A Hundred Years of Prime Numbers

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Paul T. Bateman and Harold G. Diamond

EARLY WORK ON PRIMES. This year marks the hundredth anniversary of the proof of the Prime Number Theorem (PNT), one of the most celebrated results in mathematics. The theorem is an asymptotic formula for the counting function of primes $\pi(x) := \#\{p \leq x \colon p \text{ prime}\}$ asserting that

$$\pi(x) \sim x/\log x. \qquad \text{(PNT)}$$

The twiddle notation is shorthand for the statement $\lim_{x \to \infty} \pi(x)/\{x/\log x\} = 1$. Here we shall survey early work on the distribution of primes, the proof of the PNT, and some later developments.

Since the time of Euclid, the primes, $2, 3, 5, 7, 11, 13, \ldots,$ have been known to be infinite in number. They appear to be distributed quite irregularly, and early attempts to find a closed formula for the nth prime were unsuccessful. By the end of the 18th century many mathematical tables had been computed, and examination of tables of prime numbers led C. F. Gauss and A. M. Legendre to change the question under investigation. Instead of seeking an exact formula for the nth prime, they considered the counting function $\pi(x)$ and asked for approximations to this function, evidently a new kind of question in number theory. Each of the two men conjectured the PNT, though neither did so in the form we have given. In 1808 Legendre published the formula $\pi(x) = x/(\log x + A(x))$, where $A(x)$ tends to a constant as $x \to \infty$. Gauss recorded his conjecture in one of his favorite books of tables around 1792 or 1793 but first disclosed it, in a mathematical letter, over fifty years later. He actually found a better approximation for $\pi(x)$ in terms of the logarithmic integral function, defined for $x > 0$ by

$$\operatorname{li}(x) := \lim_{\epsilon \to 0+} \left\{ \int_0^{1-\epsilon} + \int_{1+\epsilon}^{x} \right\} \frac{1}{\log t}\, dt.$$

It is easy to show that $\operatorname{li}(x) \sim x/\log x$, so either expression can be used in the asymptotic formula for $\pi(x)$. It has been shown that $\operatorname{li}(x)$ is a more accurate estimate of $\pi(x)$ than either $x/\log x$ or Legendre's proposed formula, so today $\operatorname{li}(x)$ is used in PNT error estimates. For more details about Gauss' meditations on the PNT, see **[Gol]**.

The function that we now call the Riemann zeta function, which was to play a decisive role in the proof of the PNT, was introduced by L. Euler in the 18th century. For s real and $s > 1$, define

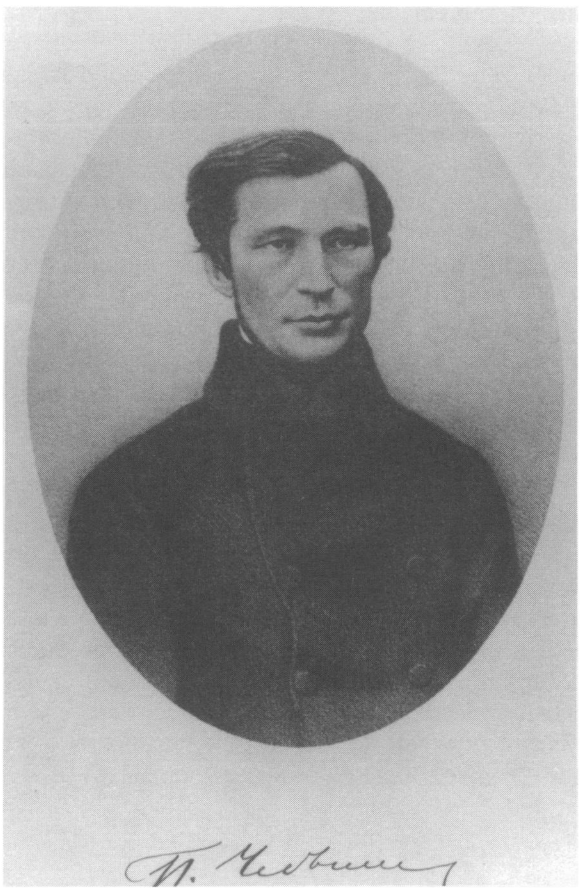
$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Using the unique factorization of positive integers, Euler proved that

