$$\frac{2K}{\pi} = \frac{2^{1/4}(kk')^{-1/6}}{\sqrt{2\pi p}} \left\{ \frac{\pi\,\Gamma\left(\frac{\alpha}{p}\right)}{\pi\,\Gamma\left(\frac{\beta}{p}\right)} \right\}^{w/4} \qquad (8)$$

where (p *is a prime*) *and* $w = 6$ *if* $p = 3$, $w = 2$ *if* $p > 3$; α *runs through the* $\frac{p-1}{2}$ *quadratic residues of* p *that lie between* 0 *and* p, *while* β *runs through the remaining* $\frac{p-1}{2}$ *numbers between* 0 *and* p.

Specializing again to the case $p = 7$ we obtain in the usual notation for hypergeometric series:

$$F(1/4, 1/4, 1; 1/64) = \sqrt{\frac{2}{7\pi}} \left\{ \frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{\Gamma(3/7)\Gamma(5/7)\Gamma(6/7)} \right\}^{1/2}$$

5. Let $G_d(s)$ denote the analytical continuation of the function defined for $\sigma > 3/2$ by the series

$$\sum{}'(x^2 + y^2 + dz^2)^{-s}$$

From a formula similar to (4) it is deduced that

THEOREM: *There exists a real number* θ_d *such that*

$$G_d(\theta_d) = 0 \qquad [d > d_0]$$

where $\theta_d \to 0$ *as* $d \to \infty$, *but* $\theta_d \neq 0$.

ON A NEW METHOD IN ELEMENTARY NUMBER THEORY WHICH LEADS TO AN ELEMENTARY PROOF OF THE PRIME NUMBER THEOREM

BY P. ERDÖS

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY

Communicated by P. A. Smith, April 16, 1949

1. *Introduction.*—In the course of several important researches in elementary number theory A. Selberg[1] proved some months ago the following asymptotic formula:

$$\sum_{p \leq x} (\log p)^2 + \sum_{pq \leq x} \log p \log q = 2x \log x + O(x), \qquad (1)$$

where p and q run over the primes. This is of course an immediate consequence of the prime number theorem. The point is that Selberg's ingenious proof of (1) is completely elementary. Thus (1) can be used as a starting point for elementary proofs of various theorems in analytical number theory, which previously seemed inaccessible by elementary methods.

Using (1) I proved that $p_{n+1}/p_n \to 1$ as $n \to \infty$. In fact, I proved the following slightly stronger result: To every c there exists a positive $\delta(c)$, so that for x sufficiently large we have

$$\pi[x(1 + c)] - \pi(x) > \delta(c)x/\log x \qquad (2)$$

where $\pi(x)$ is the number at primes not exceeding x.

I communicated this proof of (2) to Selberg, who, two days later, using (1), (2) and the ideas of the proof of (2), deduced the prime number theorem $\lim_{x \to \infty} \frac{\pi(x)\log x}{x} = 1$ or, equivalently[2]

$$\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1, \text{ where } \vartheta(x) = \sum_{p \leqq x} \log p. \qquad (3)$$

In a few more days, Selberg simplified my proof of (2), and later we jointly simplified the proof of the prime number theorem. The new proof no longer required (2), but used the same ideas as in the proof of (2) and (3). I was also able to prove the prime number theorem for arithmetic progressions. My proof of the latter was helped by discussions with Selberg and it utilizes ideas of Selberg's previous elementary proof of Dirichlet's theorem,[3] according to which every arithmetic progression whose first term and difference are relatively prime contains infinitely many primes. This proof will be given in a separate paper.

Selberg has now a more direct proof of (3), which is not yet published. It is possible, therefore, that the present method may prove to be only of historical interest.

I now proceed to give the proofs as they occurred in chronological order. (It should be remarked that we never utilize the full strength of (1), indeed an error term $o(x \log x)$ is all that is used in the following proofs.)

