FINITE ELEMENT METHODS FOR THE NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

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PREFACE

This is the current version of notes that I have used for the past thirty-five years in graduate courses at the University of Tennessee, Knoxville, the University of Crete, the National Technical University of Athens, and the University of Athens. They have evolved, and aged, over the years but I hope they may still prove useful to students interested in learning the basic theory of Galerkin - finite element methods and some facts about Sobolev spaces.

I first heard about approximation with cubic Hermite functions and splines from George Fix in the Numerical Analysis graduate course at Harvard in the fall of 1971, and also, subsequently, from Garrett Birkhoff of course. But most of the basic techniques of the analysis of Galerkin methods I learnt from courses and seminars that Garth Baker taught at Harvard during the period 1973-75.

Over the years I was fortunate to be associated with and learn more about Galerkin methods from Max Gunzburger, Ohannes Karakashian, Larry Bales, Bill McKinney, George Akrivis, Vidar Thomée and my students; for my debt to the latter it is apt to say that $\delta\iota\delta\acute{\alpha}\sigma\kappa\omega$ $\grave{\alpha}\epsilon\grave{\iota}$ $\delta\iota\delta\alpha\sigma\kappa\acute{o}\mu\epsilon\nu o\varsigma$.

I would like to thank very much Dimitri Mitsoudis and Gregory Kounadis for transforming the manuscript into TeX.

V. A. Dougalis

Athens, March 2012

In the revised 2013 edition, two new chapters 6 and 7 on Galerkin finite element methods for parabolic and second-order hyperbolic equations were added. These had previously existed in hand-written form, and I would like to thank Gregory Kounadis and Leetha Saridakis for writing them in TeX.

V. A. Dougalis

Athens, February 2013

In the 2019 edition I revised Chapter 2 and added Appendix A with proofs of some facts about function spaces (mainly density and approximation results stated in Section 2.2). In this way, Chapter 2 is now fairly self-contained. I also revised Chapter 3 and added sections 3.4–3.7 to it; these contain well-known results on Galerkin methods for two-point boundary-value-problems that I have been mentioning over the years in reading courses and seminars. My thanks go again to Gregory Kounadis for writing the hand-written notes in TeX.

V. A. Dougalis

Athens, November 2019.

Contents

Chapter 1

Some Elements of Hilbert Space Theory

1.1 Vector spaces

A set V of elements u, v, w, ... is called a *vector space* (over the complex numbers) if

1. For every pair of elements $u \in V$, $v \in V$ we define a new element $w \in V$, their *sum*, denoted by $w = u + v$.

2. For every complex number λ and every $u \in V$ we define an element $w = \lambda\, u \in V$, the *product* of λ and u.

3. Sum and product obey the following laws:

   i. $\forall u, v \in V :\ u + v = v + u$.

   ii. $\forall u, v, w \in V :\ (u + v) + w = u + (v + w)$.

   iii. $\exists\, 0 \in V$ such that $u + 0 = u$, $\forall u \in V$.

   iv. $\forall u \in V \ \exists\, (-u) \in V$ such that $u + (-u) = 0$.

   v. $1 \cdot u = u$, $\forall u \in V$.

   vi. $\lambda(\mu u) = (\lambda\mu)u$, $\forall u \in V$ and complex λ, μ.

   vii. $(\lambda + \mu)u = \lambda u + \mu u$, for $u \in V$, λ, μ complex.

   viii. $\lambda(u + v) = \lambda u + \lambda v$, for u, $v \in V$, λ complex.

The elements u, v, w, ... of V are called *vectors*. It is clear that we may also consider vector spaces over the *real numbers*. In fact we will later consider only such *real* vector spaces.

An expression of the form

$$\lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_n u_n,$$

where λ_i complex numbers and $u_i \in V$, is called a *linear combination* of the vectors u_i, $1 \leq i \leq n$. The vectors $u_1, \ldots, u_n$ are called *linearly dependent* if there exist complex numbers λ_i, not all zero, for which:

$$\lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_n u_n = 0.$$

They are called *linearly independent* if they are not linearly dependent, i.e. if $\lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_n u_n = 0$ holds only in the case $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$.

A vector space V is called *finite-dimensional* (of dimension n) if V contains n linearly independent vectors and if any $n + 1$ vectors in V are linearly dependent. As a consequence, a set of n linearly independent vectors forms a *basis* of V, i.e. it is a set of linearly independent vectors that *spans* V, i.e. such that any u in V can be written uniquely as a linear combination of the basis vectors.

1.2   Inner product, Norm

A vector space V is called an *inner product space* if for every pair of elements $u \in V$, $v \in V$ we can define a complex number, denoted by (u, v) and called the *inner product* of u and v, with the following properties:

1. $\forall u \in V : \ (u, u) \geq 0$. If $(u, u) = 0$ then $u = 0$.

2. $(u, v) = \overline{(v, u)}$, $\forall u, v \in V$, where $\overline{z}$ is the complex conjugate of z.

3. $(\lambda u + \mu v, w) = \lambda(u, w) + \mu(v, w)$ for $u, v, w \in V$, λ, μ complex.

As a consequence of (2) and (3) $(u, \lambda v) = \overline{\lambda}\,(u, v)$ for $u, v \in V$ and λ complex. The vectors u, v are called *orthogonal* if $(u, v) = 0$.

For every $u \in V$ we define the nonnegative number $\|u\|$ by

$$\|u\| = (u, u)^{\frac{1}{2}},$$

which is called the *norm* of u. As a consequence of the properties of the inner product we see that:

i. $\|u\| \geq 0$ and if $\|u\| = 0$, then $u = 0$

ii. $\forall$ complex λ, $u \in V$: $\|\lambda u\| = |\lambda| \, \|u\|$

iii. $\forall u, v \in V : \|u + v\| \leq \|u\| + \|v\|$ (*Triangle inequality*).

To prove (iii) we first prove the *Cauchy–Schwarz Inequality*:

iv. $|(u, v)| \leq \|u\| \, \|v\|$, $\forall u, v \in V$.

To prove (iv) we may assume that $(u, v) \neq 0$. We let now $\theta = \frac{(u,v)}{|(u,v)|}$. We find then for any real λ that

$$0 \leq (\bar{\theta} u + \lambda v, \bar{\theta} u + \lambda v) = \lambda^2 (v, v) + 2\lambda \, |(u, v)| + (u, u).$$

Hence for any real λ the quadratic on the right hand side of this inequality is nonnegative. Hence, necessarily,

$$|(u, v)|^2 \leq (u, u) \, (v, v),$$

which gives (iv). To prove now the triangle inequality (iii) we see that

$$\begin{aligned}
\|u + v\|^2 &= (u + v, u + v) = (u, u) + (v, v) + (u, v) + (v, u) \\
&\leq \|u\|^2 + \|v\|^2 + 2 \, |(u, v)| \leq \|u\|^2 + \|v\|^2 + 2 \, \|u\| \, \|v\| \\
&= (\|u\| + \|v\|)^2,
\end{aligned}$$

from which (iii) follows.

Supplied only with a norm that just satisfies properties (i)–(iii), V becomes a *normed vector space*. In an inner product space the norm is induced by the inner product according to the formula $\|u\| = (u, u)^{1/2}$.

1.3 Some topological concepts

In a normed vector space V we define the *distance* ρ of two vectors u and v as $\rho(u, v) = \|u - v\|$. It is straightforward to see that ρ defines a *metric* on V. If u_0 is a fixed vector in V and δ a given positive number, the set of vectors v in V which satisfy $\|v - u_0\| < \delta$ is called the *open ball with center u_0 and radius δ*. The set $\|v - u_0\| \leq \delta$ is the *closed* ball with center u_0 and radius δ. We say that a sequence of vectors (u_i)

in V is *convergent* if there exists a vector $u \in V$ such that, given $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ for which:

$$\|u_n - u\| < \epsilon, \text{ for all } n \geq N.$$

We call u the *limit* of the sequence (u_i) and write $\lim u_n = u$ or $u_n \to u$ in V as $n \to \infty$. It is easy to see that a convergent sequence has only one limit. Obviously $u_n \to u$ in $V \Leftrightarrow \|u_n - u\| \to 0$ as $n \to \infty$.

A sequence of vectors (u_i) in V is said to be a *Cauchy sequence* if given any $\epsilon > 0$, there exists an integer $N = N(\epsilon)$ such that

$$\|u_n - u_m\| < \epsilon, \text{ for all } m, n \geq N.$$

It is easy to see that every convergent sequence is Cauchy. The converse is not always true. We will say that V is *complete* whenever every Cauchy sequence in V is convergent. A subset A of V is called a *dense* subset of V if for every $u \in V$ there exists a sequence (u_i) of vectors in A such that $u_n \to u$ as $n \to \infty$.

Exercise: If $u_n \to u$, $v_n \to v$ in V then:

(a) $\lim (\lambda u_n + \mu v_n) = \lambda u + \mu v$ for all complex λ, μ.

(b) $\lim (u_n, v_n) = (u, v)$ (and as a consequence we say that the inner product is a continuous function of its arguments).

(c) $\lim \|u_n\| = \|u\|$.

(d) $\lim \lambda_n u_n = \lambda u$ for every convergent sequence $\lambda_n \to \lambda$ of complex numbers. $\Diamond$

1.4   Hilbert space

A complete inner product space V is called a Hilbert space. In other words, a Hilbert space is an inner product space in which any Cauchy sequence (with respect to the norm induced by the inner product) is convergent. We will usually denote Hilbert spaces by H.

A subset S of a Hilbert space H is called a *subspace* of H if $u \in S$, $v \in S$ imply that $\lambda u + \mu v \in S$, for any complex numbers λ, μ. S is said to be a *dense subspace* of H

if it is a dense subset of H and a subspace of H. S is said to be a *closed subspace* of H if S is a subspace of H with the following property: let (u_n) be a convergent sequence in H such that $u_n \in S$. Then $u = \lim u_n$ belongs to S too.

Exercise: A dense, closed subspace of H coincides with H. ◊

Any inner product space V (or generally, any normed vector space) may be *completed* in the sense that it can be identified with a dense subspace M of a Hilbert space H (a dense subspace of a complete normed space). H is the set of all equivalence classes of Cauchy sequences (u_n) of vectors of V. (Two Cauchy sequences (u_n), (v_n) will be equivalent if $\lim (u_n - v_n) = 0$). See for example [2.7], [2.8].

1.5 Examples of Hilbert spaces

A. $H = \mathbb{C}^n$ with the Euclidean inner product $(u, v) = \sum_{i=1}^{n} u_i \overline{v_i}$ and norm $\|u\| = (\sum_{i=1}^{n} |u_i|^2)^{1/2}$. (Show that the coefficients with respect to the canonical basis of $\mathbb{C}^n$ of a Cauchy sequence in $\mathbb{C}^n$ form Cauchy sequences of complex numbers.)

B. l_2

We denote by l_2 the set of all complex sequences $u = (u_j)$ which satisfy the inequality

$$\sum_{j=1}^{\infty} |u_j|^2 < \infty.$$

In l_2 we define $u + v$, λu and (u, v) in the following way:

$$w = u + v \text{ with } w = (w_i), \ w_i = u_i + v_i.$$

$$z = \lambda u \ (\lambda \text{ complex number}) \ z = (z_i), \ z_i = \lambda u_i.$$

$$(u, v) = \sum_{j=1}^{\infty} u_j \overline{v_j}.$$

Then clearly it follows that $\sum_{j=1}^{\infty} |z_j|^2 < \infty$, and $\sum_{j=1}^{\infty} |w_j|^2 < \infty$ since

$$|w_j|^2 = |u_j + v_j|^2 \leq 2|u_j|^2 + 2|v_j|^2.$$

The convergence of the series defining the inner product follows from he inequalities

$$|u_j \overline{v_j}| = |u_j||v_j| \leq \frac{1}{2}\{|u_j|^2 + |v_j|^2\}.$$

By verifying the axioms one by one we easily conclude that ℓ_2 is an inner product space. (The zero vector is the sequence $\mathbf{0} = \{0, 0, \ldots\}$).

We remark that the Cauchy–Schwarz inequality $|(u, v)| \leq \|u\|\|v\|$ becomes

$$|\sum_{j=1}^{\infty} u_j \overline{v_j}| \leq \left(\sum_{j=1}^{\infty} |u_j|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} |v_j|^2\right)^{\frac{1}{2}}.$$

The triangle inequality $\|u + v\| \leq \|u\| + \|v\|$ becomes

$$\left(\sum_{j=1}^{\infty} |u_j + v_j|^2\right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^{\infty} |u_j|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{\infty} |v_j|^2\right)^{\frac{1}{2}}.$$

The space l_2 is a *complete* inner product space and hence a Hilbert space. To show that let $\{u^{(1)}, u^{(2)}, \ldots\}$ be a Cauchy sequence in l_2 with

$$u^{(n)} = \{u_1^{(n)}, u_2^{(n)}, \ldots\}.$$

Then given $\epsilon > 0$

$$\|u^{(n)} - u^{(m)}\| = (u^{(n)} - u^{(m)}, u^{(n)} - u^{(m)})^{\frac{1}{2}} = \left(\sum_{j=1}^{\infty} |u_j^{(n)} - u_j^{(m)}|^2\right)^{\frac{1}{2}} < \epsilon \qquad (1.1)$$

for all $n, m \geq N(\epsilon)$. In particular it follows that

$$|u_j^{(n)} - u_j^{(m)}| < \epsilon \text{ for all } n, m \geq N(\epsilon) \text{ and every } j = 1, 2, 3, \ldots.$$

Fix j. Then the sequence $u_j^{(1)}, u_j^{(2)}, \ldots$ is convergent, since it is a Cauchy sequence of complex numbers. We denote the limit of this sequence by u_j, i.e.

$$\lim_{n \to \infty} u_j^{(n)} = u_j \text{ for } j = 1, 2, 3, \ldots. \qquad (1.2)$$

Now it follows from (1.1) that for every positive integer k

$$\sum_{j=1}^{k} |u_j^{(n)} - u_j^{(m)}|^2 < \epsilon^2 \text{ for all } n, m \geq N(\epsilon).$$

Letting $m \to \infty$ in the above, since it is a finite sum, we obtain by (1.2) that

$$\sum_{j=1}^{k} |u_j^{(n)} - u_j|^2 \leq \epsilon^2 \text{ for all } n, m \geq N(\epsilon).$$

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The sequence of the sums in the left-hand side of this inequality is increasing. Hence its limit as $k \to \infty$ exists and we have

$$\sum_{j=1}^{\infty} |u_j^{(n)} - u_j|^2 \le \epsilon^2 \text{ for all } n \ge N(\epsilon). \tag{1.3}$$

We set now $u = (u_i)$. By (1.3), $u - u^{(n)} \in l_2$. Hence $u = (u - u^{(n)}) + u^{(n)} \in l_2$. By (1.3) it also follows that

$$\|u^{(n)} - u\| = \left(\sum_{j=1}^{\infty} |u_j^{(n)} - u_j|^2\right)^{\frac{1}{2}} < \epsilon \text{ for all } n \ge N(\epsilon).$$

Hence there exists $u \in l_2$ such that $u^{(n)} \to u$ in ℓ_2 as $n \to \infty$. We conclude that the (arbitrary) Cauchy sequence $\{u^{(1)}, u^{(2)}, \ldots\}$ converges. Therefore l_2 is complete and hence a Hilbert space.

C. $L_2(\Omega)$

Let Ω be an open set of $\mathbb{R}^d$. (An open interval in $\mathbb{R}$.) We denote the points in $\mathbb{R}^d$ by d-tuples $x = (x_1, x_2, \ldots, x_d)$ and denote the (Euclidean) length of the vector x by:

$$|x| = \left(\sum_{i=1}^{d} x_i^2\right)^{\frac{1}{2}}.$$

We consider the set of complex-valued continuous functions $u(x) = u(x_1, \ldots, x_d)$ defined on Ω. Addition $u + v$ and multiplication λu by a complex number λ are defined, as usual, by:

$$w = u + v, \quad w(x) = u(x) + v(x),$$
$$z = \lambda u, \quad z(x) = \lambda u(x)$$

We define now an *inner product* for such functions by:

$$(u, v) = \int_{\Omega} u(x)\overline{v(x)} \, dx, \tag{1.4}$$

where dx is the volume element in Ω, i.e. $dx = dx_1 dx_2 \ldots dx_d$ and $\int_{\Omega} \ldots dx$ is the multiple integral (in the Riemann sense)

$$\int_{\Omega} \ldots dx = \underbrace{\int \int \int \ldots \int}_{\Omega} \ldots dx_1 dx_2 \ldots dx_d.$$

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Since Ω is an arbitrary open set of $\mathbb{R}^d$, the integral in (1.4) defining the inner product may not exist. We restrict therefore our attention to complex-valued continuous functions $u(x)$, defined on Ω, with the property that

$$\int_\Omega |u(x)|^2 \, dx < \infty.$$

Let now V be the above-described vector space, i.e. let

$$V = \{u \mid u \text{ continuous on } \Omega, \int_\Omega |u(x)|^2 \, dx < \infty\}.$$

V is an inner product space with the inner product defined by (1.4). To see this, note that

$$|u(x) + v(x)|^2 \le 2(|u(x)|^2 + |v(x)|^2).$$

It follows that $u \in V$, $v \in V \Rightarrow u + v \in V$ and, easily, $\lambda u \in V$ for λ complex. Finally the existence of the integral in (1.4) for $u, v \in V$ is proved by noting that $\forall \, x \in \Omega$:

$$2|u(x)||v(x)| \le |u(x)|^2 + |v(x)|^2.$$

Hence, integrating:

$$
\begin{aligned}
|(u, v)| &= |\int_\Omega u(x)\overline{v(x)} \, dx| \le \int_\Omega |u(x)||v(x)| \, dx \\
&\le \frac{1}{2} \int_\Omega |u(x)|^2 \, dx + \frac{1}{2} \int_\Omega |v(x)|^2 \, dx.
\end{aligned}
$$

By verifying the axioms we may confirm that V is an inner product space. The norm on V is given by

$$\|u\| \equiv (u, u)^{\frac{1}{2}} = \left(\int_\Omega |u(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

The zero element in V is the function $u(x) \equiv 0$, $x \in \Omega$. For $u, v \in V$, the Cauchy–Schwarz and the triangle inequalities take the form:

$$\left| \int_\Omega u(x)\overline{v(x)} \, dx \right| \le \left(\int_\Omega |u(x)|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |v(x)|^2 \, dx \right)^{\frac{1}{2}}$$

$$\left(\int_\Omega |u(x) + v(x)|^2 \, dx \right)^{\frac{1}{2}} \le \left(\int_\Omega |u(x)|^2 \, dx \right)^{\frac{1}{2}} + \left(\int_\Omega |v(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

The functions $u_1, u_2, \ldots \in V$ form a *Cauchy sequence* in V if $\forall \, \epsilon > 0$

$$\|u_m - u_n\| = \left(\int_\Omega |u_m(x) - u_n(x)|^2 \, dx \right)^{\frac{1}{2}} < \epsilon,$$

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for all $m, n \geq N = N(\epsilon)$. The sequence is *convergent* if there exists a function $u \in V$ such that for every $\epsilon > 0$ there exists an integer $N = N(\epsilon)$ such that

$$\|u_n - u\| = \left(\int_\Omega |u_n(x) - u(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} < \epsilon$$

holds for all $n \geq N(\epsilon)$.

The space V is *not* complete. To see that let for example $\Omega = (-1, 1)$ and V be the set of continuous functions u defined on $(-1, 1)$ such that $\int_{-1}^{1} |u(x)|^2 \, \mathrm{d}x < \infty$. Let (u, v) be defined as $\int_{-1}^{1} u(x)\overline{v(x)} \, \mathrm{d}x$ and let $\|u\| = (u, u)^{\frac{1}{2}}$. Consider the sequence (u_i), where

$$u_j(x) = \begin{cases} -1 & \text{for} \quad -1 < x < -\frac{1}{j} \\ jx & \text{for} \quad -\frac{1}{j} \leq x \leq \frac{1}{j} \\ 1 & \text{for} \quad \frac{1}{j} < x < 1 \,. \end{cases}$$

Exercise: Prove that (u_j) is a Cauchy sequence in V. $\Diamond$

However, there is *no* continuous function u on $(-1, 1)$ for which $\|u_n - u\| \to 0$ as $n \to \infty$. It is easy to see that $\|u_n - f\| \to 0$ as $n \to \infty$ where e.g. f is the discontinuous function

$$f(x) = \begin{cases} -1 & \text{for} \quad -1 < x < 0 \\ 0 & \text{for} \quad x = 0 \\ 1 & \text{for} \quad 0 < x < 1 \,. \end{cases}$$

Thus, V is a non–complete inner product space. By our remark at the end of §1.4 we see that we may *complete* the space V. This extended *complete* space we call $L_2(\Omega)$.

It is well known that $L_2(\Omega)$ is isometrically isomorphic to the set of (equivalence classes of) complex–valued Lebesgue measurable functions u on Ω for which the (Lebesgue) integral $\int_\Omega |u(x)|^2 dx$ is finite. The inner product (understood in the Lebesgue sense) is given again by (1.4).

As a consequence of the process of completion of V:

(i) $L_2(\Omega)$ is a complete inner product space (a Hilbert space).

(ii) V is dense in $L_2(\Omega)$, i.e. for every $u \in L_2(\Omega)$ there exists a sequence (u_n) of functions in V such that $\|u_n - u\| = \left(\int_\Omega |u_n(x) - u(x)|^2 dx \right)^{\frac{1}{2}} \to 0$ as $n \to \infty$.

(iii) Strictly speaking the elements of $L_2(\Omega)$ are equivalence classes of measurable functions; two functions f, g are equivalent (and hence identified in $L_2(\Omega)$) if

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$f(x) = g(x)$ almost everywhere (a.e.) in Ω. However the term 'function' is frequently used to denote elements of $L_2(\Omega)$ with the identification noted above.

1.6   The Projection Theorem

The following result provides important information about geometric properties of a Hilbert space.

Theorem 1.1 (Projection Theorem). *Let G be a closed subspace of a Hilbert space H, properly included in H. Then, given $h \in H$ there exists a unique element $g \in G$ such that:*

$$(i) \quad \|h - g\| = \inf_{\phi \in G} \|h - \phi\|.$$

Moreover

$$(ii) \quad (h - g, \phi) = 0 \ \text{for each } \phi \in G.$$
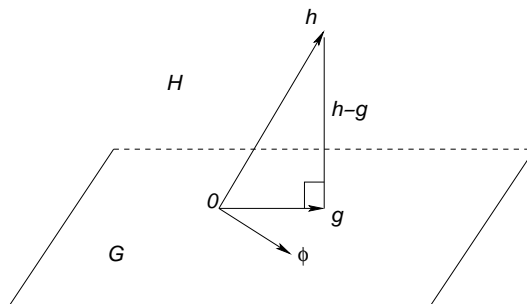
Figure 1.1

Proof. Let $h \in H$ such that $h \notin G$ (if $h \in G$ pick $g = h$ and the theorem is proved). Since for all $\phi \in G$, $\|h - \phi\| \geq 0$, there exists a sequence $\phi_n \in G$ such that

$$\lim_{n \to \infty} \|h - \phi_n\| = \inf_{\phi \in G} \|h - \phi\| \equiv \delta > 0. \tag{1.5}$$

We first show that (ϕ_n) is a Cauchy sequence. Let $f_1, f_2 \in H$. Then the *parallelogram law* holds:

$$2\|f_1\|^2 + 2\|f_2\|^2 = \|f_1 + f_2\|^2 + \|f_1 - f_2\|^2.$$

(Proof: **Exercise**). Set $f_1 = h - \phi_m$, $f_2 = h - \phi_n$. Then

$$\|\phi_m - \phi_n\|^2 = 2\|h - \phi_m\|^2 + 2\|h - \phi_n\|^2 - 4\|h - \frac{\phi_m + \phi_n}{2}\|^2. \tag{1.6}$$

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Now
$$\|h - \frac{\phi_m + \phi_n}{2}\| \le \frac{1}{2}\|h - \phi_m\| + \frac{1}{2}\|h - \phi_n\|$$

and by (1.5),
$$\limsup_{m,n\to\infty} \|h - \frac{\phi_m + \phi_n}{2}\| \le \frac{1}{2}\delta + \frac{1}{2}\delta = \delta.$$

But by definition of δ,
$$\liminf_{m,n\to\infty} \|h - \frac{\phi_m + \phi_n}{2}\| \ge \delta.$$

Hence $\lim_{m,n\to\infty} \|h - \frac{\phi_m+\phi_n}{2}\| = \delta$. Then by (1.6) and (1.5) we conclude that

$$\lim_{m,n\to\infty} \|\phi_m - \phi_n\|^2 = 2\delta^2 + 2\delta^2 - 4\delta^2 = 0.$$

Hence (ϕ_n) is a Cauchy sequence in G. Since G is closed, the sequence is convergent in G, i.e. there exists $g \in G$ such that $\phi_n \to g$ as $n \to \infty$. We show that this g satisfies $\|h - g\| = \inf_{\phi \in G} \|h - \phi\| = \delta$. Obviously $\|h - g\| \le \|h - \phi_n\| + \|g - \phi_n\|$, and taking limits as $n \to \infty$ we get that $\|h - g\| \le \delta$. By definition of δ, $\|h - g\| \ge \delta$. Hence $\|h - g\| = \delta$ as required. We also show that g is *unique*. Indeed, let $g_1, g_2 \in G$, $g_1 \ne g_2$ have the property that

$$\delta = \inf_{g \in G} \|h - g\| = \|h - g_1\| = \|h - g_2\|.$$

Then, since $\frac{1}{2}(g_1 + g_2) \in G \Rightarrow \delta \le \|h - \frac{1}{2}(g_1 + g_2)\|$. But by the triangle inequality

$$\|h - \frac{1}{2}(g_1 + g_2)\| \le \frac{1}{2}\|h - g_1\| + \frac{1}{2}\|h - g_2\| = \frac{1}{2}\delta + \frac{1}{2}\delta = \delta.$$

Hence
$$\|h - \frac{1}{2}(g_1 + g_2)\| = \delta = \frac{1}{2}\|h - g_1\| + \frac{1}{2}\|h - g_2\|,$$

i.e. the triangle inequality holds as *equality*. Now for any $\chi, \psi \in H$ such that $\chi \ne 0$, $\psi \ne 0$, $\|\chi + \psi\| = \|\chi\| + \|\psi\| \Leftrightarrow \chi = \lambda\psi$, for some $\lambda > 0$. (Proof: **Exercise**.) Hence there exists λ such that $h - g_1 = \lambda(h - g_2)$, i.e. $h(1 - \lambda) = g_1 - \lambda g_2$. If $\lambda = 1$, $g_1 = g_2$ (contradiction). If $\lambda \ne 1$, $h = \frac{(g_1 - \lambda g_2)}{1 - \lambda}$ i.e. $h \in G$ (contradiction). Hence g is unique (i) is proved.

To prove (ii), with g constructed as above, suppose that there exists $\phi_* \ne 0$ in G for which (ii) fails, i.e $(h - g, \phi_*) \ne 0$. Define the element $g_* \in G$ by:

$$g_* = g + \frac{(h - g, \phi_*)}{(\phi_*, \phi_*)}\phi_*.$$

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Then

$$
\begin{aligned}
\|h - g_*\|^2 &= (h - g - \frac{(h - g, \phi_*)}{\|\phi_*\|^2}\phi_*, h - g - \frac{(h - g, \phi_*)}{\|\phi_*\|^2}\phi_*) \\
&= (h - g, h - g) - \frac{(h - g, \phi_*)}{(\phi_*, \phi_*)}(\phi_*, h - g) \\
&\quad - \frac{\overline{(h - g, \phi_*)}}{(\phi_*, \phi_*)}(h - g, \phi_*) + \frac{|(h - g, \phi_*)|^2}{\|\phi_*\|^4}(\phi_*, \phi_*) \\
&= \|h - g\|^2 - \frac{|(h - g, \phi_*)|^2}{\|\phi_*\|^2}.
\end{aligned}
$$

Hence

$$
\|h - g_*\| < \|h - g\| = \inf_{\phi \in G} \|h - \phi\| \quad \text{(contradiction)}.
$$

Hence $(h - g, \phi) = 0$, $\forall \phi \in G$ and we have (ii). $\qquad \square$

Exercise: Prove that if for some $g \in G$, $(h - g, \phi) = 0 \ \forall \phi \in G$, then (i) in Theorem 1.1 holds. $\qquad \diamondsuit$

Given $h \in H$ we call g, the existence and uniqueness of which is guaranteed by Theorem 1.1, the *orthogonal projection* of h on the closed subspace G or the *best approximation* of h in G. If we let $f = h - g$, we can write

$$
h = f + g \quad \text{where } g \in G \text{ and } (f, \phi) = 0, \ \forall \phi \in G.
$$

Hence f is *orthogonal* to all vectors of the closed subspace G. Let $G^\perp$ be the set of all such vectors, i.e. let

$$
G^\perp = \{u \in H : (u, \phi) = 0 \ \forall \phi \in G\}.
$$

$G^\perp$ is called the *orthogonal complement* of G and it is a *closed subspace* of H. It is clearly a subspace. To see that it is closed let $u_n \in G^\perp$ such that $u_n \to u$. Then $(u, \phi) = (u, \phi) - (u_n, \phi)$, for all $\phi \in G$ since $(u_n, \phi) = 0$. Hence for all $\phi \in G$ $|(u, \phi)| = |(u - u_n, \phi)| \leq \|u - u_n\| \|\phi\| \to \infty$ as $n \to \infty$, i.e. $(u, \phi) = 0 \Rightarrow u \in G^\perp$. Hence $G^\perp$ is closed. It is easy to see that $G \cap G^\perp = \{0\}$. Hence we can write H as the *direct sum* of G and $G^\perp$

$$
H = G \oplus G^\perp,
$$

meaning that there exist two disjoint closed subspaces G and $G^\perp$ such that every element $h \in H$ can be written uniquely as the sum of a $g \in G$ and a $f \in G^\perp$, i.e.

$h = g + f$, as above.

Exercise: With g, f defined as above, prove the Pythagorean theorem:

$$\|h\|^2 = \|g\|^2 + \|f\|^2.$$

$\diamond$

A particular case of importance occurs when G is *finite–dimensional*. Then G is *closed* (Proof: **Exercise**). Let $\{\varphi_1, \varphi_2, \ldots, \varphi_s\}$ be a basis of G. Given $h \in H$ we can *explicitly* construct the best approximation g of h in G as follows: By (ii), g satisfies:

$$(h - g, \varphi) = 0 \quad \forall \varphi \in G.$$

Hence

$$(h - g, \varphi_i) = 0, \quad i = 1, 2, \ldots, s. \tag{1.7}$$

Let $g = \sum_{i=1}^{s} c_i \varphi_i$. We seek the coefficients $\{c_i\}_{i=1}^{s}$. By (1.7), the c_i's satisfy the following linear system of equations:

$$\sum_{j=1}^{s} M_{ij} c_j = (h, \varphi_i) \quad 1 \leq i \leq s, \tag{1.8}$$

where $M = (M_{ij})$ is the $s \times s$ *Gram matrix* (or *mass matrix*) associated with the basis $\{\varphi_i\}_{i=1}^{s}$ of G defined by

$$M_{ij} = (\varphi_j, \varphi_i), \quad 1 \leq i, j \leq s.$$

To see that M is invertible, suppose that for some complex s–vector $d = [d_1, \ldots d_s]^T$ we have that $M d = 0$. Then for each i

$$\sum_{j=1}^{s} M_{ij} d_j = 0 \Rightarrow (\sum_{j=1}^{s} d_j \varphi_j, \varphi_i) = 0.$$

Therefore the vector $u = \sum_{j=1}^{s} d_j \varphi_j \in G$ is orthogonal to all $\varphi \in G$, i.e. $u \in G^\perp \cap G \Rightarrow u = \sum_{j=1}^{s} d_j \varphi_j = 0$ and by the linear independence of the φ_j's $\Rightarrow d_j = 0$, $\forall j \Rightarrow d = 0$. Hence $M d = 0 \Rightarrow d = 0$, i.e. M is invertible. Actually, M is positive-definite (**Exercise**).

Example 1.2. Let $\Omega = (0, 1)$ and $H = L^2(0, 1)$ (real–valued). Suppose f is a given element of $L^2(0, 1)$ and let G be the subspace of H consisting of all real–valued polynomials of degree $\leq n - 1$, $n > 1$. Find the best approximation to f in G.

Solution. A basis for G obviously consists of the functions $\varphi_j(x) = x^{j-1}$, $1 \le j \le n$. Let g be the best approximation (orthogonal projection) of f in G. Suppose that $g = \sum_{j=1}^n a_j \varphi_j$. Then g satisfies:

$$(g - f, \varphi_i) = 0\,, \quad 1 \le i \le n,$$

from which

$$\sum_{j=1}^n M_{ij} a_j = (f, \varphi_i)\,, \quad 1 \le i \le n, \tag{1.9}$$

where M is the $n \times n$ symmetric Gram matrix,

$$M_{ij} = (\varphi_i, \varphi_j) = \int_0^1 \varphi_i \varphi_j = \int_0^1 x^{i+j-2} dx = \frac{1}{i+j-1}\,.$$

M is positive-definite but very ill conditioned. $\qquad\qquad\square$

1.7 Bounded linear functionals on a Hilbert space

Let H be a Hilbert space. By a *functional F* on H we mean a function from H into the complex numbers $\mathbb{C}$, i.e. a map which assigns to every $\phi \in H$ a unique complex number $F(\phi)$,

$$F : H \to \mathbb{C}\,, \quad \phi \mapsto F(\phi).$$

A functional F on H is called a *linear functional* if for every $\phi, \psi \in H$ and $\lambda, \mu \in \mathbb{C}$:

$$F(\lambda\phi + \mu\psi) = \lambda F(\phi) + \mu F(\psi).$$

A functional F on H is called *bounded* if there exists a $M > 0$ such that $|F(\phi)| \le M\|\phi\|$ for all ϕ in H.

If a functional F on H is bounded we define its *norm*, denoted by $\|F\|$ (do not confuse with the norm of $\phi \in H$, $\|\phi\|$) by

$$\|F\| = \sup_{0 \ne \phi \in H} \frac{|F(\phi)|}{\|\phi\|}\,. \tag{1.10}$$

Let F be a bounded linear functional on H. Then it is easy to see that F is a continuous function of its argument. Indeed, let $\phi_n \to \phi$ in H. Then

$$|F(\phi_n) - F(\phi)| = |F(\phi_n - \phi)| \le \|F\|\|\phi_n - \phi\| \longrightarrow 0.$$

Hence $F(\phi_n) \to F(\phi)$ in $\mathbb{C}$, i.e. F is continuous. (Note that the inequality

$$|F(\phi)| \leq \|F\|\|\phi\|$$

follows from the definition of the norm (1.10) $\|F\|$ of F.)

Conversely, let F be a linear functional on H and suppose that F is a continuous function at some point of H. (It follows that F will be everywhere continuous on H). We shall show that F is bounded. Indeed, if F is continuous at $\phi_0 \in H$, then for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|F(\phi_0) - F(h)| < \epsilon$ for $\|h - \phi_0\| \leq \delta$. Let $\phi \neq 0$ be an arbitrary element of H. By the linearity of F we obtain that

$$F(\phi) = \frac{\|\phi\|}{\delta} F\left(\frac{\delta\phi}{\|\phi\|}\right) = \frac{\|\phi\|}{\delta}\left\{F\left(\frac{\delta\phi}{\|\phi\|} + \phi_0\right) - F(\phi_0)\right\}.$$

Since the vector $\frac{\delta\phi}{\|\phi\|} + \phi_0 = h$, satisfies the relation $\|h - \phi_0\| \leq \delta$ we have $|F(\phi)| < \frac{\epsilon}{\delta}\|\phi\|$, i.e. $\frac{|F(\phi)|}{\|\phi\|} < \frac{\epsilon}{\delta}$. Fix $\epsilon = \epsilon_0 > 0$. Then $\delta = \delta(\epsilon_0) \equiv \delta_0$, and since ϵ_0, δ_0 are independent of ϕ we see that

$$\|F\| = \sup_{0 \neq \phi \in H} \frac{|F(\phi)|}{\|\phi\|} \leq \frac{\epsilon_0}{\delta_0} < \infty,$$

i.e. F is bounded.

Hence we proved that for a linear functional F on H, boundedness $\Leftrightarrow$ continuity. We speak thus of a *bounded* or *continuous linear functional* (b.l.f.).

Exercise: Let F be a b.l.f. on H. With the norm $\|F\|$ defined by (1.10) show that

$$\|F\| = \sup_{\phi \in H:\, \|\phi\| \leq 1} |F(\phi)| = \sup_{\phi \in H:\, \|\phi\| = 1} |F(\phi)|.$$

$\diamond$

Now let F, G be b.l.f.'s on a Hilbert space H. We can define their sum $F + G = L$ by $L(\phi) = F(\phi) + G(\phi)$ for each $\phi \in H$ and the scalar product λF as $\lambda F = G$, $G(\phi) = \lambda F(\phi)$.

Exercise: We denote by H' the space of bounded linear functionals on a Hilbert space H. (H' is called the *dual* of H). With addition and scalar multiplication defined as above show that H' forms a vector space. Then $\|F\|$, defined by (1.10), is a *norm* on H', i.e. H' is a normed linear space. Finally show that H' is *complete*, i.e. every Cauchy sequence in H' converges to an element in H'. $\diamond$

An example of a bounded linear functional on H is furnished by the *inner product* of the elements of H with a fixed element $f \in H$. Given $f \in H$, define for every $\phi \in H$

$F(\phi)$ by

$$F(\phi) = (\phi, f).$$

Clearly F is a linear functional on H. To see that it is bounded observe that

$$|F(\phi)| = |(\phi, f)| \leq \|\phi\| \, \|f\|.$$

So for all $\phi \neq 0$:

$$\frac{|F(\phi)|}{\|\phi\|} \leq \|f\| < \infty.$$

Hence $\|F\| \leq \|f\|$. In fact, since $F(f) = \|f\|^2$ we see that the sup in the definition of $\|F\|$ is attained for $\phi = f \in H$. Hence $\|F\| = \|f\|$.

It turns out that the *converse* of the above statement is also true, namely that *every* bounded linear functional on H has the form (ϕ, f) for *some* $f \in H$. This is the content of:

Theorem 1.3 (Riesz Representation Theorem)**.** *Every bounded linear functional F on a Hilbert space H can be expressed in the form $F(\phi) = (\phi, f)$ for each $\phi \in H$, where f is an element of H which is uniquely determined by F. Moreover, $\|F\| = \|f\|$.*

Proof. We denote by G the set of all elements $g \in H$ such that $F(g) = 0$, i.e. $G = \mathrm{Ker}F$. Obviously G is a subspace of H. Moreover G is a *closed* subspace of H. To see that let $g_n \to g$ with $g_n \in G$. Then $F(g) = F(g - g_n) + F(g_n) = F(g - g_n)$. Hence

$$|F(g)| \leq \|F\| \|g - g_n\| \to 0 \quad \text{as } n \to \infty, \text{ i.e. } F(g) = 0 \Leftrightarrow g \in G.$$

There are two possibilities now: either $G = H$ or $G \subsetneq H$ (properly included in H). In the first case F is the *zero functional* on H and the theorem is proved with $f = 0$. Hence, assume that $G \subsetneq H$. In this case $G^\perp$ contains non–zero elements by the Projection Theorem. Let $f_0 \in G^\perp$, $f_0 \neq 0$. For $\phi \in H$, consider the vector $F(\phi)f_0 - F(f_0)\phi$. This vector belongs to G because $F(F(\phi)f_0 - F(f_0)\phi) = F(\phi)F(f_0) - F(f_0)F(\phi) = 0$. Hence, since $f_0 \in G^\perp$, we see that for all $\phi \in H$:

$$(F(\phi)f_0 - F(f_0)\phi, f_0) = 0,$$

i.e. $F(\phi)\|f_0\|^2 = F(f_0)\,(\phi, f_0)$ from which

$$F(\phi) = \left(\phi, \frac{\overline{F(f_0)}}{\|f_0\|^2} f_0\right), \quad \text{for all } \phi \in H.$$

We set now

$$f = \frac{\overline{F(f_0)}}{\|f_0\|^2} \, f_0$$

and the equality above provides the required representation, i.e. we have *existence* of f.

To prove that f is *unique*, suppose that there exist two vectors $f_1 \neq f_2$ such that for all $\phi \in H$: $f(\phi) = (\phi, f_1) = (\phi, f_2)$. Hence $(\phi, f_1 - f_2) = 0$ for all $\phi \in H$. In particular, set $\phi = f_1 - f_2$ from which it follows that $f_1 - f_2 = 0$. It remains to prove that $\|F\| = \|f\|$. We immediately obtain from $F(\phi) = (\phi, f)$ that

$$|F(\phi)| = |(\phi, f)| \leq \|\phi\|\|f\| \overset{\phi \neq 0}{\Longrightarrow} \frac{|F(\phi)|}{\|\phi\|} \leq \|f\|.$$

Hence

$$\|F\| = \sup_{0 \neq \phi \in H} \frac{|F(\phi)|}{\|\phi\|} \leq \|f\|.$$

On the other hand taking $\phi = f$ we see that $F(f) = (f, f) = \|f\|^2$, from which

$$\|f\|^2 \leq \|F\|\|f\|, \quad \text{i.e. } \|f\| \leq \|F\|.$$

Hence $\|F\| = \|f\|$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

1.8 Bounded linear operators on a Hilbert space

Let H be a Hilbert space as usual. Let M be a subspace of H. By a *linear operator* $T : M \to H$ we mean a function defined on M with values in H which assigns to the vector $u \in M$ the (unique) vector $Tu \in H$ and which satisfies:

$$T(\lambda u + \mu v) = \lambda T(u) + \mu T(v) \quad \text{for } u, v \in M, \ \lambda, \mu \in \mathbb{C} \,.$$

The subspace M of H on which T is defined is called the *domain* of T and is denoted by $D(T)$. The *range* of T is the set of vectors $v \in H$ to each one of which there corresponds at least one $u \in D(T)$ such that $Tu = v$, i.e.

$$\text{Range}(T) \equiv \text{Ran}(T) = \{v \in H : \exists\, u \in D(T) \text{ such that } Tu = v\}.$$

We also define

$$\text{Ker}(T) \equiv \{u \in D(T) : Tu = 0\}.$$

The operator T is called *one–to–one* (1–1) if $u_1 \neq u_2 \Rightarrow Tu_1 \neq Tu_2$. Equivalently, T is one–to–one if $Tu_1 = Tu_2 \Rightarrow u_1 = u_2$. T is called *onto* if $\text{Ran}(T) = H$, i.e. if for every $v \in H$ we can find a $u \in D(T)$ such that $Tu = v$.

Exercise: $\text{Ran}(T)$ is a subspace of H. So is $\text{Ker}(T)$. $\diamondsuit$

A linear operator T defined on the whole of H, (i.e. $D(T) = H$) will be called a linear operator *on H*. Unless otherwise indicated we will assume henceforth that $D(T) = H$. A linear operator T on H is said to be *bounded* if

$$\sup_{0 \neq \phi \in H} \frac{\|T\phi\|}{\|\phi\|} < \infty.$$

Let T be a bounded linear operator (b.l.op.) on H. We define the *norm* of T by

$$\|T\| = \sup_{0 \neq \phi \in H} \frac{\|T\phi\|}{\|\phi\|}. \tag{1.11}$$

(It follows that $\|T\phi\| \leq \|T\|\|\phi\| \quad \forall \phi \in H$).

A linear operator T on H is *continuous* at $f \in H$ if whenever $f_n \to f$ in H then $\|Tf_n - Tf\| \to 0$. As in the case of bounded linear functionals we can prove (**Exercise**) that a linear operator T on H is continuous at all $f \in H$ if it is continuous at some $f_0 \in H$ and it is bounded if and only if it is continuous. As in the case of bounded linear functionals we can show (**Exercise**) that

$$\|T\| = \sup_{0 \neq \phi \in H: \|\phi\| \leq 1} \|T\phi\| = \sup_{\phi \in H: \|\phi\| = 1} \|T\phi\|.$$

Now, let T, S be b.l.op's on H. We define their sum $T + S$ as that (linear) operator W on H, such that $W\phi = T\phi + S\phi$, $\forall \phi \in H$. Similarly $\lambda T = S$, if $S\phi = \lambda T\phi$. It is easy to see that with these definitions of addition and scalar multiplication, the set of b.l.op's on H forms a vector space.

Exercise: With the *norm* defined by (1.11) the vector space of b.l.op's on H becomes a *normed linear space*. $\diamondsuit$

We denote this normed linear space by $\mathcal{B}(H)$.

Exercise: $\mathcal{B}(H)$ is a complete normed linear space. $\diamondsuit$

Now, let $T, S \in \mathcal{B}(H)$. Their product TS is defined as the function on H which maps the element $u \in H$ on the element $T(Su)$. It is easily seen that TS is a linear operator on H. Moreover $T(S_1 + S_2) = TS_1 + TS_2$ etc, while in general $TS \neq ST$.

Since $\|TSu\| = \|T(Su)\| \leq \|T\|\|Su\| \leq \|T\|\|S\|\|u\|$, we see that $\|TS\| \leq \|T\|\|S\|$, i.e. $TS \in \mathcal{B}(H)$.

Referring to the Projection Theorem 1.1, set $g = Ph$, where g is the orthogonal projection (best approximation) of h on a closed subspace G of H. P is called the (orthogonal) *projection operator onto* G.

Exercise: Show that

(i) P is a linear operator on H.

(ii) P is a bounded linear operator on H.

(iii) $\mathrm{Ran}P = G$, $\mathrm{Ker}P = G^\perp$, $\mathrm{Ran}(I - P) = G^\perp$, $\mathrm{Ker}(I - P) = G$, where I is the identity operator $Iu = u$, $\forall u \in H$ (obviously $I \in \mathcal{B}(H)$ with $\|I\| = 1$).

(iv) $P^2 = P$, $\|P\| = 1$.

(v) $I - P$ is the projection operator onto $G^\perp$. $\diamondsuit$

1.9 The Lax–Milgram and the Galerkin theorems

Henceforth, to simplify the analysis, we will usually consider *real Hilbert spaces*, i.e. complete inner product spaces over the real numbers with $(\lambda f, \mu g) = \lambda\mu(f, g)$, $\forall \lambda, \mu$ real, $f, g \in H$, and $(f, g) = (g, f)$, $\forall f, g \in H$.

A (real) *bilinear form* on a real Hilbert space H is a map from $H \times H$ into $\mathbb{R}$ denoted by $B(f, g)$ for $f, g \in H$, which satisfies:

$$B(\lambda_1 f_1 + \lambda_2 f_2, g) = \lambda_1 B(f_1, g) + \lambda_2 B(f_2, g)$$
$$B(f, \mu_1 g_1 + \mu_2 g_2) = \mu_1 B(f, g_1) + \mu_2 B(f, g_2)$$

for $f_i, f, g_i, g \in H$, $\mu_i, \lambda_i \in \mathbb{R}$. In general, $B(f, g) \neq B(g, f)$, i.e. B is not symmetric.

The following theorem will be central in the sequel:

Theorem 1.4 (Lax–Milgram Theorem). *Let H be a (real) Hilbert space and let $B(.,.) : H \times H \to \mathbb{R}$ be a bilinear form on H for which there exist constants $c_1 \geq 0$, $c_2 > 0$*

such that

$$(i) \quad |B(\phi, \psi)| \leq c_1 \|\phi\| \|\psi\|, \quad \forall \phi, \psi \in H,$$

$$(ii) \quad B(\phi, \phi) \geq c_2 \|\phi\|^2, \qquad \forall \phi \in H.$$

Let $F : H \to \mathbb{R}$ be a given (real-valued) bounded linear functional on H. Then there exists a unique $u \in H$ satisfying

$$B(u, v) = F(v) \quad \text{for all } v \in H.$$

Moreover,

$$\|u\| \leq \frac{1}{c_2} \|F\|.$$

Proof. Let $\phi \in H$ be fixed. Then $\Phi : H \to \mathbb{R}$, defined for every $v \in H$ by $\Phi(v) = B(\phi, v)$, defines a continuous linear functional on H. (Linearity follows from the fact that B is a bilinear form. For boundedness observe that for each $v \in H$

$$|\Phi(v)| = |B(\phi, v)| \leq c_1 \|\phi\| \|v\|.$$

Hence $\|\Phi\| \leq c_1 \|\phi\| < \infty$).

By the Riesz Representation Theorem 1.3 therefore, there exists a unique element $\tilde{\phi} \in H$ such that

$$\Phi(v) = B(\phi, v) = (v, \tilde{\phi}) \quad \text{for every } v \in H. \tag{1.12}$$

Hence for every $\phi \in H$, we define a $\tilde{\phi} \in H$ by (1.12) and denote the correspondence $\phi \mapsto \tilde{\phi}$ by $\tilde{\phi} = A\phi$, i.e. put

$$B(\phi, v) = (v, A\phi), \quad \forall \phi \in H, \ \forall v \in H. \tag{1.13}$$

Now A is a linear operator defined on H. To show linearity, observe that, given $\phi, \psi \in H$, for every $v \in H$ and λ, μ real we have

$$(v, A(\lambda\phi + \mu\psi)) = B(\lambda\phi + \mu\psi, v) = \lambda B(\phi, v) + \mu B(\psi, v) =$$

$$= \lambda(v, A\phi) + \mu(v, A\psi) = (v, \lambda A\phi + \mu A\psi).$$

Hence $A(\lambda\phi + \mu\psi) = \lambda A\phi + \mu A\psi \iff A$ is linear.

We claim now that the range of A, $\text{Ran}(A)$, is a *closed* subspace of H. It is (easily) a subspace. To show that it is closed, let $\hat{\phi}_n = A\phi_n$ be a convergent sequence, such

that $\hat{\phi}_n \to \hat{\phi}$. Now, since $B(\phi_n, v) = (v, A\phi_n) \, \forall \, v \in H \Rightarrow B(\phi_n - \phi_m, v) = (v, A\phi_n - A\phi_m) \, \forall \, v \in H$. Choose $\phi_n - \phi_m = v$ and using (ii) get $\|\phi_n - \phi_m\| \leq \frac{1}{c_2}\|A\phi_n - A\phi_m\|$. Hence (ϕ_n) is a Cauchy sequence in H, i.e. there exists $\phi \in H$ such that $\phi_n \to \phi$. We now show that $\hat{\phi} = A\phi$, thus showing that $\hat{\phi} \in \mathrm{Ran}(A)$, i.e. that $\mathrm{Ran}(A)$ is closed.

Indeed, $|B(\phi_n, v) - B(\phi, v)| \leq c_1\|\phi_n - \phi\|\|v\|$ gives that

$$\lim_{n\to\infty} B(\phi_n, v) = B(\phi, v), \quad \forall \, v \in H.$$

Also $(A\phi_n, v) = (\hat{\phi}_n, v) \to (\hat{\phi}, v)$ since $|(\hat{\phi}_n, v) - (\hat{\phi}, v)| \leq \|\hat{\phi}_n - \hat{\phi}\|\|v\|$. Since $B(\phi_n, v) = (A\phi_n, v) \, \forall \, v \in H \Rightarrow B(\phi, v) = (\hat{\phi}, v) \, \forall \, v \in V$, i.e. $\hat{\phi} = A\phi$, by definition of A. Hence $\mathrm{Ran}(A)$ is closed. We now claim that $\mathrm{Ran}(A) = H$. Suppose that $\mathrm{Ran}(A)$ is properly included in H, so that $\exists \, z \neq 0 \in (\mathrm{Ran}(A))^\perp$. Hence $(z, v) = 0 \, \forall v \in \mathrm{Ran}(A)$. In particular $\forall \, \phi \in H$, $B(\phi, z) = (A\phi, z) = 0$. Hence for $\phi = z$, $0 = B(z, z) \geq c_2\|z\|^2 \Rightarrow z = 0$ (contradiction). So $\mathrm{Ran}(A) = H$.

Now, given F, a b.l.f. on H, by Riesz representation, $\exists ! \, \chi \in H$ such that $F(v) = (\chi, v) \, \forall \, v \in H$. Since $\mathrm{Ran}(A) = H$, $\exists \, u \in H$ such that $Au = \chi$. Hence $\exists \, u$ such that

$$F(v) = (Au, v) = B(u, v), \quad \forall \, v \in H$$

and we have existence of u as claimed in the statement of the theorem.

For uniqueness, suppose that $\exists \, u_1 \neq u_2$ such that $B(u_1, v) = F(v) = B(u_2, v)$ $\forall \, v \in H$. Hence

$$B(u_1 - u_2, v) = 0 \, \forall v \in H \Rightarrow 0 = B(u_1 - u_2, u_1 - u_2) \geq c_2\|u_1 - u_2\|^2 \Rightarrow u_1 = u_2.$$

Finally since $B(u, u) = F(u)$, (i), (ii) give that $(u \neq 0)$ $c_2\|u\|^2 \leq |F(u)|$, from which $\|u\| \leq \frac{1}{c_2}\frac{|F(u)|}{\|u\|}$. Hence

$$\|u\| \leq \sup_{v \neq 0} \frac{1}{c_2}\frac{|F(v)|}{\|v\|} = \frac{1}{c_2}\|F\|.$$

$\square$

We finally present a basic theorem for the *Galerkin approximation* (see below for definition) u_h to the solution u of $B(u, v) = F(v)$ guaranteed by the Lax–Milgram theorem.

Theorem 1.5 (Galerkin). *Let H be a real Hilbert space and let $B(.,.) : H \times H \to \mathbb{R}$ be a bilinear form on H which satisfies:*

$$(i) \quad |B(\phi, \psi)| \leq c_1 \|\phi\| \|\psi\|, \quad \forall \, \phi, \psi \in H,$$

$$(ii) \quad B(\phi, \phi) \geq c_2 \|\phi\|^2, \qquad \forall \, \phi \in H,$$

for some constants $c_1 \geq 0$, $c_2 > 0$ independent of ϕ, $\psi \in H$. Let F be a given real–valued b.l.f. on H and let u be the unique element of H, guaranteed by the Lax–Milgram theorem, satisfying $B(u, v) = F(v)$, $\forall \, v \in H$.

Let $\{S_h\}$, for $0 < h \leq 1$, be a family of finite–dimensional subspaces of H. For every h there exists a unique u_h such that

$$B(u_h, v_h) = F(v_h), \quad \forall \, v_h \in S_h. \tag{1.14}$$

We call u_h the Galerkin approximation *of u in S_h.*

Moreover we have the error estimate

$$\|u - u_h\| \leq \frac{c_1}{c_2} \inf_{\chi \in S_h} \|u - \chi\|. \tag{1.15}$$

Proof. The existence–uniqueness of $u_h \in S_h$ is guaranteed by the Lax–Milgram theorem applied to the Hilbert space $(S_h, \|\cdot\|)$. Alternatively, let $\{\phi_j\}_{j=1}^m$ be a basis for S_h, where $m = m(h) = \dim S_h$, and try to find $u_h \in S_h$ in the form $u_h = \sum_{j=1}^m c_j \phi_j$. By (1.14) u_h satisfies

$$B(u_h, \phi_i) = F(\phi_i) \ 1 \leq i \leq m, \quad \text{i.e.}$$

$$B\left(\sum_{j=1}^m c_j \phi_j, \phi_i\right) = F(\phi_i) \ 1 \leq i \leq m \Longrightarrow \sum_{j=1}^m c_j B(\phi_j, \phi_i) = F(\phi_i), \ 1 \leq i \leq m.$$

Hence if A is the $m \times m$ matrix given by $A_{ij} = B(\phi_j, \phi_i)$, $1 \leq i, j \leq m$, the c_j's are the solution of the linear system

$$\sum_{j=1}^m A_{ij} c_j = F(\phi_i), \ 1 \leq i \leq m. \tag{1.16}$$

The associated homogeneous system $\sum_{j=1}^m A_{ij} \tilde{c}_j = 0$, $1 \leq i \leq m$, has only the zero solution. (Since $\sum_{j=1}^m A_{ij} \tilde{c}_j = 0 \Rightarrow B(\sum_{j=1}^m \tilde{c}_j \phi_j, \phi_i) = 0$, $1 \leq i \leq m$, $\Rightarrow B(v_h, v_h) = 0$, where $v_h = \sum_{j=1}^m \tilde{c}_j \phi_j$. Hence, by (ii) $v_h = 0 \Rightarrow \tilde{c}_i = 0$.) Therefore A is invertible and

(1.16) has a unique solution, i.e. (1.14) has a unique solution $u_h \in S_h$. (**Exercise:** Show that A is positive definite.)

For the error estimate observe that by (ii)

$$c_2 \|u - u_h\|^2 \leq B(u - u_h, u - u_h) = B(u - u_h, u) \tag{1.17}$$

(since $B(u_h, \psi) = F(\psi) = B(u, \psi) \ \forall \psi \in S_h \Rightarrow B(u - u_h, \psi) = 0 \ \forall \psi \in S_h$). For the same reason, for any $\chi \in S_h$

$$B(u - u_h, u) = B(u - u_h, u - \chi) \leq c_1 \|u - u_h\| \ \|u - \chi\|,$$

using (i).

By (1.17) we conclude therefore that $c_2 \|u - u_h\|^2 \leq c_1 \|u - u_h\| \ \|u - \chi\|$, i.e.

$$\|u - u_h\| \ \leq \ \frac{c_1}{c_2} \|\chi - u\| \ \forall \chi \in S_h, \quad \text{i.e.}$$

$$\|u - u_h\| \ \leq \ \frac{c_1}{c_2} \inf_{\chi \in S_h} \|\chi - u\| = \frac{c_1}{c_2} \|P_h u - u\|,$$

where P_h is the projection operator on S_h. Hence (1.15) is proved. $\qquad\square$

Here is an immediate corollary to Galerkin's Theorem 1.5.

Corollary 1.6. *With notation introduced in Theorem 1.5, suppose that the family S_h of subspaces satisfies*

$$\lim_{h \to 0} \inf_{\chi \in S_h} \|u - \chi\| = 0.$$

Then $\lim_{h \to 0} \|u - u_h\| = 0$. $\qquad\square$

Finally we mention that in the case of a *symmetric, bilinear form B*, i.e. when (in addition to (i), (ii) of Theorem 1.4)

(iii) $B(u, v) = B(v, u), \quad \forall u, v \in H,$

we can obtain an alternative of the problem of finding $u \in H$ such that

$$B(u, v) = F(v), \quad \forall v \in H,$$

where F is a bounded linear functional on H.

For $v \in H$ consider the (nonlinear) functional $J : H \to \mathbb{R}$ defined by

$$J(v) = \frac{1}{2} B(v, v) - F(v), \tag{1.18}$$

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and the associated optimization problem of finding $z \in H$ such that

$$J(z) = \min_{v \in H} J(v). \tag{1.19}$$

Then, the following holds:

Theorem 1.7 (Rayleigh–Ritz). *Suppose B is a symmetric, bilinear form which satisfies the hypotheses (i), (ii) of Theorem 1.4. Then, the problem (1.19) of minimizing over H the functional J defined by (1.18) has a unique solution which coincides with u, the existence–uniqueness of which was guaranteed by the Lax–Milgram Theorem 1.4.*

Proof. Let u be the solution of the problem $B(u, v) = F(v), \forall v \in H$. Then $\forall w \in H$:

$$
\begin{aligned}
J(u + w) &= \frac{1}{2} B(u + w, u + w) - F(u + w) = \text{ (due to the symmetry of } B) = \\
&= \left(\frac{1}{2} B(u, u) - F(u) \right) + \left(\frac{1}{2} B(w, w) \right) + (B(u, w) - F(w)) \\
&= J(u) + \frac{1}{2} B(w, w), \quad \text{since } B(u, w) = F(w) \text{ by Theorem 1.4.}
\end{aligned}
$$

Hence

$$J(u + w) = J(u) + \frac{1}{2} B(w, w) \geq J(u) + \frac{c_2}{2} \|w\|^2 \quad \text{by (ii).}$$

Therefore $\forall w \in H, w \neq 0$: $J(u + w) > J(u)$ i.e.

$$J(u) = \min_{v \in H} J(v) \quad \text{and } J(v) > J(u) \text{ if } v \neq u.$$

$\square$

Immediately, we have the following corollary, which is the analog of Theorem 1.5.

Corollary 1.8 (Rayleigh–Ritz, Galerkin). *With notation introduced in Theorem 1.5 and the additional hypothesis of symmetry of B, the problem of minimizing the functional J defined by (1.18), over S_h, i.e. finding $u_h \in S_h$ such that*

$$J(u_h) = \min_{\chi \in S_h} J(\chi)$$

has a unique solution u_h, which coincides with the Galerkin approximation in S_h of u, constructed in Theorem 1.5.

Chapter 2

Elements of the Theory of Sobolev Spaces and Variational Formulation of Boundary–Value Problems in One Dimension

This chapter is based on the analogous material in the books [2.2] by H. Brezis and [2.7] by H. Triebel. We assume the reader is acquainted with the basic theory of Lebesgue measure and integration, as e.g. in Royden, [2.6].

2.1   Motivation

We consider the following "two–point" boundary–value problem in one dimension. Find a real–valued function $u(x)$, defined for $x \in [a, b]$ and satisfying

$$(*) \quad \begin{cases} -(p(x)u'(x))' + q(x)u(x) = f(x), & a \leq x \leq b. \\ u(a) = u(b) = 0. \end{cases}$$

Here $p(x), q(x), f(x)$ are real–valued functions defined on $[a, b]$ such that $p \in C^1([a, b])$, $p(x) \geq \alpha > 0$ for $x \in [a, b]$, $q \in C([a, b])$, $q(x) \geq 0 \ \forall x \in [a, b]$, $f \in C([a, b])$. A *classical* solution of the boundary–value problem (b.v.p.) $(*)$ is a function u of class $C^2([a, b])$ which satisfies $(*)$ in the usual sense.

If we multiply the equation in $(*)$ by a function $\phi \in C^1([a, b])$, such that $\phi(a) = \phi(b) = 0$ and integrate by parts we obtain

$$(**) \quad \int_a^b pu'\phi' \, dx + \int_a^b qu\phi \, dx = \int_a^b f\phi \, dx, \quad \forall \, \phi \in C^1([a, b]), \ \phi(a) = \phi(b) = 0.$$

Note that $(**)$ makes sense for $u \in C^1([a, b])$, as opposed to $(*)$ which requires $u \in C^2([a, b])$. In fact $(**)$ just requires that u, u' be integrable functions. One may say (vaguely) that a solution $u \in C^1([a, b])$ (such that $u(a) = u(b) = 0$) of $(**)$ is (one kind of) a *weak* or *generalized* solution of $(*)$.

The *variational method* for solving, (i.e. proving existence and uniqueness of solutions of) problems such as $(*)$ – and also boundary–value problems for *partial differential equations* proceeds roughly as follows:

(i) We define precisely what we mean by a *weak solution* of $(*)$. Typically it will be the solution of a *weak* (or *variational*) form of the problem $(*)$, such as $(**)$, or, equivalently the solution of an appropriate minimization problem. It turns out that spaces of continuous functions such as $C^1([a, b])$ etc. are not suitable spaces of weak solutions because they are not complete in L^2. Hence we must work in Hilbert spaces in which generalized (weak) derivatives may be defined. In this, the *Sobolev spaces* will play a central role.

(ii) We show existence and uniqueness of the weak solution, for example by the *Lax–Milgram theorem*; note that $(**)$ suggests the variational problem
$B(u, v) \equiv \int_a^b (pu'v' + quv) = F(v) \equiv \int_a^b fv$, for u, v in an appropriate Hilbert space. Note that the Lax-Milgram theorem holds in a *Hilbert* space, i.e. it needs *completeness* of the inner product space to be valid.

(iii) We then prove that the weak solution is sufficiently *regular*. For example, under our hypotheses on p, q and f it turns out that the weak solution is in $C^2([a, b])$ and satisfies the zero boundary conditions.

(iv) We finally prove that a *weak solution* is a *classical* solution of $(*)$.

Note that a weak formulation of the problem provides us also with a Galerkin method for approximating its weak solution in a suitably chosen finite–dimensional subspace of the Hilbert space in which the problem is posed. This space must be chosen so that it has good approximation properties, i.e. that in $\inf_{\chi \in S_h} \|u - \chi\|$ in (1.15) is small, and so that the linear system (1.16) may be solved in an efficient manner.

2.2 Notation and preliminaries

We now introduce some notation on function spaces that will be used in the sequel and list some useful density and approximation results in the spaces L^p.

We let Ω denote an open subset of $\mathbb{R}^d$, not necessarily bounded. For simplicity we shall consider only *real–valued functions* defined on Ω or $\mathbb{R}^d$. We let

$C(\Omega) =$ space of continuous functions on Ω.

$C^k(\Omega) =$ space of k–times differentiable functions on Ω, i.e the space of those functions $f(x)$, $x \in \Omega$ such that $\frac{\partial^{\alpha_1 + \cdots + \alpha_d} f(x)}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$ are continuous functions on Ω for all integers $0 \leq \alpha_i \leq k$, $1 \leq i \leq d$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_d \leq k$. Let $C^0(\Omega) \equiv C(\Omega)$.

$C^\infty(\Omega) = \cap_{k \geq 0} C^k(\Omega)$.

$C(\overline{\Omega}) =$ space of continuous functions on $\overline{\Omega}$. Analogously define $C^k(\overline{\Omega})$, $C^\infty(\overline{\Omega})$.

$C_c(\Omega) =$ space of functions in $C(\Omega)$ whose support is a compact set included in Ω. (If $f \in C(\Omega)$, support of $f = \mathrm{supp} f = \overline{\{x \in \Omega : \ f(x) \neq 0\}}$). Hence these functions vanish outside a compact set included in Ω.

$C_c^k(\Omega) = C^k(\Omega) \cap C_c(\Omega)$.

$C_c^\infty(\Omega) = C^\infty(\Omega) \cap C_c(\Omega)$. Often the notation $C_0^\infty(\Omega)$ is used instead of $C_c^\infty(\Omega)$.

We recall a few facts about the spaces $L^p(\Omega)$, cf. e.g. [2.6]. Let $\mathrm{d}x$ denote the Lebesgue measure in $\mathbb{R}^d$. By $L^1(\Omega)$ we denote the space of (Lebesgue) integrable functions f on Ω, i.e. the functions for which

$$\|f\|_{L^1} = \|f\|_{L^1(\Omega)} = \int_\Omega |f(x)| \, dx < \infty.$$

(We denote usually $\int_\Omega f = \int_\Omega f(x) \, dx$).

Let $1 \leq p < \infty$. Then

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \, ; |f|^p \in L^1(\Omega)\}.$$

We put

$$\|f\|_{L^p} = \|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

For $p = \infty$ we define $L^\infty(\Omega) = \{f : \Omega \to \mathbb{R}, \ f$ measurable such that there exists a constant $C < \infty$ such that $|f(x)| \le C$ a.e. (almost everywhere) in Ω, i.e. such that $|f(x)| \le C$ for all $x \in \Omega$ except possibly for some x belonging to a subset of Ω of Lebesgue measure zero$\}$. We put

$$\|f\|_{L^\infty} = \|f\|_{L^\infty(\Omega)} = \inf\{C : \ |f(x)| \le C \text{ a.e. in } \Omega\}$$

and note that $|f(x)| \le \|f\|_{L^\infty}$ for every $x \in \Omega - O$, where O has measure 0. The quantities $\|f\|_{L^p}$, $1 \le p \le \infty$ are *norms* on the respective spaces L^p, which are complete under these norms. We have already introduced the Hilbert space $L^2 = L^2(\Omega))$ in §1.5 C.

We note once more that what are usually referred to as "functions" $f(x) \in L^p(\Omega)$ are really *equivalence classes* of functions, where the equivalence $f \sim g$ holds if and only if $f(x) = g(x)$ a.e. in Ω. For example, when we say that $f = 0$ as an element of $L^p(\Omega)$, we mean that $f(x) = 0$ for all x outside a set of measure zero in Ω, i.e. that $f(x) = 0$ a.e. in Ω; contrast with the situation $f \in C(\Omega)$, $f = 0 \Rightarrow f(x) = 0 \ \forall x \in \Omega$.

We list below some density and approximation results in L^p, $1 \le p < \infty$, wherein elements of L^p are approximated by continuous (or smoother) functions, and that will be frequently used in the sequel, mainly for $p = 2$. Their proofs may be found in the Appendix.

Proposition 2.1. *The step functions are dense in $L^p(\Omega)$, $1 \le p < \infty$.*

Theorem 2.2. *The space $C_c(\Omega)$ is dense in $L^p(\Omega)$, $1 \le p < \infty$.*

In the sequel $L^1_{\text{loc}}(\Omega)$ will denote the functions f on Ω for which $\int_K |f(x)| \, dx < \infty$ for every compact set $K \subset \Omega$. For example, $f(x) = \frac{1}{x}$ belongs to $L^1_{\text{loc}}((0,1))$ but not to $L^1((0,1))$.

Proposition 2.3. *If $f \in L^1_{loc}(\Omega)$ is such that*

$$\int_\Omega fu = 0, \quad \forall u \in C_c(\Omega),$$

then $f = 0$ a.e. in Ω.

Definition 2.4. A regularizing sequence (or a sequence of mollifiers) is a sequence of functions (ρ_n), such that:

$$\rho_n \in C_c^\infty(\mathbb{R}^d),$$

$$\rho_n \geq 0 \text{ on } \mathbb{R}^d,$$

$$\operatorname{supp} \rho_n \subset B(0, \frac{1}{n}) \equiv \{x \in \mathbb{R}^d \ : \ |x| \equiv \left(\sum_{i=1}^d x_i^2\right)^{\frac{1}{2}} \leq \frac{1}{n}\},$$

$$\int_{\mathbb{R}^d} \rho_n \, dx = 1.$$

Such functions clearly exist. E.g. in $\mathbb{R}$, let

$$\rho(x) = \begin{cases} e^{\frac{1}{x^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Clearly $\rho(x)$ is continuous on $\mathbb{R}$ and

$$\rho^{(j)}(x) = \frac{\Pi_j(x)}{(1-x^2)^{2j}} \, e^{-\frac{1}{1-x^2}}, \ |x| < 1,$$

where $\Pi_j(x)$ are polynomials of degree $3j - 2$. Since $y^j e^{-y} \to 0$ as $y \to +\infty$ we see that $\rho(x) \in C_c^\infty(\mathbb{R})$. Of course $\operatorname{supp}\rho = [-1, 1]$. Let

$$C = \left(\int_{-\infty}^\infty \rho(x) \, dx\right)^{-1}.$$

Define

$$\rho_n(x) = C \, n \, \rho(nx).$$

Then

$$\rho_n \in C_c^\infty(\mathbb{R}), \ \rho_n(x) \geq 0 \text{ on } \mathbb{R}, \ \operatorname{supp}\rho_n = [-\frac{1}{n}, \frac{1}{n}], \ \int_{-\infty}^\infty \rho_n \, dx = 1.$$

In $\mathbb{R}^d$ define

$$\rho(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

(where $|x| = (\sum_{i=1}^d x_i^2)^{1/2}$) and

$$\rho_n(x) = C \, n^d \, \rho(nx), \ \ C = \left(\int_{\mathbb{R}^d} \rho(x) \, dx\right)^{-1}.$$

Then it is straightforward to see that the sequence (ρ_n) satisfies the requirements of Definition 2.4.

Denote the *convolution* of two functions $f(x)$, $g(x)$ defined on $\mathbb{R}^d$ as the function

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)\, g(y)\, \mathrm{d}y$$

(provided the integral exists). The following results show that we may approximate continuous and L^p functions by the "regularized means" or "mollifications" $\rho_n * f$ introduced by Sobolev. Their proofs may be found in the Appendix.

Lemma 2.5. *(i) Let $f \in C(\mathbb{R}^d)$. Then $\rho_n * f \in C^\infty(\mathbb{R}^d)$ and $\rho_n * f \to f$ uniformly on every compact set $K \subset \mathbb{R}^d$, i.e.*

$$\sup_{x \in K} |f(x) - (\rho_n * f)(x)| \to 0, \quad n \to \infty$$

$\forall K$ compact $\subset \mathbb{R}^d$.

(ii) Let $f \in C_c(\Omega)$. Extend f by zero on the whole of $\mathbb{R}^d$. Then, for sufficiently large n, $\rho_n * f \in C_c^\infty(\Omega)$ and

$$\sup_{x \in \Omega} |f(x) - (\rho_n * f)(x)| \to 0, \quad n \to \infty.$$

(iii) Let $f \in L^p(\Omega)$, $1 \le p < \infty$. Extend f by zero to the whole of $\mathbb{R}^d$. Then $\rho_n * f \in C^\infty(\mathbb{R}^d)$. Moreover $\rho_n * f \in L^p(\Omega)$, $\|\rho_n * f\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)}$, and $\|f - \rho_n * f\|_{L^p(\Omega)} \to 0$, $n \to \infty$.

Finally we mention the following basic result whose proof follows from the above lemmas and may be found in the Appendix.

Proposition 2.6. *The space $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, if $1 \le p < \infty$.*

2.3 The Sobolev space $H^1(I)$

Let $I = (a, b)$ be an open interval in $\mathbb{R}$. (We will mainly have a bounded interval (a, b) in the applications but here we will suppose that I could be unbounded in general, i.e. that possibly $a = -\infty$ and/or $b = \infty$, unless otherwise stated).

Definition 2.7. The Sobolev space $H^1(I)$ is defined by

$$H^1(I) = \{ u \in L^2(I) : \exists g \in L^2(I) \text{ such that } \int_I u\phi' = -\int_I g\phi, \ \forall \phi \in C_c^1(I) \}.$$

For $u \in H^1(I)$ we denote $g = u'$ and call g the weak (generalized) derivative of u (in the L^2 sense).

Remarks 2.8.

(i) When there is no reason for confusion we shall denote $H^1 = H^1(I)$, $L^2 = L^2(I)$, etc.

(ii) It is clear that the generalized derivative g in the above definition is *unique*. For suppose $\exists\, g_1, g_2 \in L^2(I)$ such that $\int_I (g_1 - g_2)\phi = 0$, $\forall \phi \in C_c^1(I)$. Since $C_c^\infty(I) \subset C_c^1(I) \subset L^2(I)$ and since (by Proposition 2.6) $C_c^\infty(I)$ is dense in $L^2(I)$, it follows that $C_c^1(I)$ is dense in $L^2(I)$. It follows that $g_1 - g_2 = 0$ in $L^2(I)$. (N.B. In general in a Hilbert space H, where $D \subset H$ dense in H, we prove that $(g, \phi) = 0 \,\forall \phi \in D \Rightarrow g = 0$, since $\exists\, \phi_i \in D$ such that $\phi_i \to g$, $i \to \infty$ in H. Therefore $0 = (g, \phi_i) \to (g, g) \Rightarrow g = 0$). We emphasize again that $g_1 = g_2$ in L^2 means that $g_1(x) = g_2(x)$ a.e. in I.

(iii) The functions $\phi \in C_c^1(I)$ in the definition of H^1 are called *test functions*. One could take $C_c^\infty(I)$ to be the set of test functions instead of $C_c^1(I)$. (The only thing to show is that if $\int_I u\phi' = -\int_I g\phi$, for $u, g \in L^2(I)$, holds for every $\phi \in C_c^\infty(I)$, then it will hold for every $\phi \in C_c^1(I)$. This follows from the facts that $\phi \in C_c^1(I) \Rightarrow \rho_n * \phi \in C_c^\infty(I)$ for sufficient large n and $\rho_n * \phi \to \phi$ in $L^2(I)$, and also that $(\rho_n * \phi)' = \rho_n * \phi' \in C_c^\infty(I)$ and $\rho_n * \phi' \to \phi'$ in $L^2(I)$, see e.g. the proofs of Lemma 2.5, (ii), (iii).

(iv) It is clear that if $u \in C^1(I) \cap L^2(I)$ and if the (classical) derivative u' of u belongs to $L^2(I)$, then integration by parts gives that $\int_I u\phi' = -\int_I u'\phi \,\forall \phi \in C_c^1(I)$, i.e. that u' is the weak derivative of u, i.e. that $u \in H^1(I)$. Of course, if I is bounded, then $u \in C^1(\overline{I}) \Rightarrow u, u' \in L^2(I)$ and we have $C^1(\overline{I}) \subset H^1(I)$.

(v) There are other ways of defining the Sobolev space H^1. Using e.g. the *theory of distributions* we may conclude that every $u \in L^2(I)$ has a distributional derivative u'. We say that $u \in H^1(I)$ if u' is represented, as a distribution, by a function $u' \in L^2(I)$. If $I = \mathbb{R}$ we may also define H^1 using Fourier transforms. $\square$

Examples 2.9.

(i) Consider $u(x) = |x|$ on $I = (-1, 1)$. Clearly $u \in C(\overline{I})$, $u \in L^2(I)$, but u fails to

have a classical derivative at $x = 0$. Consider the function

$$g(x) = \begin{cases} -1 & \text{if } -1 < x \leq 0 \\ 1 & \text{if } \quad 0 < x < 1. \end{cases}$$

Clearly, $g \in L^2(I)$. In addition for each $\phi \in C_c^1(I)$,

$$
\begin{aligned}
-\int_{-1}^{1} g(x)\phi(x)\,dx &= -\int_{-1}^{0} (-1)\,\phi(x)\,dx - \int_{0}^{1} 1\,\phi(x)\,dx \\
&= -\int_{-1}^{0} \phi(x)\,d(-x) - \int_{0}^{1} \phi(x)\,dx = -\left[(-x)\phi(x)\right]_{-1}^{0} \\
&\quad - \left[x\phi(x)\right]_{0}^{1} + \int_{-1}^{0} (-x)\phi'(x)\,dx + \int_{0}^{1} x\phi'(x)\,dx \\
&= \int_{-1}^{1} |x|\phi'(x)\,dx = \int_{-1}^{1} u(x)\phi'(x)\,dx.
\end{aligned}
$$

It follows that $u(x) = |x| \in H^1((-1,1))$ and $u' = g$ is the weak derivative of u.

(ii) More generally, if I is a bounded interval and $u \in C(\overline{I})$ with u' (classical derivative) piecewise continuous on $\overline{I}$ (as would be the case e.g. if u is a piecewise polynomial, continuous function on $\overline{I}$), then $u \in H^1(I)$ and its weak derivative coincides with the classical derivative a.e. in I.

(iii) As in (i) the function

$$u(x) = \frac{1}{2}(|x| + x) = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

on $I = (-1, 1)$ belongs to H^1 and its weak derivative u' is the function

$$h(x) = \begin{cases} 0 & \text{if } -1 < x \leq 0 \\ 1 & \text{if } \quad 0 < x < 1, \end{cases}$$

which is called *Heaviside's function*. Clearly $h \in L^2(I)$. Does h belong to $H^1(I)$? The answer is no: Suppose that $h \in H^1(I)$. Then there must exist $v \in L^2(I)$ such that $\int_I h\phi' = -\int_I v\phi$, $\forall \phi \in C_c^1(I)$, i.e. a $v \in L^2(I)$ such that $\int_{-1}^{1} v\phi = -\int_{-1}^{1} h\phi' = -\int_0^1 \phi' = -\phi(1) + \phi(0) = \phi(0)$, $\forall \phi \in C_c^1(I)$. Take then such a ϕ with support in the interval $(-1, 0)$. It follows that $\int_{-1}^{0} v\phi = \int_{-1}^{1} v\phi = 0$ $\forall \phi \in C_c^1((-1,0))$. Since $C_c^1((-1,0))$ is dense in $L^2((-1,0))$, as in Remark 2.8 (ii), we see that $v(x) = 0$ a.e. in $(-1,0)$. Analogously, taking $\phi \in C_c^1(0,1)$ we

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prove that $v(x) = 0$ a.e. in $(0,1)$. We conclude therefore that $v = 0$ a.e. in $(-1,1)$. But that would contradict $\int_{-1}^{1} v\phi = \phi(0)$, $\forall \phi \in C_c^1(I)$. (Of course, as a *distribution*, $h(x)$ has a distributional derivative which coincides with the δ–"function", $h' = \delta_0$. We just proved that $\delta_0 \notin L^2(I)$). $\qquad\qquad\square$

It is clear that $H^1(I)$ is a linear subspace of $L^2(I)$, since if u, $v \in H^1(I)$ and u', v' are their weak derivatives, then $\lambda u' + \mu v'$ is the weak derivative of $\lambda u + \mu v$. Hence $\lambda u + \mu v \in H^1(I)$, and $(\lambda u + \mu v)' = \lambda u' + \mu v'$ for λ, $\mu \in \mathbb{R}$. We denote by $(\cdot, \cdot)$, $\|\cdot\|$, respectively, the inner product, resp. norm of $L^2 = L^2(I)$, i.e. we let, for u, $v \in L^2(I)$, $(u,v) = \int_I u(x)v(x)\,dx$, $\|u\| = (u,u)^{\frac{1}{2}}$. Then, for u, $v \in H^1(I)$ we define

$$
\begin{aligned}
(u,v)_1 &\equiv (u,v) + (u',v'), \\
\|u\|_1 &\equiv \left(\|u\|^2 + \|u'\|^2\right)^{\frac{1}{2}} = (u,u)_1^{\frac{1}{2}}.
\end{aligned}
$$

It is clear that $(\cdot, \cdot)_1$ is an *inner product* on $H^1 = H^1(I)$ and $\|\cdot\|_1$ the induced *norm* on $H^1(I)$. (To be precise, sometimes we shall denote $\|\cdot\|_1 = \|\cdot\|_{H^1(I)}$ etc.). Hence $H^1(I)$ becomes an inner product space.