$$\zeta(s) = \prod_p \{1 + p^{-s} + p^{-2s} + \cdots\} = \prod_p \{1 - p^{-s}\}^{-1},$$

where the product extends over all primes p. Further, he gave another proof of the infinitude of the primes by observing that if the number of primes were finite, then the product for $\zeta(1)$ would converge, while in fact the sum for ζ at $s = 1$ is the harmonic series, which diverges. Euler's proof shows further that the primes are sufficiently numerous that the sum of their reciprocals diverges.

Legendre conjectured and incorrectly believed he had proved that there are an infinite number of primes in each arithmetic progression for which the first term and common difference are relatively prime. This theorem was established by P. L. Dirichlet in 1837 by greatly extending the method of Euler described above. In two papers, Dirichlet introduced characters (periodic completely multiplicative arithmetic functions) to select the elements of an arithmetic progression; he generalized the ζ function by multiplying terms of the series for ζ by characters to make what we today call Dirichlet L functions; he related the value of an L function $L(1, \chi)$ with the class number of quadratic forms of a given discriminant, and from the positivity of the class number he deduced his key lemma that each of the L functions is nonzero at the point $s = 1$. The subject of analytic number theory is generally considered to have begun with Dirichlet.



P. L. Chebyshev

The first person to establish the true order of $\pi(x)$ was P. L. Chebyshev. In the middle of the 19th century he found an ingenious elementary method to estimate

$\pi(x)$ and established the bounds $.921x/\log x < \pi(x) < 1.106x/\log x$ for all sufficiently large values of x. Chebyshev's work was based on use of the arithmetic identity

$$\sum_{d|n} \Lambda(d) = \log n,$$

where von Mangoldt's function Λ is a weighted prime and prime power counting function defined by $\Lambda(d) = \log p$ if $d = p^{\alpha}$ for some prime p and positive integer α and $\Lambda(d) = 0$ otherwise. Chebyshev's formula is the arithmetic equivalent of the zeta function identity $\{-\zeta'(s)/\zeta(s)\} \cdot \zeta(s) = -\zeta'(s)$. Chebyshev showed also that if $\pi(x)/\{x/\log x\}$ had a limit as $x \to \infty$, then its value would be 1. Attempts at improving Chebyshev's methods led to slightly sharper estimates and much more elaborate calculations, but the PNT was not to be established by an elementary method for another hundred years.

A few years after the appearance of Chebyshev's paper, a path to the proof of the PNT was laid out by G. F. B. Riemann **[Edw]**, **[Lan]** in his only published paper on number theory. Riemann's revolutionary idea was to consider ζ as a function of a complex variable and express $\pi(x)$ in terms of a complex integral involving ζ. By formally deforming the integration contour, Riemann achieved an explicit

B. Riemann

formula for $\pi(x)$ as an infinite series whose leading term was $\mathrm{li}(x)$ and that involved the zeros of $\zeta(s)$. However, there was not enough analysis available at that time to rigorously deduce the PNT following Riemann's program. It was not until the end of the 19th century that the missing essential ingredient was supplied: this was the theory of entire functions of finite order, which was developed by J. Hadamard for the purpose of proving the PNT.

Riemann proved that the ζ function has an analytic continuation to **C** with just one singularity, a simple pole with residue 1 at the point $s = 1$ and that ζ satisfies a functional equation connecting its values at complex arguments s and $1 - s$. Incidentally, we owe to Riemann the unusual notation for a complex number $s = \sigma + it$ that has become standard in analytic number theory. Riemann recognized the key role that zeros of the ζ function play in prime number theory. He conjectured several properties of these zeros, all but one of which were proved around the end of the 19th century by Hadamard and H. von Mangoldt. The one conjecture that remains to this day, and is generally considered to be the most famous unsolved problem in mathematics, is the so-called Riemann hypothesis:

$$\text{All nonreal zeros of the } \zeta \text{ function have real part } 1/2. \tag{RH}$$