We introduce the following notation:

$$A = \limsup_{x \to \infty} \frac{\vartheta(x)}{x}, \quad a = \liminf_{x \to \infty} \frac{\vartheta(x)}{x}.$$

First, we state a few elementary facts about primes which will be used subsequently. Of these, I, II and IV are well known in elementary prime number theory, while III is shown to be a simple consequence of (1).

I. $a > 0$.

II. $\sum_{p \leqq x} \frac{\log p}{p} = [1 + o(1)]\log x,$

III. Let $x_2 > x_1$. Then

$$\vartheta(x_2) - \vartheta(x_1) < 2(x_2 - x_1) + o(x_2).$$

Thus, in particular, if $x_1 = 0$, we obtain $A \leqq 2$.

Put in (1) $x = x_2$ and $x = x_1$ and subtract. Then we obtain

$$\sum_{x_1 < p \leqq x_2} (\log p)^2 \leqq 2x_2 \log x_2 - 2x_1 \log x_1 + o(x_2 \log x_2) \leqq 2(x_2 - x_1)\log x_2 + o(x_2 \log x_2). \quad (4)$$

We distinguish two cases: (A) $x_1 \geqq x_2/(\log x_2)^2$. Then clearly $\log x_1 = (1 + o(1))\log x_2$ and III follows from (4) on dividing both sides by $\log x_2$. (B) $x_1 < x_2/(\log x_2)^2$. Then we have by (A)

$$\vartheta(x_2) - \vartheta(x_1) < \vartheta(x_2) - \vartheta(x_2/(\log x_2)^2) + \frac{x_2}{(\log x_2)^2}\log x_2 < 2\left(x_2 - \frac{x_2}{(\log x_2)^2}\right) + o(x_2) = 2(x_2 - x_1) + o(x_2), \text{ q. e. d.}$$

IV. $A \leqq 1.5$. This is a consequence of the known result $\vartheta(x) \leqq 1.5x$.

2. Proof of (2).—It is equivalent to prove that to every positive c there exists a positive $\delta(c)$ such that $\vartheta[(1 + c)x] - \vartheta(x) > \delta(c)x$ for x sufficiently large.

Suppose this not true, then there exist positive constants c' and corresponding arbitrarily large x so that

$$\vartheta[x(1 + c')] - \vartheta(x) = o(x). \quad (5)$$

Put $C = \sup c'$. It easily follows from I and the finiteness of A that $C < \infty$.

First we show that C satisfies (5), in other words, that there are arbitrarily large values of x for which

$$\vartheta[x(1 + C)] - \vartheta(x) = o(x). \quad (6)$$

Choose $c' > C - {}^1/_2\epsilon$ and let $x \to \infty$ through values satisfying (5). Then by III we have

$$\vartheta[x(1 + C)] - \vartheta(x) = \vartheta[x(1 + C)] - \vartheta[x(1 + c')] + \vartheta[x(1 + c')] - \vartheta(x) \leqq 2(C - c')x + o(x) < \epsilon x + o(x),$$

which (since ϵ can be chosen arbitrarily small) proves (6).

Now we shall show that (6) leads to a contradiction. From (1) we obtain by subtraction

$$\sum_{x < p \leqq x(1 + C)} (\log p)^2 + \sum_{x < pq \leqq x(1 + C)} \log p \log q = 2Cx \log x + o(x \log x).$$

From (6) we have for suitable x since $\sum_{x < p \leqq x(1 + C)} (\log p)^2 = o(x \log x)$

$$\sum_{p \leq x(1+X)} \log p\left(\vartheta\left[\frac{x}{p}(1 + C)\right] - \vartheta\left(\frac{x}{p}\right)\right) = 2Cx \log x + o(x \log x) \quad (7)$$

Now we deduce the following fundamental lemma.