Theorem 2.10. *The space* $(H^1, \|\cdot\|_1)$ *is a Hilbert space.*

Proof. We only need to show that $H^1(I)$ is complete with respect to the norm $\|\cdot\|_1$. Let $(u_n) \in H^1(I)$ be a Cauchy sequence in the norm $\|\cdot\|_1$, i.e. let

$$
\lim_{m,n\to\infty} \|u_m - u_n\|_1 = 0.
$$

By the definition of $\|\cdot\|_1$ it follows that (u_n) and (u_n') are Cauchy sequences in L^2. Since L^2 is complete, there exist $u, g \in L^2(I)$ such that $u_n \to u$ in L^2, $u_n' \to g$ in L^2. Now, by definition, $(u_n, \phi') = -(u_n', \phi)$, $\forall \phi \in C_c^1(I)$, for $n = 1, 2, 3, \ldots$.

Since $\forall \phi \in C_c^1(I)$

$$
|(u_n, \phi') - (u, \phi')| \leq \|u_n - u\|\|\phi'\| \to 0, \quad n \to \infty
$$

and

$$
|(u_n', \phi) - (g, \phi)| \leq \|u_n' - g\|\|\phi\| \to 0, \quad n \to \infty,
$$

it follows that $(u, \phi') = -(g, \phi)$, $\forall \phi \in C_c^1(I)$, i.e. that $u \in H^1(I)$ and $u' = g$.

It remains to show that $u_n \to u$ as $n \to \infty$ in H^1. But this follows from

$$\|u_n - u\|_1^2 = \|u_n - u\|^2 + \|u_n' - u'\|^2 = \|u_n - u\|^2 + \|u_n' - g\|^2 \to 0, \ \text{ as } n \to \infty.$$

Hence $\exists\, u \in H^1(I)$ such that $u_n \to u$ in H^1, i.e. $H^1(I)$ is complete. $\qquad\square$

Remark 2.11. Consider the map $T : H^1 \to L^2 \times L^2$ given by $Tu = [u, u']$, $u \in H^1$. Equipping $L^2 \times L^2$ with the norm $((u, u) + (v, v))^{1/2}$ we see that T is an isometry of H^1 onto a closed subspace of $L^2 \times L^2$. It follows that H^1 is *separable*, since L^2 is. $\quad\square$

The following result will be very important in the sequel:

Theorem 2.12. *If $u \in H^1(I)$, then $\exists\, \tilde{u} \in C(\overline{I})$ such that $u = \tilde{u}$ a.e. in I and*

$$\tilde{u}(x) - \tilde{u}(y) = \int_y^x u'(t)\, dt, \ \ \forall x, y \in \overline{I}.$$

Before proving the theorem we make some comments on its content. Note first that if $u \in H^1(I)$ and $u = v$ a.e. on I, then $v \in H^1(I)$. Then Theorem 2.12 tells us that in the equivalence class of an element $u \in H^1(I)$ there is one (and *only* one since $u, v \in C(\overline{I})$, $u = v$ a.e. on $I \Rightarrow u(x) = v(x), \forall x \in I$) continuous "representative" of u, denoted in the theorem by $\tilde{u}$. Hence, when there is need to do so, we shall use instead of u its continuous representative $\tilde{u}$. For example as the *value $u(x)$* for some $x \in \overline{I}$ (not well-defined if $u \in L^2$) we mean the *value* of $\tilde{u}$ at that x. Sometimes we shall replace u by $\tilde{u}$ with no special mention or by just noting that u is continuous, "upon modification on a set of measure zero in I". We emphasize that the statement "$\exists\, \tilde{u} \in C(\overline{I})$ such that $u = \tilde{u}$ a.e. in I" is different from the statement that "u is continuous a.e. in I".

For the proof of Theorem 2.12 we shall need two lemmata.

Lemma 2.13. *Let $f \in L_{loc}^1(I)$ such that*

$$\int_I f\phi' = 0, \quad \forall \phi \in C_c^1(I).$$

Then, there exists a constant C such that $f = C$ a.e. in I.

Proof. Let ψ be a fixed function in $C_c(I)$ such that $\int_I \psi = 1$. We shall show that, given $w \in C_c(I)$, there exists $\phi \in C_c^1(I)$, such that $\phi' = w - (\int_I w)\psi$. Indeed, given $w \in C_c(I)$, consider $h(x) = w(x) - (\int_I w)\psi(x)$. Clearly $h \in C_c(I)$. Put $\phi(x) = \int_a^x h(x)\, dx$. Let

supp $h \subset [c,d] \subset (a,b) = I$. Clearly, for $a < y < c$, $\phi(y) = \int_a^y h(x)\,dx = 0$, and for $d < z < b$

$$
\begin{aligned}
\phi(z) &= \int_a^z h(x)\,dx = \int_a^b h(x)\,dx = \int_I h = \int_I \left(w - \left(\int_I w \right)\psi \right) \\
&= \int_I w - \left(\int_I w \right)\left(\int_I \psi \right) = \int_I w - \int_I w = 0.
\end{aligned}
$$

It follows that $\phi \in C_c(I)$. Also $\phi'(x) = h(x) \in C_c(I)$, i.e. $\phi \in C_c^1(I)$, and $\phi' = h = w - (\int_I w)\psi$. Now, by hypothesis, $\int_I f\phi' = 0$. In particular, for each $w \in C_c(I)$,

$$
\int_I f\left(w - \left(\int_I w \right)\psi \right) = 0 \;\Rightarrow\; \int_I fw - \int_I f\psi \int_I w = 0 \Rightarrow \int_I fw - \int_I \left(\int_I f\psi \right) w = 0
$$

$$
\Rightarrow\; \int_I \left(f - \int_I f\psi \right) w = 0.
$$

By Proposition 2.3 we conclude that $f(x) = \int_I f\psi$ a.e. on I. i.e. $f(x) = C \equiv \int_I f\psi$ a.e. on I. $\qquad\square$

Lemma 2.14. *Let $g \in L_{loc}^1(I)$. For $y_0 \in I$ fixed, put*

$$
v(x) = \int_{y_0}^x g(t)\,dt, \quad x \in I.
$$

Then $v \in C(I)$ and

$$
\int_I v\phi' = -\int_I g\phi, \quad \forall \phi \in C_c^1(I).
$$
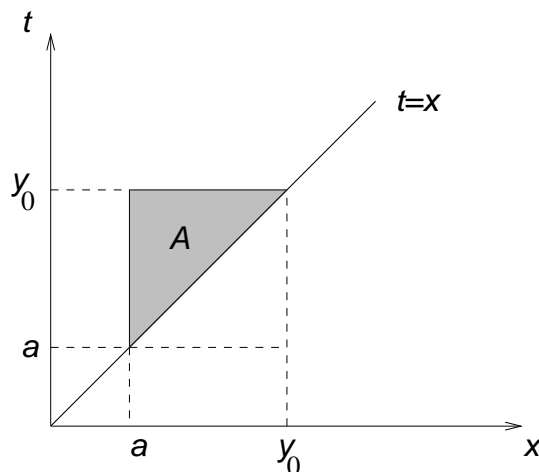
Proof. That $v \in C(I)$ when $g \in L_{loc}^1(I)$, is a well–known fact from measure theory. We now have for $\phi \in C_c^1(I)$ that

$$
\begin{aligned}
\int_I v\phi' &= \int_I \left(\int_{y_0}^x g(t)\,dt \right)\phi'(x)\,dx \\
&= -\int_a^{y_0} dx \left(\int_x^{y_0} g(t)\,dt \right)\phi'(x) + \int_{y_0}^b dx \left(\int_{y_0}^x g(t)\,dt \right)\phi'(x).
\end{aligned}
$$

Now

$$-\int_a^{y_0} dx \left(\int_x^{y_0} g(t)\, dt\right)\phi'(x) = -\int_a^{y_0} dx \int_x^{y_0} dt(g(t)\phi'(x))$$

$$= \text{(since } g(t)\phi'(x) \text{ is integrable on } A, \text{ see figure)}$$

$$= -\int_A g(t)\phi'(x)\, dxdt = -\int_a^{y_0} dt \int_a^t dx g(t)\phi'(x)$$

$$= -\int_a^{y_0} g(t)\, dt \int_a^t \phi'(x)\, dx = -\int_a^{y_0} g(t)\phi(t)\, dt.$$

Similarly we may prove that

$$\int_{y_0}^b dx \left(\int_{y_0}^x g(t)\, dt\right)\phi'(x) = -\int_{y_0}^b g(t)\phi(t)\, dt.$$

We conclude that

$$\int_I v\phi' = -\int_a^{y_0} g\phi - \int_{y_0}^b g\phi = -\int_I g\phi, \quad \forall \phi \in C_c^1(I).$$

$\square$

Proof of Theorem 2.12. Fix $y_0 \in I$. Note that since $u \in H^1(I) \Rightarrow u' \in L^2(I) \Rightarrow u' \in L^1_{loc}(I)$. Put

$$\bar{u}(x) = \int_{y_0}^x u'(t)\, dt.$$

By Lemma 2.14 $\bar{u} \in C(I)$ and $\int_I \bar{u}\phi' = -\int_I u'\phi$, $\forall \phi \in C_c^1(I)$. But by definition of u', $-\int_I u'\phi = \int_I u\phi'$, $\forall \phi \in C_c^1(I)$. Therefore

$$\int_I (\bar{u} - u)\phi' = 0, \quad \forall \phi \in C_c^1(I).$$

By Lemma 2.13, we conclude that there exists a constant C such that $\bar{u} - u = C$ a.e. on I. Define now $\tilde{u}(x) = \bar{u}(x) - C$. It follows that $\tilde{u} \in C(\bar{I})$ and $\tilde{u} = u$ a.e. on I. Moreover for $x, y \in \bar{I}$,

$$\tilde{u}(x) - \tilde{u}(y) = \bar{u}(x) - \bar{u}(y) = \int_{y_0}^x u' - \int_{y_0}^y u'$$

$$= \int_{y_0}^x u' + \int_y^{y_0} u' = \int_y^x u'(t)\, dt$$

$\square$

Remark 2.15. Lemma 2.14 gives in particular that the primitive (antiderivative) v of a function $g \in L^2(I)$ is in $H^1(I)$ provided $v \in L^2(I)$. (The latter fact is always true if I is bounded). $\square$

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The following theorem gives a technical tool that will be often used in sequel.

Theorem 2.16 (Extension theorem)**.** *There exists an extension operator*

$$E : H^1(I) \to H^1(\mathbb{R}),$$

linear and continuous, such that

(i) $Eu|_I = u \ \forall u \in H^1(I)$, $(f|_I$ *denotes the restriction of* f *to* I*).*

(ii) $\|Eu\|_{L^2(\mathbb{R})} \le C\|u\|_{L^2(I)} \ \forall u \in H^1(I)$.

(iii) $\|Eu\|_{H^1(\mathbb{R})} \le C\|u\|_{H^1(I)} \ \forall u \in H^1(I)$.

(In (ii) we can take $C = 2\sqrt{2}$ *and in (iii)* $C = C_0(1+1/\mu(I))$, *where* C_0 *some constant, independent of* u *and* I, *and* $\mu(I)$ *the length of* I *– possibly* $\mu(I) = \infty$*).*

Proof. We begin with the case $I = (0, \infty)$. We will show that the extension operator defined by *even reflection* about $x = 0$, i.e. by

$$(Eu)(x) \equiv u^*(x) = \begin{cases} u(x) & \text{if } x \ge 0 \\ u(-x) & \text{if } x < 0, \end{cases}$$

$u \in H^1(I)$, solves the problem. Indeed

$$\|u^*\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^0 (u(-x))^2 \, \mathrm{d}x + \int_0^\infty (u(x))^2 \, \mathrm{d}x = 2\|u\|_{L^2(I)}.$$

So (ii) is satisfied. (Obviously E is linear and satisfies (i)). Now put

$$v(x) = \begin{cases} u'(x) & \text{if } x > 0 \\ -u'(-x) & \text{if } x < 0. \end{cases}$$

Clearly $v \in L^2(\mathbb{R})$ since $\|v\|_{L^2(\mathbb{R})}^2 = 2\|u'\|_{L^2(I)}^2$. By Theorem 2.2 we also have that

$$u^*(x) - u(0) = \int_0^x u'(t) \, \mathrm{d}t = \int_0^x v(t) \, \mathrm{d}t, \text{ for } x \ge 0.$$

Also, for $x < 0$,

$$u^*(x) - u(0) = \int_0^{-x} u'(t) \, \mathrm{d}t = \int_0^x -u'(-t) \, \mathrm{d}t = \int_0^x v(t) \, \mathrm{d}t.$$

Hence, $u^*(x) - u(0) = \int_0^x v(t)\,dt$, $x \in \mathbb{R}$. Since $u^* \in L^2(\mathbb{R})$ and $v \in L^2(\mathbb{R})$, it follows by Lemma 2.14 (see Remark 2.15) that $u^* \in H^1(\mathbb{R})$ and $(u^*)' = v$. Hence

$$\|u^*\|_{H^1(\mathbb{R})}^2 = \|u^*\|_{L^2(\mathbb{R})}^2 + \|v\|_{L^2(\mathbb{R})}^2 = 2\left(\|u\|_{L^2(I)}^2 + \|u'\|_{L^2(I)}^2\right) = 2\|u\|_{H^1(I)}^2.$$

Hence in the case $I = (0, \infty)$, with $Eu = u^*$, (ii) and (iii) are satisfied (as equalities) with $C = \sqrt{2}$. (The proof holds for any unbounded interval of the form (a, ∞) or $(-\infty, a)$, $a \in \mathbb{R}$. For example, for $u \in H^1((a, \infty))$, define Eu by reflection evenly about $x = a$, i.e. as

$$(Eu)(x) = \begin{cases} u(x) & \text{if } x > a \\ u(2a - x) & \text{if } x \le a, \end{cases}$$

and the proof follows – with the same constants C – mutatis mutandis).

We now turn to the case of a bounded interval. It suffices to consider $I = (0, 1)$. Consider a fixed function $\eta \in C^1(\mathbb{R})$, $0 \le \eta(x) \le 1$, $\forall x \in \mathbb{R}$, such that

$$\eta(x) = \begin{cases} 1 & \text{if } x < \frac{1}{4} \\ 0 & \text{if } x > \frac{3}{4}, \end{cases}$$

and for every f defined on $(0, 1)$ denote by $\tilde{f}$ its extension by zero to $(0, \infty)$, i.e. put

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in (0, 1) \\ 0 & \text{if } x \ge 1. \end{cases}$$

Now if $u \in H^1(I)$ it follows that $\eta\tilde{u} \in H^1((0, \infty))$ and that $(\eta\tilde{u})' = \eta'\tilde{u} + \eta\tilde{u'}$, where by $\tilde{u'}$ we mean the extension by zero to $(0, \infty)$ of $u' \in L^2((0, 1))$. To see this, note first that $\eta\tilde{u} \in L^2((0, \infty))$ since

$$\int_0^\infty \eta^2 \tilde{u}^2 \le \int_0^{\frac{3}{4}} u^2 \le \|u\|_{L^2((0,1))}^2.$$

Moreover, for any $\phi \in C_c^1((0, \infty))$ we have that

$$\begin{aligned}
\int_0^\infty \eta\tilde{u}\phi' &= \int_0^1 u\eta\phi' = \int_0^1 u\left((\eta\phi)' - \eta'\phi\right) = \int_0^1 u(\eta\phi)' - \int_0^1 u\eta'\phi \\
&= \left(\text{since } \phi \in C_c^1((0, \infty)) \Rightarrow \eta\phi \in C_c^1((0, 1)), \text{ and } u \in H^1((0, 1))\right) \\
&= -\int_0^1 u'(\eta\phi) - \int_0^1 u\eta'\phi = -\int_0^1 (u'\eta + u\eta')\phi = \int_0^\infty g\phi,
\end{aligned}$$

where

$$g(x) = \begin{cases} u'\eta + u\eta' & \text{if } x \in (0, 1) \\ 0 & \text{if } x \ge 1. \end{cases}$$

Now $g \in L^2((0, \infty))$ since

$$\|g\|^2_{L^2((0,\infty))} \quad = \quad \int_0^1 (u'\eta + u\eta')^2 \leq 2 \left(\int_0^1 \eta^2 (u')^2 + \int_0^1 (\eta')^2 u^2 \right)$$

$$\leq \quad 2 \left(\int_0^1 (u')^2 + \max_{0 \leq x \leq 1} |\eta'(x)|^2 \int_0^1 u^2 \right) \leq C_1 \|u\|^2_{H^1(I)}.$$

(Note that we can easily arrange that $\max_{0 \leq x \leq 1} |\eta'(x)|$ be equal to e.g. 2.5). Moreover $g = \eta'\tilde{u} + \eta\tilde{u}'$. It follows that $\eta\tilde{u} \in H^1((0, \infty))$ and $(\eta\tilde{u})' = g = \eta'\tilde{u} + \eta\tilde{u}'$. Returning to the proof of the theorem, for $u \in H^1(I)$, $I = (0, 1)$, write u as

$$u = \eta u + (1 - \eta)u, \quad \eta \text{ as above.}$$

The function ηu can be extended to $(0, \infty)$ by $\eta\tilde{u}$ as before. Clearly $\eta\tilde{u} \in H^1((0, \infty))$ and $\|\eta\tilde{u}\|^2_{L^2((0,\infty))} \leq \|u\|^2_{L^2((0,1))}$. Also, as above

$$\|(\eta\tilde{u})'\|^2_{L^2((0,\infty))} \quad = \quad \int_0^\infty g^2 \leq 2 \left(\|u'\|^2_{L^2(I)} + \max_{0 \leq x \leq 1} |\eta'(x)|^2 \|u\|^2_{L^2((0,1))} \right)$$

$$\leq \quad C_1 \|u\|^2_{H^1(I)}.$$

It follows that

$$\|\eta\tilde{u}\|^2_{H^1((0,\infty))} \leq C_2 \|u\|_{H^1(I)}.$$

Now, extend $\eta\tilde{u} \in H^1((0, \infty))$ as in the first part of the proof to a function $v_1(x) \in H^1(\mathbb{R})$ by even reflection about $x = 0$. It follows that

$$\|v_1\|_{L^2(\mathbb{R})} = \sqrt{2}\|\eta\tilde{u}\|_{L^2((0,\infty))} \leq \sqrt{2}\|u\|_{L^2(I)}$$

and that

$$\|v_1\|_{H^1(\mathbb{R})} = \sqrt{2}\|\eta\tilde{u}\|_{H^1((0,\infty))} \leq \sqrt{2}C_2\|u\|_{H^1(I)}.$$

(It is clear that $v_1|_I = \eta u$ and that the operation $\eta u \mapsto v_1$ is linear in u).

Analogously the function $(1 - \eta)u$ (for which $(1 - \eta)u = 0$ for $0 \leq x \leq 1/4$), can be extended to $(-\infty, 1)$ by $(1 - \eta)\tilde{\tilde{u}}$ where

$$\tilde{\tilde{u}}(x) = \begin{cases} u(x) & \text{if } 0 < x < 1 \\ 0 & \text{if } -\infty < x \leq 0. \end{cases}$$

We obtain again $(1 - \eta)\tilde{\tilde{u}} \in H^1((-\infty, 1))$ with

$$\|(1 - \eta)\tilde{\tilde{u}}\|_{L^2((-\infty,1))} \leq \|u\|_{L^2(I)}$$

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and

$$\|(1 - \eta)\tilde{\tilde{u}}\|_{H^1((-\infty,1))} \le C_2 \|u\|_{H^1(I)}.$$

Extend now $(1 - \eta)\tilde{\tilde{u}}$ to a function $v_2 \in H^1(\mathbb{R})$ by (even) reflection about $x = 1$. It follows that

$$\|v_2\|_{L^2(\mathbb{R})} = \sqrt{2}\|(1 - \eta)\tilde{\tilde{u}}\|_{L^2((-\infty,1))} \le \sqrt{2}\|u\|_{L^2(I)}$$

and that

$$\|v_2\|_{H^1(\mathbb{R})} = \sqrt{2}\|(1 - \eta)\tilde{\tilde{u}}\|_{H^1((-\infty,1))} \le \sqrt{2}C_2 \|u\|_{H^1(I)},$$

that $v_2|_I = (1 - \eta)u$ and that $(1 - \eta)u \mapsto v_2$ is linear.

We define now the operator E as $Eu = v_1 + v_2$. Clearly E satisfies (i) and is linear; (ii) and (iii) follow by the above and the triangle inequality. (Note that when $I = (a, b)$, η must be redefined as

$$\eta_{a,b}(x) = \eta(\frac{x - a}{x - b}),$$

so that

$$\eta'_{a,b}(x) = \frac{1}{b - a}\,\eta'(\frac{x - a}{x - b}).)$$

$\square$

The following result is a basic *density theorem* for $H^1(I)$ and will be used very often in sequel.

Theorem 2.17. *Let $u \in H^1(I)$. There exists a sequence (u_n) of functions in $C_c^\infty(\mathbb{R})$ such that*

$$u_n|_I \to u \text{ in } H^1(I), \quad n \to \infty.$$

Comment: The theorem asserts that if $I = \mathbb{R}$, then $C_c^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$. Otherwise, $C_c^\infty(I)$ is *not* dense in $H^1(I)$ – in fact we shall see later that the closure of $C_c^\infty(I)$ in $H^1(I)$ is the space $\overset{0}{H}{}^1$ consisting of those functions of $H^1(I)$ which are zero at the boundary of I. If I is bounded, Theorem 2.17 asserts that there is a sequence of functions $u_n \in C^\infty(\overline{I})$ such that $u_n \to u$ in $H^1(I)$.

Proof of Theorem 2.17. First note that it suffices to consider the case $I = \mathbb{R}$. For suppose that the result holds for $\mathbb{R}$. If $I \subset \mathbb{R}$ extend u to Eu in $H^1(\mathbb{R})$ as in Theorem 2.16. Then there exists a sequence $u_n \in C_c^\infty(\mathbb{R})$ such that $\|u_n - Eu\|_{H^1(\mathbb{R})} \to 0$,

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$n \to \infty$. But then

$$\|u_n|_I - u\|_{H^1(I)} = \|u_n - Eu\|_{H^1(I)} \le \|u_n - Eu\|_{H^1(\mathbb{R})} \to 0, \ n \to \infty,$$

i.e. the result holds for I.

Hence consider the case $I = \mathbb{R}$. The approximating sequence is constructed by *regularization* and *truncation* as $u_n = \zeta_n(\rho_n * u)$. Here (ρ_n), $n = 1, 2, \ldots$ is the regularizing sequence defined in Def. 2.4 and $\zeta_n = \zeta_n(x) \in C_c^\infty(\mathbb{R})$ is a truncation function, defined for $n = 1, 2, \ldots$ by

$$\zeta_n(x) = \zeta\left(\frac{x}{n}\right),$$

where $\zeta(x)$ is a fixed function in $C_c^\infty(\mathbb{R})$ such that

$$\zeta(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

Hence $\zeta_n(x) = 1$ for $|x| \le n$ and $\zeta_n(x) = 0$ for $|x| \ge 2n$. Moreover,

$$|\zeta_n'(x)| = \frac{1}{n}\left|\zeta'\left(\frac{x}{n}\right)\right| \le \frac{C}{n}, \ \text{where } C = \max_{x \in \mathbb{R}}|\zeta'(x)|.$$

Note, by Lebesgue's dominated convergence theorem, that we have $\zeta_n f \to f$ as $n \to \infty$ in L^2 for every $f \in L^2(\mathbb{R})$. Let now $u_n = \zeta_n(\rho_n * u)$. Clearly $u_n \in C_c^\infty(\mathbb{R})$ since $\zeta_n \in C_c^\infty(\mathbb{R})$ and $\rho_n * u \in C^\infty(\mathbb{R})$, cf. Lemma 2.5 (iii) We have

$$u_n - u = \zeta_n(\rho_n * u) - u = \zeta_n[(\rho_n * u) - u] + (\zeta_n u - u).$$

It follows that

$$\begin{aligned} \|u_n - u\|_{L^2(\mathbb{R})} &\le \|\zeta_n[(\rho_n * u) - u]\|_{L^2(\mathbb{R})} + \|\zeta_n u - u\|_{L^2(\mathbb{R})} \\ &\le \|\rho_n * u - u\|_{L^2(\mathbb{R})} + \|\zeta_n u - u\|_{L^2(\mathbb{R})} \to 0, \ n \to \infty \end{aligned}$$

by the above and Lemma 2.5 (iii). Now $u_n' = \zeta_n(\rho_n * u)' + \zeta_n'(\rho_n * u)$. (Note that for $u \in H^1(\mathbb{R})$, $\rho_n * u \in H^1(\mathbb{R})$ and $(\rho_n * u)' = \rho_n * u'$: The interested reader may verify that for $\phi \in C_c^1(\mathbb{R})$ the following equalities hold

$$\begin{aligned} \int_{-\infty}^{\infty}(\rho_n * u)\phi' &= \int_{-\infty}^{\infty} u(\rho_n(-x) * \phi') = \int_{-\infty}^{\infty} u(\rho_n(-x) * \phi)' = -\int_{-\infty}^{\infty} u'(\rho_n(-x) * \phi) \\ &= -\int_{-\infty}^{\infty}(\rho_n * u')\phi.) \end{aligned}$$

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Hence, since $u'_n - u' = \zeta_n(\rho_n * u') + \zeta'_n(\rho_n * u) - u'$, we have

$$
\begin{aligned}
\|u'_n - u'\|_{L^2(\mathbb{R})} \;\leq\; & \|\zeta'_n(\rho_n * u)\|_{L^2(\mathbb{R})} + \|\zeta_n[(\rho_n * u') - u']\|_{L^2(\mathbb{R})} \\
+ \;& \|\zeta_n u' - u'\|_{L^2(\mathbb{R})} \;\leq\; \max_{x \in \mathbb{R}} |\zeta'_n(x)| \; \|\rho_n * u\|_{L^2(\mathbb{R})} \\
+ \;& \|\rho_n * u' - u'\|_{L^2(\mathbb{R})} + \|\zeta_n u' - u'\|_{L^2(\mathbb{R})} \to 0 \text{ as } n \to \infty,
\end{aligned}
$$

by Lemma 2.5 (iii), since $\|\rho_n * u\|_{L^2(\mathbb{R})}$, $n = 1, 2, 3, \ldots$ is bounded. $\qquad \square$

We are able now to prove *Sobolev's imbedding theorem* for $H^1(I)$.

Theorem 2.18 (Sobolev)**.** *There exists a constant C (depending only on $\mu(I) \leq \infty$) such that*

$$
\|u\|_{L^\infty(I)} \leq C \, \|u\|_{H^1(I)}, \quad \forall u \in H^1(I). \tag{2.1}
$$

(We say that $H^1(I) \subset L^\infty(I)$, i.e. that $H^1(I) \subset C(\overline{I})$ – in view of Theorem 2.12 – if I bounded, with continuous imbedding).

Proof. Again it suffices to prove the result for $I = \mathbb{R}$. (For suppose it holds for $\mathbb{R}$ and let $u \in H^1(I)$. Extend u to Eu in $H^1(\mathbb{R})$ as in Theorem 2.16. Then

$$
\|u\|_{L^\infty(I)} \leq \|Eu\|_{L^\infty(\mathbb{R})} \leq C \, \|Eu\|_{H^1(\mathbb{R})} \leq C' \, \|u\|_{H^1(I)},
$$

using (iii) in Theorem 2.16 and (2.1) for $I = \mathbb{R}$). Suppose first that $v \in C_c^\infty(\mathbb{R})$. Then for every $x \in \mathbb{R}$:

$$
\begin{aligned}
v^2(x) \;=\; & \int_{-\infty}^x (v^2)' = 2 \int_{-\infty}^x v v' \leq 2\|v\|_{L^2(\mathbb{R})} \|v'\|_{L^2(\mathbb{R})} \\
\leq \;& \|v\|_{L^2(\mathbb{R})}^2 + \|v'\|_{L^2(\mathbb{R})}^2 = \|v\|_{H^1(\mathbb{R})}^2.
\end{aligned}
$$

Hence $\|v\|_{L^\infty(\mathbb{R})} \leq \|v\|_{H^1(\mathbb{R})}$, $\forall v \in C_c^\infty(\mathbb{R})$, i.e. (2.1) holds on $C_c^\infty(\mathbb{R})$. Now given $u \in H^1(\mathbb{R})$, since $C_c^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$ (Theorem 2.17), we can find a sequence (u_n) in $C_c^\infty(\mathbb{R})$ such that $u_n \to u$ in $H^1(\mathbb{R})$. It follows that $u_n \to u$ in $L^2(\mathbb{R})$. Therefore there is a subsequence of u_n (denote it again by u_n, i.e. consider that subsequence to be the original sequence) such that $u_n(x) \to u(x)$ a.e. on $\mathbb{R}$ as $n \to \infty$. By (2.1), which was established on $C_c^\infty(\mathbb{R})$, we have that u_n is Cauchy in $L^\infty(\mathbb{R})$ (since it is Cauchy in $H^1(\mathbb{R})$). Therefore it converges in L^∞ to some element $\tilde{u} \in L^\infty(\mathbb{R})$. It follows that $u = \tilde{u}$ a.e. on $\mathbb{R}$, i.e. that $u \in L^\infty(\mathbb{R})$. Taking limits in $\|u_n\|_{L^\infty(\mathbb{R})} \leq \|u_n\|_{H^1(\mathbb{R})}$ we obtain (2.1). $\qquad \square$

Remarks 2.19.

(i) In the case of a finite interval $I = (a, b)$ an alternative proof for Sobolev's theorem and (2.1) is the following: Let $u \in H^1(I)$. Then, by Theorem 2.12, there exists $\tilde{u} \in C(\bar{I})$ with $u = \tilde{u}$ a.e. in I such that

$$\tilde{u}(x) - \tilde{u}(y) = \int_y^x u'(t)\, dt, \quad \forall x, y \in \bar{I}.$$

Therefore, $\forall x, y \in \bar{I}$

$$|\tilde{u}(x)| \le |\tilde{u}(y)| + \int_a^b |u'| \le |\tilde{u}(y)| + \sqrt{b - a}\, \|u'\|_{L^2(I)}.$$

Fix now $y \in \bar{I}$. Then

$$\max_{x \in \bar{I}} |\tilde{u}(x)| \le |\tilde{u}(y)| + \sqrt{b - a}\, \|u'\|_{L^2(I)}.$$

This holds for all $y \in \bar{I}$. Integrating with respect to y we find

$$(b - a) \max_{\bar{I}} |\tilde{u}| \le \int_I |u| + (b - a)^{3/2} \|u'\|_{L_2(I)} \le \sqrt{b - a}\, \|u\|_{L^2(I)} + (b - a)^{3/2} \|u'\|_{L^2(I)}.$$

Hence

$$\max_{\bar{I}} |\tilde{u}| = \|u\|_{L^\infty(I)} \le \frac{1}{(b - a)^{1/2}} \|u\|_{L^2(I)} + (b - a)^{1/2} \|u'\|_{L^2(I)}$$

$$\le \sqrt{2} \left(\frac{1}{b - a} \|u\|_{L^2(I)}^2 + (b - a) \|u'\|_{L^2(I)}^2 \right)^{1/2},$$

i.e. $\|u\|_{L^\infty(I)} \le C(I) \|u\|_{H^1(I)}$, where $C(I) = \sqrt{2} \max\left(b - a, \frac{1}{b-a}\right)$.

(ii) If I is bounded, the imbedding $H^1(I) \subset C(\bar{I})$ is *compact*. This follows from Theorem 2.12: If $u \in \mathcal{N}$ (=the unit ball in $H^1(I)$ with center zero), we have

$$|u(x) - u(y)| = \left| \int_y^x u'(t)\, dt \right| \le \|u'\|_{L^2(I)} |x - y|^{\frac{1}{2}}, \quad \forall x, y \in I.$$

Hence $\forall u \in \mathcal{N}$, $|u(x) - u(y)| \le |x - y|^{1/2}$ and the conclusion follows from the Arzela–Ascoli theorem.

(iii) The inequality (2.1) for $I = \mathbb{R}$ implies that if $u \in H^1(\mathbb{R})$, then

$$\lim_{|x| \to \infty} u(x) = 0.$$

For if $C_c^\infty(\mathbb{R}) \ni u_n \to u$ in $H^1(\mathbb{R})$, then $\|u_n - u\|_{L^\infty(\mathbb{R})} \to 0$, $n \to \infty$. Hence $\forall \epsilon > 0 \, \exists N$ such that $\|u_N - u\|_{L^\infty(\mathbb{R})} < \epsilon \Rightarrow |u(x)| < \epsilon$ for $|x|$ sufficiently large, since $u_N \in C_c^\infty(\mathbb{R})$, i.e. $\lim_{|x| \to \infty} u(x) = 0$. $\qquad \square$

The following two propositions follow from Theorem 2.18 and are useful in the applications.

Proposition 2.20. *Let $u, v \in H^1(I)$. Then $uv \in H^1(I)$ and $(uv)' = u'v + uv'$. Moreover we can integrate by parts:*

$$\int_y^x u'v = u(x)v(x) - u(y)v(y) - \int_y^x uv' \quad \forall\, x, y \in \overline{I}.$$

Proof. Since $u \in H^1(I) \overset{\text{Th.2.18}}{\Longrightarrow} u \in L^\infty(I)$. Hence $v \in L^2(I) \Rightarrow uv \in L^2(I)$. Now let u_n, v_n, $n = 1, 2, \ldots$ be sequences in $C_c^\infty(\mathbb{R})$ such that $u_n|_I \to u$ in $H^1(I)$ and $v_n|_I \to v$ in $H^1(I)$ as $n \to \infty$ (Theorem 2.17). It follows by Theorem 2.18 that $u_n|_I \to u$ in $L^\infty(I)$ and $v_n|_I \to v$ in $L^\infty(I)$. Now

$$
\begin{aligned}
\|u_n v_n - uv\|_{L^2(I)} &\leq \|u_n v_n - u_n v\|_{L^2(I)} + \|u_n v - uv\|_{L^2(I)} \\
&\leq \|u_n\|_{L^\infty(I)} \|v_n - v\|_{L^2(I)} + \|v\|_{L^\infty(I)} \|u_n - u\|_{L^2(I)} \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Hence $u_n v_n|_I \to uv$ in $L^2(I)$.

In addition $(u_n v_n)' = u_n' v_n + u_n v_n' \to u'v + uv'$ ($\in L^2(I)$) in $L^2(I)$. (To see this note e.g. that

$$
\begin{aligned}
\|u_n' v_n - u'v\|_{L^2(I)} &\leq \|u_n' v_n - u' v_n\|_{L^2(I)} + \|u' v_n - u'v\|_{L^2(I)} \\
&\leq \|v_n\|_{L^\infty(I)} \|u_n' - u'\|_{L^2(I)} + \|u'\|_{L^2(I)} \|v_n - v\|_{L^\infty(I)} \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Similarly $u_n v_n' \to uv'$ in $L^2(I)$).

We now have a sequence $\phi_n \equiv u_n v_n|_I$ in $H^1(I)$ such that $\phi_n \overset{L^2}{\to} \phi \equiv uv$, and such that $\phi_n' \overset{L^2}{\to} \psi \equiv uv' + u'v$, $\psi \in L^2(I)$. Hence ϕ_n is Cauchy in L^2 and ϕ_n' is Cauchy in L^2 $\Rightarrow \phi_n$ is Cauchy in $H^1 \Rightarrow \phi_n \to w$ in H^1. Hence $\phi_n \to w$ in $L^2 \Rightarrow w = \phi$ and $\phi_n' \to w'$ in $L^2 \Rightarrow w' = \psi$; thus $\phi' = (uv)' = \psi = u'v + uv'$. The integration by parts formula follows by integrating both members of $(uv)' = u'v + uv'$ and using, since $uv \in H^1(I)$, Theorem 2.12. $\qquad\square$

Remark 2.21. It is well known that $u, v \in L^2(I) \not\Rightarrow uv \in L^2(I)$. (Take e.g. $I = (0, 1)$, $u = v = x^{-\frac{1}{4}}$). Hence $\{L^2(I), \|\cdot\|_{L^2(I)}\}$ is not a Banach algebra with respect to multiplication of functions, whereas $\{H^1(I), \|\cdot\|_{H^1(I)}\}$ is. $\qquad\square$

Proposition 2.22. *Let $G \in C^1(\mathbb{R})$ such that $G(0) = 0$ and let $u \in H^1(I)$. Then $G(u) \in H^1(I)$ and $(G(u(x)))' = G'(u(x))u'(x)$.*

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Proof. Since $u \in H^1(I) \Rightarrow u \in L^\infty(I)$. Let $M = \|u\|_{L^\infty(I)}$. Since $G \in C^1(\mathbb{R})$ and $G(0) = 0$, by the mean-value theorem, given s there exists $\theta : G(s) = G'(\theta)s$. Hence, given $\delta > 0$, there exists a constant $C = C(M, G, \delta)$ such that

$$|G(s)| \leq C|s| \text{ for } s \in [-M - \delta, M + \delta]. \qquad (2.2)$$

Since $|u(x)| \leq \|u\|_{L^\infty(I)}$ a.e. on I, it follows that $-M \leq u(x) \leq M$ a.e. on I, i.e. that $|G(u(x))| \leq C|u(x)|$ a.e. on I. Since $u \in L^2(I)$ it follows that $G(u) \in L^2(I)$. Also by (2.2) we have that $|G'(u(x))| \leq C$ a.e. on I. It follows that $|G'(u)||u'| \leq C|u'|$ a.e. $\Rightarrow G'(u)u' \in L^2(I)$ since $u' \in L^2(I)$. It remains to show that

$$\int_I G(u)\phi' = -\int_I G'(u)u'\phi, \quad \forall \phi \in C_c^1(I).$$

Since $u \in H^1(I)$, it follows by Theorem 2.17 that there exists a sequence $\{u_n\} \in C_c^\infty(\mathbb{R})$ such that $u_n \to u$ in $H^1(I)$. Moreover, by Theorem 2.18 we have that $u_n \to u$ in $L^\infty(I)$. By continuity, $G(u_n) \to G(u)$ in $L^\infty(I)$. Since $\|u_n\|_{L^\infty(I)} \to \|u\|_{L^\infty(I)}$ it follows that for n large enough, $\|u_n\| \leq M + \delta$. Hence (2.2) gives that for n large enough $|G(u_n)| \leq C|u_n| \Rightarrow G(u_n) \in L^2(I)$ and $\|G(u_n)\|_{L^2(I)} \leq C'\|u\|_{L^2(I)}$. By the dominated convergence theorem it follows that $G(u_n) \to G(u)$ in $L^2(I)$. Hence $\forall \phi \in C_c^1(I)$ $\int_I G(u_n)\phi' \to \int_I G(u)\phi'$, as $n \to \infty$. Now

$$\begin{aligned}
\|G'(u_n)u_n' - G'(u)u'\|_{L^2(I)} &\leq \|G'(u_n)u_n' - G'(u_n)u'\|_{L^2(I)} + \\
+\|G'(u_n)u' - G'(u)u'\|_{L^2(I)} &\leq \|G'(u_n)\|_{L^\infty(I)}\|u_n' - u'\|_{L^2(I)} + \\
&\quad + \|u'\|_{L^2(I)}\|G'(u_n) - G'(u)\|_{L^\infty(I)}.
\end{aligned}$$

Now, $\|u_n' - u'\|_{L^2(I)} \to 0$ since $u_n \to u$ in $H^1(I)$. Also, for n large enough $\|u_n\|_{L^\infty(I)} \leq M + \frac{\delta}{2}$ and therefore $\|G'(u_n)\|_{L^\infty(I)} \leq C$. Since $u_n \to u$ in $L^\infty(I)$, by continuity we have that $G'(u_n) \to G'(u)$ in $L^\infty(I)$. It follows that $G'(u_n)u_n' \to G'(u)u'$ in $L^2(I)$; hence $-\int_I G'(u_n)u_n'\phi \to -\int_I G'(u)u'\phi$, $\forall \phi \in C_c^1(I)$. Since $\phi \in C_c^1(I)$, $u_n \in C_c^\infty(\mathbb{R})$, we have that

$$\int_I G(u_n)\phi' = -\int_I G'(u_n)u_n'\phi, \ n = 1, 2, 3, \ldots.$$

Letting $n \to \infty$ we obtain the desired equality

$$\int_I G(u)\phi' = -\int_I G'(u)u'\phi.$$

$\square$

We close this section by providing an important result, *Rellich's compact imbedding theorem* for H^1, which we will have occasion to use in the sequel. The proof presupposes some familiarity with operator theory in Hilbert space.

Theorem 2.23 (Rellich). *Let $I = (a, b)$ be a finite interval. Then the imbedding $H^1(I) \subset L^2(I)$ is compact, i.e. if $E : H^1(I) \to L^2(I)$ is the imbedding operator of $H^1(I)$ into $L^2(I)$ (defined for $v \in H^1(I)$ as $Ev = v$), then E is a compact linear operator from $H^1(I)$ to $L^2(I)$.*

Proof. Let $v \in H^1(I)$. By Theorem 2.12 we have

$$v(x) = v(\alpha) + \int_a^x v'(t)\, \mathrm{d}t \quad \text{a.e. in } \overline{I}. \tag{2.3}$$

We define the following linear operators: for $x \in \overline{I}$

$$(Av)(x) = v(a),$$

$$(Bv)(x) = \int_a^x v(y)\, \mathrm{d}t,$$

$$(Cv)(x) = v'(x).$$

These are well defined for $v \in H^1(I)$. (In the sequel we write as usual H^1 and L^2, instead of $H^1(I)$, $L^2(I)$, respectively, and denote $\|\cdot\| = \|\cdot\|_{L^2}$, $\|\cdot\|_1 = \|\cdot\|_{H^1}$.)

Since $A : H^1 \to \mathbb{R}$, we may consider A as an operator $A : H^1 \to L^2$ by identifying $v(a)$ by the constant function $v(a) \cdot 1$. Since, by Sobolev's theorem, for $v \in H^1$

$$\|Av\| = |v(a)| \left(\int_a^b 1^2 \right)^{\frac{1}{2}} = \sqrt{b-a}\, |v(a)| \leq c\|v\|_1,$$

for some constant $c = c(I)$, we conclude that $A \in \mathcal{B}(H^1, L^2)$, where, if X, Y are Hilbert spaces, $\mathcal{B}(X, Y)$ denotes the space of bounded linear operators $T : X \to Y$. Similarly, since for $v \in H^1$, $\|Cv\| = \|v'\| \leq \|v\|_1$, we conclude that $C \in \mathcal{B}(H^1, L^2)$ as well.

Now since $\dim(\operatorname{Ran} A) = 1$, $A : H^1 \to L^2$ is a compact operator (see e.g. [2.5]). Also, since $(Bv)(x) = \int_a^x v(t)\, \mathrm{d}t = \int_a^b K(x,t)v(t)\, \mathrm{d}t$, where

$$K(x,t) = \begin{cases} 1, & a \leq t \leq x \leq b, \\ 0, & a \leq x < t \leq b, \end{cases}$$

and $\int_a^b \int_a^b K^2(x,t)\,\mathrm{d}x\,\mathrm{d}t = \frac{(b-a)^2}{2}$, we conclude by the Cauchy-Schwarz inequality that $B \in \mathcal{B}(L^2, L^2)$ and that B is a Hilbert-Schmidt (Volterra) operator; therefore $B : L^2 \to L^2$ is compact (cf. e.g. [2.5]).

From (2.3) and the definitions of the operators A, B, C we have that for $v \in H^1$, $v = (A + BC)v$. Hence, if $E : H^1 \to L^2$ is the imbedding operator of H^1 into L^2 defined for $v \in H^1$ by $Ev = v$, we have $E = A + BC$. Since $B : L^2 \to L^2$ is compact and $C : H^1 \to L^2$ is bounded, $BC : H^1 \to L^2$ is compact. Therefore E is compact as sum of the compact operators A and BC. $\qquad\square$

2.4 The Sobolev spaces $H^m(I)$, $m = 2, 3, 4, \ldots$

In analogy to $H^1(I)$ we define for $m \geq 2$ integer the space

$$
\begin{aligned}
H^m(I) \;=\; & \{u \in L^2(I) : \exists\, g_i \in L^2(I),\ i = 1, \ldots, m,\ \text{such that} \\
& \int_I u\phi^{(i)} = (-1)^i \int_I g_i\phi,\ 1 \leq i \leq m,\ \ \forall \phi \in C_c^\infty(I)\}.
\end{aligned}
$$

Here

$$
\phi^{(i)} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i \phi.
$$

It follows easily that $H^m(I) \subset H^1(I)$ and that $g_1 = u'$, that $u' \in H^1(I)$, and that $(u')' = g_2$ etc..., and that finally

$$
u^{(m-1)} = (((u\ \underbrace{')')'\ldots'}_{m-1\ \text{times}}) \in H^1(I) \text{ and that } u^{(m)} = (u^{(m-1)})' = g_m.
$$

We call g_i the (uniqueness easy) *weak (generalized) derivative of order i* (in the L^2 sense) of $u \in H^m(I)$ and define

$$
D^i u \equiv u^{(i)} \equiv g_i,\ \ 1 \leq i \leq m.
$$

It follows that for $m \geq 1$

$$
H^m(I) = \{u \in H^{m-1}(I) : u' \in H^{m-1}(I)\}.
$$

Here $H^0(I) \equiv L^2(I)$. We can easily construct examples of functions in $H^m(I)$. For example, on a bounded interval if $u \in C^1(\overline{I})$ with u'' (classical derivative) piecewise continuous on $\overline{I}$, then $u \in H^2(I)$ and its weak second derivative coincides a.e. with u''.

We equip $H^m(I)$ with the inner product $(\cdot, \cdot)_m$, where

$$(u, v)_m = \sum_{j=0}^{m} (D^j u, D^j v), \quad (u^{(0)} \equiv D^0 u = u), \quad \forall u, v \in H^m(I).$$

This inner product induces the norm

$$\|u\|_m = (u, u)_m^{\frac{1}{2}} = \left(\sum_{i=0}^{m} \|D^i u\|^2 \right)^{\frac{1}{2}}.$$

An obvious modification of Theorem 2.10 shows that $\{H^m(I), (\cdot, \cdot)_m\}$ is a Hilbert space. By definition and Theorem 2.12, it follows that if $u \in H^m(I)$, then there exists $\tilde{u} \in C^{m-1}(\overline{I})$ such that $u = \tilde{u}$ a.e. on I and

$$D^i \tilde{u}(x) - D^i \tilde{u}(y) = \int_y^x D^{i+1} u(t)\, dt, \quad \forall x, y \in \overline{I}, \quad i = 0, 1, 2, \ldots, m-1.$$

Also, given $u \in H^m(I)$, there exists a sequence $\{u_n\} \in C_c^\infty(\mathbb{R})$ such that $u_n|_I \to u$ in $H^m(I)$ (density) and that $H^m(I) \subset C^{m-1}(I)$ with

$$\sum_{j=0}^{m-1} \|D^j u\|_{L^\infty(I)} \leq C \|u\|_{H^m(I)}, \quad \forall u \in H^m(I).$$

If $u, v \in H^m(I)$, then $uv \in H^m(I)$ and

$$D^m(uv) = \sum_{j=0}^{m} \binom{m}{j} D^j u\, D^{m-j} v, \quad m \geq 1 \text{ (Leibniz's rule)}.$$

These results follow easily with techniques similar to the ones used in their H^1 counterparts.

We finally mention without proof the following *interpolation* result. If $1 \leq j \leq m-1$, then $\forall \epsilon > 0 \; \exists C_\epsilon = C(\epsilon, \mu(I) \leq \infty)$ such that

$$\|D^j u\| \leq \epsilon \|D^m u\| + C_\epsilon \|u\|, \quad \forall u \in H^m(I).$$

It follows that the quantity $\|u\| + \|D^m u\|$ for $u \in H^m(I)$ is a *norm* on $H^m(I)$, equivalent to $\|u\|_m$.

2.5   The space $\overset{o}{H}^1(I)$

Definition 2.24. We define $\overset{o}{H}^1(I)$ to be the closure of $C_c^1(I)$ in $H^1(I)$, i.e. the (closed) subspace of $H^1(I)$ whose elements are limits in $H^1(I)$ of sequences of functions in $C_c^1(I)$.

It follows that $\{\overset{0}{H}{}^1(I), (\cdot, \cdot)_1\}$ is a Hilbert space (separable).

Remarks 2.25.

(i) If $I = \mathbb{R}$, since $C_c^\infty(\mathbb{R}) \subset C_c^1(\mathbb{R}) \subset H^1(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$ (cf. Theorem 2.17), it follows that $C_c^1(\mathbb{R})$ is dense in $H^1(\mathbb{R}) \Rightarrow \overset{0}{H}{}^1(\mathbb{R}) = H^1(\mathbb{R})$. However if $I \neq \mathbb{R}$, then $\overset{0}{H}{}^1(I) \subset H^1(I)$. For example, on a bounded interval I consider $u(x) = c_1 e^x + c_2 e^{-x} \in C^\infty(\overline{I})$ for which $u'' = u$. Hence

$$0 = (u'' - u, \phi) = \int_I (u'' - u)\phi = -\int_I u'\phi' - \int_I u\phi = -(u, \phi)_1, \quad \forall \phi \in C_c^1(I).$$

Hence u is orthogonal in $H^1(I)$ to $C_c^1(I)$, which therefore cannot be dense in $H^1(I)$.

(ii) In fact $C_c^\infty(I)$ is dense in $\overset{0}{H}{}^1(I)$. To see this, let, for $u \in \overset{0}{H}{}^1(I)$, $\epsilon > 0$, $\phi \in C_c^1(I)$ be such that $\|u - \phi\|_1 \leq \frac{\epsilon}{2}$. Now extending $\phi \in C_c^1(I)$ by zero outside its support to the whole of $\mathbb{R}$, we have $\phi \in C_c^1(\mathbb{R})$ and $\rho_n * \phi \in C_c^\infty(I)$ for sufficiently large n (cf. Lemma 2.5 (ii)). Moreover, (cf. Remark 2.8 (iii)) $\|\phi - \rho_n * \phi\|_{H^1(I)} \to 0$ as $n \to \infty$. Therefore choose n so that $\|\phi - \rho_n * \phi\|_1 \leq \frac{\epsilon}{2}$, from which it follows that $\|u - \rho_n * \phi\|_1 \leq \epsilon$, i.e. that $C_c^\infty(I)$ is dense in $\overset{0}{H}{}^1(I)$.

(iii) Let us also remark that $u \in H^1(I) \cap C_c(I) \Rightarrow u \in \overset{0}{H}{}^1(I)$. In fact, if $u \in H^1(I) \cap C_c(I)$, extending u by zero outside I to the whole of $\mathbb{R}$, we have, by Lemma 2.5 (ii), that, for n sufficiently large, $\rho_n * u \in C_c^\infty(I)$ and (cf. Proof of Theorem 2.17), $\|u - \rho_n * u\|_1 \to 0$ as $n \to \infty$. Hence by (ii) above, $u \in \overset{0}{H}{}^1(I)$.

$\square$

These remarks prepare the ground for the following result which characterizes the functions in $\overset{0}{H}{}^1(I)$ in a very useful way.

Theorem 2.26. *Let $u \in H^1(I)$. Then $u \in \overset{0}{H}{}^1(I)$ if and only if $u = 0$ on ∂I (= the boundary of I).*

Comments: Again for $u \in H^1(I)$ the statement "$u = 0$ on ∂I" is well understood, in the sense of Theorem 2.12 – see the remarks there. Theorem 2.26 makes then $\overset{0}{H}{}^1(I)$ a very useful space, in which homogeneous (zero) boundary conditions are automatically satisfied for $u|_{\partial I}$, i.e. a space in which "weak" solutions of boundary–value problems

such as $(*)$ in Section 2.1 may be naturally sought. In the sequel assume $I \neq \mathbb{R}$, since $H^1(\mathbb{R}) = \overset{0}{H}{}^1(\mathbb{R})$.

Proof. If $u \in \overset{0}{H}{}^1(I)$, there exists a sequence $(u_n) \in C_c^1(I)$ such that $u_n \to u$ in $H^1(I)$. By Sobolev's Theorem 2.18 $u_n \to u$ in $L^\infty(I)$, i.e., if $\tilde{u}(x) \in C(\overline{I})$, $\tilde{u} = u$ a.e. on I $\Rightarrow \sup_{x \in I} |u_n(x) - \tilde{u}(x)| \to 0$, $n \to \infty$. Since $u_n|_{\partial I} = 0 \Rightarrow \tilde{u}|_{\partial I} = 0$, i.e. the values of u on ∂I are zero.

Suppose now that $u \in H^1(I)$ and $u|_{\partial I} = 0$. We shall show that $u \in \overset{0}{H}{}^1(I)$.

First, let us examine the case of a bounded interval and take, with no loss of generality, $I = (0, 1)$. Consider the intervals

$$K_1 = \left(-\frac{1}{3}, \frac{1}{3}\right), \quad I = (0, 1), \quad K_2 = \left(\frac{2}{3}, \frac{4}{3}\right).$$

Then, we may find functions $\phi_1 \in C_c^\infty(K_1)$, $\phi_0 \in C_c^\infty(I)$, $\phi_2 \in C_c^\infty(K_2)$ such that $0 \leq \phi_i \leq 1$ and (extending them by zero outside their intervals of definition) $\phi_1(x) + \phi_0(x) + \phi_2(x) = 1$, $\forall x \in I$. (The functions ϕ_i form a *partition of unity* corresponding to the open cover $\{K_1, I, K_2\}$ of I, and can be constructed e.g. as follows:

It is clear that given any interval (a, b) we can find $\psi \in C_c^\infty(a, b)$ such that (for $\delta > 0$
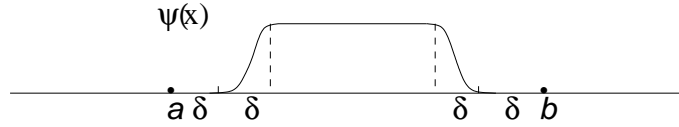
small enough) $\psi(x) = 1$ for $a + 2\delta \leq x \leq b - 2\delta$ and $\psi(x) = 0$ for $x \in (a, a + \delta]$ and $x \in [b - \delta, b)$. With the same δ (take any $0 < \delta < \frac{1}{12}$) construct such functions $\psi_1(x)$ for $(-\frac{1}{3}, \frac{1}{3})$, $\psi_0(x)$ for $(0, 1)$, $\psi_2(x)$ for $(\frac{2}{3}, \frac{4}{3})$. It is clear then that $\psi_0(x) + \psi_1(x) + \psi_2(x) \geq 1$, $x \in [-\frac{1}{3} + 2\delta, \frac{4}{3} - 2\delta]$.

Let $\omega(x)$ be such a function (with the same e.g. δ) for the interval $[-2\delta, 1 + 2\delta]$. Then

let

$$\phi_i(x) = \frac{\psi_i(x)}{\sum \psi_i(x)} \omega(x), \quad i = 0, 1, 2.$$

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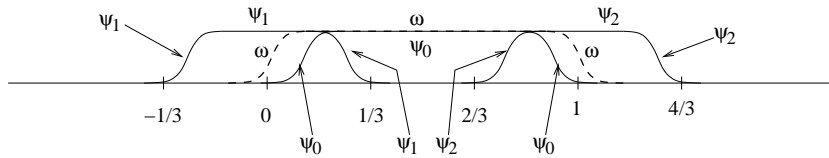
It is easily seen that the ϕ_i's satisfy the desired properties.

Let now $u_i(x) = u(x)\phi_i(x)$, $i = 0, 1, 2$, so that

$$u(x) = \sum_{i=0}^{2} u_i(x), \quad x \in I.$$

Consider $u_0(x) = u(x)\phi_0(x)$. Since $u \in H^1(I)$, $\phi_0 \in C_c^\infty(I) \Rightarrow u_0 \in H^1(I) \cap C_c(I)$ (identify u_0 with its continuous representative). By Remark 2.25 (iii), $u_0 \in \overset{0}{H}{}^1(I)$.

We consider now $u_1(x) = u(x)\phi_1(x)$. Extending ϕ_1 by zero to the whole of I, we see that $u \in H^1(I)$, $\phi_1 \in C_c^\infty(-\frac{1}{3}, 1) \Rightarrow u_1 \in H^1(I)$. By hypothesis we also have that
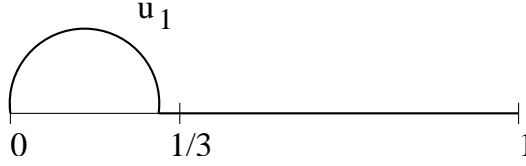
$u_1(0) = u(0)\phi_1(0) = 0 \cdot \phi_1(0) = 0$. Also $\mathrm{supp}\, u_1 \subset [0, \frac{1}{3})$. Extend now u_1 by zero to $\mathbb{R}$, i.e. consider the function

$$\tilde{u}_1(x) = \begin{cases} u_1(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I. \end{cases}$$

This function belongs to $H^1(\mathbb{R})$, since, for any $\phi \in C_c^\infty(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \tilde{u}_1 \phi' = \int_0^1 u_1 \phi' = \underbrace{u_1(1)}_{=0}\phi(1) - \underbrace{u_1(0)}_{=0}\phi(0) - \int_0^1 u_1' \phi$$

$$= -\int_{-\infty}^{\infty} \tilde{u'_1}\phi,$$

where $\tilde{u'_1}$, the extension by zero outside I of u_1', is in $L^2(\mathbb{R})$ since $u_1 \in H^1(I)$.

Now, for any function f on $\mathbb{R}$ let for $h > 0$ $\tau_h f$ denote the right h–translate of f, i.e. let $(\tau_h f)(x) \equiv f(x - h)$. Since $\tilde{u}_1 \in H^1(\mathbb{R})$ it follows that $\tau_h \tilde{u}_1 \in H^1(\mathbb{R})$ and

$$\lim_{h \to 0} \|\tau_h \tilde{u}_1 - \tilde{u}_1\|_{H^1(\mathbb{R})} = 0.$$

(To see this, given $\epsilon > 0$ let $\phi \in C_c^\infty(\mathbb{R})$ be such that $\|\tilde{u}_1 - \phi\|_{H^1(\mathbb{R})} < \frac{\epsilon}{3}$. It follows, for any $h > 0$, that

$$\|\tau_h \tilde{u}_1 - \tau_h \phi\|_{H^1(\mathbb{R})} = \|\tilde{u}_1 - \phi\|_{H^1(\mathbb{R})} < \frac{\epsilon}{3}.$$

But it is obvious, since $\phi, \tau_h \phi \in C_c^\infty(\mathbb{R})$, that there exists h_0 such that

$$0 \le h < h_0 \Rightarrow \|\tau_h \phi - \phi\|_{H^1(\mathbb{R})} \le \frac{\epsilon}{3}.$$

Hence, given $\epsilon > 0 \; \exists \, h_0$ such that

$$0 \leq h < h_0 \;\Rightarrow\; \|\tau_h \tilde{u}_1 - \tilde{u}_1\|_{H^1(\mathbb{R})} < \epsilon$$

by the triangle inequality). Now, the restriction $\tau_h \tilde{u}_1|_I$, for h sufficiently small, belongs to $H^1(I) \cap C_c(I)$ (possibly upon its modification on a set of measure zero). Hence, by Remark 2.25 (iii), $\tau_h \tilde{u}_1|_I \in \overset{0}{H}{}^1(I)$. Therefore, given $\epsilon > 0$ we have that $\exists \, \phi \in C_c^\infty(I)$ such that $\|\tau_h \tilde{u}_1 - \phi\|_{H^1(I)} < \frac{\epsilon}{2}$. Since $\lim_{h \to 0} \|\tau_h \tilde{u}_1 - \tilde{u}_1\|_{H^1(\mathbb{R})} = 0 \Rightarrow \exists \, h$ such that

$$\|\tau_h \tilde{u}_1 - \tilde{u}_1\|_{H^1(\mathbb{R})} = \|\tau_h \tilde{u}_1 - u_1\|_{H^1(I)} < \frac{\epsilon}{2}.$$

It follows that $\|u_1 - \phi\|_{H^1(I)} < \epsilon$, i.e. that $u_1 \in \overset{0}{H}{}^1(I)$.

Entirely analogous considerations show that $u_2 \in \overset{0}{H}{}^1(I)$. Since $u = u_0 + u_1 + u_2$ it follows that $u \in \overset{0}{H}{}^1(I)$. $\qquad\qquad\square$

For a semi–infinite interval the proof follows in the analogous manner. Let $I = (0, \infty)$, with no loss of generality. Construct $\phi_1 \in C_c^\infty(-\frac{1}{3}, \frac{1}{3})$, $\phi_0 \in C_c^\infty(0, \infty)$ with $\mathrm{supp}\,\phi_0 \subset [\alpha, \infty)$, $\alpha > 0$ sufficiently small, so that $0 \leq \phi_i \leq 1$ and so that (extending ϕ_1 by zero to $[\frac{1}{3}, \infty)$) $\phi_1(x) + \phi_0(x) = 1$ for $x \in [0, \infty)$. (This can be achieved as previously, by taking ψ_1 as before and extending $\psi_0(x)$ and $w(x)$ by setting them equal to 1 for $x \geq \frac{1}{2}$). Again with $u_i = u\phi_i$ we have as before that, since $u_0 = 0$, $u_1 \in \overset{0}{H}{}^1(0, \infty)$. Consider $u_0 = u\phi_0$. Extend it by zero to the whole of $\mathbb{R}$, i.e. put

$$\tilde{u}_0(x) = \begin{cases} u_0(x) & \text{if} \quad 0 \leq x < \infty \\ 0 & \text{if} \quad -\infty < x < 0. \end{cases}$$

It is clear that $\tilde{u}_0 \in H^1(\mathbb{R})$ (since $u_0 \in H^1(I)$). Since u_0 it has support in $[\alpha, \infty)$, $\alpha > 0$, it is not hard to see that for n sufficiently large, $\rho_n * \tilde{u}_0$ has support in $[\alpha', \infty)$, $\alpha' > 0$, belongs to $C^\infty(\mathbb{R}) \cap H^1(\mathbb{R})$ and, of course it holds that

$$\|\tilde{u}_0 - \rho_n * \tilde{u}_0\|_{H^1(\mathbb{R})} = \|u_0 - (\rho_n * \tilde{u}_0)|_I\|_{H^1(I)} \to 0 \text{ as } n \to \infty.$$

Consider now the functions

$$u_{0,n} = \zeta_n|_I (\rho_n * \tilde{u}_0)|_I,$$

where $\zeta_n(x)|_I$ is the restriction to $I = [0, \infty)$ of the smooth truncation function $\zeta_n(x)$ introduced in the proof of Theorem 2.17. It follows that, for n sufficiently large,

$u_{0,n} \in C_c^\infty(I)$. A similar calculation to the one used in the proof of Theorem 2.17 shows finally that

$$\|u_{0,n} - u_0\|_{H^1(I)} \to 0 \text{ as } n \to \infty.$$

Hence $u_0 \in \overset{0}{H}{}^1(I)$. $\qquad\square$

Remark 2.27. Essentially the proof above shows the following characterization of $\overset{0}{H}{}^1(I)$ which is of interest by itself. For $u \in L^2(I)$ let $\tilde{u}(x)$ be its extension by zero to the whole real line, i.e. let

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in I \\ 0 & \text{if } x \in \mathbb{R} - I. \end{cases}$$

Then, $u \in \overset{0}{H}{}^1(I)$ if and only if $\tilde{u} \in H^1(\mathbb{R})$. $\qquad\square$

We finally mention a result, the *inequality of Pointcaré–Friedrichs*, which will be very useful in the sequel.

Proposition 2.28 (Poincaré–Friedrichs)**.** *Suppose that I is a bounded interval. Then, there exists a constant C_* (depending on $\mu(I)$) such that*

$$\|u\|_1 \leq C_*\|u'\|, \quad \forall u \in \overset{0}{H}{}^1(I). \tag{2.4}$$

In other words, the quantity $\|u'\|$ is a norm on $\overset{0}{H}{}^1(I)$, equivalent to $\|u\|_1$.

Proof. If $u \in \overset{0}{H}{}^1(I) \equiv \overset{0}{H}{}^1(a,b)$ we have, by Theorems 2.12 and 2.26 that for $x \in I$

$$|u(x)| = |\int_a^x u'(t)\,dt| \leq \int_a^b |u'(t)|\,dt \leq (b-a)^{\frac{1}{2}}\|u'\|.$$

Hence

$$\|u\|^2 = \int_a^b u^2(t)\,dt \leq (b-a)^2\|u'\|^2,$$

and (2.3) holds with $C_* = (1 + (b-a)^2)^{1/2}$. (This is not the best constant.) $\qquad\square$

Remarks 2.29.

(i) It follows that on $\overset{0}{H}{}^1(I)$, I bounded, the expression (u', v') defines an inner product.

(ii) Entirely analogously one may define, for $m \geq 2$ integer, the spaces $\overset{0}{H}{}^m(I)$ as completions of $C_c^\infty(I)$ in $H^m(I)$. One may show that

$$\overset{0}{H}{}^m(I) = \{u \in H^m(I) : u = Du = \cdots = D^{m-1}u = 0 \text{ on } \partial I\},$$

where $D = \frac{\mathrm{d}}{\mathrm{d}x}$. We should keep in mind the distinction between e.g.

$$\overset{0}{H}{}^2(I) = \{u \in H^2(I) : u = Du = 0 \text{ on } \partial I\}$$

and

$$H^2(I) \cap \overset{0}{H}{}^1(I) = \{u \in H^2(I) : u = 0 \text{ on } \partial I\}.$$

(iii) Finding the best constant in the Poincaré-Friedrichs inequality is of some importance and allows us to obtain more precise bounds of quantities useful in practice. The best constant may be found as follows. On the interval (a, b) consider the Sturm-Liouville eigenvalue problem $-u'' = \lambda u$, $u(a) = u(b) = 0$, whose eigenvalues are given by $\lambda_n = \frac{n^2 \pi^2}{(b-a)^2}$, $n = 1, 2, 3, \ldots$, with associated eigenfunctions $u_n = A_n \sin \frac{n\pi(x-a)}{b-a}$, were $A_n \neq 0$ are constants. We will first show that the minimization problem

$$m = \min_{0 \neq w \in \overset{0}{H}{}^1} \frac{\|w'\|^2}{\|w\|^2}$$

is solved when $w = u_1$, with the minimum value given by $m = \lambda_1 = \frac{\pi^2}{(b-a)^2}$. (Note that the last inequality in the proof of Proposition 2.28 implies that $\frac{\|w'\|}{\|w\|^2}$ is bounded below when $w \in \overset{0}{H}{}^1$.) Let us minimize first the quotient $\frac{\|w'\|^2}{\|w\|^2}$ for $w \in H^2 \cap \overset{0}{H}{}^1$. Let $\hat{u} \in H^2 \cap \overset{0}{H}{}^1$ be a minimizer and $\hat{m} = \frac{\|\hat{u}'\|^2}{\|\hat{u}\|^2}$. Then, given any $v \in H^2 \cap \overset{0}{H}{}^1$ the function $f_v : \mathbb{R} \to \mathbb{R}$ given by $f_v(\varepsilon) = \frac{\|(\hat{u}+\varepsilon v)'\|^2}{\|\hat{u}+\varepsilon v\|^2}$ must satisfy $f_v'(0) = 0$. Since $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\|\hat{u}+\varepsilon v\|^2 = 2\varepsilon\|v\|^2 + 2(\hat{u}, v)$, $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\|(\hat{u}+\varepsilon v)'\|^2 = 2\varepsilon\|v'\|^2 + 2(\hat{u}', v')$, we see that $f_v'(0) = 0$ is equivalent to the equation $\|\hat{u}\|^2(\hat{u}', v') = \|\hat{u}'\|^2(\hat{u}, v)$. Therefore, $\hat{u} \in H^2 \cap \overset{0}{H}{}^1$ satisfies $(\hat{u}', v') = \hat{m}(\hat{u}, v)$, $\forall v \in H^2 \cap \overset{0}{H}{}^1$, which implies that $(-\hat{u}'' - \hat{m}\hat{u}, v) = 0$, $\forall v \in H^2 \cap \overset{0}{H}{}^1$. Since $H^2 \cap \overset{0}{H}{}^1$ is dense in L^2 (as it contains $C_c^\infty(a, b)$), we see that $\hat{u}$ is an eigenfunction of the eigenvalue problem $-u'' = \lambda u$, $a \leq x \leq b$, $u(a) = u(b) = 0$, and $\hat{m}$ is the associated eigenvalue. Therefore $\hat{u} = u_1$ and $\hat{m} = \lambda_1$, the smallest eigenvalue, since if (u_n, λ_n) is any eigenpair then $\hat{m} = \frac{\|\hat{u}'\|^2}{\|\hat{u}\|^2} \leq \frac{\|u_n'\|^2}{\|u_n\|^2} = \frac{(-u_n'', u_n)}{\|u_n\|^2} = \lambda_n$. Finally, it is straightforward

to see that $\hat{u}$ is actually the minimizer of $\frac{\|w'\|^2}{\|w\|^2}$ over $\overset{0}{H}{}^1$, so that $\hat{m} = m$. Indeed, given $w \in \overset{0}{H}{}^1$, there exists a sequence $w_n \in H^2 \cap \overset{0}{H}{}^1$, such that $w_n \to w$ in H^1, $n \to \infty$, since $H^2 \cap \overset{0}{H}{}^1$ (since contains $C_c^\infty(a,b)$) is dense in $\overset{0}{H}{}^1$. Therefore $w_n' \to w'$, $w_n \to w$ in L^2, so that $\frac{\|w_n'\|^2}{\|w_n\|^2} \to \frac{\|w'\|^2}{\|w\|^2}$. Since $\hat{m} \le \frac{\|w_n'\|^2}{\|w_n\|^2}$, it follows that $\hat{m} \le \frac{\|w'\|^2}{\|w\|^2}$, $\forall w \in \overset{0}{H}{}^1$, so that $\hat{m} = m$ and the minimum of $\frac{\|w'\|^2}{\|w\|^2}$ over $\overset{0}{H}{}^1$ occurs when $w = u_1$.

Since $\frac{\|u\|_1^2}{\|u'\|^2} = 1 + \frac{\|u\|^2}{\|u'\|^2}$, we see that the best constant C_* in (2.4) is given by

$$C_* = \left(1 + \max_{0 \ne u \in \overset{0}{H}{}^1} \frac{\|u\|^2}{\|u'\|^2}\right)^{1/2} = \left(1 + \lambda_1^{-1}\right)^{1/2} = \left(1 + \frac{(b-a)}{\pi^2}\right)^{1/2}. \qquad \square$$

2.6 Two–point boundary–value problems

We return now to the two–point boundary–value problem ($*$) of Section 2.1. We shall mainly discuss homogeneous (zero) Dirichlet and Neumann boundary conditions and follow the "variational method" outlined in Section 2.1.

2.6.1 Zero Dirichlet boundary conditions.

We let $I = (a,b)$ be a bounded interval. We consider the problem of finding $u(x)$, $x \in [a,b]$ such that

$$-(pu')' + qu = f \quad \text{in } (a,b), \tag{2.5}$$

$$u(a) = u(b) = 0, \tag{2.6}$$

where p, q, f are given functions on I such that $p \in C^1(\overline{I})$, $p(x) \ge \alpha > 0$, $\forall x \in \overline{I}$, $q \in C(\overline{I})$ such that $q(x) \ge 0$, $\forall\, x \in \overline{I}$, and where we shall assume $f \in C(\overline{I})$ (sometimes just $f \in L^2(I)$). Based on the discussion on in Section 2.1 we make the following

Definition 2.30. Let $f \in C(\overline{I})$. Then a *classical solution* of (2.5), (2.6), is a function $u(x) \in C^2(\overline{I})$ which satisfies the D.E. (2.5) in the usual sense for each $x \in I$ and which also satisfies the boundary conditions (b.c.) (2.6) in the usual sense. (We say that "u satisfies (2.5) and (2.6) classically or in the usual sense"). If $f \in L^2(I)$, then a *weak solution* of (2.5), (2.6), is a function $u \in \overset{0}{H}{}^1(I)$ which satisfies

$$\int_I pu'v' + \int_I quv = \int_I fv \qquad \forall\, v \in \overset{0}{H}{}^1(I). \tag{2.7}$$

Following the plan outlined in Section 2.1 we prove a series of results:

(i) A classical solution of (2.5)**,** (2.6) **is a weak solution of** (2.5)**,** (2.6) **as well.**

Let u satisfy (2.5), (2.6) classically. Since $u \in C^2(\overline{I})$ and $u(a) = u(b) = 0 \Rightarrow u \in H^2(I) \cap \overset{0}{H}{}^1(I)$. We multiply the D.E. (2.5) by any $v \in \overset{0}{H}{}^1(I)$ and integrate on I. By Proposition 2.20, since $pu' \in H^1$ we can integrate by parts and obtain

$$\int_I fv \;=\; -\int_I (pu')'v + \int_I quv = -pu'v|_a^b + \int_I pu'v' + \int_I quv$$

$$=\; \int_I pu'v' + \int_I quv,$$

since $v \in \overset{0}{H}{}^1 = \overset{0}{H}{}^1(I)$. (We will usually suppress I from the symbols of the function spaces). Hence u is a weak solution.

(ii) Existence and uniqueness of the weak solution.

Let $f \in L^2(I)$ and let $B(v,w) = \int_I pv'w' + \int_I qvw$. Clearly, by our hypotheses $B(\cdot,\cdot)$ is a bilinear, symmetric form on $\overset{0}{H}{}^1 \times \overset{0}{H}{}^1$ (i.e. on $H^1 \times H^1$). Moreover, for $v, w \in H^1$ we have

$$
\begin{aligned}
|B(v,w)| &\le \int_I |p||w'||v'| + \int_I |q||w||v| \\
&\le \max_{x\in\overline{I}} |p(x)| \, \|w'\| \, \|v'\| + \max_{x\in\overline{I}} |q(x)| \, \|w\| \, \|v\| \\
&\le c_1 \|v\|_1 \, \|w\|_1,
\end{aligned}
\tag{2.8}
$$

where $c_1 = \max_{x\in\overline{I}} |p(x)| + \max_{x\in\overline{I}} |q(x)|$. Hence B is continuous on $H^1 \times H^1$.

Now, for $v \in \overset{0}{H}{}^1$ we have, by our hypotheses on p and q that

$$B(v,v) = \int_I p(v')^2 + \int_I qv^2 \ge \alpha \int_I (v')^2 \ge c_2 \|v\|_1^2,
\tag{2.9}$$

where $c_2 = \alpha/C_*^2$, with C_* being the constant in the Poincaré–Friedrichs inequality (2.4), which was used (since $v \in \overset{0}{H}{}^1$) in the last step.

Hence the bilinear form B satisfies, on the Hilbert space $\{\overset{0}{H}{}^1, \|\cdot\|_1\}$, the hypotheses (i) and (ii) of the Lax–Milgram Theorem 1.4. In addition, it is straightforward to see that in the r.h.s. of (2.7), $F(v) = \int_I fv$ is a continuous, linear functional on $\overset{0}{H}{}^1$ since $|F(v)| \le \|f\|\|v\| \le \|f\|\|v\|_1 \; \forall v \in \overset{0}{H}{}^1$. (Note that $|||\,F\,||| \le \|f\|$ where by $|||\,F\,|||$ we denote the norm of the b.l.f. F on $\overset{0}{H}{}^1$, i.e. the $\sup_{0\ne v\in\overset{0}{H}{}^1} \frac{|F(v)|}{\|v\|_1}$). Hence the Lax–Milgram theorem applies and shows that the problem

$$\int_I pu'v' + quv = \int_I fv, \quad \text{i.e.} \;\; B(u,v) = F(v), \;\; \forall v \in \overset{0}{H}{}^1,$$

has a unique solution $u \in \overset{0}{H}{}^1$. Moreover

$$\|u\|_1 \leq \frac{1}{c_2} \|f\|. \tag{2.10}$$

Since a classical solution is a weak solution and we have shown now uniqueness of the weak solution, it follows that there is at most one classical solution.

(iii) Regularity of the weak solution.

Let $f \in L^2(I)$. Then (2.7) gives that if u is the weak solution, then

$$\int_I pu'v' = \int_I (f - qu)v, \quad \forall v \in \overset{0}{H}{}^1.$$

Hence

$$\int_I pu'\phi' = \int_I (f - qu)\phi, \quad \forall \phi \in C_c^\infty(I),$$

which shows that $pu' \in H^1$, since $pu' \in L^2$ and since there exists $g \ (= qu - f)$, $g \in L^2$, such that $\int_I pu'\phi' = -\int_I g\phi$, $\forall \phi \in C_c^\infty(I)$. It follows that $pu' \in H^1$ and $(pu')' = qu - f$. i.e. $-(pu')' + qu = f$ holds in L^2, (which means that it holds pointwise a.e. on I).

Now, since $p(x) \geq \alpha > 0$, $x \in \overline{I}$, $p \in C^1(\overline{I})$, it follows that $\frac{1}{p(x)} \in C^1(\overline{I})$. Hence $u' = \frac{1}{p}(pu') \in H^1$, since $pu' \in H^1$ and $p^{-1} \in H^1$. It follows that the weak solution u (shown to be in $\overset{0}{H}{}^1$ by the Lax–Milgram theorem) actually belongs to $H^2 \cap \overset{0}{H}{}^1$. Moreover, since $-(pu')' + qu = f$ in L^2, we have now $pu'' = -p'u' + qu - f$ in L^2. Hence

$$\alpha\|u''\| \leq \max_{x \in \overline{I}} |p'(x)|\|u'\| + \max_{x \in \overline{I}} |q(x)|\|u\| + \|f\|.$$

This estimate coupled with (2.10), shows that there exists a constant $c_3 = c_3(p, q, I)$, such that for the weak solution $u \in H^2 \cap \overset{0}{H}{}^1$ of (2.5), (2.6) we also have

$$\|u\|_2 \leq c_3\|f\|. \tag{2.11}$$

(Estimates such as (2.11) are called "elliptic regularity" estimates $(H^2 - L^2)$ for the problem (2.5)–(2.6).)

Now, let $f \in C(\overline{I})$. We shall show now that the weak solution u is in $C^2(\overline{I})$ ($u \in \overset{0}{H}{}^1(I)$ guarantees that $u(a) = u(b) = 0$ if we identify u with its continuous representative $\tilde{u}$). Since $u \in H^2(I) \Rightarrow u' \in C(\overline{I})$ (take the continuous representative of u'). Hence $u \in C^1(\overline{I})$. But $pu'' = -p'u' + qu - f$ (in L^2). Since $f \in C(\overline{I})$, the l.h.s. is continuous $\Rightarrow u'' \in C(\overline{I})$ by our hypotheses on $p(x)$. Hence $u \in C^2(\overline{I})$.

(iv) If $f \in C(\bar{I})$, the weak solution is a classical solution.

Since $f \in C(\bar{I})$ the above shows that $u \in C^2(\bar{I})$. Now (2.7) gives

$$\int_I (pu')\phi' + \int_I qu\phi = \int_I f\phi, \quad \forall \phi \in C_c^\infty(I).$$

Integrating by parts, since $u \in C^2(\bar{I})$, we have

$$\int_I (-(pu')' + qu - f)\phi = 0, \quad \forall \phi \in C_c^\infty(I).$$

Since $C_c^\infty(I)$ is dense in $L^2(I) \Rightarrow -(pu')' + qu - f = 0$ a.e. on I. (This also follows from the fact that $-(pu')' + qu - f = 0$ in L^2). But $-(pu')' + qu - f \in C(\bar{I})$. Therefore $-(pu')' + qu = f$, $\forall x \in I$. Since $u(a) = u(b) = 0$ as well, it follows that for $f \in C(\bar{I})$ the weak solution is also classical.

Remarks 2.31.

(i) Since $B(u, v)$ is *symmetric*, the weak solution of (2.5), (2.6), i.e. the solution of the variational problem

$$B(u, v) = (f, v,), \quad \forall v \in \overset{0}{H}{}^1,$$

may be characterized as the (unique) solution of the minimization problem

$$J(u) = \min_{v \in \overset{0}{H}{}^1} J(v),$$

where J is the *energy functional*

$$J(v) = \frac{1}{2} B(v, v) - F(v) = \frac{1}{2} \int_I p(v')^2 + qv^2 - \int_I fv.$$

This follows from the Rayleigh–Ritz Theorem 1.7 and is known in the present context as *Dirichlet's principle*.

(ii) It is evident that the weak solution of (2.5), (2.6), i.e. the solution of (2.7), exists under much weaker conditions on p and q than those assumed. For example, the two integrals appearing in the definition of $B(\cdot, \cdot)$ exist if e.g. $p, q \in L^\infty(I)$. Similarly (2.8) and (2.9) hold only under the additional assumptions $p \geq \alpha > 0$, $q \geq 0$ a.e. on I. Moreover the right–hand side $F(v)$ makes sense (interpreting (f, v) properly) as a bounded linear functional on $\overset{0}{H}{}^1$ with much more general

f than $f \in L^2$. More precisely, f may belong to the *dual of* $\overset{0}{H}{}^1$, the so called "negative" Sobolev space $H^{-1}(I)$; e.g. the delta function is an H^{-1} "function" in 1 dimension. It is evident that such weak conditions guarantee only existence–uniqueness of u in $\overset{0}{H}{}^1$. To obtain more smoothness for u, i.e. to show that $u \in H^2$ or that $u \in C^2(\overline{I})$, it is clear that one has to assume more smoothness for the coefficients p, q and f. In the same vein of thought one may allow $q(x)$ to take on negative values, for example, as long as $q(x) \geq \beta$, $\forall x \in \overline{I}$, where β is such that in (2.9), $B(v,v) \geq c_2 \|v\|_1^2$ for some $c_2 > 0$. (A lower bound on β may be easily found in terms of $0 < \alpha = \min_{x \in I} p(x)$ and the constant μ of Poincaré's inequality $\int_I v^2 \leq \mu \int_I (v')^2$. By Remark 2.29 (iii) we see that the best such constant μ is equal to $\frac{1}{\lambda_1}$, where λ_1 is the smallest eigenvalue of the problem $-u'' = \lambda u$ with zero Dirichlet b.c. at a and b, i.e. $\lambda_1 = \frac{\pi^2}{(b-a)^2}$).

