) = np$$

$$Var(R) = np(1-p)$$

$$\theta = p^2$$

$$\hat{\theta} = \left(\frac{R}{n}\right)^2$$

$$E(\hat{\theta}) = E\left(\left(\frac{R}{n}\right)^2\right) = Var\left(\frac{R}{n}\right) + \left[E\left(\frac{R}{n}\right)\right]^2 =$$

$$= \frac{1}{n^2} np(1-p) + p^2 = \frac{p(1-p)}{n} + p^2$$

$$\hat{\theta} = \left(\frac{R}{n}\right)^2 \qquad \hat{\theta}_{(i)} = \left(\frac{R - X_i}{n-1}\right)^2 =$$

$$= \frac{R^2 - 2RX_i + X_i^2}{(n-1)^2}$$

$$\bar{\hat{\theta}}_{(\cdot)} = \frac{1}{n} \sum_{i=1}^{n} \frac{R^2 - 2RX_i + X_i^2}{(n-1)^2} =$$

$$= \frac{1}{n} \, \frac{nR^2 - 2R\sum_{i=1}^{n} X_i + \sum_{i=1}^{n} X_i^2}{(n-1)^2} \qquad \underline{\underline{R = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_i^2}}$$

$$= \frac{(n-2)R^2 + R}{n(n-1)^2}$$

$$\hat{\theta}_J = n\hat{\theta} - (n-1)\bar{\hat{\theta}}_{(\cdot)} =$$

$$= \cancel{n} \cdot \frac{R^2}{n^{\cancel{2}}} - (\cancel{n-1}) \cdot \frac{(n-2)R^2 + R}{n(n-1)^{\cancel{2}}} =$$

$$= \frac{R^2}{n} - \frac{nR^2 - 2R^2 + R}{n(n-1)} =$$

$$= \frac{\cancel{nR^2} - R^2 - \cancel{nR^2} + 2R^2 - R}{n(n-1)} =$$

$$= \frac{R^2 - R}{n(n-1)} = \frac{R(R-1)}{n(n-1)}$$

$$\boxed{\hat{\theta}_J = \frac{R(R-1)}{n(n-1)}}$$

$$\hat{\theta}_j = \frac{R(R-1)}{n(n-1)} \quad \theta \text{ s.o.o } \quad E(\hat{\theta}_j) = p^2$$

$$E(\hat{\theta}_j) = \frac{1}{n(n-1)} E(R^2 - R) =$$

$$= \frac{E(R^2) - E(R)}{n(n-1)} =$$

$$E(R^2) = Var(R) + [E(R)]^2 =$$

$$= np(1-p) + (np)^2$$

$$= \frac{\cancel{np} - np^2 + n^2p^2 - \cancel{np}}{n(n-1)} =$$

$$= \frac{\cancel{n}p^2\cancel{(n-1)}}{\cancel{n}\cancel{(n-1)}} = p^2$$

$$p_i - \bar{p} \qquad \underline{\underline{p_i = n\hat{\theta} - (n-1)\hat{\theta}_{(i)}}}$$

$$= n\hat{\theta} - (n-1)\hat{\theta}_{(i)} - \frac{1}{n}\sum \left(n\hat{\theta} - (n-1)\hat{\theta}_{(i)}\right) =$$

$$= n\hat{\theta} - (n-1)\hat{\theta}_{(i)} - \frac{n^{\cancel{2}}\hat{\theta}}{\cancel{n}} + (n-1)\frac{\sum_{i=1}^{n}\hat{\theta}_{(i)}}{n} \quad = \bar{\hat{\theta}}_{(\cdot)}$$

$$= (n-1)\left(\bar{\hat{\theta}}_{(\cdot)} - \hat{\theta}_{(i)}\right)$$

---

$$Var(\hat{\theta}_j) = Var\left[n\hat{\theta} - (n-1)\bar{\hat{\theta}}_{(\cdot)}\right] =$$

$$= n^2 Var(\hat{\theta}) + (n-1)^2 Var(\bar{\hat{\theta}}_{(\cdot)}) - 2n(n-1)Cov(\hat{\theta}, \bar{\hat{\theta}}_{(\cdot)})$$

$$Var(\hat{\theta}_j) \approx Var(\hat{\theta})$$