LEMMA 1. *Let $x \to \infty$ through values satisfying (6), then for all primes $p \leqq x(1 + C)$, except possibly for a set of primes for which*

$$\sum \frac{\log p}{p} = o(\log x) \quad (8)$$

we have

$$\vartheta\left[\frac{x}{p}(1 + C)\right] - \vartheta\left(\frac{x}{p}\right) = 2C\frac{x}{p} + o\left(\frac{x}{p}\right). \quad (9)$$

Suppose the lemma is not true. Then there exist two positive constants b_1 and b_2 so that for arbitrarily large x (satisfying (6)) we have for a set of primes satisfying $\sum_{p \leq x(1+C)} \frac{\log r}{r} \sim b_1 \log x$

$$\vartheta\left[\frac{x}{r}(1 + C)\right] - \vartheta\left(\frac{x}{p}\right) < (2C - b_2)\frac{x}{r}. \quad (10)$$

But then from II, III and (10), since (9) holds at best for a set of primes satisfying $\sum \frac{\log r}{r} \sim (1 - b_1) \log x$ we have

$$\sum_{p \leq x(1+C)} \log p\left(\vartheta\left[\frac{x}{r}(1 + C)\right] - \vartheta(x)\right) \leqq b_1(2C - b_2)x \log x +$$
$$2C(1 - b_1)x \log x + o(x \log x) = (2C - b_1b_2)x \log x + o(x \log x)$$

But this contradicts (7), hence the lemma is established.

The primes satisfying (9) we shall call good primes, the other primes we shall call bad primes (of course the goodness and badness of a prime depends on x).

We shall prove the existence of a sequence of good primes $p_1 < p_2 < \ldots p_k$ satisfying the following conditions:

$$10p_1 < p_k < 100p_1,\ (1 + C)(1 + t)^2p_i > p_{i+1} >$$
$$(1 + t)p_i,\ i = 1, 2, \ldots, k - 1 \quad (11)$$

where t is a small but fixed number (small compared to C). Since $(1 + t)^k < 100$ it is clear that $k < k_o$ with constant $k_o = k_o(t)$.

Suppose we already established the existence of a sequence satisfying (11). Then we prove (2) as follows: Consider the two intervals

$$\left[\frac{x}{p_{i+1}}, \frac{x}{p_{i+1}}(1 + C)\right], \left[\frac{x}{p_i}, \frac{x}{p_i}(1 + C)\right] \tag{12}$$

If they overlap, then by (11)

$$\frac{x}{p_{i+1}}(1 + t) < \frac{x}{p_i} < \frac{x}{p_{i+1}}(1 + C)$$

Clearly

$$\vartheta\left(\frac{x}{p_i}\right) - \vartheta\left(\frac{x}{p_{i+1}}\right) = 2\left(\frac{x}{p_i} - \frac{x}{p_{i+1}}\right) + o\left(\frac{x}{p_i}\right) \tag{13}$$

since otherwise

$$\vartheta\left(\frac{x}{p_i}\right) - \vartheta\left(\frac{x}{p_{i+1}}\right) < (2 - c_1)\left(\frac{x}{p_i} - \frac{x}{p_{i+1}}\right)$$

with $c_1 > 0$ and we would have from (9)

$$\vartheta\left[\frac{x}{p_{i+1}}(1 + C)\right] - \vartheta\left(\frac{x}{p_i}\right) > (2 + c_2)\left[\frac{x}{p_{i+1}}(1 + C) - \frac{x}{p_i}\right]$$

which contradicts III. Adding (13) and (9) with $p = p_i$ we obtain

$$\vartheta\left[\frac{x}{p_i}(1 + C)\right] - \vartheta\left(\frac{x}{p_{i+1}}\right) = 2\left[\frac{x}{p_i}(1 + C)\right] - \frac{x}{p_{i+1}} + o\left(\frac{x}{p_i}\right) \tag{14}$$