(iii) Using (2.10) and Sobolev's inequality we conclude that under our hypotheses the weak solution u belongs to $L^\infty(I)$ and that $\|u\|_{L^\infty} \leq c\|f\|$. Using the elliptic regularity estimate (2.11) and Sobolev's inequality we can conclude that $u' \in L^\infty$ and $\|u'\|_{L^\infty} \leq c\|f\|$. Finally, using the equation (which holds a.e.) we see that $\|u''\|_{L^\infty} \leq c\|f\|$, i.e. that the weak solution $u \in H^2 \cap \overset{0}{H}{}^1$ actually belongs to a space of functions with bounded generalized derivatives. For $u \in C^2(\overline{I})$, the classical solution, we then have the maximum norm estimate

$$\max_{x \in \overline{I}} \left(|u| + |u'| + |u''| \right) \leq c \max_{x \in \overline{I}} |f|.$$

(iv) The case of the b.v.p. with non homogeneous Dirichlet boundary conditions $u(a) = a_1$, $u(b) = a_2$ easily reverts to the problem (2.5), (2.6). Let $\psi(x)$ be a linear function (or any other function in $C^2(\overline{I})$) such that $\psi(a) = a_1$, $\psi(b) = a_2$. Then if u is the solution of the nonhomogeneous b.v.p., the function $v = u - \psi$ satisfies the homogeneous b.c. $v(a) = v(b) = 0$ and the D.E.

$$-(pv')' + qv = g := f - L\psi = f + (p\psi')' - q\psi,$$

i.e. a D.E. of the same form with new r.h.s. $g = f - L\psi \in C(\overline{I})$. $\qquad \square$

2.6.2   Neumann boundary conditions.

We now consider the problem

$$-(pu')' + qu = f \quad \text{in } (a,b), \tag{2.12}$$

$$u'(a) = u'(b) = 0, \tag{2.13}$$

i.e. the (homogeneous) *Neumann b.c.* problem. Again we let $p \in C^1(\bar{I})$, $p(x) \geq \alpha > 0$ $\forall x \in \bar{I}$, $f \in C(\bar{I})$ (or $f \in L^2(I)$), but now we assume that $q \in C(\bar{I})$ with $q(x) \geq \beta > 0$ $\forall x \in \bar{I}$. (Note possible nonuniqueness if $q = 0$: e.g. consider $p = 1$, $f = 0$, $q = 0$, i.e. $-u'' = 0$, $u'(a) = 0$, $u'(b) = 0$, which has any constant as its solution). It is not hard to motivate the following

Definition 2.32. Let $f \in C(\bar{I})$. Then a *classical solution* of the Neumann b.v.p. (2.12), (2.13), is a function $u(x) \in C^2(\bar{I})$ which satisfies (2.12), (2.13) in the usual sense. If $f \in L^2(I)$, then a *weak solution* of (2.12), (2.13), is a function $u \in H^1(I)$ such that

$$\int_I pu'v' + \int_I quv = \int_I fv, \quad \forall v \in H^1(I). \tag{2.14}$$

Following the steps of the proof in the Dirichlet b.c. case we have:

(i) A classical solution of (2.12), (2.13) **is a weak solution as well.**

Proof obvious as before (now multiply by $v \in H^1$ and use the "natural" b.c. $u'(a) = u'(b) = 0$ on u; (2.14) follows).

(ii) Existence and uniqueness of the weak solution.

Let $f \in L^2(I)$ and $B(v,w) = \int_I pv'w' + \int_I qvw$. By our hypotheses $B(\cdot,\cdot)$ is a bilinear, symmetric form on $H^1 \times H^1$. The inequality (2.8) holds with the same constant c_1. For $v \in H^1$ we have

$$B(v,v) = \int_I p(v')^2 + \int_I qv^2 \geq \alpha \int_I (v')^2 + \beta \int_I v^2 \geq c_2\|v\|_1^2, \tag{2.15}$$

where $c_2 = \min\{\alpha, \beta\} > 0$. Since $F(v) = \int_I fv$ is a b.l.f. on H^1 there follows, from the Lax–Milgram theorem, that there exists a unique weak solution $u \in H^1$, i.e. a solution of (2.14), and that $\|u\|_1 \leq \frac{1}{c_2}\|f\|$. (Note that since $u \in H^1$ we cannot give meaning to the point values $u'(a)$, $u'(b)$ unless we prove that the weak solution is in H^2, something

that we do in the next step).

(iii) Regularity of the weak solution.

For $f \in L^2(I)$ the weak solution u satisfies

$$\int_I pu'v' = \int_I (f - qu)v, \quad \forall v \in H^1. \tag{2.16}$$

Hence, a fortiori,

$$\int_I pu'\phi' = \int_I (f - qu)\phi, \quad \forall \phi \in C_c^\infty(I).$$

It follows, as in the Dirichlet b.c. case, that $pu' \in H^1$ and that $-(pu')' + qu = f$ in L^2. Since $u' = \frac{1}{p}(pu')$, it follows, as before, that $u \in H^2$. Returning now to (2.16), since $pu' \in H^1$ and $v \in H^1$, integration by parts gives (note that we can assign now meaning to the values $u'(a)$, $u'(b) \equiv$ the values of the continuous representative of $u' \in H^1$ at a, b) that

$$\int_I (-(pu')'v + quv - fv)\,dx + p(b)u'(b)v(b) - p(a)u'(a)v(a) = 0$$

holds, for each $v \in H^1$. Since we already saw that $-(pu')' + qu = f$ in L^2, it follows that $p(b)u'(b)v(b) - p(a)u'(a)v(a) = 0$, $\forall v \in H^1 \Rightarrow u'(b) = 0$, $u'(a) = 0$. (Choose e.g. $v(x) = x - a$ or $v(x) = x - b$). Hence the weak solution of (2.12), (2.13), which exists uniquely by the Lax–Milgram in H^1, actually belongs to H^2, satisfies $-(pu')' + qu = f$ in L^2 and $u'(a) = u'(b) = 0$. Moreover, exactly as before, we obtain $\|u\|_2 \leq c\|f\|$ ("elliptic regularity" estimate).

Assume now $f \in C(\overline{I})$. Then exactly as in the Dirichlet b.c. case, it follows that $u \in C^2(\overline{I})$ for the weak solution.

(iv) If $f \in C(\overline{I})$, the weak solution is classical.

The proof that $-(pu')' + qu = f$ a.e. $\Rightarrow -(pu')' + qu = f$ everywhere in I, i.e. that the weak solution (which is C^2 if $f \in C(\overline{I})$) is classical, is identical to the one in the Dirichlet b.c. case. Of course $u'(a) = u'(b) = 0$ holds already for the weak solution (in H^2).

Remarks 2.33.

(i) The weak solution is characterized now (by the Rayleigh–Ritz theorem) as the solution of the minimization problem

$$J(u) = \min_{v \in H^1} J(v), \quad J(v) = \frac{1}{2}\int_I p(v')^2 + qv^2 - \int_I fv, \quad v \in H^1.$$

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This follows from the symmetry of B.

(ii) Analogous remarks to the Dirichlet b.c. remarks 2.31 (ii)–(iv) follow, mutatis mutandis.

(iii) We can consider of course other homogeneous b.v. problems as well for the equation $-(pu')' + qu = f$ on (a, b). For example the problem with b.c. (assume Dirichlet b.c. assumptions on p, q)

$$u(a) = 0, \ u'(b) = 0, \qquad (\text{"mixed" b.c.}),$$

has the following weak formulation: Let

$$\overset{0}{H}_a \equiv \{v \in H^1(I) : \ v(a) = 0\}.$$

Clearly $\{\overset{0}{H}_a, \|\cdot\|_1\}$ is a Hilbert space (a closed subspace of $H^1(I)$ that includes $\overset{0}{H}{}^1$). We then seek $u \in \overset{0}{H}_a$ such that

$$B(u, v) \equiv \int_I pu'v' + \int_I quv = \int_I fv, \quad \forall v \in \overset{0}{H}_a .$$

The Lax–Milgram theorem shows existence–uniqueness of the above weak solution since $B(\cdot, \cdot)$ is bilinear, bounded and coercive on $\overset{0}{H}_a \times \overset{0}{H}_a$. The rest $(f \in C(\overline{I}) \Rightarrow u$ classical etc.) follows easily.

The problem with an "elastic" b.c. at $x = a$, i.e. with

$$u'(a) - k\, u(a) = 0 \ (k \text{ const.}), \ u(b) = 0,$$

has the following weak formulation on $\overset{0}{H}_b \equiv \{v \in H^1(I) : \ v(b) = 0\}$:
Seek $v \in \overset{0}{H}_b$ such that

$$B_k(u, v) \equiv \int_I pu'v' + \int_I quv + p(a)k\, u(a)v(a) = \int_I fv, \quad \forall v \in \overset{0}{H}_b .$$

It is straightforward to see that $B_k(\cdot, \cdot)$ is bilinear, symmetric, continuous on $\overset{0}{H}_b \times \overset{0}{H}_b$ and coercive, provided $k \geq 0$ (or $k < 0$ with $|k|$ sufficiently small). Hence a weak solution $u \in \overset{0}{H}_b$ exists uniquely and the existence-uniqueness of its classical analog follows.

The *periodic boundary condition* problem, i.e. the problem with b.c.'s

$$u(a) = u(b), \ u'(a) = u'(b),$$

may be similarly treated with the following weak formulation: Seek $u \in H^1_\pi$ (assume $p(a) = p(b)$) such that

$$\int_I (pu'v' + quv) = \int_I fv, \quad \forall v \in H^1_\pi,$$

where $H^1_\pi \equiv \{v \in H^1(I) : v(a) = v(b)\}$, the so called "periodic" H^1. $\quad\square$

2.6.3 Sturm-Liouville eigenvalue problems

In this section we consider eigenvalue problems for the second-order differential operator $Lu = -(pu)' + qu$ on a finite interval (we take $I = (0,1)$). Such eigenvalue problems are known as *Sturm-Liouville problems*. For simplicity we consider only the case of zero Dirichlet boundary conditions. We presuppose some familiarity with operator theory in Hilbert space.

Theorem 2.34. *Let $p \in C^1(\overline{I})$ with $p(x) \geq \alpha > 0$ on $\overline{I}$ for some positive constant α and suppose that $q \geq 0$ on $\overline{I}$. Then, there exists a sequence $(\lambda_n)_{n \geq 1}$ in $\mathbb{R}$ of eigenvalues of L and an orthonormal basis $\{\phi_n\}_{n \geq 1}$ on $L^2(I)$ of eigenvectors, such that $\phi_n \in C^2(\overline{I})$ and*

$$\begin{aligned} L\phi_n &= \lambda_n \phi_n, \quad x \in \overline{I}, \\ \phi_n(0) &= \phi_n(1) = 0. \end{aligned} \tag{2.17}$$

Moreover $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, with $\lambda_n \to \infty$, $n \to \infty$.

Proof. (i) We begin the proof by making some remarks on the operator L and its inverse T.

The domain of definition of L may be taken e.g. as $C^2(\overline{I})$ with zero boundary conditions when classical solutions of $Lu = f$ are considered, but also, as we saw in Section 2.6.1, the space $H^2 \cap \overset{0}{H}{}^1$ with the derivatives understood in the weak sense. We proved in Section 2.6.1 that given $f \in L^2$, there exists a unique $u \in H^2 \cap \overset{0}{H}{}^1$ (the weak solution of the two-point bvp), that satisfies

$$\begin{aligned} Lu &= f \text{ in } I \text{ (in the } L^2 \text{ sense)}, \\ u(0) &= u(1) = 0. \end{aligned} \tag{2.18}$$

Consider the map $f \mapsto u$, that we write as $u = Tf$, where T, the "solution operator" of the problem (2.18), is a linear operator $T : L^2 \to H^2 \cap \overset{0}{H}{}^1$. Note that if $B(v,w) =$

$\int_I pv'w' + qvw$, $v, w \in \overset{0}{H}{}^1$ is the bilinear form defined in Section 2.6.1, then $B(Tf, v) = (f, v)$, $\forall v \in \overset{0}{H}{}^1$, and T satisfies $\|Tf\|_1 \leq \frac{1}{c_2}\|f\|$, cf. (2.10), and the "elliptic regularity" estimate $\|Tf\|_2 \leq C\|f\|$, cf. (2.11). Hence T, considered as a linear operator $T : L^2 \to H^2 \cap \overset{0}{H}{}^1$ is bounded, and of course $T \in \mathcal{B}(L^2, L^2)$ since $\|Tf\| \leq \|Tf\|_1$. Since $TLu = u$ for $u \in H^2 \cap \overset{0}{H}{}^1$, and $LTf = f$ for $f \in L^2$, T is the inverse of L. (T is an integral operator with the Green's function for the problem (2.18) as its kernel, and the properties of T may also be studied in this way.)

(ii) We now prove that $T \in \mathcal{B}(L^2, L^2)$ is a selfadjoint, compact operator with $\operatorname{Ker} T = \{0\}$. Let $f, g \in L^2$ and $u, v \in H^2 \cap \overset{0}{H}{}^1$ be weak solutions of $Lu = f$, $Lv = g$, respectively. Then $u = Tf$, $v = Tg$ and $B(u, w) = (f, w)$, $B(v, z) = (f, z)$, $\forall w, z \in \overset{0}{H}{}^1$. Take $w = v$, $z = u$. Then by symmetry of $B(\cdot, \cdot)$, it follows that $(f, v) = (g, u)$, i.e. $(f, Tg) = (Tf, g)$ proving that T is selfadjoint.

To prove that T is compact, recall from Theorem 2.23 that the imbedding of H^1 into L^2 is compact, i.e. that if the sequence $(v_n) \in H^1$ is bounded in H^1, then there exists a subsequence (v'_n) of (v_n) that converges in L^2. Let now (f_n) be a bounded sequence in L^2, $\|f_n\| \leq M$, $\forall n$. Then, since $T \in \mathcal{B}(L^2, \overset{0}{H}{}^1)$ it follows that $\|Tf_n\|_1 \leq \frac{1}{c_2}\|f_n\| \leq \frac{M}{c_2}$. Hence (Tf_n) is a bounded sequence in H^1 and by Theorem 2.23 there exists a subsequence of (Tf_n), that we call (Tf'_n), which converges in L^2. Hence T, viewed as an operator on L^2, is compact. To prove the last assertion note that if $u = Tf$ for $f \in L^2$, then $Tf = 0 \Rightarrow u = 0 \Rightarrow f = Lu = 0$, i.e. $\operatorname{Ker} T = \{0\}$.

(iii) Proof of Theorem 2.34: By the spectral theorem for the selfadjoint, compact operator T on L^2, it is well known that since $\operatorname{Ker} T = \{0\}$, there exists an orthonormal basis $\{\phi_n\}_{n \geq 1}$ of L^2 and a corresponding sequence of real numbers $\mu_n \neq 0$, such that $|\mu_1| \geq |\mu_2| \geq \ldots$, with $\mu_n \to 0$, $n \to \infty$, so that $T\phi_n = \mu_n \phi_n$. In our case $\mu_n > 0$. To see this, note that $\phi_n \in \operatorname{Ran} T$ implies that $\phi_n \in H^2 \cap \overset{0}{H}{}^1$. Hence $c_2\|\phi_n\|_1^2 \leq B(\phi_n, \phi_n) = \frac{1}{\mu_n} B(T\phi_n, \phi_n) = \frac{1}{\mu_n}(\phi_n, \phi_n)$. Therefore $\mu_n > 0$, and we have the useful identity that $\frac{1}{\mu_n}$ is equal to the *Rayleigh quotient* $\frac{B(\phi_n, \phi_n)}{\|\phi_n\|^2}$. Putting now $u_n = T\phi_n = \mu_n \phi_n$ we see that $u_n \in H^2 \cap \overset{0}{H}{}^1$ and $Lu_n = LT\phi_n = \phi_n$ in the L^2 sense. Therefore $L\phi_n = \frac{1}{\mu_n}\phi_n$, $n \geq 1$, and if $\lambda_n = \frac{1}{\mu_n}$ it follows that $L\phi_n = \lambda_n \phi_n$, $n \geq 1$, and that the λ_n have the desired properties. Since ϕ_n is the weak solution of the bvp $L\phi_n = \lambda_n \phi_n$, $\phi_n(0) = \phi_n(1) = 0$, and $\phi_n \in \overset{0}{H}{}^1 \Rightarrow \phi_n \in C(\overline{I})$, it follows that ϕ_n is a classical solution, i.e. that $\phi_n \in C^2(\overline{I})$.

(If p, q have higher regularity, e.g. if $p, q \in C^\infty(\overline{I})$, then $\phi \in C^\infty(\overline{I})$.) $\qquad\square$

Chapter 3

Galerkin Finite Element Methods for Two–Point Boundary–Value Problems

3.1   Introduction

We consider again the two–point boundary–value problem on a bounded interval (a, b) with homogeneous Dirichlet b.c.'s:

$$-(pu')' + qu \;=\; f \quad \text{in } (a,b) \equiv I, \tag{3.1}$$

$$u(a) = u(b) \;=\; 0, \tag{3.2}$$

for which we assume, as in §2.6, that $p \in C^1(\overline{I})$, $p(x) \geq \alpha > 0$ in $\overline{I}$, $q \in C(\overline{I})$, $q(x) \geq 0$ on $\overline{I}$. We recall that if $f \in L^2(I)$, there exists a unique weak solution of (3.1), (3.2) satisfying

$$u \in \overset{0}{H}{}^1(I), \quad B(u,v) = (f, v) \quad \forall v \in \overset{0}{H}{}^1(I) \;, \tag{3.3}$$

where

$$B(v,w) = \int_a^b (pv'w' + qvw), \; v, w \in H^1(I), \; (v,w) = \int_a^b vw.$$

In fact, $u \in H^2 \cap \overset{0}{H}{}^1$ (suppressing I in the notation $\overset{0}{H}{}^1(I)$ etc.) and we have the elliptic regularity estimate

$$\|u\|_2 \leq C \; \| f \|, \tag{3.4}$$

where C is a nonnegative constant independent of u and f. If $f \in C(\overline{I})$, then the (unique) weak solution is in fact in $C^2(\overline{I})$ and solves (3.1), (3.2) in the classical sense.

The *Galerkin method* for the approximation of the weak solution, i.e. of the solution of (3.3), follows the abstract framework of §1.9. We note that B satisfies (see §2.6)

$$|B(v, w)| \leq C_1 \|v\|_1 \| w \|_1 \quad \forall v, w \in H^1(I), \tag{3.5}$$

$$B(v, v) \geq C_2 \|v\|_1^2 \quad \forall v \in \overset{0}{H}{}^1, \tag{3.6}$$

where in fact e.g. $C_1 = \| p \|_\infty + \| q \|_\infty$, $C_2 = \alpha/C_*^2$, where C_* is the constant in the Poincaré–Friedrichs inequality. Hence, if S_h, $0 < h \leq 1$, is a family of finite–dimensional subspaces of $\overset{0}{H}{}^1$, Galerkin's Theorem 1.5 gives that for each h, there exists a unique element $u_h \in S_h$, *the Galerkin approximation of u in S_h* satisfying

$$B(u_h, v_h) = (f, v_h), \quad \forall v_h \in S_h. \tag{3.7}$$

Moreover, u_h satisfies the H^1–*error estimate*

$$\| u - u_h \|_1 \leq \frac{C_1}{C_2} \inf_{\phi \in S_h} \| u - \phi \|_1 . \tag{3.8}$$

The *discrete problem* (3.7) (discrete version of (3.3)) is equivalent to a linear system of equations. Let $N = N_h = \dim S_h$ and $\{\phi_i\}_{i=1}^N$ be a *basis* of S_h. Then, as we saw in Ch. 1, the coefficients $\{c_i\}$ of u_h with respect to the basis ϕ_i, i.e. the numbers c_i:

$$u_h(x) = \sum_{i=1}^N c_i \phi_i(x), \tag{3.9}$$

satisfy the linear system

$$A c = f_h, \tag{3.10}$$

where A is the $N \times N$ matrix with elements $A_{ij} = B(\phi_j, \phi_i) = \int_a^b (p \phi_j' \phi_i' + q \phi_j \phi_i)$, $1 \leq i, j \leq N$, $c = [c_1, \ldots, c_N]^T$, $f_h = [(f, \phi_1), \ldots, (f, \phi_N)]^T$. The matrix A is symmetric and positive definite on $\mathbb{R}^N$ due to our assumptions on B (i.e. on p and q).

The question is, of course, how to choose the finite–dimensional subspace S_h of $\overset{0}{H}{}^1(I)$. The choice should be such that

- $\inf_{\phi \in S_h} \| u - \phi \|_1$ is *small* (cf. (3.8)), i.e. that we can *approximate well* elements u of $H^2 \cap \overset{0}{H}{}^1$ by elements of S_h.

- The system (3.10) is *easy to solve*.

A classical choice (Ritz, Galerkin, "spectral methods") is to let $S_h \equiv S_N$ be the span of the first N orthonormal eigenfunctions ϕ_j of the operator $Lu = -(pu')' + qu$ with zero b.c.'s at $x = a$ and $x = b$. If λ_n, ϕ_n are the eigenvalues, resp. eigenfunctions, of this problem (cf. §2.6.3), then the system (3.10) may be solved explicitly by

$$c_i = \frac{(f, \phi_i)}{\lambda_i}, \;\; 1 \leq i \leq N \;,$$

and the Galerkin approximation $u_h \equiv u_N = \sum_{i=1}^{N} c_i \phi_i(x)$ has good approximation properties; in particular

$$\inf_{\phi \in S_N} \| u - \phi \|_1 \to 0, \text{ as } N \to \infty \text{ for } u \in H^2 \cap \overset{0}{H}{}^1.$$

The difficulty in this approach is of course that it requires the explicit knowledge of the eigenpairs (λ_i, ϕ_i), $1 \leq i \leq N$, which are, in general, not easy to find analytically.

Another obvious choice that will also guarantee good approximation properties is to choose S_h as the vector space of the polynomials of a fixed degree that vanish at the endpoints. The problem with this approach is that, in general, the *condition* of the linear system (3.10) will be bad. Moreover, the matrix A will be, in general, full, since the polynomial basis functions ϕ_j will not be of small support in $\overline{I}$. Since we expect N to be large for approximability, we conclude that, in general, the system (3.10) will be very hard to solve accurately. Moreover, since $N \sim$ degree of polynomials in S_h, we will run into problems trying to compute u_h as a polynomial function of large degree.

A *good* choice turns out to be *piecewise polynomial functions* (consisting of polynomials – of small degree – on each subinterval of a partition of I), continuous on I, and endowed with *basis functions of small support*. In this way, we obtain various *finite element* subspaces S_h of $\overset{0}{H}{}^1(I)$.

3.2 The Galerkin finite element method with piecewise linear, continuous functions

Let $a = x_0 < x_1 < x_2 < \ldots < x_N < x_{N+1} = b$ define an arbitrary partition of $[a, b]$ with N interior points x_i, $1 \leq i \leq N$. Let $h_i = x_{i+1} - x_i$, $0 \leq i \leq N$ and put $h = \max_i h_i$. (For *uniform* partitions $h = h_i = (b-a)/N+1$). Let S_h be the vector space of functions

defined by

$$S_h = \{\phi \ : \ \phi \in C[a, b], \ \phi(a) = \phi(b) = 0, \ \phi \text{ is a linear}$$

$$\text{polynomial on each } (x_i, x_{i+1}), \ 0 \leq i \leq N\}. \qquad (3.11)$$

(We write the last condition as $\phi \mid_{(x_i, x_{i+1})} \in \mathbb{P}_1$).

It is not hard to see that S_h is a finite dimensional subspace of $\overset{0}{H}{}^1$ and that $\dim S_h = N$. The latter follows e.g. from the fact that the set of functions $\{\phi_i\}_{i=1}^N$ defined by

$$\{\phi_i \in S_h, \ \phi_i(x_j) = \delta_{ij}\},$$

or, explicitly, for $1 \leq i \leq N$, as

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & \text{if} \quad x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & \text{if} \quad x_i \leq x \leq x_{i+1} \\ 0, & \text{if} \quad x \in \overline{I} \setminus (x_{i-1}, x_{i+1}) \end{cases} \qquad (3.12)$$

forms a *basis* of S_h: Obviously $\phi_i \in S_h$.
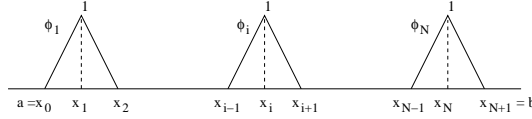
Moreover, if $\sum_{i=1}^N c_i \phi_i(x) = 0, \ \forall \, x \in I$, putting $x = x_j, \ 1 \leq j \leq N$, we see that $c_j = 0$, i.e. that $\{\phi_i\}$ are linearly independent. In addition, since for each $\phi \in S_h$,

$$\phi(x) = \sum_{i=1}^N c_i \phi_i(x), \ \text{where } c_i = \phi(x_i),$$

i.e. since each ϕ in S_h is uniquely determined by its *nodal values* $\phi(x_i), \ 1 \leq i \leq N$, we conclude that $\{\phi_i\}$ spans S_h.
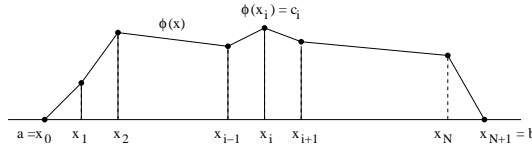
Evaluating the elements $A_{ij} = \int_a^b (p\phi_j' \phi_i' + q\phi_j \phi_i)$ of the matrix A of the system (3.10) yields that (since $\text{supp}\phi_i = [x_{i-1}, x_{i+1}]$) $A_{ii} \neq 0, \ 1 \leq i \leq N, \ A_{i,i+1} \neq 0,$

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$A_{i+1,i} \neq 0$, $1 \leq i \leq N - 1$, while all other $A_{ij} = 0$. Hence, due to the small support of the basis functions, the matrix A is *sparse*; in this case *tridiagonal*. Since it is also symmetric and positive definite, the system (3.10) may be efficiently solved e.g. by a *banded* (tridiagonal) *Cholesky* algorithm. In the special case $p = q = 1$ and a uniform partition of $I = (0, 1)$ with $h = 1/(N + 1)$, A is the sum of the *stiffness* matrix S, where $S_{ij} = \int_0^1 \phi_j' \phi_i'$, and the *mass* matrix M, where $M_{ij} = \int_0^1 \phi_j \phi_i$. Indeed, S and M are then the tridiagonal, symmetric, and positive definite matrices

$$
S = \frac{1}{h} \begin{pmatrix} 2 & -1 & & & \mathbf{0} \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \qquad M = \frac{h}{6} \begin{pmatrix} 4 & 1 & & & \mathbf{0} \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & 1 & 4 & 1 \\ & & & 1 & 4 \end{pmatrix}.
$$

We now return to the error estimate (3.8). As $h \to 0$, i.e. as $\dim S_h \to \infty$, we would like to prove that $\inf_{\phi \in S_h} \|u - \phi\|_1 \to 0$, where $u \in H^2 \cap \overset{0}{H}{}^1$ is the weak solution of (3.1), (3.2). In fact, we shall prove that there is a constant C, independent of h and v, such that $\inf_{\phi \in S_h} \|v - \phi\|_1 \leq C h \|v''\|$ for any $v \in H^2 \cap \overset{0}{H}{}^1$.

A simple and convenient way to do this is to study the properties of the *interpolant of v in the space S_h.*

The interpolant $(I_h v)(x)$ of a continuous function $v(x)$ (such that $v(a) = v(b) = 0$) in the space S_h is defined as the (unique) element of S_h satisfying

$$(I_h v)(x_i) = v(x_i), \quad 1 \leq i \leq N, \tag{3.13}$$
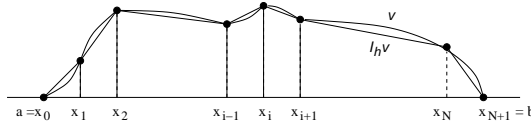
i.e. as the element $(I_h v)(x) = \sum_{i=1}^N v(x_i) \phi_i(x)$ of S_h. Hence $I_h : \overset{0}{H}{}^1 \to S_h$ is a linear operator on $\overset{0}{H}{}^1$ (the *interpolation* operator onto S_h).

Lemma 3.1. Let $v \in \overset{0}{H}{}^1$. Then $I_h v$ satisfies

$$(i) \quad ((I_h v - v)', \varphi') = 0, \quad \forall \varphi \in S_h, \tag{3.14}$$

$$(ii) \quad \|v - I_h v\| \leq C h \|(v - I_h v)'\|, \tag{3.15}$$

for some constant C independent of h and v.

Proof. (i) Let $\varphi \in S_h$. Then

$$((I_h v - v)', \varphi') \;=\; \int_a^b (I_h v - v)' \, \varphi' \, dx = \sum_{i=0}^N \int_{x_i}^{x_{i+1}} (I_h v - v)' \, \varphi' \, dx =$$

$$=\; \sum_{i=0}^N \left\{ [(I_h v - v) \, \varphi']_{x=x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} (I_h v - v) \, \varphi'' \, dx \right\} = 0,$$

since $(I_h v)(x_i) = v(x_i)$, $0 \le i \le N+1$, and $\varphi'' \mid_{[x_i, x_{i+1}]} = 0$.

(ii) Since $(I_h v - v)(x_i) = 0$, $0 \le i \le N+1$, $I_h v - v \in \overset{0}{H}{}^1(x_i, x_{i+1})$ for each $i = 0, 1, \ldots, N$. By the proof of Proposition 2.28 (the Poincaré–Friedrichs inequality), we conclude that there is a constant C, independent of the x_i or v such that

$$\|I_h v - v\|_{L^2(x_i, x_{i+1})} \;\le\; C \, (x_{i+1} - x_i) \, \|(I_h v - v)'\|_{L^2(x_i, x_{i+1})}$$

for all i (in fact the best value of C is $1/\pi$). We conclude that

$$\|I_h v - v\|^2 \;=\; \sum_{i=0}^N \|I_h v - v\|_{L^2(x_i, x_{i+1})}^2 \le C^2 \, h^2 \sum_{i=0}^N \|(I_h v - v)'\|_{L^2(x_i, x_{i+1})}^2 =$$

$$=\; C^2 \, h^2 \, \|(I_h v - v)'\|^2.$$

$\square$

Using Lemma 3.1 we may now prove the following approximation properties of the interpolant $I_h v$:

Proposition 3.2. There is a constant C independent of h such that

$$(i) \quad \|v - I_h v\| + h \|(v - I_h v)'\| \;\le\; C \, h \, \|v'\|, \quad \forall v \in \overset{0}{H}{}^1, \tag{3.16}$$

$$(ii) \quad \|v - I_h v\| + h \|(v - I_h v)'\| \;\le\; C \, h^2 \, \|v''\|, \quad \forall v \in H^2 \cap \overset{0}{H}{}^1. \tag{3.17}$$

Proof. (i) For $v \in \overset{0}{H}{}^1$, $\|(v - I_h v)'\| \le \|v'\| + \|(I_h v)'\|$. But, by (3.14), $((I_h v)', \varphi') = (v', \varphi') \; \forall \varphi \in S_h$. Put $\varphi = I_h v$ and obtain

$$\|(I_h v)'\|^2 = (v', (I_h v)') \;\le\; \|v'\| \|(I_h v)'\|, \text{ i.e. } \|(I_h v)'\| \le \|v'\|.$$

We conclude that $\|(v - I_h v)'\| \le 2\|v'\|$. Hence, by (3.15)

$$\|v - I_h v\| \;\le\; C \, h \, \|(v - I_h v)'\| \le 2 \, C \, h \, \|v'\|,$$

proving (3.16).

(ii) Let now $v \in H^2 \cap \overset{0}{H}{}^1$. Then,

$$\|(v - I_h v)'\| \leq C h \|v''\| \tag{3.18}$$

for some constant C independent of h and v. To see this, note that

$$
\begin{aligned}
\|(v - I_h v)'\|^2 &= ((v - I_h v)', (v - I_h v)') = (v', (v - I_h v)') \quad \text{(why?)}\\
&= \int_a^b v' (v - I_h v)' = [v' (v - I_h v)]_a^b - \int_a^b v'' (v - I_h v)\\
&\quad (\text{ since } v' \in H^1, \ v - I_h v \in H^1)\\
&= -\int_a^b v'' (v - I_h v) \leq \|v''\|\|v - I_h v\| \overset{\text{by}(3.15)}{\leq} C h \|v''\|\|(v - I_h v)'\|.
\end{aligned}
$$

We conclude therefore that $\|(v - I_h v)'\| \leq C h \|v''\|$. This gives, in view of (3.15), $\|v - I_h v\| \leq C h^2 \|v''\|$. Hence (3.17) holds. $\qquad\square$

In view of Proposition 3.1, since

$$\inf_{\phi \in S_h} (\|v - \phi\| + h \|(v - \phi)'\|) \leq \|v - I_h v\| + h \|(v - I_h v)'\|,$$

it follows that

$$\inf_{\phi \in S_h} (\|v - \phi\| + h \|(v - \phi)'\|) \leq C_k h^k \|D^k v\| \tag{3.19}$$

for $k = 1, 2$ if $v \in H^k \cap \overset{0}{H}{}^1$.

Hence,

$$\inf_{\phi \in S_h} \|v - \phi\|_1 \leq C h \|v''\| \tag{3.20}$$

for $v \in H^2 \cap \overset{0}{H}{}^1$.

We are now ready to prove *error estimates* in the H^1 and the L^2 norms for the error $u - u_h$ of the Galerkin approximation $u_h \in S_h$ of u, the weak solution of (3.1), (3.2).

Theorem 3.3. Let u be the weak solution of (3.1), (3.2) and $u_h \in S_h$ its Galerkin approximation in S_h. Then,

$$\|u - u_h\|_1 \leq C h \|u''\|, \tag{3.21}$$

$$\|u - u_h\| \leq C h^2 \|u''\|, \tag{3.22}$$

for some constant C independent of h and u.

Proof. (3.21) follows from (3.8) and (3.20). The proof of (3.22) follows from the following *duality* argument (Nitsche trick). Set $e = u - u_h \in \overset{0}{H}{}^1$. Let ψ be the solution of the problem

$$B(\psi, v) = (e, v), \quad \forall v \in \overset{0}{H}{}^1, \tag{3.23}$$

i.e. the weak solution of the problem $-(p\psi')' + q\psi = e$ in (a, b), $\psi(a) = \psi(b) = 0$. We know that ψ exists uniquely in $\overset{0}{H}{}^1$ and in fact $\psi \in H^2 \cap \overset{0}{H}{}^1$ and that it satisfies (elliptic regularity)

$$\|\psi\|_2 \leq C \|e\|. \tag{3.24}$$

Put $v = e$ in (3.23); then

$$\|e\|^2 = (e, e) = B(\psi, e) = B(e, \psi) = B(u - u_h, \psi) = B(e, \psi - \chi)$$

for **any** $\chi \in S_h$ (why?). Take $\chi = I_h \psi$. Then

$$\|e\|^2 = B(e, \psi - I_h \psi) \leq C_1 \|e\|_1 \|\psi - I_h \psi\|_1 \overset{\text{by}(3.17)}{\leq}$$
$$C_1 \|e\|_1 \, C \, h \, \|\psi''\| \leq C' \, h \, \|e\|_1 \|e\|, \text{ (by (3.24))}.$$

Hence, $\|e\| \leq C' \, h \, \|e\|_1$ and (3.22) follows from (3.21). $\qquad\square$

Remarks 3.4.

(i) Property (3.14) characterizes uniquely the interpolant as an element of S_h. I.e. the equation

$$((v_h - v)', \varphi') = 0, \quad \forall \varphi \in S_h, \tag{3.25}$$

(given $v \in \overset{0}{H}{}^1$), has a unique solution $v_h \in S_h$, which, by Lemma 3.1 (i), coincides with the interpolant $I_h v$ of v. To prove uniqueness, note that (3.25) may be written as $(v_h', \varphi') = (v', \varphi')$, $\forall \varphi \in S_h$. Hence, if there existed two such elements v_h^i, we would have

$$((v_h^1 - v_h^2)', \varphi') = 0, \quad \forall \varphi \in S_h \overset{\varphi = v_h^1 - v_h^2}{\Longrightarrow} \|(v_h^1 - v_h^2)'\| = 0 \Rightarrow v_h^1 = v_h^2,$$

by the Poincaré–Friedrichs inequality.

Equation (3.25) also states that in our case, $I_h v$ is the *projection* of v on S_h with respect to the inner product (u', v') on $\overset{0}{H}{}^1$. We conclude that

$$\|(I_h v - v)'\| = \inf_{\varphi \in S_h} \|v' - \varphi'\|.$$

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Hence, by Poincaré–Friedrichs

$$\|I_h v - v\|_1 \leq C \|(I_h v - v)'\| \leq C \inf_{\varphi \in S_h} \|v - \varphi\|_1.$$

Since, obviously, $\inf_{\varphi \in S_h} \|v - \varphi\|_1 \leq \|I_h v - v\|_1$, we finally obtain that

$$\inf_{\varphi \in S_h} \|v - \varphi\|_1 \leq \|I_h v - v\|_1 \leq C \inf_{\varphi \in S_h} \|v - \varphi\|_1,$$

i.e. that $I_h v$ is a *quasi–optimal* approximation of v in S_h (just as the Galerkin solution u_h is a quasi–optimal approximation of u in S_h). Hence, nothing essentially was lost by using the upper bound $\|I_h u - u\|_1$ of $\inf_{\varphi \in S_h} \|u - \varphi\|_1$ in (3.8) – which led to (3.21).

(ii) It may be proved that the exponents of h in the estimates (right-hand sides)

$$\|I_h u - u\|_1 \leq C_1 h \|u''\|,$$
$$\|I_h u - u\| \leq C_2 h^2 \|u''\|,$$

$$\|u_h - u\|_1 \leq C_1' h \|u''\|,$$
$$\|u_h - u\| \leq C_2' h^2 \|u''\|,$$

i.e. the powers 1 (for H^1 norms) and 2 (for L^2 norms) in the errors of $I_h u$ and u_h (viewed as approximations of $u \in H^2 \cap \overset{o}{H}{}^1$), are *optimal*, i.e. they cannot be increased in general.

We may also derive error estimates in *other norms*. For example, if the solution u of (3.1), (3.2) belongs to $C^2[a, b]$ (if it is a classical solution, that is), we may prove that there exists a constant C, independent of h and u such that

$$\|u_h - u\|_\infty + h\|u_h' - u'\|_\infty \leq C h^2 \|u''\|_\infty.$$

(Again, the interpolant satisfies a similar estimate. E.g. it is obvious (Lagrange interpolation) that $\|I_h u - u\|_\infty \leq (h^2/8) \|u''\|_\infty$).

(iii) Other boundary conditions. Let us consider for example (3.1) with the *Neumann* b.c.'s

$$u'(a) = u'(b) = 0, \tag{3.26}$$

where we assume now $q(x) \geq \beta > 0$, $x \in [a, b]$, for uniqueness. As the space in which the weak solution lies now is H^1, we consider subspaces of H^1 in which we seek the Galerkin approximation u_h of u.

Let $a = x_0 < x_1 < \ldots < x_{N+1} = b$ be an arbitrary partition of $[a,b]$ and let $h = \max_i(x_{i+1} - x_i)$ as before. The finite–dimensional space

$$\tilde{S}_h = \{\phi : \phi \in C[a,b], \ \phi \mid_{(x_i, x_{i+1})} \in \mathbb{P}_1\}$$

is a subspace of H^1 and has the basis $\{\phi_i\}_{i=0}^{N+1}$ consisting of the "hat" functions ϕ_i, $1 \le i \le N$, that were defined by (3.12), plus the endpoint "half–hat" functions ϕ_0, ϕ_{N+1}
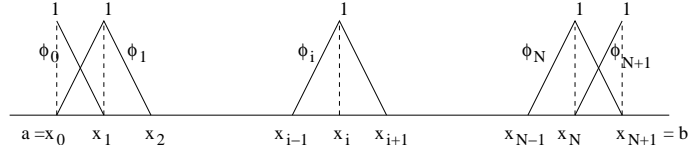
defined as follows:

$$\phi_0 \in \tilde{S}_h, \ \phi_0(x_0) = 1, \ \phi_0(x_j) = 0, \ j \ne 0, \ \phi_{N+1}(x_{N+1}) = 1, \ \phi_{N+1}(x_j) = 0, \ j \ne N+1.$$

Hence, $\dim \tilde{S}_h = N + 2$.

Defining again the interpolant $\tilde{I}_h v$ of a continuous function v on $[a,b]$ by the formula

$$(\tilde{I}_h v)(x) = \sum_{i=0}^{N+1} v(x_i)\,\phi_i(x),$$

i.e. as the unique element of $\tilde{S}_h$ that satisfies

$$(\tilde{I}_h v)(x_i) = v(x_i), \ 0 \le i \le N+1,$$

we may prove again that:

$$\text{(i)} \ \ ((\tilde{I}_h v)', \phi') = (v', \phi') \ \ \forall v \in H^1, \ \phi \in \tilde{S}_h. \tag{3.27}$$

(The proof is identical to that of Lemma 3.1 (i))

$$\text{(ii)} \ \ \|(\tilde{I}_h v)'\|^2 + \|(v - \tilde{I}_h v)'\|^2 = \|v'\|^2, \ \ \forall v \in H^1. \tag{3.28}$$

(This follows from (i) by putting $\phi = \tilde{I}_h v$).

With these in mind we may now see that

$$\|(v - \tilde{I}_h v)'\| \le \|v'\|, \ \ \forall v \in H^1. \tag{3.29}$$

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(A simple consequence of (3.28)). Since for each i, $v - \tilde{I}_h v \in \overset{0}{H}{}^1((x_i, x_{i+1}))$, we have again, as in Lemma 3.1, that

$$\|v - \tilde{I}_h v\| \leq h \|(v - \tilde{I}_h v)'\|, \quad \forall v \in H^1. \tag{3.30}$$

As a consequence of (3.29) and (3.30),

$$\|v - \tilde{I}_h v\| \leq h \|v'\|, \quad \forall v \in H^1. \tag{3.31}$$

Now we have that for $v \in H^2$,

$$\|(v - \tilde{I}_h v)'\|^2 = -(v - \tilde{I}_h v, v''). \tag{3.32}$$

The proof of (3.32) is identical to that given for I_h after the estimate (3.18).

As a consequence of (3.32), we have

$$\|v' - (\tilde{I}_h v)'\|^2 \leq \|v - \tilde{I}_h v\| \, \|v''\|,$$

which, when combined with (3.30), yields

$$\|v' - (\tilde{I}_h v)'\| \leq h \|v''\|, \quad \forall v \in H^2. \tag{3.33}$$

Using (3.33) in (3.30) we see that for $v \in H^2$

$$\|v - \tilde{I}_h v\| \leq h^2 \|v''\|. \tag{3.34}$$

We conclude therefore that

$$\|v - \tilde{I}_h v\| + h \|(v - \tilde{I}_h v)'\| \leq C_1 h \|v'\|, \quad \forall v \in H^1, \tag{3.35}$$

and

$$\|v - \tilde{I}_h v\| + h \|(v - \tilde{I}_h v)'\| \leq C_2 h^2 \|v''\|, \quad \forall v \in H^2. \tag{3.36}$$

These inequalities are the analogs of (3.16) and (3.17) and lead e.g. to the estimate

$$\inf_{\phi \in S_h} \|v - \phi\|_1 \leq \|v - \tilde{I}_h v\|_1 \leq C h \|v''\|, \quad v \in H^2. \tag{3.37}$$

The Galerkin approximation u_h of u in $\tilde{S}_h$ is defined again as the (unique) element of $\tilde{S}_h$ that satisfies

$$B(u_h, \phi) = (f, \phi) \quad \forall \phi \in \tilde{S}_h,$$

and may be found again in the form $u_h = \sum_{i=0}^{N+1} c_i \phi_i$, where $c = [c_0, \ldots, c_{N+1}]^T$ is the solution of the linear system $A c = f_h$, where the $(N+2) \times (N+2)$ (symmetric, positive definite) tridiagonal matrix A is given by $A_{ij} = B(\phi_j, \phi_i)$, and f_h is the vector $[(f, \phi_0), \ldots, (f, \phi_{N+1})]^T$. As before, if u is the weak solution of (3.1), (3.26), then

$$\|u - u_h\|_1 \leq C \inf_{\phi \in S_h} \|u - \phi\|_1,$$

which yields, in view of (3.37), the H^1–error estimate

$$\|u - u_h\|_1 \leq C h \|u''\|, \tag{3.38}$$

since $u \in H^2$. The error bound $(O(h))$ is of optimal rate.

The L^2-error estimate (of optimal rate as well)

$$\|u - u_h\| \leq C h^2 \|u''\| \tag{3.39}$$

follows again by a duality argument, exactly as in the proof of (3.22) in Theorem 3.3. (Use is made of the elliptic regularity inequality for the Neumann problem).

Analogously, we may approximate the solution of other two–point boundary–value problems with (3.1) as the underlying differential equation. For example, with the boundary conditions $u(a) = 0$, $u'(b) = 0$, we use the finite element space (defined on our usual partition):

$$\{\phi \in C[a, b], \ \phi(a) = 0, \ \phi\,|_{(x_i, x_{i+1})} \in \mathbb{P}_1\},$$

which is a finite–dimensional subspace of the Hilbert space $\{v : \ v \in H^1(a, b), \ v(a) = 0\}$ etc.

(iv) A note on implementation. (From Brenner–Scott, [3.2], Section 0.6)

A basic computational problem in the finite element method is the evaluation, or *assembly*, of the matrix and the right–hand side of the system $A c = f_h$ of the Galerkin equations, and in general, of the inner product (f, v) and the bilinear form

$$B(v, w) = \int_a^b (p(x)v'(x)w'(x) + q(x)v(x)w(x))\, dx$$

given $v, w \in S_h$. We will discuss this problem in its present simple one–dimensional context; the strategy extends in a natural way to the multidimensional case.

Given the partition $a = x_0 < x_1 < x_2 < \ldots < x_N < x_{N+1} = b$ of $[a, b]$, we will simply refer to a subinterval $I_k = [x_{k-1}, x_k]$ as an *element*. Hence in our problem we have $N + 1$ elements,
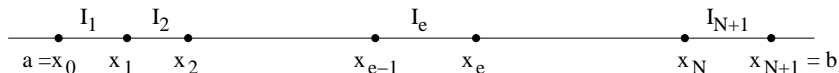
the subintervals $I_e = [x_{e-1}, x_e]$, $e = 1, 2, \ldots, N + 1$. Each element I_e
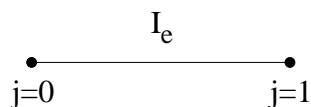
has two *nodes*, its endpoints, corresponding to the values of the *local node index* $j = 0$ (for the left endpoint x_{e-1}) and $j = 1$ (for the right endpoint x_e). The *element number* e and the local index j determine the *global index* i for the node x_i. In our case we have

$$i = i(e, j) = e + j - 1, \quad e = 1, 2, \ldots, N + 1, \ j = 0, 1.$$

Thus the left endpoint of the element I_{15} has global index $i = 15 + 0 - 1 = 14$, i.e. it is the node x_{14}, which of course coincides with the right endpoint of the element I_{14} (given by e=14, j=1).

Let us consider the basis $\{\varphi_i\}_{i=0}^{N+1}$ of "hat" functions, introduced as basis of the subspace $\tilde{S}_h$ used for the Neumann problem, i.e. the basis for the finite element space of piecewise linear, continuous functions with no boundary conditions imposed. Each function $v(x)$ in this subspace may be written as

$$v(x) = \sum_{i=0}^{N+1} v(x_i)\varphi_i(x) = \sum_{i=0}^{N+1} v_i\varphi_i(x), \quad v_i \equiv v(x_i).$$

There is a particularly effective way of describing $v(x)$ (for the purposes of efficient assembly of (f, v) and $B(v, w)$) in terms of its nodal values $v(x_i)$ and the *"local" basis functions* φ_j^e, $j = 0, 1$. The latter are simply the restrictions on $[x_{e-1}, x_e]$ of the basis functions $\varphi_{e-1}(x)$, $\varphi_e(x)$, respectively, and may be described in terms of two fixed
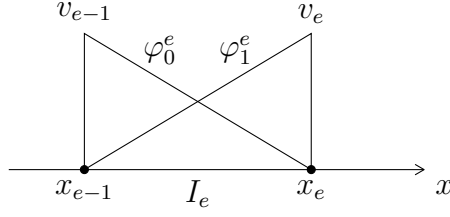
functions Φ_0, Φ_1 on the "reference" element $[0,1]$, as follows: First define Φ_0, Φ_1 as

$$\Phi_0(y) = \begin{cases} 1 - y & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}, \quad \Phi_1(y) = \begin{cases} y & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Associated with $I_e = [x_{e-1}, x_e]$ (defined by the *affine* transformation $x = h_e y + x_{e-1}$, $0 \leq y \leq 1$, where $h_e = x_e - x_{e-1}$, that maps the reference element $[0,1]$ onto I_e), define the local basis functions φ_0^e, φ_1^e as

$$\varphi_0^e(x) = \Phi_0(y), \ \varphi_1^e(x) = \Phi_1(y), \ \text{whenever } x = h_e y + x_{e-1},$$

i.e. as

$$\varphi_j^e(x) = \Phi_j \left(\frac{x - x_{e-1}}{h_e} \right), \quad j = 0, 1.$$

(Consequently, $\varphi_j^e(x) = 0$ for $x \notin I_e$).

A function $v(x)$ in the finite element subspace may be described now as

$$v(x) = \sum_{e=1}^{N+1} \sum_{j=0}^{1} v_{i(e,j)} \, \varphi_j^e(x). \tag{3.40}$$

If $x \in (x_{e-1}, x_e)$, (3.40) reduces to

$$v(x) = v_{e-1}\varphi_0^e(x) + v_e\varphi_1^e(x),$$
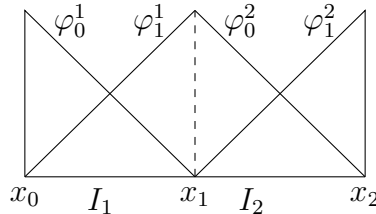
which is the correct description of $v(x)$ restricted to I_e. However, (3.40) is not a correct expression at the nodes. For example, at $x = x_1$, the right–hand side of (3.40) is equal

to

$$v_{i(1,1)}\varphi_1^1(x_1) + v_{i(2,0)}\varphi_0^2(x_1) = v_1 + v_1 = 2v_1,$$

instead of correct value $v_1 = v(x_1)$. This inconsistency will not affect the assembly of $B(v, w)$ or (f, v), which involve *integrals* over the I_e's.

The assembly e.g. of $B(v, w)$ (for v, w in the finite element subspace) is now done as follows. We write

$$B(v, w) = \sum_e B_e(v, w), \tag{3.41}$$

where $B_e(v, w)$ is a locally defined bilinear expression given by

$$B_e(v, w) = \int_{I_e} pv'w' + qvw = \int_{x_{e-1}}^{x_e} (p(x)v'(x)w'(x) + q(x)v(x)w(x)) \, dx$$

where $' = \frac{d}{dx}$.

Using (3.40) we write

$$
\begin{aligned}
B_e(v, w) &= \int_{x_{e-1}}^{x_e} p(x) \left(\sum_j v_{i(e,j)}\varphi_j^e(x) \right)' \left(\sum_j w_{i(e,j)}\varphi_j^e(x) \right)' \, dx \\
&+ \int_{x_{e-1}}^{x_e} q(x) \left(\sum_j v_{i(e,j)}\varphi_j^e(x) \right) \left(\sum_j w_{i(e,j)}\varphi_j^e(x) \right) \, dx.
\end{aligned}
$$

Using the map $x \mapsto y = (x - x_{e-1})/h_e$ we may transform the integrals above to integrals over the reference element $[0,1]$:

$$
\begin{aligned}
B_e(v, w) &= \frac{1}{h_e} \int_0^1 p(h_e y + x_{e-1}) \left(\sum_j v_{i(e,j)}\Phi_j(y) \right)' \left(\sum_j w_{i(e,j)}\Phi_j(y) \right)' \, dy \\
&+ h_e \int_0^1 q(h_e y + x_{e-1}) \left(\sum_j v_{i(e,j)}\Phi_j(y) \right) \left(\sum_j w_{i(e,j)}\Phi_j(y) \right) \, dy,
\end{aligned}
$$

where now the prime $'$ denotes differentiation with respect to y. Letting $\tilde{p}_e(y) = p(h_e y + x_{e-1})$, $\tilde{q}_e(y) = q(h_e y + x_{e-1})$ ($\tilde{p}_e$ and $\tilde{q}_e$ depend on e), we may rewrite the above in matrix–vector form

$$B_e(v, w) = \frac{1}{h_e} (v_{i(e,0)}, v_{i(e,1)}) S_e \begin{pmatrix} w_{i(e,0)} \\ w_{i(e,1)} \end{pmatrix} + h_e (v_{i(e,0)}, v_{i(e,1)}) M_e \begin{pmatrix} w_{i(e,0)} \\ w_{i(e,1)} \end{pmatrix}, \tag{3.42}$$

where S_e is the 2×2 *local stiffness matrix* given by

$$S_e = \begin{pmatrix} \int_0^1 \tilde{p}_e(\Phi_0')^2 & \int_0^1 \tilde{p}_e\Phi_0'\Phi_1' \\ \int_0^1 \tilde{p}_e\Phi_1'\Phi_0' & \int_0^1 \tilde{p}_e(\Phi_1')^2 \end{pmatrix} = \int_0^1 \tilde{p}_e(y) \, dy \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and M_e is the 2×2 *local mass matrix* defined as

$$M_e = \begin{pmatrix} \int_0^1 \tilde{q}_e \Phi_0^2 & \int_0^1 \tilde{q}_e \Phi_0 \Phi_1 \\ \int_0^1 \tilde{q}_e \Phi_1 \Phi_0 & \int_0^1 \tilde{q}_e \Phi_1^2 \end{pmatrix}.$$

The formula (3.42) is quite effective for computing the local bilinear form $B_e(v, w)$. It reduces to a few matrix – vector operations and the evaluation of the integrals in the entries of S_e and M_e. These integrals are evaluated on the reference finite element $[0, 1]$, in practice by some *numerical integration (quadrature) rule*. For example, use of the *trapezoidal rule*

$$\int_0^1 f(y)\, dy \approx \frac{f(0) + f(1)}{2}$$

yields an approximate local stiffness matrix $\tilde{S}_e$ given by

$$\tilde{S}_e = \frac{p(x_{e-1}) + p(x_e)}{2} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and a *diagonal* ("lumped") local mass matrix $\tilde{M}_e$ (why?)

$$\tilde{M}_e = \frac{1}{2} \cdot \begin{pmatrix} q(x_{e-1}) & 0 \\ 0 & q(x_e) \end{pmatrix}.$$

(It may be proved that using e.g. the trapezoidal rule in computing the elements of the matrix and the right–hand side of the Galerkin system $A\, c = f_h$, yields a new Galerkin approximation $\tilde{u}_h$ that satisfies again $\|\tilde{u}_h - u\|_1 = O(h)$, $\|\tilde{u}_h - u\| = O(h^2)$, i.e. the same type optimal–rate error estimates. See § 3.5.2 for an analysis of the error of numerical integration.)

3.3   Approximation by Hermite cubic functions and cubic splines

In this section we will construct two examples of finite dimensional subspaces of $H^1(I)$ (or $\overset{0}{H}{}^1(I)$), $I = (a, b)$, consisting of piecewise *cubic* polynomials. The Galerkin approximation u_h of u, the solution of (3.1),(3.2) or of (3.1),(3.24), will have higher order of accuracy. For example, its L^2 error $\|u - u_h\|$ will have an $O(h^4)$ bound, provided u is in $H^4(I)$.

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3.3.1 Hermite, piecewise cubic functions

On $I = (a, b)$ we consider, for simplicity, the uniform partition of meshlength $h = \frac{b-a}{N+1}$, defined by $x_i = a + i\,h$, $i = 0, \ldots, N+1$, and the associated vector space of functions

$$\mathcal{H} = \{\phi : \phi \in C^1[a, b], \ \phi \text{ is a cubic polynomial on each } (x_i, x_{i+1}), \ 0 \le i \le N\}.$$

The space $\mathcal{H}$ is called the vector space of *Hermite, piecewise cubic functions* on I, relative to the partition $\{x_i\}$.

It is not hard to see that $\mathcal{H}$ is a $(2N + 4)$-dimensional subspace of $H^2(I)$. To construct a suitable *basis* for $\mathcal{H}$ we define two functions, V and S on $[-1, 1]$ as follows:

$$
\begin{array}{ll}
V \in C^1[-1, 1], & S \in C^1[-1, 1], \\
V\big|_{[-1,0]}, V\big|_{[0,1]} \in \mathbb{P}_3, & S\big|_{[-1,0]}, S\big|_{[0,1]} \in \mathbb{P}_3, \\
V(-1) = 0, \ V'(-1) = 0, \ V(0) = 1, & S(-1) = 0, \ S'(-1) = 0, \ S(0) = 0, \\
V'(0) = 0, \ V(1) = 0, \ V'(1) = 0, & S'(0) = 1, \ S(1) = 0, \ S'(1) = 0.
\end{array}
$$

Since a cubic polynomial is uniquely defined by its values and the values of its derivative at two points, it follows that the functions V and S exist uniquely. In fact, they are given by the following formulas:

$$
V(x) = \begin{cases} (x + 1)^2(-2x + 1) & -1 \le x \le 0, \\ (x - 1)^2(2x + 1) & 0 \le x \le 1, \end{cases}
$$
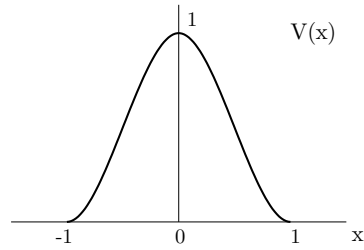
$$
S(x) = \begin{cases} x(x + 1)^2 & -1 \le x \le 0, \\ x(x - 1)^2 & 0 \le x \le 1, \end{cases}
$$
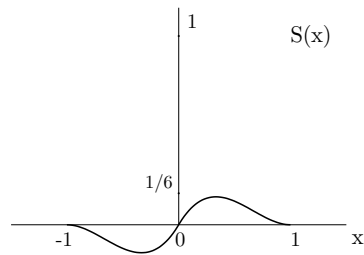
Using V and S we define the functions $\{V_j, S_j\}$, $0 \le j \le N + 1$, on $\bar{I}$ as follows: For $j = 1, \ldots, N$ we let:

$$
V_j(x) = \begin{cases} V\left(\frac{x - x_j}{h}\right) & x_{j-1} \le x \le x_{j+1} \\ 0 & \text{otherwise} \end{cases}, \quad
S_j(x) = \begin{cases} hS\left(\frac{x - x_j}{h}\right) & x_{j-1} \le x \le x_{j+1} \\ 0 & \text{otherwise} \end{cases}.
$$

We also define V_0, V_{N+1}, S_0, S_{N+1} as

$$V_0(x) = \begin{cases} V\left(\frac{x-a}{h}\right) & a \leq x \leq x_1 \\ 0 & \text{otherwise} \end{cases}, \quad S_0(x) = \begin{cases} hS\left(\frac{x-a}{h}\right) & a \leq x \leq x_1 \\ 0 & \text{otherwise} \end{cases},$$

$$V_{N+1}(x) = \begin{cases} V\left(\frac{x-b}{h}\right) & x_N \leq x \leq b \\ 0 & \text{otherwise} \end{cases}, \quad S_{N+1}(x) = \begin{cases} hS\left(\frac{x-b}{h}\right) & x_N \leq x \leq b \\ 0 & \text{otherwise} \end{cases},$$

We may easily verify that V_j, $S_j \in \mathcal{H}$, $0 \leq j \leq N+1$, and that $V_j(x_k) = \delta_{jk}$, $V_j'(x_k) = 0$, $S_j(x_k) = 0$, $S_j'(x_k) = \delta_{jk}$, $0 \leq j, k \leq N+1$.
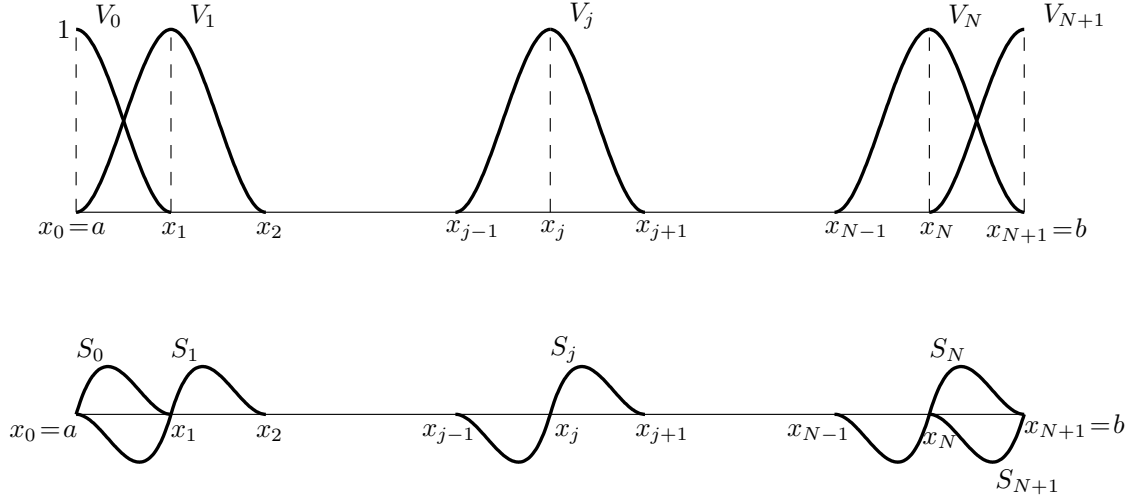
The $2N + 4$ functions $\{V_j, S_j\}$, $0 \leq j \leq N + 1$, form a *basis* of $\mathcal{H}$. Indeed, any function $\phi \in \mathcal{H}$ may be written in the form

$$\phi(x) = \sum_{j=0}^{N+1} \phi(x_j)V_j(x) + \phi'(x_j)S_j(x), \quad x \in \overline{I}, \tag{3.43}$$

since, on the interval $[x_k, x_{k+1}]$ the cubic polynomial $\phi\big|_{[x_k, x_{k+1}]}$ is uniquely determined by the values $\phi(x_k)$, $\phi'(x_k)$, $\phi(x_{k+1})$ and $\phi'(x_{k+1})$. On the other hand, the right-hand side of (3.43) is a cubic polynomial on $[x_k, x_{k+1}]$, whose values at x_k, x_{x+1} are $\phi(x_k)$, $\phi(x_{k+1})$, respectively, and whose derivative values are $\phi'(x_k)$ and $\phi'(x_{k+1})$ as well. In addition, since the relation

$$\sum_{j=0}^{N+1} c_j V_j(x) + d_j S_j(x) = 0,$$

implies (set $x = x_k$, $0 \leq k \leq N + 1$) that $c_k = 0$ all k, and that

$$\sum_{j=0}^{N+1} c_j V_j'(x) + d_j S_j'(x) = 0.$$

(from which, similarly, $d_j = 0$ all j), we conclude that the set $\{V_j, S_j\}$ is linearly independent.

Note that $\text{supp}(V_j) = \text{supp}(S_j) = [x_{j-1}, x_{j+1}]$ for $1 \leq j \leq N$. Hence, the functions V_j, S_j have the minimal support possible in $\mathcal{H}$: A function in $\mathcal{H}$ with support in *one* interval $[x_k, x_{k+1}]$ is identically zero.

If we are given now a function $f \in C^1[a, b]$, there is a uniquely determined element $\tilde{f} \in \mathcal{H}$ such that

$$f(x_j) = \tilde{f}(x_j), \quad f'(x_j) = \tilde{f}'(x_j), \quad 0 \leq j \leq N+1. \tag{3.44}$$

$\tilde{f}$ is called the *cubic Hermite interpolant of f* (relative to the partition $\{x_i\}$ of I) and is given by the formula

$$\tilde{f}(x) = \sum_{j=0}^{N+1} f(x_j)V_j(x) + f'(x_j)S_j(x), \quad x \in \overline{I}. \tag{3.45}$$

We shall study the approximation properties of the Hermite interpolant. Denote as usual by $\|\cdot\|_k$ the norm on the Sobolev space $H^k = H^k(I)$. Then, the following theorem holds:

Theorem 3.5. There exists $C > 0$ (independent of h), such that for each $f \in H^m$, $m = 2, 3$ or 4, we have

$$\|f - \tilde{f}\|_k \leqq C \, h^{m-k} \, \|f^{(m)}\| \quad k = 0, 1, 2. \tag{3.46}$$

Before proving it, let us remark that $m \geqq 2$ since, otherwise, $f'(x_k)$ may not have a meaning. In addition, since $\tilde{f} \in H^2$ but $\tilde{f} \notin H^3$ in general, we take $k \leq 2$. If f is taken in H^4, then we have the best order of accuracy, $O(h^{4-k})$ in H^k, $k = 0, 1, 2$. This rate cannot be improved even if f is smoother.

Theorem 3.5 follows from two lemmas:

Lemma 3.6. Let $f \in H^2$ and $k = 0$ or 1. Then we have for $1 \leq j \leq N+1$

$$\|D^k(f - \tilde{f})\|_{L^2(x_{j-1}, x_j)} \leq h \, \|D^{k+1}(f - \tilde{f})\|_{L^2(x_{j-1}, x_j)} \tag{3.47}$$

Proof. Since for $k = 0$ or 1, $D^k(f - \tilde{f})(x_j) = 0$, all j, we have $D^k(f - \tilde{f})(x) = \int_{x_{j-1}}^{x} \left(D^{k+1}(f - \tilde{f})\right)(t)dt, \quad x_{j-1} \leq x \leq x_j$.

Hence, for $x \in [x_{j-1}, x_{j+1}]$, from the Cauchy-Schwartz inequality:

$$|D^k(f - \tilde{f})(x)| \leqq \int_{x_{j-1}}^{x_j} |D^{k+1}(f - \tilde{f})| \, dt \leqq \sqrt{h}\|D^{k+1}(f - \tilde{f})\|_{L^2(x_{j-1}, x_j)}.$$

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Therefore

$$\|D^k(f-\tilde{f})\|^2_{L^2(x_{j-1},x_j)} \lesseqgtr \int_{x_{j-1}}^{x_j} \left(D^k(f-\tilde{f})\right)^2 dt \lesseqgtr h^2 \|D^{k+1}(f-\tilde{f})\|^2_{L^2(x_{j-1},x_j)}.$$

$\square$

Lemma 3.7. Let $f \in H^m$, $m = 2, 3,$ or 4. Then

$$\|D^2(f-\tilde{f})\| \le h^{m-2} \|D^m f\|. \tag{3.48}$$

Proof. For $1 \le j \le N+1$ we have the 'orthogonality' relation $\int_{x_{j-1}}^{x_j} D^2(f-\tilde{f})D^2\tilde{f} = 0$. (This follows from the observation that $\int_{x_{j-1}}^{x_j} D^2(f-\tilde{f})D^2\tilde{f} =$ (integrating by parts and using $D(f-\tilde{f})(x_j) = 0$ all j) $= -\int_{x_{j-1}}^{x_j} D(f-\tilde{f})D^3\tilde{f} = -\left[(f-\tilde{f})(x)D^3\tilde{f}(x)\right]_{x=x_{j-1}}^{x_j} + \int_{x_{j-1}}^{x_j}(f-\tilde{f})D^4\tilde{f} = 0$, since $(f-\tilde{f})(x_j) = 0$, all j, and $D^4\tilde{f}\big|_{[x_{j-1},x_j]} = 0$ since $\tilde{f} \in \mathbb{P}_3[x_{j-1}, x_j]$.)

This 'orthogonality' relation implies, for $f \in H^m$, $m = 2, 3,$ or 4

$$\|D^2(f-\tilde{f})\|^2_{L^2(x_{j-1},x_j)} \le \|D^{4-m}(f-\tilde{f})\|_{L^2(x_{j-1},x_j)}\|D^m f\|_{L^2(x_{j-1},x_j)}, \quad 1 \le j \le N+1 \tag{3.49}$$

To see (3.49), consider

$$\|D^2(f-\tilde{f})\|^2_{L^2(x_{j-1},x_j)} = \int_{x_{j-1}}^{x_j} D^2(f-\tilde{f})D^2(f-\tilde{f}) = (\text{'orthogonality'}) =$$

$$= \int_{x_{j-1}}^{x_j} D^2(f-\tilde{f})D^2 f = (-1)^m \int_{x_{j-1}}^{x_j} D^{4-m}(f-\tilde{f})D^m f,$$

where the last equality is trivial for $m = 2$, requires one integration by parts and use of the fact that $D(f-\tilde{f})(x_j) = 0$, for all j, if $m = 3$, and one more integration by parts and the fact that $(f-\tilde{f})(x_j) = 0$ all j, if $m = 4$. The Cauchy-Schwarz inequality yields now (3.49).

If $m = 2$, (3.49) gives $\|D^2(f-\tilde{f})\|_{L^2(x_{j-1},x_j)} \le \|D^2 f\|_{L^2(x_{j-1},x_j)}$. Hence, squaring and summing over j we get

$$\|D^2(f-\tilde{f})\|^2 = \sum_{j=1}^{N+1} \|D^2(f-\tilde{f})\|^2_{L^2(x_{j-1},x_j)} \le \sum_{j=1}^{N+1} \|D^2 f\|^2_{L^2(x_{j-1},x_j)} = \|D^2 f\|^2,$$

which is (3.48) for $m = 2$. If $m = 3$, (3.49) and (3.47) for $k = 1$, give

$$\|D^2(f-\tilde{f})\|^2_{L^2(x_{j-1},x_j)} \le \|D(f-\tilde{f})\|_{L^2(x_{j-1},x_j)}\|D^3 f\|_{L^2(x_{j-1},x_j)}$$

$$\le h\|D^2(f-\tilde{f})\|_{L^2(x_{j-1},x_j)}\|D^3 f\|_{L^2(x_{j-1},x_j)}.$$

Hence, $\|D^2(f-\tilde{f})\|_{L^2(x_{j-1},x_j)} \le h\|D^3 f\|_{L^2(x_{j-1},x_j)}$, which gives (3.48) after squaring and summing over j. Finally, if $m = 4$, (3.49), and (3.47) for $k = 0$ yield

$$
\begin{aligned}
\|D^2(f-\tilde{f})\|_{L^2(x_{j-1},x_j)}^2 &\le \|f-\tilde{f}\|_{L^2(x_{j-1},x_j)}\|D^4 f\|_{L^2(x_{j-1},x_j)} \\
&\le h\|D(f-\tilde{f})\|_{L^2(x_{j-1},x_j)}\|D^4 f\|_{L^2(x_{j-1},x_j)}.
\end{aligned}
$$

Again, by (3.47) for $k = 1$, we obtain

$$
\|D^2(f-\tilde{f})\|_{L^2(x_{j-1},x_j)}^2 \le h^2\|D^2(f-\tilde{f})\|_{L^2(x_{j-1},x_j)}\|D^4 f\|_{L^2(x_{j-1},x_j)},
$$

i.e. $\|D^2(f-\tilde{f})\|_{L^2(x_{j-1},x_j)} \le h^2\|D^4 f\|_{L^2(x_{j-1},x_j)}$. Squaring and summing we get (3.48). $\square$

Proof of Theorem 3.5. It follows from Lemma 3.6 and 3.7 easily. For example, if $m = 4$, we have:

$$
\begin{aligned}
k = 0 : \quad \|f - \tilde{f}\| \;&\le\; \text{(Lemma 3.6)} \;\le\; h\|D(f-\tilde{f})\| \;\le\; \text{(Lemma 3.6)} \\
&\le\; h^2\|D^2(f-\tilde{f})\| \;\le\; \text{(Lemma 3.7)} \;\le\; h^4\|D^4 f\|. \\
k = 1 : \quad \|(f-\tilde{f})'\| \;&\le\; \text{(Lemma 3.6)} \;\le\; h\|D^2(f-\tilde{f})\| \;\le\; \text{(Lemma 3.7)} \\
&\le\; h^3\|D^4 f\|. \\
&\text{Hence } \|f-\tilde{f}\|_1^2 \le \left((h^4)^2 + (h^3)^2\right)\|D^4 f\|^2 \le C\, h^6\|D^4 f\|^2. \\
k = 2 : \quad \|D^2(f-\tilde{f})\| \;&\le\; \text{(Lemma 3.7)} \;\le\; h^2\|D^4 f\| \;\Rightarrow \\
&\|f-\tilde{f}\|_2 \le C\, h^2\|D^4 f\|.
\end{aligned}
$$

The cases $m = 2$, $0 \le k \le 2$, $m = 3$, $0 \le k \le 2$, also follow analogously. $\square$

We conclude that given $f \in H^m$ ($m = 2, 3$, or 4), there exists an element $\chi \in \mathcal{H}$ (we verified it for $\chi = \tilde{f}$) such that

$$
\sum_{k=0}^{2} h^k \|f - \chi\|_k \le C\, h^m \|D^m f\|. \tag{3.50}
$$

We turn now to the Galerkin solution of the 2-pt. b.v.p. for the d.e. (3.1). Considering first Neumann boundary conditions, i.e. the b.c. (3.24), we see that u_h is the unique element of $\mathcal{H}$ that satisfies

$$
B(u_h, \phi) = (f, \phi), \quad \forall \phi \in \mathcal{H},
$$

for which, as usual (taking as our Hilbert space $H^1(I)$), there follows that

$$\|u - u_h\|_1 \leq C \inf_{\phi \in \mathcal{H}} \|u - \phi\|_1.$$

Hence, by (3.67) we conclude that if $u \in H^m$, $m = 2, 3,$ or 4, then

$$\|u - u_h\|_1 \leq C\, h^{m-1}\|D^m u\|. \tag{3.51}$$

By a duality argument ('Nitsche trick') it follows, just as in the proof of Thm 3.1 (with H^1 instead of $\overset{0}{H}{}^1$), that in L^2 we have

$$\|u - u_h\| \leq C\, h^m\|D^m u\|. \tag{3.52}$$

Hence, the (optimal) rate of accuracy one may achieve for u_h is 4 in L^2 (3 in H^1), provided u, the solution of (3.1), (3.24), is in H^4. (If for example $p \in C^{m-1}$, $q \in C^{m-2}$, $f \in H^{m-2}$ for some $m \leq 2$, it may be shown that the elliptic regularity estimate (3.4) generalizes and gives that $u \in H^m$ and

$$\|u\|_m \leqq C_m\|f\|_{m-2}\, .\Big)$$

In the case of homogeneous Dirichlet boundary conditions, i.e. the problem (3.1)-(3.2), we may take as our subspace of $\overset{0}{H}{}^1(I)$ the vector space $\overset{0}{\mathcal{H}} = \{\phi : \phi \in \mathcal{H},\ \phi(a) = \phi(b) = 0\}$. It is easily seen that $\overset{0}{\mathcal{H}}$ is a $(2N+2)$-dimensional subspace of $H^2 \cap \overset{0}{H}{}^1$. Its basis consists of the elements V_j, $1 \leq j \leq N$, and S_j, $0 \leq j \leq N+1$. We may define the interpolant $\tilde{f}$ of a function $f \in H^2 \cap \overset{0}{H}{}^1$ in the natural way, and show as before that

$$\|f - \tilde{f}\|_k \leq C\, h^{m-k}\|D^m f\|, \quad k = 0, 1, 2\,,$$

provided $f \in H^m \cap \overset{0}{H}{}^1$, and m is $2, 3,$ or 4. The analogous estimates to (3.51) and (3.52) still hold, i.e. we have

$$\|u - u_h\|_j \leq C\, h^{m-j}\|D^m u\|, \quad j = 0, 1\,,$$

provided u, the solution of (3.1),(3.2), belongs to $H^m \cap \overset{0}{H}{}^1$.

The system that defines the Galerkin equations is now of size (say, for the Neumann problem) $(2N + 4) \times (2N + 4)$. Ordering the basis vectors $\{\phi_1, \ldots, \phi_{2N+4}\}$ as $\{V_0, S_0, V_1, S_1, \ldots, V_{N+1}, S_{N+1}\}$, we see that e.g. the Gram matrix (with elements $\int_a^b \phi_i \phi_j$) is *block-tridiagonal* with 2x2 blocks. Its general 'line' of blocks is:

$$
\boxed{\begin{matrix} \int V_j V_{j-1} & \int V_j S_{j-1} \\ \int S_j V_{j-1} & \int S_j S_{j-1} \end{matrix}} \quad \boxed{\begin{matrix} \int (V_j)^2 & \int V_j S_j \\ \int S_j V_j & \int (S_j)^2 \end{matrix}} \quad \boxed{\begin{matrix} \int V_j V_{j+1} & \int V_j S_{j+1} \\ \int S_j V_{j+1} & \int S_j S_{j+1} \end{matrix}}
$$

Hence, the cubic Hermite elements will give an accuracy of $O(h^4)$ in L^2 (if $u \in H^4$) at the expense of solving a $(2N+4) \times (2N+4)$ 7-diagonal linear system.

We close this section with two comments:

(i) We may define Hermite piecewise cubic functions on a general partition $a = x_0 < x_1 < \ldots < x_{N+1} = b$ of $[a,b]$. The associated basis $\{V_j, S_j\}$, $0 \le j \le N+1$, may be defined in a straightforward way; its members have support in $[x_{j-1}, x_{j+1}]$ for $1 \le j \le N$, etc. An interpolant may be defined in the natural way. The estimate (3.46) still holds with $h := \max_j (x_{j+1} - x_j)$.

(ii) The cubic Hermite functions are a special case of the space $\mathcal{H}_m = \left\{ \phi \in C^{m-1}[a,b], \ \phi\big|_{[x_i, x_{i+1}]} \in \mathbb{P}_{2m-1} \right\}$, on which the error of the Galerkin approximation is of $O(h^{2m})$ in L^2.

3.3.2 Cubic splines

The dimension of $\mathcal{H}$, the space of Hermite cubics is, as we saw, $2N+4$. This, in particular, leads to larger linear systems that have to be solved for the Galerkin approximation u_h. A natural idea is to lower the dimension of the space by requiring more continuity at the nodes x_j. In the cubic polynomial case this leads to the space

$$
\mathcal{S} := \{ \phi : \ \phi \in C^2[a,b], \ \phi\big|_{[x_i, x_{i+1}]} \in \mathbb{P}_3, \ 0 \le i \le N \},
$$

of the so-called *cubic splines*. (We suppose that the $\{x_i\}$ define a uniform partition of $[a,b]$ with $x_0 = a$, $x_{N+1} = b$, of meshlength $h = (b-a)/(N+1)$.)

A count of the free parameters and the constraints of functions in $\mathcal{S}$ leads to the conjecture that $\mathcal{S}$ is a $(N+4)$-dimensional subspace of $H^3((a,b))$. To prove this fact, we shall construct a basis of $\mathcal{S}$, in fact a basis with elements of *minimal support*.

It is not hard to see that there are no nontrivial elements of $\mathcal{S}$ vanishing outside less than four adjacent mesh intervals. (For example, prove that the only element of $\mathcal{S}$ which vanishes identically outside $[x_{j-1}, x_{j+2}]$ is the zero element.) In addition, it is easy to see that there is a unique function $S \in C^2[-2,2]$, such that $supp(S) = [-2,2]$,

$S\big|_{[k,k+1]} \in \mathbb{P}_3$ for $k = -2, -1, 0, 1$, $S(\pm 2) = S'(\pm 2) = S''(\pm 2) = 0$, $S(0) = 1$. This function is given by the formula

$$S(x) = \begin{cases} \frac{1}{4}(x+2)^3 & -2 \le x \le -1, \\ \frac{1}{4}\left[1 + 3(x+1) + 3(x+1)^2 - 3(x+1)^3\right] & -1 \le x \le 0, \\ \frac{1}{4}\left[1 + 3(1-x) + 3(1-x)^2 - 3(1-x)^3\right] & 0 \le x \le 1, \\ \frac{1}{4}(2-x)^3 & 1 \le x \le 2 \\ 0 & x \in \mathbb{R} - [-2, 2]. \end{cases} \qquad (3.53)$$
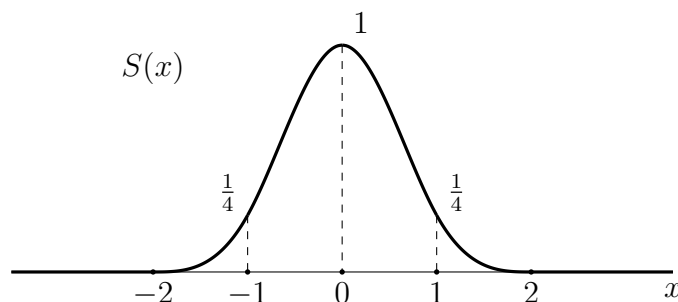
Using $S(x)$, we define the functions $\{\phi_j\}$, $j = -1, 0, 1, \ldots, N + 2$, on $[a, b]$ as follows: We introduce the extra nodes $x_{-1} := a - h$, $x_{N+2} := b + h$ and put

$$\phi_j(x) = S\left(\frac{x - x_j}{h}\right)\bigg|_{[a,b]}, \quad -1 \le j \le N + 2,$$

(the *restriction* of $S\left(\frac{x-x_j}{h}\right)$ to $[a, b]$). It may be seen immediately that each ϕ_j, $-1 \le j \le N + 2$, is a cubic polynomial on each interval $[x_k, x_{k+1}]$, $0 \le k \le N$, belongs to $C^2[a, b]$, and hence $\phi_j \in \mathcal{S}$, $1 \le j \le N + 2$.