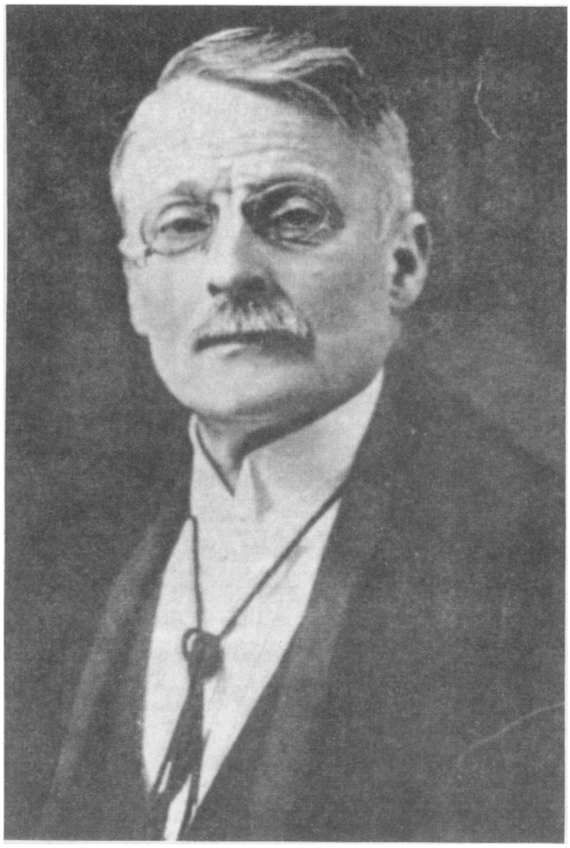
Riemann evidently perceived the greater difficulty of the RH, for while he stated his other conjectures with no qualification, he prefaced the statement of the RH with the phrase "it is very likely that [es ist sehr wahrscheinlich dass]... ."

Activity in prime number theory increased toward the end of the 19th century. The term "Prime Number Theorem" appears to have originated at this time in the Göttingen dissertation of H. von Schaper "Über die Theorie der Hadamardschen Funktionen und ihre Anwendung auf das Problem der Primzahlen," 1898. There were several false starts before correct proofs of the PNT were given. For example, in 1885 Stieltjes **[Stl]** claimed to have proved the RH. With this result one could establish the PNT with an essentially optimal error term

$$\pi(x) - \mathrm{li}(x) = O(x^{\frac{1}{2}+\epsilon}). \tag{1}$$

Here we have used the notation $f(x) = O(g(x))$, where g is a positive function for all x from some point onward, if $|f(x)|/g(x) < B$ holds for some positive constant B and all sufficiently large positive values of x. The deduction of (1) under the assumption of the RH was later carried out by von Koch. Stieltjes died in 1894 without having either substantiated or retracted his claim of having proved the RH.

FIRST PROOFS OF THE PRIME NUMBER THEOREM. The PNT was established in 1896 by Jacques Hadamard and by Charles-Jean de la Vallée Poussin. It was the first major achievement for each at the start of long and distinguished careers. Hadamard was born at Versailles, France, in 1865. After studies at the Ecole Normale Supérieure he obtained his doctorate in 1892. He spent most of his career in Paris, working principally in complex function theory, partial differential equations, and differential geometry. He died in 1963, within two months of his 98th birthday. De la Vallée Poussin was born in 1866 in Louvain, Belgium, where his father was a professor of mineralogy and geology at the University. After studying at Louvain, he too joined the faculty of the University, at the age of 26, as Professor of Mathematics. His elegant and lucid *Cours d'Analyse* has educated generations of mathematicians in the methods of Borel and Lebesgue. De la Vallée Poussin died in 1962, in his 96th year.



Ch. J. de la Vallée Poussin

The arguments of both Hadamard and de la Vallée Poussin followed the scheme laid out by Riemann. Both papers made essential use of Riemann's functional equation for the zeta function, several other properties of ζ conjectured by Riemann and established by Hadamard, and Hadamard's new theory of entire functions.