If the intervals (12) do not overlap we obtain by a simple calculation (using (9) and the fact that t is small)

$$\vartheta\left[\frac{x}{p_i}(1 + C)\right] - \vartheta\left(\frac{x}{p_{i+1}}\right) > 1.9\left[\frac{x(1 + C)}{p_i} - \frac{x}{p_{i+1}}\right] \tag{15}$$

Adding all the equations (14) and (15) (for $i = 1, 2, \ldots, k$) we clearly obtain

$$\vartheta\left[\frac{x}{p_1}(1 + C)\right] - \vartheta\left(\frac{x}{p_k}\right) > 1.9\left[\frac{x}{p_1}(1 + C) - \frac{x}{p_k}\right] \tag{16}$$

Since $p_k > 10p_1$ we obtain from (16)

$$\vartheta\left[\frac{x}{p_1}(1 + C)\right] > 1.6\frac{x}{p_1}(1 + C). \tag{17}$$

But (17) contradicts IV.

Thus to complete the proof of (2) it will suffice to show the existence of a sequence of good primes satisfying (11).

Consider the intervals

$$I_r = (B^{2r}, B^{2r+1}),\ r = 0, 1, \ldots, \left[\frac{\log x}{2 \log B}\right] - 1,$$

where B is a fixed, sufficiently large number. Clearly all the intervals I_r lie in the interval $(0, x)$. First we show that with the exception of $o(\log x)$ r's the interval I_r contains good primes. From I and IV it easily follows that for sufficiently large B we have (since $\vartheta(Bx) - \vartheta(x) > cx$)

$$\sum_{p \text{ in } I_r} \frac{\log p}{p} > c_1 (c_1 > 0 \text{ independent of } r)$$

Thus if there were $c_2 \log x$ with $c_2 > 0$ of the I_r's without good primes, we would have

$$\sum_{p \text{ bad}} \frac{\log p}{p} > c_1 c_2 \log x$$

which contradicts (8).

Let now $p_1^{(r)}$ be the smallest good prime in I_3 (if it exists), and suppose that a sequence $p_1^{(r)}, p_2^{(r)}, \ldots, p_t^{(r)}$ satisfying (11) exists, but no $p_{t+1}^{(r)}$ satisfying (11) can be found. Thus, all the primes in

$$J_t^{(r)} = [p_t^{(r)} (1 + t), p_t^{(r)}(1 + t)^2(1 + C)]$$

are bad. We have, by the definition of C,

$$\sum_{p \text{ in } J^{(r)}} \log p > \eta p_t^{(r)}(1 + t)^2(1 + C), (\eta \text{ absolute constant}).$$

Thus

$$\sum_{p \text{ in } J^{(r)}} \frac{\log p}{p} > \eta \tag{18}$$

Clearly for $B > 100$ we have $p_t^{(r)}(1 + t)^2(1 + C) < B^{2r+2}$. Thus the intervals $J_{t_1}^{r_1}, J_{t_2}^{r_2}, \ldots$ do not overlap. Hence from (18), since the number of r's with $p_1^{(r)}$ existing is $> \dfrac{\log x}{4 \log B}$,

$$\sum_{p \text{ bad}} \frac{\log p}{p} > \frac{\eta \log x}{4 \log B}$$

which contradicts (8) and establishes (2).

3. *Selberg's deduction of the prime number theorem from* (2).—Assume $a < A$. First we prove the following lemmas.

LEMMA 2. *$a + A = 2$.*

Choose $x \to \infty$ so that $\vartheta(x) = Ax + o(x)$. Then a simple computation (as in the proof of III) shows that

$$\sum_{p \leq x} (\log p)^2 = Ax \log x + o(x \log x).$$

Thus from (1)

$$\sum_{p \leqq x} (\log p)\vartheta\left(\frac{x}{r}\right) = (2 - A)x \log x + o(x \log x). \tag{19}$$

By the definition of a and by II we obtain by a simple computation

$$\sum_{p \leqq x} (\log p)\vartheta\left(\frac{x}{r}\right) \geqq ax \sum_{p \leqq x} \frac{\log p}{p} + o(x \log x) = ax \log x + o(x \log x)$$

Thus from (19), $2 \geqq a + A$. We obtain $a + A \leqq 2$ similarly, by choosing x so that $\vartheta(x) = ax + o(x)$. Thus lemma 2 is proved.