Moreover, $\mathrm{supp}(\phi_j) = [x_{j-2}, x_{j+2}]$, for, $2 \le j \le N - 1$, and

$$\phi_{-1} \quad \text{vanishes for } x \in [a, b] - [x_0, x_1],$$

$$\phi_0 \quad \text{vanishes for } x \in [a, b] - [x_0, x_2],$$

$$\phi_1 \quad \text{vanishes for } x \in [a, b] - [x_0, x_3],$$

$$\phi_N \quad \text{vanishes for } x \in [a, b] - [x_{N-2}, x_{N+1}],$$

$$\phi_{N+1} \quad \text{vanishes for } x \in [a, b] - [x_{N-1}, x_{N+1}],$$

$$\phi_{N+2} \quad \text{vanishes for } x \in [a, b] - [x_N, x_{N+1}],$$

In addition, $\phi_j(x_j) = 1$ for $0 \le j \le N + 1$.
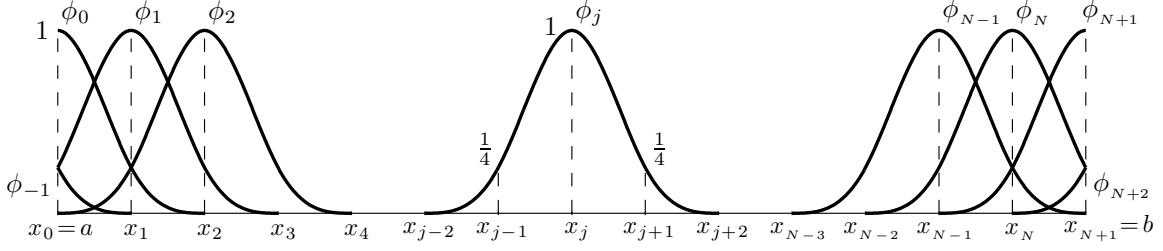
The $N + 4$ functions ϕ_j, $-1 \leq j \leq N + 2$, are known as *B-splines*, since they form a *basis* of $\mathcal{S}$. We shall show this in a series of lemmata:

Lemma 3.8. The $\{\phi_j\}_{j=-1}^{N+2}$ are linearly independent.

Proof. Suppose that for real constants c_j, $-1 \leq j \leq N + 2$, we have

$$\sum_{j=-1}^{N+2} c_j \phi_j(x) = 0, \quad x \in [a, b] \, .$$

Then, in particular we have

$$\sum_{j=-1}^{N+2} c_j \phi_j(x_k) = 0, \quad k = 0, \ldots, N + 1 \, , \tag{3.54a}$$

$$\text{and} \quad \sum_{j=-1}^{N+2} c_j \phi_j'(x_k) = 0, \quad k = 0, \ldots, N + 1 \, . \tag{3.54b}$$

Using the facts that

$$\phi_j(x_k) = \begin{cases} 1 & \text{if } j = k \, , \\ \frac{1}{4} & \text{if } |j - k| = 1 \, , \\ 0 & \text{if } |j - k| \geqq 2 \, , \end{cases}$$

we may write the relation (3.54a) as

$$\tfrac{1}{4} c_{k-1} + c_k + \tfrac{1}{4} c_{k+1} = 0, \quad k = 0, \ldots, N + 1. \tag{3.55}$$

Now, using the fact that S is even (about zero) and therefore that S' is an odd function, we conclude from the definition of ϕ_j that

$$\phi_j'(x_j) = 0, \quad \phi_j'(x_{j-1}) = -\phi_j'(x_{j+1}), \quad \phi_j'(x_k) = 0, \; |j - k| \geq 2 \, .$$

Therefore, (3.54b) for $k = 0$ gives

$$c_{-1} \phi_{-1}'(x_0) + c_1 \phi_1'(x_0) = 0 \; \Rightarrow \; c_{-1} = c_1 \, . \tag{3.56a}$$

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In addition, (3.54b) for $k = N + 1$ gives

$$c_N \phi'_N(x_{N+1}) + c_{N+2} \phi'_{N+2}(x_{N+1}) = 0 \implies c_N = c_{N+2}. \tag{3.56b}$$

The relations (3.56a) and (3.56b) are used to eliminate the unknowns c_{-1} and c_{N+2} from the linear system (3.55), which takes the form

$$Ac = 0, \tag{3.57}$$

where $c = [c_0, \ldots, c_{N+1}]^T$ and A is the $(N+2) \times (N+2)$ tridiagonal matrix

$$A = \begin{pmatrix} 4 & 2 & & & \mathbf{0} \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ \mathbf{0} & & & 2 & 4 \end{pmatrix}.$$

The matrix A is strictly diagonally dominant and therefore is invertible by Gerschgorin's lemma. Therefore, $c_j = 0$, $0 \le j \le N+1$, since c solves the homogeneous problem (3.57). By (3.56a) and (3.56b) $c_{N+2} = 0$, $c_{-1} = 0$. We conclude that the $\{\phi_j\}$, $-1 \le j \le N+2$, are linearly independent. $\square$

We let now $M := \langle \phi_{-1}, \ldots, \phi_{N+2} \rangle$ be the subspace of $\mathcal{S}$ which is *spanned* by the elements of the set $\{\phi_j\}_{j=-1}^{N+2}$. We shall show that in fact $M = \mathcal{S}$, thus completing the proof that $\{\phi_j\}_{j=-1}^{N+2}$ is a basis for $\mathcal{S}$. This will be accomplished by two intermediate results of *interpolation*:

Lemma 3.9. Let $f \in C^1[a, b]$. Then, there exists a unique element $\tilde{f} \in M$ satisfying the *interpolation conditions*

$$\begin{cases} \tilde{f}(x_j) = f(x_j), & 0 \le j \le N+1, \\ \tilde{f}'(a) = f'(a), \\ \tilde{f}'(b) = f'(b) \end{cases} \tag{3.58}$$

Proof.

The functions $\{\phi_j\}_{j=-1}^{N+2}$ form a basis for M. We seek therefore a function $\tilde{f} = \sum_{j=-1}^{N+2} c_j \phi_j \in M$

which satisfies (3.58), i.e. the linear system (for its coefficients c_j):

$$\left.\begin{array}{l} \sum_{j=-1}^{N+2} c_j \phi_j(x_k) = f(x_k), \quad 0 \le k \le N+1, \\[2mm] c_{-1}\phi'_{-1}(a) + c_1\phi'_1(a) = f'(a), \\[2mm] c_N\phi'_N(b) + c_{N+2}\phi'_{N+2}(b) = f'(b). \end{array}\right\} \tag{3.59}$$

The linear system (3.59) has a unique solution, since the associated homogeneous system (obtained by putting $f(x_k) = 0$, $0 \le k \le N+1$, $f'(a) = f'(b) = 0$ in (3.59)), has only the trivial solution as we argued in the proof of Lemma 3.8. We conclude that there is a unique element $\tilde{f} \in M$ satisfying the interpolation conditions (3.58). $\qquad\square$

Lemma 3.10. Let $f \in C^1[a,b]$. Then, there is a unique element $\tilde{s} \in \mathcal{S}$ which satisfies the analogous to (3.58) interpolation conditions

$$\left.\begin{array}{l} \tilde{s}(x_j) = f(x_j), \quad 0 \le j \le N+1, \\[2mm] \tilde{s}'(a) = f'(a), \\[2mm] \tilde{s}'(b) = f'(b). \end{array}\right\} \tag{3.60}$$

Proof. See [6.1, Thm. 4.7]. The function $\tilde{s}(x)$ is explicitly constructed by the values of its second derivative at the points x_j, $0 \le j \le N+1$. $\qquad\square$

We may now conclude that the $\{\phi_i\}$ span $\mathcal{S}$ i.e. that $M = \mathcal{S}$:

Theorem 3.11. The $\{\phi_j\}_{j=-1}^{N+2}$ are a basis of $\mathcal{S}$.

Proof. In view of Lemma 3.8 we need only to show that $\mathcal{S} \subset M$. Let $f \in \mathcal{S}$. Then, by Lemma 3.9, there is a unique element $\tilde{f} \in M$ that satisfies the interpolation conditions (3.58). But $M \subset \mathcal{S}$, hence $\tilde{f} \in \mathcal{S}$. However, by Lemma 3.10 there is a unique element of $\mathcal{S}$ that satisfies the interpolation conditions (3.60) which are the same as (3.58). This element coincides with f since $f \in \mathcal{S}$. We conclude that $f = \tilde{f}$. Hence $f \in M$, i.e. $\mathcal{S} \subset M$. $\qquad\square$

Given $f \in C^1[a,b]$, we call $\tilde{f}(= \tilde{s})$ its *cubic spline interpolant* (with "first derivative" end–point conditions $\tilde{f}'(a) = f'(a)$, $\tilde{f}'(b) = f'(b)$.) It is not hard to see that if we are given $f''(a)$, $f''(b)$, we may construct a cubic spline interpolant $\tilde{\tilde{f}}$ satisfying, in addition to the conditions $\tilde{\tilde{f}}(x_j) = f(x_j)$, $0 \le j \le N+1$, the endpoint "second derivative"

boundary conditions $\tilde{\tilde{f}}''(a) = f''(a)$, $\tilde{\tilde{f}}''(b) = f''(b)$. Similarly, for *periodic* $f \in C^1[a, b]$ we may construct the "periodic spline" interpolant etc. All these interpolant functions satisfy error estimates of $O(h^4)$ accuracy in L^2 provided $f \in H^4(I)$. For example, we shall prove the following result for the interpolant with first-derivative boundary conditions:

Theorem 3.12. There exists a constant $C > 0$ (independent of h), such that for each $f \in H^m$, $m = 2$, 3, or 4, we have

$$\|f - \tilde{f}\|_k \leq C\, h^{m-k} \|f^{(m)}\|, \quad k = 0,\ 1,\ 2. \quad \square \tag{3.61}$$

We shall prove the theorem using two intermediate results. First, we prove the analog of Lemma 3.6; but now the norms are *global*, i.e. they are the $L^2(a, b)$ norms.

Lemma 3.13. Let $f \in H^2$ and $k = 0$ or 1. Then, there exists a constant C independent of h and f, such that

$$\|D^k(f - \tilde{f})\| \leq C\, h\, \|D^{k+1}(f - \tilde{f})\|. \tag{3.62}$$

Proof. For $k = 0$ we may prove the local result as well. Since f coincides with $\tilde{f}$ at the nodes x_j, $0 \leq j \leq N + 1$, we have, for $x \in [x_{j-1}, x_j]$, for some $1 \leq j \leq N + 1$, that

$$f(x) - \tilde{f}(x) = \int_{x_{j-1}}^x D(f - \tilde{f})(y)\, dy.$$

Hence, for $x_{j-1} \leq x \leq x_j$, using the Cauchy-Schwarz inequality,

$$
\begin{aligned}
|(f - \tilde{f})(x)| &\leq \int_{x_{j-1}}^x |D(f - \tilde{f})(y)|\, dy \leq \sqrt{(x_j - x_{j-1})} \left(\int_{x_{j-1}}^{x_j} \left(D(f - \tilde{f})(y) \right)^2 dy \right)^{\frac{1}{2}} \\
&\leq \sqrt{h}\, \|D(f - \tilde{f})\|_{L^2(x_{j-1}, x_j)}.
\end{aligned}
$$

Therefore,

$$\|f - \tilde{f}\|_{L^2(x_{j-1}, x_j)} \leqq h\, \|D(f - \tilde{f})\|_{L^2(x_{j-1}, x_j)}, \tag{3.63}$$

from which, by squaring and summing, we get (3.62) for $k = 0$ with $C = 1$.

To prove (3.62) for $k = 1$, we note that since

$$\phi(x_j) = 0, \quad 0 \leq j \leq N + 1,$$

where $\phi := f - \tilde{f}$, there follows, by Rolle's theorem, that there exist $\xi_j \in (x_{j-1}, x_j)$, $1 \leq j \leq N + 1$, where $\phi'(\xi_j) = 0$.

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(Note that $\phi \in H^2[a, b]$; hence, $\phi \in C^1[a, b]$). Note also that, by definition of $\tilde{f}$, $\phi'(x_0) = \phi'(x_{N+1}) = 0$. Introduce then $\xi_0 := x_0$ and $\xi_{N+2} := x_{N+1}$, so that $\phi'(\xi_j) = 0$, $0 \leq j \leq N + 2$. Therefore, for $x \in [\xi_{j-1}, \xi_j]$, $\phi'(x) = \int_{\xi_{j-1}}^x \phi''(y)\, dy$. Hence $|\phi'(x)| \leq \sqrt{(x - \xi_{j-1})}\|\phi''\|_{L^2(\xi_{j-1}, \xi_j)} \leq \sqrt{(\xi_j - \xi_{j-1})}\|\phi''\|_{L^2(\xi_{j-1}, \xi_j)} \leq \sqrt{2h}\|\phi''\|_{L^2(\xi_{j-1}, \xi_j)}$. We conclude that

$$\|\phi'\|_{L^2(\xi_{j-1}, \xi_j)} \leq (\xi_j - \xi_{j-1})(2h)\|\phi''\|^2_{L^2(\xi_{j-1}, \xi_j)} \leq 4h^2\|\phi''\|^2_{L^2(\xi_{j-1}, \xi_j)}.$$

Hence,

$$\|\phi'\|^2 = \sum_{j=1}^{N+2} \|\phi'\|^2_{L^2(\xi_{j-1}, \xi_j)} \leq 4h^2 \sum_{j=1}^{N+2} \|\phi''\|^2_{L^2(\xi_{j-1}, \xi_j)} = 4h^2\|\phi''\|^2.$$

We conclude that (3.62) is valid for $k = 1$ as well, with $C = 2$. $\qquad\square$

We proceed now to the analog of Lemma 3.7:

Lemma 3.14. Let $f \in H^m$, $m = 2$, 3 or 4. Then, for some C independent of h and f:

$$\|D^2(f - \tilde{f})\| \leqq C\, h^{m-2}\|D^m f\|. \tag{3.64}$$

Proof. First we prove the *orthogonality relation*

$$\int_a^b D^2(f - \tilde{f})D^2\tilde{f} = 0. \tag{3.65}$$

We have, using the endpoint condition $f'(a) = \tilde{f}'(a)$, $f'(b) = \tilde{f}'(b)$,

$$\int_a^b D^2(f - \tilde{f})D^2\tilde{f} = \left[(f - \tilde{f})'\tilde{f}''\right]_a^b - \int_a^b (f - \tilde{f})'\tilde{f}'''$$

$$= -\int_a^b (f - \tilde{f})'\tilde{f}'''. \quad \text{(Note that } \tilde{f} \in H^3(a, b)).$$

Now $\int_a^b (f - \tilde{f})'\tilde{f}''' = \sum_{j=0}^N \int_{x_j}^{x_{j+1}} (f - \tilde{f})'\tilde{f}''' = \sum_{j=0}^N \left\{[(f - \tilde{f})\tilde{f}''']_{x_j}^{x_{j+1}} - \int_{x_j}^{x_{j+1}} (f - \tilde{f})D^4\tilde{f}\right\} = 0$, since $(f - \tilde{f})(x_j) = 0$, $0 \leq j \leq N + 1$, and $\tilde{f} \in \mathbb{P}_3$ in each $[x_j, x_{j+1}]$. Thus, (3.65) is proved. We have now by Cauchy-Schwarz

$$\|D^2(f - \tilde{f})\|^2 = \int_a^b D^2(f - \tilde{f})D^2(f - \tilde{f}) = \int_a^b D^2(f - \tilde{f})D^2 f \leq \|D^2(f - \tilde{f})\|\|D^2 f\|.$$

Hence, (3.64) holds, if $m = 2$, with $C = 1$. If now $m = 3$, we have, integrating by parts once and using the endpoint derivative conditions:

$$\begin{aligned}
\|D^2(f - \tilde{f})\|^2 &= \int_a^b D^2(f - \tilde{f})D^2 f = \left[D(f - \tilde{f})D^2 f\right]_a^b - \int_a^b (f - \tilde{f})' D^3 f \\
&= -\int_a^b ((f - \tilde{f})' D^3 f \le \|(f - \tilde{f})'\| \, \|D^3 f\| \, ,
\end{aligned}$$

from which, by (3.62) $k = 1, \|D^2(f - \tilde{f})\|^2 \le C\,h\,\|D^2(f - \tilde{f})\| \, \|D^3 f\|$ that shows (3.64) for $m = 3$. Finally, if $m = 4$, we may integrate by parts once more; using the interpolation conditions $f(a) = \tilde{f}(a)$, $f(b) = \tilde{f}(b)$, we have:

$$\begin{aligned}
\|D^2(f - \tilde{f})\|^2 &= -\int_a^b (f - \tilde{f})' D^3 f = -\left[(f - \tilde{f})D^3 f\right]_a^b + \int_a^b (f - \tilde{f})D^4 f \\
&= \int_a^b (f - \tilde{f})D^4 f \le \|f - \tilde{f}\| \, \|D^4 f\| \le C\,h^2\,\|D^2(f - \tilde{f})\| \, \|D^4 f\| \, .
\end{aligned}$$

(In the last step, we used Lemma 3.13 twice). We conclude that (3.64) holds as well for $m = 4$. $\qquad\square$

Proof of Theorem 3.12. It is straightforward to see that (3.61) follows from the results of Lemmata 3.13 and 3.14 . $\qquad\square$

We conclude that given $f \in H^m$, ($m = 2$, 3, or 4), there exists an element $\chi \in \mathcal{S}$ (take $\chi = \tilde{f}$), such that

$$\sum_{k=0}^2 h^k \|f - \chi\|_k \le C\,h^m\,\|D^m f\| \, . \tag{3.66}$$

As before, it follows for the Galerkin approximation u_h to u, the solution e.g. of the b.v.p. (3.1),(3.2), that

$$\|u - u_h\|_k \le C\,h^{m-k}\|D^m u\| \, , \tag{3.67}$$

for $k = 0$, 1, provided $u \in H^m$, $m = 2$, 3 or 4. In the case of the problem (3.1),(3.2), we take as subspace of $\overset{0}{H}{}^1(I)$ the vector space $\overset{0}{\mathcal{S}} = \{\phi : \phi \in \mathcal{S}, \ \phi(a) = \phi(b) = 0\}$. It is easily seen that $\overset{0}{\mathcal{S}}$ is a $(N+2)$-dimensional subspace of $\overset{0}{H}{}^1 \cap H^3$. A *basis* of this subspace consists of the previously defined B-splines ϕ_j, $2 \le j \le N - 1$, plus four functions $\tilde{\phi}_0$, $\tilde{\phi}_1$, $\tilde{\phi}_N$, $\tilde{\phi}_{N+1} \in \overset{0}{\mathcal{S}}$, taken as (independent) linear combinations of the $\phi_{-1}, \phi_0, \phi_1$ and $\phi_N, \phi_{N+1}, \phi_{N+2}$ B-splines, which are such that $\tilde{\phi}_0(a) = 0$, $\tilde{\phi}_1(a) = 0$, $\tilde{\phi}_N(b) = 0$, $\tilde{\phi}_{N+1}(b) = 0$. For example, we may take

$$\tilde{\phi}_0 := \phi_0 - 4\phi_{-1}, \quad \tilde{\phi}_1 := \phi_1 - \phi_{-1} \quad \text{etc .}$$

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Given $f \in \overset{\scriptscriptstyle 0}{H^1} \cap H^2$, we construct again an interpolant $\tilde{f} \in \overset{\scriptscriptstyle 0}{\mathcal{S}}$ satisfying $\tilde{f}(x_i) = f(x_i)$, $0 \leq i \leq N+1$, and $\tilde{f}'(x_0) = f'(x_0)$, $\tilde{f}'(x_{N+1}) = f'(x_{N+1})$, for example. The error estimates are entirely analogous. (The linear system that defines the Galerkin equations has now a seven-diagonal, banded matrix $B(\phi_i, \phi_j)$.)

One may now prove the error estimates (3.66), (3.67) for cubic splines defined on a general partition $\{x_j\}$ of $[a, b]$. In addition one may define *higher-order smooth* spline spaces: For $m \geqq 2$ define

$$\mathcal{S}_{(m)} = \left\{ \phi : \; \phi \in C^m[a, b], \; \phi\big|_{[x_i, x_{i+1}]} \in \mathbb{P}_{2m-1} \right\},$$

in which one may prove L^2 error estimates for the Galerkin approximations of $O(h^{2m})$ accuracy.

3.4   More general finite element spaces and their approximation properties

In this section we will study more general finite element spaces in comparison to the concrete ones we saw in the previous two sections. For simplicity of notation we assume that our functions are defined on the interval $I = (0, 1)$. Let r, μ be integers and suppose that $r \geq 2$ and $0 \leq \mu \leq r - 2$. Consider a partition of $\bar{I}$ defined by $0 = x_0 < x_1 < x_2 < \ldots < x_{N+1} = 1$ with $h = \max_j(x_{j+1} - x_j)$, and the associated vector spaces

$$S_h^{r,\mu} = \left\{ \phi \in C^\mu(\bar{I}), \; \phi\big|_{[x_i, x_{i+1}]} \in \mathbb{P}_{r-1}, \; 0 \leq i \leq N \right\}$$

consisting of C^μ functions on $\bar{I}$ that are piecewise polynomials of degree $r - 1$. We have already seen examples of such spaces corresponding to

$$\mu = 0, \, r = 2 : \text{Piecewise linear, continuous functions,}$$
$$\mu = 1, \, r = 4 : \text{Hermite } (C^1) \text{ piecewise cubic functions,}$$
$$\mu = 2, \, r = 4 : \text{Cubic splines } (C^2).$$

It is not hard to see (cf. Example 2.9 (ii)) that these vector spaces are finite-dimensional subspaces of $H^{\mu+1}(I)$ with $\dim S_h^{r,\mu} = (r - \mu - 1)N + r$. The subspace of $S_h^{r,\mu}$, denoted by $\overset{\scriptscriptstyle 0}{S}_h^{r,\mu}$ and consisting of the functions in $S_h^{r,\mu}$ that vanish at $x = 0$ and

$x = 1$ is a subspace of $\overset{0}{H}{}^1(I) \cap H^{\mu+1}$ of dimension $(r - \mu - 1)N + r - 2$. The spaces $S_h^{r,\mu}$ are called *finite element spaces* since they can all be equipped with basis functions of small, i.e. $O(h)$, support. Moreover, they have the following approximation property that we will call *property* Π:

Given $v \in H^s(I)$, where s an integer such that $1 \leq s \leq r$,

there exists a $\chi \in S_h^{r,\mu}$ such that

$$\|v - \chi\| + h\|v' - \chi'\| \leq Ch^s \|v\|_s, \qquad (\Pi)$$

for some constant C indepedent of v and the partition $\{x_i\}$.

We have already seen in Section 3.2 that (Π) holds for piecewise linear, continuous functions ($r = 2$, $\mu = 0$), wherein χ is e.g. the interpolant $\tilde{I}_h v$ of v defined by the conditions $(\tilde{I}_h v)(x_i) = v(x_i)$, $0 \leq i \leq N + 1$. In Section 3.3.1, in the case of Hermite cubic functions ($r = 4$, $\mu = 1$), we proved in Theorem 3.5 that (Π) holds if e.g. $\chi = \tilde{v}$, where $\tilde{v}$ is the interpolant of $v \in H^s(I)$, $s \geq 2$, satisfying $\tilde{v}(x_i) = v(x_i)$, $\tilde{v}'(x_i) = v'(x_i)$, $0 \leq i \leq N + 1$; we also established the additional H^2-estimate $\|v - \tilde{v}\|_2 \leq Ch^{s-2}\|v\|_s$, for $s = 2, 3, 4$.

We also saw in Section 3.3.2 (Theorem 3.12), in the case of cubic splines ($r = 4$, $\mu = 2$), that (Π) holds for $s = 2, 3, 4$ if e.g. χ is the cubic spline interpolant $\tilde{v}$ of $v \in H^s(I)$, defined by the conditions $\tilde{v}(x_i) = v(x_i)$, $0 \leq i \leq N + 1$, $\tilde{v}'(x_0) = v'(x_0)$, $\tilde{v}'(x_{N+1}) = v'(x_{N+1})$. In addition we proved that $\|v - \tilde{v}\|_2 \leq Ch^{s-2}\|v\|_s$ for $s = 2, 3$, or 4. (These estimates were established in Sections 3.3.1 and 3.3.2 for uniform meshes. They also hold for nonuniform meshes x_i as well.)

In the sequel we will prove (Π) for general spaces $S_h^{r,\mu}$ when $\mu = 0$ and $r \geq 2$, and when $\mu = 1$ and $r \geq 4$, for $2 \leq s \leq r$. For the general case of (Π) we refer the interested reader to the paper of C. de Boor and G.J. Fix., "Spline approximation by quasiinterpolants", J. Approx. Theory, 8(1973), 19–45.

We should also point out that the conclusion of (Π) also holds e.g. for functions $v \in H^s(I) \cap \overset{0}{H}{}^1(I)$ approximated by elements of $\overset{0}{S}{}_h^{r,\mu}$.

We may use (Π) to derive *error estimates* for the approximation of the solutions of the two-point boundary-value problems of Section 3.1 by elements of $S_h^{r,\mu}$ or its subspaces. For example, consider the two-point bvp (3.1)–(3.2), whose weak solution $u \in \overset{0}{H}{}^1$ is defined by (3.3). Let u_h be its Galerkin approximation in $\overset{0}{S}{}_h^{r,\mu}$ defined by

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the equations

$$B(u_h, \varphi) = (f, \varphi), \quad \forall \varphi \in \overset{0}{S}_h^{r,\mu}, \tag{3.68}$$

that reduce as usual to solving a $\dim \overset{0}{S}_h^{r,\mu} \times \dim \overset{0}{S}_h^{r,\mu}$ linear system of equations for the coefficients of u_h with respect to a basis $\{\varphi_i\}$ of $\overset{0}{S}_h^{r,\mu}$, with a positive-definite matrix $B(\phi_j, \phi_i)$ and right-hand side (f, φ_i). Arguing as in Theorem 3.3 and using (Π) gives the error estimate, for $1 \le s \le r$

$$\|u - u_h\| + h\|u - u_h\|_1 \le Ch^s\|u\|_s, \tag{3.69}$$

provided that $u \in H^s(I) \cap \overset{0}{H}^1(I)$.

A basic tool for proving approximation properties like (Π) is the *Bramble-Hilbert Lemma* (also called the *Deny-Lions* Lemma), whose general form in $\mathbb{R}^d$ we will see in Section 5.2. Here we will prove it in the case of a finite interval just using Taylor's theorem. (For the rest of this section we follow the exposition of analogous results in the notes by G. Akrivis, [6.4].)

Proposition 3.15. Let $k \ge 1$ be an integer and F be a nonnegative functional defined on $H^k = H^k(I)$ and satisfying

   (i) $F(v + w) \le F(v) + F(w)$, $\forall v, w \in H^k$,

   (ii) $\exists$ constant $C > 0$ such that $F(v) \le C\|v\|_k$, $\forall v \in H^k$,

   (iii) $F(p) = 0$, $\forall p \in \mathbb{P}_{k-1}$.

Then, there exists a positive constant c such that $F(v) \le c|v|_k$, $\forall v \in H^k$ where $|v|_k$ denotes the *seminorm* $|v|_k = \|v^{(k)}\|$.

Proof. Suppose that $k \ge 2$. Let $v \in H^k$ and let $p_{k-1}(x)$ be it's Taylor polynomial of degree at most $k - 1$ about $x = 0$, i.e. $p_{k-1}(x) = \sum_{j=0}^{k-1} \frac{x^j}{j!} v^{(j)}(0)$. ($p_{k-1}$ is well defined since the values of all derivatives $v^{(j)}(0)$, $0 \le j \le k - 1$, exist: Take the corresponding values of the derivatives of the C^{k-1} representative of v.) Then by the properties of F it follows that

$$F(v) \le F(v - p_{k-1}) + F(p_{k-1}) = F(v - p_{k-1}) \le C\|v - p_{k-1}\|_k.$$

Therefore, it suffices to prove that $\|v - p_{k-1}\|_k \le C'|v|_k$ for some constant C' independent of v. Using the integral form of the remainder of the Taylor polynomial we have now

$$v(x) = p_{k-1}(x) + \frac{1}{(k-1)!} \int_0^x (x-t)^{k-1} v^{(k)}(t)\, \mathrm{d}t, \quad \forall x \in \bar{I}.$$

(The usual hypothesis for this formula to hold is that $v \in C^k(\bar{I})$. More generally, it is valid if $v^{(k-1)}$ is absolutely continuous on I, which is the case if $v \in H^k$, as we may see by applying Theorem 2.12 with $u = v^{(k-1)}$.) It follows, for $x \in \bar{I}$ by the Cauchy-Schwartz inequality, that

$$|v(x) - p_{k-1}(x)|^2 \le \frac{1}{((k-1)!)^2} \int_0^x (x-t)^{2(k-1)}\, \mathrm{d}t \int_0^x \left(v^{(k)}(t)\right)^2 \mathrm{d}t$$

$$\le \frac{1}{((k-1)!)^2} \left(\int_0^1 s^{2(k-1)}\mathrm{d}s\right) \|v^{(k)}\|^2 = \frac{1}{((k-1)!)^2} \frac{1}{2k-1} \|v^{(k)}\|^2,$$

so that

$$\|v - p_{k-1}\|^2 \le \frac{1}{((k-1)!)^2} \frac{1}{2k-1} \|v^{(k)}\|^2. \tag{3.70}$$

Since $p'_{k-1}(x) = \sum_{j=1}^{k-1} \frac{x^{j-1}}{(j-1)!} v^{(j)}(0) = \sum_{j=0}^{k-2} \frac{x^j}{j!} v^{(j+1)}(0)$, it follows that $p'_{k-1}(x)$ is the Taylor polynomial of degree at most $k-2$ of $v'(x)$ about $x = 0$. In general, for $i = 0, 1, \ldots, k-1$, $p^{(i)}_{k-1}(x)$ is the Taylor polynomial of degree at most $k-1-i$ of $v^{(i)}(x)$ about $x = 0$, and in analogy to (3.70) we have

$$\|v^{(i)} - p^{(i)}_{k-1}\|^2 \le \frac{1}{((k-i-1)!)^2} \cdot \frac{1}{2(k-i)-1} \|v^{(k)}\|^2, \quad i = 0, \ldots, k-1.$$

Since $p^{(k)}_{k-1} = 0$, it follows that

$$\|v - p_{k-1}\|_k^2 \le \left[1 + \sum_{i=0}^{k-1} \frac{1}{((k-1-i)!)^2} \cdot \frac{1}{2(k-i)-1}\right] \|v^{(k)}\|^2,$$

i.e. that

$$\|v - p_{k-1}\|_k \le C_k |v|_k,$$

for a constant C_k depending only on k, which is the desired conclusion. If $k = 1$, p_0 is a constant, equal to $v(0)$. Using $v(x) = v(0) + \int_0^x v'(t)\, \mathrm{d}t$, $x \in \bar{I}$, gives the result trivially. $\qquad\square$

We remark that the Bramble-Hilbert is valid in other function spaces as well. For example it is straightforward to check that it holds if we substitute $H^k = H^k(I)$

by the Banach space $C^k(\bar{I})$ equipped with the norm $\|v\|_{C^k} = \sum_{j=0}^{k} \|v^{(j)}\|_\infty$, where $\|v\|_\infty = \max_{x \in \bar{I}} |v(x)|$ for $v \in C(\bar{I})$. The analogous seminorm $|v|_{C^k}$ is $\|v^{(k)}\|_\infty$.

The following result, whose proof follows from the Bramble-Hilbert Lemma, gives a useful bound of the L^2 error of the Lagrange interpolating polynomial of degree $r-1$ of a function $v \in H^s(a,b)$, where $r \geq s \geq 2$, and a corresponding bound of the L^2 error of its derivative.

Lemma 3.16. Let $r \geq 2$ be an integer, x_i, $1 \leq i \leq r$ be r points in an interval $[a,b]$ such that $x_i \neq x_j$, if $i \neq j$, and s be an integer satisfying $r \geq s \geq 2$. Given any $v \in H^s(a,b)$, let $p \in \mathbb{P}_{r-1}$ be its Lagrange interpolating polynomials at the points $\{x_i\}$. Then there is a constant c, independent of a, b, and v such that

$$\|v - p\|_{L^2(a,b)} \leq c\,(b-a)^s \|v^{(s)}\|_{L^2(a,b)}, \tag{3.71}$$

$$\|v' - p'\|_{L^2(a,b)} \leq c\,(b-a)^{s-1} \|v^{(s)}\|_{L^2(a,b)}. \tag{3.72}$$

Proof. Consider the affine, 1-1 map of $[a,b]$ onto $[0,1]$, defined by $x \in [a,b] \mapsto \hat{x} = \frac{x-a}{b-a} \in [0,1]$, let $\hat{x}_i \in [0,1]$ be the images of x_i under this map, and define $\hat{v}(\hat{x}) = v(x)$, $\hat{p}(\hat{x}) = p(x)$. Then $\hat{v} \in H^s(0,1)$ and $\hat{p}(\hat{x})$ is its Lagrange interpolating polynomial of degree $\leq r-1$ at the points $\hat{x}_i$, $1 \leq i \leq r$. Let $F : H^s(0,1) \to [0,\infty)$ be defined by $F(\hat{v}) := \|\hat{v} - \hat{p}\|_{L^2(0,1)}$ for $\hat{v} \in H^s(0,1)$. Then F satisfies the hypotheses of the Bramble-Hilbert Lemma for $k = s$. To see this, note that if $\hat{q} \in \mathbb{P}_{r-1}$ is the Lagrange interpolating polynomial at $\{\hat{x}_i\}$ of a function $\hat{w} \in H^s(0,1)$, by the uniqueness of the Lagrange interpolating polynomial we conclude that $\hat{\pi} = \hat{p} + \hat{q}$ is the Lagrange interpolating polynomial of degree $\leq r-1$ of $\hat{v} + \hat{w}$. Hence

$$F(\hat{v} + \hat{w}) = \|\hat{v} + \hat{w} - \hat{\pi}\|_{L^2(0,1)} = \|\hat{v} + \hat{w} - (\hat{p} + \hat{q})\|_{L^2(0,1)}$$

$$\leq \|\hat{v} - \hat{p}\|_{L^2(0,1)} + \|\hat{w} - \hat{q}\|_{L^2(0,1)} = F(\hat{v}) + F(\hat{w}),$$

i.e. (i) in Proposition 3.15 is satisfied. Let now $\hat{\theta}$ be any polynomial in $\mathbb{P}_{s-1}$ and let $\hat{\pi} \in \mathbb{P}_{r-1}$ be its Lagrange interpolating polynomial at $\{\hat{x}_i\}$. Since $s \leq r$, by the uniqueness of $\hat{\pi}$ it follows that $\hat{\pi} = \hat{\theta}$. Hence $F(\hat{\theta}) = \|\hat{\theta} - \hat{\pi}\|_{L^2(0,1)} = 0$, and (iii) holds. To verify (ii) recall that $\hat{p}$ may be represented as

$$\hat{p}(\hat{x}) = \sum_{i=1}^{r} \hat{v}(\hat{x}_i)\hat{L}_i(\hat{x}),$$

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where the Lagrange cardinal polynomials $\hat{L}_i \in \mathbb{P}_{r-1}$ satisfy $\hat{L}_i(\hat{x}_j) = \delta_{ij}$, $1 \le i, j \le r$. It follows that

$$\|\hat{p}\|_{L^2(0,1)} \le \sum_{i=1}^{r} |\hat{v}(\hat{x}_i)| \|\hat{L}_i\|_{L^2(0,1)} \le C_{\text{Sob}} \left(\sum_{i=1}^{r} \|\hat{L}_i\|_{L^2(0,1)} \right) \|\hat{v}\|_{H^1(0,1)},$$

where C_{Sob} is the constant in Sobolev's inequality on $[0,1]$. (By Remark 2.19 (i), $C_{\text{Sob}} \le \sqrt{2}$.) Hence

$$
\begin{aligned}
F(\hat{v}) = \|\hat{v} - \hat{p}\|_{L^2(0,1)} &\le \|\hat{v}\|_{L^2(0,1)} + \|\hat{p}\|_{L^2(0,1)} \\
&\le \left[1 + C_{\text{Sob}} \sum_{i=1}^{r} \|\hat{L}_i\|_{L^2(0,1)} \right] \|\hat{v}\|_{H^1(0,1)} \\
&\le C \|\hat{v}\|_{H^s(0,1)},
\end{aligned}
$$

where the constant $C = 1 + C_{\text{Sob}} \sum_{i=1}^{r} \|\hat{L}_i\|_{L^2(0,1)}$ depends only on the $\hat{x}_i$. We conclude that (ii) in Proposition 3.15 holds, and so there exists a constant c independent of v such that

$$\|\hat{v} - \hat{p}\|_{L^2(0,1)} \le c \|\hat{v}^{(s)}\|_{L^2(0,1)}, \tag{3.73}$$

where $\hat{v}^{(s)}(\hat{x}) = \left(\frac{\mathrm{d}}{\mathrm{d}\hat{x}} \right)^s \hat{v}(\hat{x})$. Now, by the change of variables $x \mapsto \hat{x}$ and (3.73) (i.e. by a "scaling" argument), we have

$$
\begin{aligned}
\|v - p\|^2_{L^2(0,1)} = \int_a^b (v(x) - p(x))^2 \, \mathrm{d}x &= (b-a) \int_0^1 (\hat{v}(\hat{x}) - \hat{p}(\hat{x}))^2 \, \mathrm{d}\hat{x} \le c^2 (b-a) \|\hat{v}^{(s)}\|^2_{L^2(0,1)} \\
&= \frac{b-a}{b-a} c^2 \int_a^b (b-a)^{2s} \left(\left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^s v(x) \right)^2 \mathrm{d}x = c^2 (b-a)^{2s} \|v^{(s)}\|^2_{L^2(a,b)},
\end{aligned}
$$

which gives (3.71). The inequality (3.72) follows in a similar way by considering the functional $\hat{v} \mapsto \|\hat{v}' - \hat{p}'\|_{L^2(0,1)}$. $\qquad \square$

The estimates (3.71), (3.72) also hold for a Hermite-type interpolating polynomial. Here is a result along this direction.

Lemma 3.17. Let $r \ge 4$ be an integer and let $a = x_1 < x_2 < \ldots < x_{r-2} = b$. If s is an integer such that $r \ge s \ge 2$ and $v \in H^s(a,b)$, consider the Hermite-type interpolating polynomial $p \in \mathbb{P}_{r-1}$ that satisfies

$$p(x_i) = v(x_i), \quad 1 \le i \le r-2, \quad p'(x_i) = v'(x_i), \quad i = 1, r-2, \tag{3.74}$$

Then, the estimates (3.71), (3.72) hold in this case as well.

Proof. The polynomial p in the statement of the lemma exist and is unique: The conditions (3.74) form a $r \times r$ system of linear equations for the coefficients of p. The associated homogeneous system has only the trivial solution, since, otherwise, $p \in \mathbb{P}_{r-1}$ would have the simple roots x_i, $2 \le i \le r-3$, and the double roots x_1, x_{r-2}, i.e. a total of r roots counting multiplicities, which is impossible. Hence the nonhomogeneous system defined by (3.74) has a unique solution.

Define now the polynomials $L_i \in \mathbb{P}_{r-1}$, $0 \le i \le r-1$, by the conditions $L_i(x_j) = \delta_{ij}$, $0 \le i \le r-1$, $0 \le j \le r-2$, $L_i'(x_1) = \delta_{i0}$, $L_i'(x_{r-2}) = \delta_{i,r-1}$, $0 \le i \le r-1$. It is easily seen that $p(x)$ may be represented as

$$p(x) = \sum_{i=1}^{r-2} v(x_i) L_i(x) + v'(x_1) L_0(x) + v'(x_{r-2}) L_{r-1}(x),$$

since the right-hand side is a polynomial of degree $\le r-1$ satisfying the condition (3.74). Consider again the map $\hat{x} \mapsto x = a + (b-a)\hat{x}$ of $[0,1]$ onto $[a,b]$, and let $\hat{x}_i \in [0,1]$ be the inverse images of the x_i. Consider the change of dependent variable $\hat{w}(\hat{x}) = w(x)$ for a function w defined on $[a,b]$. Then, if $\hat{p}$ is the transformed p, it is given by

$$\hat{p}(\hat{x}) = \sum_{i=1}^{r-2} \hat{v}(\hat{x}_i) \hat{L}_i(\hat{x}) + \hat{v}'(\hat{x}_1) \hat{L}_0(\hat{x}) + \hat{v}'(\hat{x}_{r-2}) \hat{L}_{r-1}(\hat{x}),$$

where now $'$ denotes $\frac{\mathrm{d}}{\mathrm{d}x}$. Consider, as in the proof of Lemma 3.16, the functional $F : H^s(0,1) \to [0,\infty)$ given by $F(\hat{v}) = \|\hat{v} - \hat{p}\|_{L^2(0,1)}$. It is easily seen, as in the proof of Lemma 3.16, that F satisfies the conditions (i), (iii) of the Bramble-Hilbert Lemma (Proposition 3.15) for $k = s$. Moreover,

$$\|\hat{p}\|_{L^2(0,1)} \le \sum_{i=1}^{r-2} |\hat{v}(\hat{x}_i)| \|\hat{L}_i\|_{L^2(0,1)} + |\hat{v}'(\hat{x}_1)| \|\hat{L}_0\|_{L^2(0,1)}$$

$$+ |\hat{v}'(\hat{x}_{r-2})| \|\hat{L}_{r-1}\|_{L^2(0,1)} \le C_{\text{Sob}} \left(\sum_{i=0}^{r-2} \|\hat{L}_i\|_{L^2(0,1)} \right) \|\hat{v}\|_{H^2(0,1)},$$

where we denoted again by C_{Sob} the constant in Sobolev's inequality on $[0,1]$. It follows, since $r \ge 2$, that

$$F(\hat{v}) = \|\hat{v} - \hat{p}\|_{L^2(0,1)} \le \|\hat{v}\|_{L^2(0,1)} + \|\hat{p}\|_{L^2(0,1)}$$

$$\le C\|\hat{v}\|_{H^s(0,1)},$$

where C depends only on the $\{\hat{x}_i\}$. Thus (iii) is satisfied as well. Using the conclusion of Proposition 3.15 and continuing as in the proof of Lemma 3.16 we obtain again (3.71) and (3.72) *mutatis mutandis*. □

We now have the ingredients to prove the approximation property (Π) under some restrictions on μ and r mentioned in the beginning of the section.

Theorem 3.18. Let $r \geq 2$ be an integer and suppose that $\mu = 0$ or that $\mu = 1$ and $r \geq 4$. Let $0 = x_0 < x_1 < \ldots < x_{N+1} = 1$, $I = (0,1)$, and suppose that

$$S_h^{r,\mu} = \left\{ \phi \in C^\mu(\bar{I}),\ \phi\big|_{[x_i,x_{i+1}]} \in \mathbb{P}_{r-1},\ 0 \leq i \leq N \right\}, \quad h = \max_i (x_{i+1} - x_i).$$

If s is an integer such that $2 \leq s \leq r$ and $v \in H^s(I)$, there exists a function $\chi \in S_h^{r,\mu}$ such that

$$\|v - \chi\| + h\|v' - \chi'\| \leq Ch^s\|v^{(s)}\|, \tag{3.75}$$

for some constant C independent of v and h.

Proof. (i) Let $\mu = 0$, $r \geq 2$. Let $0 = \tau_1 < \tau_2 < \ldots < \tau_r = 1$ and $h_i = x_{i+1} - x_i$, $0 \leq i \leq N$. Put $x_{ij} = x_i + h_i \tau_j$, $0 \leq i \leq N$, $1 \leq j \leq r$, and for $i = 0, 1, \ldots, N$, let $p_i : [x_i, x_{i+1}] \to \mathbb{R}$ be the Lagrange interpolating polynomial of degree $\leq r - 1$ of the function v at the points x_{ij}, $j = 1, \ldots, r$, thus satisfying $p_i(x_{ij}) = v(x_{ij})$, $1 \leq j \leq r$. Let χ be the piecewise polynomial function on $[0,1]$ defined by $\chi(x) = p_i(x)$ if $x \in [x_i, x_{i+1}]$. Then $\chi \in S_h^{r,0}$ since $\chi\big|_{[x_i,x_{i+1}]} = p_i \in \mathbb{P}_{r-1}$ and $\chi \in C^0(\bar{I})$. (To check the continuity of χ at the interior nodes x_i, $1 \leq i \leq N$, note that $\chi(x_i^-) = p_{i-1}(x_i) = v(x_i) = p_i(x_i) = \chi(x_i^+)$. Note that in particular χ is an interpolant of v in $S_h^{r,0}$ at $x = x_i$, $0 \leq i \leq N+1$). By Lemma 3.16 and (3.71) we have for $0 \leq i \leq N$

$$\|v - \chi\|^2_{L^2(x_i,x_{i+1})} = \|v - p_i\|^2_{L^2(x_i,x_{i+1})} \leq c^2 h_i^{2s}\|v^{(s)}\|^2_{L^2(x_i,x_{i+1})} \leq c^2 h^{2s}\|v^{(s)}\|^2_{L^2(x_i,x_{i+1})},$$

for some constant c independent of h and v. Summing over i we conclude that $\|v - \chi\|^2 = \|v - \chi\|^2_{L^2(0,1)} \leq c^2 h^{2s}\|v^{(s)}\|^2$, i.e. $\|v - \chi\| \leq ch^s\|v^{(s)}\|$. Using (3.72) we may similarly conclude that $\|v' - \chi'\| \leq ch^{s-1}\|v^{(s)}\|$ and (3.75) follows.

(ii) Let $\mu = 1$ and $r \geq 4$. Suppose now that $0 \leq \tilde{\tau}_1 < \tilde{\tau}_2 < \ldots < \tilde{\tau}_{r-2} = 1$ and let $x_{ij} = x_i + h_i \tilde{\tau}_j$, $0 \leq i \leq N$, $1 \leq j \leq r - 2$. For $i = 0, 1, \ldots, N$ let $\tilde{p}_i : [x_1, x_{i+1}] \to \mathbb{R}$ be the Hermite-type interpolating polynomial of degree $\leq r - 1$ of v satisfying the

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conditions $\tilde{p}_i(x_{ij}) = v(x_{ij})$, $1 \leq j \leq r - 2$, and $\tilde{p}'_i(x_{ij}) = v'(x_{ij})$, $j = 1, r - 2$. Let χ be the piecewise polynomial function on $[0, 1]$ defined by $\chi(x) = \tilde{p}_i(x)$, if $x \in [x_i, x_{i+1}]$. Then $\chi \in S_h^{r,1}$ since $\chi\big|_{[x_i, x_{i+1}]} = \tilde{p}_i \in \mathbb{P}_{r-1}$, and $\chi \in C^1(\bar{I})$. (Note that $\chi(x_i) = v(x_i)$ and $\chi'(x_i) = v'(x_i)$, $0 \leq i \leq N + 1$, i.e. that χ is a Hermite-type interpolant of v in $S_h^{r,1}$). From Lemma 3.17 we conclude now as in the case (i) that (3.75) holds in this case as well. $\qquad\square$

Remark 3.19. As noted in the proof of Theorem 3.18 the functions χ that were constructed in both cases (i) and (ii) are *interpolants* of v at x_i in the respective finite element spaces $S_h^{r,\mu}$. Moreover these interpolants are defined *locally* in each $[x_i, x_{i+1}]$, i.e. $\chi\big|_{[x_i, x_{i+1}]}$ is independent of the values of v outside the interval $[x_i, x_{i+1}]$. Contrast with the cubic spline interpolant (an element of $S_h^{4,2}$) defined in section 3.3.2, which is *nonlocal*, since at each point x it depends on the values of v at all x_i as e.g. the relations (3.59) imply. $\qquad\square$

Remark 3.20 (*Higher-order local approximation*)**.** With notation as in the proof of Lemma 3.16 consider more generally the functionals $\hat{v} \mapsto \|\hat{v}^{(j)} - \hat{p}^{(j)}\|_{L^2(0,1)}$ for $0 \leq j \leq s$, where $\hat{p}$ is the Lagrange interpolating polynomial of $\hat{v}$, of degree $\leq r - 1$, at the points $\hat{x}_i$, $1 \leq i \leq r$. It is straightforward to see that they satisfy the hypotheses of the Bramble-Hilbert Lemma (Proposition 3.15) for $k = s$. Therefore, $\|\hat{v}^{(j)} - \hat{p}^{(j)}\|_{L^2(0,1)} \leq C\|\hat{v}^{(s)}\|_{L^2(0,1)}$ for $0 \leq j \leq s$, and the change of variables $x \mapsto \hat{x}$ allows us to generalize the estimates (3.71) and (3.72) and prove

$$\|v^{(j)} - p^{(j)}\|_{L^2(a,b)} \leq c(b-a)^{s-j}\|v^{(s)}\|_{L^2(a,b)}, \quad 0 \leq j \leq s. \tag{3.76}$$

Arguing in a similar fashion we may extend the proof of Lemma 3.17 and prove that (3.76) also holds if $v \in H^s(a, b)$ and $p \in \mathbb{P}_{r-1}$ is the Hermite-type interpolating polynomial of v satisfying the condition (3.74).

From these observations it follows that one may derive *higher-order local error estimates* (i.e. in every interval $[x_i, x_{i+1}]$) for the approximations χ constructed in the course of the proof of Theorem 3.18. Specifically, in the case $\mu = 0$, we have for $2 \leq s \leq r$, using the notation established in Theorem 3.18 and (3.76), the *local* approximation

estimate

$$\|v^{(j)} - \chi^{(j)}\|^2_{L^2(x_i, x_{i+1})} = \|v^{(j)} - p_i^{(j)}\|^2_{L^2(x_i, x_{i+1})} \le Ch_i^{2(s-j)}\|v^{(s)}\|^2_{L^2(x_i, x_{i+1})}$$

$$\le Ch^{2(s-j)}\|v^{(s)}\|^2_{L^2(x_i, x_{i+1})}, \quad 0 \le j \le s. \quad (3.77)$$

It follows that

$$\left(\sum_{i=0}^{N} \|v^{(j)} - \chi^{(j)}\|^2_{L^2(x_i, x_{i+1})}\right)^{1/2} \le Ch^{s-j}\|v^{(s)}\|, \quad 0 \le j \le s, \quad (3.78)$$

which generalizes (3.75) in the higher-order cases $j = 2, \ldots, s$, if we use the *mesh-dependent norm* $\left(\sum_{j=0}^{N} \|v^{(j)} - \chi^{(j)}\|^2_{L^2(x_i, x_{i+1})}\right)^{1/2}$. In the case $\mu = 1$ it follows from (3.76) *mutatis mutandis* that (3.77) and (3.78) hold for the Hermite polynomial spaces as well. Note that in this case for $j = 2$, $\left(\sum_{i=0}^{N} \|v^{(2)} - \chi^{(2)}\|^2_{L^2(x_i, x_{i+1})}\right)^{1/2} = \|v^{(2)} - \chi^{(2)}\|_{L^2(a,b)}$, since $\chi \in S_h^{r,1} \subset H^2(a,b)$, so that, in addition to (3.75), we have in the Hermite case $\mu = 1$, $r \ge 4$

$$\sum_{j=0}^{2} h^j \|v - \chi\|_j \le ch^s \|v^{(s)}\|, \quad 2 \le s \le r. \quad (3.79)$$

It may be shown that (3.79) holds for the general case of $S_h^{r,\mu}$, $\mu \ge 2$ as well; the approximation χ is now a *quasiinterpolant* of v, cf. the work of de Boor and Fix previously cited. $\qquad \square$

We conclude this section by deriving *inverse inequalities* in the finite element spaces $S_h^{r,\mu}$. Consider first a finite interval (λ, μ), with $\mu - \lambda < 1$, and let $\mathbb{P}_k(\lambda, \mu)$ denote the polynomials of degree $\le k$ on (λ, μ). Then, there exists a positive constant C_k depending only on k such that

$$\|\varphi\|_{H^1(\lambda, \mu)} \le \frac{C_k}{\mu - \lambda}\|\varphi\|_{L^2(\lambda, \mu)}, \quad \forall \varphi \in \mathbb{P}_k(\lambda, \mu). \quad (3.80)$$

The inequality (3.80) is an example of an *inverse inequality* since it bounds above $\|\varphi\|_{H^1(\lambda, \mu)}$ in terms of a weaker norm $\|\varphi\|_{L^2(\lambda, \mu)}$; of course this holds for a special class of functions (here $\mathbb{P}_k(\lambda, \mu)$), and at the expense of a multiplicative constant $\frac{C_k}{\mu - \lambda}$ that grows as $\mu - \lambda$ diminishes. (Recall that the reverse, trivial inequality $\|\varphi\|_{L^2(\lambda, \mu)} \le \|\varphi\|_{H^1(\lambda, \mu)}$ holds for all $\varphi \in H^1(\lambda, \mu)$ on any (λ, μ).) To prove (3.80), consider the polynomial space $\mathbb{P}_k(0, 1)$. Then there exists a constant C_k such that

$$\|\hat{\varphi}\|_{H^1(0,1)} \le C_k \|\hat{\varphi}\|_{L^2(0,1)}, \quad \forall \hat{\varphi} \in \mathbb{P}_k(0, 1), \quad (3.81)$$

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since $\mathbb{P}_k(0,1)$ is a finite-dimensional space and consequently all norms on it are equivalent. Then (3.80) follows by the following *scaling* argument: Consider the change of variables $\hat{x} \mapsto x = (\mu - \lambda)\hat{x} + \lambda$, that maps $[0,1]$ 1-1 onto the interval $[\lambda, \mu]$, and let $\varphi(x) = \hat{\varphi}(\hat{x})$ for $\hat{x} \in [0,1]$ and $x = (\mu - \lambda)\hat{x} + \lambda$. Then $\varphi \in \mathbb{P}_k(\lambda, \mu)$ if and only if $\hat{\varphi} \in \mathbb{P}_k(0,1)$. Changing variables in (3.81) i.e. in

$$\int_0^1 \Big[(\hat{\varphi}(\hat{x}))^2 + \Big(\frac{\mathrm{d}}{\mathrm{d}\hat{x}}\hat{\varphi}(\hat{x})\Big)^2\Big]\mathrm{d}\hat{x} \le C_k^2 \int_0^1 (\hat{\varphi}(\hat{x}))^2 \, \mathrm{d}\hat{x},$$

for $\hat{\varphi} \in \mathbb{P}_k(0,1)$, since $\mathrm{d}\hat{x} = \frac{\mathrm{d}x}{\mu - \lambda}$ and $\frac{\mathrm{d}\hat{\varphi}}{\mathrm{d}\hat{x}} = (\mu - \lambda)\frac{\mathrm{d}\varphi}{\mathrm{d}x}$, we have

$$\int_\lambda^\mu \Big[\varphi^2(x) + (\mu - \lambda)^2\Big(\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x)\Big)^2\Big]\mathrm{d}x \le C_k^2 \int_\lambda^\mu \varphi^2(x) \, \mathrm{d}x.$$

Since $\mu - \lambda < 1$ this gives

$$(\mu - \lambda)^2 \|\varphi\|_{H^1(\lambda,\mu)}^2 \le C_k^2 \|\varphi\|_{L^2(\lambda,\mu)}^2, \quad \forall \varphi \in \mathbb{P}_k(\lambda, \mu),$$

from which (3.80) follows.

The inverse inequality (3.80) may be generalized in various ways. For example, if $m \ge 1$ is an integer we have

$$\|\varphi\|_{H^m(\lambda,\mu)} \le \frac{C_{k,m}}{(\mu - \lambda)^m} \|\varphi\|_{L^2(\lambda,\mu)}, \quad \forall \varphi \in \mathbb{P}_k(\lambda, \mu), \tag{3.82}$$

where $C_{k,m}$ is a positive constant depending only on k and m. The proof of (3.82) follows along the lines of the proof of (3.80) if we note that the change of variables $x = (\mu - \lambda)\hat{x} + \lambda$, $\hat{\varphi}(\hat{x}) = \varphi(x)$ implies that $\frac{\mathrm{d}^j \hat{\varphi}}{\mathrm{d}\hat{x}^j} = (\mu - \lambda)^j \frac{\mathrm{d}^j \varphi}{\mathrm{d}x^j}$. We also have e.g. that

$$\|\varphi\|_{H^2(\lambda,\mu)} \le \frac{C_k'}{\mu - \lambda} \|\varphi\|_{H^1(\lambda,\mu)}, \quad \forall \phi \in \mathbb{P}_k(\lambda, \mu), \tag{3.83}$$

where C_k' a positive constant depending only on k. To see this, note that if $\varphi \in \mathbb{P}_k(\lambda, \mu)$ with $k \ge 1$, then $\varphi' \in \mathbb{P}_{k-1}(\lambda, \mu)$ and it follows from (3.80) that $\|\varphi'\|_{H^1(\lambda,\mu)} \le \frac{C_{k-1}}{\mu - \lambda} \|\varphi'\|_{L^2(\lambda,\mu)}$, i.e. that $\|\varphi'\|_{L^2(\lambda,\mu)}^2 + \|\varphi''\|_{L^2(\lambda,\mu)}^2 \le \frac{C_{k-1}}{(\mu - \lambda)^2} \|\varphi'\|_{L^2(\lambda,\mu)}$, from which we get

$$\|\varphi\|_{H^2(\lambda,\mu)}^2 \le \frac{C_{k-1}^2}{(\mu - \lambda)^2} \|\varphi'\|_{L^2(\lambda,\mu)}^2 + \|\varphi\|_{L^2(\lambda,\mu)}^2 \le \max\Big(\frac{C_{k-1}^2}{(\mu - \lambda)^2}, 1\Big) \|\varphi\|_{H^2(\lambda,\mu)}^2.$$

Since we may always suppose that $C_{k-1} \ge 1$, we have $\frac{C_{k-1}}{\mu - \lambda} > 1$ and (3.83) follows. Similar considerations lead to the general inverse inequality

$$\|\varphi\|_{H^m(\lambda,\mu)} \le \frac{C_{m,\ell,k}}{(\mu - \lambda)^{m-\ell}} \|\varphi\|_{H^\ell(\lambda,\mu)}, \quad \forall \varphi \in P_k(\lambda, \mu), \tag{3.84}$$

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where $m \geq \ell \geq 0$, and $C_{m,\ell,k}$ is a positive constant depending only on m, ℓ, and k.

A different type of inverse inequality is

$$\|\varphi\|_{L^\infty(\lambda,\mu)} \leq \frac{C_k}{(\mu-\lambda)^{1/2}} \|\varphi\|_{L^2(\lambda,\mu)}, \quad \forall \varphi \in \mathbb{P}_k(\lambda,\mu). \tag{3.85}$$

This follows again from the change of variables $x = (\mu-\lambda)\hat{x} + \lambda$, $\varphi(x) = \hat{\varphi}(\hat{x})$, and the inequality

$$\|\hat{\varphi}\|_{L^\infty(0,1)} \leq C_k \|\hat{\varphi}\|_{L^2(0,1)}, \quad \forall \hat{\varphi} \in \mathbb{P}_k(0,1),$$

i.e. the equivalence of the L^2 and L^∞ norms on $\mathbb{P}_k(0,1)$, since

$$\max_{\lambda \leq x \leq \mu} |\varphi(x)| = \max_{0 \leq \hat{x} \leq 1} |\hat{\phi}(\hat{x})| \leq C_k \|\hat{\varphi}\|_{L^2(0,1)} \leq \frac{C_k}{(\mu-\lambda)^{1/2}} \|\varphi\|_{L^2(\lambda,\mu)}.$$

The inequality (3.85) may be easily generalized to

$$\|\varphi\|_{C^j(\lambda,\mu)} \leq \frac{C_{k,j}}{(\mu-\lambda)^{j+1/2}} \|\varphi\|_{L^2(\lambda,\mu)}, \quad \forall \varphi \in \mathbb{P}_k(\lambda,\mu), \tag{3.86}$$

where, for $j \geq 0$ integer, $\|\varphi\|_{C^j} = \sum_{i=0}^{j} \|\varphi^{(i)}\|_{L^\infty}$, with $\|\varphi\|_{C^0} = \|\varphi\|_{L^\infty}$.

We may now convert these inequalities to inverse inequalities in finite element spaces. Consider the partition $0 = x_0 < x_1 < \ldots < x_{N+1} = 1$ of e.g. $[0,1]$ and let $h_i = x_{i+1} - x_i$ with $h = \max_i h_i$. Let $r \geq 1$ and let S_h be a finite element space consisting of piecewise polynomials of degree $\leq r-1$ on each subinterval $[x_i, x_{i+1}]$. Then, e.g. the inequality (3.84) gives, for $0 \leq \ell \leq m$ and a constant $C = C_{m,\ell,r}$ the *local inverse inequality*

$$\|\varphi\|_{H^m(I_i)} \leq \frac{C}{h_i^{m-\ell}} \|\varphi\|_{H^\ell(I_i)}, \quad 0 \leq i \leq N, \ \forall \varphi \in S_h, \tag{3.87}$$

from which it follows that

$$\left(\sum_{i=0}^{N} \|\varphi\|_{H^m(I_i)}^2\right)^{1/2} \leq C \left(\sum_{i=0}^{N} \frac{1}{h_i^{2(m-\ell)}} \|\varphi\|_{H^\ell(I_i)}^2\right)^{1/2}, \quad \forall \varphi \in S_h, \tag{3.88}$$

which is an $H^m - H^\ell$ type inverse inequality on S_h for the *mesh-dependent norms* that appear in both sides of (3.88). If the mesh x_i is *quasiuniform*, i.e. if there exists a constant $\sigma > 0$ such that

$$\frac{h}{h_i} \leq \sigma, \quad 0 \leq i \leq N, \tag{3.89}$$

then from (3.88) it follows that

$$\left(\sum_{i=0}^{N} \|\varphi\|_{H^m(I_i)}^2\right)^{1/2} \leq \frac{C'}{h^{m-\ell}} \left(\sum_{i=0}^{N} \|\varphi\|_{H^\ell(I_i)}^2\right)^{1/2}, \tag{3.90}$$

where $C' = C\sigma^{m-\ell}$. From (3.90) we see that if $S_h \subset H^m(0,1)$ (if $S_h = S_h^{r,j}$ this holds for $m = j + 1$), then we may convert this inequality into one with global norms, i.e. into

$$\|\varphi\|_{H^m(0,1)} \leq \frac{C'}{h^{m-\ell}}\|\varphi\|_{H^\ell(0,1)}, \quad \forall \varphi \in S_h, \tag{3.91}$$

where $0 \leq \ell \leq m$. In a similar manner we get from (3.86) that if $S_h \subset C^j(0,1)$, $\mu \geq 0$ and (3.89) holds, then

$$\|\varphi\|_{C^j(0,1)} \leq \frac{C_{r,j}}{h^{j+1/2}}\|\varphi\|_{L^2(0,1)}, \quad \forall \varphi \in S_h. \tag{3.92}$$

We will use the inverse inequalities in the next section and also in stability conditions for explicit fully discrete schemes for time-dependent problems in Chapters 6 and 7.

3.5   Superconvergence and effect of quadrature

In this section we will make an introductory study of two important issues associated with the Galerkin finite element method for two-point boundary-value problems, namely of the phenomenon of *superconvergence*, i.e. the convergence of the error of the Galerkin approximation u_h at a higher than optimal rate at specific points, and of the effect of *numerical integration* with a *quadrature rule*; this is used in practice to approximate the integrals in the elements of the stiffness and mass matrix in the case of variable coefficients, and also in the right-hand side of the linear system defining the finite element method, cf. e.g. the end of Remark 3.4 (iv). To fix ideas we consider again the problem (3.1), (3.2) on the interval $I = (0,1)$, that we rewrite here for convenience

$$\begin{aligned} -(p(x)u')' + q(x)u &= f(x), \quad x \in I, \\ u(0) = u(1) &= 0, \end{aligned} \tag{3.93}$$

assuming as usual that $p \in C^2(\bar{I})$ with $p(x) \geq \alpha \geq 0$, $x \in \bar{I}$, and $q \in C(\bar{I})$ with $q \geq 0$ on $\bar{I}$. Let $0 < x_0 < x_1 \ldots < x_{N+1} = 1$ be a partition of $[0,1]$ with $h = \max_i(x_{i+1} - x_i)$ and for $r \geq 2$ consider the finite element space $S_h = \overset{0}{S}_h^{r,0}$ (in the notation of section 3.4), i.e. the space

$$S_h = \left\{ \varphi \in C^0(\bar{I}), \; \varphi\big|_{[x_i,x_{i+1}]} \in \mathbb{P}_{r-1}, \; 0 \leq i \leq N, \; \varphi(0) = \varphi(1) = 0 \right\}.$$

Recall that S_h satisfies the approximation property (Π) of section 3.4; in (Π) we suppose that $v \in H^s \cap \overset{0}{H}{}^1$ because of the b.c. in (3.93). Let $u_h \in S_h$ be the Galerkin approximation of the solution of (3.93), defined as usual by

$$(pu_h', \varphi') + (qu_h, \varphi) = (f, \varphi), \quad \forall \varphi \in S_h. \tag{3.94}$$

As a consequence of (Π), u_h satisfies the error estimate

$$\|u - u_h\| + h\|u - u_h\|_1 \leq Ch^r\|u\|_r, \tag{3.95}$$

provided $u \in H^r \cap \overset{0}{H}{}^1$.

3.5.1 Superconvergence at the meshpoints

In addition to the error estimate (3.95) of u_h one may also derive error estimates in the *maximum norm*. For example, as shown in the papers by Wheeler[1] and Douglas *et al*[2], if $u \in C^r[0,1]$ and $e = u - u_h$, then

$$\max_{0 \leq x \leq 1} |e(x)| \leq Ch^r \max_{0 \leq x \leq 1} |u^{(r)}(x)|. \tag{3.96}$$

In addition, one may prove that this error estimate is of optimal order, i.e. that the power r of h^r cannot be taken larger in general.

However, it has been observed that there exist points $\xi \in \bar{I}$ where the error $e(\xi)$ is of higher order of accuracy than $\mathcal{O}(h^r)$. As an example, we will prove in this section that, under our hypotheses of S_h, at the meshpoints x_i of the partition it holds that $|e(x_i)| \leq Ch^{2r-2}\|u\|_r$, where C is a constant independent of h and u. Hence, if $r \geq 3$, i.e. if we use piecewise quadratic or higher degree polynomials, we will obtain increased accuracy at the mesh points. This is an instance of superconvergence of the method, in this case at the meshpoints. For the proof of this result we will follow Douglas and Dupont[3]. The reader who is interested in superconvergence for general finite element spaces in one and higher dimensions is referred to the monograph [3.8] by L. Wahlbin.

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[1]M. F. Wheeler, An optimal L_∞ error estimate for Galerkin approximations to solutions of two point boundary value problems, SIAM J. Num. Anal., 10(1973), 914–917.

[2]J. Douglas, Jr., T. Dupont, and L.Wahlbin, Optimal L_∞ error estimates for Galerkin approximations to solutions of two-point boundary value problems, Math. Comp., 29(1975), 475–483.

[3]J. Douglas, Jr., and T. Dupont, Galerkin approximations for the two point boundary value problem using continuous, piecewise polynomial spaces, Numer. Math., 22(1974), 99–109.

Proposition 3.21. With the notation introduced above, if p and q are sufficiently smooth, then

$$\max_{0 \le i \le N+1} |e(x_i)| \le Ch^{2r-2} \|u\|_r. \tag{3.97}$$

Proof. We first recall a few well-known facts from o.d.e. theory. Let u_1, u_2 be two linearly independent solutions of the homogeneous equation $-(pu')' + qu = 0$ satisfying $u_1(0) = 0$, $u_2(1) = 0$, and let $W(x) = u_1(x)u_2'(x) - u_1'(x)u_2(x)$ be their Wronskian. Then, $W(x) \ne 0$, $x \in \bar{I}$. In addition let

$$G(x, \xi) = \begin{cases} \frac{u_1(x)u_2(\xi)}{-p(\xi)W(\xi)}, & 0 \le x \le \xi \le 1, \\ \frac{u_2(x)u_1(\xi)}{-p(\xi)W(\xi)}, & 0 \le \xi \le x \le 1, \end{cases}$$

be the associated Green's function. Note that G is continuous for $0 \le x, \xi \le 1$ and that for each $x \in [0, 1]$ $G(x, \cdot) \in \overset{0}{H}{}^1$ since $G(x, \cdot)$ is continuously differentiable in $[0, x]$ and $[x, 1]$ and satisfies $G(x, 0) = G(x, 1) = 0$. Moreover, if $f \in C(\bar{I})$, the classical solution of (3.93) is given by the integral representation

$$u(x) = \int_0^1 G(x, \xi)f(\xi)\,\mathrm{d}\xi, \quad x \in \bar{I}. \tag{3.98}$$

It is also straightforward to prove that for each x the integral in the right-hand side of (3.98) exists if $f \in L^2(I)$ and, in that case, the function u given by (3.98) is the weak solution, belonging to $H^2 \cap \overset{0}{H}{}^1$, of the bvp (3.93). Moreover for $f \in L^2$ we have

$$\begin{aligned} u(x) &= \int_0^1 G(x, \xi)f(\xi)\,\mathrm{d}\xi = \text{(since the ode } -p(u')' + qu = f \text{ holds in } L^2) \\ &= \int_0^1 G(x, \xi)(-(pu')' + qu)(\xi)\,\mathrm{d}\xi = \text{ (by integraion by parts)} \\ &= \int_0^1 p(\xi)G_\xi(x, \xi)u'(\xi)\,\mathrm{d}\xi + \int_0^1 q(\xi)G(x, \xi)u(\xi)\,\mathrm{d}\xi. \end{aligned}$$

Therefore for $x \in \bar{I}$

$$u(x) = B(u, G(x, \cdot)), \tag{3.99}$$

where $B(v, w) = \int_0^1 (pv'w' + qvw)\,\mathrm{d}x$ for $v, w \in \overset{0}{H}{}^1$. The representation (3.99) holds for *any* $u \in H^2 \cap \overset{0}{H}{}^1$: Let $u \in H^2 \cap \overset{0}{H}{}^1$ be given and define g by the equation $-(pu')' + qu = g$. Then $g \in L^2$ and u is the weak solution of the bvp $-(pu')' + qu = g$, $u(0) = u(1) = 0$, which leads to its representation by (3.99). In fact (3.99) golds for *any* $u \in \overset{0}{H}{}^1$. (The right-hand side of (3.99) exists for such u.) To see this, suppose

$u \in \overset{0}{H}{}^1$ and let (φ_n) be a sequence of elements of $C_c^\infty(I)$ such that $\varphi_n \to u$ in H^1, cf. Remark 2.25 (ii). By Sobolev's theorem we have $\max_{x \in \bar{I}} |\varphi_n(x) - u(x)| \to 0$, $n \to \infty$. Since now, by (3.99), $\varphi_n(x) = B(\varphi_n, G(x, \cdot))$ and since for $x \in \bar{I}$ $|B(\varphi_n, G(x, \cdot)) - B(u, G(x, \cdot))| = |B(\varphi_n - u, G(x, \cdot))| \le C_1(p, q)\|\varphi_n - u\|_1 \|G(x, \cdot)\|_1 \to 0$, $n \to \infty$, it follows that $u(x) = B(u, G(x, \cdot))$.

Let now $e = u - u_h$, where u is the solution of (3.93). By our remark above, since $e \in \overset{0}{H}{}^1$, we have $e(x) = B(e, G(x, \cdot))$. Recall that $B(e, \varphi) = 0$, $\forall \varphi \in S_h$. Therefore $e(x) = B(e, G(x, \cdot) - \varphi)$, $\forall \varphi \in S_h$, from which, for $0 \le i \le N + 1$, it follows that

$$|e(x_i)| \le C_1 \|e\|_1 \|G(x_i, \cdot) - \varphi\|_1, \quad \forall \varphi \in S_h. \tag{3.100}$$

Since it was assumed that p, q are sufficiently smooth, it follows by the definition of the Green's function that $\forall i \ge 1$, $G(x_i, \xi) \in H^r(0, x_i) \cap H^r(x_i, 1)$ and that $G(x_i, \xi)$ is continuous at $\xi = x_i$. Take now $\phi = \chi$, the approximant (e.g. an interpolant) of $G(x_i, \cdot)$ in S_h guaranteed by (Π). Then

$$\|G(x_i, \cdot) - \chi\|_{H^1(I)} = \left(\|G(x_i, \cdot) - \chi\|_{H^1(0, x_i)}^2 + \|G(x_i, \cdot) - \chi\|_{H^1(x_i, 1)}^2 \right)^{1/2}$$
$$\le \|G(x_i, \cdot) - \chi\|_{H^1(0, x_i)} + \|G(x_i, \cdot) - \chi\|_{H^1(x_i, 1)}$$
$$\le C_* h^{r-1} \left(\|G(x_i, \cdot)\|_{H^r(0, x_i)} + \|G(x_i, \cdot)\|_{H^r(x_i, 1)} \right)$$

where C_* is independent of h and i. It follows that

$$\|G(x_i, \cdot) - \chi\|_{H^1(I)} \le C h^{r-1}, \tag{3.101}$$

where C is independent of h, since both terms $\|G(x_i, \cdot)\|_{H^r(0, x_i)}$, $\|G(x_i, \cdot)\|_{H^r(x_i, 1)}$ can be bounded above by constants independent of i. By (3.100) and (3.101) we get

$$|e(x_i)| \le C h^{r-1} \|u\|_r h^{r-1} = C h^{2r-2} \|u\|_r,$$

as claimed. □

3.5.2 The effect of numerical integration

Finding the solution u_h of the Galerkin equations (3.94) requires computing the integrals $\int_0^1 p\varphi'\psi'$, $\int_0^1 q\varphi\psi$, $\int_0^1 f\varphi$, for $\varphi, \psi \in S_h$. (Actually φ, ψ are elements of a basis of S_h.) This is done in practice by a numerical integration (or quadrature) rule. Such rules

should be accurate enough to ensure that the linear system defining u_h is still invertible and that the convergence rate of the method in L^2, i.e. the exponent of the right-hand side of (3.95), is not diminished. We will analyze the effects of quadrature following the work by Dougals and Dupont that we cited in the previous subsection 3.5.1. For the effect of numerical integration in the case of more general finite element methods in higher dimensions cf. [3.3], [3.5]. We will use the notation and the assumptions on S_h specified in the beginning of the section, making the additional assumption that the mesh $\{x_i\}$ is quasiuniform, cf. (3.89).

We first specify the properties of the quadrature rule to be used. Let $0 \leq \xi_1 < \xi_2 < \ldots < \xi_s \leq 1$ be s distinct points (the quadrature nodes) in $[0, 1]$ and $g \in C([0, 1])$. We consider a quadrature rule of the form

$$\int_0^1 g(x)\,\mathrm{d}x \simeq \sum_{i=1}^s w_i g(\xi_i), \tag{3.102}$$

where, for simplicity, we assume that the quadrature *weights* w_i are all positive. (For a review of the theory of numerical integration see e.g. Ch. 6 of [6.1].) We also suppose that (3.102) is exact when g is a polynomial of degree at most $t \geq 0$. Examples of well-known such quadrature rules include:

- The *trapezoidal rule*, for which

  $s = 2$, $\xi_1 = 0$, $\xi_2 = 1$, $w_1 = w_2 = \frac{1}{2}, t = 1$.

- The *midpoint rule*, for which

  $s = 1$, $\xi_1 = \frac{1}{2}$, $w_1 = 1$, $t = 1$.

- The two-point *Gauss rule*, for which

  $s = 2$, $\xi_1 = \frac{1}{2} - \frac{1}{2\sqrt{3}}$, $\xi_2 = \frac{1}{2} + \frac{1}{2\sqrt{3}}$, $w_1 = w_2 = \frac{1}{2}$, $t = 3$.

- The two-point Gauss is a special case of the family of *s-point Gauss* (or *Gauss-Legendre*) rules, for which the ξ_i are the roots of the Legendre polynomial of degree s on $[0, 1]$, and $w_i = \int_0^1 (L_i(x))^2\,\mathrm{d}x$, where L_i is the ith Lagrange cardinal polynomial corresponding to the points ξ_i, i.e. the unique $L_i \in \mathbb{P}_{s-1}$ such that $L_i(\xi_j) = \delta_{ij}$. For these rules $t = 2s - 1$.

We will use the quadrature rule (3.102) changing variables in order to approximate the integrals occurring in the finite element method on every subinterval $I_j = [x_{j-1}, x_j]$,

$1 \leq j \leq N+1$, of the partition. Denote by $h_j = x_j - x_{j-1}$ the length of I_j, and put as usual $h = \max_j h_j$. On each I_j denote the local quadrature nodes by $\xi_{ij} = x_{j-1} + h_i \xi_j$, $i = 1, \ldots, N+1$, $j = 1, \ldots, s$. Given $v, w \in C[0,1]$ let $(v,w)_i := \int_{I_i} vw \, dx$, so that $(v,w) = \int_0^1 vw \, dx = \sum_{i=1}^{J+1} (v,w)_i$. Changing the dependent variable in (3.102) we see that on I_i the rule becomes $\int_{I_i} g(x) \, dx \simeq h_i \sum_{j=1}^s w_j g(\xi_{ij})$. Letting $\langle v, w \rangle_i := h_i \sum_{j=1}^s w_j (vw)(\xi_{ij})$, we see that (v,w) is approximated by the bilinear symmetric form $\langle v, w \rangle = \sum_{i=1}^{J+1} \langle v, w \rangle_i$. Note that our assumption about the accuracy of the quadrature rule implies that $\langle v, w \rangle_i = (v,w)_i$ if $vw \in \mathbb{P}_t(I_i)$.

Using the quadrature rule to approximate the integrals in (3.94) we see that u_h is approximated by $z_h \in S_h$, where z_h satisfies

$$\langle p z_h', \varphi' \rangle + \langle q z_h, \varphi \rangle = \langle f, \varphi \rangle, \quad \forall \varphi \in S_h. \tag{3.103}$$

It is not obvious that the linear system represented by the equations (3.103) has a unique solution. The following Lemma shows that, since $w_i > 0$, this happens provided the quadrature rule is accurate enough.

Lemma 3.22. Let $t \geq 2r - 4$. Then, there exists a positive constant C such that

$$\langle p \varphi', \varphi' \rangle + \langle q \varphi, \varphi \rangle \geq C \|\varphi\|_1^2, \quad \forall \varphi \in S_h. \tag{3.104}$$

Proof. Let $\varphi \in S_h$. Since $p(x) \geq \alpha > 0$ and $q(x) > 0$ on $[0,1]$ we have

$$
\begin{aligned}
\langle p \varphi', \varphi' \rangle + \langle q \varphi, \varphi \rangle &= \sum_{i=1}^{N+1} \langle p \varphi', \varphi' \rangle_i + \langle q \varphi, \varphi \rangle_i \\
&= \sum_{i=1}^{N+1} h_i \left\{ \sum_{j=1}^s \left[w_j p(\xi_{ij}) (\varphi'(\xi_{ij}))^2 + w_j q(\xi_{ij}) \varphi^2(\xi_{ij}) \right] \right\} \\
&\geq \alpha \sum_{i=1}^{N+1} h_i \sum_{j=1}^s w_j (\varphi'(\xi_{ij}))^2 = \alpha \sum_{i=1}^{N+1} \langle \varphi', \varphi' \rangle_i \\
&= \left(\text{since } \varphi \in \mathbb{P}_{r-1}(I_i) \text{ implies that } (\varphi')^2 \in \mathbb{P}_{2r-4}(I_i) \right) \\
&= \alpha \sum_{i=1}^{N+1} (\varphi', \varphi')_i = \alpha \|\varphi'\|^2,
\end{aligned}
$$

and the conclusion follows from the Poincaré-Friedrichs inequality. $\square$

The inequality (3.104) implies that z_h may be found by solving a linear system, represented by (3.103), with a symmetric, positive definite matrix.

In the sequel we will use the *Peano Kernel theorem*, see e.g. [6.1], to estimate the error of the quadrature rule (3.102) on I_i. By the change of variable $x \mapsto y = (b-a)x+a$ we first apply the rule on a general interval $[a,b]$. Given $g \in C[a,b]$ it takes the form

$$\int_a^b g(y)\,\mathrm{d}y \simeq (b-a)\sum_{i=1}^s g(\hat{\xi}_i), \tag{3.105}$$

where $\hat{\xi}_i = a + (b-a)\xi_i \in [a,b]$. If (3.102), and consequently (3.105), is exact for polynomials of degree $\leq t$ and $g \in C^{t+1}[a,b]$ the Peano Kernel theorem asserts that

$$\int_a^b g(y)\,\mathrm{d}y - (b-a)\sum_{i=1}^s w_i g(\hat{\xi}_i) = \frac{1}{t!}\int_a^\beta k_t(\tau)g^{(t+1)}(\tau)\,\mathrm{d}\tau, \tag{3.106}$$

where the *Peano Kernel* $k_t(\tau)$ is given for $a \leq \tau \leq b$ by

$$k_t(\tau) = \int_a^b (x-\tau)_+^t\,\mathrm{d}x - (b-a)\sum_{i=1}^s w_i(\hat{\xi}_i - \tau)_+^t. \tag{3.107}$$

where

$$(x-\tau)_+^t = \begin{cases} (x-\tau)^t, & \text{if } x \geq \tau, \\ 0, & \text{if } \tau < x, \end{cases} \quad x, \tau \in [a,b].$$

In the following Lemma we estimate the remainder term, i.e. the analog of the right-hand side of (3.106), when the quadrature rule is applied to the subinterval $I_i = [x_{i-1}, x_i]$.

Lemma 3.23. Under our hypotheses on the quadrature rule (3.102) and if $g \in C^{t+1}(I_i)$ for some $1 \leq i \leq N+1$, then it holds that

$$\left| \int_{I_i} g(x)\,\mathrm{d}x - h_i \sum_{j=1}^s w_j g(\xi_{ij}) \right| \leq C h_i^{t+1} \|g^{(t+1)}\|_{L^1(I_i)}, \tag{3.108}$$

where $h_i = x_i - x_{i-1}$ and C is a positive constant that depends only on t.