Hadamard's paper on the PNT **[Had]** consists of two parts. Here are the opening paragraphs of Part I, "On the distribution of zeros of the zeta function" (in our translation). It is interesting to see how he treats Stieltjes' claim.

The Riemann zeta function is defined, when the real part of s is greater than 1, by the formula

$$\log \zeta(s) = -\sum_p \log(1 - 1/p^s), \tag{2}$$

where p runs over the prime numbers,... . [Translators' remark: Use the principal branch for the logarithms on the right side of (2).] It is holomorphic in the entire plane, except at the point $s = 1$, which is a simple pole. It does not vanish for any value of s with real part greater than 1, since the right-hand side of (2) is finite. But it admits an infinity of complex zeros with real part between 0 and 1. Stieltjes proved, in accordance with Riemann's expectations, that these zeros are all of the form $\frac{1}{2} + it$ (where t is real); but his proof has never been published, and it has not even been established that the function ζ has no zeros on the line $\Re s = 1$.

It is this last assertion that I propose to prove here.

J. Hadamard

Hadamard's proof that $\zeta \neq 0$ on the line $L\{\sigma = 1\} := \{s \in \mathbf{C}\colon \Re s = 1\}$ used formula (2) for $\log \zeta(s)$, where $s = \sigma + it$ with $\sigma > 1$ and t real, and the representation

$$-\Re \log(1 - p^{-s}) = \Re \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} = \sum_{m=1}^{\infty} \frac{\cos(mt \log p)}{mp^{m\sigma}}.$$

Thus

$$\log|\zeta(s)| = \sum_{p} \sum_{m=1}^{\infty} \frac{\cos(mt \log p)}{mp^{m\sigma}}. \tag{3}$$

In the analysis of ζ, one can ignore the contribution of the higher prime powers, because that part of the series is uniformly bounded for $\sigma > 1$, while the sum over just the primes in (3) is not. Hadamard observed first that, because of the simple pole of ζ at $s = 1$,

$$\sum_p p^{-\sigma} \sim \log \zeta(\sigma) \sim \log \frac{1}{\sigma - 1} \quad (\sigma \to 1+). \tag{4}$$

He next noted that if $1 + it_0$ were a zero of ζ, necessarily simple, then it would follow that

$$\sum_p p^{-\sigma} \cos(t_0 \log p) \sim -\log \frac{1}{\sigma - 1} \quad (\sigma \to 1+). \tag{5}$$

Comparing (4) and (5), he concluded in succession that (a) $\cos(t_0 \log p) \approx -1$ for most primes p, (b) hence $\cos(2t_0 \log p) \approx +1$ for most primes p, and (c) finally $1 + 2it_0$ would be a pole of ζ, contradicting the fact that ζ has no singularities in **C** other than at $s = 1$. We have omitted some details that Hadamard gave to make this argument complete; they can be found also in [**THB**, Ch. 3]. Today it is customary to use a cleaner method, due to F. Mertens, that combines formula (3) with a trigonometric inequality to get an inequality for ζ that expresses Hadamard's idea. For example, the choice $3 + 4\cos\theta + \cos 2\theta \geq 0$ yields

$$\zeta(\sigma)^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| > 1 \quad (\sigma > 1).$$

Part II of Hadamard's 1896 paper, "Arithmetic Consequences," contains his deduction of the PNT. It begins with the following modest words:

> As one can see, we are quite far from having proved the assertion of Riemann-Stieltjes; we have not even been able to exclude the hypothesis of an infinity of zeros of $\zeta(s)$ approaching arbitrarily close to the limiting line $\Re s = 1$. However, the result which we have obtained suffices by itself to prove the principal arithmetic consequences which people have, up to now, sought to deduce from the properties of $\zeta(s)$.