LEMMA 3. *Let $x \rightarrow \infty$ so that $\vartheta(x) = Ax + o(x)$. Then for any prime $p_i \leqq x$ except possible for a set of primes satisfying*

$$\sum \frac{\log p}{p} = o(\log x) \tag{20}$$

we have

$$\vartheta\left(\frac{x}{p_i}\right) = a\frac{x}{p_i} + o\left(\frac{x}{p_i}\right) \tag{21}$$

Suppose the lemma is false. Then as in the proof of lemma 1 there exist two positive constants b_1 and b_2 so that for arbitrarily large x, satisfying $\vartheta(x) = Ax + o(x)$, and for a set of primes satisfying $\sum \frac{\log p}{p} > b_1 \log x$, *we have*

$$\vartheta\left(\frac{x}{p}\right) > (a + b_2)\frac{x}{p} \tag{22}$$

But then we have from (22), lemma 2, (19) and II (as in the proof of lemma 1)

$$ax \log x + o(x \log x) = \sum_{p \leqq x} (\log p)\vartheta\left(\frac{x}{p}\right) > b_1(a + b_2)x \log x + (1 - b_1)ax$$
$$\log x + o(x \log x) = ax \log x + b_1 b_2 x \log x + o(x \log x),$$

an evident contradiction. This proves lemma 3.

LEMMA 4. *Let p_1 be the smallest prime satisfying (21). Then $p_1 < x^\epsilon$, and for all primes $p_j < x/p_1$ except possible for a set of primes satisfying*

$$\sum \frac{\log p}{p} = o(\log x) \tag{23}$$

we have

$$\vartheta\left(\frac{x}{p_1p_j}\right) = A\frac{x}{p_1p_j} + o\left(\frac{x}{p_1p_j}\right) \tag{24}$$

$p_1 < x^\epsilon$ follows immediately from (20) and II. The second part of lemma 4 follows by applying the argument of lemma 3 to x/p_1 instead of x and interchanging A and a.

Now the deduction of the prime number theorem. Let p_i be any prime satisfying (21). Assume $\frac{x}{p_1p_j} < \frac{x}{p_i}$. Then (since $\vartheta(x)$ is non-decreasing) from (21) and (24)

$$a\frac{x}{p_i} + o\left(\frac{x}{p_i}\right) \leqq A\frac{x}{p_1p_j} + o\left(\frac{x}{p_1p_j}\right)$$

or p_j cannot lie in the interval

$$I_i = \left[\frac{p_i}{p_1}, \frac{p_i}{p_1}\left(\frac{A}{a} - \delta\right)\right],$$

where $\delta > 0$ is an arbitrary fixed number. Hence all primes in I_i must be "bad," i.e., they do not satisfy (24). But it immediately follows from (2) that

$$\sum_{p \text{ in } I_i} \frac{\log p}{p} > \eta$$

To obtain a contradiction to (23) it suffices to construct $c \log x$ disjoint intervals I_i. This can be accomplished in the same way as in the end of the proof of (2) (where the disjoint intervals $J_i^{(r)}$ were constructed). This completes the first elementary proof of the prime number theorem.