Proof. Using (3.105)–(3.107) on $[a,b] = I_i$ we obtain

$$\left| \int_{I_i} g(x)\,\mathrm{d}x - h_i \sum_{j=1}^s w_j g(\xi_{ij}) \right| \leq \frac{1}{t!} \max_{\tau \in I_i} |k_{t,i}(\tau)| \|g^{(t+1)}\|_{L^1(I_i)}, \tag{3.109}$$

where for $\tau \in I_i$

$$k_{t,i}(\tau) = \int_{I_i} (x-\tau)_+^t\,\mathrm{d}x - h_i \sum_{j=1}^s w_j(\xi_{ij} - \tau)_+^t. \tag{3.110}$$

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First, for $\tau \in I_i$, we have

$$\left| \int_{I_i} (x - \tau)_+^t \, dx \right| \leq \int_\tau^{x_i} (x - \tau)^t \, dx = \frac{1}{t+1}(x_i - \tau)^{t+1} \leq \frac{h_i^{t+1}}{t+1}. \qquad (3.111)$$

Also, by the definition of $(\xi_{ij} - \tau)_+^t$ for $\tau \in I_i$, we see that the maximum of $(\xi_{ij} - \tau)_+^t$ occurs at $\tau = x_{i-1}$ and is equal to $h_i^t \xi_j^t$. Since $w_j > 0$ and $\sum_{j=1}^s w_j = 1$ we obtain

$$\left| h_i \sum_{j=1}^s w_j (\xi_{ij} - \tau)_+^t \right| \leq h_i \sum_{j=1}^s w_j h_i^t \xi_j^t \leq h_i^{t+1} \sum_{j=1}^s w_j = h_i^{t+1}. \qquad (3.112)$$

The desired error estimate (3.108) follows now from (3.109)–(3.112). $\qquad \square$

The following result shows that the integrals in $B(u_h, \varphi)$, $\varphi \in S_h$, are approximated well by the quadrature rule if t is large enough.

Lemma 3.24. Let $t \geq r - 2$, suppose that p, q are smooth enough ($p, q \in C^{t+1}$ will suffice), and that the solution of (3.93) belongs to $H^{r-1} \cap \overset{0}{H}{}^1$. Then, there exists a constant C independent of u and h, such that for $\varphi \in S_h$

$$\begin{aligned} |(pu_h', \varphi') - \langle pu_h', \varphi' \rangle| &\leq Ch^{t-r+2} \|u\|_{r-1} \|\varphi\|, \\ |(qu_h, \varphi) - \langle qu_h, \varphi \rangle| &\leq Ch^{t-r+2} \|u\|_{r-1} \|\varphi\|. \end{aligned} \qquad (3.113)$$

Proof. We have

$$(pu_h', \varphi') = \langle pu_h', \varphi' \rangle + \sum_{i=1}^{N+1} \left[(pu_h', \varphi')_i - \langle pu_h', \varphi' \rangle_i \right]. \qquad (3.114)$$

By Lemma 3.23 for each i, $1 \leq i \leq J + 1$, it holds that

$$|(pu_h', \varphi') - \langle pu_h', \varphi' \rangle_i| \leq Ch_i^{t+1} \|(pu_h' \varphi')^{(t+1)}\|_{L^1(I_i)}. \qquad (3.115)$$

Since, by Leibniz's rule, for $v \in C^{t+1}(I_i)$ we have $(pv)^{(t+1)} = \sum_{j=0}^{t+1} \binom{t+1}{j} p^{(t+1-j)} v^{(j)}$ on I_i, it follows that $\|(pu_h' \varphi')^{(t+1)}\|_{L^1(I_i)} \leq C \sum_{j=0}^{t+1} \|(u_h' \varphi')^{(j)}\|_{L^1(I_i)}$. But $u_h^{(j)}$, $\varphi^{(j)}$ are zero on I_i if $j \geq r$. Therefore $\|(pu_h' \varphi')^{(t+1)}\|_{L^1(I_i)} \leq C \sum_{k,\ell=1}^{r-1} \int_{I_i} |u_h^{(k)}| |\varphi^{(\ell)}| \leq C \sum_{k,\ell=1}^{r-1} \|u_h^{(k)}\|_{L^2(I_i)} \|\varphi^{(\ell)}\|_{L^2(I_i)} \leq C\|u_h\|_{H^{r-1}(I_i)} \|\varphi\|_{H^{r-1}(I_i)}$. Finally, since by a local inverse inequality on I_i we have $\|\varphi\|_{H^{r-1}(I_i)} \leq Ch_i^{-(r-1)} \|\varphi\|_{L^2(I_i)}$, we get from (3.115) and the above that

$$|(pu_h', \varphi')_i - \langle pu_h', \varphi' \rangle_i| \leq Ch_i^{t-r+2} \|u_h\|_{H^{r-1}(I_i)} \|\varphi\|_{L^2(I_i)},$$

and therefore, by (3.114) and the Cauchy-Schwarz inequality on $\mathbb{R}^{J+1}$, that

$$|(pu_h', \varphi') - \langle pu_h', \varphi' \rangle| \leq Ch^{t-r+2} \left(\sum_{i=1}^{N+1} \|u_h\|_{H^{r-1}(I_i)}^2 \right)^{1/2} \|\varphi\|. \qquad (3.116)$$

Let χ be the approximant of u (i.e. an interpolant) guaranteed by property (Π). For each i we have by the triangle inequality

$$\|u_h\|_{H^{r-1}(I_i)} \leq \|u_h - \chi\|_{H^{r-1}(I_i)} + \|\chi - u\|_{H^{r-1}(I_i)} + \|u\|_{H^{r-1}(I_i)}. \qquad (3.117)$$

By a local inverse inequality on I_i we see that

$$\|u_h - \chi\|_{H^{r-1}(I_i)} \leq Ch_i^{-(r-1)} \|u_h - \chi\|_{L^2(I_i)} \leq Ch_i^{-(r-1)} \left(\|u_h - u\|_{L^2(I_i)} + \|u - \chi\|_{L^2(I_i)} \right)$$

Therefore, by the quasiuniformity of the mesh, and (Π) and (3.70) for $j = r - 1$, it follows that

$$\left(\sum_{i=1}^{N+1} \|u_h - \chi\|_{H^{r-1}(I_i)}^2 \right)^{1/2} \leq Ch^{-(r-1)} (\|u_h - u\| + \|u - \chi\|)$$

$$\leq Ch^{-(r-1)} (Ch^{r-1} \|u\|_{r-1}) \leq C \|u\|_{r-1}. \qquad (3.118)$$

Moreover, by (3.78) for $s = j = r - 1$ we obtain

$$\left(\sum_{i=1}^{N+1} \|\chi - u\|_{H^{r-1}(I_i)}^2 \right)^{1/2} \leq C \|u\|_{r-1}, \qquad (3.119)$$

(which holds for $r = 2$ as well). Hence, (3.117)–(3.119) give the H^{r-1} "stability" estimate

$$\left(\sum_{i=1}^{N+1} \|u_h\|_{H^{r-1}(I_i)}^2 \right)^{1/2} \leq C \|u\|_{r-1}, \qquad (3.120)$$

that yields the first inequality of (3.113) in view of (3.116).

In order to prove the second inequality in (3.113) note that for $\varphi \in S_h$

$$(qu_h, \varphi) = \langle qu_h, \varphi \rangle + \sum_{i=1}^{N+1} [(qu_h, \varphi)_i - \langle qu_h, \varphi \rangle_i].$$

By the Peano Kernel theorem we see again that for each i

$$|(qu_h, \varphi)_i - \langle qu_h, \varphi \rangle_i| \leq Ch_i^{t+1} \|(qu_h\varphi)^{(t+1)}\|_{L^1(I_i)}.$$

Since q is smooth enough and $u_h^{(j)} - \varphi^{(j)} = 0$ on I_i for $j \geq r$ we obtain

$$\|(qu_h\varphi)^{(t+1)}\|_{L^1(I_i)} \leq C \sum_{k,\ell=0}^{r-1} \int_{I_i} |u_h^{(k)} \varphi^{(\ell)}| \leq C \|u_h\|_{H^r(I_i)} \|\varphi\|_{H^{r-1}(I_i)},$$

and we may continue as in the proof of the first inequality of (3.113). $\qquad \square$

We examine now the error of numerical integration in the right-hand side (f, φ) of (3.94). This exercise is harder; it entails introducing a suitable local projection of f, and assuming that the accuracy of the quadrature rule be somewhat higher.

Lemma 3.25. Let $t \geq 2r - 3$ and $f \in H^{r-1}$. Then there exists a constant C independent of f and h such that for $\varphi \in S_h$

$$|(f, \varphi) - \langle f, \varphi \rangle| \leq C h^{r-1} \|f\|_{r-1} \|\varphi\|. \tag{3.121}$$

Proof. Let $\tilde{f}$ be a piecewise polynomial function relatively to the partition $\{x_i\}$ of $[0, 1]$ such that $\tilde{f}\big|_{I_i}$ is the L^2 projection of $f\big|_{I_i}$ onto $\mathbb{P}_{r-2}(I_i)$, i.e. so that $\tilde{f}\big|_{I_i} \in \mathbb{P}_{r-2}(I_i)$ and

$$(f, w)_i = (\tilde{f}, w)_i, \quad \forall w \in \mathbb{P}_{r-2}(I_i), \quad 1 \leq i \leq N + 1. \tag{3.122}$$

For $\varphi \in S_h$ we write $(f, \varphi) = (\tilde{f}, \varphi) + (f - \tilde{f}, \varphi)$. Since $\tilde{f}\varphi\big|_{I_i} \in \mathbb{P}_{2r-3}$ and $t \geq 2r - 3$, it follows that $(\tilde{f}, \varphi) = \sum_{i=1}^{N+1} (\tilde{f}, \varphi)_i = \sum_{i=1}^{N+1} \langle \tilde{f}, \varphi \rangle_i = \langle \tilde{f}, \varphi \rangle$. Therefore

$$(f, \varphi) = \langle \tilde{f}, \varphi \rangle + (f - \tilde{f}, \varphi), \quad \forall \varphi \in S_h. \tag{3.123}$$

Let $p \in \mathbb{P}_{r-2}(I_i)$ be the Lagrange interpolant of f at $r - 1$ distinct points in I_i. Then, by (3.122)

$$\|f - \tilde{f}\|_{L^2(I_i)} = \inf_{w \in \mathbb{P}_{r-2}(I_i)} \|f - w\|_{L^2(I_i)} \leq \|f - p\|_{L^2(I_i)} \leq C h_i^{r-1} \|f^{(r-1)}\|_{L^2(I_i)},$$

where in the last step we used the inequality (3.71) (which holds for $r = 2$ as well). Therefore in (3.123)

$$|(f - \tilde{f}, \varphi)| \leq \|f - \tilde{f}\| \|\varphi\| = \left(\sum_{i=1}^{N+1} \|f - \tilde{f}\|_{L^2(I_i)}^2 \right)^{1/2} \|\varphi\|$$

$$\leq C h^{r-1} \|f\|_{r-1} \|\varphi\|, \quad \forall \varphi \in S_h, \tag{3.124}$$

and hence

$$|(f, \varphi) - \langle \tilde{f}, \varphi \rangle| \leq C h^{r-1} \|f\|_{r-1} \|\varphi\|, \quad \forall \varphi \in S_h. \tag{3.125}$$

In addition, it holds that

$$|\langle f - \tilde{f}, \varphi \rangle| \leq C h^{r-1} \|f\|_{r-1} \|\varphi\|, \quad \forall \varphi \in S_h. \tag{3.126}$$

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To see this we write

$$\langle f - \tilde{f}, \varphi \rangle = \sum_{i=1}^{N+1} \langle f - \tilde{f}, \varphi \rangle_i, \quad \varphi \in S_h, \tag{3.127}$$

and use for each term in the right-hand side the Peano Kernel theorem on each I_i as in Lemma 3.23, not going up to the full order of accuracy $t+1$ as in (3.108), but only up to $r-1$. In this way we have for $1 \le i \le J+1$, $\varphi \in S_h$

$$|\langle f - \tilde{f}, \varphi \rangle_i - (f - \tilde{f}, \varphi)_i| \le C h_i^{r-1} \|((f - \tilde{f})\varphi)^{(r-1)}\|_{L^1(I_i)}$$

$$\text{(Leibniz's rule)} \le C h_i^{r-1} \sum_{j=0}^{r-1} \|(f - \tilde{f})^{(r-1-j)} \varphi^{(j)}\|_{L^1(I_i)}$$

$$\text{(Cauchy-Schwarz)} \le C h_i^{r-1} \sum_{j=0}^{r-1} \|(f - \tilde{f})^{(r-1-j)}\|_{L^2(I_i)} \|\varphi^{(j)}\|_{L^2(I_i)}$$

$$\text{(inverse inequality in each } I_i) \le C \sum_{j=0}^{r-1} h_i^{r-1-j} \|f - \tilde{f}\|_{H^{r-1-j}(I_i)} \|\varphi\|_{L^2(I_i)}.$$

Recall now that $\tilde{f}|_{I_i}$ is the L^2 projection of $f|_{I_i}$ in $\mathbb{P}_{r-2}(I_i)$. Therefore, for $0 \le j \le r-1$, $\|f - \tilde{f}\|_{H^{r-1-j}(I_i)} \le C h^{r-1-(r-1-j)} \|f\|_{H^{r-1-j}(I_i)} = C h^j \|f\|_{H^{r-1-j}(I_i)}$ by (3.77) (which holds for $s=1$ as well when $r=2$). We conclude that for $1 \le i \le J+1$, $\varphi \in S_h$

$$|\langle f - \tilde{f}, \varphi \rangle_i - (f - \tilde{f}, \varphi)_i| \le C h_i^{r-1} \|f\|_{H^{r-1}(I_i)} \|\varphi\|_{L^2(I_i)}.$$

Therefore, for $\varphi \in S_h$

$$|\langle f - \tilde{f}, \varphi \rangle| = \left| \sum_{i=1}^{J+1} \langle f - \tilde{f}, \varphi \rangle_i \right| \le \left| \sum_{i=1}^{N+1} \langle f - \tilde{f}, \varphi \rangle_i - (f - \tilde{f}, \varphi)_i \right| + \left| \sum_{i=1}^{J+1} (f - \tilde{f}, \varphi)_i \right|$$

$$\le C h^{r-1} \sum_{i=1}^{N+1} \|f\|_{H^{r-1}(I_i)} \|\varphi\|_{L^2(I_i)} + |(f - \tilde{f}), \varphi)|.$$

Now, using in the above the Cauchy-Schwarz inequality in the $\mathbb{R}^{J+1}$ and (3.124) we conclude that (3.126) holds. Finally, from (3.126) and (3.125) we get for $\varphi \in S_h$

$$|(f, \varphi) - \langle f, \varphi \rangle| \le |(f, \varphi) - \langle \tilde{f}, \varphi \rangle| + |\langle \tilde{f} - f, \varphi \rangle|$$

$$\le C h^{r-1} \|f\|_{r-1} \|\varphi\|,$$

which is the desired inequality (3.121). □

We now have all the ingredients to prove the main result of this section, namely an $\mathcal{O}(h^{r-1})$ H^1-error bound for the solution z_h of the finite element method (3.103) with quadrature.

Proposition 3.26. Let $t \geq 2r - 3$, p, q be sufficiently smooth and $f \in H^{r-1}$. Then

$$\|u - z_h\|_1 \leq Ch^{r-1}\|f\|_{r-1}. \tag{3.128}$$

Proof. We know that $\|u - u_h\|_1 \leq Ch^{r-1}\|u\|_{r-1}$. Since $\|u\|_{r-1} \leq C\|f\|_{r-1}$, (3.128) will follow if we prove that $\|u_h - z_h\|_1 \leq Ch^{r-1}\|f\|_{r-1}$. (Note that an elliptic regularity estimate would give $\|u\|_{r-1} \leq C\|f\|_{r-3}$, which for $r = 2$ needs the introduction of the "H^{-1} norm" $\|f\|_{-1}$. However, in order to prove (3.128) we may restrict ourselves, in view of (3.121), to the 'suboptimal' estimate $\|u\|_{r-1} \leq C\|f\|_{r-1}$.)

Let $e = u_h - z_h$. Then, by (3.94) and (3.103), for $\varphi \in S_h$, it follows that we have

$$\langle pe', \varphi' \rangle + \langle qe, \varphi \rangle = \langle pu_h', \varphi' \rangle - \langle pz_h', \varphi \rangle + \langle qu_h, \varphi \rangle - \langle qz_h, \varphi \rangle$$
$$= \langle pu_h', \varphi' \rangle - (pu_h', \varphi') + \langle qu_h, \varphi \rangle - (qu_h, \varphi) + (f, \varphi) - \langle f, \varphi \rangle.$$

Therefore, by (3.113) and (3.121), since $t \geq 2r - 3$, we obtain

$$\langle pe', \varphi' \rangle + \langle qe, \varphi \rangle \leq Ch^{r-1}\|u\|_{r-1}\|\varphi\| + Ch^{r-1}\|f\|_{r-1}\|\varphi\|$$
$$\leq Ch^{r-1}\|f\|_{r-1}\|\varphi\|.$$

Hence, taking $\varphi = e$ and using (3.104), we get $\|e\|_1^2 \leq Ch^{r-1}\|f\|_{r-1}\|e\|$, from which (3.128) follows. $\qquad\square$

In the paper of Douglas and Dupont that we followed, it is also proved that for $t \geq 2r - 3$ z_h satisfies in addition the optimal-order L^2 error estimate $\|u - z_h\| \leq Ch^r\|f\|_r$ and the superconvergence estimate $|(u - z_h)(x_i)| \leq Ch^{2r-2}\|f\|_r$. (We leave the checking of the proofs to the reader, as nontrivial exercises in Green's functions –see § 3.5.1– and the Peano Kernel theorem!) Note that the orders of convergence in these estimates are optimal, like those of the "exact" Galerkin approximation u_h. They require though, higher regularity of f and of the coefficients p and q, compared to the estimates for u_h, since the latter satisfies $\|u - u_h\|_1 \leq Ch^{r-1}\|u\|_r$, $\|u - u_h\| \leq Ch^r\|u\|_r$, $|(u - u_h)(x_i)| \leq Ch^{2r-2}\|u\|_r$ (as we saw in § 3.5.1), where $\|u\|_r \leq C\|f\|_{r-2}$.)

From the practical point of view, we conclude that in order to preserve the order of accuracy of the Galerkin method one should use a quadrature rule which is exact for polynomials of degree up to $t = 2r - 3$. Hence for continuous, piecewise linear approximations it suffices that $t = 1$, for piecewise quadratics that $t = 3$, for piecewise

cubics that $t = 5$, etc. In particular, if one uses the s-point Gauss rules (for which $t = 2s - 1$), taking $s = 1, 2$, and 3, for piecewise linear, quadratic, and cubic approximations, respectively, will suffice. The results also hold for smoother finite element approximations, e.g. with C^1 or C^2 elements.

3.6   *A posteriori* error estimates and mesh adaptivity

The error estimates for the Galerkin finite element methods that we have derived thusfar, for example the L^2 and H^1 error estimates

$$\|u - u_h\| + h\|u - u_h\|_1 \leq Ch^r\|u\|_r \tag{3.129}$$

for the two-point boundary-value problem (bvp) (3.1)–(3.2), when $u \in H^r \cap \overset{0}{H}{}^1$ and the finite element subspace S_h satisfies property (Π) of section 3.4, are called *a priori* error estimates, in the sense that the error bounds are not expressed in terms of the computable numerical solution u_h but only in terms of the unknown, exact solution u of the bvp. This type of error estimate gives us information about the asymptotic behavior of the errors as $h \to 0$ and some idea of the order of magnitude of the errors. If however we are interested in estimating precisely the error itself, which is of course important in practice, bounds like the one in the right-hand side of (3.129) are not useful in general. If the constant C is known, one could argue that estimating $\|u\|_r$ in terms of the data p, q, f of the bvp would give a computable bound. However such estimates are usually pessimistic. Moreover since the a priori error bounds are expressed in terms of the maximum mesh length $h = \max(x_{i+1} - x_i)$, in order to make the error smaller than a preassigned tolerance one would have to compute with smaller h, i.e. to refine the mesh globally by taking *all* $h_i = x_{i+1} - x_i$ smaller, which is costly, especially in bvp's in higher dimensions.

Another type of error estimates are called *a posteriori* since they may be used after computing the numerical solution. They are of the general form

$$\|u - u_h\| \leq \eta(u_h),$$

where $\eta(u_h)$ is a computable quantity, a functional of the numerical solution u_h, called the *error estimator* of the $\|\|\cdot\|\|$ norm of the error. Such bounds are useful if $\eta(u_h)$ is an accurate estimate of the error, is straightforward to compute, and gives some guidance on how to refine efficiently the mesh in order to achieve smaller errors.

In this section we will derive such *a posteriori* error estimates for the two-point bvp (3.1)–(3.2), for which we will assume that $0 \leq \alpha = \min_{0 \leq x \leq 1} p(x)$ and $q(x) \geq 0$ on $[0,1]$. Let $0 = x_0 < x_1 < \ldots < x_{N+1} = 1$ be a partition of $[0,1]$ and $h_i = x_{i+1} - x_i$, $h = \max_i h_i$, $I_i = [x_i, x_{i+1}]$, $0 \leq i \leq N$. For integer $r \geq 2$ consider the finite element space $S_h = \{\varphi \in C[0,1], \ \varphi|_{I_i} \in \mathbb{P}_{r-1}, \ \varphi(0) = \varphi(1) = 0\}$ and assume that S_h satisfies the approximation (Π) of section 3.4. Let $u_h \in S_h$ be the Galerkin approximation of u, computed as usual as solution of the problem

$$B(u_h, \varphi) = (f, \varphi), \quad \forall \varphi \in S_h, \tag{3.130}$$

where $B(v,w) = \int_0^1 pv'w' + qvw$, for $v, w \in \overset{0}{H}{}^1$. The *a posteriori* error estimates that we will construct for the errors $\|u' - u'_h\|$, $\|u - u_h\|$, are based on the (local) *residual* of the finite element method, defined on each I_i as the computable function

$$r_h = -(pu'_h)' + qu_h - f, \quad x \in I_i. \tag{3.131}$$

Specifically, we will derive bounds of $\|u' - u'_h\|$, $\|u - u_h\|$ in terms of quantities of the form $h_i^s \|r_h\|_{L^2(I_i)}$, $s > 0$, so that it would be possible to refine the mesh by taking h_i smaller for those I_i where the L^2 norm of the local residual $\|r_h\|_{L^2(I_i)}$ is large, i.e. in regions where u_h is not close enough to the solution u (whose residual is of course zero.) We will start with such *a posteriori* estimates of $\|u' - u'_h\|$. (The results are standard; we follow the exposition in [6.2].)

Proposition 3.27. Let $u \in H^2 \cap \overset{0}{H}{}^1$ be the solution of (3.1)–(3.2). Then

$$\|u' - u'_h\| \leq \frac{1}{\pi \alpha} \left(\sum_{i=0}^N h_i^2 \|r_h\|_{L^2(I_i)}^2 \right)^{1/2}. \tag{3.132}$$

Proof. Let $e = u - u_h$. Then, since $B(v,v) = \int_0^1 p(v')^2 + qv^2 \geq \alpha \|v'\|^2$ for $v \in H^1$, we have $\alpha \|e'\|^2 \leq B(e,e)$. Recalling that $B(e, \chi) = 0$, $\forall \chi \in S_h$, we obtain therefore

$$\alpha \|e'\|^2 \leq B(e, e - \chi), \quad \forall \chi \in S_h. \tag{3.133}$$

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Let now $v \in \overset{0}{H}{}^1(0,1)$ satisfy $v(x_i) = 0$, $0 \le i \le N+1$. Then

$$B(e,v) = \int_0^1 pe'v' + qev = \sum_{i=0}^N \int_{I_i} pe'v' + (qe,v) = -\sum_{i=0}^N \int_{I_i} (pe')'v + (qe,v)$$

$$= \sum_{i=0}^N \int_{I_i} [-(p(u' - u_h'))'v + q(u - u_h)v] = \text{(since the ode (3.1) holds in } L^2)$$

$$= (-(pu')' + qu, v)$$

$$= -\sum_{i=0}^N \int_{I_i} [-(pu_h')'v + qu_h v] = \text{(using the definition of } r_h)$$

$$= (f,v) - \sum_{i=0}^N \int_{I_i} r_h v - (f,v).$$

Therefore for $v \in \overset{0}{H}{}^1(0,1)$ with $v(x_i) = 0$, $0 \le i \le N+1$, it holds that

$$B(e,v) = -\sum_{i=0}^N \int_{I_i} r_h v. \tag{3.134}$$

Let $\hat{\chi} \in S_h$, $\hat{\chi}\big|_{I_i} \in \mathbb{P}_1$ be the piecewise linear interpolant of e at the nodes x_i i.e. satisfy $\hat{\chi}(x_i) = e(x_i)$, $0 \le i \le N+1$. Since, for each i $\int_{I_i} (e' - \hat{\chi}')\hat{\chi}' = [(e - \hat{\chi})\hat{\chi}']_{x_i}^{x_{i+1}} - \int_{I_i} (e - \hat{\chi})\hat{\chi}'' = 0$, we have $\int_{I_i} (e' - \hat{\chi}')^2 = \int_{I_i} (e' - \hat{\chi}')(e' - \hat{\chi}') = \int_{I_i} (e' - \hat{\chi}')e' \le \|e' - \hat{\chi}'\|_{L^2(I_i)} \|e'\|_{L^2(I_i)}$, i.e.

$$\|e' - \hat{\chi}'\|_{L^2(I_i)} \le \|e'\|_{L^2(I_i)}, \quad 0 \le i \le N. \tag{3.135}$$

Using now the Poincaré-Friedrichs inequality with the best constant (see Remark 2.29 (iii)) on I_i we have, since $e - \hat{\chi} \in \overset{0}{H}{}^1(I_i)$, that $\|e - \hat{\chi}\|_{L^2(I_i)} \le \frac{h_i}{\pi} \|e' - \hat{\chi}'\|_{L^2(I_i)}$, and therefore (3.135) implies that

$$\|e - \hat{\chi}\|_{L^2(I_i)} \le \frac{h_i}{\pi} \|e'\|_{L^2(I_i)}, \quad 0 \le i \le N. \tag{3.136}$$

Hence, from (3.133) with $\chi = \hat{\chi}$, (3.134) with $v = e - \chi$, and (3.136) we see that

$$\alpha \|e'\|^2 \le B(e, e - \hat{\chi}) = -\sum_{i=0}^N \int_{I_i} r_h(e - \hat{\chi}) \le \frac{1}{\pi} \sum_{i=0}^N \|r_h\|_{L^2(I_i)} h_i \|e'\|_{L^2(I_i)}$$

$$\le \frac{1}{\pi} \left(\sum_{i=0}^N h_i^2 \|r_h\|_{L^2(I_i)}^2 \right)^{1/2} \left(\sum_{i=0}^N \|e'\|_{L^2(I_i)}^2 \right)^{1/2} = \frac{1}{\pi} \left(\sum_{i=0}^N h_i^2 \|r_h\|_{L^2(I_i)}^2 \right)^{1/2} \|e'\|,$$

from which the conclusion (3.132) of the proposition follows. $\qquad\square$

We proceed now with an analogous *a posteriori* estimate for the L^2 norm $\|u - u_h\|$ of the error.

Proposition 3.28. Let $u \in H^2 \cap \overset{0}{H}{}^1$ be the solution of (3.1)–(3.2). Then

$$\|u - u_h\| \leq \frac{\gamma}{\pi^2} \left(\sum_{i=0}^{N} h_i^4 \|r_h\|_{L^2(I_i)}^2 \right)^{1/2}, \tag{3.137}$$

where $\gamma = \frac{1}{\pi\alpha^2} \|p'\|_\infty + \frac{1}{\pi^2\alpha^2} \|q\|_\infty + \frac{1}{\alpha}$, $\alpha = \min_{0 \leq x \leq 1} p(x)$.

Proof. We will use Nitsche's trick as we did in the *a priori* estimate. Let $e = u - u_h$ and let w be the solution of the problem

$$\begin{aligned}
-(pw')' + qw &= e, \\
w(0) = w(1) &= 0.
\end{aligned} \tag{3.138}$$

Then $w \in H^2 \cap \overset{0}{H}{}^1$ and

$$B(w, z) = (e, z), \quad \forall z \in \overset{0}{H}{}^1. \tag{3.139}$$

Hence $\alpha\|w'\|^2 \leq B(w, w) = (e, w) \leq \frac{1}{\pi}\|e\|\|w'\|$, where in the last inequality we used the Poincaré-Friedrichs inequality for w with the best constant on $(0, 1)$. Therefore

$$\|w'\| \leq \frac{1}{\pi\alpha}\|e\|,$$

which implies that

$$\|w\| \leq \frac{1}{\pi}\|e'\| \leq \frac{1}{\pi^2\alpha}\|e\|.$$

In addition, since $\alpha = \min_{[0,1]} p$, using the ode in (3.138) we have

$$\|w''\| \leq \frac{1}{\alpha} \left(\|p'\|_\infty \|w'\| + \|q\|_\infty \|w\| + \|e\| \right),$$

which, together with the estimates of $\|w\|$ and $\|w'\|$ above, gives

$$\|w''\| \leq \gamma\|e\|, \tag{3.140}$$

where γ is the computable constant defined in the statement of the proposition.

From (3.139) and (3.130) we have, as usual, for each $\varphi \in S_h$, $\|e\|^2 = B(w, e) = B(e, w - \varphi)$. Taking $\varphi = \chi$, where χ is the piecewise linear interpolant of w at the nodes

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x_i, we see from (3.134) for $v = w - \chi$ that $\|e\|^2 = B(e, w - \chi) = -\sum_{i=0}^{N} \int_{I_i} r_h(w - \chi)$. Therefore by the Poincaré-Friedrichs inequality for $w - \chi$ in I_i with the best constant,

$$\|e\|^2 \leq \frac{1}{\pi} \sum_{i=0}^{N} \|r_h\|_{L^2(I_i)} h_i \|w' - \chi'\|_{L^2(I_i)}. \tag{3.141}$$

Now, since

$$\|w' - \chi'\|^2_{L^2(I_i)} = \int_{x_i}^{x_{i+1}} (w' - \chi')(w' - \chi') = -\int_{x_i}^{x_{i+1}} (w'' - \chi'')(w - \chi) = -\int_{x_i}^{x_{i+1}} w''(w - \chi)$$

$$\leq \|w''\|_{L^2(I_i)} \|w - \chi\|_{L^2(I_i)} \leq \|w''\| \frac{h_i}{\pi} \|w' - \chi'\|_{L^2(I_i)},$$

(where in the last inequality we used the Poincaré-Friedrichs inequality again) we get

$$\|w' - \chi'\|_{L^2(I_i)} \leq \frac{h_i}{\pi} \|w''\|_{L^2(I_i)}.$$

Using this estimate in the right-hand side of (3.141) we finally obtain, in view of (3.140) that

$$\|e\|^2 \leq \frac{1}{\pi^2} \sum_{i=0}^{N} h_i^2 \|r_h\|_{L^2(I_i)} \|w''\|_{L^2(I_i)} \leq \frac{1}{\pi^2} \left(\sum_{i=0}^{N} h_i^4 \|r_h\|^2_{L^2(I_i)} \right)^{1/2} \|w''\|$$

$$\leq \frac{\gamma}{\pi^2} \|e\| \left(\sum_{i=0}^{N} h_i^4 \|r_h\|^2_{L^2(I_i)} \right)^{1/2},$$

from which the conclusion (3.137) of the propositions follows. $\qquad\square$

How good are the estimates (3.132) and (3.137)? One way to answer this question is to check the rates of convergence to zero of the right-hand sides of these inequalities as $h \to 0$. We know from the *a priori* estimates that under our hypotheses on S_h, $\|u - u_h\| = \mathcal{O}(h^r)$ and $\|u' - u'_h\| = \mathcal{O}(h^{r-1})$, provided $u \in H^r \cap \overset{0}{H}{}^1$. We will now prove that if the partition $\{x_i\}$ is quasiuniform, cf. (3.89), then the right-hand sides of (3.132), (3.137) are also of $\mathcal{O}(h^{r-1})$ and $\mathcal{O}(h^r)$, respectively, if $u \in H^r \cap \overset{0}{H}{}^1$, i.e. that asymptotically they behave like the *a priori* estimates. By the definition of (3.131) of the residual r_h we have on each I_i

$$r_h = -(pu'_h)' + qu_h - f = -(pu'_h)' + qu_h + (pu')' - qu =$$

$$= -p(u''_h - u'') - p'(u'_h - u') + q(u_h - u).$$

Therefore,

$$\|r_h\|_{L^2(I_i)} \leq C\|u_h - u\|_{H^2(I_i)}, \tag{3.142}$$

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for some constant C independent of h_i. Let χ be the approximation of u in S_h satisfying property (Π). (As in Section 3.4 –see the proof of Theorem 3.18– $\chi\big|_{I_i} \in \mathbb{P}_{r-1}(I_i)$ may be taken as the Lagrange interpolant of u at r distinct points in I_i two of which are the endpoints of I_i.) Using a local $H^2 - L^2$ inverse inequality on I_i we have

$$\|u_h - u\|_{H^2(I_i)} \leq \|u_h - \chi\|_{H^2(I_i)} + \|\chi - u\|_{H^2(I_i)} \leq Ch_i^{-2}\|u_h - \chi\|_{L^2(I_i)} + \|\chi - u\|_{H^2(I_i)}$$

$$\leq Ch_i^{-2}\left(\|u_h - u\|_{L^2(I_i)} + \|u - \chi\|_{L^2(I_i)}\right) + \|u - \chi\|_{H^2(I_i)}. \qquad (3.143)$$

We recall now the local estimates for $u - \chi$ given in (3.77) and obtain

$$h_i^{-2}\|u - \chi\|_{L^2(I_i)} + \|u - \chi\|_{H^2(I_i)} \leq Ch_i^{r-2}\|u^{(r)}\|_{L^2(I_i)}.$$

Therefore, by (3.142), (3.143) we have

$$\|r_h\|_{L^2(I_i)}^2 \leq Ch_i^{-4}\|u_h - u\|_{L^2(I_i)}^2 + Ch_i^{2r-4}\|u^{(r)}\|_{L^2(I_i)}^2.$$

This gives, in view of the quasiuniformity of the mesh and (3.129), that

$$\left(\sum_{i=0}^{N} h_i^2\|r_h\|_{L^2(I_i)}^2\right)^{1/2} \leq C\left(\sum_{i=0}^{N} h_i^{-2}\|u_h - u\|_{L^2(I_i)}^2\right)^{1/2} + C\left(\sum_{i=0}^{N} h_i^{2r-2}\|u^{(r)}\|_{L^2(I_i)}^2\right)^{1/2}$$

$$\leq Ch^{-1}\|u_h - u\| + Ch^{r-1}\|u^{(r)}\| \leq Ch^{r-1}\|u\|_r,$$

i.e. that the right-hand side of (3.132) is of $\mathcal{O}(h^{r-1})$ if $u \in H^r \cap \overset{0}{H}{}^1$ as claimed. Since $\left(\sum_{i=0}^{N} h_i^4\|r_h\|_{L^2(I_i)}^2\right) \leq h\left(\sum_{i=0}^{N} h_i^2\|r_h\|_{L^2(I_i)}^2\right)^{1/2}$, we see that the right-hand side of (3.137) is of $\mathcal{O}(h^r)$.

The form of the *a posteriori* error bounds in (3.132) and (3.137) allows one to refine the mesh *locally* in order to achieve errors $\|u - u_h\|$ or $\|u' - u_h'\|$ smaller than preassigned tolerances. For example if we want the error $\|u' - u_h'\|$ not to exceed a given tolerance ε, the inequality (3.132) implies that it is sufficient to require for all i that

$$h_i\|r_h\|_{L^2(I_i)}^2 \leq \pi^2 \alpha^2 \varepsilon^2. \qquad (3.144)$$

(Indeed, if this holds, then $\|u' - u_h'\| \leq \frac{1}{\pi\alpha}\left(\sum_{i=0}^{N} h_i^2\|r_h\|_{L^2(I_i)}^2\right)^{1/2} \leq \varepsilon\sqrt{\sum_{i=0}^{N} h_i} = \varepsilon$.) To achieve (3.144) it is sufficient to compute $\|r_h\|_{L^2(I_i)}$ for all i, refine the mesh locally, i.e. take intervals of smaller width h_i for those i that (3.144) happens to be violated, and recompute the solution u_h using the new mesh. This is much less expensive than

reducing $h = \max(x_{i+1} - x_i)$ as an *a priori* bound would suggest. (Note that we may also take larger h_i if the left-hand side of (3.144) is much smaller than the right-hand side.) This *adaptive* variation of the mesh in order to achieve errors of desired magnitude at low cost is absolutely essential for solving problems with the finite element method in higher dimensions.

3.7 Non-self-adjoint and indefinite problems

In this section we will generalize the error estimates obtained for the standard Galerkin finite element method for the two-point bvp (3.1)–(3.2) in order to derive analogous estimates in the case of a more general second-order linear differential operator L which is *not self-adjoint*, i.e. does not satisfy $(Lu, v) = (u, Lv)$ for $u, v \in H^2 \cap \overset{0}{H}{}^1$, and is not necessarily positive definite (i.e. is *indefinite*), i.e. does not satisfy $B(v, v) \geq c\|v\|_1^2$ for some positive constant $c > 0$, for all $v \in \overset{0}{H}{}^1$. Here, and in the rest of this section, B will denote the bilinear form on $\overset{0}{H}{}^1 \times \overset{0}{H}{}^1$ that is obtained by integrating by parts in the integral $\int_0^1 Luv \, dx$, for $v \in \overset{0}{H}{}^1$. Consider the two-point bvp on $I = (0, 1)$ given by

$$
\begin{aligned}
&Lu \equiv -(au')' + bu' + cu = f, \quad 0 \leq x \leq 1, \\
&u(0) = u(1) = 0,
\end{aligned}
\tag{3.145}
$$

and assume that $a \in C^1(\bar{I})$ is such that $a(x) \geq a_0 > 0$, $x \in \bar{I}$, for some constant a_0, that $b \in C(\bar{I})$, and $c \in C(\bar{I})$. In addition let $f \in C(\bar{I})$ when classical solutions of (3.145) are considered and $f \in L^2(I)$ when generalized solutions are in view. (Note that the ode $p_0 u'' + p_1 u' + p_2 u = f$ (with $p_0 < 0$, $x \in \bar{I}$), apparently more general, may nevertheless be written in the form given in (3.145) if $a = -p_0$, $b = p_1 - p_0'$, $c = p_2$.) Note that if $f \in C(\bar{I})$, L may be considered as a linear operator in $C(\bar{I})$ with domain $D(L) = \{u \in C^2(\bar{I}), \ u(0) = u(1) = 0\}$, while if $f \in L^2 = L^2(I)$, it may be considered as a linear operator in L^2 with domain $D(L) = H^2 \cap \overset{0}{H}{}^1$; in the latter case the solutions of (3.145) are generalized (weak) solutions in $H^2 \cap \overset{0}{H}{}^1$ and the ode $Lu = f$ holds in L^2 and pointwise a.e. in I.

In Chapter 2 we saw a variational theory of existence, uniqueness, and regularity of solutions of two-point bvp's in the case of self-adjoint positive-definite operators corresponding to the case $b = 0$, $c \geq 0$ in $\bar{I}$. Here, in order to study analogous issues

for the bvp (3.145) we will adopt a more classical, ode approach and construct the solution of (3.145) by means of the *Green's function* for the operator L under the b.c.'s $u(0) = u(1) = 0$, following e.g. Kato [2.5, Section III.2.3].

It is well known that the homogeneous ode $Lu = 0$ has two linearly independent solutions v_1, $v_2 \in C^2(\bar{I})$. Consider u_1, u_2 two nontrivial solutions of $Lu = 0$ such that $u_1(0) = 0$, $u_2(1) = 0$. (That such solutions exist is evident from the fact that if $u_1 = c_1 v_1 + c_2 v_2$ with $u_1(0) = 0$ and e.g. $u_1'(0) = 1$, then the numbers c_1, c_2 exist uniquely since the Wronskian $v_1 v_2' - v_1' v_2$ is non-zero; we argue similarly for u_2 for which $u_2(1) = 0$, and e.g. $u_2'(0) = 1$.) Now note that the Wronskian $W(x) = (u_1 u_2' - u_2 u_1')(x)$ is non-zero in $\bar{I}$ if and only if $u_1(1) \neq 0$, since $W(x) \neq 0 \iff 0 \neq W(1) = u_1(1) u_2'(1) - u_2(1) u_1'(1) = u_1(1)$.

A simple sufficient condition that ensures that $u_1(1) \neq 0$ is that $c(x) \geq 0$, $x \in \bar{I}$. To see this, suppose that $c(x) \geq 0$ and $u_1(1) = 0$. Then u_1 has either a positive maximum or a negative minimum in $(0, 1)$. (Indeed, if $u_1 > 0$ in $(0, 1)$ it has a positive maximum, if $u_1 < 0$ it has a negative minimum, and if it changes signs the conclusion also holds; note that there does not exist a point in $(0, 1)$ where $u_1 = u_1' = 0$ since, by the uniqueness of solutions of the initial-value problem, this would imply that $u_1 \equiv 0$ in $\bar{I}$). If u_1 has a positive maximum at $x = x_0 \in I$, then $u_1''(x_0) < 0$, $u_1'(x_0) = 0$, $u_1(x_0) > 0$, and $0 = Lu(x_0) = -a(x_0)u''(x_0) + (b - a')u'(x_0) + c(x_0)u(x_0) > 0$, a contradiction. Similarly one may also rule out that u_1 has a negative minimum. We conclude that the condition $c(x) \geq 0$, $x \in \bar{I}$, implies that $u_1(1) \neq 0$.

We will assume then that $W \neq 0$ for $x \in \bar{I}$ for the continuation of this proof. Consider the Green's function of the operator L with zero b.c., given by

$$g(x, y) = \begin{cases} \frac{u_1(x)\, u_2(y)}{-a(y)\, W(y)}, & 0 \leq x \leq y \leq 1, \\ \frac{u_2(x)\, u_1(y)}{-a(y)\, W(y)}, & 0 \leq y \leq x \leq 1. \end{cases}$$

Suppose that $f \in C(\bar{I})$. Then the classical solution of the problem (3.145) is given by the formula

$$u(x) = \int_0^1 g(x, y) f(y)\, \mathrm{d}y, \quad 0 \leq x \leq 1. \tag{3.146}$$

To show this we should check that $u \in C^2(\bar{I})$, $Lu = f$, $u(0) = u(1) = 0$; uniqueness of solution follows. (This is equivalent to showing that if $W \neq 0$, L is invertible in

$C(\bar{I})$ and its inverse, the "solution" operator $T : C(\bar{I}) \to D(L) = \{u \in C^2(\bar{I}),\ u(0) = u(1) = 0\}$, is the integral operator given by $Tf = u$, where u satisfies (3.146).)

We write (3.146) as

$$u(x) = u_2(x)I_2(x) + u_1(x)I_1(x), \quad \forall x \in \bar{I}, \tag{3.147}$$

where I_2, I_1 are the integrals given for $x \in \bar{I}$ by

$$I_2(x) = \int_0^x \frac{u_1(y)f(y)}{-a(y)W(y)}\mathrm{d}y, \quad I_1(x) = \int_x^1 \frac{u_2(y)f(y)}{-a(y)W(y)}\mathrm{d}y.$$

Under our hypotheses, I_2 and I_1 are continuous in $\bar{I}$. Therefore, u given by (3.147) is continuous and satisfies $u(0) = u(1) = 0$. It is easy to check that u is differentiable and its derivative is given by $u' = u_2'I_2 + u_1'I_1$. It follows that u' is differentiable and

$$u'' = u_2''I_2 + u_1''I_1 - \frac{f}{a}.$$

We conclude that $u \in C^2(\bar{I})$ and by the above

$$\begin{aligned}
Lu = -au'' + (b - a')u' + cu &= -au_2''I_2 - au_1''I_1 + f \\
&\quad + (b - a')u_2'I_2 + (b - a')u_1'I_1 + cu_2I_2 + cu_1I_1 \\
&= I_2(Lu_2) + I_1(Lu_1) + f = f,
\end{aligned}$$

i.e. u is a classical solution of (3.145). If u, $\tilde{u}$ are two such solutions, then, if $w = u - \tilde{u}$, $Lw = 0$, $w(0) = w(1) = 0$, and w vanishes since it satisfies (3.146) with $f = 0$.

If $f \in L^2(I)$ the formula (3.146) yields a weak solution u of (3.146) that belongs to $H^2 \cap \overset{0}{H}{}^1$. (Hence if L is now considered as an operator in L^2 with $D(L) = H^2 \cap \overset{0}{H}{}^1$, then, if $W \neq 0$, L is invertible and its inverse, the "solution" operator defined on L^2 by $Tf(x) = \int_0^1 g(x,y)f(y)\,\mathrm{d}y$, maps L^2 into $H^2 \cap \overset{0}{H}{}^1$.) We will prove this, i.e. that $u = Tf$ for $f \in L^2$ belongs to $H^2 \cap \overset{0}{H}{}^1$ and satisfies $Lu = f$ in L^2. Moreover the *elliptic regularity* estimate

$$\|u\|_2 \leq C\|f\|, \tag{3.148}$$

holds, i.e. the linear operator $T : L^2 \to H^2 \cap \overset{0}{H}{}^1$ is bounded. Note that the kernel g is continuous on $[0,1] \times [0,1]$ and hence bounded. Therefore if u is given by (3.146), then for $x \in \bar{I}$ $|u(x)| \leq \int_0^1 |g(x,y)||f(y)|\,\mathrm{d}y \leq C \int_0^1 |f| \leq C\|f\|$. Therefore $u \in L^\infty$. Writing $u = u_2I_2 + u_1I_1$, where I_2, I_1 are the integrals defined as before, and taking

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into account that I_2, I_1 are continuous on $\bar{I}$ since f is integrable on I, we conclude that $u \in C(\bar{I})$, that $u(0) = u(1) = 0$, and, by the above, $\max_{x \in \bar{I}} |u(x)| \leq C\|f\|$, so that $\|u\| \leq C\|f\|$. Since f in integrable, the functions I_2 and I_1 are differentiable a.e. in I and $I_2'(x) = \frac{u_1(x) f(x)}{-a(x) W(x)}$, $I_1'(x) = \frac{u_2(x) f(x)}{a(x) W(x)}$ with the equalities understood a.e. in I. We conclude that $u' = u_2' I_2 + u_1' I_1$ a.e. and therefore u' is continuous. We also have $|u'|_\infty \leq C\|f\|$ and therefore $\|u'\| \leq C\|f\|$. Finally $u'' = u_2'' I_2 + u_1'' I_1 - \frac{f}{a}$ a.e. in I, which shows that $u'' \in L^2$ and $\|u''\| \leq C\|f\|$. We conclude that now $Lu = f$ in L^2 (i.e. a.e. pointwise), that $u \in H^2 \cap \overset{0}{H}{}^1$, and that the estimate (3.148) holds.

As a final remark, to be used later, note that the adjoint L^* of L is given by

$$L^* v = -(av')' - (bv)' + cv. \tag{3.149}$$

One may indeed easily show by integrating by parts that

$$(Lv, w) = (v, L^* w), \quad \forall v, w \in H^2 \cap \overset{0}{H}{}^1.$$

As done in the case of L, one may show that if e.g. $c - b' \geq 0$ L^*, viewed as an operator in L^2, is invertible, $D(L^*) = H^2 \cap \overset{0}{H}{}^1$ and its inverse (solution operator of the problem $L^* u = f$) is bounded from L^2 into $H^2 \cap \overset{0}{H}{}^1$. (Hence, a *sufficient* condition for the invertibility of both L and L^* is that $c \geq \max(0, b')$ in $\bar{I}$.)

Even if this condition is not satisfied one may still have existence of a classical or a weak solution of (3.145) if one is able to show that $W \neq 0$. An example is given by the two-point bvp for the 1d Helmholtz equation

$$-u'' - k^2 u = f(x), \quad 0 \leq x \leq 1,$$
$$u(0) = u(1) = 0, \tag{3.150}$$

where k is a positive constant. The operator $L = -\frac{d^2}{dx^2} - k^2$ is self-adjoint under the b.c. of (3.150) but indefinite. In order to construct the Green's function as we did previously, we have to find nontrivial solutions u_1, u_2 of the homogeneous ode such that $u_1(0) = 0$, $u_2(1) = 0$. We take for example $u_1 = \sin kx$, $u_2 = \sin k(1 - x)$, and compute their Wronskian $W = u_1 u_2' - u_1' u_2 = -k \sin k$. Hence $W \neq 0$ if and only if $\sin k \neq 0$, i.e. $k \neq n\pi$, $n = 1, 2, 3, \ldots$, i.e. as expected if $k^2 \neq \lambda_n$, where $\lambda_n = n^2 \pi^2$, $n \geq 1$, are the eigenvalues of the Sturn-Liouville problem given by (3.150) with $f = 0$,

which is equivalent to assuming that the bvp (3.150) has only the trivial solution $u = 0$ when $f = 0$. Therefore the Green's function is given, for $k \neq n\pi$, by

$$g(x, y) = \begin{cases} \frac{\sin kx \, \sin k(1-y)}{k \sin k}, & 0 \leq x \leq y \leq 1, \\ \frac{\sin k(1-x) \, \sin kx}{k \sin k}, & 0 \leq y \leq x \leq 1, \end{cases}$$

and $u(x) = \int_0^1 g(x, y) f(y) \, dy$ is the solution of (3.150), valid for $k \neq n\pi$; it satisfies $\|u\|_2 \leq C(k)\|f\|$, $k \neq n\pi$.

We proceed now to the numerical solution of (3.145) with Galerkin finite element methods. We assume again that $a \in C^1(\bar{I})$ with $a(x) \geq a_0 > 0$, $x \in \bar{I}$, and that $b \in C^1(\bar{I})$, and $c \in C(\bar{I})$. We let $f \in L^2$ and assume that (3.145) has a unique solution $u \in H^2 \cap \overset{0}{H}{}^1$ satisfying $\|u\|_2 \leq C\|f\|$. We also assume that the associated two-point bvp for the adjoint operator L^* on $\bar{I}$ with boundary conditions $u(0) = u(1) = 0$ and right-hand side $f \in L^2(I)$ has a unique solution u in $H^2 \cap \overset{0}{H}{}^1$ satisfying $\|u\|_2 \leq C\|f\|$ as well. The weak form of (3.143) is given as usual by

$$B(u, v) = (f, v), \quad \forall v \in \overset{0}{H}{}^1, \tag{3.151}$$

where, for $v, w \in H^1$, the nonsymmetric bilinear form B is given by

$$B(v, w) := \int_0^1 (av'w' + bv'w + cvw) \, dx. \tag{3.152}$$

Obviously, for some constant C_1 B satisfies

$$|B(v, w)| \leq C_1\|v\|_1\|w\|_1, \quad \forall v, w \in H^1, \tag{3.153}$$

but it is not in general positive definite, as for $v \in \overset{0}{H}{}^1$

$$\begin{aligned} B(v, v) &= \int_0^1 \left[a(v')^2 + bv'v + cv^2 \right] dx \\ &= \int_0^1 \left[a(v')^2 + \tfrac{1}{2} b\partial_x(v^2) + cv^2 \right] dx = \int_0^1 \left[a(v')^2 - \tfrac{1}{2} b'v^2 + cv^2 \right] dx \\ &= \int_0^1 \left[a(v')^2 + \left(c - \tfrac{b'}{2} \right) v^2 \right] dx. \end{aligned}$$

Therefore $B(v, v)$ satisfies a *Gårding's inequality*, i.e. an inequality of the form

$$B(v, v) \geq \alpha\|v'\|^2 - \beta\|v\|^2, \quad \forall v \in \overset{0}{H}{}^1, \tag{3.154}$$

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where in our case $\alpha = a_0 > 0$, $\beta = -\min_{0 \leq x \leq 1} \left(c - \frac{b'}{2} \right) \in \mathbb{R}$. We conclude that B is positive definite in $\overset{0}{H}{}^1$ if $\beta \leq 0$, or if β is positive and small enough as we may easily see by applying the Poincaré-Friedrichs inequality $\|v'\| \geq C_I \|v\|$ in the right-hand side of (3.154). Otherwise, B is indefinite and we cannot apply the Lax-Milgram theorem to it.

In order to approximate the solution of (3.145) by the standard Galekrin finite element method we let as usual S_h be a finite-dimensional subspace of $\overset{0}{H}{}^1$ with the approximation property that if $r \geq 2$, then, given $v \in H^2 \cap \overset{0}{H}{}^1$, where $2 \leq s \leq r$,

$$\inf_{\chi \in S_h} \left(\|v - \chi\| + h\|v - \chi\|_1 \right) \leq C h^s \|v\|_s. \tag{3.155}$$

We try to define the Galerkin approximation $u_h \in S_h$ as usual by

$$B(u_h, \phi) = (f, \phi), \quad \forall \phi \in S_h. \tag{3.156}$$

However we do not know if u_h exists as a solution of the linear system represented by (3.156) since B is in general indefinite. However, following Schatz[4] we may prove that if h is sufficiently small, then u_h exists uniquely and satisfies our usual L^2 and H^1 error estimates.

Theorem 3.29. Suppose that u, the solution (3.145) belongs to $H^r \cap \overset{0}{H}{}^1$ and let h be sufficiently small. Then the solution u_h of the Galerkin equations (3.156) exists uniquely and satisfies

$$\|u - u_h\|_j \leq C h^{r-j} \|u\|_r, \quad j = 0, 1. \tag{3.157}$$

Proof. Assume for the time being that the equation (3.156) has a solution $u_h \in S_h$. Then, by (3.151)

$$B(u - u_h, \phi) = 0, \quad \forall \phi \in S_h. \tag{3.158}$$

Let C_* be the constant in the Poincaré-Friedrichs inequality written in the form

$$\|v\|_1 \leq C_* \|v'\|, \quad \forall v \in \overset{0}{H}{}^1. \tag{3.159}$$

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[4]Math. Comp. 28(1974), 959–962.

(Recall from Remark 2.29 (iii) that the best such C_* is equal to $\left(1 + \frac{1}{\pi^2}\right)^{1/2}$.) From (3.159), (3.154) and (3.158) we have

$$\frac{\alpha}{C_*^2}\|u - u_h\|_1^2 \leq \alpha\|(u - u_h)'\|^2 \leq B(u - u_h, u - u_h) + \beta\|u - u_h\|^2$$

$$= B(u - u_h, u - \chi) + \beta\|u - u_h\|^2, \quad \forall \chi \in S_h.$$

Therefore, by (3.153) and the approximation property (3.155) that give $|B(u - u_h, u - \chi)| \leq Ch^{r-1}\|u - u_h\|_1\|u\|_r$, we conclude that

$$\frac{\alpha}{C_*^2}\|u - u_h\|_1^2 \leq Ch^{r-1}\|u - u_h\|_1\|u\|_r + \beta\|u - u_h\|^2. \tag{3.160}$$

Let now L^* be the adjoint of L, introduced previously. Consider the problem

$$L^*w \equiv -(aw')' - (bw)' + c = u - u_h, \quad x \in [0, 1], \tag{3.161}$$
$$w(0) = w(1) = 0,$$

whose solution has been assumed to exist uniquely in $H^2 \cap \overset{0}{H}{}^1$ and satisfy the elliptic regularity estimate. Hence

$$\|w\|_2 \leq C\|u - u_h\|. \tag{3.162}$$

Now, form (3.161), for $v \in \overset{0}{H}{}^1$,

$$(u - u_h, v) = (L^*w, v) = (w, Lv) = (Lv, w) = B(v, w).$$

Taking $v = u - u_h$ we have by (3.158), (3.153), for any $\chi \in S_h$

$$\|u - u_h\|^2 = B(u - u_h, w) = B(u - u_h, w - \chi) \leq C_1\|u - u_h\|_1\|w - \chi\|_1.$$

Apply now the approximation property (3.155) for $s = 2$ and obtain from the above and (3.162), for some positive constant C_3

$$\|u - u_h\|^2 \leq C_3 h\|u - u_h\|\|u - u_h\|_1,$$

i.e.

$$\|u - u_h\| \leq C_3 h\|u - u_h\|_1. \tag{3.163}$$

Let us examine (3.160). If $\beta \leq 0$ or if e.g. $0 < \beta < \frac{\alpha}{C_*^2}$ we have that

$$\|u - u_h\|_1 \leq Ch^{r-1}\|u\|_r. \tag{3.164}$$

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For general positive β, (3.163) and (3.160) give

$$\frac{\alpha}{C_*^2}\|u - u_h\|_1^2 \leq Ch^{r-1}\|u - u_h\|_1\|u\|_r + \beta C_3^2 h^2 \|u - u_h\|_1^2.$$

So, if $\frac{\alpha}{C_*^2} - \beta C_3^2 h^2 > 0$, i.e. if h is small enough, this gives (3.162) again, for a different constant C.

Let now $f = 0$. Then $u = 0$, since we assumed that (3.145) has a unique solution. But if $u = 0$ (3.164) gives $u_h = 0$, i.e. that the homogeneous linear system represented by (3.156) has only the trivial solution. We conclude that the matrix of this system is invertible and therefore that the solution u_h of (3.156) exists uniquely. The estimate (3.164) holds therefore, and (3.163) gives then that $\|u - u_h\| \leq Ch^r \|u\|_r$; the theorem has been proved. $\qquad\square$

Chapter 4

Results from the Theory of Sobolev Spaces and the Variational Formulation of Elliptic Boundary–Value Problems in $\mathbb{R}^N$

In this chapter we let Ω be a bounded domain in $\mathbb{R}^N$ (i.e. an open, connected, bounded point set in $\mathbb{R}^N$). Many of the results that we shall quote hold for arbitrary open sets in Ω but we do not strive for generality here, having in mind applications in the case of boundary–value problems on bounded domains. Let $x = (x_1, \ldots, x_N)$ denote a generic point of Ω or $\mathbb{R}^N$. All functions are real–valued.

4.1 The Sobolev space $H^1(\Omega)$.

Definition. *The Sobolev space $H^1 = H^1(\Omega)$ is defined by*

$$
\begin{aligned}
H^1(\Omega) \;=\; & \{u \in L^2(\Omega) : \exists\, g_1, g_2, \ldots, g_N \in L^2(\Omega), \text{such that} \\
& \int_\Omega u\, \frac{\partial \phi}{\partial x_i} = -\int_\Omega g_i \phi, \;\; \forall \phi \in C_c^\infty(\Omega),\, \forall i : \; 1 \le i \le N\}.
\end{aligned}
$$

For $u \in H^1$ we denote $g_i = \frac{\partial u}{\partial x_i}$ and call g_i the weak (generalized) partial derivative of u with respect to x_i.

(Note e.g. that this definition does not need that Ω be bounded).

Remarks.

(i) Using Lemma 2.4 we conclude that each generalized partial derivative g_i in the above definition is unique, as in the case of one dimension.

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(ii) Again, $C_c^1(\Omega)$ may be used in place of $C_c^\infty(\Omega)$ for the *test functions* ϕ.

(iii) It is clear that if $u \in C^1(\Omega) \cap L^2(\Omega)$ and if the classical partial derivatives $\frac{\partial u}{\partial x_i}$ belong to $L^2(\Omega)$ for $i = 1, 2, \ldots, N$, then the weak derivatives exist and coincide with the classical; in particular $u \in H^1(\Omega)$. Of course $C^1(\overline{\Omega}) \subset H^1(\Omega)$. It can be shown with some care that, inversely, if $u \in H^1(\Omega) \cap C(\Omega)$ and if the generalized derivatives $\frac{\partial u}{\partial x_i}$ belong to $C(\Omega)$ for $1 \le i \le N$, then $u \in C^1(\Omega)$.

(iv) Since $u \in L^2(\Omega) \Rightarrow u \in L^1_{\text{loc}}(\Omega)$, we can define the *distributional* derivatives of u, $\frac{\partial u}{\partial x_i}$, in the sense of the theory of distributions. We can say that H^1 is the set of elements of $L^2(\Omega)$ whose distributional derivatives $\frac{\partial u}{\partial x_i}$, $1 \le i \le N$, are represented by functions in L^2.

It is clear that H^1 is a subspace of L^2. Denoting by $(\cdot, \cdot)$, $\| \cdot \|$ the inner product, respectively the norm, on $L^2(\Omega)$, we introduce the quantities:

$$(u, v)_1 \;=\; (u, v) + \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right), \quad u, v \in H^1,$$

$$\|u\|_1 \;=\; \left(\|u\|^2 + \sum_{i=1}^{N} \| \frac{\partial u}{\partial x_i} \|^2 \right)^{\frac{1}{2}}, \quad u \in H^1,$$

which clearly define an *inner product*, resp. (the induced) *norm*, on H^1. Hence H^1 is a normed linear space.

Theorem 4.1. The space $(H^1(\Omega), \| \cdot \|_1)$ is a Hilbert space.

Proof. Adapt the 1–dim. proof. $\square$

Remark. $(H^1(\Omega), \| \cdot \|_1)$ is separable.

In the case of one dimension we had shown (Theorem 2.4) that the restrictions on $\Omega = I$ of functions in $C_c^\infty(\mathbb{R})$ form a dense set in H^1. In more than one dimensions this is not true for an arbitrary Ω. We list below several "density" results:

1. (**Friedrichs**) Let $u \in H^1(\Omega)$. Then, there exists a sequence $\{u_n\} \in C_c^\infty(\mathbb{R}^N)$ such that:

   (α) $\quad u_n|_\Omega \to u$ in $L^2(\Omega)$

   (β) $\quad \forall i : \frac{\partial u_n}{\partial x_i}|_\omega \to \frac{\partial u}{\partial x_i}|_\omega$ in $L^2(\omega)$ for every precompact $\omega \subset \Omega$.

2. (**Meyers** − **Serrin**) If $u \in H^1(\Omega)$, then $\exists$ sequence $\{u_n\} \in C^\infty(\Omega) \cap H^1(\Omega)$ such that $u_n \to u$ in $H^1(\Omega)$.

3. If Ω is an arbitrary open (or even open and bounded) set and if $u \in H^1(\Omega)$, in general there does not exist a sequence $u_n \in C_c^1(\mathbb{R}^N)$ such that $u_n|_\Omega \to u$ in $H^1(\Omega)$. (The problem is at the boundary; however, compare with Theorem 4.2 below).

With the aid e.g. of Friedrichs' result (1), above, we can show the following analog of Proposition 2.1:

Proposition 4.1. If $u, v \in H^1(\Omega) \cap L^\infty(\Omega)$, then $uv \in H^1 \cap L^\infty$ and

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i}, \quad 1 \le i \le N.$$

As in the case of one dimension, many properties of H^1 are established easier if $\Omega = \mathbb{R}^N$. Then an extension result is needed to establish the same property for an $\Omega \subset \mathbb{R}^N$. Such extensions (of functions i.e. of $H^1(\Omega)$ to functions of $H^1(\mathbb{R}^N)$)are not always possible to construct, unless the set Ω has a "regular" boundary $\partial\Omega$ in a certain sense.

Definition. *Let Ω be a bounded domain in $\mathbb{R}^N$. We say that Ω is of class C^1 if there exist finitely many open balls $B_i \subset \mathbb{R}^N$, $i = 1, 2, \ldots, M$ such that*

(i) *$\partial\Omega \subset \cup_{i=1}^M B_i$, $B_i \cap \partial\Omega \ne \emptyset$.*

(ii) *There is for each i, $1 \le i \le M$, a function $y = f^{(i)}(x)$ in $C^1(\overline{B_i})$ which maps the ball B_i in an one–to–one and onto way onto a domain in $\mathbb{R}^N$ so that $\partial\Omega \cap \overline{B_i}$ gets mapped onto a subset of the hyperplane $y_N = 0$ and $\Omega \cap B_i$ into a simply connected domain in the half–space $\{y : \ y_N > 0\}$. Moreover the Jacobian determinant $\det\left(\frac{\partial f_k^{(i)}}{\partial x_l}\right)$ does not vanish for $x \in \overline{B_i}$.*

The C^1 property of Ω permits for example the advertized "extension" result:

Proposition 4.2. Suppose that Ω is of class C^1 (or that $\Omega = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$). Then there exists a linear extension operator $P : H^1(\Omega) \to H^1(\mathbb{R}^N)$ such that,

(i) $Pu|_\Omega = u \ \forall u \in H^1(\Omega)$.

(ii) $\exists C$ such that $\| Pu \|_{L^2(\mathbb{R}^N)} \leq C \| u \|_{L^2(\Omega)}, \forall u \in H^1(\Omega)$.

(iii) $\exists C'$ such that $\| Pu \|_{H^1(\mathbb{R}^N)} \leq C' \| u \|_{H^1(\Omega)} \forall u \in H^1(\Omega)$.

Using e.g. this extension one may show the following important density result:

Theorem 4.2. If Ω is of class C^1 and $u \in H^1(\Omega)$, then, there exists a sequence $u_n \in C_c^\infty(\mathbb{R}^N)$ such that $u_n|_\Omega \to u$ in H^1. That is to say, the restriction to Ω of functions in $C_c^\infty(\mathbb{R}^N)$ are dense in $H^1(\Omega)$.

For the purposes of studying boundary–value problems, it is important to study the behavior of functions in $H^1(\Omega)$ at the boundary $\partial\Omega$ of Ω. *We suppose to this effect that Ω is of class C^1.* Then we can "measure" (in the sense of the above definition of the C^1 domain) the content of pieces on the "hypersurface" $\partial\Omega$ (through measuring "plane" surface pieces on the hyperplane $y_N = 0$, i.e. measuring the "area" of the images of the pieces of $\partial\Omega$ that are mapped on $y_N = 0$ by the functions $y = f^{(i)}(x)$). One may define open sets on $\partial\Omega$ as intersections of $\partial\Omega$ and open sets in $\mathbb{R}^N$. These open sets on $\partial\Omega$ one can then complete into a σ–algebra and then extend the elementary "content measure" into the Lebesque measure on $\partial\Omega$. With respect to this measure we may define the surface integral $\int_\Omega g(y)dy$. Consequently, one may consider the Hilbert space $L^2(\partial\Omega)$ of functions defined on $\partial\Omega$ with norm

$$\| g \|_{L^2(\partial\Omega)} \equiv \left(\int_{\partial\Omega} g^2(y) \, dy \right)^{\frac{1}{2}} \text{ for } g \in L^2(\partial\Omega).$$

One may show first that the following result holds for *smooth* functions:

Lemma 4.1. Let Ω be of class C^1. Then there exists a constant C such that for all functions $f \in C^\infty(\overline{\Omega})$ we have

$$\| f \|_{L^2(\partial\Omega)} \leq C \| f \|_{H^1(\Omega)} .$$

This lemma, along with Theorem 4.2, permits us to define boundary values for functions in $H^1(\Omega)$ for C^1 domains Ω. Let $f \in H^1(\Omega)$. Then $\exists f_n \in C^\infty(\overline{\Omega})$ such that $f_n \to f$ in $H^1(\Omega)$ (Theorem 4.2). By Lemma 4.1 $\{f_n\}$ is Cauchy in $L^2(\partial\Omega)$. So $f_n \to g$ in $L^2(\partial\Omega)$. (It is easily seen that this g is independent of the chosen sequence f_n). This limiting function g we denote by $f|_{\partial\Omega}$ and call it "boundary value" on $\partial\Omega$ of f, or *trace* of f on $\partial\Omega$ (sometimes we say that $g = f|_{\partial\Omega}$ is the boundary value of f *in the sense of trace*). We summarize in the following theorem.

Theorem 4.3 (Trace theorem)**.** Let Ω be of class C^1. Then, every $f \in H^1(\Omega)$ possesses boundary values in the above sense (also denoted by f), which belong to the Hilbert space $L^2(\partial\Omega)$. Moreover there exists a constant C such that

$$\| f \|_{L^2(\partial\Omega)} \leq C \| f \|_{H^1(\Omega)} \ \forall \ f \in H^1(\Omega).$$

Remarks.

(i) It can be shown that under our hypotheses on Ω, the boundary value $f|_{\partial\Omega}$ of a function $f \in H^1(\Omega)$ actually belongs to the so–called "fractional–order" space $H^{1/2}(\partial\Omega)$, an intermediate space defined by interpolation between the spaces $H^0(\partial\Omega) \equiv L^2(\partial\Omega)$ and $H^1(\partial\Omega)$.

(ii) Let Ω be a C^1 domain. Then we may show that *Green's formula* (*Gauss's theorem*) holds: $\forall i : \ 1 \leq i \leq N$

$$\int_\Omega \frac{\partial u}{\partial x_i} \, v = - \int_\Omega u \, \frac{\partial v}{\partial x_i} + \int_{\partial\Omega} u \, v \, \nu_i \, dy$$

for $u, v \in H^1(\Omega)$. Here dy is the surface Lebesgue measure constructed on $\partial\Omega$ as above and $\nu_i = \vec{n} \cdot \vec{e_i}$ is the ith component of the unit outward normal $\vec{n}(y)$ defined on the boundary of the C^1 domain Ω. (Note that since $u, v \in H^1(\Omega)$ $\Rightarrow \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \in L^2(\Omega)$, $u, v \in L^2(\partial\Omega)$ and all terms in the above equality make sense).

4.2 The Sobolev space $\overset{o}{H}{}^1(\Omega)$.

Definition. *We define the space $\overset{o}{H}{}^1(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to the $H^1(\Omega)$ norm.*

Hence $\left(\overset{o}{H}{}^1(\Omega), \| \cdot \|_1 \right)$ is a Hilbert space (a closed subspace of H^1). (We may show that the completion of $C_c^\infty(\mathbb{R}^N)$ under the $\| \cdot \|_{H^1(\mathbb{R}^N)}$ norm is $H^1(\mathbb{R}^N)$ itself, i.e. that $\overset{o}{H}{}^1(\mathbb{R}^N) = H^1(\mathbb{R}^N)$. But for $\Omega \subset \mathbb{R}^N$ we have in general that $\overset{o}{H}{}^1(\Omega) \subset H^1(\Omega)$). More precisely, we may show that for sufficiently smooth $\partial\Omega$ (e.g. C^1) then $\overset{o}{H}{}^1(\Omega)$ consists precisely of those functions in $H^1(\Omega)$ which vanish (in the sense of trace) on $\partial\Omega$.

Theorem 4.4. Let Ω be of class C^1. Then

$$\overset{o}{H}{}^1(\Omega) = \{v \in H^1(\Omega) : \ v|_{\partial\Omega} = 0\},$$

(where by $v|_{\partial\Omega} = 0$ we mean that the trace of v on $\partial\Omega$ (a function in $L^2(\partial\Omega)$), is equal to the zero function in $L^2(\partial\Omega)$).

On $\overset{0}{H}{}^1(\Omega)$ we also have the analog of Proposition 2.3:

Proposition 4.3 (Inequality of Poincaré–Friedrichs). Let Ω be a bounded domain. Then, there exists a constant $C_* = C_*(\Omega)$ such that

$$\|u\| \leq C_* \left(\sum_{i=1}^{N} \| \frac{\partial u}{\partial x_i} \|^2 \right)^{\frac{1}{2}} \quad \forall u \in \overset{0}{H}{}^1(\Omega).$$

In particular the expression $\left(\sum_{i=1}^{N} \| \frac{\partial u}{\partial x_i} \|^2 \right)^{1/2}$ is a norm on $\overset{0}{H}{}^1(\Omega)$, equivalent to the norm $\| \cdot \|_1$ on $\overset{0}{H}{}^1(\Omega)$. The quantity $\int_{\Omega} \left(\sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right) dx$ is an inner product on $\overset{0}{H}{}^1(\Omega)$.

Remark. As in the 1–dim case, if Ω is of class C^1, then $u \in \overset{0}{H}{}^1(\Omega)$ if and only if the extension

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega \end{cases}$$

belongs to $H^1(\mathbb{R}^N)$. (In such a case also $\overline{\frac{\partial u}{\partial x_i}} = \frac{\partial \bar{u}}{\partial x_i}$).

4.3 The Sobolev spaces $H^m(\Omega)$, $m = 2, 3, 4, \ldots$

For $m \geq 2$ an integer we can define the spaces $H^m(\Omega)$ recursively by

$$H^m = H^m(\Omega) = \{u \in H^{m-1}(\Omega) : \frac{\partial u}{\partial x_i} \in H^{m-1}(\Omega), \ \forall i = 1, 2, \ldots, N\}.$$

We introduce some notation. A *multiindex* $\alpha = (\alpha_1, \ldots, \alpha_N)$ is an N–vector of non-negative integers $\alpha_i \geq 0$, $1 \leq i \leq N$. If α is a multiindex we let $| \alpha |= \sum_{i=1}^{N} \alpha_i$. Then, the partial derivatives of a function of N variables may be denoted by

$$D^\alpha \varphi = \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}.$$

It follows that H^m is the set:

$$H^m = H^m(\Omega) = \{u \in L^2(\Omega) : \ \forall \alpha \text{ with } | \alpha |\leq m, \ \exists g_\alpha \in L^2(\Omega) \text{ such that}$$
$$\int_{\Omega} u \, D^\alpha \varphi = (-1)^{|\alpha|} \int_{\Omega} g_\alpha \, \varphi, \ \forall \varphi \in C_c^\infty(\Omega)\}.$$

We call g_α the generalized partial derivative of order α of u and denote $g_\alpha = D^\alpha u$.

The space H^m with the norm

$$\|u\|_m = \left(\sum_{0 \leq |\alpha| \leq m} \| D^\alpha u \|^2 \right)^{\frac{1}{2}},$$

induced by the inner product

$$(u, v)_m = \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v)$$

is a Hilbert space. One may show that if $\partial\Omega$ is sufficiently regular (C^1 will certainly suffice), then the norm $\| \cdot \|_m$ on $H^m(\Omega)$ is equivalent to the norm

$$\left(\|u\|^2 + \sum_{|\alpha|=m} \| D^\alpha u \|^2 \right)^{\frac{1}{2}}.$$

In effect one may show that $\forall \alpha$, $0 <| \alpha | \leq m$ and $\epsilon > 0$, there exists a constant $C = C(\alpha, \epsilon, \Omega)$ such that the *interpolation inequality*

$$\| D^\alpha u \| \leq \epsilon \sum_{|\beta|=m} \| D^\beta u \| + C \|u\|, \quad \forall u \in H^m(\Omega)$$

holds.

Now, since $u \in H^m \Rightarrow D^\alpha u \in H^1(\Omega)$ for each α: $0 \leq | \alpha | \leq m-1$, we can define by the trace theorem, boundary values on $\partial\Omega$ (for Ω, say, of class C^1) for all derivatives $D^\alpha u$, $0 \leq | \alpha | \leq m-1$ of u. In this sense we can define e.g. the *normal derivative* on $\partial\Omega$ of a function $u \in H^2(\Omega)$ as the linear combination

$$\frac{\partial u}{\partial n} = \sum_{i=1}^N \frac{\partial u}{\partial x_i}|_{\partial\Omega} n_i,$$

where $\vec{n}(y)$ is the unit outer normal on $\partial\Omega$. For $u \in H^2(\Omega)$, $\frac{\partial u}{\partial n} \in L^2(\partial\Omega)$. One has another formula of Green's too:

$$-\int_\Omega \Delta u \, v = \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, dy, \quad \forall u, v \in H^2(\Omega).$$

One may define again $\overset{0}{H}{}^m(\Omega)$ as the completion of $C_c^\infty(\Omega)$ in the $\| \cdot \|_m$ norm. For $\partial\Omega$ sufficiently smooth (e.g. for Ω a domain of class C^m – replace C^1 by C^m in the definition

of C^1 domain) one may show that $\overset{0}{H}{}^m(\Omega)$ is equal to the subspace of $H^m(\Omega)$ for which

$u \in \overset{0}{H}{}^m(\Omega) \Leftrightarrow u \in H^m(\Omega)$ and $D^\alpha u|_{\partial\Omega} = 0$ (in the sense of trace), $0 \leq |\alpha| \leq m - 1$.

Again we note the difference between the spaces

$$H^2 \cap \overset{0}{H}{}^1 = \{u \in H^2 : u|_{\partial\Omega} = 0\}$$

and

$$\overset{0}{H}{}^2 = \{u \in H^2 : u|_{\partial\Omega} = \frac{\partial u}{\partial x_i}|_{\partial\Omega} = 0\}.$$

4.4   Sobolev's inequalities.

In one dimension we had proved that $H^1(I) \subset C(\overline{I})$ for a bounded interval I and that $\|u\|_{L^\infty(I)} \leq C \|u\|_{H^1(I)}$. In more than one dimensions this is no longer true. There is a wealth of imbedding theorems of which we quote two results:

Theorem 4.5 (Sobolev). Let Ω be of class C^1. Then

(a) If $N = 2$, $H^1 \subset L^p$ $\forall p$, $1 \leq p < \infty$.

If $N > 2$, $H^1 \subset L^p$ where $1 \leq p \leq \frac{2N}{N-2}$.

(b) If $m > \frac{N}{2}$ we have $H^m(\Omega) \subset C^k(\overline{\Omega})$ where $0 \leq k < m - \frac{N}{2}$ and

$$\sup_{x \in \overline{\Omega}, 0 \leq |\alpha| \leq k} |D^\alpha u(x)| \leq C \|u\|_m \quad \forall u \in H^m(\Omega).$$

This theorem tells us that if $N > 1$ the functions in $H^1(\Omega)$ are no longer continuous (in the sense of a.e. equality as usual). For example if $N = 2$ we need to go to $H^2(\Omega)$ to obtain continuous functions in $\overline{\Omega}$. As a counterexample in this direction we may verify that the function

$$u = \left(\log \frac{1}{|x|}\right)^\alpha \text{ with } 0 < \alpha < \frac{1}{2}, \quad \Omega = \{x \in \mathbb{R}^2 : |x| < \frac{1}{2}\}$$

belongs to $H^1(\Omega)$ but it is not bounded because of the singularity at $x = 0$.

4.5 Variational formulation of some elliptic boundary–value problems.

4.5.1 (a) Homogeneous Dirichlet boundary conditions.

We consider the following problem. Let $\Omega \subset \mathbb{R}^N$ be a C^1 domain. We seek a function $u : \overline{\Omega} \to \mathbb{R}$ satisfying

$$-\Delta u + u \;=\; f \;\; \text{in} \;\; \Omega, \;\; \Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \tag{4.1}$$

$$u \;=\; 0 \;\; \text{on} \;\; \partial\Omega, \tag{4.2}$$

where f is a given function on Ω. The boundary condition $u = 0$ on $\partial\Omega$ is called *homogeneous (zero) Dirichlet b.c.*

Definition *Let $f \in C(\overline{\Omega})$. Then, a classical solution of (4.1), (4.2) is a function $u \in C^2(\overline{\Omega})$ satisfying the P.D.E. (4.1) and the b.c. (4.2) in the usual (pointwise) sense. Let now $f \in L^2(\Omega)$. Then, a weak solution of (4.1), (4.2) is a function $u \in \overset{0}{H}{}^1$ which satisfies the weak form of (4.1), (4.2), i.e. the relation*

$$\int_{\Omega} \left(\sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + u\,v \right) dx = \int_{\Omega} f\,v \quad \forall v \in \overset{0}{H}{}^1. \tag{4.3}$$

(i) A classical solution of (4.1), (4.2) is a weak solution.

Let $f \in C(\overline{\Omega})$ and let $u \in C^2(\overline{\Omega})$ be a classical solution of (4.1), (4.2). Then, since $u \in H^1(\Omega)$ and $u = 0$ on $\partial\Omega$, it follows that $u \in \overset{0}{H}{}^1$. Multiplying $-\Delta u + u = f$ by a function $\varphi \in C_c^\infty(\Omega)$ we have, using Green's theorem (cf. p. 88) that

$$\int_{\Omega} \left(\sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} + u\,\varphi \right) dx = \int_{\Omega} f\,\varphi$$

holds. Since now $C_c^\infty(\Omega)$ is dense in $(\overset{0}{H}{}^1, \| \cdot \|_1)$ (4.3) follows by the above by approximating in H^1, $v \in \overset{0}{H}{}^1(\Omega)$ by a sequence $\varphi_i \in C_c^\infty(\Omega)$. Hence u is a weak solution.

(ii) Existence and uniqueness of the weak solution.

Let $f \in L^2(\Omega)$ and consider the bilinear form $a(v, w)$ defined on $\overset{0}{H}{}^1 \times \overset{0}{H}{}^1$ by

$$a(v, w) = \int_{\Omega} \left(\sum_{i=1}^{N} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} + v\,w \right) dx. \tag{4.4}$$

By our hypotheses, $a(\cdot, \cdot)$ is a bilinear, symmetric form on $\overset{0}{H}{}^1 \times \overset{0}{H}{}^1$. Moreover, for $v, w \in \overset{0}{H}{}^1$ we have

$$
\begin{aligned}
a(v, w) \ &\leq \ \sum_{i=1}^{N} \int_{\Omega} |\frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i}| + \int_{\Omega} |v\, w| \\
&\leq \ \sum_{i=1}^{N} \|\frac{\partial v}{\partial x_i}\| \, \|\frac{\partial w}{\partial x_i}\| + \|v\| \, \|w\| \\
&\leq \ \left(\sum_{i=1}^{N} \|\frac{\partial v}{\partial x_i}\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} \|\frac{\partial w}{\partial x_i}\|^2\right)^{\frac{1}{2}} + \|v\|_1 \, \|w\|_1 \\
&\leq \ 2 \, \|v\|_1 \, \|w\|_1, \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (4.5)
\end{aligned}
$$

i.e. that $a(v, w)$ is continuous on $\overset{0}{H}{}^1 \times \overset{0}{H}{}^1$. Also

$$
\begin{aligned}
a(v, v) \ &= \ \sum_{i=1}^{N} \int_{\Omega} (\frac{\partial v}{\partial x_i})^2 + \int_{\Omega} v^2 \geq \sum_{i=1}^{N} \int_{\Omega} (\frac{\partial v}{\partial x_i})^2 \\
&\geq \ c\,\|v\|_1^2, \quad c > 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (4.6)
\end{aligned}
$$

using the Poincaré–Friedrichs inequality. Hence $a(\cdot, \cdot)$ satisfies the hypotheses of the Lax–Milgram theorem on the Hilbert space $\overset{0}{H}{}^1$. Since $f \mapsto \int_{\Omega} f\, v$ is a bounded linear functional on $\overset{0}{H}{}^1$, it follows that (4.3) has a unique solution $u \in \overset{0}{H}{}^1$ that satisfies

$$
\|u\|_1 \ \leq \ C \ \|f\| \, .
$$

(Note that by the symmetry of $a(\cdot, \cdot)$, the weak solution can be also characterized as the (unique) element of $\overset{0}{H}{}^1$ that solves the minimization problem

$$
J(u) = \inf_{v \in \overset{0}{H}{}^1(\Omega)} J(v),
$$

where

$$
J(v) = \frac{1}{2} \int_{\Omega} \left(\sum_{i=1}^{N} (\frac{\partial v}{\partial x_i})^2 + v^2\right) - \int_{\Omega} f\, v.
$$

This is "Dirichlet's principle").