Here are the main ingredients in Hadamard's deduction of the PNT. He firs established the following "smoothed" form of the Mellin inversion formula,

$$\sum_{n<x} a_n \log(x/n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^s}{s^2} \sum_{n=1}^{\infty} \frac{a_n}{n^s}\, ds,$$

valid for x positive and $\Sigma a_n n^{-s}$ a Dirichlet series that is absolutely convergent fo $\Re s > 1$. The arithmetic function to which he applied the formula was the vo Mangoldt function $\Lambda(n)$ that appeared in Chebyshev's work. The associate Dirichlet series satisfies the zeta function formula

$$\sum_{n=1}^{\infty} \Lambda(n) n^{-s} = -\zeta'(s)/\zeta(s),$$

which is shown by differentiating formula (2) for $\log \zeta(s)$. Using the Weierstrass Hadamard product representation for $(s-1)\zeta(s)$, the convergence of $\Sigma|\rho|^-$ (where ρ runs over the nonreal zeros of ζ), and a contour deformation an estimation of the above Mellin integral, Hadamard deduced that

$$\sum_{n \leq x} \Lambda(n) \log(x/n) \sim x.$$

From this relation the PNT follows quite easily.

Like Hadamard, de la Vallée Poussin [**VaP**] began his proof by establishing that ζ has no zeros with real part 1 (by a rather more complicated argument than that of Hadamard). He also used a smoothed form of the Mellin inversion formula, but with an expression $x^s/(s-u)(s-v)$ in place of x^s/s^2. In 1899, de la Vallée Poussin published another article in which he obtained the PNT with an error estimate

$$\pi(x) - \mathrm{li}(x) = O(x \exp\{-c \log^{\alpha} x\}), \qquad (6)$$

where $\alpha = 1/2$ and c is some positive constant. In the last paper he made use of Mertens' trigonometric inequality. A quarter of a century went by before de la Vallée Poussin's error bound was improved.

We note that the estimate (6) with a fixed positive value of α is superior to any estimate of the form

$$\pi(x) - \mathrm{li}(x) = O(x/\log^k x)$$

with fixed $k > 0$. The RH implies that (6) holds with $\alpha = 1$, $c = \frac{1}{2} - \epsilon$, as stated in (1).

LATER DEVELOPMENTS. In just over a decade after the proof of the PNT, prime number theory moved from obscurity to mainstream. So little was known on the subject in England at the turn of the century that J. E. Littlewood was assigned the task of proving the RH by E. W. Barnes, his Cambridge research supervisor, and at one point, according to G. H. Hardy, it was believed that the RH had been

E. Landau

proved. The publication of E. Landau's *Handbuch der Lehre von der Verteilung der Primzahlen* **[Lan]** in 1909 quickly changed the status of the subject. Landau's book presented in accessible form nearly everything that was then known about the distribution of primes. Incidentally, the O notation we use was popularized by Landau.

In addition to writing about prime number theory, Landau made significant contributions to the subject, including the simplification of some of the main arguments and extension of the results. For example, he was the first to prove the PNT without making use of the functional equation of ζ. His idea was to combine an analytic continuation of the zeta function a bit to the left of $L\{\sigma = 1\}$, e.g., via

$$\zeta(s) - \frac{s}{s-1} = s\int_1^\infty \frac{[x]-x}{x^{s+1}}\,dx, \quad \Re s > 0,$$

with an upper bound for the logarithmic derivative of the zeta function in a suitable zero-free region. With the aid of his new methods, Landau was able to treat some related problems, such as estimating the number of prime ideals of norm at most x in the ring of integers of an arbitrary algebraic number field **[Lan**, Sec. 242]. This result solved part of the eighth problem posed in Hilbert's famous 1900 address to the International Congress of Mathematicians.