4. *Sketch of Selberg's simplification of the proof of* (2).—If we can find two good primes satisfying

$$(1 + c)p_1 > p_2 > (1 + t)p_1, \quad c > \frac{C}{1 + t} \tag{25}$$

then (2) follows easily. The intervals $\left[\frac{x}{p_1}, \frac{x}{p_1}(1 + c)\right], \left[\frac{x}{p_2}, \frac{x}{p_2}(1 + c)\right],$ overlap. Thus (13), with $i = 1$, holds. But then exactly as in lemma 1 there exists a prime p so that

$$\vartheta\left[\frac{x}{p_1p}(1 + c)\right] - \vartheta\left(\frac{x}{p_2p}\right) = o\left(\frac{x}{p_2p}\right).$$

But this is impossible (by the definition of C) since

$$\frac{x}{p_1p}(1 + c)/\frac{x}{p_2p} = \frac{p_2(1 + c)}{p_1} > 1 + C.$$

Thus we only have to show that good primes satisfying (24) exist, and this can be accomplished by using III (a contradiction with $\sum_{p \text{ good}} \frac{\log p}{p} = [1 + o(1)]\log x$ can be established similarly as in the previous proof).

5. *The joint simplified proof of the prime number theorem.*—

LEMMA 5. *Let $x_2 > x_1$ and $x_1 \to \infty$. Assume that $\vartheta(x_1) = Ax_1 + o(x_1)$ and $\vartheta(x_2) = ax_2 + o(x_2)$, or $\vartheta(x_1) = ax_1 + o(x_1)$ and $\vartheta(x_2) = Ax_2 + o(x_2)$. Then*

$$x_2/x_1 \leqq A/a + o(1).$$

Since $\vartheta(x)$ is non-decreasing we have in the first case

$$ax_2 + o(x_2) \leqq Ax_1 + o(x_1) \text{ or } x_2/x_1 \leqq A/a + o(1)$$

In the second case we have by III $\vartheta(x_2) - \vartheta(x_1) \leqq 2(x_2 - x_1) + o(x_2)$

$$ax_1 + 2(x_2 - x_1) \leqq Ax_2 + o(x_2) \text{ or } (2 - A)x_2 \leqq (2 - a)x_1 + o(x_2).$$

Hence by lemma 2, $ax_2 \leqq Ax_1 + o(x_2)$. Thus again $x_2/x_1 \leqq A/a + o(1)$. q.e.d.

Put $1 + D = \frac{A}{a} + \delta$ where δ is sufficiently small, and will be determined later. Next we prove the following result.

LEMMA 6.

$$\sum_{y \leqq p \leqq (1+D)y} \frac{\log p}{p} > \eta (\eta \text{ independent of } y).$$

First we show that

$$\sum_{y \leqq p \leqq (1+D)y} \log p > \eta(1 + D)y. \qquad (26)$$

If (26) is false then for a suitable sequence of y's we have $\vartheta[(1 + D)y] - \vartheta(y) = o(y)$. But then for these y's

$$\frac{\vartheta[(1 + D)y]}{(1 + D)y} = \frac{\vartheta(y) + o(y)}{(1 + D)y} \leqq \frac{Ay + o(y)}{(1 + D)y} < a - c_1,$$

which contradicts the definition of a. Thus (26) holds and lemma 6 follows immediately.

Choose now x so that $\vartheta(x) = Ax + \rho(x)$. Then by lemmas 3 and 4 we obtain (p_1, p_i and p_j having the same meaning as in lemmas 3 and 4)

$$\vartheta\left(\frac{x}{p_1p_j}\right) = A\frac{x}{p_1p_j} + o\left(\frac{x}{p_1p_j}\right), \quad \vartheta\left(\frac{x}{p_i}\right) = a\frac{x}{p_i} + o\left(\frac{x}{p_i}\right)$$

From lemma 5 we obtain that for any fixed ϵ and sufficiently large x (satisfying $\vartheta(x) = Ax + o(x)$)

$$\text{either } \frac{x}{p_i} > \left(\frac{A}{a} - \epsilon\right)\frac{x}{p_1 p_j} \text{ or } \frac{x}{p_i} < \left(\frac{a}{A} + \epsilon\right)\frac{x}{p_1 p_j}$$