(Note also that simply $a(v, w) = (v, w)_1$ on $\overset{0}{H}{}^1 \times \overset{0}{H}{}^1$. Hence an appeal to the Riesz theorem would solve the problem. Of course, $a(v, w) \leq \|v\|_1 \, \|w\|_1$ and $a(v, v) = \|v\|_1^2$ too. But the proof of (4.5), (4.6) above indicates the general way of proving (4.5) and (4.6) in the case e.g. of a positive definite form with variable coefficients).

(iii) Regularity of the weak solution.

If $f \in L^2(\Omega)$ and Ω is a domain of class C^2 and if $u \in \overset{0}{H}{}^1$ is a weak solution of (4.1), (4.2), i.e. a solution of (4.3), then one may show that in fact $u \in H^2(\Omega)$ and that $\|u\|_2 \le C \parallel f \parallel$ ("elliptic regularity" estimate). Here C is a constant depending only on Ω. More generally, if Ω is of class C^{m+2} and if $f \in H^m(\Omega)$, then $u \in H^{m+2}(\Omega)$ and $\|u\|_{m+2} \le C_m \parallel f \parallel_m$. (In particular, using Sobolev's Theorem 4.5 we may conclude that if $m > \frac{N}{2}$, then $u \in C^2(\overline{\Omega})$).

(iv) If the weak solution is in $C^2(\overline{\Omega})$, then it is classical.

Let u, the weak solution of (4,1), (4.2) be in $C^2(\overline{\Omega})$ and let $f \in C(\overline{\Omega})$. Since $u \in \overset{0}{H}{}^1 \cap C^2(\overline{\Omega})$ we conclude that $u|_{\partial\Omega} = 0$ (in the classical sense). Applying Green's formula we have now from (4.3)

$$\int_\Omega (-\Delta u + u)\, v\, dx = \int_\Omega f v \quad \forall v \in C_c^\infty(\Omega)$$

from which $-\Delta u + u - f = 0$ a.e. in Ω since $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$. We conclude, by our hypotheses that $-\Delta u + u = f \ \forall x \in \Omega$. Hence u is a classical solution.

Remarks

(i) The above discussion extends with no extra difficulties to the case of a linear, self–adjoint elliptic operator with variable coefficients. Consider the problem of finding $u : \overline{\Omega} \to \mathbb{R}$ such that

$$-\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + a_0 u = f \text{ in } \Omega \qquad (4.7)$$

$$u = 0 \text{ on } \partial\Omega, \qquad (4.8)$$

where we suppose that $a_{ij}(x) = a_{ji}(x)$ are functions in $C^1(\overline{\Omega})$ such that the matrix a_{ij} is symmetric and uniformly positive definite, i.e. that the *ellipticity condition*

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \ge \alpha \sum_{i=1}^N \xi_i^2$$

holds for some $\alpha > 0 \ \forall x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$. We also suppose that $a_0 \in C(\overline{\Omega})$ and that $a_0(x) \ge 0$ on $\overline{\Omega}$. We define a classical solution of (4.7), (4.8) to be a function $u \in C^2(\overline{\Omega})$ satisfying (4.7), (4.8) in the usual sense, while a *weak solution* is an element of $\overset{0}{H}{}^1$ satisfying

$$A(u,v) = \int_\Omega \left(\sum_{i,j} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + a_0 uv\right) = (f,v) \quad \forall v \in \overset{0}{H}{}^1. \qquad (4.9)$$

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For $f \in L^2(\Omega)$ we may show that (4.9) has a unique solution $u \in \overset{0}{H}{}^1$: It is easy to see that $A(u,v) \leq C_1 \|u\|_1 \|v\|_1 \ \forall u,v \in \overset{0}{H}{}^1$ and $A(u,v) \geq C_2 \|v\|_1^2 \ \forall v \in \overset{0}{H}{}^1$ for $C_1, C_2 > 0$. The regularity in H^2 also holds under our hypotheses on a_{ij}, a_0. In general, for $m \geq 1$ if $f \in H^m(\Omega)$, $a_{ij} \in C^{m+1}(\overline{\Omega})$, $a_0 \in C^m(\overline{\Omega})$ yields $u \in H^{m+2}(\Omega)$ (elliptic regularity).

(ii) The fact that the weak solution of (4.1), (4.2) is in $C^2(\overline{\Omega})$ if $f \in H^m$ with $m > \frac{N}{2}$ follows from Sobolev's theorem. There is a sharper theory based on *Schauder's estimates* which states that if Ω is of class $C^{2,a}$ (Hölder spaces) with $0 < a < 1$ and $f \in C^{0,a}(\overline{\Omega})$, then $\exists \ u \in C^{2,a}(\overline{\Omega})$, unique solution of (4.1), (4.2) in the classical sense. Moreover of Ω is of class $C^{m+2}(\overline{\Omega})$ ($m \geq 1$ integer) and $f \in C^{m,a}(\overline{\Omega})$, then $u \in C^{m+2,a}(\overline{\Omega})$ and an analogous elliptic regularity result holds. Here

$$
\begin{aligned}
C^{0,a}(\overline{\Omega}) &= \{u \in C(\overline{\Omega}), \ \sup_{x,y \in \Omega, x \neq y} \frac{|\,u(x) - u(y)\,|}{|\,x - y\,|^a} < \infty\} \\
C^{m,a}(\overline{\Omega}) &= \{u \in C^m(\overline{\Omega}), \ D^\beta u \in C^{0,a}(\overline{\Omega}) \ \forall \beta : |\,\beta\,| = m\}.
\end{aligned}
$$

4.5.2 (b) Homogeneous Neumann boundary conditions.

We now consider the problem of finding $u : \overline{\Omega} \to \mathbb{R}$ such that

$$
\begin{aligned}
-\Delta u + u &= f \ \text{ in } \ \Omega & (4.10) \\
\frac{\partial u}{\partial n} &= 0 \ \text{ on } \ \partial\Omega & (4.11)
\end{aligned}
$$

with f given on Ω. As usual $\frac{\partial u}{\partial n} = \vec{\nabla} u \cdot \vec{n}$ is the normal derivative at the boundary $\partial\Omega$ (again Ω is of class C^1). A *classical solution* of (4.10), (4.11) (for $f \in C(\overline{\Omega})$)is a $C^2(\overline{\Omega})$ function u satisfying (4.10), (4.11) in the classical sense. A *weak solution* is an element $u \in H^1(\Omega)$ satisfying (for $f \in L^2(\Omega)$ say)

$$
a(u,v) \equiv \int_\Omega \left(\sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + u\,v \right) dx = \int_\Omega f\,v \quad \forall v \in H^1(\Omega). \qquad (4.12)
$$

(i) Every classical solution is weak.

Let $u \in C^2(\overline{\Omega})$ be a classical solution of (4.10), (4.11). Then Green's formula gives

$$
\int_\Omega \Delta u\,v = -\int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + \int_{\partial\Omega} \frac{\partial u}{\partial n} v\,dy \quad \forall v \in C^\infty(\overline{\Omega}).
$$

Hence (4.10), (4.11) yields that

$$\int_\Omega \left(\sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + u\,v \right) = \int_\Omega f\,v \quad \forall v \in C^\infty(\overline{\Omega})$$

and density of $C^\infty(\overline{\Omega})$ in $H^1(\Omega)$ yields then that (4.12) is satisfied.

(ii) An immediate application of the Lax–Milgram theorem yields that *there exists a unique weak solution.*

(iii) More refined theory yields again the *regularity of the weak solution.* Exactly the same results as in the Dirichlet b.c. case hold.

(iv) If the weak solution is in $C^2(\overline{\Omega})$, then it is classical.

For if this case we have by (4.12) that ($f \in C(\overline{\Omega})$ here)

$$\int_\Omega (-\Delta u + u)\,v + \int_{\partial\Omega} \frac{\partial u}{\partial n} v\,dy = \int_\Omega f\,v \quad \forall v \in C^\infty(\overline{\Omega}).$$

Choosing $v \in C_c^\infty(\Omega)$ we obtain as in the Dirichlet b.c. case that $-\Delta u + u = f$ in Ω. It follows that $\int_{\partial\Omega} \frac{\partial u}{\partial n} v\,dy = 0 \; \forall v \in C^\infty(\overline{\Omega}) \Rightarrow \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$.

Chapter 5

The Galerkin Finite Element Method for Elliptic Boundary–Value Problems

5.1   Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$. Consider the boundary–value problem of finding $u = u(x)$, $x \in \overline{\Omega}$, such that

$$\left. \begin{array}{ll} Lu = f, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{array} \right\} \tag{5.1}$$

where L is the elliptic operator given by

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + a_0(x)\, u.$$

As in the previous chapter, we assume that for $1 \le i, j \le N$ $a_{ij}(x) = a_{ji}(x)$, $x \in \overline{\Omega}$, that $\exists\, c > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge c \sum_{i=1}^{N} \xi_i^2 \ \ \forall \xi \in \mathbb{R}^N,$$

that a_{ij}, a_0 are sufficiently smooth functions of x, and that $a_0 \ge 0$ in $\overline{\Omega}$. Under these hypotheses, and if $f \in L^2(\Omega)$, we have shown that there exists a weak solution $u \in \overset{0}{H}{}^1(\Omega)$ of (5.1) satisfying

$$B(u, v) = (f, v) \quad \forall v \in \overset{0}{H}{}^1, \tag{5.2}$$

where

$$B(u, v) = \int_\Omega \left(\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0(x)uv \right) dx.$$

We shall assume that the data are such that the unique solution u of (5.2) belongs to $H^2 \cap \overset{\scriptscriptstyle 0}{H}{}^1$ and satisfies the elliptic regularity estimate that for some $C > 0$, independent of f and u, we have

$$\| u \|_2 \ \leq \ C \, \| \, f \, \| \, . \tag{5.3}$$

As we have seen in Chapters 1 and 3, the *(standard) Galerkin method* for approximating the solution u of (5.2) amounts to constructing a family of finite–dimensional subspaces S_h of $\overset{\scriptscriptstyle 0}{H}{}^1$, say for $0 < h < 1$, and seeking $u_h \in S_h$ satisfying the linear system of equations

$$B(u_h, v_h) = (f, v_h) \quad \forall v_h \in S_h. \tag{5.4}$$

Under our hypotheses, we have seen that a unique solution u_h of (5.4) exists and satisfies

$$\| \, u - u_h \, \|_1 \leq \ C \inf_{\chi \in S_h} \| \, u - \chi \, \|_1 \tag{5.5}$$

for some constant C independent of h. Assuming e.g. that

$$\inf_{\chi \in S_h} \left(\| \, v - \chi \, \| + h \, \| \, v - \chi \, \|_1 \right) \leq \ C \, h^2 \|v\|_2 \quad \forall v \in H^2 \cap \overset{\scriptscriptstyle 0}{H}{}^1, \tag{5.6}$$

we obtain from (5.5) the *optimal–rate H^1–error estimate*

$$\| \, u - u_h \, \|_1 \leq \ C \, h \|u\|_2. \tag{5.7}$$

The L^2–error estimate is obtained again by the "Nitsche trick", by letting $e = u - u_h$ and considering $w \in H^2 \cap \overset{\scriptscriptstyle 0}{H}{}^1$, the solution of the problem

$$B(w, v) = (e, v) \quad \forall v \in \overset{\scriptscriptstyle 0}{H}{}^1. \tag{5.8}$$

Then, $\| \, e \, \|^2 = (e, e) = B(w, e) = B(e, w) = B(e, w - \chi)$ for any $\chi \in S_h$ – we used (5.2) and (5.4) –. By the continuity of B in $H^1 \times H^1$ we have then

$$\| \, e \, \|^2 \leq C \, \| \, e \, \|_1 \| \, w - \chi \, \|_1 \overset{\text{by}(5.6)}{\leq} C \, \| \, e \, \|_1 \, h \, \| \, w \, \|_2 \leq C \, h \, \| \, e \, \|_1 \| \, e \, \| \, .$$

Hence $\| \, e \, \| \leq \ C \, h \, \| \, e \, \|_1 \leq \ C \, h^2 \|u\|_2$ by (5.7)

In general, assuming that for some $r \geq 2$ (integer) we have

$$\inf_{\chi \in S_h} \left(\| \, v - \chi \, \| + h \, \| \, v - \chi \, \|_1 \right) \leq \ C \, h^r \|v\|_r \quad \forall v \in H^r \cap \overset{\scriptscriptstyle 0}{H}{}^1, \tag{5.9}$$

and that the weak solution u of (5.2) is in $H^r \cap \overset{0}{H}{}^1$, we see that (5.5) and the Nitsche argument give

$$\| \, u - u_h \, \| + h \, \| \, u - u_h \, \|_1 \leq C \, h^r \|u\|_r. \qquad (5.10)$$

Hence, our task is to construct subspaces of $\overset{0}{H}{}^1$ (endowed with bases of small support so that the linear system (5.4) is sparse) so that (5.6) or, in general, (5.9) holds. In what follows we shall consider the subspace of piecewise linear, continuous functions on a polygonal domain in $\mathbb{R}^2$ (subdivided into triangular elements).

5.2 Piecewise linear, continuous functions on a triangulation of a plane polygonal domain

(This section is largely based on Ciarlet (1978), chapter 3).

We consider a *convex* polygonal domain Ω in $\mathbb{R}^2$ and the elliptic b.v.p. (5.1) associated with it. Although Ω is not C^1, it is known that (5.3) still holds.

We subdivide Ω into triangles τ_i, $1 \leq i \leq M$, $M = M(h)$, forming a triangulation $\mathcal{T}_h = \{\tau_i\}$ of Ω. We assume that the τ_i are open and disjoint, that $\max_i(\operatorname{diam} \tau_i) \leq h$, $0 < h < 1$, and that $\Omega = \operatorname{Int}(\cup_{i=1}^{M} \overline{\tau_i})$. The vertices of the triangles are called *nodes* of the triangulation. We shall assume that $\mathcal{T}_h$ is such that there are no nodes in the interior of the interior sides of triangles:
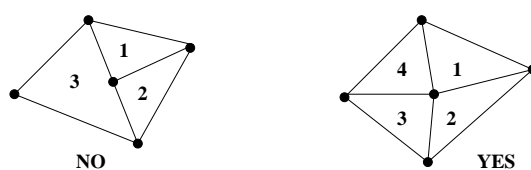
We let now S_h be the vector space of continuous functions on $\overline{\Omega}$ that are linear on each τ_i and vanish on $\partial\Omega$, i.e. let

$$S_h = \{\phi \in C(\overline{\Omega}), \ \phi \mid_{\tau_i} = \alpha_i + \beta_i x + \gamma_i y \quad (\text{i.e. } \phi \in \mathbb{P}_1(\tau_i)), \ \phi \mid_{\partial\Omega} = 0\}.$$

Let $N = N(h)$ be the number of *interior* nodes P_i of the triangulation (i.e. those nodes are <u>not</u> on $\partial\Omega$). Since these points which are not collinear define a single plane, it follows that an element $\phi \in S_h$ is uniquely defined by its values $\phi(P_j)$, $1 \leq j \leq N$.

(This means that $\dim S_h = N$). A suitable basis of S_h (for our purposes) consists of the functions $\phi_i \in S_h$, $1 \le i \le N$, such that

$$\phi_i(P_j) = \delta_{ij}, \quad 1 \le i, j \le N.$$
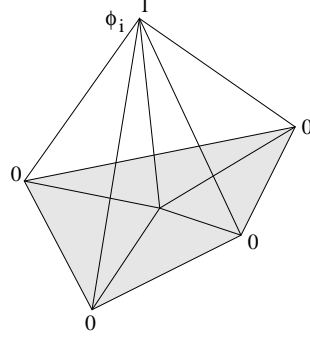
It is clear that the support of ϕ_i consists exactly of those triangles of $\mathcal{T}_h$ that share P_i as a vertex. The ϕ_i are linearly independent, since if $\sum_{i=1}^N c_i \phi_i(x) = 0$, then putting $x = x_j$ yields $c_j = 0$, $1 \le j \le N$. Moreover, given $\psi \in S_h$, we can write

$$\psi(x) = \sum_{j=1}^N \psi(P_j)\, \phi_j(x), \quad x \in \overline{\Omega}, \tag{5.11}$$

since both sides of (5.11) are elements of S_h that coincide at the interior nodes P_j, $1 \le j \le N$. Hence, $\{\phi_i\}_{i=1}^N$ form a basis for S_h.

S_h is a subspace of $\overset{0}{H}^1(\Omega)$: Obviously, $S_h \subset L^2(\Omega)$ and the elements of S_h vanish (pointwise) on $\partial\Omega$. Hence, to show that $v \in S_h$ belongs to $H^1(\Omega)$, it suffices to prove that there exist $g_i \in L^2(\Omega)$, $i = 1, 2$, such that

$$\int_\Omega v \frac{\partial \phi}{\partial x_i} = -\int_\Omega g_i\, \phi \quad \forall \phi \in C_c^\infty(\Omega), \ \ i = 1, 2. \tag{5.12}$$

Let $v^{(\tau)}$ be the restriction of v to $\tau \in \mathcal{T}_h$. For $i = 1, 2$ let $g_i \in L^2(\Omega)$ be defined as the piecewise constant function given by

$$g_i = \frac{\partial}{\partial x_i}(v^{(\tau)}) \quad i = 1, 2, \ \ \text{if } x \in \tau. \tag{5.13}$$

There follows that for $\phi \in C_c^\infty(\Omega)$

$$
\begin{aligned}
\int_\Omega v \frac{\partial \phi}{\partial x_i}\, dx &= \sum_{\tau \in \mathcal{T}_h} \int_\tau v \frac{\partial \phi}{\partial x_i}\, dx = \sum_{\tau \in \mathcal{T}_h} \int_\tau v^{(\tau)} \frac{\partial \phi}{\partial x_i}\, dx \overset{\text{(Gauss theorem on τ)}}{=} \\
&= -\sum_{\tau \in \mathcal{T}_h} \int_\tau g_i\, \phi\, dx + \sum_{\tau \in \mathcal{T}_h} \int_{\partial \tau} v^{(\tau)}\, \phi\, \nu_i^{(\tau)}\, dy,
\end{aligned}
$$

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where $\nu^{(\tau)} = (\nu_1^{(\tau)}, \nu_2^{(\tau)})^T$ is the unit outward normal on the boundary $\partial \tau$ of τ.
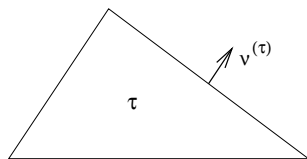
The first term of the right–hand side of the above equality is equal to $-\int_\Omega g_i \phi \, dx$, where $g_i \in L^2(\Omega)$ was defined (piecewise) by (5.13). The second term vanishes. To see this, note that the second term is eventually the sum of terms

$$\int_s v^{(\tau)} \phi \, \nu_i^{(\tau)} \, dy$$

where s any side of any triangle τ. If $s \subset \partial \Omega$, then the corresponding term is zero since $\phi = 0$ on $\partial \Omega$. If $s = AB$ is an interior side, suppose it is the common side of two adjacent triangles τ_1 and τ_2.
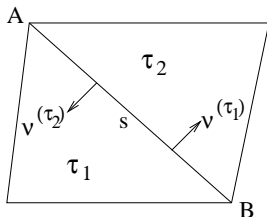
In that case, there are precisely two terms in the sum $\sum_{s:\, s \in \partial \tau} \int_s \ldots$ involving AB, namely

$$\int_{AB} v^{(\tau_1)} \phi \, \nu_i^{(\tau_1)} \, dy \quad \text{and} \quad \int_{AB} v^{(\tau_2)} \phi \, \nu_i^{(\tau_2)} \, dy,$$

which cancel each other since $v^{(\tau_1)} \mid_{AB} = v^{(\tau_2)} \mid_{AB}$ (v is continuous in $\overline{\Omega}$ as an element of S_h), $\phi \in C_c^\infty(\Omega)$, and $\nu_i^{(\tau_1)} = -\nu_i^{(\tau_2)}$, $i = 1, 2$. We conclude then that (5.12) holds, i.e. that $v \in H^1(\Omega)$, q.e.d.

Given $v \in C(\overline{\Omega})$, $v \mid_{\partial \Omega} = 0$, we define the *interpolant* $I_h v$ of v in S_h, as the unique element $I_h v$ of S_h that coincides with v at the interior nodes P_i, $1 \leq i \leq N$, of the triangulation $\mathcal{T}_h$, i.e. as

$$(I_h v)(x) = \sum_{i=1}^N v(P_i) \, \phi_i(x), \quad x \in \overline{\Omega}.$$

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It will be the objective of this section to show that for $v \in H^2(\Omega) \cap \overset{0}{H}{}^1(\Omega)$ (note that $v \in C(\overline{\Omega})$, $v \mid_{\partial\Omega} = 0$), we have, for some constant C independent of h and v:

$$\| v - I_h v \| + h \| v - I_h v \|_1 \leq C h^2 \mid v \mid_{2,\Omega}. \tag{5.14}$$

(Given $v \in H^m(\omega)$, where ω is a subdomain of Ω, we define

$$
\begin{aligned}
\mid v \mid_{0,\omega} &= \| v \|_{L^2(\omega)} \\
\mid v \mid_{1,\omega} &= \left(\| \frac{\partial v}{\partial x_1} \|^2_{L^2(\omega)} + \| \frac{\partial v}{\partial x_2} \|^2_{L^2(\omega)} \right)^{\frac{1}{2}} \\
&\vdots \\
\mid v \mid_{m,\omega} &= \left(\sum_{|\alpha|=m} \| D^\alpha v \|^2_{L^2(\omega)} \right)^{\frac{1}{2}}, \quad D^\alpha = \frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}.
\end{aligned}
$$

Then $\mid v \mid_{m,\omega}$ is in general a semi–norm on $H^m(\Omega)$).

If (5.14) is established, then (5.6) holds and, as a consequence, we have our optimal–order L^2 and H^1 error estimates for $u - u_h$.

The estimate (5.14) will be proved as a consequence of two facts:

(i) A local L^2 and H^1 estimate for the interpolant:

Given a function $v \in C(\overline{\tau})$, where τ is the triangle with vertices P_1, P_2, P_3 define the (local) interpolant $I_\tau v \in \mathbb{P}_1(\tau)$ as the unique linear polynomial in x_1, x_2 on τ such that

$$(I_\tau v)(P_i) = v(P_i), \quad i = 1, 2, 3.$$

Note that if $v \in C(\overline{\Omega})$, then $I_h v \mid_\tau = I_\tau v$.

Following Ciarlet, we shall prove that there exists a constant C, independent of τ and $\mathcal{T}_h$, such that for each $v \in H^2(\tau)$, $\tau \in \mathcal{T}_h$

$$\mid v - I_\tau v \mid_{m,\tau} \leq C \frac{h_\tau^2}{\rho_\tau^m} \mid v \mid_{2,\tau}, \quad m = 0, 1, \tag{5.15}$$

where $h_\tau = \text{diam}\,\tau \equiv$ the length of the largest side of the triangle τ, and ρ_τ is the diameter of the inscribed circle in the triangle τ.

(ii) The regularity of the triangulation:

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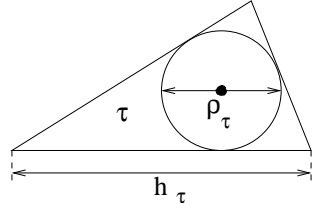
We shall assume that the triangulation $\mathcal{T}_h$ is *regular*, in the sense that there exists a constant $\sigma > 0$, independent of τ and $\mathcal{T}_h$, such that

$$\frac{h_\tau}{\rho_\tau} \leq \sigma \quad \forall \tau \in \mathcal{T}_h. \tag{5.16}$$

(The regularity condition (5.16) essentially states that $h = \max_\tau h_\tau \to 0$, i.e. the triangulation is refined, if and only if $\max \rho_\tau \to 0$ i.e. all triangles tend to become 'points' and not 'needles', (for which $\frac{h_\tau}{\rho_\tau}$ would become unbounded). It may be shown that (5.16) is equivalent to requiring that there exists a $\theta_0 > 0$ independent of $\mathcal{T}_h$ such that $\theta_\tau \geq \theta_0 \ \forall \tau \in \mathcal{T}_h$, where θ_τ is the minimum (interior) angle of τ. It is also equivalent to requiring that $\exists c_0$, independent of $\mathcal{T}_h$, such that $\mu(\tau) \geq c_0 \, h_\tau^2 \ \forall \tau \in \mathcal{T}_h$, where $\mu(\tau) = \text{area}(\tau)$).

Indeed, if (5.15) and (5.16) hold, we have

$$\frac{1}{\rho_\tau} \leq \frac{\sigma}{h_\tau} \Rightarrow \frac{1}{\rho_\tau^m} \leq \frac{\sigma^m}{h_\tau^m}, \quad m = 0, 1,$$

so that (5.15) gives

$$\mid v - I_\tau v \mid_{m,\tau} \leq C_m \, h_\tau^{2-m} \mid v \mid_{2,\tau}, \quad m = 0, 1, \quad \forall v \in H^2(\tau), \tag{5.17}$$

for constants C_0, C_1 independent of $\mathcal{T}_h$.

We conclude then for $v \in H^2(\Omega) \cap \overset{0}{H}{}^1(\Omega) \ (\Rightarrow v \in C(\overline{\Omega})$ by Sobolev's theorem)

$$
\begin{aligned}
\| v - I_h v \|^2 &\equiv \| v - I_h v \|_{L^2(\Omega)}^2 = \sum_{\tau \in \mathcal{T}_h} \mid v - I_\tau v \mid_{0,\tau}^2 \leq C_0^2 \sum_{\tau \in \mathcal{T}_h} h_\tau^4 \mid v \mid_{2,\tau}^2 \\
&\leq C_0^2 \, h^4 \sum_{\tau \in \mathcal{T}_h} \mid v \mid_{2,\tau}^2 = C_0^2 \, h^4 \mid v \mid_{2,\Omega}^2 .
\end{aligned}
$$

Hence

$$\| v - I_h v \| \leq C \, h^2 \mid v \mid_{2,\Omega} \tag{5.18}$$

for some constant C independent of $\mathcal{T}_h$.

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In addition, analogously

$$
\begin{aligned}
\mid v - I_h v \mid_{1,\Omega}^2 &= \sum_{\tau \in \mathcal{T}_h} \mid v - I_\tau v \mid_{1,\tau}^2 \le C_1^2 \sum_{\tau \in \mathcal{T}_h} h_\tau^2 \mid v \mid_{2,\tau}^2 \\
&\le C_1^2 h^2 \sum_{\tau \in \mathcal{T}_h} \mid v \mid_{2,\tau}^2 = C_1^2 h^2 \mid v \mid_{2,\Omega}^2 .
\end{aligned}
$$

Therefore

$$
\mid v - I_h v \mid_{1,\Omega} \le C h \mid v \mid_{2,\Omega}, \tag{5.19}
$$

for some C independent of $\mathcal{T}_h$.

The estimates (5.18) and (5.19) yield then (5.14) as advertized. We turn then to proving (5.15). To accomplish this we shall need a series of results and definitions:

Definition: *Let $\Omega, \hat{\Omega}$ be two bounded domains in $\mathbb{R}^N$. We say that Ω and $\hat{\Omega}$ are affinely equivalent if there exists an invertible affine map $F : \hat{x} \in \mathbb{R}^N \mapsto F(\hat{x}) = B\hat{x} + b \in \mathbb{R}^N$ such that $F(\hat{\Omega}) = \Omega$.*

In the definition of the invertible affine map B is an $N \times N$ invertible matrix of constants and $b \in \mathbb{R}^N$. Hence, if $x \in \mathbb{R}^N$, $\hat{x} = F^{-1}(x) \equiv B^{-1}x - B^{-1}b$.

Hence $\hat{x} \in \hat{\Omega} \Leftrightarrow x = F(\hat{x}) \in \Omega$ and if $\hat{v} : \hat{\Omega} \to \mathbb{R}$ is a real–valued function defined on $\hat{\Omega}$, then defining $v = \hat{v} \circ F^{-1} : \Omega \to \mathbb{R}$, we have if $x \in \Omega$, $x = F(\hat{x})$, i.e. if $\hat{x} = F^{-1}(x)$, that

$$
v(x) = (\hat{v} \circ F^{-1})(x) = \hat{v}(F^{-1}(x)) = \hat{v}(\hat{x}).
$$

(Note that if $v = \hat{v} \circ F^{-1} : \Omega \to \mathbb{R}$, then $\hat{v} = v \circ F : \hat{\Omega} \to \mathbb{R}$).

Using this notation we may prove:

Lemma 5.1. Let $\Omega, \hat{\Omega}$ be two affinely equivalent bounded domains in $\mathbb{R}^N$ and let F be the associated affine map such that $F(\hat{\Omega}) = \Omega$. Then $v \in H^m(\Omega)$, $m \ge 0$ integer, if and only if $\hat{v} = v \circ F \in H^m(\hat{\Omega})$. Moreover, there exist constants C and $\hat{C}$ depending only on m and N such that

$$
\mid \hat{v} \mid_{m,\hat{\Omega}} \le C \mid B \mid^m \mid \det B \mid^{-\frac{1}{2}} \mid v \mid_{m,\Omega} \tag{5.20}
$$

$$
\text{and} \quad \mid v \mid_{m,\Omega} \le \hat{C} \mid B^{-1} \mid^m \mid \det B \mid^{\frac{1}{2}} \mid \hat{v} \mid_{m,\hat{\Omega}} . \tag{5.21}
$$

Here $\mid B \mid$ denotes the matrix norm induced by the Euclidean vector norm in $\mathbb{R}^N$, i.e.

$$
\mid B \mid = \sup_{\mathbb{R}^N \ni x \ne 0} \frac{\mid Bx \mid}{\mid x \mid}, \text{ where } \mid x \mid = \left(\sum_{i=1}^N x_i^2 \right)^{\frac{1}{2}} .
$$

Proof: We shall prove (5.20) for $m = 0$ and $m = 1$. Recall that whenever $x = F(\hat{x}) = B\hat{x} + b$, i.e. whenever $\hat{x} = B^{-1}x + c$, $c = -B^{-1}b$, then $\hat{v}(\hat{x}) = v(x)$. Hence

$$\int_{\hat{\Omega}} \hat{v}^2(\hat{x})\, d\hat{x} = \int_{\Omega} v^2(x)\, \mid J \mid\, dx,$$

where J is the determinant of the Jacobian matrix of the transformation $\hat{x} = B^{-1}x + c$. Hence $J = \det(B^{-1}) = (\det(B))^{-1}$ and we conclude that

$$\parallel \hat{v} \parallel_{0,\hat{\Omega}}^2 = \int_{\hat{\Omega}} \hat{v}^2(\hat{x})\, d\hat{x} = \mid \det B \mid^{-1} \int_{\Omega} v^2(x)\, dx = \mid \det B \mid^{-1} \|v\|_{0,\Omega}^2,$$

which implies (5.20) with $m = 0$, $C = 1$.

Let now $m = 1$ and $\hat{v} \in H^1(\hat{\Omega})$. Then, if $x = F(\hat{x})$ and $1 \leq i \leq N$,

$$\frac{\partial \hat{v}}{\partial \hat{x}_i}(\hat{x}) = \frac{\partial}{\partial \hat{x}_i}(v(x)) = \sum_{j=1}^{N} \frac{\partial v}{\partial x_j}(x) \frac{\partial x_j}{\partial \hat{x}_i} = \sum_{j=1}^{N} \frac{\partial v}{\partial x_j}(x) \frac{\partial F_j(\hat{x})}{\partial \hat{x}_i},$$

where $x_j = F_j(\hat{x}) \equiv \sum_{k=1}^{N} B_{jk}\hat{x}_k + b_j$. Hence $\frac{\partial F_j(\hat{x})}{\partial \hat{x}_i} = B_{ji}$ and we conclude that

$$\frac{\partial \hat{v}}{\partial \hat{x}_i}(\hat{x}) = \sum_{j=1}^{N} \frac{\partial v}{\partial x_j}(x)\, B_{ji}, \quad 1 \leq i \leq N,$$

which we may write as

$$(\widehat{\nabla}\hat{v})(\hat{x}) = B^T\, (\nabla v)(x), \tag{5.22}$$

where

$$\widehat{\nabla}\hat{v} = \left(\frac{\partial \hat{v}}{\partial \hat{x}_1}, \ldots, \frac{\partial \hat{v}}{\partial \hat{x}_N} \right)^T \quad \text{and} \quad \nabla v = \left(\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_N} \right)^T.$$

From (5.22), taking Euclidean norms we see that

$$\mid \widehat{\nabla}\hat{v} \mid \,\leq\, \mid B^T \mid \mid \nabla v \mid \,=\, \mid B \mid \mid \nabla v \mid \tag{5.23}$$

since $\mid B^T \mid = \mid B \mid$. (To see this, recall that

$$\mid B \mid = \max_i \sqrt{\lambda_i(B^T B)} = \max_i \sqrt{\lambda_i(BB^T)} = \mid B^T \mid,$$

since BB^T and $B^T B$ have the same eigenvalues).

Hence, using (5.23) we have

$$
\begin{aligned}
\mid \hat{v} \mid_{1,\hat{\Omega}}^2 &= \sum_{i=1}^{N} \int_{\hat{\Omega}} \left(\frac{\partial \hat{v}}{\partial \hat{x}_i}(\hat{x}) \right)^2 d\hat{x} = \int_{\hat{\Omega}} \mid \widehat{\nabla}\hat{v} \mid^2 d\hat{x} \,\leq\, \mid B \mid^2 \int_{\Omega} \mid \nabla v \mid^2 \mid J \mid\, dx \\
&= \mid B \mid^2 \mid \det B \mid^{-1} \mid v \mid_{1,\Omega}^2,
\end{aligned}
$$

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which is (5.20) for $m = 1$.

The rest of the proof is analogous. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Given two bounded, affinely equivalent domains Ω and $\hat{\Omega}$ we let
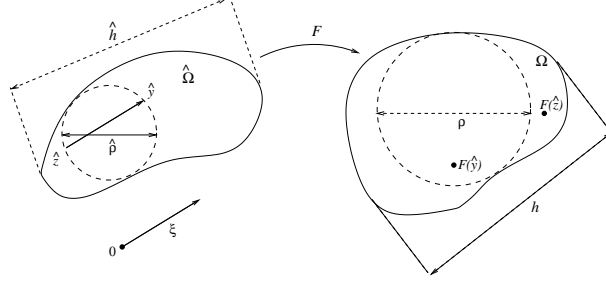
$\hat{h} = \operatorname{diam}\hat{\Omega} \equiv \sup_{\hat{x},\hat{y}\in\hat{\Omega}} \mid \hat{x} - \hat{y} \mid$, $\hat{\rho} = $ diameter of the inscribed ball in $\hat{\Omega} = \sup\{\operatorname{diam}\hat{S},\ \hat{S}$ is a ball contained in $\hat{\Omega}\}$, and h, ρ be the corresponding quantities for Ω.

Lemma 5.2. Let Ω, $\hat{\Omega}$ be two affinely equivalent bounded domains in $\mathbb{R}^N$ such that $\Omega = F(\hat{\Omega})$, $F(\hat{x}) = B\hat{x} + b$. Then

$$\mid B \mid \le \frac{h}{\hat{\rho}}, \quad \mid B^{-1} \mid \le \frac{\hat{h}}{\rho}. \tag{5.24}$$

Proof: An easy scaling argument yields that

$$\mid B \mid = \frac{1}{\hat{\rho}} \sup_{\xi\in\mathbb{R}^N, |\xi|=\hat{\rho}} \mid B\xi \mid .$$

Now given $\xi \in \mathbb{R}^N$ such that $\mid \xi \mid = \hat{\rho}$, we may find two points $\hat{y}, \hat{z} \in \overline{\hat{\Omega}}$ such that $\hat{y} - \hat{z} = \xi$ (see Fig. 5.6). For these points we have that $F(\hat{y}) - F(\hat{z}) = B\hat{y} - B\hat{z} = B\xi$, with $F(\hat{y})$, $F(\hat{z}) \in \overline{\Omega}$. Hence $\sup_{\xi\in\mathbb{R}^N, |\xi|=\hat{\rho}} \mid B\xi \mid \le h$. We conclude that $\mid B \mid \le \frac{h}{\hat{\rho}}$. $\square$

Let us consider now our triangulation $\mathcal{T}_h = \{\tau\}$ of the polygonal domain Ω in $\mathbb{R}^2$.
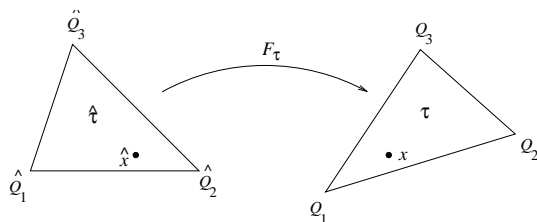
It is clear that any triangle τ of the triangulation is affinely equivalent to a fixed *reference triangle* $\hat{\tau}$. (E.g. we can take $\hat{\tau}$ to be the triangle with vertices (0,0), (0,1), (1,0). Indeed we have for each $\tau \in \mathcal{T}_h$

$$F_\tau(\hat{\tau}) = \tau$$

for some invertible affine map $F_\tau(\hat{x}) = B_\tau \hat{x} + b_\tau$, $\hat{x} \in \hat{\tau}$, depending on τ. (We construct the map F_τ by requiring that $Q_i = F_\tau(\hat{Q}_i)$, $1 \le i \le 3$, where $\hat{Q}_i$, Q_i, $i = 1, 2, 3$, are the vertices of $\hat{\tau}$, τ, respectively. These three 2–vector equations determine uniquely the six constants which are the entries of the matrix B_τ and the vector b_τ. Then we may easily check that $Q_1 Q_2 = F_\tau(\hat{Q}_1 \hat{Q}_2)$ etc., and that each point $\hat{x}$ in $\hat{\tau}$ is mapped onto a uniquely defined point x in τ, and that each point $\hat{x} \in \partial\hat{\tau}$ is mapped onto a corresponding point x on $\partial\tau$. The idea is to work on $\hat{\tau}$ and obtain corresponding estimates on τ (such as the required (5.15)) by using the properties of the interpolant in spaces of piecewise linear continuous functions as well as the scaling and transformation inequalities of Lemmata 5.1 and 5.2.

To this effect, define the interpolant $I_{\hat{\tau}}$ on the reference triangle $\hat{\tau}$ as the map $I_{\hat{\tau}} : C(\hat{\tau}) \to \mathbb{P}_1(\hat{\tau})$ such that

$$(I_{\hat{\tau}}\hat{v})(\hat{Q}_i) = \hat{v}(\hat{Q}_i), \quad 1 \le i \le 3. \tag{5.25}$$

for any continuous real–valued function $\hat{v}$ defined on $\hat{\tau}$. Our basic step towards proving (5.15) is the following

Lemma 5.3. Let $\hat{v} \in H^2(\hat{\tau})$. Then, there exists a constant $C(\hat{\tau})$ such that for $m = 0, 1, 2$

$$\mid \hat{v} - I_{\hat{\tau}}\hat{v} \mid_{m,\hat{\tau}} \le C(\hat{\tau}) \mid \hat{v} \mid_{2,\hat{\tau}} . \tag{5.26}$$

Proof: The estimates (5.26) will be proved as a consequence of three important facts:

(i) The linear map $I_{\hat{\tau}}$ preserves linear polynomials, i.e. $\forall \hat{p} \in \mathbb{P}_1(\hat{\tau})$, $I_{\hat{\tau}}\hat{p} = \hat{p}$. This is obvious and implies that for any $\hat{v} \in H^2(\hat{\tau})$ and *any $\hat{p} \in \mathbb{P}_1(\hat{\tau})$*

$$\hat{v} - I_{\hat{\tau}}\hat{v} = \hat{v} - \hat{p} - I_{\hat{\tau}}\hat{v} + I_{\hat{\tau}}\hat{p} = (\hat{v} - \hat{p}) - I_{\hat{\tau}}(\hat{v} - \hat{p}). \tag{5.27}$$

(ii) $I_{\hat{\tau}}$ is "stable" on $H^2(\hat{\tau})$. By this we mean that there exists a constant $\tilde{C} = \tilde{C}(\hat{\tau})$ such that

$$\| I_{\hat{\tau}}\hat{w} \|_{2,\hat{\tau}} \leq \tilde{C}(\hat{\tau}) \| \hat{w} \|_{2,\hat{\tau}} \quad \forall \hat{w} \in H^2(\hat{\tau}). \tag{5.28}$$

(To see this, let $\hat{\phi}_i \in \mathbb{P}_1(\hat{\tau})$ be the "hat" basis functions associated with the vertices $\hat{Q}_i$ of $\hat{\tau}$, i.e. let $\hat{\phi}_i \in \mathbb{P}_1(\hat{\tau})$ be defined for $i = 1, 2, 3$ by the relations

$$\hat{\phi}_i(\hat{Q}_j) = \delta_{ij}, \quad 1 \leq i, j \leq 3.$$

Then, for $\hat{w} \in H^2(\hat{\tau})$

$$\begin{aligned}
\| I_{\hat{\tau}}\hat{w} \|_{2,\hat{\tau}} &= \| \hat{w}(\hat{Q}_1)\hat{\phi}_1 + \hat{w}(\hat{Q}_2)\hat{\phi}_2 + \hat{w}(\hat{Q}_3)\hat{\phi}_3 \|_{2,\hat{\tau}} \leq \sum_{i=1}^{3} | \hat{w}(\hat{Q}_i) | \, \| \hat{\phi}_i \|_{2,\hat{\tau}} \\
&\leq C'(\hat{\tau}) \max_{\hat{x} \in \bar{\hat{\tau}}} | \hat{w}(\hat{x}) | \leq \tilde{C}(\hat{\tau}) \| \hat{w} \|_{2,\hat{\tau}} , \text{ by Sobolev's theorem in } \mathbb{R}^2).
\end{aligned}$$

As a consequence of (5.27) and (5.28) note that for $m = 0, 1, 2$ and $\hat{v} \in H^2(\hat{\tau})$

$$\begin{aligned}
| \hat{v} - I_{\hat{\tau}}\hat{v} |_{m,\hat{\tau}} &\leq | \hat{v} - \hat{p} |_{m,\hat{\tau}} + | I_{\hat{\tau}}(\hat{v} - \hat{p}) |_{m,\hat{\tau}} \leq \| \hat{v} - \hat{p} \|_{2,\hat{\tau}} + \| I_{\hat{\tau}}(\hat{v} - \hat{p}) \|_{2,\hat{\tau}} \\
&\leq \| \hat{v} - \hat{p} \|_{2,\hat{\tau}} + \tilde{C}(\hat{\tau}) \| \hat{v} - \hat{p} \|_{2,\hat{\tau}} \\
&\leq \tilde{\tilde{C}}(\hat{\tau}) \| \hat{v} - \hat{p} \|_{2,\hat{\tau}} \quad \forall \hat{p} \in \mathbb{P}_1(\hat{\tau}). \tag{5.29}
\end{aligned}$$

We invoke now the

(iii) Bramble–Hilbert Lemma, which in our case asserts that there exists a constant $C^*(\hat{\tau})$ such that:

$$\min_{\hat{p} \in \mathbb{P}_1(\hat{\tau})} \| \hat{v} - \hat{p} \|_{2,\hat{\tau}} \leq C^*(\hat{\tau}) | \hat{v} |_{2,\hat{\tau}} \quad \forall \hat{v} \in H^2(\hat{\tau}). \tag{5.30}$$

(We postpone for the moment the proof of this important result; we shall prove it later in more generality).

Putting together (5.29) and (5.30) yields now

$$| \hat{v} - I_{\hat{\tau}}\hat{v} |_{m,\hat{\tau}} \leq \tilde{\tilde{C}}(\hat{\tau}) \min_{\hat{p} \in \mathbb{P}_1(\hat{\tau})} \| \hat{v} - \hat{p} \|_{2,\hat{\tau}} \leq C(\hat{\tau}) | \hat{v} |_{2,\hat{\tau}},$$

which is the advertized result (5.26). □

Before turning to the proof of (5.30) we first complete the argument and show how (5.26) implies (5.15). To do this, recall that if $\hat{w}$ is defined on $\hat{\tau}$, then for $x = F_\tau(\hat{x})$ we have $\hat{w}(\hat{x}) = w(x)$, where w is the image function, i.e. where $w = \hat{w} \circ F_\tau^{-1}$, defined on τ. Note that the ^ operation is linear, in the sense that $\widehat{\mu u + \lambda v} = \mu \hat{u} + \lambda \hat{v}$, μ, $\lambda \in \mathbb{R}$. (Indeed $(\widehat{\mu u + \lambda v})(\hat{x}) = (\mu u + \lambda v)(x) = \mu u(x) + \lambda v(x) = \mu \hat{u}(\hat{x}) + \lambda \hat{v}(\hat{x})$).

Hence, if $I_\tau v$ is the (local) linear interpolant on $\mathbb{P}_1(\tau)$ of $v \in H^2(\tau)$, we have

$$(v - I_\tau v)^\wedge = \hat{v} - \widehat{I_\tau v}. \tag{5.31}$$

Now, $\widehat{I_\tau v} = I_{\hat{\tau}} \hat{v}$. To see this, note that for $i = 1, 2, 3$

$$(\widehat{I_\tau v})(\hat{Q}_i) = (I_\tau v)(Q_i) = v(Q_i) = \hat{v}(\hat{Q}_i) = (I_{\hat{\tau}} \hat{v})(\hat{Q}_i)$$

i.e. $\widehat{I_\tau v}$ and $I_{\hat{\tau}} \hat{v}$ (which are real–valued functions defined on $\overline{\hat{\tau}}$) coincide at the vertices $\hat{Q}_i$ of $\hat{\tau}$. However $I_{\hat{\tau}} \hat{v} \in \mathbb{P}_1(\hat{\tau})$ and $(\widehat{I_\tau v})(\hat{x}) = (I_\tau v)(x) = (I_\tau v)(F_\tau(\hat{x})) \in \mathbb{P}_1(\hat{\tau})$. Hence $(\widehat{I_\tau v})(\hat{x}) = (I_{\hat{\tau}} \hat{v})(\hat{x}) \; \forall \hat{x} \in \hat{\tau}$. Therefore, (5.31) gives that

$$(v - I_\tau v)^\wedge = \hat{v} - I_{\hat{\tau}} \hat{v}. \tag{5.32}$$

Now, using (5.21) for $m = 0, 1, 2$, we have ($h_{\hat{\tau}} = \text{diam}(\hat{\tau})$)

$$
\begin{aligned}
| v - I_\tau v |_{m,\tau} &\leq \tilde{C} \; | B_\tau^{-1} |^m | \det B_\tau |^{\frac{1}{2}} | (v - I_\tau v)^\wedge |_{m,\hat{\tau}} \\
\text{(by (5.24), (5.26), (5.32))} &\leq \tilde{C} \frac{h_{\hat{\tau}}^m}{\rho_\tau^m} | \det B_\tau |^{\frac{1}{2}} C(\hat{\tau}) | \hat{v} |_{2,\hat{\tau}} \\
&\leq C'(\hat{\tau}) \frac{1}{\rho_\tau^m} | \det B_\tau |^{\frac{1}{2}} | \hat{v} |_{2,\hat{\tau}} \\
\text{(using again (5.20))} &\leq C''(\hat{\tau}) \frac{1}{\rho_\tau^m} | \det B_\tau |^{\frac{1}{2}} | B_\tau |^2 | \det B_\tau |^{-\frac{1}{2}} | v |_{2,\tau} \\
\text{(by (5.24))} &\leq C''(\hat{\tau}) \frac{h_\tau^2}{\rho_\tau^m} \frac{1}{\rho_{\hat{\tau}}^2} | v |_{2,\tau} \\
&\leq C'''(\hat{\tau}) \frac{h_\tau^2}{\rho_\tau^m} | v |_{2,\tau},
\end{aligned}
$$

which is (5.15), since $C'''(\hat{\tau}) \equiv C$ is a constant depending only on the fixed reference triangle $\hat{\tau}$ and is, hence, independent of $\mathcal{T}_h$.

We finally turn to proving (5.30). This will be done in some generality in the following Proposition in which Ω is assumed to be a bounded, Lipschitz domain in $\mathbb{R}^N$.

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Prposition 5.1 (Bramble–Hilbert/Deny–Lions)**.** For $k \geq 1$ there exists a constant $C_k = C_k(\Omega)$ such that for every $u \in H^k(\Omega)$

$$\min_{p \in \mathbb{P}_{k-1}} \| u + p \|_k \leq C_k \mid u \mid_k . \tag{5.33}$$

Remark: (5.33) may be viewed as a type of "Taylor's theorem". Also, note that obviously $\mid u \mid_k \leq \min_{p \in \mathbb{P}_{k-1}} \| u + p \|_k$. Hence (5.33) implies that $\mid \cdot \mid_k$ is a norm, equivalent to the quotient norm $\min_{p \in \mathbb{P}_{k-1}} \| \cdot + p \|_k$ on the space $H^k / \mathbb{P}_{k-1}$. (From now on $H^k = H^k(\Omega)$).

Proof. The estimate (5.33) is a direct consequence of two facts:

(i) For each $u \in H^k$ there exists a unique $q \in \mathbb{P}_{k-1}$ such that

$$\forall \alpha : \mid \alpha \mid \leq k - 1, \quad \int_\Omega D^\alpha q \, dx = \int_\Omega D^\alpha u \, dx. \tag{5.34}$$

(ii) There exists $C_k = C_k(\Omega)$ such that $\forall u \in H^k$:

$$\|u\|_k \leq C_k \left\{ \mid u \mid_k^2 + \sum_{|\alpha| < k} \left(\int_\Omega D^\alpha u \, dx \right)^2 \right\}^{\frac{1}{2}}. \tag{5.35}$$

Indeed, if (i) and (ii) hold, then for $u \in H^k$, with q as in (i),

$$\min_{p \in \mathbb{P}_{k-1}} \| u + p \|_k \leq \| u - q \|_k \overset{\text{(ii)}}{\leq} C_k \left\{ \mid u - q \mid_k^2 + \sum_{|\alpha| < k} \left(\int_\Omega (D^\alpha u - D^\alpha q) \, dx \right)^2 \right\}^{\frac{1}{2}}$$

$$\overset{\text{(i)}}{=} C_k \mid u - q \mid_k = C_k \mid u \mid_k$$

since $q \in \mathbb{P}_{k-1} \Rightarrow D^\alpha q = 0$ for $\mid \alpha \mid = k$.

Therefore, (5.33) is a consequence of (i) and (ii).

We now prove (i) and (ii).

(i). Let $u \in H^k$ be given. We shall construct a polynomial $q \in \mathbb{P}_{k-1}$ such that the relations (5.34) hold. Let q be of the form

$$q(x) = \sum_{m=0}^{k-1} \sum_{|\alpha|=m} c_\alpha x^\alpha \equiv \sum_{m=0}^{k-1} \sum_{|\alpha|=m} c_{\alpha_1 \ldots \alpha_N} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N} .$$

We shall determine the unknown coefficients c_α, $\mid \alpha \mid \leq k - 1$, from the relations (5.34) which represent a linear system of equations for the c_α of size $M_k \times M_k$ where M_k is the

number of multiindices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ with $\mid \alpha \mid \leq k - 1$. We first determine the coefficients c_α multiplying the terms x^α of highest degree, i.e. the c_α with $\mid \alpha \mid = k - 1$. For a multiindex α with $\mid \alpha \mid = k - 1$ we have

$$D^\alpha q = D^\alpha \left(\sum_{\beta: |\beta| = k-1} c_\beta x^\beta \right) + D^\alpha \underbrace{\left(\sum_{\beta: |\beta| < k-1} c_\beta x^\beta \right)}_{\in \mathbb{P}_{k-2}} = c_\alpha \, \alpha!,$$

where $\alpha! \equiv \alpha_1! \alpha_2! \ldots \alpha_N!$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$.

(To see the last equality, consider any one term $D^\alpha(C_\beta x^\beta)$ of the sum

$$D^\alpha \left(\sum_{\beta: |\beta| = k-1} c_\beta x^\beta \right).$$

Hence, for such a term, $\mid \alpha \mid = \mid \beta \mid = k - 1$. If $\alpha \neq \beta$ then $D^\alpha(C_\beta x^\beta) = 0$. For if $\alpha \neq \beta$ there must exist an index j, $1 \leq j \leq N$, such that $\beta_j \neq \alpha_j$. Then for the corresponding factor in $D^\alpha x^\beta$ we will have $\left(\frac{\partial}{\partial x_j} \right)^{\alpha_j} x_j^{\beta_j} = 0$. Now, if $\alpha = \beta$ we have $D^\alpha(x^\beta) = D^\alpha(x^\alpha) = \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} x_1^{\beta_1} \ldots \left(\frac{\partial}{\partial x_N} \right)^{\alpha_N} x_N^{\beta_N} = \alpha_1! \alpha_2! \ldots \alpha_N! \equiv \alpha!$).

Hence, the relation (5.34) for α such that $\mid \alpha \mid = k - 1$ yield

$$\int_\Omega D^\alpha u \, dx = \int_\Omega D^\alpha q \, dx = \int_\Omega c_\alpha \, \alpha! \, dx = c_\alpha \, \alpha! \, \mu(\Omega) \Rightarrow$$
$$c_\alpha = \frac{\int_\Omega D^\alpha u \, dx}{\alpha! \, \mu(\Omega)}, \quad \mid \alpha \mid = k - 1, \tag{5.36}$$

i.e. the coefficients of q with $\mid \alpha \mid = k - 1$ have been determined.

Write now $q_{k-1}(x) = \sum_{|\alpha| = k-1} c_\alpha x^\alpha$ (q_{k-1} is now known). Hence

$$q(x) = \sum_{|\alpha| = k-2} c_\alpha x^\alpha + q_{k-1} + s(x),$$

where $s \in \mathbb{P}_{k-3}$, from which, for $\mid \alpha \mid = k - 2$, as before

$$D^\alpha q = c_\alpha \, \alpha! + D^\alpha q_{k-1}.$$

Therefore, (5.34) for $\mid \alpha \mid = k - 2$ yield

$$\int_\Omega D^\alpha u \, dx = c_\alpha \, \alpha! \, \mu(\Omega) + \int_\Omega D^\alpha q_{k-1} \, dx$$

from which the c_α, $\mid \alpha \mid = k - 2$ are determined. We continue in the same fashion to determine all c_α.

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Note that the polynomial q satisfying (5.34) is necessarily unique: if two such $q_1, q_2 \in \mathbb{P}_{k-1}$ exist, then $q = q_1 - q_2$ will satisfy $\int_\Omega D^\alpha q \, dx = 0$, $\mid \alpha \mid \leq k - 1$, which is a homogeneous linear system for the associated c_α. The formulas (5.36) would yield now $c_\alpha = 0$, $\mid \alpha \mid = k - 1$. Therefore $q_{k-1} = 0$, i.e. $c_\alpha = 0$, $\mid \alpha \mid = k - 2$, and so on, implying finally that $q = 0$.

(ii). We argue by contradiction: Suppose (5.36) does not hold. This means that for any constant $C > 0$ there exists a $u \in H^k$ such that

$$\|u\|_k > C \left\{ \mid u \mid_k^2 + \sum_{|\alpha|<k} \left(\int_\Omega D^\alpha u \, dx \right)^2 \right\}^{\frac{1}{2}},$$

i.e. such that

$$C \left\{ \frac{\mid u \mid_k^2}{\|u\|_k^2} + \frac{\sum_{|\alpha|<k} \left(\int_\Omega D^\alpha u \, dx \right)^2}{\|u\|_k^2} \right\}^{\frac{1}{2}} < 1.$$

This implies that for any constant $C > 0$ there exists a $v \in H^k$ with $\|v\|_k = 1$ (take $v = u/\|u\|_k$) such that

$$C \left\{ \mid v \mid_k^2 + \sum_{|\alpha|<k} \left(\int_\Omega D^\alpha v \, dx \right)^2 \right\}^{\frac{1}{2}} < 1.$$

Take $C = n$, $n = 1, 2, \ldots$. Hence, there exists a sequence $\{u_n\}$ of functions in H^k with $\| u_n \|_k = 1$, such that

$$\mid u_n \mid_k^2 + \sum_{|\alpha|<k} \left(\int_\Omega D^\alpha u_n \, dx \right)^2 < \frac{1}{n^2}. \tag{5.37}$$

We now use the fact ("Rellich's theorem" cf. Adams) that for a domain such as Ω (in fact for any bounded, Lipschitz domain) and for $k \geq 1$, H^k may be compactly imbedded in H^{k-1}, in the sense that every bounded subset of H^k is relatively compact when viewed as a subset of H^{k-1}. This means that every bounded sequence in H^k has a subsequence which converges in the H^{k-1} norm. Therefore the bounded sequence $u_n \in H^k$ ($\| u_n \|_k = 1$) has a subsequence, which we denote again by u_n without loss of generality, and which converges in H^{k-1}. But (5.37) yields that $\mid u_n \mid_k \to 0$, $n \to \infty$, i.e. that $D^\alpha u_n \to 0$ in L^2 for $\mid \alpha \mid = k$. Since u_n converges in H^{k-1} already, we conclude that u_n converges in H^k. Let the limit of $\{u_n\}$ in H^k be denoted by w. Since $\| u_n \|_k = 1$ $\Rightarrow \| w \|_k = 1$. But since $D^\alpha u_n \to 0$ in L^2, $\mid \alpha \mid = k$, we conclude that $D^\alpha w = 0$, $\mid \alpha \mid = k$, i.e. $w \in \mathbb{P}_{k-1}$.

Now, (5.37) also yields $\sum_{|\alpha|<k} \left(\int_\Omega D^\alpha u_n \, dx \right)^2 \to 0$, $n \to \infty$, i.e. that

$$\int_\Omega D^\alpha u_n \, dx \to 0, \quad \text{for } \mid \alpha \mid \leq k-1.$$

We conclude, since $u_n \to w$ in H^k, that $\int_\Omega D^\alpha w = 0$ for $\mid \alpha \mid \leq k-1$. Since the polynomial $w \in \mathbb{P}_{k-1}$ satisfies $\int_\Omega D^\alpha w = 0$ for $\mid \alpha \mid \leq k-1$, the construction in (i) yields that $w = 0$, a contradiction, since $\| w \|_k = 1$; q.e.d. $\qquad \square$

5.3 Implementation of the finite element method with P_1 triangles

In this section we shall study the details of the implementation of the standard Galerkin / finite element method for a simple elliptic boundary–value problem on a polygonal plane domain, based on the ideas of MODULEF, cf. Bernadou et al., 1985. We seek $u(x) = u(x_1, x_2)$ defined on $\overline{\Omega}$, where Ω is a convex, polygonal domain in $\mathbb{R}^2$, and satisfying

$$\left. \begin{array}{ll} -\Delta u + a(x)u = f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{array} \right\} \tag{5.38}$$

Here f, a are given, say continuous, functions on $\overline{\Omega}$ with $a \geq 0$. The weak formulation of the problem is, as usual, to seek $u \in \overset{\circ}{H}{}^1 = \overset{\circ}{H}{}^1(\Omega)$, such that

$$\int_\Omega (\nabla u \cdot \nabla v + a(x)\, u\, v)\, dx = \int_\Omega f\, v\, dx, \quad \forall v \in \overset{\circ}{H}{}^1. \tag{5.39}$$

Let S_h be a finite–dimensional subspace of $\overset{\circ}{H}{}^1$. The *standard Galerkin method* for the approximation of the solution of (5.39) consists in seeking $u_h \in S_h$ such that

$$\int_\Omega (\nabla u_h \cdot \nabla v_h + a(x)\, u_h\, v_h)\, dx = \int_\Omega f\, v_h\, dx, \quad \forall v_h \in S_h. \tag{5.40}$$

We take S_h to be the space of continuous functions on $\overline{\Omega}$ that vanish on $\partial\Omega$ and are polynomials of degree at most 1 on each triangle τ of a triangulation $\mathcal{T}_h$ of Ω. (For notation cf. Section 5.2). Accordingly, we refer to (5.40) as the "standard Galerkin / finite element method with P_1 triangles".

(i) Local degrees of freedom.
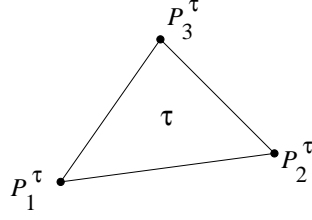
On each triangle $\tau \in \mathcal{T}_h$, u_h is represented in terms of the *local basis functions* $\varphi_j^\tau(x)$, $j = 1, 2, 3$, (which are $\mathbb{P}_1$ polynomials on τ such that $\varphi_j^\tau(P_i^\tau) = \delta_{ij}$, $1 \le i, j \le 3$) as

$$u_h(x) = \sum_{j=1}^{3} \varphi_j^\tau(x)\, u_h(P_j^\tau), \quad x \in \tau. \tag{5.41}$$

The values $\{u_h(P_j^\tau)\}$, $1 \le j \le 3$, coefficients of $\varphi_j^\tau(x)$ in the linear combination of the $\{\varphi_j^\tau(x)\}$ in the r.h.s. of (5.41), are, in our case, the "local degrees of freedom" that determine $u_h(x)$ uniquely on τ. (In general, there are N_τ degrees of freedom – not all function values of u_h necessarily – on each triangle τ. In our case $N_\tau = 3 \ \forall \tau$). Introducing the 1×3 *matrix of local basis functions* $\Phi^\tau = \Phi^\tau(x)$ by

$$\Phi^\tau = [\varphi_1^\tau(x), \varphi_2^\tau(x), \varphi_3^\tau(x)] \tag{5.42}$$

and the 3×1 vector

$$U^\tau = [u_h(P_1^\tau), u_h(P_2^\tau), u_h(P_3^\tau)]^T \tag{5.43}$$

of the *local degrees of freedom*, we may rewrite (5.41) as

$$u_h(x) = \Phi^\tau(x)\, U^\tau. \tag{5.44}$$

Let $\nabla u_h = [\frac{\partial u_h}{\partial x_1} \ \frac{\partial u_h}{\partial x_2}]^T$ denote the gradient of u_h. Then, (5.41) gives

$$\frac{\partial u_h}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial \varphi_j^\tau}{\partial x_i}\, u_h(P_j^\tau), \quad i = 1, 2,$$

i.e. that

$$\nabla u_h(x) = D\Phi^\tau(x) \cdot U^\tau, \quad x \in \tau, \tag{5.45}$$

where $D\Phi^\tau = D\Phi^\tau(x)$, $x \in \tau$, denotes the 2×3 matrix

$$D\Phi^\tau = \begin{pmatrix} \frac{\partial \varphi_1^\tau}{\partial x_1} & \frac{\partial \varphi_2^\tau}{\partial x_1} & \frac{\partial \varphi_3^\tau}{\partial x_1} \\ \frac{\partial \varphi_1^\tau}{\partial x_2} & \frac{\partial \varphi_2^\tau}{\partial x_2} & \frac{\partial \varphi_3^\tau}{\partial x_2} \end{pmatrix}. \tag{5.46}$$

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The Galerkin equations (5.40) become, since $\overline{\Omega} = \cup_{\tau \in \mathcal{T}_h} \overline{\tau}$,

$$\sum_{\tau \in \mathcal{T}_h} \int_\tau (\nabla u_h \cdot \nabla v_h + a(x)\, u_h\, v_h)\, dx = \sum_{\tau \in \mathcal{T}_h} \int_\tau f(x)\, v_h\, dx, \quad \forall v_h \in S_h. \tag{5.47}$$

We shall write (5.47) in matrix–vector form in terms of the local basis functions and the local degrees of freedom $\{U^\tau\}$ and $\{V^\tau\}$ of u_h and v_h, respectively. Writing

$$u_h = \Phi^\tau U^\tau, \quad v_h = \Phi^\tau V^\tau, \quad x \in \tau,$$

we have

$$\begin{aligned}
u_h v_h &= (\Phi^\tau U^\tau)(\Phi^\tau V^\tau) = (V^\tau)^T (\Phi^\tau)^T \Phi^\tau U^\tau, \quad x \in \tau. \\
\nabla u_h \cdot \nabla v_h &= (D\Phi^\tau U^\tau) \cdot (D\Phi^\tau V^\tau) = (D\Phi^\tau V^\tau)^T (D\Phi^\tau U^\tau) \\
&= (V^\tau)^T (D\Phi^\tau)^T D\Phi^\tau U^\tau, \quad x \in \tau. \\
f v_h &= f \Phi^\tau V^\tau = (\Phi^\tau V^\tau)^T = (V^\tau)^T (\Phi^\tau)^T f, \quad x \in \tau.
\end{aligned}$$

Using these expressions in (5.47) we have

$$\sum_{\tau \in \mathcal{T}_h} \int_\tau \left\{ (V^\tau)^T (D\Phi^\tau)^T D\Phi^\tau U^\tau + a(x)(V^\tau)^T (\Phi^\tau)^T \Phi^\tau U^\tau \right\} dx =$$

$$= \sum_{\tau \in \mathcal{T}_h} \int_\tau (V^\tau)^T (\Phi^\tau)^T f(x)\, dx, \quad \forall v_h \in S_h. \tag{5.48}$$

Let K^τ, M^τ denote, respectively, the 3×3 local *stiffness* and *mass* matrix. These are given by the formulas

$$K^\tau := \int_\tau (D\Phi^\tau)^T D\Phi^\tau\, dx, \tag{5.49}$$

$$M^\tau := \int_\tau a(x)\, (\Phi^\tau)^T \Phi^\tau\, dx. \tag{5.50}$$
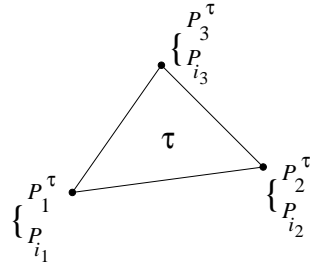
Let also $A^\tau := K^\tau + M^\tau$, and b^τ be the 3×1 vector

$$b^\tau := \int_\tau f(x)\, (\Phi^\tau)^T\, dx. \tag{5.51}$$

Using these local quantities in (5.48) yields the desired matrix–vector form of (5.47):

$$\sum_{\tau \in \mathcal{T}_h} (V^\tau)^T A^\tau U^\tau = \sum_{\tau \in \mathcal{T}_h} (V^\tau)^T b^\tau, \quad \forall v_h \in S_h. \tag{5.52}$$

(ii) The global - to - local degrees of freedom map.

Suppose that the triangulation $\mathcal{T}_h$ consists of N nodes (vertices) (including the nodes on the boundary $\partial\Omega$), that are denoted by P_i, $i = 1, 2, \ldots, N$, in some *global indexing scheme*. Then the vertices $P_1^\tau, P_2^\tau, P_3^\tau$ of of a given triangle $\tau \in \mathcal{T}_h$ correspond to the points $P_{i_1}, P_{i_2}, P_{i_3}$, respectively, in the global enumeration. We would like to find an efficient way of expressing the correspondence $P_{i_k} \to P_k^\tau$. Specifically, we are realy interested in expressing the local degrees of freedom, i.e. in our case the values $u_h(P_k^\tau)$, $k = 1, 2, 3$, in terms of the global degrees of freedom, i.e. the values $u_h(P_i)$, $i = 1, 2, \ldots, N$.

Let us consider an example: let Ω be the rectangle $(0, \alpha) \times (0, \beta)$. Subdivide it in 20 rectangles of size $\Delta x_1 \times \Delta x_2$, where $\Delta x_1 = \frac{\alpha}{5}$, $\Delta x_2 = \frac{\beta}{4}$ and then in 40 equal triangles as shown, by bisecting the rectangles.
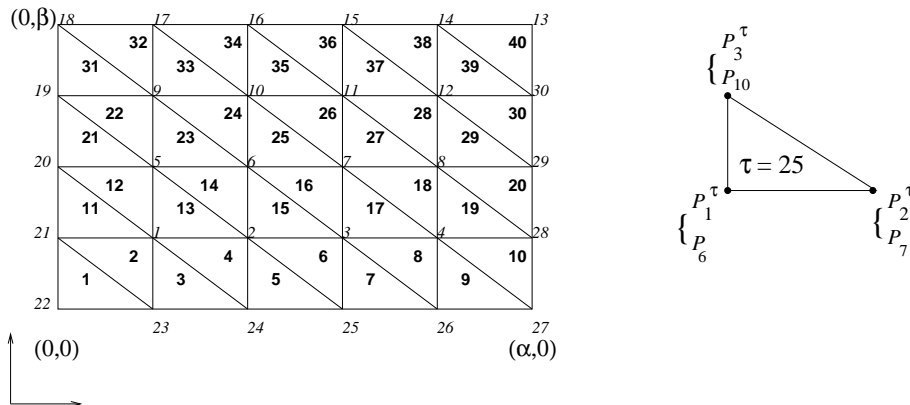
We number the triangles τ from 1 to 40 as shown and introduce the following global indexing scheme for the nodes: The interior nodes are the points P_i shown, with $i = 1, 2, \ldots, 12$, and the boundary nodes are the points P_i, $i = 13, \ldots, 30$. Here $N = 30$,

therefore. For example, the triangle with $\tau = 25$ is defined by the nodes P_6, P_7 and P_{10} in the global numbering. These coincide with the local nodes P_1^τ, P_2^τ, P_3^τ, $(\tau = 25)$, respectively. For $\tau = 25$, let the local degrees of freedom be represented by the 3×1 vector $U^\tau = [U_1^\tau, U_2^\tau, U_3^\tau]^T$ $(U_j^\tau = u_h(P_j^\tau), j = 1, 2, 3)$. The global degrees of freedom $u_h(P_i)$, $i = 1, 2, \ldots, N = 30$, are arranged in the 30×1 vector $U = [U_1, \ldots, U_{30}]^T$. Clearly, we have $U^\tau = \mathcal{G}^\tau U$, where $\mathcal{G}^\tau$ is a 3×30 matrix whose elements are 0 or 1 (Boolean matrix). In our case, i.e. for $\tau = 25$, we have

$$
\begin{pmatrix} U_1^\tau \\ U_2^\tau \\ U_3^\tau \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \overset{6\text{th}}{1} & \overset{7\text{th}}{0} & 0 & 0 & \overset{10\text{th}}{0} & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_{30} \end{pmatrix}.
$$

In general, let $\mathcal{T}_h$ be the triangulation consisting of triangles $\{\tau\}$ that we label, abusing notation a bit, as $\tau = 1, 2, \ldots, J$. For each τ, let

$$
i = g(\tau, j)
$$

be the map that associates the local index j, $1 \le j \le 3$, of the local vertices P_j^τ to the global index i, $1 \le i \le 30$, of the corresponding points P_i. Then g is a function defined on the set $\{1, 2, \ldots, J\} \times \{1, 2, 3\}$ with values onto the set $\{1, 2, \ldots, N\}$ that can be easily stored. In our example, the values of i for $\tau = 25$ and $j = 1, 2, 3$ are

$$
\begin{aligned} g(25, 1) &= 6, \\ g(25, 2) &= 7, \\ g(25, 3) &= 10. \end{aligned}
$$

Let $\mathcal{G}^\tau$, for each τ, denote the $3 \times N$ matrix whose elements are given by

$$
\mathcal{G}_{kl}^\tau = \delta_{g(\tau,k),l} \quad 1 \le k \le 3, \ 1 \le l \le N, \tag{5.53}
$$

where $\delta_{i,j}$ is the Kronecker delta, i.e. $\delta_{i,j} = 1$ if $i = j$, $\delta_{i,j} = 0$ if $i \ne j$. Then, the relation between the global degrees of freedom vector U, $U_i = u_h(P_i)$, $1 \le i \le N$, and the local degrees of freedom vector U^τ on τ, $U_j^\tau = u_h(P_j^\tau)$, $j = 1, 2, 3$, is expressed as

$$
U^\tau = \mathcal{G}^\tau U. \tag{5.54}
$$

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Substituting (5.54), and the corresponding expression $V^\tau = \mathcal{G}^\tau V$, into (5.52), we have

$$\sum_{\tau \in \mathcal{T}_h} V^T (\mathcal{G}^\tau)^T A^\tau \mathcal{G}^\tau U = \sum_{\tau \in \mathcal{T}_h} V^T (\mathcal{G}^\tau)^T b^\tau, \quad \forall v_h \in S_h$$

or, since V, U do not depend on τ,

$$V^T \left\{ \sum_{\tau \in \mathcal{T}_h} (\mathcal{G}^\tau)^T A^\tau \mathcal{G}^\tau \right\} U = V^T \sum_{\tau \in \mathcal{T}_h} (\mathcal{G}^\tau)^T b^\tau, \quad \forall v_h \in S_h.$$

Hence, (5.52) in terms of global degrees of freedom may be written as

$$V^T A U = V^T b, \quad \forall V \in \mathbb{R}^N \text{ such that } v_h \in S_h, \tag{5.55}$$

where A is the $N \times N$ matrix defined by

$$A := \sum_{\tau \in \mathcal{T}_h} (\mathcal{G}^\tau)^T A^\tau \mathcal{G}^\tau, \tag{5.56}$$

and b the $N \times 1$ vector given by

$$b := \sum_{\tau \in \mathcal{T}_h} (\mathcal{G}^\tau)^T b^\tau. \tag{5.57}$$

(iii) Assembly of A and b.

The matrix A and the vector b defined by (5.56) and (5.57) should be *assembled* from their local contributions A^τ and b^τ. In doing this we should *not* form $\mathcal{G}^\tau$ and perform the indicated matrix–matrix and matrix–vector operations, since this would be very costly in terms of storage and number of operations. Instead, recalling the definition (5.53) of $\mathcal{G}^\tau$, we have, for the vector $b = \{b_i\}$, $1 \le i \le N$:

$$b_i = \sum_{\tau \in \mathcal{T}_h} \left((\mathcal{G}^\tau)^T b^\tau \right)_i = \sum_{\tau \in \mathcal{T}_h} \left(\sum_{j=1}^3 \mathcal{G}_{ji}^\tau b_j^\tau \right) = \sum_{\tau \in \mathcal{T}_h} \left(\sum_{j=1}^3 \delta_{g(\tau,j),i} b_j^\tau \right).$$

We conclude that the following algorithm computes the b_i:

> For $i = 1, 2, \ldots, N$ do:
>> $b_i = 0$
>
> For $i = 1, 2, \ldots, N$ do:
>> For $\tau \in \mathcal{T}_h$ do:
>>> For $j = 1, 2, 3$ do:
>>>> $b_i = b_i + \delta_{g(\tau,j),i} \, b_j^\tau.$

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Since $\delta_{g(\tau,j),i} = 0$ unless $i = g(\tau, j)$, the last term contributes only to the b_i with $i = g(\tau, j)$. Hence, the above algorithm can be written, more compactly, in the form:

$$
\left[
\begin{array}{l}
\text{For } i = 1, 2, \ldots, N \text{ do:} \\
b_i = 0 \\
\text{For } \tau \in \mathcal{T}_h \text{ do:} \\
\quad \left[
\begin{array}{l}
\text{For } j = 1, 2, 3 \text{ do:} \\
b_{g(\tau,j)} = b_{g(\tau,j)} + b_j^\tau,
\end{array}
\right.
\end{array}
\right.
\tag{5.58}
$$

i.e. the b_j^τ is added to the previous value of that b_i that has $i = g(\tau, j)$. Similarly by (5.56) we have

$$
A_{ij} = \sum_{\tau \in \mathcal{T}_h} \left(\sum_{k,l=1}^{3} \mathcal{G}_{ki}^\tau A_{kl}^\tau \mathcal{G}_{lj}^\tau \right) = \sum_{\tau \in \mathcal{T}_h} \left(\sum_{k,l=1}^{3} \delta_{g(\tau,k),i} A_{kl}^\tau \delta_{g(\tau,l),j} \right),
$$

i.e. the term A_{kl}^τ contributes to the element A_{ij} if $i = g(\tau, k)$ and $j = g(\tau, l)$. Consequently, the following algorithm may be used to assemble the matrix A:

$$
\left[
\begin{array}{l}
\text{For } i = 1, 2, \ldots, N \text{ do:} \\
\quad \left[
\begin{array}{l}
\text{For } j = 1, 2, \ldots, N \text{ do:} \\
A_{ij} = 0
\end{array}
\right. \\
\text{For } \tau \in \mathcal{T}_h \text{ do:} \\
\quad \left[
\begin{array}{l}
\text{For } k = 1, 2, 3 \text{ do:} \\
\quad \left[
\begin{array}{l}
\text{For } l = 1, 2, 3 \text{ do:} \\
A_{g(\tau,k),g(\tau,l)} = A_{g(\tau,k),g(\tau,l)} + A_{kl}^\tau.
\end{array}
\right.
\end{array}
\right.
\end{array}
\right.
\tag{5.59}
$$

The algorithms (5.58) and (5.59) implement the *assembly* of the (global) matrix A and vector b in the equations (5.55) from their local parts A^τ and b^τ.

(iv) Reduction of (5.55) to a linear system of equations.

The components of the global vector of degrees of freedom $U \in \mathbb{R}^N$, may be ordered (although this is not necessary always) so that U be of the form

$$
U = [U_1, U_2, \ldots, U_{N-N_0}, U_{N-N_0+1}, \ldots, U_N]^T,
$$

so that the degrees of freedom $U_1, U_2, \ldots, U_{N-N_0}$ are the values of u_h at the interior nodes $P_1, P_2, \ldots, P_{N-N_0}$ and $U_{N-N_0+1}, \ldots, U_N$ are the values of u_h at the N_0 boundary nodes $P_{N-N_0+1}, \ldots, P_N$. (In the example on p. 114, $N_0 = 18$, $N = 30$).

Then, writing

$$U = \begin{bmatrix} U_I \\ U_{II} \end{bmatrix}, \quad \text{with } U_I \in \mathbb{R}^{N-N_0}, \ U_{II} \in \mathbb{R}^{N_0},$$

and partitioning conformably V, A and b in (5.55), we can write it in the form

$$[V_I^T \ V_{II}^T] \begin{bmatrix} A_{I,I} & A_{I,II} \\ A_{II,I} & A_{II,II} \end{bmatrix} \begin{bmatrix} U_I \\ U_{II} \end{bmatrix} = [V_I^T \ V_{II}^T] \begin{bmatrix} b_I \\ b_{II} \end{bmatrix}, \quad \forall v_h \in S_h. \qquad (5.60)$$

Since S_h is such that $v_h \in S_h \Leftrightarrow V_I \in \mathbb{R}^{N-N_0}$, $V_{II} = 0$ in $\mathbb{R}^{N_0}$, and since $u_h \in S_h$ as well, we rewrite the above as

$$V_I^T \ A_{I,I} \ U_I \ = \ V_I^T \ b_I \quad \forall V_I \in \mathbb{R}^{N-N_0},$$

which is of course equivalent to the $(N - N_0) \times (N - N_0)$ system

$$A_{I,I} \ U_I \ = \ b_I. \qquad (5.61)$$

The reduction of the $N \times N$ system (5.60) to the system (5.61) is usually referred to as "taking into account the boundary conditions of the problem".

(v) Computing K^τ, M^τ and b^τ.

There remains one important issue of implementation, namely the computation of the local stiffness and mass matrices K^τ and M^τ, as well as the computation of the (local) b^τ, cf. (5.49)–(5.51). This can be accomplished efficiently by letting τ be the affine map of a fixed *reference triangle* $\hat{\tau}$ and transforming the integrals in the formulas (5.49)–(5.51) to $\hat{\tau}$. To this end we will use the notation introduced in Section 5.2.
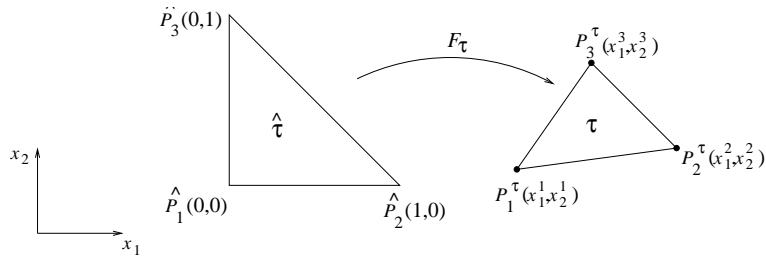
Let $\hat{P}_1\hat{P}_2\hat{P}_3$ be the unit right triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$. Let $\tau \in \mathcal{T}_h$ be an arbitrary triangle in the triangulation with vertices P_j^τ, $1 \leq j \leq 3$, where

$P_j^\tau = (x_1^j, x_2^j)$, $j = 1, 2, 3$. Consider the affine map F_τ that maps $\hat{\tau}$ onto τ and is defined by the requirements that $P_j^\tau = F_\tau(\hat{P}_j)$, $j = 1, 2, 3$. Then F_τ is of the form

$$x = F_\tau(\hat{x}) = B_\tau\, \hat{x}\, +\, c_\tau, \tag{5.62}$$

where B_τ is the 2×2 invertible matrix

$$B_\tau = \begin{pmatrix} x_1^2 - x_1^1 & x_1^3 - x_1^1 \\ x_2^2 - x_2^1 & x_2^3 - x_2^1 \end{pmatrix}$$

and $c_\tau = (x_1^1, x_2^1)^T$. Notice that $\mid \det B_\tau \mid = 2 \mid \tau \mid := 2\, \text{area}(\tau)$.