It had long been noted, possibly already by Gauss, that

$$\pi(x) - \mathrm{li}(x) < 0 \tag{7}$$

for $x = 2, 3, \ldots$ to whatever point it was checked. In addition to this empirical evidence, theoretical support for the conjecture that (7) holds for all $x \geq 2$ was

J. E. Littlewood

provided by Riemann, who observed that his formula for $\pi(x)$ begins with the terms $\mathrm{li}(x) - \mathrm{li}(\sqrt{x})/2$. However, this conjecture was disproved by Littlewood **[Ing]**, who used almost periodic functions and diophantine ideas to show that in fact the difference changes sign infinitely often. Littlewood's proof did not provide an estimate of where the first change of sign might be, and this question attracted further attention. The suggestion was raised that the question might be undecidable. However, it was proved by S. Skewes that there is a number $x < \exp\exp\exp\exp 7.705$ for which (7) does not hold. Skewes' number, which is among the largest that have occurred in mathematics, has subsequently been replaced by a more modest number with fewer than 400 decimal digits. There is a moral here: vast amounts of empirical evidence together with a "philosophical" explanation for a mathematical phenomenon are not the same as a proof.

What is the relation between the PNT and the nonvanishing of the Riemann zeta function on $L\{\sigma = 1\}$? It is quite easy to see that the PNT implies that ζ has no zeros on the line. Proofs of the PNT were given first by Landau [**Lan**, Sec. 241] and then by Hardy and Littlewood that used, besides the nonvanishing of ζ on $L\{\sigma = 1\}$, only very weak growth conditions for $\zeta(\sigma + it)$ for $\sigma > 1$ and $|t| \to \infty$. The question arose whether the PNT could be proved using just the fact that ζ has no zeros on $L\{\sigma = 1\}$. This was answered affirmatively around 1930 by work of N. Wiener using Fourier analysis. Wiener created an approximate integral formula for $\pi(x)$ involving a compactly supported smoothing function. The following tauberian theorem **[Ch1]** provides one of the most direct proofs now known for the PNT.

Wiener-Ikehara Theorem. *Suppose f is a non-decreasing real-valued function on $[1, \infty)$ such that*

$$\int_1^{\infty} |f(u)| u^{-\sigma-1}\, du < \infty$$

for each real $\sigma > 1$. Suppose further that

$$\int_1^{\infty} f(u) u^{-s-1}\, du = \frac{\alpha}{s-1} + g(s), \quad \Re s > 1,$$

where $\alpha \in \mathbf{R}$ and g is the restriction to $\{s\colon \Re s > 1\}$ of a continuous function on the closed half plane $\{s\colon \Re s \geq 1\}$. Then

$$\lim_{u \to \infty} u^{-1} f(u) = \alpha.$$

In 1937 A. Beurling introduced an abstraction of prime number theory in which multiplicative structure was preserved but the additive structure of integers was dropped. A sequence of real numbers $p_1 \leq p_2 \leq p_3 \leq \cdots$, called "generalized primes," was introduced, and the free abelian semigroup generated from them under multiplication was called the associated sequence of "generalized integers." From the assumption that the counting function of the generalized integers satisfies the condition

$$I(x) = Ax + O(x \log^{-\gamma} x), \quad \gamma > 3/2,$$

an analogue of the PNT was established. Moreover, the condition that $\gamma > 3/2$ was shown to be best possible. A form of the Wiener-Ikehara theorem with a weaker hypothesis on the behavior of the function g near $L\{\sigma = 1\}$ can be used in the proof of Beurling's theorem.

N. Wiener

Generalized prime number theory has several applications, and it has raised interesting new problems. For example, Landau's prime ideal theorem is easily deduced from Beurling's result. Also, there are generalized prime models for which the counting function of generalized integers is quite close to that of the usual integers, but for which the analogue of the RH is false. This means that a successful proof of the RH will require more than just the facts that the positive integers are a multiplicative semigroup and that the counting function of positive integers $[x]$ is close to x; presumably, the additive structure of the integers must be taken into account. More on this topic can be found in the authors' survey article **[BaD]**.