Hence p_j cannot lie in the interval

$$I_i = \left[\left(\frac{a}{A} + \right)\frac{p_i}{p_1}\left(\frac{A}{a} - \epsilon\right)\frac{p_i}{p_1}\right].$$

Now if δ is small enough then $1 + D \leqq \left(\frac{A}{a} - \epsilon\right)/\left(\frac{a}{A} + \epsilon\right)$. Hence by lemma 6

$$\sum_{p \text{ in } I_i} \frac{\log p}{p} > \eta$$

But by what has been said before all the primes in I_i are bad (i.e., they do not satisfy (24)). Thus to arrive at a contradiction with (23) it will suffice as in the proof of (2) to construct $c \log x$ disjoint intervals I_i. This can be accomplished as in the proof of (2), which completes the proof of the prime number theorem.

6. Perhaps this last step can be carried out slightly more easily as follows: Put

$$S = \sum \frac{\log p_i}{p_i} \sum_{p \text{ in } I_i} \frac{\log p}{p} \tag{27}$$

where p_i runs through the primes satisfying (21). As stated before all the primes in I_i are bad (i.e., they do not satisfy (24)). Thus we have from (27)

$$S > \eta \sum \frac{\log p_i}{p_i} > \frac{\eta}{2} \log x \tag{28}$$

since by II and (20) $\Sigma \frac{\log p_i}{p_i} > {}^1/_2 \log x$ for large x.

On the other hand by interchanging the order of summation we obtain

$$S = \sum \frac{\log p}{p} \sum_{p \text{ in } J_p} \frac{\log p_i}{p_i}$$

where p runs through all the primes of all the intervals I_i (each p is, of course, counted only once) and p_i runs through the primes satisfying (21) of the interval

$$J_p = \left[pp_1\left(\frac{A}{a} - \epsilon\right)^{-1}, \quad pp_1\left(\frac{a}{A} + \epsilon\right)^{-1}\right].$$

We evidently have from $A < \infty$

$$\sum_{p_i \text{ in } J_p} \frac{\log p_i}{p_i} < c$$

Hence

$$\sum \frac{\log p}{p} > S/c$$

or from (27)

$$\sum \frac{\log p}{p} > \frac{\eta}{2c} \log x,$$

which contradicts (22) and completes the proof.

[1] Selberg's proof of (1) is not yet published.

[2] See, for example, Landau, E., *Handbuch der Lehre von der Verteilung der Primzahlen*, § 19, or Ingham, A. E., *The Distribution of Prime Numbers*, p. 13.

[3] An analogous result is used in Selberg's proof of Dirichlet's theorem.

[4] See, for example, Landau, E., op. cit., §§18 and 26, or Ingham, A. E., op. cit., pp. 14, 15 and 22.

ON THE STRUCTURE OF LOCALLY COMPACT GROUPS

BY ANDREW M. GLEASON

SOCIETY OF FELLOWS, HARVARD UNIVERSITY

Communicated by Hassler Whitney, May 7, 1949

1. *Introduction*—Locally compact groups have attracted a great deal of study in the years since the introduction of invariant integration by Haar.[1] It has been shown that their structure is closely related to that of Lie groups in certain important cases (compact,[2] abelian[3] and solvable[4] groups), and it is widely conjectured that similar results are valid in general. We shall state here certain theorems which strengthen this conjecture and reduce its verification to the study of simple groups.

2. *The Extension Theorem for Lie Groups.*—THEOREM 1. *Let G be a topological group. Suppose that G has a closed normal subgroup N such that both N and G/N are Lie groups. Then G is itself a Lie group.*

In case N is abelian, Kuranishi[5] has proved this theorem under the additional hypothesis that there is a local cross-section for the cosets of N; that is, a closed set having exactly one point in common with each coset of N near the identity. The author has shown[6] that such a cross-section set always exists for abelian Lie groups. Hence our theorem is true for the special case that N is abelian.