Recall that functions on $\hat{\tau}$ are transformed to the corresponding functions on τ by $\hat{v}(\hat{x}) = v(x)$, whenever $x = F_\tau(\hat{x})$. I.e. $\hat{v}(\hat{x}) = v(F_\tau(\hat{x}))$ and $v(x) = \hat{v}(F_\tau^{-1}(x))$. (Under our assumption on B_τ, the map F_τ is invertible, with $\hat{x} = F_\tau^{-1}(x) = B_\tau^{-1}x - B_\tau^{-1}c_\tau$). (In our example (p. 114) the triangle $\tau = 25$ has

$$B_\tau = \begin{pmatrix} \Delta x_1 & 0 \\ 0 & \Delta x_2 \end{pmatrix}, \quad c_\tau = \begin{pmatrix} 2\,\Delta x_1 \\ 2\,\Delta x_2 \end{pmatrix}, \quad \text{where } \Delta x_1 = \frac{\alpha}{5},\ \Delta x_2 = \frac{\beta}{4}\ ).$$

Let $\hat{\varphi}_i(\hat{x})$, $1 \le i \le 3$ be the local basis functions on the triangle $\hat{\tau}$, i.e. let $\hat{\varphi}_i \in P_1$ be defined by the relations $\hat{\varphi}_i(\hat{P}_j) = \delta_{ij}$. Then, we easily see that

$$\hat{\varphi}_1(\hat{x}_1, \hat{x}_2) = 1 - \hat{x}_1 - \hat{x}_2,$$
$$\hat{\varphi}_2(\hat{x}_1, \hat{x}_2) = \hat{x}_1,$$
$$\hat{\varphi}_3(\hat{x}_1, \hat{x}_2) = \hat{x}_2.$$

It is easy to see that the corresponding local basis functions on τ, i.e. the elements of P_1 that satisfy $\varphi_i^\tau(P_j^\tau) = \delta_{ij}$, $1 \le i, j \le 3$, are given by the relations

$$\varphi_i^\tau(x) = \hat{\varphi}_i(\hat{x}), \quad \text{whenever } x = F_\tau(\hat{x}).$$

Defining the 1×3 *matrix of reference basis functions* $\hat{\Phi}$ by

$$\hat{\Phi} = \hat{\Phi}(\hat{x}) = [\hat{\varphi}_1(\hat{x}), \hat{\varphi}_2(\hat{x}), \hat{\varphi}_3(\hat{x})]^T, \tag{5.63}$$

we see that $\Phi^\tau(x) = \hat{\Phi}(\hat{x})$, whenever $x = F_\tau(\hat{x})$, where $\Phi^\tau(x)$ is the matrix of the local basis functions on τ defined by (5.42).

We now compute the quantities b^τ, M^τ and K^τ in terms of integrals of functions defined on $\hat\tau$. We already seen that the area element transforms so that $dx = \mid \det B_\tau \mid d\hat x$ i.e. $dx = 2 \mid \tau \mid d\hat x$. Then we have

$$b^\tau = \int_\tau f(x)\,(\Phi^\tau(x))^T\,dx = 2 \mid \tau \mid \int_\tau \hat f(\hat x)\,(\hat\Phi(\hat x))^T\,d\hat x. \tag{5.64}$$

Hence, to compute b^τ we must evaluate the integral on $\hat\tau$ of the vector–valued function $f(F_\tau(\hat x))(\hat\Phi(\hat x))^T$, where $\hat\Phi$ is defined in (5.63). Unless $\hat f(\hat x)$ is a very simple function, such integrals are evaluated numerically by an integration rule on $\hat\tau$. A simple but effective rule is the *barycenter rule*, which is exact for P_1 polynomials and states that

$$\int_{\hat\tau} \hat v(\hat x)\,d\hat x \cong \mid \hat\tau \mid \hat v(\hat M),$$

where $\mid \hat\tau \mid = \mathrm{area}(\hat\tau) = 1/2$ and $\hat M = (1/3, 1/3)$ is the barycenter of $\hat\tau$. Hence using

$$\int_{\hat\tau} \hat v(\hat x)\,d\hat x \cong \frac{1}{2}\,\hat v(1/3, 1/3)$$

in (5.64) we see that

$$b^\tau \cong \mid \tau \mid \hat f(1/3, 1/3) \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \text{ i.e. } b_i^\tau \cong \frac{\mid \tau \mid}{3} f(F_\tau(1/3, 1/3)), \ 1 \le i \le 3.$$

Similarly, we may easily compute the elements of M^τ. Since

$$M^\tau = \int_\tau a(x)\,(\Phi^\tau(x))^T\,\Phi^\tau(x)\,dx = 2 \mid \tau \mid \int_{\hat\tau} \hat a(\hat x)\,(\hat\Phi(\hat x))^T\,\hat\Phi(\hat x)\,d\hat x,$$

we have, for $1 \le i, j \le 3$

$$M_{ij}^\tau = 2 \mid \tau \mid \int_{\hat\tau} a(F_\tau(\hat x))\,\hat\varphi_i(\hat x)\,\hat\varphi_j(\hat x)\,d\hat x.$$

(If numerical integration with the barycenter rule is used, we have that

$$M_{ij}^\tau \cong \frac{\mid \tau \mid}{9}\,a(F_\tau(1/3, 1/3)), \ \ 1 \le i, j \le 3).$$

The computation of

$$K^\tau = \int_\tau [D\Phi^\tau(x)]^T\,[D\Phi^\tau(x)]\,dx$$

requires transforming the matrix $D\Phi^\tau$. We have, for $1 \le i, j \le 3$

$$(D\Phi^\tau(x))_{ij} = \frac{\partial\varphi_j^\tau(x)}{\partial x_i} = \frac{\partial}{\partial x_i}(\hat{\varphi}_j(\hat{x})) = \sum_{k=1}^{3} \frac{\partial}{\partial\hat{x}_k}(\hat{\varphi}_j(\hat{x}))\frac{\partial\hat{x}_k}{\partial x_i}.$$

In analogy to (5.46) let $\hat{D}\hat{\Phi}(\hat{x})$ be the 2×3 matrix with elements

$$(\hat{D}\hat{\Phi}(\hat{x}))_{ij} = \frac{\partial\hat{\varphi}_j(\hat{x})}{\partial\hat{x}_i}. \tag{5.65}$$

In addition, note that from $\hat{x} = B_\tau^{-1}x - B_\tau^{-1}c_\tau$, we infer that

$$\frac{\partial\hat{x}_k}{\partial x_i} = (B_\tau^{-1})_{ki}.$$

Hence,

$$(D\Phi^\tau(x))_{ij} = \sum_{k=1}^{3}(B_\tau^{-1})_{ki}(\hat{D}\hat{\Phi}(\hat{x}))_{kj}, \quad 1 \le i, j \le 3,$$

i.e.

$$D\Phi^\tau(x) = (B_\tau^{-1})^T\,\hat{D}\hat{\Phi}(\hat{x}). \tag{5.66}$$

We conclude that

$$K^\tau = 2\,|\,\tau\,|\int_{\hat{\tau}}(\hat{D}\hat{\Phi}(\hat{x}))^T\,B_\tau^{-1}\,(B_\tau^{-1})^T\,\hat{D}\hat{\Phi}(\hat{x})\,d\hat{x}.$$

The quantities inside the integral are independent of $\hat{x}$. Indeed,

$$\hat{D}\hat{\Phi}(\hat{x}) := J = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

and therefore

$$K^\tau = |\,\tau\,|\,J^T\,(B_\tau^T B_\tau)^{-1}\,J.$$

Chapter 6

The Galerkin Finite Element Method for the Heat Equation

6.1 Introduction. Elliptic projection

In this chapter we shall construct and analyze Galerkin finite element methods for the following model *parabolic* initial-boundary-value problem. Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 1$, 2, or 3. We seek a real function $u = u(x, t)$, $x \in \overline{\Omega}$, $t \geq 0$, such that

$$\begin{cases} u_t - \Delta u = f, & x \in \Omega, \ t \geq 0, \\ u = 0, & x \in \partial\Omega, \ t \geq 0, \\ u(x, 0) = u^0(x), & x \in \overline{\Omega}. \end{cases} \tag{6.1}$$

Here $f = f(x, t)$ and u^0 are given real functions on $\Omega \times [0, \infty)$ and $\overline{\Omega}$, respectively. We shall assume that the initial-boundary-value problem (ibvp) (6.1) has a unique solution which is sufficiently smooth for the purposes of the analysis of its numerical approximation. For the theory of existence-uniqueness and regularity of problems like (6.1) see [2.2] - [2.5]. In this chapter we will just introduce some basic issues of approximating ibvp's like (6.1) with Galerkin methods. The reader is referred to [3.6] and [3.7] for many other related topics.

The spatial approximation of functions defined in $\overline{\Omega}$ will be effected by a Galerkin finite element method. For this purpose we suppose that for $h > 0$ we have a finite-dimensional subspace S_h of $\overset{0}{H}{}^1 = \overset{0}{H}{}^1(\Omega)$ such that, for integer $r \geqq 2$ and h sufficiently

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small, there holds

$$\inf_{\chi \in S_h} \{\|v - \chi\| + h\|\nabla(v - \chi)\|\} \leq Ch^s \|v\|_s \quad \text{for} \quad v \in H^s \cap \overset{0}{H}{}^1, \quad 2 \leq s \leq r, \quad (6.2)$$

where C is a positive constant independent of h and v. (In (6.2) we have used the notation $\|\nabla v\| \equiv |v|_1 \equiv \left(\sum_{i=1}^d \|\frac{\partial v}{\partial x_i}\|^2\right)^{1/2} = \left(\int_\Omega \nabla v \cdot \nabla v \, dx\right)^{1/2} \equiv (\nabla v, \nabla v)^{1/2})$. For example, (6.2) holds in $\mathbb{R}^2$ when $S_h = \left\{\phi \in C(\overline{\Omega}), \ \phi|_\tau \in \mathbb{P}_1 \ \forall \tau \in \mathcal{T}_h, \ \phi|_{\Omega - \Omega_h} = 0\right\}$, where Ω_h is a polygonal domain included in Ω and $\mathcal{T}_h$ a regular triangulation of Ω_h with triangles τ with $h = \max(\text{diam }\tau)$ whose vertices on $\partial\Omega_h$ lie on $\partial\Omega$, cf. [3.5].

Given $v \in H^1$, we define $R_h v$, the *elliptic projection* of v in S_h, by the linear mapping $R_h : H^1 \to S_h$, such that

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi), \qquad \forall \chi \in S_h . \quad (6.3)$$

Given $v \in H^1$ it is easy to see that $R_h v$ exists uniquely in S_h and satisfies $\|\nabla R_h v\| \leq \|\nabla v\|$. The following error estimates follow from (6.2) and (6.3). (We have essentially seen their proof in Section 5.1 but we repeat it here for the convenience of the reader).

Proposition 6.1. Suppose that $v \in H^s \cap \overset{0}{H}{}^1$, where $2 \leq s \leq r$. Then, there exists a constant C independent if v and h such that

$$\|R_h v - v\| + h\|\nabla(R_h v - v)\| \leq C h^s \|v\|_s, \quad 2 \leq s \leq r. \quad (6.4)$$

Proof. We have, by (6.3)

$$\|\nabla(R_h v - v)\|^2 = (\nabla(R_h v - v), \nabla R_h v - \nabla v)$$
$$= -(\nabla(R_h v - v), \nabla v) = (\nabla(R_h v - v), \nabla \chi - \nabla v)$$

for any $\chi \in S_h$. Hence, by (6.2)

$$\|\nabla(R_h v - v)\|^2 \leq \|\nabla(R_h v - v)\| \, \|\nabla(v - \chi)\| \leq C\|\nabla(R_h v - v)\| \, h^{s-1} \|v\|_s ,$$

from which the H^1 estimate in (6.4) follows. For the L^2 estimate we use *Nitsche's trick*. Given $g \in L^2 = L^2(\Omega)$, consider the bvp

$$\begin{cases} -\Delta \psi = g & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega . \end{cases}$$

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This problem has a unique solution $\psi \in H^2 \cap \overset{0}{H}{}^1$ such that $\|\psi\|_2 \leq C \|\Delta \psi\| = C \|g\|$ (Elliptic regularity). For $v \in H^s \cap \overset{0}{H}{}^1$, $2 \leq s \leq r$, we have by Gauss's theorem and (6.3)

$$(R_h v - v, g) = -(R_h v - v, \Delta \psi) = (\nabla(R_h v - v), \nabla \psi) = (\nabla(R_h v - v), \nabla(\psi - \chi))$$

for any $\chi \in S_h$. Hence, the H^1 estimate in (6.4), (6.2), and elliptic regularity imply that

$$(R_h v - v, g) \leqq Ch^{s-1} \|v\|_s h \|\psi\|_2 \leq Ch^s \|v\|_s \|g\| .$$

Taking $g = R_h v - v$ gives that the L^2 estimate in (6.4). $\qquad\square$

6.2   Standard Galerkin semidiscretization

(In this and in the sections 6.2 and 6.3 we generally follow Thomée, [3.6, Ch.1])

Multiplying the pde in (6.1) by a function $v \in \overset{0}{H}{}^1$, and integrating over Ω using Gauss's theorem, we see that

$$(u_t, v) + (\nabla u, \nabla v) = (f, v), \quad t \geq 0 . \tag{6.5}$$

Motivated by (6.5), for each $t \geq 0$ we approximate $u(t) = u(\cdot, t)$ by a function $u_h(t) = u_h(\cdot, t)$ in S_h, called the (*standard Galerkin*) *semidiscrete approximation* (or *spatial discretization*) *of* u *in* S_h, and defined by the equations

$$\begin{cases} (u_{ht}, \phi) + (\nabla u_h, \nabla \phi) = (f, \phi), & \forall \phi \in S_h, \quad t \geq 0, \\ u_h(0) = u_h^0 , \end{cases} \tag{6.6}$$

where u_h^0 is an approximation of u^0 in S_h to be specified later.

The equations (6.6), that will be called the (*standard Galerkin*) *semidiscretization* of (6.1) in S_h, are equivalent to a linear system of ordinary differential equations (ode's). To see this, let $\{\phi_j\}_{j=1}^{N_h}$ be a basis of S_h, where $N_h = \dim S_h$, and let

$$u_h(x, t) = \sum_{j=1}^{N_h} \alpha_j(t) \phi_j(x)$$

be the unknown semidiscrete approximation of u. Substituting this expression for u_h in (6.6) and taking $\phi = \phi_k$, $k = 1, \ldots, N_h$, we see that

$$
\begin{cases}
\displaystyle\sum_{j=1}^{N_h} \alpha'_j(t)(\phi_j, \phi_k) + \alpha_j(t)(\nabla\phi_j, \nabla\phi_k) = (f, \phi_k), \quad 1 \leq k \leq N_h, \quad t \geq 0, \\
\alpha_j(0) = \alpha_j^0, \quad 1 \leq j \leq N_h,
\end{cases}
$$

where $u_h^0 = \sum_{j=1}^{N_h} \alpha_j^0 \phi_j$. Hence the vector of unknowns $\alpha = \alpha(t) = [\alpha_1, \ldots, \alpha_{N_h}]^{\mathrm{T}}$ satisfies the initial-value problem

$$
\begin{cases}
G\dot\alpha + S\alpha = F(t), \quad t \geq 0, \\
\alpha(0) = \alpha^0,
\end{cases}
\tag{6.7}
$$

where $G = (G_{ij})$ is the $N_h \times N_h$ *mass (Gram)* matrix defined by $G_{ij} = (\phi_j, \phi_i)$, $S = (S_{ij})$ the $N_h \times N_h$ *stiffness* matrix given by $S_{ij} = (\nabla\phi_j, \nabla\phi_i)$, and $F_i = (f, \phi_i)$. As we know, G and S are real, symmetric, positive definite matrices. In particular G is invertible and the ivp (6.7) has a unique solution $\alpha(t)$ for all $t \geqq 0$. We conclude that the Galerkin semidiscrete approximation u_h exists uniquely for all $t \geq 0$.

For each $t > 0$ choose $\phi = u_h$ in (6.6). Then

$$
(u_{ht}, u_h) + \|\nabla u_h\|^2 = (f, u_h) \, .
$$

Since $(u_{ht}, u_h) = \frac{1}{2}\int_\Omega \partial_t\left(u_h^2(\cdot, t)\right)\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h(t)\|^2$, we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2 + \|\nabla u_h\|^2 = (f, u_h) \leqq \|f\|\,\|u_h\|, \quad t \geq 0 \, .
$$

Recall the Pointcarè-Friedrichs inequality, i.e. that

$$
\|v\| \leq C_p\,\|\nabla v\|, \quad v \in \overset{0}{H}{}^1(\Omega),
\tag{6.8}
$$

valid for some $C_p = C_p(\Omega)$. Using (6.8) in the above gives

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2 + \|\nabla u_h\|^2 \leq C_p\,\|f\|\,\|\nabla u_h\| \leq \frac{C_p^2}{2}\|f\|^2 + \frac{1}{2}\|\nabla u_h\|^2,
$$

from which

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2 + \|\nabla u_h\|^2 \leq C_p^2\,\|f\|^2, \quad t \geq 0 \, .
$$

We conclude that for any $t > 0$ there holds that

$$
\|u_h(t)\|^2 + \int_0^t \|\nabla u_h(s)\|^2\mathrm{d}s \leqq \|u_h^0\|^2 + C_p^2\int_0^t \|f(s)\|^2\mathrm{d}s \, .
\tag{6.9}
$$

In particular, for $f = 0$, we get $\|u_h(t)\| \leq \|u_h^0\|$ for $t \geqq 0$, i.e. that u_h is stable in L^2.

We now prove the main error estimate of this section.

Theorem 6.1. Let u_h, u be the solutions of (6.6), (6.1), respectively. Then, there exists a constant $C > 0$, independent of h such that

$$\|u_h(t) - u(t)\| \leqq \|u_h^0 - u^0\| + Ch^r \left(\|u^0\|_r + \int_0^t \|u_t\|_r \mathrm{d}s \right), \quad t \geq 0 . \tag{6.10}$$

Proof. Following Wheeler (SIAM J. Numer. Anal., 10 (1973), 723-759), we write

$$u_h - u = \theta + \rho , \tag{6.11}$$

with $\theta = u_h - R_h u$, where R_h is the elliptic projection operator onto S_h defined by (6.3), and $\rho = R_h u - u$. Note that $\theta \in S_h$ for $t \geq 0$, and in order to estimate $\|u_h - u\|$ we should estimate $\|\theta\|$ and $\|\rho\|$. For the latter, we have

$$\|\rho(t)\| = \|R_h u(t) - u(t)\| \leq Ch^r \|u(t)\|_r, \quad t \geq 0 ,$$

by (6.4). Since $u(x,t) = u^0(x) + \int_0^t u_t(x,s)\, \mathrm{d}s$, assuming $u^0 \in H^r \cap \overset{0}{H}{}^1$ and $u_t \in H^r \cap \overset{0}{H}{}^1$ for $t \geq 0$ with $\int_0^t \|u_t\|_r \mathrm{d}s < \infty$, we see that $u \in H^r \cap \overset{0}{H}{}^1$ for $t \geq 0$ and $\|u(t)\|_r \leq \|u^0\|_r + \int_0^t \|u_t\|_r\, \mathrm{d}s$.

Therefore

$$\|\rho(t)\| \leqq Ch^r \left(\|u^0\|_r + \int_0^t \|u_t\|_r \mathrm{d}s \right) \quad \text{for } t \geq 0 . \tag{6.12}$$

In order to get an equation for θ note that for $t \geq 0$ and any $\chi \in S_h$ we have

$$(\theta_t, \chi) + (\nabla\theta, \nabla\chi) = (u_{ht}, \chi) + (\nabla u_h, \nabla\chi) - ((R_h u)_t, \chi) - (\nabla R_h u, \nabla\chi) .$$

Hence, using (6.6), (6.3), and the fact that $((R_h u)_t, \chi) = (R_h u_t, \chi)$ for $\chi \in S_h$ (this follows from (6.3) by differentiating both sides with respect to t), we have

$$(\theta_t, \chi) + (\nabla\theta, \nabla\chi) = (f, \chi) - (R_h u_t, \chi) - (\nabla u, \nabla\chi) .$$

Therefore, by (6.5) and the definition of ρ, we see that

$$(\theta_t, \chi) + (\nabla\theta, \nabla\chi) = -(\rho_t, \chi), \quad \forall \chi \in S_h, \quad t \geq 0 . \tag{6.13}$$

Given $t > 0$, take $\chi = \theta$ in the above to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2 + \|\nabla\theta\|^2 = -(\rho_t, \theta) . \tag{6.14}$$

We would like to argue now that $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2 = \|\theta\|\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|$ and conclude from (6.14) that $\|\theta(t)\| \leq \|\theta(0)\| + \int_0^t \|\rho_t\| \mathrm{d}s$, but we don't know whether $\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|$ exists if $\theta = 0$ for some

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t. Therefore we argue as follows: For all $\varepsilon > 0$ we have from (6.14) $\frac{1}{2}\frac{d}{dt}\|\theta\|^2 = \frac{1}{2}\frac{d}{dt}(\|\theta\|^2 + \varepsilon^2) \leqq \|\rho_t\|\,\|\theta\|$. Hence $\sqrt{\|\theta\|^2 + \varepsilon^2}\frac{d}{dt}\sqrt{\|\theta\|^2 + \varepsilon^2} \leqq \|\rho_t\|\,\|\theta\| \leq \|\rho_t\|\sqrt{\|\theta\|^2 + \varepsilon^2}$. Therefore, $\frac{d}{dt}\sqrt{\|\theta\|^2 + \varepsilon^2} \leq \|\rho_t\|$ for all $t > 0$, from which $\sqrt{\|\theta(t)\|^2 + \varepsilon^2} \leq \int_0^t \|\rho_t\|ds + \sqrt{\|\theta(0)\|^2 + \varepsilon^2}$. Letting $\varepsilon \to 0$ we obtain the desired inequality

$$\|\theta(t)\| \leq \int_0^t \|\rho_t\|ds + \|\theta(0)\|, \quad t \geq 0 . \tag{6.15}$$

Since $\rho_t = R_h u_t - u_t$, we have $\int_0^t \|\rho_t\|ds \leq Ch^r \int_0^t \|u_t\|_r ds$ form (6.4). On the other hand,

$$\|\theta(0)\| \leqq \|u_h^0 - u^0\| + \|R_h u^0 - u^0\| \leq \|u_h^0 - u^0\| + Ch^r\|u^0\|_r .$$

Therefore, (6.15), (6.11) and (6.12) yield the desired estimate (6.10). $\qquad\square$

Remarks

a. Theorem 6.1, and subsequent error estimates, depend on assumptions of sufficient *regularity* of the solution u of the continuous problem. Such assumptions will not normally be explicitly made in the statements of theorems but will appear in the conclusions or in the course of proofs of the error estimates. For example, in the case of the estimate at hand the proof requires that $u^0 \in H^r \cap \overset{0}{H}{}^1$ and $u_t(\cdot, s) \in H^r \cap \overset{0}{H}{}^1$ for $0 \leq s \leq t$. These assumptions guarantee in particular that the bound of $\|u_h(t) - u(t)\|$ in (6.10) is of the form $\|u_h^0 - u^0\| + O(h^r)$ where the $O(h^r)$ term is of *optimal order of convergence* in L^2 for S_h, as evidenced by the approximation property (6.2).

b. The initial value u_h^0 may be chosen in various ways so that $\|u_h^0 - u^0\| = O(h^r)$. For example, we could choose it as $u_h^0 = R_h u^0$ or $u_h^0 = P u^0$ (the L^2 projection of u^0 onto S_h) or equal to an interpolant of u^0 in S_h. For example, if one of the first two choices is made, (6.2) and (6.4) yield that $\|u_h^0 - u^0\| \leq Ch^r\|u^0\|_r$, provided $u^0 \in H^r \cap \overset{0}{H}{}^1$, and the overall optimal-order accuracy $O(h^r)$ is preserved in the right-hand side of (6.10).

Exercise 1. In the proof of Theorem 6.1 take in (6.11) $u_h - u = \theta + \rho$, where, for example, $\theta = u_h - Pu$, where P is the L^2-projection operator onto S_h, or $\theta = u_h - I_h u$, where $I_h u$ is an interpolant of u in S_h, satisfying $\|v - I_h v\| + h\|\nabla(v - I_h v)\| \leqq Ch^r\|v\|_r$ for $v \in H^r \cap \overset{0}{H}{}^1$. For these choices show that the best L^2-error estimate that one could obtain is of $O(h^{r-1})$, i.e. of suboptimal order. Try to understand from these

considerations why Wheeler's choice $\theta = u_h - R_h u$ is crucial.

We now prove an optimal-order H^1 estimate for u_h.

Proposition 6.2. Under the hypotheses of Theorem 6.1, there holds

$$\|\nabla(u_h(t)-u(t))\| \leqq \|\nabla(u_h^0-u^0)\|+Ch^{r-1}\left[\|u^0\|_r + \|u(t)\|_r + \left(\int_0^t \|u_t\|_{r-1}^2 \mathrm{d}s\right)^{1/2}\right], t \geq 0.$$

(6.16)

Proof: As before, we write $\nabla(u_h - u) = \nabla\theta + \nabla\rho$, where $\theta = u_h - R_h u$, $\rho = R_h u - u$. Note that $\|\nabla\rho\| \leq Ch^{r-1}\|u\|_r$. From (6.13) with $\chi = \theta_t$ it follows that

$$\|\theta_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|^2 = -(\rho_t, \theta_t) \leqq \frac{1}{2}\|\rho_t\|^2 + \frac{1}{2}\|\theta_t\|^2 .$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|^2 \leqq \|\rho_t\|^2, \quad t \geq 0$$

form which

$$\|\nabla\theta(t)\| \leq \|\nabla\theta(0)\| + \left(\int_0^t \|\rho_t\|^2\mathrm{d}s\right)^{1/2}, \quad t \geq 0 .$$

We conclude that

$$\|\nabla(u_h - u)\| \leq \|\nabla\theta(0)\| + \left(\int_0^t \|\rho_t\|^2\mathrm{d}s\right)^{1/2} + \|\nabla\rho\|$$

$$\leq \|\nabla(u_h^0 - u^0)\| + \|\nabla(R_h u^0 - u^0)\| + \|\nabla\rho\| + Ch^{r-1}\left(\int_0^t \|u_t\|_{r-1}^2\right)^{1/2},$$

from which (6.16) follows. □

Exercise 2. Consider instead of (6.1) the initial-boundary value problem with *Neumann* boundary conditions:

$$\begin{cases} u_t - \Delta u = f, & x \in \Omega, \quad t \geq 0, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \quad t \geq 0, \\ u(x,0) = u^0(x), & x \in \overline{\Omega} . \end{cases}$$

Construct the standard Galerkin semidiscretization for this problem in a finite-dimensional subspace S_h of H^1 and prove an analog of Theorem 6.1. (Define now an elliptic projection $R_h : H^1 \to S_h$ by the equations $a(R_h v, \chi) = a(v, \chi)$, $\forall\chi \in S_h$ for $v \in H^1$, where

$a(v, w) = (\nabla v, \nabla w) + (v, w)$ for $v, w \in H^1$.)

Exercise 3. Generalize the results of this section to the case of the initial-boundary value problem with variable coefficients

$$\begin{cases} u_t - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u = f(x,t), & x \in \Omega, \quad t \geq 0, \\ u = 0 \text{ on } \Omega, \quad t \geq 0, \\ u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \end{cases}$$

where the functions a_{ij} satisfy $a_{ij}(x) = a_{ji}(x)$, $x \in \overline{\Omega}$ and $\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \geq c_0 \sum_{i=1}^{d} \xi_i^2$, $\forall x \in \overline{\Omega}$, $\forall \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, for some positive constant c_0 independent of x and ξ, i.e. when the matrix-valued function a_{ij} is symmetric and uniformly positive definite for $x \in \overline{\Omega}$, and where $a_0(x) \geq 0$, $x \in \overline{\Omega}$. Assume that the coefficients a_{ij}, a_0 are smooth enough on $\overline{\Omega}$. The weak formulation of this ibvp is to find $u \in \overset{0}{H}{}^1$ for $t \geq 0$ such that

$$\begin{cases} (u_t, v) + a(u, v) = (f(t), v), & \forall v \in \overset{0}{H}{}^1, \quad t \geq 0, \\ u(0) = u_0, \end{cases}$$

where $a(u, v) := \sum_{i,j=1}^{d} \int_{\Omega} a_{ij}uv\mathrm{d}x + \int_{\Omega} a_0uv\,\mathrm{d}x$. (Establish first that there exist positive constants C_1, C_2 such that $|a(v, w)| \leq C_1\|v\|_1\|w\|_1$ $\forall v, w \in \overset{0}{H}{}^1$, and $a(v, v) \geq C_2\|v\|_1^2$ $\forall v \in \overset{0}{H}{}^1$, and introduce now the elliptic projection of $v \in \overset{0}{H}{}^1$ onto S_h by $a(R_hv, \chi) = a(v, \chi) \ \forall \chi \in S_h$. Use the Lax-Milgram theorem and Nitche's trick to prove analogous properties of R_h to those of Section 6.1, assuming elliptic regularity, i.e. that $\|u\|_2 \leq C\|f\|$ holds for the associated elliptic bvp

$$\begin{cases} -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)\,u = f(x), & x \in \widetilde{\Omega}, \\ u = 0 \text{ on } \partial\Omega \,. \end{cases}$$

)

6.3 Full discretization with the implicit Euler and the Crank-Nicolson method

To solve the o.d.e. system (6.6) (or (6.7)) we need to discretize it in t and obtain a *fully discrete method*. This may be done by various time-stepping techniques. In this section we shall examine two of them, the implicit Euler and the Crank-Nicolson methods, that are 'unconditionally stable' in a sense that we shall make precise.

Let $\Delta t = k$ be the length of the (uniform) timestep and $t^n = nk, \quad n = 0, 1, 2, \ldots$. The *implicit Euler* full discretization of (6.16) is defined as follows. We seek for $n = 0, 1, 2, \ldots$ approximations $U^n \in S_h$ of $u_h(t^n)$ that satisfy

$$\begin{cases} \left(\dfrac{U^n - U^{n-1}}{k}, \chi \right) + (\nabla U^n, \nabla \chi) = (f^n, \chi), \quad \forall \chi \in S_h, \quad n \geq 1, \\ U^0 = u_h^0, \end{cases} \tag{6.17}$$

where $f^n = f(\cdot, t^n)$.

Finding U^n for $n \geq 1$, given U^{n-1}, requires solving a linear system for the coefficients of U^n with respect to the basis $\{\phi_j\}_{j=1}^{N_h}$ of S_h. Let $U^n = \sum_{i=1}^{N_h} \alpha_i^n \phi_i$, where $\alpha^n = (\alpha_1^n, \ldots, \alpha_{N_h}^n) \in \mathbb{R}^{N_h}$. Then, putting $\chi = \phi_j, \quad 1 \leq j \leq N_h$, in (6.17) gives

$$(G + kS)\alpha^n = G\alpha^{n-1} + kF^n, \quad n \geq 1, \tag{6.18}$$

where $F_i^n = (f^n, \phi_i), \quad 1 \leq i \leq N_h$. Thus, computing α^n requires forming the right-hand side of (6.18) and solving a $N_h \times N_h$ linear system with the matrix $G + kS$, which is symmetric positive definite, and has the sparsity structure of G and S. If a direct method, like Cholesky's method, is used to solve this linear system, the LL^{T} analysis of $G + kS$ may be done only once and α^n computed for each n using two backsolves with L and L^{T}. In more than one spatial dimensions such linear systems are usually solved by a preconditioned conjugate-gradient type method.

Putting $\chi = U^n$ in (6.17) gives

$$\|U^n\|^2 + k \|\nabla U^n\|^2 = (U^{n-1}, U^n) + k(f^n, U^n) \leqq \left(\|U^{n-1}\| + k\|f^n\| \right) \|U^n\| .$$

Hence, $\|U^n\| \leq \|U^{n-1}\| + k\|f^n\|$, for $n = 1, 2, \ldots$. This implies that

$$\|U^n\| \leq \|U^0\| + k \sum_{j=1}^{n} \|f^j\|, \qquad n = 1, 2, \ldots$$

If $f = 0$ we see that $\|U^n\| \le \|U^0\|$, i.e. that the implicit Euler scheme is L^2-*stable*. In fact for each n we have $\|U^n\| \le \|U^{n-1}\|$, i.e. the L^2 norm of U^n is non-increasing. In particular, if $f^n = 0$, $U^{n-1} = 0$, we see that $U^n = 0$, i.e. that the homogenous linear system of equations of the form (6.18) has only the trivial solution, implying that (6.18) has a unique solution; this is an alternative to the matrix argument used previously to show the same result. Note that the L^2 stability of the scheme was proved without any assumption on the timestep k, i.e. that the scheme (6.17) is *unconditionally stable* in L^2.

As expected, the implicit Euler method is first-order accurate in the time variable as the following estimate suggests.

Theorem 6.2. Let U^n, $u(t) = u(\cdot, t)$, be the solutions of (6.17), (6.1), respectively. Then, there exists a positive constant C, independent of h, k, and n, such that for $n \ge 0$

$$\|U^n - u(t^n)\| \le \|u_h^0 - u^0\| + Ch^r \left[\|u^0\|_r + \int_0^{t^n} \|u_t\|_r \mathrm{d}s \right] + k \int_0^{t^n} \|u_{tt}\| \mathrm{d}s . \quad (6.19)$$

Proof: As in the proof of Theorem 6.1 we write $U^n - u(t^n) = \theta^n + \rho^n$, where $\theta^n = U^n - R_h u(t^n)$, $\rho^n = R_h u(t^n) - u(t^n)$.

Using (6.4) we see that

$$\|\rho^n\| \le C\, h^r \|u(t^n)\|_r \le C\, h^r \left[\|u^0\|_r + \int_0^{t^n} \|u_t\|_r \mathrm{d}s \right], \quad (6.20)$$

and it remains to estimate $\|\theta^n\|$. Let $\overline{\partial} U^n := \frac{1}{k}(U^n - U^{n-1})$. By (6.17), (6.5) and for $\chi \in S_h$ we have

$$\begin{aligned}
(\overline{\partial}\theta^n, \chi) + (\nabla\theta^n, \nabla\chi) &= (\overline{\partial} U^n, \chi) + (\nabla U^n, \nabla\chi) - (\overline{\partial} R_h u(t^n), \chi) - (\nabla R_h u(t^n), \nabla\chi) \\
&= (f(t^n), \chi) - (\overline{\partial} R_h u(t^n), \chi) - (\nabla u(t^n), \nabla\chi) \\
&= (u_t(t^n) - R_h \overline{\partial} u(t^n), \chi),
\end{aligned}$$

i.e.

$$(\overline{\partial}\theta^n, \chi) + (\nabla\theta^n, \nabla\chi) = -(\omega^n, \chi), \quad \forall \chi \in S_h, \ n \ge 1, \quad (6.21)$$

where

$$\begin{aligned}
\omega^n &= R_h \overline{\partial} u(t^n) - u_t(t^n) \\
&= (R_h - I)\overline{\partial} u(t^n) + (\overline{\partial} u(t^n) - u_t(t^n)) =: \omega_1^n + \omega_2^n. \quad (6.22)
\end{aligned}$$

Putting $\chi = \theta^n$ in (6.21) gives

$$(\overline{\partial}\theta^n, \theta^n) + \|\nabla\theta^n\|^2 \le \|\omega^n\| \, \|\theta^n\| \ , \ \text{i.e.}$$

$$\|\theta^n\|^2 - (\theta^{n-1}, \theta^n) \le h \, \|\omega^n\| \, \|\theta^n\| \ , \ \text{from which}$$

$$\|\theta^n\| \le \|\theta^{n-1}\| + k \, \|\omega^n\|, \quad n \ge 1 \ .$$

Therefore, in view of (6.22), summation with respect to n gives

$$\|\theta^n\| \le \|\theta^0\| + k \sum_{j=1}^{n} \|\omega_1^j\| + k \sum_{j=1}^{n} \|\omega_2^j\|, \quad n \ge 1. \tag{6.23}$$

Now

$$\omega_1^j = (R_h - I)\overline{\partial}u(t^j) = (R_h - I)\frac{1}{k}(u(t^j) - u(t^{j-1}))$$

$$= (R_h - I)\frac{1}{k}\int_{t^{j-1}}^{t^j} u_t(s)\mathrm{d}s = \frac{1}{2}\int_{t^{j-1}}^{t^j}(R_h - I)u_t(x)\mathrm{d}s \ .$$

Therefore by (6.4)

$$\|\omega_j^1\| \le C\frac{h^r}{k}\int_{t^{j-1}}^{t^j}\|u_t\|_r\mathrm{d}s, \ \text{giving}$$

$$k\sum_{j=1}^{n}\|\omega_1^j\| \le C\,h^r\int_0^{t^n}\|u_t\|_r\mathrm{d}s \ . \tag{6.24}$$

For the last term in (6.23) we note

$$\omega_2^j = \overline{\partial}u(t^j) - u_t(t^j) = \frac{1}{k}\big(u(t^j) - u(t^{j-1})\big) - u_t(t^j) \ .$$

For a real function $v = v(t)$ recall Taylor's theorem with integral remainder:

$$v(t) = v(a) + (t - a)v'(a) + \ldots + \frac{(t-a)^p}{p!}v^{(p)}(a) + \frac{1}{p!}\int_a^t(t-s)^pv^{(p+1)}(s)\mathrm{d}s \ .$$

Hence

$$u(t^{j-1}) = u(t^j) - ku_t(t^j) + \int_{t^j}^{t^{j-1}}(t^{j-1} - s)u_{tt}(s)\mathrm{d}s \ .$$

We conclude

$$\omega_2^j = -\frac{1}{k}\int_{t^j}^{t^{j-1}}(t^{j-1} - s)u_{tt}(s)\mathrm{d}s \ ,$$

from which

$$\|\omega_2^j\| \le \frac{1}{k}\int_{t^{j-1}}^{t^j}(s - t^{j-1})\|u_{tt}(s)\|\mathrm{d}s \le \int_{t^{j-1}}^{t^j}\|u_{tt}(s)\|\mathrm{d}s \ ,$$

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yielding

$$k \sum_{j=1}^{n} \|\omega_2^j\| \leqq k \int_0^{t^n} \|u_{tt}(s)\| \mathrm{d}s \,. \qquad (6.25)$$

Since $\|\theta^0\| = \|U^0 - R_h u^0\| \leq \|u_h^0 - u^0\| + C\,h^r \|u^0\|_r$, we conclude from (6.23), (6.24), (6.25) that

$$\|\theta^n\| \leqq \|u_h^0 - u^0\| + C\,h^r \left[\|u^0\|_r + \int_0^{t^n} \|u_t\|_r \mathrm{d}s \right] + k \int_0^{t^n} \|u_{tt}(s)\| \mathrm{d}s, \quad n \geq 0,$$

which, in view of the inequality $\|U^n - u(t^n)\| \leq \|\theta^n\| + \|\rho^n\|$ and (6.20) gives (6.19).

Remark. The inequality (6.23) is essentially a *stability* inequality for the 'error' equation (6.21), whereas the estimates (6.24) and (6.25) are bounds on the spatial and temporal 'truncation' errors in the right-hand side of (6.23) and express the *consistency* of the fully discrete scheme in that the tend to zero as $h \to 0$, $k \to 0$ under the implied regularity assumptions on u. In fact, they imply that the spatial accuracy of the scheme in L^2 is of $O(h^r)$ and the temporal accuracy of $O(k)$. Thus the proof of Theorem 6.2 is an illustration of the general principle that 'stability + consistency $\Rightarrow$ convergence'. Such a statement has to be verified in any given particular case and depends on the choice of norms and the regularity of solutions.

We turn now to the *Crank - Nicolson* scheme for discretizing (6.6) in t with second-order accuracy and retaining unconditional stability. We seek for $n \geq 0$ approximations $U^n \in S_h$ of $u_h(t^n)$ satisfying

$$\begin{cases} \left(\dfrac{U^n - U^{n-1}}{k}, \chi \right) + \dfrac{1}{2}(\nabla(U^n + U^{n-1}), \nabla\chi) = (f^{n-1/2}, \chi), \quad \forall \chi \in S_h,\ n \leq 1, \\ U^0 = u_h^0, \end{cases} \qquad (6.26)$$

where $f^{n-1/2} = f\left(\,\cdot\, , t^n - \frac{k}{2} \right)$. Using our previous notation, we see that for $n \geq 1$ the matrix-vector representation of (6.26) is

$$\left(G + \frac{k}{2}S \right) \alpha^n = \left(G - \frac{k}{2}S \right) \alpha^{n-1} + k\,F^{n-1/2}, \quad n \geq 1, \qquad (6.27)$$

where again α^n is the vector of coefficients of U^n with respect to the basis of S_h. The matrix $G + \frac{k}{2}S$ is again sparse, symmetric and positive definite, and similar remarks hold for computing α^n as in the case of the implicit Euler scheme. Putting $\chi = U^n + U^{n-1}$

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in (6.26) gives for $n \geq 1$

$$\|U^n\|^2 - \|U^{n-1}\|^2 + \frac{k}{2}\|\nabla(U^n + U^{n-1})\|^2 = k(f^{n-1/2}, U^{n+1} + U^n)$$

$$\leqq k\,\|f^{n-1/2}\|\left(\|U^n\| + \|U^{n-1}\|\right).$$

Hence $\|U^n\| \leqq \|U^{n-1}\| + k\|f^{n-1/2}\|$, for $n \geqq 1$, implying that

$$\|U^n\| \leq \|U^0\| + k\sum_{j=1}^{n}\|f^{j-1/2}\|, \quad n = 1, 2, \dots .$$

From these relations, if $f = 0$, we see that the Crank-Nicolson method is L^2-stable and that $\|U^n\|$ is non-increasing, unconditionally. In particular, if $f^{n-1/2} = 0$, $U^{n-1} = 0$, it follows that $U^n = 0$, which means that the homogenous linear system of the form (6.27) has only the trivial solution, implying that (6.27) has a unique solution given α^{n-1} and $F^{n-1/2}$. We proceed with an error estimate for the scheme.

Theorem 6.3. Let U^n, $u(t) = u(\,\cdot\,, t)$, be the solutions of (6.26), (6.1), respectively. Then, there exists a positive constant C, independent of h, k and n, such that for $n \geq 0$

$$\|U^n - u(t^n)\| \leq \|u_h^0 - u^0\| + C\,h^r\left[\|u^0\|_r + \int_0^{t^n}\|u_t\|_r \mathrm{d}s\right] + C\,k^2\int_0^{t^n}\left(\|u_{ttt}\| + \|\Delta u_{tt}\|\right)\mathrm{d}s.$$
$$(6.28)$$

Proof: We write again $U^n - u(t^n) = \theta^n + \rho^n$, where $\theta^n = U^n - R_h u(t^n)$, $\rho^n = R_h u(t^n) - u(t^n)$ and note that

$$\|\rho^n\| \leq C\,h^r\left[\|u^0\|_r + \int_0^{t^n}\|u_t\|_r \mathrm{d}s\right].\qquad(6.29)$$

From (6.26), (6.5) and for $\chi \in S_h$ we have

$$(\bar{\partial}\theta^n, \chi) + \frac{1}{2}\left(\nabla(\theta^n + \theta^{n-1}), \nabla\chi\right)$$

$$= (f(t^{n-1/2}), \chi) - (R_h\bar{\partial}u(t^n), \chi) - \frac{1}{2}\left(\nabla(u(t^n) + u(t^{n-1})), \nabla\chi\right)$$

$$= (f(t^{n-1/2}), \chi) - (u_t(t^{n-1/2}), \chi) + (\Delta u(t^{n-1/2}), \chi)$$

$$+ (u_t(t^{n-1/2}) - R_h\bar{\partial}u(t^n), \chi) - (\Delta u(t^{n-1/2}) - \frac{1}{2}\Delta(u(t^n) + u(t^{n-1})), \chi),$$

where we used Gauss's theorem (integration by parts) in the last term. Hence

$$(\bar{\partial}\theta^n, \chi) + \frac{1}{2}(\nabla(\theta^n + \theta^{n-1}), \nabla\chi) = -(\omega^n, \chi), \quad \forall\chi \in S_h,\ n \geq 1,\qquad(6.30)$$

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where

$$\omega^n = \omega_1^n + \omega_2^n + \omega_3^n := (R_h - I)\overline{\partial}u(t^n) + \left(\overline{\partial}u(t^n) - u_t(t^{n-1/2})\right)$$

$$+ \Delta\left(u(t^{n-1/2}) - \frac{1}{2}(u(t^n) + u(t^{n-1}))\right). \quad (6.31)$$

If we put $\chi = \theta^n + \theta^{n-1}$ in (6.30), we obtain, as in the stability proof of the scheme,

$$\|\theta^n\| \leqq \|\theta^0\| + k\sum_{j=1}^n \|\omega_1^j\| + k\sum_{j=1}^n \|\omega_2^j\| + k\sum_{j=1}^n \|\omega_3^j\|, \quad n \geq 1. \quad (6.32)$$

As in the case of the implicit Euler scheme (cf. (6.24)), we have

$$k\sum_{j=1}^n \|\omega_1^j\| \leq C h^r \int_0^{t^n} \|u_t\|_r \mathrm{d}s. \quad (6.33)$$

To estimate ω_2^j we note that for $j \geq 1$

$$\omega_2^j = \overline{\partial}u(t^j) - u_t(t^{j-1/2}) = \frac{1}{k}\left(u(t^j) - u(t^{j-1})\right) - u_t(t^{j-1/2}).$$

Using Taylor's theorem gives

$$u(t^j) = u(t^{j-1/2}) + \frac{k}{2}u_t(t^{j-1/2}) + \frac{k^2}{2!\,4}u_{tt}(t^{j-1/2}) + \frac{1}{2!}\int_{t^{j-1/2}}^{t^j}(t^j - s)^2 u_{ttt}(s)\mathrm{d}s,$$

$$u(t^{j-1}) = u(t^{j-1/2}) - \frac{k}{2}u_t(t^{j-1/2}) + \frac{k^2}{2!\,4}u_{tt}(t^{j-1/2}) + \frac{1}{2!}\int_{t^{j-1/2}}^{t^{j-1}}(t^{j-1} - s)^2 u_{ttt}(s)\mathrm{d}s,$$

so that

$$\omega_2^j = \frac{1}{2k}\left[\int_{t^{j-1/2}}^{t^j}(t^j - s)^2 u_{ttt}(s)\mathrm{d}s + \int_{t^{j-1}}^{t^{j-1/2}}(s - t^{j-1})^2 u_{ttt}(s)\mathrm{d}s\right].$$

Therefore, for $1 \leq j$

$$\|\omega_2^j\| \leqq \frac{1}{2k}\left(\frac{k^2}{4}\int_{t^{j-1/2}}^{t^j}\|u_{ttt}\|\mathrm{d}s + \frac{k^2}{4}\int_{t^{j-1}}^{t^{j-1/2}}\|u_{ttt}\|\mathrm{d}s\right) = \frac{k}{8}\int_{t^{j-1}}^{t^j}\|u_{ttt}\|\mathrm{d}s,$$

i.e.

$$k\sum_{j=1}^n \|\omega_2^j\| \leq \frac{k^2}{8}\int_0^{t^n}\|u_{ttt}\|\mathrm{d}s. \quad (6.34)$$

For ω_3^j, $j \leq 1$, we have

$$\omega_3^j = \Delta\left(u(t^{j-1/2}) - \frac{1}{2}(u(t^j) + u(t^{j-1}))\right).$$

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By Taylor's theorem

$$u(t^j) = u(t^{j-1/2}) + \frac{k}{2}u_t(t^{j-1/2}) + \int_{t^{j-1/2}}^{t^j} (t^j - s)u_{tt}(s)\mathrm{d}s\,,$$

$$u(t^{j-1}) = u(t^{j-1/2}) - \frac{k}{2}u_t(t^{j-1/2}) + \int_{t^{j-1/2}}^{t^{j-1}} (t^{j-1} - s)u_{tt}(s)\mathrm{d}s\,,$$

so that

$$\omega_3^j = -\frac{1}{2}\int_{t^{j-1/2}}^{t^j} (t^j - s)\Delta u_{tt}\mathrm{d}s - \frac{1}{2}\int_{t^{j-1}}^{t^{j-1/2}} (s - t^{j-1})\Delta u_{tt}\mathrm{d}s\,.$$

Therefore

$$k\sum_{j=1}^{n} \|\omega_3^j\| \leqq \frac{k^2}{4}\int_0^{t^n} \|\Delta u_{tt}\|\mathrm{d}s\,. \tag{6.35}$$

The desired estimate (6.28) follows now from the inequalities (6.29), (6.32)-(6.35), and the fact that $\|\theta^0\| = \|U^0 - R_h u_0\| \leqq \|u_h^0 - u^0\| + \|u^0 - R_h u^0\| \leqq \|u_h^0 - u^0\| + C\,h^r\|u^0\|_r\,. \ \square$

Exercise 1. Let $\frac{1}{2} \leqq \alpha \leqq 1$ and consider the following family of fully discrete schemes

$$\begin{cases} \left(\dfrac{U^n - U^{n-1}}{k}, \chi\right) + \left(\nabla(\alpha U^n + (1-\alpha)U^{n-1}), \nabla\chi\right) \\[2mm] \qquad\qquad = \left(\alpha f(t^n) + (1-\alpha)f(t^{n-1}), \chi\right) \quad \forall \chi \in S_h,\ n \geq 1, \\[2mm] U^0 = u_h^0\,. \end{cases}$$

Show that the schemes are L^2-stable and prove error estimates of the form

$\|U^n - u(t^n)\| \leq \|u_h^0 - u^0\| + O(k^p + h^r)$, where $p = 1$ if $1/2 < \alpha \leq 1$ and $p = 2$ if $a = 1/2$.

Exercise 2. Consider the implicit Euler method with *variable* step k_n, where $k_n = t^n - t^{n-1}$, $n \geq 1$:

$$\begin{cases} \left(\dfrac{U^n - U^{n-1}}{k_n}, \chi\right) + (\nabla U^n, \nabla\chi) = (f(t^n), \chi), \quad \forall \chi \in S_h,\ n \geq 1, \\[2mm] U^0 = u_h^0\,. \end{cases}$$

Show that the scheme is L^2-stable and prove an error estimate of the form (6.19) with $k = \max_n k_n$.

6.4 The explicit Euler method. Inverse inequalities and stiffness

The *explicit Euler* full discretization of (6.6) is the following scheme: We seek for $n = 0, 1, 2, \dots$ approximations $U^n \in S_h$ of $u_h(t^n)$ that satisfy

$$\begin{cases} \left(\dfrac{U^n - U^{n-1}}{k}, \chi \right) + (\nabla U^{n-1}, \nabla \chi) = (f^{n-1}, \chi), \quad \forall \chi \in S_h, \ n \geq 1, \\ U^0 = u_h^0. \end{cases} \quad (6.36)$$

Finding U^n for $n \geq 1$, given U^{n-1}, requires solving the linear system

$$G \, \alpha^n = (G - k \, S) \, \alpha^{n-1} + k \, F^{n-1}, \quad n \geq 1, \quad (6.37)$$

where $G, \ S, \ F^n$ have been previously defined and α^n is as usual the vector of coefficients of U^n with respect to the basis of S_h. (Note that although the time-stepping method is explicit as a scheme for solving initial-value problems for ode's, we still have to solve linear systems with the mass matrix G at each time step.) Obviously the linear system (6.37) has a unique solution α^n given α^{n-1} and F^{n-1}.

In order to study the stability of the scheme put $\chi = U^n$ in (6.36). Then, for $n \geq 1$ we have

$$\|U^n\|^2 - (U^{n-1}, U^n) + k \, (\nabla U^{n-1}, \nabla U^n) = k \, (f^{n-1}, U^n).$$

Use now the identities

$$-(U^{n-1}, U^n) = \frac{1}{2} \left(\|U^n - U^{n-1}\|^2 - \|U^n\|^2 - \|U^{n-1}\|^2 \right),$$

$$(\nabla U^{n-1}, \nabla U^n) = \frac{1}{4} \|\nabla(U^n + U^{n-1})\|^2 - \frac{1}{4} \|\nabla(U^n - U^{n-1})\|^2$$

to obtain

$$\|U^n - U^{n-1}\|^2 + \|U^n\|^2 - \|U^{n-1}\|^2 + \frac{k}{2} \|\nabla(U^n + U^{n-1})\|^2 - \frac{k}{2} \|\nabla(U^n - U^{n-1})\|^2 =$$

$$2 \, k \, (f^{n-1}, U^n), \quad n \geq 1. \quad (6.38)$$

In the left-hand side of this identity the troublesome term is $-\frac{k}{2} \|\nabla(U^n - U^{n-1})\|^2$ which is non-positive. To resolve this problem we write (6.38) in the form

$$\|U^n - U^{n-1}\|^2 + \|U^n\|^2 - \|U^{n-1}\|^2 + \frac{k}{2} \|\nabla(U^n + U^{n-1})\|^2$$

$$= \frac{k}{2} \|\nabla(U^n - U^{n-1})\|^2 + 2 \, k \, (f^{n-1}, U^n)$$

and use the *inverse inequality* (valid for quasiuniform partitions of Ω)

$$\|\nabla\chi\| \leq \frac{C_*}{h}\|\chi\|, \quad \chi \in S_h,\tag{6.39}$$

where C_* is a constant independent of h and χ, in the first term of the right-hand side to get

$$\|U^n - U^{n-}\|^2 + \|U^n\|^2 - \|U^{n-1}\|^2 \leq \frac{k}{2}\frac{C_*^2}{h^2}\|U^n - U^{n-1}\|^2 + 2\,k\,\|f^{n-1}\|\,\|U^n\|\,,$$

i.e.

$$\left(1 - \frac{k}{h^2}\frac{C_*^2}{2}\right)\|U^n - U^{n-1}\|^2 + \|U^n\|^2 - \|U^{n-1}\|^2 \leq 2\,k\,\|f^{n-1}\|\,\|U^n\|\,.$$

Therefore, if

$$\frac{k}{h^2} \leq \frac{2}{C_*^2}\,,\tag{6.40}$$

the above inequality gives

$$\|U^n\|^2 - \|U^{n-1}\|^2 \leq 2\,k\,\|f^{n-1}\|\,\|U^n\| \leq 2\,k\,\|f^{n-1}\|\,\left(\|U^n\| + \|U^{n-1}\|\right)\,,$$

from which

$$\|U^n\| \leq \|U^{n-1}\| + 2\,k\,\|f^{n-1}\|, \quad n \geq 1\,,$$

and finally

$$\|U^n\| \leq \|U^0\| + 2\,k\sum_{j=0}^{n-1}\|f^j\|\,,$$

which is the required L^2-stability inequality, analogous to those that were derived for the implicit Euler and the Crank-Nicolson schemes. However, whereas such inequalities in the case of the previous schemes were valid unconditionally, the explicit Euler scheme needs a *stability condition* of the form (6.40). This condition is very restrictive in that it requires taking $k = O(h^2)$, i.e. very small time steps. It can be shown that such a condition is also necessary for stability. (A simple numerical experiment with piecewise linear functions in 1D gives a clear indication !).

We postpone for the time being the proof of the inverse inequality (6.39) in order to prove an L^2 error estimate for the fully discrete scheme (6.36):

Theorem 6.4. Suppose that (6.39) and (6.40) hold and let U^n, u be the solutions of (6.36), (6.1) respectively. Then, there exists a positive constant C, independent of h, k and u, such that for $n \geq 0$

$$\|U^n - u(t^n)\| \leq \|u_h^0 - u^0\| + C\,h^r\left[\|u^0\|_r + \int_0^{t^n}\|u_t\|_r\mathrm{d}s\right] + 2\,k\int_0^{t^n}\|u_{tt}\|\,\mathrm{d}s\,.\tag{6.41}$$

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Proof: Writing again $U^n - u(t^n) = \theta^n + \rho^n$, $\theta^n = U^n - R_h u(t^n)$, $\rho^n = R_h u(t^n) - u(t^n)$, and noting that (6.29) holds for ρ^n, we obtain for $n \geq 1$ and any $\chi \in S_h$

$$(\bar{\partial}\theta^n, \chi) + (\nabla\theta^{n-1}, \nabla\chi) = -(\omega^n, \chi),$$

where

$$\omega^n = \omega_1^n + \omega_2^n := (R_h - I)\bar{\partial}u(t^n) + \left(\bar{\partial}u(t^n) - u_t(t^{n-1})\right).$$

Putting $\chi = \theta^n$ we see, as in the derivation of (6.38), that

$$\|\theta^n\|^2 - \|\theta^{n-1}\|^2 + \|\theta^n - \theta^{n-1}\|^2 + \frac{k}{2}\|\nabla(\theta^n + \theta^{n-1})\|^2 = \frac{k}{2}\|\nabla(\theta^n - \theta^{n-1})\|^2 - 2\,k\,(\omega^n, \theta^n)$$

$$\underset{(6.39)}{\leq} \frac{k\,C_*^2}{2\,h^2}\|\theta^n - \theta^{n-1}\|^2 - 2\,k\,(\omega^n, \theta^n).$$

Therefore, using (6.40), as in the stability proof

$$\|\theta^n\| \leq \|\theta^0\| + 2\,k\sum_{j=1}^n \|\omega^j\| \leq \|\theta^0\| + 2\,k\sum_{j=1}^n \|\omega_1^j\| + 2\,k\sum_{j=1}^n \|\omega_2^j\|. \qquad (6.42)$$

For the ω_1^j term we have as in (6.24)

$$k\sum_{j=1}^n \|\omega_1^j\| \leq C\,h^r \int_0^{t^n} \|u_t\|_r \mathrm{d}s.$$

Since by Taylor's theorem

$$k\,\omega_2^j = u(t^j) - u(t^{j-1}) - k\,u_t(t^{j-1}) = \int_{t^{j-1}}^{t^j} (t^j - s)u_{tt}(s)\mathrm{d}s, \quad j \geq 1,$$

we see that

$$k\sum_{j=1}^n \|\omega_2^j\| \leqq k \int_0^{t^n} \|u_{tt}\|\mathrm{d}s,$$

and (6.41) follows from (6.42). $\qquad\square$

We proceed now with verifying the inverse inequality (6.39). This is straightforward to do in 1D. Consider an interval (λ, μ) with $\mu - \lambda < 1$ and let $\mathbb{P}_k$ be the polynomials of degree $\leq k$. Then for some constant $C = C(k)$ independent of λ and μ it holds that

$$\|\phi\|_{H^1(\lambda,\mu)} \leq \frac{C(k)}{\mu - \lambda}\|\phi\|_{L^2(\lambda,\mu)}, \quad \forall\phi \in \mathbb{P}_k. \qquad (6.43)$$

To see this, observe that there exists a constant $C = C(k)$ such that

$$\|\phi\|_{H^1(0,1)} \leq C\,\|\phi\|_{L^2(0,1)}, \quad \forall\phi \in \mathbb{P}_k, \qquad (6.44)$$

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as a consequence of the fact that $\mathbb{P}_k$ is a finite-dimensional vector space and the norms $\|\cdot\|_{H^1(0,1)}$ and $\|\cdot\|_{L^2(0,1)}$ are equivalent on $\mathbb{P}_k(0,1)$. To derive (6.43) from (6.44) requires just a change of scale. Write the inequality (6.44) for $\phi \in \mathbb{P}_k$ as

$$\int_0^1 \left(\phi^2(x) + (\phi'(x))^2 \right) dx \leq C^2 \int_0^1 \phi^2(x) dx\,,$$

and make in the integrals the change of variable $x \mapsto y$, $y = (\mu - \lambda)x + \lambda$, that maps $[0, 1]$ onto $[\lambda, \mu]$. Then the above inequality becomes

$$\frac{1}{\mu - \lambda} \int_\lambda^\mu \left[\phi^2(y) + (\mu - \lambda)^2 \left(\frac{d\phi}{dy}(y) \right)^2 \right] dy \leq \frac{C^2}{\mu - \lambda} \int_\lambda^\mu \phi^2(y) dy\,,$$

giving

$$(\mu - \lambda)^2 \int_\lambda^\mu \left[\phi^2(y) + \left(\frac{d\phi}{dy}(y) \right)^2 \right] dy \leq C^2 \int_\lambda^\mu \phi^2(y) dy\,,$$

since we assumed $\mu - \lambda < 1$. Hence (6.43) follows.

Let now (a, b) be any fixed finite interval and let $a = x_0 < x_1 < \ldots < x_{J+1} = b$ be an arbitrary partition of $[a, b]$. Letting $h_i = x_{i+1} - x_i$, $0 \leq i \leq J$, we obtain from (6.43) that

$$\|\phi\|_{H^1(x_i, x_{i+1})} \leq \frac{C(k)}{h_i} \|\phi\|_{L^2(x_i, x_{i+1})}, \quad \forall \phi \in \mathbb{P}_k(x_i, x_{i+1})\,. \tag{6.45}$$

Let now $S_h = \left\{ \phi \in C[a, b] : \phi\big|_{[x_i, x_{i+1}]} \in \mathbb{P}_k \right\}$. Since $S_h \subset H^1$, using (6.45) we get for any $\phi \in S_h$

$$\|\phi\|_{H^1(a,b)}^2 = \sum_{i=0}^J \|\phi\|_{H^1(x_i, x_{i+1})}^2 \leqq C^2(k) \sum_{i=0}^J \frac{1}{h_i^2} \|\phi\|_{L^2(x_i, x_{i+1})}^2\,. \tag{6.46}$$

We now assume that the partition $\{x_i\}$ of $[a, b]$ is *quasiuniform*, i.e. that there is a constant ν independent of the partition (in the sense that as the partition is refined ν does not change) such that

$$\frac{h}{h_i} \leqq \nu, \quad \forall i\,, \tag{6.47}$$

where $h = \max_i h_i$. In view of (6.47), (6.46) gives

$$\|\phi\|_{H^1(a,b)} \leqq \frac{C_*}{h} \|\phi\|_{L^2(a,b)}, \quad \forall \phi \in S_h\,, \tag{6.48}$$

with $C_* = C(k)\,\nu$, from which (6.39) follows in 1D.

In order to prove (6.39) in 2D, we assume that Ω is a polygonal domain and let $\mathcal{T}_h = \{\tau\}$ be a regular (cf. (5.16)) triangulation of Ω. We recall from section 5.2 that

any triangle τ of the triangulation is affinely equivalent to a fixed reference triangle $\widehat{\tau}$. As in the case of 1D, there exists a constant $C_1 = C_1(\widehat{\tau}, k)$ such that

$$\|\widehat{\phi}\|_{1,\widehat{\tau}} \leq C_1 \|\widehat{\phi}\|_{0,\widehat{\tau}}, \quad \forall \widehat{\phi} \in \mathbb{P}_k(\widehat{\tau}), \tag{6.49}$$

as a consequence again of the fact that $\mathbb{P}_k(\widehat{\tau})$ is a finite-dimensional space and that $\| \cdot \|_{1,\widehat{\tau}} = \| \cdot \|_{H^1(\widehat{\tau})}$ and $\| \cdot \|_{0,\widehat{\tau}} = \| \cdot \|_{L^2(\widehat{\tau})}$ are norms on $\mathbb{P}_k(\widehat{\tau})$. Suppose now that $S_h = \left\{ \phi \in C(\overline{\Omega}) : \phi\big|_{\tau} \in \mathbb{P}_k(\tau) \right\}$ and let F_τ be the affine map that maps $\widehat{\tau}$ one-one onto τ. Following the notation of section 5.2 we write $F_\tau(\widehat{x}) = B_\tau \widehat{x} + b_\tau$, for $\widehat{x} = (\widehat{x}_1, \widehat{x}_2) \in \widehat{\tau}$, where B_τ is a 2×2 invertible matrix and b_τ a 2-vector. If $\phi \in \mathbb{P}_k(\tau)$ we define $\widehat{\phi} \in \mathbb{P}_k(\widehat{\tau})$ by $\phi(x) = \widehat{\phi}(\widehat{x})$, where $x = (x_1, x_2)$ and $x = F_\tau(\widehat{x})$ as usual. Then for $\phi \in \mathbb{P}_k(\tau)$, using the transformation norm inequalities (5.20), (5.21) and (6.49) we get

$$\begin{aligned} |\phi|_{1,\tau} &\leqq \widehat{C}(k) |B_\tau^{-1}| \, |\det B_\tau|^{1/2} \, |\widehat{\phi}|_{1,\widehat{\tau}} \\ &\leqq C_1 \widehat{C}(k) |B_\tau^{-1}| \, |\det B_\tau|^{1/2} \, \|\widehat{\phi}\|_{0,\widehat{\tau}} \\ &\leqq C(k, \widehat{\tau}) |B_\tau^{-1}| \, |\det B_\tau|^{1/2} \cdot |\det B_\tau|^{-1/2} \, \|\phi\|_{0,\tau} \, . \end{aligned}$$

Therefore, using the regularity of the triangulation and (5.24) we see that there exists a constant C independent if τ such that

$$|\phi|_{1,\tau} \leq \frac{C}{h_\tau} \|\phi\|_{0,\tau}, \quad \forall \phi \in \mathbb{P}_k(\tau),$$

where $h_\tau = \text{diam}(\tau)$. Since $S_h \subset H^1(\Omega)$ we see that

$$|\phi|_{1,\Omega}^2 = \sum_{\tau \in \mathcal{T}_h} |\phi|_{1,\tau}^2 \leqq C^2 \sum_{\tau \in \mathcal{T}_h} \frac{1}{h_\tau^2} \|\phi\|_{0,\tau}^2, \quad \forall \phi \in S_h \, .$$

If the triangulation is *quasiuniform* in the sense that for some constant ν independent of the partition

$$\frac{h}{h_\tau} \leqq \nu, \quad \forall \tau \in \mathcal{T}_h, \tag{6.50}$$

where $h = \max_\tau h_\tau$, we see that

$$|\phi|_{1,\Omega} \leq \frac{C\nu}{h} \|\phi\|_{0,\Omega}, \quad \forall \phi \in S_h, \tag{6.51}$$

from which (6.39) (and also $\|\phi\|_1 \leq Ch^{-1}\|\phi\|$) follows.

We mention that similar scaling arguments yield, for quasiuniforms partitions, the more general inverse inequalities

$$\|\chi\|_\alpha \leqq Ch^{\beta - \alpha} \|\chi\|_\beta, \quad \forall \chi \in S_h,$$

provided α, β are nonnegative integers so that $\beta < \alpha$ and $S_h \subset H^\alpha(\Omega)$, and

$$\|\chi\|_\infty \leq Ch^{-d/2}\|\chi\|, \quad \forall \chi \in S_h$$

for $\Omega \in \mathbb{R}^d$. Of interest is also the nonstandard inverse "almost Sobolev" inequality

$$\|\chi\|_\infty \leq C|\ln h|^{1/2}\|\chi\|_1, \quad \forall \chi \in S_h,$$

valid for $\Omega \subset \mathbb{R}^2$, cf. [3.6, p.68].

We close this section with a few remarks about stiff systems of ode's and the interpretation of the stability condition (6.40) as a restriction on the time step related to the size of the eigenvalues of the matrix $G^{-1}S$. Let A be a symmetric and positive definite real $m \times m$ matrix and consider the ode ivp for $y = y(t) \in \mathbb{R}^m$

$$
\begin{aligned}
\dot{y} + Ay &= 0, \quad t \geq 0, \\
y(0) &= y_0 .
\end{aligned}
\tag{6.52}
$$

Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ be the eigenvalues of A. Then from the theory of numerical solution of ode's, we know that if a method for the numerical solution of ivp's has interval of absolute stability $[-\alpha, 0]$, where $\alpha > 0$ ($a = +\infty$ if the method is A_0-stable), it will give stable approximations, when applied to (6.52), provided the time step Δt is chosen so that $(-\lambda_i)\Delta t \in [-\alpha, 0]$ for all i, i.e. so that $\Delta t \leqq \frac{\alpha}{\lambda_m}$. We recall that for the explicit Euler method $\alpha = 2$, while $\alpha = +\infty$ for the implicit Euler and the trapezoidal method. Hence, the latter two methods are suitable for stiff systems, i.e. systems for which $\lambda_1 = O(1)$ and $\lambda_m \gg 1$.

Consider now the ode system (6.7) corresponding to the Galerkin semidiscretization of the ibvp (6.1). For $f = 0$ we write the system as

$$
\begin{aligned}
G\dot{y} + Sy &= 0, \quad t \geq 0, \\
y(0) &= y_0 ,
\end{aligned}
\tag{6.53}
$$

where $y \in \mathbb{R}^m$ with $m = N_h = \dim S_h$. This system is of the form (6.52) with $A = G^{-1}S$ but $G^{-1}S$ is not symmetric. To transform the system into the form (6.52) with a symmetric positive definite matrix, consider the matrix $G^{1/2}$. Since G is symmetric and positive definite, $G^{1/2}$ is defined e.g. using the spectral representation of G and is

symmetric and positive definite. Multiplying by $G^{-1/2}$ both sides of the ode system in (6.53) we have

$$G^{1/2}\dot{y} + G^{-1/2}S\,G^{-1/2}\,G^{1/2}y = 0\,,$$

or

$$\dot{z} + Az = 0\,, \quad t \geq 0\,, \tag{6.53$'$}$$

where $z(t) = G^{1/2}y(t)$, and $A = G^{-1/2}S\,G^{-1/2}$ is easily seen to be symmetric and positive definite. Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ be the eigenvalues of A. Then they satisfy

$$G^{-1/2}S\,G^{-1/2}x = \lambda_i x\,, \tag{6.54}$$

where $x \in \mathbb{R}^m$ is a corresponding to λ_i eigenvector. It follows that $S\,G^{-1/2}x = \lambda_i G^{1/2}x$. Therefore, with $G^{-1/2}x = w$, i.e. $Gw = G^{1/2}x$, we see that the eigenvalue problem (6.54) is equivalent to the generalized eigenvalue problem

$$Sw = \lambda_i Gw \tag{6.55}$$

and so the λ_i are eigenvalues of $G^{-1}S$. It follows from (6.55) that

$$\lambda_i = \frac{w^{\mathrm{T}}Sw}{w^{\mathrm{T}}Gw}\,.$$

Let now $\phi = \sum_{i=1}^m w_i\phi_i \in S_h$, where $\{\phi_i\}$ is the chosen basis of S_h. Then $\|\phi\|^2 = w^{\mathrm{T}}Gw$, $\|\nabla\phi\|^2 = w^{\mathrm{T}}Sw$, and $\lambda_i = \frac{\|\nabla\phi\|^2}{\|\phi\|^2}$. Therefore, if the inverse inequality (6.39) holds, we have that

$$\lambda_m = \lambda_{\max}(G^{-1}S) \leqq \frac{C_*^2}{h^2}\,. \tag{6.56}$$

Hence, a sufficient condition for the stability of the explicit Euler scheme for the ivp (6.53$'$) or, equivalently, for the ivp (6.53), is, since $\left(-\frac{C_*^2}{h^2}\right)k \leqq (-\lambda_m)k$, $k = \Delta t$, that $-\frac{C_*^2}{h^2}k \geqq -2$, i.e. $\frac{k}{h^2} \leq \frac{2}{C_*^2}$, which is precisely the restriction (6.40) found by the energy method. For the implicit Euler or the Crank-Nicolson (i.e. the trapezoidal) scheme for which $\alpha = +\infty$, there is no restriction.

Chapter 7

The Galerkin Finite Element Method for the Wave Equation

7.1   Introduction

In this chapter we shall consider Galerkin finite element methods for the following model *second-order hyperbolic* initial-boundary-value problem. Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 1, 2$, or 3. We seek a real function $u = u(x, t), x \in \overline{\Omega}, t \geq 0$, such that

$$
\begin{cases}
u_{tt} - \Delta u = f, \ x \in \Omega, \ t \geq 0, \\
u = 0, \ x \in \partial\Omega, \ t \geq 0, \\
u(x, 0) = u^0(x), \ u_t(x, 0) = u_t^0(x), \ x \in \overline{\Omega}.
\end{cases}
\tag{7.1}
$$

Here $f = f(x, t)$ and u^0, u_t^0 are given real functions on $\Omega \times [0, \infty)$, and $\overline{\Omega}$ respectively. We shall assume that the ibvp (7.1) has a unique solution, which is sufficiently smooth for the purposes of the analysis of its numerical approximation. For the theory of existence-uniqueness and regularity of (7.1) see [2.2]-[2.5]. We just make two remarks here for the homogeneous problem, i.e. when $f = 0$ in (7.1).

i. Multiplying both sides of the pde in (7.1) by u_t, integrating over Ω using Gauss's theorem and the boundary and initial conditions, we easily get the *energy* conservation identity

$$
\|u_t\|^2 + \|\nabla u\|^2 = \|u_t^0\|^2 + \|\nabla u^0\|^2, \ t \geq 0.
\tag{7.2}
$$

ii. The regularity theory for (7.1) requires smoothness and compatibility conditions at

$\partial\Omega$ of the initial data u^0, u_t^0, and establishes corresponding smoothness and compatibility properties in the same spaces of the solutions u and u_t for $t > 0$. Thus, in the case of (7.1) we do not have any smoothing effect on the solutions for $t > 0$ as in the case of the heat equation, cf. e.g [2.4].

7.2 Standard Galerkin semidiscretization

Multiplying the pde in (7.1) by $v \in \overset{0}{H}{}^1$ and integrating over Ω using Gauss's theorem we obtain

$$(u_{tt}, v) + (\nabla u, \nabla v) = (f, v), \quad t \geq 0. \tag{7.3}$$

Let S_h be a finite-dimensional subspace of $\overset{0}{H}{}^1$ satisfying (6.2). Motivated by (7.3) we define the *(standard Galerkin) semidiscrete approximation u_h* of u as a function $u_h(t)$ in $S_h, t \geq 0$, such that

$$\begin{cases} (u_{htt}, \varphi) + (\nabla u_h, \nabla \varphi) = (f, \varphi), \ \forall \varphi \in S_h, \ t \geq 0, \\ u_h(0) = u_h^0, \\ u_{ht}(0) = u_{t,h}^0, \end{cases} \tag{7.4}$$

where $u_h^0, u_{t,h}^0$ are given elements of S_h approximating u^0, u_t^0, respectively. If $\{\varphi_j\}_{j=1}^{N_h}$ is a basis of S_h and we let

$$u_h(x, t) = \sum_{j=1}^{N_h} \alpha_j(t)\varphi_j(x),$$

we see that

$$\begin{cases} G\ddot{\alpha} + S\alpha = F(t), \quad t \geq 0, \\ \alpha(0) = \beta, \\ \dot{\alpha}(0) = \gamma, \end{cases} \tag{7.5}$$

where $G_{ij} = (\varphi_j, \varphi_i)$, $S_{ij} = (\nabla\varphi_j, \nabla\varphi_i)$, $1 \leq i, j \leq N_h$, $F_i = (f, \varphi_i)$, $1 \leq i \leq N_h$, $\alpha = \alpha(t) = [\alpha_1, \ldots, \alpha_{N_h}]^T$, and $\beta = (\beta_i)$, $\gamma = (\gamma_i)$ are the coefficients of $u_h^0, u_{t,h}^0$, with respect to the basis $\{\varphi_i\}$. The ivp (7.5) may also be written as

$$\begin{cases} \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \alpha \\ \dot{\alpha} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -S & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \dot{\alpha} \end{pmatrix} + \begin{pmatrix} 0 \\ F \end{pmatrix}, \quad t \geq 0, \\ (\alpha(0), \dot{\alpha}(0))^T = (\beta, \gamma)^T, \end{cases}$$

i.e. as an ivp for a first-order system of ode's; it is then evident that (7.5) has a unique solution for $t \geq 0$. If we put $\varphi = u_h$ in (7.4) with $f = 0$ it is straightforward to prove that

$$\|u_{ht}(t)\|^2 + \|\nabla u_h(t)\|^2 = \|u_{t,h}^0\|^2 + \|\nabla u_h^0\|^2, \quad t \geq 0, \tag{7.6}$$

which is the discrete analog of (7.2) and expresses the stability (conservation) of (u_h, u_{ht}) in $\overset{0}{H}{}^1 \times L^2$.

We recall now the definition of the *elliptic projection* of a function $v \in \overset{0}{H}{}^1$ as the element $R_h v \in S_h$ such that $(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi)$ for all $\chi \in S_h$. Following Dupont, SIAM J. Numer. Anal., 10(1973), 880-889, we have

Theorem 7.1. Let u, u_h be the solutions of the ivp's (7.1) and (7.4) respectively. Then, given $T > 0$, there exists a positive constant $C = C(T)$ such that

$$\|u_h(t) - u(t)\| \leq C\Big\{\|\nabla(u_h^0 - R_h u^0)\| + \|u_{t,h}^0 - R_h u_t^0\|$$
$$+ h^r \Big[\|u(t)\|_r + \Big(\int_0^t \|u_{tt}\|_r^2 ds\Big)^{1/2}\Big]\Big\}, \quad 0 \leq t \leq T. \tag{7.7}$$

Proof. We write as usual $u_h - u = \theta + \varrho$, where $\theta = u_h - R_h u$, $\varrho = R_h u - u$. We have as before

$$\|\varrho(t)\| \leq Ch^r\|u(t)\|_r, \quad t \geq 0. \tag{7.8}$$

Now, for $t \geq 0$ and any $\chi \in S_h$ we have using (7.4) and (7.3)

$$(\theta_{tt}, \chi) + (\nabla\theta, \nabla\chi) = (u_{htt}, \chi) + (\nabla u_h, \nabla\chi) - (R_h u_{tt}, \chi) - (\nabla R_h u, \nabla\chi)$$
$$= (f, \chi) - (R_h u_{tt}, \chi) - (\nabla u, \nabla\chi)$$
$$= (u_{tt} - R_h u_{tt}, \chi) = -(\varrho_{tt}, \chi).$$

Given $t > 0$ take $\chi = \theta_t$ in the above to obtain

$$\frac{1}{2}\frac{d}{dt}(\|\theta_t\|^2 + \|\nabla\theta\|^2) = -(\varrho_{tt}, \theta_t) \leq \frac{1}{2}\|\varrho_{tt}\|^2 + \frac{1}{2}\|\theta_t\|^2.$$

Hence

$$\frac{d}{dt}(\|\theta_t\|^2 + \|\nabla\theta\|^2) \leq \|\varrho_{tt}\|^2 + (\|\theta_t\|^2 + \|\nabla\theta\|^2), \quad t \geq 0. \tag{7.9}$$

We recall now *Gronwall's lemma*: If $\sigma'(t) \leq \alpha(t) + \sigma(t)$, $t \geq 0$, then $\sigma(t) \leq e^t\sigma(0) + \int_0^t e^{t-s}\alpha(s)ds$. To see this, note that for $t \geq 0$

$$e^{-t}\sigma'(t) - e^{-t}\sigma(t) \leq e^{-t}\alpha(t).$$

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Therefore

$$\frac{d}{dt}(e^{-t}\sigma(t)) \leq e^{-t}\alpha(t), \quad \text{i.e.}$$

$$e^{-t}\sigma(t) - \sigma(0) \leq \int_0^t e^{-s}\alpha(s)ds,$$

from which the conclusion follows. Using this result in (7.9) for $\sigma = \|\theta_t\|^2 + \|\nabla\theta\|^2$ we have for $0 \leq t \leq T$

$$\|\theta_t\|^2 + \|\nabla\theta\|^2 \leq C(T)\left(\|\theta_t(0)\|^2 + \|\nabla\theta(0)\|^2 + \int_0^t \|\varrho_{tt}\|^2 ds\right),$$

where $C(T) = e^T$. By the definition of θ and ϱ it follows that for $0 \leq t \leq T$

$$\|\theta_t\|^2 + \|\nabla\theta\|^2 \leq C(T)\left[\|u_{t,h}^0 - R_h u_t^0\|^2 + \|\nabla(u_h^0 - R_h u^0)\|^2 + Ch^{2r}\int_0^t \|u_{tt}\|^2 ds\right].$$

Using now the inequalities $\frac{1}{\sqrt{n}}\sum_{i=1}^n \alpha_i \leq (\sum_{i=1}^n \alpha_i^2)^{1/2} \leq \sum_{i=1}^n \alpha_i$, valid for $\alpha_i \geq 0$, we conclude that for some constant $C = C(T)$

$$\|\theta_t\| + \|\nabla\theta\| \leq C[\|u_{t,h}^0 - R_h u_t^0\| + \|\nabla(u_h^0 - R_h u^0)\|$$

$$+ h^r\left(\int_0^t \|u_{tt}\|^2 ds\right)^{1/2}], \quad 0 \leq t \leq T. \tag{7.10}$$

We note now that $\theta(t) = \int_0^t \theta_t(s)ds + \theta(0)$, from which for $t \geq 0$

$$\|\theta(t)\| \leq \|\theta(0)\| + \int_0^t \|\theta_t\|ds \leq \|\theta(0)\| + t\max_{0\leq s\leq t}\|\theta_t(s)\|.$$

Therefore, using (7.10) we have for $0 \leq t \leq T$

$$\|u_h(t) - u(t)\| \leq \|\varrho(t)\| + \|\theta(t)\|$$

$$\leq Ch^r\|u(t)\|_r + \|\theta(0)\| + t\max_{0\leq s\leq t}\|\theta_t(s)\|$$

$$\leq C(T)\{\|u_{t,h}^0 - R_h u_t^0\| + \|\nabla(u_h^0 - R_h u^0)\|$$

$$+ h^r[\|u(t)\|_r + \left(\int_0^t \|u_{tt}\|_r^2 ds\right)^{1/2}]\},$$

which is the desired estimate (7.7). (Note that we used *Poincaré's inequality* $\|\theta(0)\| \leq C\|\nabla\theta(0)\|$ in the last inequality.)