Many different proofs have been given for the PNT. A very concise argument that uses only the analyticity and nonvanishing of $(s-1)\zeta(s)$ on the closed half plane $\{s\colon \Re s \geq 1\}$ was found by D. J. Newman **[New]**. In place of the Wiener-Ikehara theorem or an application of the Mellin inversion integral, Newman's method uses basic complex function theory to estimate the integral

$$\int \frac{x^s}{s}\left\{1+\frac{s^2}{R^2}\right\}\sum_{n=1}^{\infty}\frac{a_n}{n^s}\,ds$$

over a finite contour for large values of R. Some other interesting proofs of the

PNT include that of H. Daboussi, which uses elements of sieve theory, and a method of A. Hildebrand based on the large sieve.

De la Vallée Poussin's PNT error term was improved by Littlewood, who used exponential sum methods to find bounds for Dirichlet series. These estimates led to enlarged regions on which the zeta function is guaranteed to be nonzero and consequently to better PNT estimates. The method was developed and improved by the school of I. M. Vinogradov, leading to the bound in which (6) holds with $\alpha = 3/5 - \epsilon$.

The failure of the Chebyshev methods and the success of Riemann's program in proving the PNT led to the opinion, voiced by Hardy and others, that the PNT could be proved only with the use of the Riemann zeta function. This belief was strengthened by Wiener's proof of the equivalence of the PNT and the nonvanishing of ζ on $L\{\sigma = 1\}$. Inspired by work in sieve theory, A. Selberg developed a kind of weighted analogue of Chebyshev's identity. With this formula and an argument of P. Erdös he succeeded in giving an "elementary" proof of the PNT. Subsequently, Selberg and Erdös each discovered an independent proof. Their arguments are considered elementary in the sense that they do not involve the zeta function, complex analysis, or Fourier methods; however, the methods are quite intricate. Subsequently, elementary estimates were sought for the PNT error term, and by use of higher order analogues of Selberg's formula and more elaborate tauberian arguments, error terms of type (6) with $\alpha = 1/6 - \epsilon$ were achieved. For a survey of the use of elementary methods in prime number theory, see **[Dia]**.

We conclude with a summary of what is now known about the truth of the RH. If the RH is false and ζ has even a single nonreal zero off the critical line $\{s \in \mathbf{C}: \Re s = 1/2\}$, there would be consequences for prime number theory, such as in the quality of the PNT error term. The numerical evidence in support of the RH is very great—by comparing the sign changes of a real-valued equivalent of $\zeta(\frac{1}{2} + it)$ with the zeros predicted by use of the argument principle, van de Lune, te Riele, and Winter showed that the first one and a half billion (!) nonreal zeros of zeta lie on the critical line and are simple. In the 1920's, Littlewood showed that almost all the nonreal zeros lie in any given strip of positive width that contains the critical line. Hardy proved that there were infinitely many zeros of zeta on the critical line, and later Selberg showed that a positive proportion of the nonreal zeros were on the line. Near the end of his life, N. Levinson introduced an efficient zero counting method, which B. Conrey has developed to show that more than $2/5$ of the nonreal zeta zeros are simple and lie on the critical line.

SOURCES. The theory of the distribution of prime numbers is a rich and fascinating topic. In this survey we have had to treat fleetingly or omit entirely many interesting topics. Also, it was not feasible to list the sources for all the facts cited. The following books and articles discuss further topics and provide references to original sources.

Landau's *Handbuch* **[Lan]** remains an excellent introduction to prime number theory and is a reference for virtually all early results in the area. The second edition of the *Handbuch*, edited by the first author, contains information on work up to about 1950 on the distribution of primes. The books of Chandrasekharan **[Ch1]**, **[Ch2]**, Ingham **[Ing]**, and Ellison & Mendes-France **[EMF]** provide very readable introductions to the subject. Titchmarsh & Heath-Brown **[THB]** and Ivić **[Ivc]** are standard references on the Riemann zeta function, and Edwards **[Edw]** provides a historical view of this subject. There are detailed and authoritative encyclopedia articles on prime number theory by Hadamard **[BHM]** and by Bohr and Cramér [**BoC**]. The recent survey article of W. Schwarz [**Sch**] describes the development of prime number theory in the twentieth century, including several topics that we have not treated. Finally, Ribenboim [**Rib**] provides a kind of Guinness record book about primes and includes extensive references.

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