$\square$

Remarks

a. The estimate (7.7) implies that if we choose $u_h^0 = R_h u^0$, $u_{t,h}^0 = R_h u_t^0$, then we have the optimal-order L^2-estimate $\|u_h(t) - u(t)\| = O(h^r)$. We may also choose $u_{t,h}^0$ as any optimal-order (in L^2) approximation to u_t^0 in S_h, e.g. as Pu_t, where P is the L^2-projection onto S_h. However we cannot do the same for u_h^0, which must be close to $R_h u^0$ in H^1 to $O(h^r)$ to guarantee the optimal-order bound in (7.7).

b. From (7.10) it also follows that for $0 \le t \le T$

$$\|u_{ht} - u_t\| \le \|\theta_t\| + \|\varrho_t\| \le C \left\{ \|u_{t,h}^0 - R_h u_t^0\| \right.$$
$$\left. + \|\nabla(u_h^0 - R_h u^0)\| + h^r \left[\|u_t(t)\|_r + \left(\int_0^t \|u_{tt}\|_r^2 \, ds \right)^{1/2} \right] \right\},$$

and

$$\|\nabla(u_h - u)\| \le \|\nabla\theta\| + \|\nabla\varrho\| \le C \left\{ \|u_{t,h}^0 - R_h u_t^0\| \right.$$
$$\left. + \|\nabla(u_h^0 - R_h u^0)\| + h^{r-1} \left[\|u\|_{r-1} + \left(\int_0^t \|u_{tt}\|_r^2 \, ds \right)^{1/2} \right] \right\}.$$

The latter inequality implies that initial conditions e.g. of the type $u_h^0 = Pu^0$, $u_{t,h}^0 = Pu_t^0$ will give an optimal-order estimate for $\|\nabla(u_h - u)\|$.

The following result, due to G. Baker, SIAM J. Numer. Anal. 13(1976), 564-576, relies on a 'duality' argument in time and shows that one may after all get an L^2 estimate of optimal order starting with any optimal-order L^2 approximation of u^0 and u_t^0. We state it in a form that also improves Theorem 7.1 with respect to the required regularity of the solution and the dependence of C on T.

Theorem 7.2. Let u, u_h be the solutions of the ivp' s (7.1), (7.4), respectively. Then there exists a constant C, independent of t and h, such that

$$\|u_h(t) - u(t)\| \le C\{\|u_h^0 - Pu^0\| + t\|u_{t,h}^0 - Pu_t^0\|$$
$$+ h^r[\|u^0\|_r + \int_0^t \|u_t\|_r ds]\}. \tag{7.11}$$

Proof. We put $e := u_h - u = \theta + \varrho$, where $\theta = u_h - R_h u$, $\varrho = R_h u - u$. As in the proof of Theorem 7.1 we have $(\theta_{tt}, \chi) + (\nabla\theta, \nabla\chi) = -(\varrho_{tt}, \chi)$ for all $\chi \in S_h$, $t \ge 0$. Since

$(\theta_{tt}, \chi) = \frac{d}{dt}(\theta_t, \chi) - (\theta_t, \chi_t)$ we have for any $\chi \in S_h$, $t > 0$

$$-(\theta_t, \chi_t) + (\nabla\theta, \nabla\chi) = -\frac{d}{dt}(\theta_t, \chi) - \frac{d}{dt}(\varrho_t, \chi) + (\varrho_t, \chi_t)$$

$$= -\frac{d}{dt}(e_t, \chi) + (\varrho_t, \chi_t). \tag{7.12}$$

Fix $\xi > 0$ and let $\hat{\chi}(t) := \int_t^\xi \theta(s)ds$, $t \geq 0$. Then $\hat{\chi}(t) \in S_h$ for $t \geq 0$, $\hat{\chi}(\xi) = 0$, and $\hat{\chi}_t = -\theta$. Select $\chi = \hat{\chi}$ in (7.12) to obtain

$$(\theta_t, \theta) - (\nabla\hat{\chi}_t, \nabla\hat{\chi}) = -\frac{d}{dt}(e_t, \hat{\chi}) - (\varrho_t, \theta),$$

i.e.

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 - \frac{1}{2}\frac{d}{dt}\|\nabla\hat{\chi}\|^2 = -\frac{d}{dt}(e_t, \hat{\chi}) - (\varrho_t, \theta), \quad t \geq 0.$$

Integrating both sides of the above with respect to t from 0 to ξ, we obtain

$$\|\theta(\xi)\|^2 - \|\theta(0)\|^2 - \|\nabla\hat{\chi}(\xi)\|^2 + \|\nabla\hat{\chi}(0)\|^2$$

$$= -2(e_t(\xi), \hat{\chi}(\xi)) + 2(e_t(0), \hat{\chi}(0)) - 2\int_0^\xi (\varrho_t, \theta)dt.$$

Since $\hat{\chi}(\xi) = 0$, $\hat{\chi} \in S_h$, we have for any $\xi > 0$:

$$\|\theta(\xi)\|^2 \leq \|\theta(0)\|^2 + 2(Pe_t(0), \hat{\chi}(0)) - 2\int_0^\xi (\varrho_t, \theta)dt.$$

Hence

$$\|\theta(\xi)\|^2 \leq \|\theta(0)\|^2 + 2\|Pe_t(0)\|\|\hat{\chi}(0)\| + 2\int_0^\xi \|\varrho_t\|\|\theta\|dt.$$

We recall that $\hat{\chi}(0) = \int_0^\xi \theta(t)dt$. Therefore $\|\hat{\chi}(0)\| \leq \int_0^\xi \|\theta\|dt$ and therefore

$$\|\theta(\xi)\|^2 \leq \|\theta(0)\|^2 + 2\|Pe_t(0)\|\int_0^\xi \|\theta\|dt + 2\int_0^\xi \|\varrho_t\|\|\theta\|dt$$

$$\leq \sup_{0 \leq s \leq \xi}\|\theta(s)\|(\|\theta(0)\| + 2\xi\|Pe_t(0)\| + 2\int_0^\xi \|\varrho_t\|dt).$$

Since ξ was arbitrary, the inequality above holds for all $\xi' \in [0, \xi]$. Let $\tau \in [0, \xi]$ be a point where $\|\theta(\tau)\| = \sup_{0 \leq s \leq \xi}\|\theta(s)\|$. Applying the inequality for $\xi' = \tau$ we get

$$\|\theta(\tau)\|^2 \leq \|\theta(\tau)\|\left(\|\theta(0)\| + 2\xi\|Pe_t(0)\| + 2\int_0^\xi \|\varrho_t\|dt\right).$$

Hence, since $\|\theta(\xi)\| \leqq \|\theta(\tau)\|$

$$\|\theta(\xi)\| \leq \|\theta(0)\| + 2\xi\|Pe_t(0)\| + 2\int_0^\xi \|\varrho_t\|dt,$$

for any $\xi > 0$. Therefore, for $t \geq 0$, using the definition of ϱ

$$\|\theta(t)\| \leq \|\theta(0)\| + 2t\|Pe_t(0)\| + Ch^r \int_0^t \|u_t\|_r ds, \tag{7.13}$$

and

$$\|u_h(t) - u(t)\| \leq \|\varrho(t)\| + \|\theta(t)\| \leq \|\varrho(0)\| + \int_0^t \|\varrho_t\| ds + \|\theta(t)\|$$

$$\leq C \left\{ \|u_h^0 - Pu^0\| + t\|u_{t,h}^0 - Pu_t^0\| + h^r [\|u^0\|_r + \int_0^t \|u_t\|_r ds] \right\},$$

which is (7.11). (To get the last inequality of the right-hand side we used (7.13) and that $\|\varrho(0)\| = \|R_h u^0 - u^0\| \leq Ch^r \|u^0\|_r$, $\|\theta(0)\| \leq \|u_h^0 - Pu^0\| + Ch^r\|u^0\|_r$, and $Pe_t(0) = P(u_{t,h}^0 - u_t^0) = u_{t,h}^0 - Pu_t^0$.) $\qquad\square$

It is evident that (7.11) implies that $\|u(t) - u_h(t)\| = O(h^r)$ if u_h^0, $u_{t,h}^0$ are any optimal-order L^2-approximations of u^0, u_t^0 in S_h, respectively.

Remark

It is straightforward to check that analogs of Theorems 7.1 and 7.2 hold for the ibvp

$$\begin{cases} u_{tt} - \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u = f(x,t), & x \in \Omega,\ t \geq 0, \\ u = 0 \quad \text{on } \partial\Omega,\ t \geq 0, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u_t^0(x), \quad x \in \overline{\Omega}, \end{cases}$$

where the coefficients a_{ij}, a_0 satisfy the conditions set forth in Exercise 3 of section 6.2. L.A. Bales has proved (Math. Comp. 43(1984), 383-414) that similar results hold when a_{ij} and a_0 depend also on t.

7.3   Fully discrete schemes

In this section we shall examine some simple methods for discretizing in time the semidiscrete Galerkin equations (7.4). As usual we put $t^n = nk$, $n = 0, \ldots, M$, $Mk = T > 0$. We seek $U^n \in S_h$, $0 \leq n \leq M$, approximations of the solution u of (7.1) at t^n, satisfying

$$\begin{cases} \frac{1}{k^2}(U^{n+1} - 2U^n + U^{n-1}, \chi) + (\nabla \hat{U}_\beta^n, \nabla\chi) = (\hat{f}_\beta^n, \chi), & \forall \chi \in S_h,\ 1 \leq n \leq M-1, \\ U^0, U^1 \text{ given in } S_h, \end{cases}$$

$$\tag{7.14}$$

where, for $1 \leq n \leq M - 1$ and $\beta \geq 0$, $\hat{U}_\beta^n := \beta U^{n+1} + (1 - 2\beta)U^n + \beta U^{n-1}$, $\hat{f}_\beta^n = \beta f^{n+1} + (1 - 2\beta)f^n + \beta f^{n-1}$, $f^n = f(\cdot, t^n)$. Given U^{n-1}, U^n, (7.14) defines uniquely $U^{n+1} \in S_h$ as solution of the linear system of equations

$$\left(G + \beta k^2 S\right) \alpha^{n+1} = \left(2G - (1 - 2\beta)k^2 S\right) \alpha^n - \left(G + \beta k^2 S\right) \alpha^{n-1} + k^2 \hat{F}_\beta^n,$$

where $U^n = \sum_{i=1}^{N_h} \alpha_i^n \varphi_i$, $\{\varphi_i\}$, G, S, were defined in section 7.2, and $\hat{F}_\beta^n$ is the N_h-vector with components $(\hat{f}_\beta^n, \varphi_i)$. If $\beta = 0$ (7.14) corresponds to the classical Courant-Friedrichs-Lewy two-step scheme for the wave equation.

Taking for simplicity $f = 0$ in (7.14) and putting $\chi = U^{n+1} - U^{n-1}$ we obtain for $1 \leq n \leq M - 1$

$$(U^{n+1} - 2U^n + U^{n-1}, U^{n+1} - U^{n-1}) + k^2(\nabla \hat{U}_\beta^n, \nabla(U^{n+1} - U^{n-1})) = 0. \qquad (7.15)$$

Now

$$\left(U^{n+1} - 2U^n + U^{n-1}, U^{n+1} - U^{n-1}\right)$$
$$= \left((U^{n+1} - U^n) - (U^n - U^{n-1}), (U^{n+1} - U^n) + (U^n - U^{n-1})\right)$$
$$= \|U^{n+1} - U^n\|^2 - \|U^n - U^{n-1}\|^2.$$

In addition

$$(\nabla \hat{U}_\beta^n, \nabla(U^{n+1} - U^{n-1})) = (\nabla(\beta U^{n+1} + (1 - 2\beta)U^n + \beta U^{n-1}), \nabla(U^{n+1} - U^{n-1}))$$
$$= \beta((\nabla U^{n+1}, \nabla U^{n+1}) - (\nabla U^{n-1}, \nabla U^{n-1}))$$
$$+ (1 - 2\beta)((\nabla U^{n+1}, \nabla U^n) - (\nabla U^n, \nabla U^{n-1})).$$

Hence, summing in (7.15) with respect to n from $n = 1$ to l, where $1 \leq l \leq M - 1$, we see that

$$\|U^{l+1} - U^l\|^2 + \beta k^2 (\|\nabla U^{l+1}\|^2 + \|\nabla U^l\|^2) + (1 - 2\beta)k^2(\nabla U^{l+1}, \nabla U^l)$$
$$= \|U^1 - U^0\|^2 + \beta k^2(\|\nabla U^1\|^2 + \|\nabla U^0\|^2) + (1 - 2\beta)k^2(\nabla U^1, \nabla U^0),$$

which, motivated by the identity

$$\beta(x^2 + y^2) + (1 - 2\beta)xy = \frac{(x + y)^2}{4} + (\beta - \frac{1}{4})(x - y)^2,$$

we rewrite as

$$\|U^{l+1} - U^l\|^2 + \frac{k^2}{4}\|\nabla(U^{l+1} + U^l)\|^2 + k^2(\beta - \frac{1}{4})\|\nabla(U^{l+1} - U^l)\|^2$$

$$= \|U^1 - U^0\|^2 + \frac{k^2}{4}\|\nabla(U^1 + U^0)\|^2 + k^2(\beta - \frac{1}{4})\|\nabla(U^1 - U^0)\|^2, \tag{7.16}$$

which is valid for $0 \leq l \leq M - 1$. With the aid of the identity (7.16) we may prove the following *stability* result for the fully discrete scheme (7.14).

Proposition 7.1. Suppose that there exists a constant c, independent of h and k, such that

$$\|U^1 - U^0\| \leq ck \tag{7.17}$$

and

$$\|\nabla(U^1 \pm U^0)\| \leq c. \tag{7.18}$$

Then, there exists a constant C, independent of h and k, such that for all $\beta \geq \frac{1}{4}$ the solution of (7.14) with $f = 0$ satisfies

$$\max_{0 \leq n \leq M} \|U^n\| \leq C. \tag{7.19}$$

If $0 \leq \beta < \frac{1}{4}$ and the inverse inequality (6.39) is valid in S_h, then (7.19) holds provided $\frac{k}{h} \leq \alpha$, where α is some constant depending on C_* and β.

Proof. If $\beta \geq \frac{1}{4}$, (7.16) and (7.17) - (7.18) give, for $l \geq 0$ and $k \leq 1$, $\|U^{l+1} - U^l\| \leq C$ and $\|\nabla(U^{l+1} - U^l)\| \leq C$. Hence, by Poincaré's inequality, $\|U^{l+1} \pm U^l\| \leq C$, $l \geq 0$, and (7.19) follows since $\|U^n\| = \|\left(\frac{U^{n+1}+U^n}{2}\right) - \left(\frac{U^{n+1}-U^n}{2}\right)\|$.

If $0 \leq \beta < \frac{1}{4}$, (7.16) and (7.17) - (7.18) give for $l \geq 0$

$$\|U^{l+1} - U^l\|^2 + \frac{k^2}{4}\|\nabla(U^{l+1} + U^l)\|^2 \leq k^2(\frac{1}{4} - \beta)\|\nabla(U^{l+1} - U^l)\|^2 + Ck^2.$$

Therefore, by the inverse inequality (6.39) we have

$$\|U^{l+1} - U^l\|^2 + \frac{k^2}{4}\|\nabla(U^{l+1} - U^l)\|^2 \leq \frac{k^2}{h^2}(\frac{1}{4} - \beta)C_*^2\|U^{l+1} - U^l\|^2 + Ck^2,$$

i.e.

$$[1 - C_*^2\frac{k^2}{h^2}(\frac{1}{4} - \beta)]\|U^{l+1} - U^l\|^2 + \frac{k^2}{4}\|\nabla(U^{l+1} + U^l)\|^2 \leq Ck^2, \quad l \geq 0.$$

Hence, if

$$\frac{k}{h} \leq \alpha < \frac{2}{C_*\sqrt{1 - 4\beta}} \tag{7.20}$$

we have that $\|U^{l+1} \pm U^l\| \leq C(\alpha)$ and (7.19) follows. $\qquad\square$

As we remarked in Section 6.4, a more general sufficient stability condition is

$$k \left[\lambda_{\max}(G^{-1}S) \right]^{1/2} < \frac{2}{\sqrt{1 - 4\beta}}. \tag{7.21}$$

The inverse inequality (6.39) and (7.20) imply (7.21).

We proceed now to derive L^2-error estimates for the scheme (7.14), following e.g. Dupont, op.cit. For simplicity we treat only the case $\beta = \frac{1}{4}$, i.e. the case of smallest value of β for which (7.19) holds unconditionally. The proof for the other $\beta \geq 0$ follows along similar lines. The basic step of the proof is the following result.

Proposition 7.2. Let u be the solution of (7.1), U^n the solution of the scheme (7.14) for $\beta = \frac{1}{4}$, and $\theta^n = U^n - R_h u^n$, where $u^n = u(t^n)$. Then, there exists a constant C independent of h, k and T such that for $0 \leq n \leq M - 1$

$$\max_{0 \leq l \leq n} \left(\frac{1}{k} \|\theta^{l+1} - \theta^l\| + \|\nabla(\theta^{l+1} + \theta^l)\| \right) \leq C \left\{ \frac{1}{k} \|\theta^1 - \theta^0\| + \|\nabla(\theta^1 + \theta^0)\| \right.$$

$$\left. + \sqrt{T} \left[h^r \left(\int_0^{t^{n+1}} \|u_{tt}\|_r^2 \, ds \right)^{1/2} + k^2 \left(\int_0^{t^{n+1}} \|\partial_t^4 u\|^2 \, ds \right)^{1/2} \right] \right\}. \tag{7.22}$$

Proof. For $1 \leq n \leq M - 1$ we denote $\partial_\tau^2 g^n = \frac{1}{k^2}(g^{n+1} - 2g^n + g^{n-1})$, $\hat{g}^n = \hat{g}_{1/4}^n = \frac{1}{4}(g^{n+1} + 2g^n + g^{n-1})$. If $U^n - u^n = (U^n - R_h u^n) + (R_h u^n - u^n) = \theta^n + \varrho^n$, using (7.14) with $\beta = 1/4$ and (7.1), we have for $1 \leq n \leq M - 1$, $\chi \in S_h$

$$(\partial_\tau^2 \theta^n, \chi) + (\nabla \hat{\theta}^n, \nabla \chi) = (\hat{f}^n, \chi) - (\partial_\tau^2(R_h u^n), \chi) - (\nabla \hat{u}^n, \nabla \chi)$$

$$= (\hat{u}_{tt}^n - \partial_\tau^2(R_h u^n), \chi)$$

$$= (\partial_\tau^2 u^n - \partial_\tau^2(R_h u^n) + \hat{u}_{tt}^n - \partial_\tau^2 u^n, \chi)$$

$$= (-\partial_\tau^2 \varrho^n + \omega^n, \chi),$$

where $\omega^n := \hat{u}_{tt}^n - \partial_\tau^2 u^n$. Therefore, for any $\chi \in S_h$

$$(\theta^{n+1} - 2\theta^n + \theta^{n-1}, \chi) + k^2(\nabla \hat{\theta}^n, \nabla \chi) = k^2(-\partial_\tau^2 \varrho^n + \omega^n, \chi), \quad 1 \leq n \leq M - 1. \tag{7.23}$$

We take now $\chi = U^{n+1} - U^{n-1}$ and sum both sides of (7.23) with respect to n from $n = 1$ to l, for $1 \leq l \leq M - 1$. As in the proof of Proposition 7.1 we obtain

$$\|\theta^{l+1} - \theta^l\|^2 + \frac{k^2}{4}\|\nabla(\theta^{l+1} + \theta^l)\|^2 = \|\theta^1 - \theta^0\|^2 + \frac{k^2}{4}\|\nabla(\theta^1 + \theta^0)\|^2$$

$$+ k^2 \sum_{n=1}^l (-\partial_\tau^2 \varrho^n + \omega^n, \theta^{n+1} - \theta^{n-1}). \tag{7.24}$$

For any $\varepsilon > 0$ we have

$$k^2 \sum_{n=1}^{l} (-\partial_\tau^2 \varrho^n + \omega^n, \theta^{n+1} - \theta^{n-1}) \le k^2 \sum_{n=1}^{l} \| -\partial_\tau^2 \varrho^n + \omega^n \| \| \theta^{n+1} - \theta^{n-1} \|$$

$$\le \frac{k^2}{2\varepsilon} \sum_{n=1}^{l} \| -\partial_\tau^2 \varrho^n + \omega^n \|^2 + \frac{\varepsilon k^2}{2} \sum_{n=1}^{l} \| \theta^{n+1} - \theta^{n-1} \|^2$$

$$\le \frac{k^2}{\varepsilon} \sum_{n=1}^{l} \| \partial_\tau^2 \varrho^n \|^2 + \frac{k^2}{\varepsilon} \sum_{n=1}^{l} \| \omega^n \|^2 + \frac{\varepsilon k^2}{2} \sum_{n=1}^{l} (\| \theta^{n+1} - \theta^n \| + \| \theta^n - \theta^{n-1} \|)^2$$

$$\le \frac{k^2}{\varepsilon} \sum_{n=1}^{l} \| \partial_\tau^2 \varrho^n \|^2 + \frac{k^2}{\varepsilon} \sum_{n=1}^{l} \| \omega^n \|^2 + 2\varepsilon k^2 \sum_{n=0}^{l} \| \theta^{n+1} - \theta^n \|^2. \qquad (7.25)$$

We estimate now the first two terms in the right-hand side of the above. By Taylor's theorem we have

$$\varrho^{n+1} = \varrho^n + k\varrho_t^n + \int_{t^n}^{t^{n+1}} (t^{n+1} - \tau) \varrho_{tt}(\tau) \, d\tau,$$

$$\varrho^{n-1} = \varrho^n - k\varrho_t^n + \int_{t^n}^{t^{n-1}} (t^{n-1} - \tau) \varrho_{tt}(\tau) \, d\tau.$$

Therefore

$$\partial_\tau^2 \varrho^n = \frac{1}{k^2} \left[\int_{t^n}^{t^{n+1}} (t^{n+1} - \tau) \varrho_{tt} \, d\tau - \int_{t^{n-1}}^{t^n} (t^{n-1} - \tau) \varrho_{tt} \, d\tau \right].$$

Since

$$\left(\int_{t^n}^{t^{n+1}} (t^{n+1} - \tau) \varrho_{tt} d\tau \right)^2 \le \int_{t^n}^{t^{n+1}} (t^{n+1} - \tau)^2 d\tau \int_{t^n}^{t^{n+1}} (\varrho_{tt})^2 d\tau$$

$$\le \frac{k^3}{3} \int_{t^n}^{t^{n+1}} (\varrho_{tt})^2 d\tau,$$

we have

$$(\partial_\tau^2 \varrho^n)^2 \le \frac{C}{k^4} k^3 \int_{t^{n-1}}^{t^{n+1}} (\varrho_{tt})^2 ds,$$

and we conclude by the definition of ϱ that

$$\sum_{n=1}^{l} \| \partial_\tau \varrho^n \|^2 \le \frac{C}{k} \int_0^{t^{l+1}} \| \varrho_{tt} \|^2 ds \le C k^{-1} h^{2r} \int_0^{t^{l+1}} \| u_{tt} \|_r^2 ds.$$

Now

$$\omega^n = \hat{u}_{tt}^n - \partial_\tau^2 u^n = (\hat{u}_{tt}^n - u_{tt}^n) + (u_{tt}^n - \partial_\tau^2 u^n) =: \omega_1^n + \omega_2^n.$$

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For ω_1^n by Taylor' s theorem we have

$$\omega_1^n = \frac{1}{4}\left[\int_{t^n}^{t^{n+1}}(t^{n+1}-s)\partial_t^4 u\,ds + \int_{t^n}^{t^{n-1}}(t^{n-1}-s)\partial_t^4 u\,ds\right].$$

Hence $(\omega_1^n)^2 \le ck^3\int_{t^{n-1}}^{t^{n+1}}(\partial_t^4 u)^2 ds$ and

$$\sum_{n=1}^{l}\|\omega_1^n\|^2 \le ck^3\int_0^{t^{l+1}}\|\partial_t^4 u\|^2 ds.$$

Similarly,

$$\sum_{n=1}^{l}\|\omega_2^n\|^2 \le ck^3\int_0^{t^{l+1}}\|\partial_t^4 u\|^2 ds.$$

We conclude that

$$\frac{k^2}{\varepsilon}\left(\sum_{n=1}^{l}\|\partial_\tau^2\varrho^n\|^2 + \sum_{n=1}^{l}\|\omega^n\|^2\right) \le \frac{ck}{\varepsilon}\left[h^{2r}\int_0^{t^{l+1}}\|u_{tt}\|_r^2 ds + k^4\int_0^{t^{l+1}}\|\partial_t^4 u\|^2 ds\right]$$

$$=: \frac{ck}{\varepsilon}\sigma_l, \quad 1 \le l \le M-1. \tag{7.26}$$

Therefore, by (7.24) - (7.26) we obtain for $0 \le l \le M-1$

$$\|\theta^{l+1}-\theta^l\|^2 + \frac{k^2}{4}\|\nabla(\theta^{l+1}+\theta^l)\|^2 \le \|\theta^1-\theta^0\|^2 + \frac{k^2}{4}\|\nabla(\theta^1+\theta^0)\|^2$$

$$+ \frac{ck}{\varepsilon}\sigma_l + 2\varepsilon k^2\sum_{n=0}^{l}\left(\|\theta^{n+1}-\theta^n\|^2 + \frac{k^2}{4}\|\nabla(\theta^{n+1}+\theta^n)\|^2\right)$$

$$\le \|\theta^1-\theta^0\|^2 + \frac{k^2}{4}\|\nabla(\theta^1+\theta^0)\|^2 + \frac{ck}{\varepsilon}\sigma_l$$

$$+ 2\varepsilon kT\max_{0\le n\le l}\left(\|\theta^{n+1}-\theta^n\|^2 + \frac{k^2}{4}\|\nabla(\theta^{n+1}+\theta^n)\|^2\right),$$

since $(l+1)k \le T$. Choose now $\varepsilon = \frac{1}{4Tk}$. Then, for $0 \le l \le M-1$

$$\|\theta^{l+1}-\theta^l\|^2 + \frac{k^2}{4}\|\nabla(\theta^{l+1}+\theta^l)\|^2 \le \|\theta^1-\theta^0\|^2 + \frac{k^2}{4}\|\nabla(\theta^1+\theta^0)\|^2$$

$$+ CTk^2\sigma_l + \frac{1}{2}\max_{0\le n\le l}\left(\|\theta^{n+1}-\theta^n\|^2 + \frac{k^2}{4}\|\nabla(\theta^{n+1}+\theta^n)\|^2\right). \tag{7.27}$$

Fix l for the moment and let m, $0 \le m \le l$, be an integer for which

$$\max_{0\le n\le l}\left(\|\theta^{n+1}-\theta^n\|^2 + \frac{k^2}{4}\|\nabla(\theta^{n+1}+\theta^n)\|^2\right) = \|\theta^{m+1}-\theta^m\|^2 + \frac{k^2}{4}\|\nabla(\theta^{m+1}+\theta^m)\|^2.$$

Then, from (7.27), since σ_n is increasing with n

$$\|\theta^{m+1}-\theta^m\|^2 + \frac{k^2}{4}\|\nabla(\theta^{m+1}-\theta^m)\|^2 \le \|\theta^1-\theta^0\|^2 + \frac{k^2}{4}\|\nabla(\theta^1+\theta^0)\|^2$$

$$+ CTk^2\sigma_l + \frac{1}{2}\left(\|\theta^{m+1}-\theta^m\|^2 + \frac{k^2}{4}\|\nabla(\theta^{m+1}+\theta^m)\|^2\right).$$

Therefore
$$\|\theta^{m+1} - \theta^m\|^2 + \frac{k^2}{4}\|\nabla(\theta^{m+1} - \theta^m)\|^2 \le 2[\|\theta^1 - \theta^0\|^2 + \frac{k^2}{4}\|\nabla(\theta^1 + \theta^0)\|^2] + CTk^2\sigma_l.$$

But
$$\|\theta^{l+1} - \theta^l\|^2 + \frac{k^2}{4}\|\nabla(\theta^{l+1} + \theta^l)\|^2 \le \|\theta^{m+1} - \theta^m\|^2 + \frac{k^2}{4}\|\nabla(\theta^{m+1} + \theta^m)\|^2.$$

Hence,
$$\|\theta^{l+1} - \theta^l\|^2 + \frac{k^2}{4}\|\nabla(\theta^{l+1} + \theta^l)\|^2 \le C[\|\theta^1 - \theta^0\|^2 + \frac{k^2}{4}\|\nabla(\theta^1 + \theta^0)\|^2 + Tk^2\sigma_l].$$

Dividing both sides of the above by k^2 and taking square roots we obtain (7.22) in view of (7.26). $\qquad\square$

Exercise 1. Using the technique of the proof of Proposition 7.1 show that an estimate of the form (7.22) holds for the solution of (7.14) for any $\beta \ge 0$, provided the stability condition (7.20) holds if $0 \le \beta < \frac{1}{4}$.

Exercise 2. The scheme (7.14) in the case $\beta = \frac{1}{12}$ is known as the *Störmer-Numerov method*. Show that this scheme is fourth-order accurate in time: Specifically prove that for $\beta = \frac{1}{12}$
$$\sum_{n=1}^{l} \|\omega^n\|^2 \le ck^7 \int_0^{t^{l+1}} \|\partial_t^8 u\|^2 ds,$$
and consequently that the last term in the right-hand side of (7.22) is of $O(k^4)$, provided (7.20) holds with $\beta = \frac{1}{12}$. (The Störmer-Numerov scheme is the only fourth-order accurate in k scheme of the family (7.14). For all other $\beta \ge 0$ the temporal truncation error is of $O(k^2)$.)

We present now some straightforward implications of Proposition 7.2.

Proposition 7.3. Let the hypotheses of Proposition 7.2 hold and assume in addition that for some constant C independent of h and k we have
$$\frac{1}{k}\|\theta^1 - \theta^0\| + \|\nabla(\theta^1 + \theta^0)\| \le C(k^2 + h^r). \tag{7.28}$$

Then, there is a constant $C = C(u, T)$ such that

(i) $\quad \|\frac{1}{k}(U^{n+1} - U^n) - u_t(t^{n+1/2})\| \le C(k^2 + h^r), \quad t^{n+1/2} = t^n + \frac{k}{2}, \; 0 \le n \le M - 1,$

(ii) $\quad \|\frac{1}{2}\nabla(U^{n+1} + U^n) - \nabla u(t^{n+1/2})\| \le C(k^2 + h^{r-1}), \quad 0 \le n \le M - 1,$

(iii) $\max_{0 \le n \le M} \|U^n - u^n\| \le C(k^2 + h^r).$

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Proof. (i). Let $0 \le n \le M-1$. Since

$$\frac{1}{k}(U^{n+1} - U^n) - u_t(t^{n+1/2}) = \frac{1}{k}(\theta^{n+1} - \theta^n) + \frac{1}{k}R_h(u^{n+1} - u^n) - u_t(t^{n+1/2})$$

$$= \frac{1}{k}(\theta^{n+1} - \theta^n) + R_h(\frac{u^{n+1} - u^n}{k}) - (\frac{u^{n+1} - u^n}{k}) + (\frac{u^{n+1} - u^n}{k} - u_t(t^{n+1/2}))$$

$$= \frac{1}{k}(\theta^{n+1} - \theta^n) + \frac{1}{k}\int_{t^n}^{t^{n+1}} (R_h u_t - u_t)ds + (\frac{1}{k}(u^{n+1} - u^n) - u_t(t^{n+1/2})),$$

we have

$$\|\frac{1}{k}(U^{n+1} - U^n) - u_t(t^{n+1/2})\| \le \frac{1}{k}\|\theta^{n+1} - \theta^n\| + Ch^r \max_{t^n \le s \le t^{n+1}} \|u_t(s)\|_r$$

$$+ Ck^2 \max_{t^n \le s \le t^{n+1}} \|u_{ttt}(s)\|,$$

and (i) follows from (7.22) and (7.28).

(ii). Let $0 \le n \le M-1$. Since

$$\frac{1}{2}\nabla(U^{n+1} + U^n) - \nabla u(t^{n+1/2}) = \frac{1}{2}\nabla(\theta^{n+1} + \theta^n) + \frac{1}{2}\nabla(R_h(u^{n+1} + u^n) - (u^{n+1} + u^n))$$

$$+ \nabla(\frac{1}{2}(u^{n+1} + u^n) - u(t^{n+1/2})),$$

we have

$$\|\frac{1}{2}\nabla(U^{n+1} + U^n) - \nabla u(t^{n+1/2})\| \le \frac{1}{2}\|\nabla(\theta^{n+1} + \theta^n)\| + ch^{r-1}(\|u^{n+1}\|_r + \|u^n\|_r)$$

$$+ Ck^2 \max_{t^n \le s \le t^{n+1}} \|\nabla u_{tt}(s)\|,$$

and (ii) follows from (7.22) and (7.28).

(iii). Since

$$U^l - u^l = \theta^l + \varrho^l, \quad 0 \le l \le M,$$

we have

$$\|U^l - u^l\| \le \|\theta^l\| + Ch^r\|u^l\|_r, \quad 0 \le l \le M. \tag{7.29}$$

But for $0 \le n \le M-1$, $\theta^{n+1} = \frac{\theta^{n+1} + \theta^n}{2} + \frac{k}{2}(\frac{\theta^{n+1} - \theta^n}{k})$. Hence by (7.22), (7.28) and Poincaré's inequality

$$\|\theta^{n+1}\| \le \frac{1}{2}\|\theta^{n+1} + \theta^n\| + \frac{k}{2}\frac{1}{k}\|\theta^{n+1} - \theta^n\|$$

$$\le C\|\nabla(\theta^{n+1} + \theta^n)\| + \frac{k}{2}\|\frac{\theta^{n+1} - \theta^n}{k}\| \le C(k^2 + h^r),$$

for $0 \leq n \leq M - 1$. In addition, since $\theta^0 = \frac{\theta^1 + \theta^0}{2} - \frac{k}{2}(\frac{\theta^1 - \theta^0}{k})$, we have by the Poincaré inequality and (7.28)

$$\|\theta^0\| \leq \frac{1}{2}\|\theta^1 + \theta^0\| + \frac{k}{2}\|\frac{\theta^1 - \theta^0}{k}\| \leq \frac{C}{2}\|\nabla(\theta^1 + \theta^0)\| + \frac{k}{2}\frac{\|\theta^1 - \theta^0\|}{k} \leq C(k^2 + h^r).$$

Therefore (iii) follows from (7.29) and these estimates.

$\square$

We turn now to the matter of choosing U^0 and U^1 in S_h so that (7.28) holds. We choose first

$$U^0 = R_h u^0. \tag{7.30}$$

This implies that $\theta^0 = 0$; hence, U^1 must be chosen so that

$$\frac{1}{k}\|\theta^1\| + \|\nabla\theta^1\| \leq C(k^2 + h^r). \tag{7.31}$$

A straightforward way of doing this is using Taylor expansions:

Let

$$u^*(x) = u^0(x) + ku_t^0(x) + \frac{k^2}{2}(\Delta u^0(x) + f(x, 0)), \quad x \in \overline{\Omega}. \tag{7.32}$$

Since by (7.1) $u_{tt} = \Delta u + f$, it is clear that $u^* \rceil_{\partial\Omega} = 0$ and that we have

$$u^* = u(k) + O(k^3), \quad \nabla u^* = \nabla u(k) + O(k^3).$$

We let

$$U^1 = R_h u^*. \tag{7.33}$$

Then by Poincaré's inequality

$$\|\theta^1\| = \|U^1 - R_h u(k)\| = \|R_h(u^* - u(k))\|$$
$$\leq C\|\nabla R_h(u^* - u(k))\| \leq C\|\nabla(u^* - u(k))\| \leq Ck^3,$$

and

$$\|\nabla\theta^1\| = \|\nabla R_h(u^* - u(k))\| \leq \|\nabla(u^* - u(k))\| \leq Ck^3.$$

Therefore $\frac{1}{k}\|\theta^1\| + \|\nabla\theta^1\| \leq Ck^2$ and (7.31) holds. It is then straightforward to check that the initial conditions (7.30) and (7.33) satisfy the hypotheses (7.17) and (7.18) for the L^2-stability of the scheme (7.14). Indeed, we have by Poincaré's inequality, that

$$\|U^1 - U^0\| = \|R_h(u^* - u^0)\| \leq C\|\nabla(u^* - u^0)\| \leq Ck,$$

and

$$\|\nabla(U^1 \pm U^0)\| = \|\nabla R_h(u^* \pm u_0)\| \leq C\|\nabla(u^* \pm u^0)\| \leq C.$$

Exercise 3. Consider the Störmer-Numerov method (see Exercise 2.). Take $U^0 = R_h u^0$ and $U^1 = R_h u^{**}$, where $u^{**} = u^0 + k u_t^0 + \frac{k^2}{2} u_{tt}(0) + \frac{k^3}{3!} \partial_t^3 u(0) + \frac{k^4}{4!} \partial_t^4 u(0)$. Prove that with these choices one has

$$\frac{1}{k}\|\theta^1 - \theta^0\| + \|\nabla(\theta^1 + \theta^0)\| \leq Ck^4, \tag{7.34}$$

so that by Exercises 1 and 2 and Proposition 7.3 (iii) one obtains for the Störmer-Numerov scheme the error estimate

$$\max_{0 \leq n \leq M} \|U^n - u^n\| \leq C(k^4 + h^r),$$

provided (7.17) holds for $\beta = \frac{1}{12}$. (Note that for the wave equation,

$$\partial_t^2 u(0) = \Delta u^0 + f(0), \quad \partial_t^3 u(0) = \Delta u_t^0 + f_t(0),$$

$$\partial_t^4 u(0) = \partial_t^2(\Delta u + f) \restriction_{t=0} = \Delta^2 u^0 + \Delta f(0) + f_{tt}(0),$$

so that, in principle, u^{**} can be computed by the data of (7.1).) Finally show that these choices in U^1 and U^0 also satisfy the estimates (7.17) and (7.18).

Remarks

a. The choice $U^1 = R_h u^*$ has the disadvantage that it needs the computation of Δu^0. Alternatively (see Dougalis and Serbin, Comput. Math. Appl. 7(1981), 261-279)), one may compute initial conditions for the scheme (7.14) for $\beta \neq \frac{1}{12}$ as follows: (For simplicity we consider only the homogeneous equation, $f = 0$).Take again $U^0 = R_h u^0$ and compute $U^1 \in S_h$ as solution of the problem

$$(U^1, \chi) + \beta k^2(\nabla U^1, \nabla \chi) = (U^0, \chi) + (\beta - \tfrac{1}{2})k^2(\nabla U^0, \nabla \chi) + k(u_t^0, \chi), \quad \forall \chi \in S_h.$$

It can be shown that for this choice of U^0 and U^1, (7.28) and (7.17)-(7.18) hold.

b. In the case of the Störmer-Numerov method ($\beta = \frac{1}{12}$) the choice $U^1 = R_h u^{**}$ (see Exercise 3 above) requires computing Δu^0, Δu_t^0, $\Delta^2 u^0$, $f_t(0)$, $f_{tt}(0)$, $\Delta f(0)$, which may be difficult or impossible in practice. A more efficient way (see Dougalis and Serbin, op. cit) of computing U^0, U^1, so that (7.34) and (7.17)-(7.18) hold, is the following.

(We take $f = 0$ for simplicity.) Define again $U^0 = R_h u^0$ and compute successively $U^{0,1}$, $U^{0,2}$, and U^1 in S_h by the equations

$$(U^{0,1}, \chi) + \frac{k^2}{12}(\nabla U^{0,1}, \nabla \chi) = 6(U^0, \chi) + \frac{k^2}{2}(\nabla U^0, \nabla \chi) + k(u_t^0, \chi), \quad \forall \chi \in S_h,$$

$$(U^{0,2}, \chi) + \frac{k^2}{12}(\nabla U^{0,2}, \nabla \chi) = (U^{0,1}, \chi), \quad \forall \chi \in S_h,$$

$$U^1 = U^{0,2} - 5U^0.$$

c. For higher-order accurate in time full discretizations of the second-order semidiscrete problem (7.4) one may use e.g. the so-called *cosine methods* (cf. e.g. Baker et al., Numer. Math. 35(1980), 127-142, RAIRO Anal. Numer. 13 (1979), 201-226), extended to problems with time-dependent coefficients by Bales et al., Math.Comp. 45(1985), 65-89, and to nonlinear problems by Bales and Dougalis 52(1989) 299-319, and Makridakis, Comput. Math. Appl. 19(1990), 19-34, or *linear multistep methods* (cf. e.g. Dougalis, Math. Comp. 33(1979), 563-584), etc.

Another class of fully discrete Galerkin methods for the wave equation is obtained by writing (7.1) as a *first-order system* in the temporal variable:

$$\begin{cases} q_t = \Delta u + f, & x \in \Omega, \ t \geq 0, \\ u_t = q, & x \in \Omega, \ t \geq 0, \\ u = 0, \ q = 0, & x \in \partial\Omega, \ t \geq 0, \\ u(x,0) = u^0(x), \ q(x,0) = u_t^0, & x \in \overline{\Omega}. \end{cases} \tag{7.35}$$

We may then consider the Galerkin semidiscretization of (7.35) and its temporal discretization by single-step or multistep schemes. For example, consider the *trapezoidal method* in which we seek U^n, Q^n in S_h for $0 \leq n \leq M$ satisfying

$$U^0 = Pu^0, \quad Q^0 = Pu_t^0,$$

and for $n = 0, 1, \ldots, M - 1$:

$$\begin{cases} (\frac{Q^{n+1}-Q^n}{k}, \chi) + \frac{1}{2}(\frac{U^{n+1}+U^n}{2}, \nabla \chi) = \frac{1}{2}(f^{n+1} + f^n, \chi), \quad \forall \chi \in S_h, \\ \frac{1}{k}(U^{n+1} - U^n) = \frac{1}{2}(Q^{n+1} + Q^n). \end{cases}$$

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(Here P is the L^2-projection onto S_h.) The scheme is easily solvable for Q^{n+1} by substituting U^{n+1} from the second equation into the first one. Taking $\chi = U^{n+1} - U^n$ in the first equation and the L^2-inner product of both sides of the second equation by $Q^{n+1} - Q^n$ we easily obtain the stability estimate

$$\|Q^n\|^2 + \frac{k}{2}\|\nabla U^n\|^2 = \|Q^0\|^2 + \frac{k}{2}\|\nabla U^0\|^2,$$

that may be viewed as a discrete analog of (7.2). Baker, SIAM J. Numer. Anal., 13(1976), 564-576, has analyzed the convergence of the scheme and shown that $\max_n \|U^n - u^n\| = O(k^2 + h^r)$. Higher-order temporal discretizations for the Galerkin semidiscretization of (7.35) have been studied by Baker and Bramble, RAIRO Anal. Numer. 13(1979), 75-100, and extended by Bales (Math. Comp. 43(1984), 383-414, SIAM J. Numer. Anal., 23(1986), 27-43, Comput. Math. Appl. 12A(1986), 581-604) to the case of equations with time-dependent coefficients and nonlinear terms.

Appendix A

Proofs of results about function spaces

In this Appendix we prove some results on function spaces needed in the notes. Unless stated otherwise, Ω will be an open (possibly unbounded) set in $\mathbb{R}^d$.

Proposition A 2.1. The step functions are dense in $L^p(\Omega)$, $1 \leq p < \infty$.

Proof. By a 'step function' $\phi \in L^p(\Omega)$ we mean a function such that $\phi(x) = \phi_j$ in Ω_j, where ϕ_j is a nonzero constant if $1 \leq j \leq N$, and $\phi_{N+1} = 0$. Here the Ω_j are measurable subsets of Ω such that $\Omega = \cup_{j=1}^{N+1} \Omega_j$, $\Omega_j \cap \Omega_k = \varnothing$ for $j \neq k$, and we assume that the Ω_j, $1 \leq j \leq N$, are nonempty. It follows that $\mu(\Omega_j) < \infty$ for $1 \leq j \leq N$, since $\int_\Omega |\phi|^p \geq \int_{\Omega_j} |\phi_j|^p = |\phi_j|^p \mu(\Omega_j)$. Hence only Ω_{N+1} (where $\phi = \phi_{N+1} = 0$) may be unbounded.

In what follows we let, for $\rho \geq 0$, $K_\rho = \{x \in \mathbb{R}^d : |x| \leq \rho\}$ (where, for $x \in \mathbb{R}^d$, $|x| = \left(\sum_{i=1}^d x_i^2\right)^{1/2}$). Given $f \in L^p(\Omega)$, $1 \leq p < \infty$, we let $S_r = \{x \in \Omega : |f(x)| \geq r\}$ for $r \geq 0$. We have $\Omega = \omega_1 \cup \omega_2$, where $\omega_1 = (\Omega - K_\rho) \cup S_r$, $\omega_2 = K_\rho \cap (\Omega - S_r)$. Clearly $\omega_1 \cap \omega_2 = \varnothing$. ($K_\rho$ is introduced since Ω may be unbounded, while S_r is introduced since $|f|$ may be equal to $+\infty$ on a set of measure zero in Ω. Thus $\mu(S_r) \to 0$ as $r \to \infty$. If Ω is bounded or f is e.g. continuous and bounded on Ω the proof may be simplified.)

We first prove that given $\varepsilon > 0$, there exist $\rho(\varepsilon)$ and $r(\varepsilon)$ such that for $\rho \geq \rho(\varepsilon)$, $r \geq r(\varepsilon)$, we have

$$\int_{\omega_1} |f|^p \leq \frac{\varepsilon^p}{2}. \tag{A.1}$$

To see this, note that

$$\int_{\omega_1} |f|^p \leq \int_{\Omega - K_\rho} |f|^p + \int_{S_r} |f|^p. \tag{A.2}$$

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We prove first that $\int_{S_r} |f|^p \to 0$ as $r \to \infty$. This is equivalent to showing that $\lim_{r\to\infty} \int_{\Omega-S_r} |f|^p \equiv \lim_{r\to\infty} \int_\Omega \mathcal{X}_{\Omega-S_r}|f|^p = \int_\Omega |f|^p$, where $\mathcal{X}_B$ will denote in general the characteristic function of a set B in $\mathbb{R}^d$. To show the latter fact, take for example an increasing sequence of positive numbers r_k such that $r_k \to \infty$. Then $\mu(S_{r_k}) = \mu(\cap_{j=1}^k S_{r_j})$, i.e. $\lim_{k\to\infty} \mu(S_{r_k}) = \mu(\cap_{j=1}^\infty S_{r_j}) = 0$ since $f \in L^p(\Omega)$. It follows that as $k \to \infty$, $\mathcal{X}_{\Omega-S_{r_k}}(x)|f(x)|^p \to |f(x)|^p$, a.e. in Ω, and by Lebesgue's Dominated Convergence Theorem (LDCT), $\lim_{k\to\infty} \int_\Omega \mathcal{X}_{\Omega-S_{r_k}}|f|^p = \int_\Omega |f|^p$, as claimed. We conclude that given $\varepsilon > 0$ there exists $r(\varepsilon) > 0$ such that for $r \geq r(\varepsilon)$

$$\int_{S_r} |f|^p \leq \frac{\varepsilon^p}{4}. \tag{A.3}$$

We also have $\lim_{\rho\to\infty} \int_{\Omega-K_\rho} |f|^p = 0$. (To see this note that it is equivalent to showing that $\lim_{\rho\to\infty} \int_{\Omega\cap K_\rho} |f|^p = \int_\Omega |f|^p$, i.e. that $\lim_{\rho\to\infty} \int_\Omega \mathcal{X}_{\Omega\cap K_\rho}|f|^p = \int_\Omega |f|^p$. The latter follows from the fact that $\lim_{\rho\to\infty} \mathcal{X}_{\Omega\cap K_\rho}(x) |f(x)|^p = |f(x)|^p$, $\forall x \in \Omega$, and LDCT.) We conclude that given $\varepsilon > 0$ there exists $\rho(\varepsilon) > 0$ such that for $\rho \geq \rho(\varepsilon)$

$$\int_{\Omega-K_\rho} |f|^p \leq \frac{\varepsilon^p}{4}, \tag{A.4}$$

and therefore, by (A.2), (A.3), and (A.4), that for $\rho \geq \rho(\varepsilon)$, $r \geq r(\varepsilon)$, (A.1) holds. Given now $\rho, r > 0$ we may approximate f on the bounded set $\omega_2 = K_\rho \cap (\Omega - S_r)$, where $|f(x)| < r$, by a sequence of step functions f_m that converge uniformly to f and satisfy $|f_m| \leq |f|$. This construction is classical and will be repeated here.

Suppose first that $f \geq 0$ on ω_2. For $x \in \omega_2$ define the step function

$$f_1(x) = \begin{cases} 0, & \text{if } x : 0 \leq f(x) < \frac{r}{2}, \\ \frac{r}{2}, & \text{if } x : \frac{r}{2} \leq f(x) < r. \end{cases}$$

Then, for $x \in \omega_2$, $0 \leq f_1 \leq f$ and $|f - f_1| \leq \frac{r}{2}$. Next define the step function f_2 for $x \in \omega_2$ as

$$f_2(x) = \begin{cases} 0, & \text{if } x : 0 \leq f(x) < \frac{r}{4}, \\ \frac{r}{4}, & \text{if } x : \frac{r}{4} \leq f(x) < \frac{r}{2}, \\ \frac{r}{2}, & \text{if } x : \frac{r}{2} \leq f(x) < \frac{3r}{4}, \\ \frac{3r}{4}, & \text{if } x : \frac{3r}{4} \leq f(x) < r. \end{cases}$$

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It follows that $0 \leq f_1 \leq f_2 \leq f$ and $|f - f_2| \leq \frac{r}{4}$ for $x \in \omega_2$. In general, for a positive integer m, divide $[0, r]$ into 2^m subintervals of length $\frac{r}{2^m}$ and define for $x \in \omega_2$ the step function $f_m(x)$ as

$$f_m(x) = \begin{cases} 0, & \text{if } x : 0 \leq f(x) < \frac{r}{2^m}, \\ \frac{r}{2^m}, & \text{if } x : \frac{r}{2^m} \leq f(x) < \frac{2r}{2^m}, \\ \vdots \\ \frac{(2^m-1)r}{2^m}, & \text{if } x : \frac{(2^m-1)r}{2^m} \leq f(x) < r. \end{cases}$$

It follows that $0 \leq f_1 \leq f_2 \leq \ldots \leq f_m \leq f$ and $|f_m - f| \leq \frac{r}{2^m}$, $x \in \omega_2$.

In general, for a real-valued function f on ω_2 we write

$$f(x) = f^{(1)}(x) - f^{(2)}(x), \quad x \in \omega_2,$$

where $f^{(1)} = \max(0, f)$ and $f^{(2)} = \max(0, -f)$. Obviously $f^{(i)} \geq 0$, and $f^{(1)} \cdot f^{(2)} = 0$. Therefore $|f| = \sqrt{(f^{(1)})^2 + (f^{(2)})^2}$, and since $|f(x)| < r$ on ω_2 we have $0 \leq f^{(i)}(x) \leq r$. Construct as above, for $i = 1, 2$, increasing sequences of step functions $\phi_m^{(i)}$ satisfying $0 \leq \phi_m^{(i)} \leq f^{(i)}(x)$ and $|\phi_m^{(i)}(x) - f^{(i)}(x)| \leq \frac{r}{2^m}$, $x \in \omega_2$. Define $f_m = \phi_m^{(1)} - \phi_m^{(2)}$. Then f_m is a step function, and, since $\phi_m^{(1)} \phi_m^{(2)} = 0$, it is straightforward to verify that $|f_m(x)| \leq |f(x)|$ and $|f_m(x) - f(x)| \leq \frac{r}{2^m \sqrt{2}}$ for $x \in \omega_2$. Hence, given $r, \rho > 0$, the construction of a sequence of step functions that converges uniformly to f on ω_2 is now complete. Obviously $f_m \in L^p(\omega_2)$ and

$$\int_{\omega_2} |f - f_m|^p \to 0, \quad m \to \infty. \tag{A.5}$$

In order to conclude the proof of the proposition, write $f_m = f_m^{(\rho,r)}$ to emphasize that f_m depends on ρ, r, i.e. on ω_2. Extend $f_m^{(\rho,r)}$ by zero outside ω_2 to all of Ω. Then $f_m^{(\rho,r)}$ are step functions in $L^p(\Omega)$ in the sense mentioned in the beginning of the proof. It follows that

$$\|f - f_m^{(\rho,r)}\|_{L^p(\Omega)}^p = \int_{\omega_1} |f|^p + \int_{\omega_2} |f - f_m|^p. \tag{A.6}$$

Given $\varepsilon > 0$ pick $r \geq r(\varepsilon)$, $\rho \geq \rho(\varepsilon)$ so that (A.6) holds. For these values of r, ρ pick $m \geq m(\rho, r, \varepsilon)$ such that, in view of (A.5),

$$\int_{\omega_2} |f - f_m|^p \leq \frac{\varepsilon^p}{2}. \tag{A.7}$$

Therefore by (A.6), (A.1) and (A.7) we get

$$\|f - f_m^{(\rho,r)}\|_{L^p(\Omega)} \leq \varepsilon,$$

which concludes the proof. □

Theorem A 2.2. The space $C_c(\Omega)$ is dense in $L^p(\Omega)$, $1 \leq p < \infty$.

Proof. In view of Proposition A 2.1 it suffices to prove that any step function in $L^p(\Omega)$ may be approximated arbitrarily closely in L^p by some function in $C_c(\Omega)$. With notation introduced in the beginning of the proof of Proposition A 2.1. let $\phi = \sum_{j=1}^{N+1} \phi_j \mathcal{X}_{\Omega_j}$ be such a step function. (Recall that $\mu(\Omega_j) < \infty$, $1 \leq j \leq N$, and $\phi_{N+1} = 0$) It is easily seen then that it suffices to prove our result for a characteristic function $\mathcal{X}_B$ of a measurable subset B of Ω with $\mu(B) < \infty$. By a basic property (regularity) of the Lebesgue measure, for each $\varepsilon > 0$ there exists an open set $O_\varepsilon \subset \Omega$ of finite measure such that $B \subset O_\varepsilon$ and $\mu(O_\varepsilon - B) < \varepsilon$. Since $\int_\Omega |\mathcal{X}_{O_\varepsilon} - \mathcal{X}_B|^p = \int_\Omega |\mathcal{X}_{O_\varepsilon} - \mathcal{X}_B| = \int_\Omega \mathcal{X}_{O_\varepsilon - B} = \mu(O_\varepsilon - B) < \varepsilon$, it follows that it suffices to prove our result for characteristic functions of open sets $O \subset \Omega$ of finite measure. Any such set may be written as $O = \cup_{j=1}^\infty Q_j$, where Q_j are disjoint open rectangles in Ω. Since $\mu(O) = \sum_{j=1}^\infty \mu(Q_j) < \infty$, $\forall \varepsilon > 0$ $\exists M = M(\varepsilon)$ such that $0 \leq \mu(O) - \sum_{j=1}^M \mu(Q_j) \leq \varepsilon$. Then $\int_\Omega |\mathcal{X}_O - \sum_{j=1}^M \mathcal{X}_{Q_j}|^p = \int_\Omega |\mathcal{X}_O - \sum_{j=1}^M \mathcal{X}_{Q_j}| = \int_\Omega \mathcal{X}_{O - \cup_{j=1}^M Q_j} = \mu(O) - \sum_{j=1}^M \mu(Q_j) \leq \varepsilon$. We conclude that it suffices to prove our result for characteristic functions of rectangles, i.e. to show that given a rectangle Q, $\overline{Q} \subset \Omega$, and $\varepsilon > 0$, there exists $f \in C_c(Q)$ such that

$$\|f - \mathcal{X}_Q\|_{L^p(\Omega)} \leq \varepsilon. \tag{A.8}$$

Consider such a rectangle Q. Since Q is bounded, the distance $\mathrm{d}(\overline{Q}, \partial\Omega) = \inf_{x \in \overline{Q}, y \in \partial\Omega} |x - y|$ is a positive number. Let $k_0 > 1$ be an integer such that $\mathrm{d}(\overline{Q}, \partial\Omega) > \frac{1}{k_0 - 1}$. For integer $k \geq k_0$, i.e. for which $\frac{1}{k} < \frac{1}{k_0 - 1}$, consider, for $x \in \Omega$ the function f_k defined by

$$f_k(x) = \begin{cases} 1, & \text{if } x \in \overline{Q} \text{ (i.e. when } \mathrm{d}(x, Q) = 0), \\ 1 - k\,\mathrm{d}(x, Q), & \text{if } 0 < \mathrm{d}(x, Q) < \frac{1}{k}, \\ 0, & \text{if } \mathrm{d}(x, Q) \geq \frac{1}{k}, \end{cases}$$

where $\mathrm{d}(x, Q) = \inf_{y \in Q} |x - y|$ is the distance of x from Q. It is clear that $0 \leq f_k(x) \leq 1$, $\operatorname{supp} f_k = \overline{Q} \cup \overline{\{y \in \Omega : 0 < \mathrm{d}(y, Q) \leq \frac{1}{k}\}}$, i.e. that $\operatorname{supp} f_k$ is a compact set in Ω, and that $\overline{Q} \subset \operatorname{supp} f_{k+1} \subset \operatorname{supp} f_k$ for each k. Moreover $f_k(x)$ is a continuous function of x. This follows from the fact that $\mathrm{d}(x, Q)$ is a continuous function of $x \in \Omega$. (This is true for any bounded set Q in a metric space X, ρ: Fix $x', x \in X$ and take $y \in Q$. Then

$$\rho(x', y) \leq \rho(x', x) + \rho(x, y).$$

Taking the inf in the left-hand side with respect to y, we have that $\mathrm{d}(x', Q) = \inf_{y \in Q} \rho(x', y) \leq \rho(x', x) + \rho(x, y)$. Hence $\mathrm{d}(x', Q) \leq \rho(x', x) + \mathrm{d}(x, Q)$. Reversing the role of x and x' we see that $\mathrm{d}(x, Q) \leq \rho(x', x) + \mathrm{d}(x', Q)$. We conclude that $|\mathrm{d}(x', Q) - \mathrm{d}(x, Q)| \leq \rho(x', x)$, i.e. that $\mathrm{d}(x, Q)$ is a continuous function of x.)

Returning to our proof, we conclude that $f_k \in C_c(\Omega)$ for $k \geq k_0$. Therefore, by the properties of f_k, $\int_\Omega |X_Q - f_k|^p = \int_{\operatorname{supp} f_k - \overline{Q}} |f_k|^p \leq \mu(\operatorname{supp} f_k - \overline{Q}) \to 0$ as $k \to \infty$. This concludes the proof of the Theorem. $\qquad \square$

Proposition A 2.3. If $f \in L^1_{\mathrm{loc}}(\Omega)$ is such that

$$\int_\Omega fu = 0, \quad \forall u \in C_c(\Omega), \tag{A.9}$$

then $f = 0$ a.e. in Ω.

Proof. We first prove the result under the additional hypotheses that $f \in L^1(\Omega)$ and $\mu(\Omega) < \infty$. Since by Theorem A 2.2 $C_c(\Omega)$ is dense in $L^1(\Omega)$, given $\varepsilon > 0$ there exists $f_1 \in C_c(\Omega)$ such that $\|f - f_1\|_{L^1(\Omega)} \leq \varepsilon$. Let $u \in C_c(\Omega)$. Then, using (A.9), we have

$$\left| \int_\Omega f_1 u \right| = \left| \int_\Omega (f_1 - f)u \right| \leq \|f - f_1\|_{L^1(\Omega)} \|u\|_{L^\infty(\Omega)} \leq \varepsilon \|u\|_{L^\infty(\Omega)}. \tag{A.10}$$

Now let $K_1 = \{x \in \Omega : f_1(x) \geq \varepsilon\}$, $K_2 = \{x \in \Omega : f_1(x) \leq -\varepsilon\}$. The sets K_1 and K_2 are compact, disjoint subsets of Ω. By an extension of Urysohn's theorem, there exists a function $u_0 \in C_c(\Omega)$ such that $u_0(x) = 1$, if $x \in K_1$, $u_0(x) = -1$, if $x \in K_2$, and $|u_0(x)| \leq 1$, $x \in \Omega$. Let $K = K_1 \cup K_2$. Then

$$\int_K |f_1| = \int_{K_1} |f_1| + \int_{K_2} |f_1| = \int_{K_1} f_1 + \int_{K_2} (-f_1) = \int_{K_1} f_1 u_0 + \int_{K_2} f_1 u_0 =$$

$$\int_K f_1 u_0 = \int_\Omega f_1 u_0 - \int_{\Omega - K} f_1 u_0 \leq \left| \int_\Omega f_1 u_0 \right| + \left| \int_{\Omega - K} f_1 u_0 \right| \leq$$

$$\text{(using (A.10) and the fact that } |u_0| \leq 1 \text{ on } \Omega) \leq \varepsilon + \int_{\Omega - K} |f_1|.$$

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Hence,

$$\int_\Omega |f_1| = \int_K |f_1| + \int_{\Omega-K} |f_1| \le \varepsilon + 2\int_{\Omega-K} |f_1| \le \varepsilon + 2\,\varepsilon\,\mu(\Omega - K), \qquad \text{(A.11)}$$

since $|f_1| \le \varepsilon$ on $\Omega - K$. We conclude by (A.10) with $u = u_0$ and (A.11) that

$$\|f\|_{L^1(\Omega)} \le \|f - f_1\|_{L^1(\Omega)} + \|f_1\|_{L^1(\Omega)} \le 2\,\varepsilon + 2\,\varepsilon\,\mu(\Omega - K),$$

and since $\varepsilon > 0$ was arbitrary, that $\|f\|_{L^1(\Omega)} = 0$, i.e. $f = 0$ a.e. in Ω.

We now turn to the general case wherein $f \in L^1_{\text{loc}}(\Omega)$ and Ω is not necessarily bounded. For $n = 1, 2, 3, \ldots$ let $\Omega_n = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \frac{1}{n}, \ |x| < n\}$. Obviously $\Omega = \cup_{n=1}^\infty \Omega_n$, $\overline{\Omega_n}$ compact and $\Omega_n \subset \Omega_{n+1}$. Hence $\Omega = \lim_{n\to\infty} \Omega_n$. We apply the first part of the proposition to Ω_n Indeed, if $f \in L^1_{\text{loc}}(\Omega)$ then $f \in L^1(\Omega_n)$ for some n. Also, from the hypothesis $\int_\Omega fu = 0$, $\forall u \in C_c(\Omega)$ it follows that $\int_{\Omega_n} fu = 0$, $\forall u \in C_c(\Omega_n)$. Therefore, by the first part of the proposition, $f = 0$ a.e. in Ω. (Indeed, if $f \ne 0$ everywhere on a subset $M \subset \Omega$ with $\mu(M) \ne 0$, take n large enough so that $M \cap \Omega_n \ne \varnothing$ whence $f = 0$ on $M \cap \Omega_n$, a contradiction. $\qquad\square$

Lemma A 2.5. (i) Let $f \in C(\mathbb{R}^d)$. Then $\rho_n * f \in C^\infty(\mathbb{R}^d)$ and $\rho_n * f \to f$, $n \to \infty$, uniformly on every compact set $K \subset \mathbb{R}^d$, i.e.

$$\sup_{x \in K} |f(x) - (\rho_n * f)(x)| \to 0\,, \quad n \to \infty$$

$\forall\, K$ compact $\subset \mathbb{R}^d$.

(ii) Let $f \in C_c(\Omega)$. Extend f by zero on the whole of $\mathbb{R}^d$. Then, for sufficiently large n, $\rho_n * f \in C_c^\infty(\Omega)$ and

$$\sup_{x \in \Omega} |f(x) - (\rho_n * f)(x)| \to 0, \quad n \to \infty.$$

(iii) Let $f \in L^p(\Omega)$, $1 \le p < \infty$. Extend f by zero to the whole of $\mathbb{R}^d$. Then $\rho_n * f \in C^\infty(\mathbb{R}^d)$. Moreover $\rho_n * f \in L^p(\Omega)$, $\|\rho_n * f\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)}$, and $\|f - \rho_n * f\|_{L^p(\Omega)} \to 0$, $n \to \infty$.

Proof. First note in the definition of the convolution of two functions f, g on $\mathbb{R}^d$ if for fixed $x \in \mathbb{R}^d$ the integral $\int_{\mathbb{R}^d} f(x - y)g(y)\,dy$ exists, then the change of variables $z = x - y$ gives $\int_{\mathbb{R}^d} f(x - y)g(y)\,dy = \int_{\mathbb{R}^d} g(x - z)f(z)\,dz$, i.e. that $f * g = g * f$.

(i) Let $f \in C(\mathbb{R}^d)$. Then it is clear that for each $x \in \mathbb{R}^d$ the integral $\int_{\mathbb{R}^d} \rho_n(x-y)f(y)\,dy$ exists, since $\operatorname{supp}\rho_n = B(0, \frac{1}{n})$ implies that $\rho_n(x-y) = 0$ for $|x-y| > \frac{1}{n}$, so that $\int_{\mathbb{R}^d} \rho_n(x-y)f(y)\,dy = \int_{y:|x-y|\le\frac{1}{n}} \rho_n(x-y)f(y)\,dy$.

In the sequel we set $f_n \equiv \rho_n * f$. First we show that $f_n \in C(\mathbb{R}^d)$. To this end, fix $x = (x_i) \in \mathbb{R}^d$ and for $0 < \delta < 1$ define $x' = (x_i') \in \mathbb{R}^d$ by $x_i' = x_i + \frac{\delta}{\sqrt{d}}$, so that $|x - x'| = \delta$. Then

$$f_n(x') - f_n(x) = \int_{\mathbb{R}^d} [\rho_n(x'-y) - \rho_n(x-y)]f(y)\,dy = \int_{y:|x-y|\le\frac{1}{n}+1} [\rho_n(x'-y) - \rho_n(x-y)]f(y)\,dy.$$

Therefore, letting $B(\alpha, r) = \{x \in \mathbb{R}^d : |x - \alpha| \le r\}$,

$$|f_n(x') - f_n(x)| \le \max_{y \in B(x,2)} |f(y)| \int_{y \in B(x,\frac{1}{n}+1)} |\rho_n(x'-y) - \rho_n(x-y)|\,dy. \qquad \text{(A.12)}$$

By the mean-value theorem in $\mathbb{R}^d$ it follows that $\rho_n(x'-y) - \rho_n(x-y) = \nabla\rho_n(\xi)(x'-x)$, where $\xi = \theta(x'-y) + (1-\theta)(x-y)$, for some $\theta \in [0,1]$, i.e. $|\rho_n(x'-y) - \rho_n(x-y)| \le \sup_{z\in\mathbb{R}^d} |\nabla\rho_n(z)|\,\delta$. Hence, by (A.12) $|f_n(x') - f_n(x)| \le C\,\delta$, for some positive constant $C = C(f, n, x)$, proving that $f_n \in C(\mathbb{R}^d)$.

We next show that $\frac{\partial f_n(x)}{\partial x_1}$ exists, $\frac{\partial f_n}{\partial x_1} \in C(\mathbb{R}^d)$, and $\frac{\partial f_n}{\partial x_1} = \int_{\mathbb{R}^d} \frac{\partial \rho_n}{\partial x_1}(x-y)f(y)\,dy$. Fix $x \in \mathbb{R}^d$, let $0 < \delta < 1$, and define $x' \in \mathbb{R}^d$ as $x' = (x_1+\delta, x_2, \ldots, x_d)$ so that $|x'-x| = \delta$. Then

$$\frac{1}{\delta}(f_n(x') - f_n(x)) = \int_{y \in B(x,\frac{1}{n}+1)} \frac{1}{\delta}[\rho_n(x'-y) - \rho_n(x-y)]\,f(y)\,dy,$$

and since by the mean-value theorem in the direction x_1

$$\frac{1}{\delta}|\rho_n(x'-y) - \rho_n(x-y)| \le \frac{\delta}{\delta} \max_{z\in\mathbb{R}^d} \left|\frac{\partial\rho_n}{\partial x_1}(z)\right|,$$

it follows for $y \in B(x, \frac{1}{n}+1)$ that $\left|\frac{\rho_n(x'-y)-\rho_n(x-y)}{\delta} f(y)\right| \le C$ for some constant $C = C(f, n, x)$; by LDCT, $\lim_{\delta\to 0} \frac{1}{\delta}(f_n(x') - f_n(x))$ exists and is equal to

$$\int_{y \in B(x,\frac{1}{n}+1)} \lim_{\delta\to 0} \frac{1}{\delta}(\rho_n(x'-y) - \rho_n(x-y))f(y)\,dy.$$

Therefore $\frac{\partial f_n}{\partial x_1}(x)$ exists and $\frac{\partial f_n}{\partial x_1}(x) = \int_{\mathbb{R}^d} \frac{\partial\rho_n}{\partial x_1}(x-y)f(y)\,dy$. As before, we conclude that $\frac{\partial f_n}{\partial x_1} \in C(\mathbb{R}^d)$ since $\operatorname{supp}\left(\frac{\partial\rho_n}{\partial x_1}\right) \subset B\left(0, \frac{1}{n}\right)$.

If now $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a d-vector of nonnegative integers (a so-called *multi-index*) and we let $D^\alpha = \frac{\partial^{\alpha_1+\ldots+\alpha_d}}{\partial x_1^{\alpha_1}\ldots\partial x_d^{\alpha_d}}$ denote the partial differential operator on $\mathbb{R}^d$ *of order* $(\alpha_1, \ldots, \alpha_d)$, we may easily extend the argument above, since $\rho_n \in C_0^\infty(\mathbb{R}^d)$ and

$\operatorname{supp}(\mathrm{D}^\alpha \rho_n) \subset B(0, \frac{1}{n})$, and prove that $\mathrm{D}^\alpha f_n \in C(\mathbb{R}^d)$ and $\mathrm{D}^\alpha f_n(x) = \int_{\mathbb{R}^d} \mathrm{D}_x^\alpha \rho_n(x - y) f(x)\, \mathrm{d}y$. Consequently $f_n \in C^\infty(\mathbb{R}^d)$. In order to show now the remaining assertion of (i), note that for $x \in \mathbb{R}^d$, the properties of f_n imply that

$$|f_n(x) - f(x)| = \left| \int_{\mathbb{R}^d} [f(x - y) - f(x)]\, \rho_n(y)\, \mathrm{d}y \right| = \left| \int_{y \in B(0, \frac{1}{n})} [f(x - y) - f(x)]\, \rho_n(y)\, \mathrm{d}y \right|$$

$$\leq \sup_{y \in B(0, \frac{1}{n})} |f(x - y) - f(x)|. \quad (A.13)$$

Let $K \subset \mathbb{R}^d$ be compact. Then $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $|f(x - y) - f(x)| \leq \varepsilon$ for all $x \in K$ and $y : |y| \leq \delta$. Using (A.13) for $n > \frac{1}{\delta}$ we conclude that for such n, $|f_n(x) - f(x)| \leq \varepsilon$, $\forall x \in K$, i.e. that $\sup_{x \in K} |f_n(x) - f(x)| \to 0$, $n \to \infty$.

(ii) Let $K = \operatorname{supp} f$. Then K is a compact subset of Ω. For $n = 1, 2, 3, \ldots$ define $K_n = \{x \in \mathbb{R}^d : \exists y \in K, |x - y| \leq \frac{1}{n}\}$. (Thus K_n is K increased by a band of width $\frac{1}{n}$ around K.) Since Ω is open, the distance of $\operatorname{supp} f$ from $\partial \Omega$ is positive; therefore for some integer n_0, K_{n_0} is a compact subset of Ω. Obviously, $K_n \subset K_{n_0}$ for $n \geq n_0$. Note now that $\operatorname{supp} f_n \subset K_n$. (Indeed, since $f_n(x) = \int_{\mathbb{R}^d} \rho_n(x - y)\, f(y)\, \mathrm{d}y$, if $x \notin K_n$, then $\forall y \in K$ it holds that $|x - y| > \frac{1}{n}$, i.e. $\rho_n(x - y) = 0$, implying $f_n(x) = 0$.) We conclude that for $n \geq n_0$, $\operatorname{supp} f_n \subset K_{n_0}$, and by (i) $\sup_{x \in \Omega} |f(x) - f_n(x)| = \sup_{x \in K_{n_0}} |f(x) - f_n(x)| \to 0$, as $n \to \infty$.

(iii) Let $f \in L^p(\Omega)$, $1 \leq p < \infty$, and extend f by zero to the whole of $\mathbb{R}^d$. Fix $x \in \mathbb{R}^d$. Then $f_n(x)$, i.e. the integral $\int_{\mathbb{R}^d} \rho_n(x - y)\, f(y)\, \mathrm{d}y$, exists, since $\int_{\mathbb{R}^d} |\rho_n(x - y)\, f(y)|\, \mathrm{d}y = \int_{B(x, \frac{1}{n})} \rho_n(x - y)\, |f(y)|\, \mathrm{d}y \leq \max_{z \in B(0, \frac{1}{n})} \rho_n(z) \int_{B(x, \frac{1}{n})} |f(y)|\, \mathrm{d}y = C_n \int_{B(x, \frac{1}{n})} |f(y)|\, \mathrm{d}y$, where C_n is a positive constant, and $\int_{B(x, \frac{1}{n})} |f(y)|\, \mathrm{d}y$ obviously exists if $p = 1$, while if $1 < p < \infty$ by the Hölder inequality we have for $\frac{1}{q} + \frac{1}{p} = 1$

$$\int_{B(x, \frac{1}{n})} |f(y)|\, \mathrm{d}y \leq \left(\int_{B(x, \frac{1}{n})} 1^q\, \mathrm{d}t \right)^{\frac{1}{q}} \left(\int_{B(x, \frac{1}{n})} |f(y)|^p\, \mathrm{d}y \right)^{\frac{1}{p}} \leq C\, \|f\|_{L^p(\Omega)},$$

for some positive constant $C = C(n, q, x)$.

We may also check that $f_n \in C(\mathbb{R}^d)$. To this end fix $x \in \mathbb{R}^d$ and define $x' \in \mathbb{R}^d$ by $x_i' = x_i + \frac{\delta}{\sqrt{d}}$ for $0 \leq \delta \leq 1$ so that $|x - x'| = \delta$. Then

$$|f_n(x') - f_n(x)| \leq \int_{B(x, \frac{1}{n})} |\rho_n(x' - y) - \rho_n(x - y)|\, |f_n(y)|\, \mathrm{d}y \leq$$

$$\max_{y \in B(x, \frac{1}{n})} |\rho_n(x' - y) - \rho_n(x - y)| \int_{B(x, \frac{1}{n})} |f(y)|\, \mathrm{d}y. \quad (A.14)$$

The integral in the right-hand side of this inequality may be bounded as above by $\|f\|_{L^1(\Omega)}$ if $p = 1$ and by $C(n, q, x) \|f\|_{L^p(\Omega)}$ if $1 < p < \infty$. In addition, since $\rho_n(x'-y) - \rho_n(x-y) = \nabla\rho_n(\xi)(x'-x)$ by the mean-value theorem, we have $|\rho_n(x'-y) - \rho_n(x-y)| \leq \max_{z \in \mathbb{R}^d} |\nabla\rho_n(z)| \delta$; the continuity of f_n at x follows then from (A.14).

That in fact $f_n \in C^\infty(\mathbb{R}^d)$ is straightforward to prove. Fixing $x \in \mathbb{R}^d$ and taking e.g. x' such that $x'_1 = x_1 + \delta$, $x'_i = x_i$, $2 \leq i \leq d$, for $0 < \delta < 1$, we have

$$\frac{1}{\delta}(f_n(x') - f_n(x)) = \int_{B(x, \frac{1}{n})} \frac{1}{\delta}[\rho_n(x'-y) - \rho_n(x-y)] \, f(y) \, \mathrm{d}y.$$

Arguing as above to bound $\int_{B(x, \frac{1}{n}+1)} |f(y)| \, \mathrm{d}y$ and using the mean-value theorem in the direction x_1 to bound $\frac{1}{\delta}|\rho_n(x'-y) - \rho_n(x-y)|$ for $y \in B(x, \frac{1}{n}+1)$ we may apply the LDCT to take the limit as $\delta \to 0$, and conclude that $\frac{\partial f_n}{\partial x_1}(x)$ exists and $\frac{\partial f_n}{\partial x_1}(x) = \int_{\mathbb{R}^d} \frac{\partial \rho_n}{\partial x_1}(x-y) \, f(y) \, \mathrm{d}y$ from which we infer that $\frac{\partial f_n}{\partial x_1} \in C(\mathbb{R}^d)$. (See also the proof of (i).) The fact that $\mathrm{D}^\alpha f_n(x) = \int_{\mathbb{R}^d} \mathrm{D}^\alpha_x \rho_n(x-y) \, f(y) \, \mathrm{d}y$ and, consequently, that $\mathrm{D}^\alpha f_n \in C(\mathbb{R}^d)$ for any multiindex α follows easily. Therefore $f_n \in C^\infty(\mathbb{R}^d)$.

We now prove that for all n, $f_n \in L^p(\Omega)$ and $\|f_n\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}$. For $x \in \mathbb{R}^d$, $1 < p < \infty$, and $\frac{1}{q} + \frac{1}{p} = 1$, we have by Hölder's inequality and the properties of ρ_n that

$$|f_n(x)|^p = \left| \int_{\mathbb{R}^d} f(x-y) \, \rho_n(y) \, \mathrm{d}y \right|^p \leq \left(\int_{\mathbb{R}^d} |f(x-y)| \, \rho_n^{\frac{1}{p}}(y) \, \rho_n^{\frac{1}{q}}(y) \, \mathrm{d}y \right)^p$$

$$\leq \left(\int_{\mathbb{R}^d} |f(x-y)|^p \rho_n(y) \, \mathrm{d}y \right) \left(\int_{\mathbb{R}^d} \rho_n \right)^{\frac{p}{q}} = \int_{\mathbb{R}^d} |f(x-y)|^p \rho_n(y) \, \mathrm{d}y.$$

Therefore, by Fubini's theorem, since $f = 0$ outside Ω,

$$\|f_n\|^p_{L^p(\Omega)} \leq \|f_n\|^p_{L^p(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)|^p \, \rho_n(y) \, \mathrm{d}y \, \mathrm{d}x =$$

$$\int_{\mathbb{R}^d} \rho_n(y) \left(\int_{\mathbb{R}^d} |f(x-y)|^p \, \mathrm{d}x \right) \mathrm{d}y = \left(\int_{\mathbb{R}^d} \rho_n(y) \, \mathrm{d}y \right) \|f\|^p_{L^p(\mathbb{R}^d)} = \|f\|^p_{L^p(\Omega)},$$

which proves the result. The case $p = 1$ is obvious.

Finally we show that $f_n \to f$ in $L^p(\Omega)$ as $n \to \infty$. First assume that $f \in C_c(\Omega)$ and recall from the proof of (ii) that in such a case supp $f_n \subset K_{n_0}$ for $n \geq n_0$, where K_{n_0} is a compact set included in Ω such that supp $f = K \subset K_{n_0}$, and $\max_{x \in K_{n_0}} |f(x) - f_n(x)| \to 0$. Therefore, since $\|f - f_n\|^p_{L^p(\Omega)} = \int_{K_{n_0}} |f - f_n|^p \leq \max_{x \in K_{n_0}} |f - f_n|^p \, \mu(K_{n_0})$, we see that $\|f - f_n\|_{L^p(\Omega)} \to 0$, $n \to \infty$. Hence the result is true for $f \in C_c(\Omega)$. By

Theorem A 2.2 we know that $C_c(\Omega)$ is dense in $L^p(\Omega)$. Therefore, given $f \in L^p(\Omega)$ and $\varepsilon > 0$, there exists $g_\varepsilon \in C_c(\Omega)$ such that $\|g_\varepsilon - f\|_{L^p(\Omega)} \leq \varepsilon$. Hence, putting $(g_\varepsilon)_n = \rho_n * g_\varepsilon$ we have $\|(g_\varepsilon)_n - f_n\|_{L^p(\Omega)} = \|(g_\varepsilon - f)_n\|_{L^p(\Omega)} \leq \|g_\varepsilon - f\|_{L^p(\Omega)} \leq \varepsilon$. Therefore $\|f - f_n\|_{L^p(\Omega)} \leq \|f - g_\varepsilon\|_{L^p(\Omega)} + \|g_\varepsilon - (g_\varepsilon)_n\|_{L^p(\Omega)} + \|(g_\varepsilon)_n - f_n\|_{L^p(\Omega)} \leq 2\varepsilon + \|g_\varepsilon - (g_\varepsilon)_n\|_{L^p(\Omega)}$. Now, since for each $\varepsilon > 0$, $g_\varepsilon \in C_c(\Omega)$, we have proved above that $(g_\varepsilon)_n \to g_\varepsilon$ in $L^p(\Omega)$ for each $\varepsilon > 0$, as $n \to \infty$. Thus, there exists $N = N(\varepsilon)$ such that $\|g_\varepsilon - (g_\varepsilon)_n\|_{L^p(\Omega)} \leq \varepsilon$ for $n \geq N(\varepsilon)$. We conclude that for $n \geq N(\varepsilon)$, $\|f - f_n\|_{L^p(\Omega)} \leq 3\varepsilon$, i.e. that $f_n \to f$ in $L^p(\Omega)$ as $n \to \infty$. $\qquad\square$

Proposition A 2.6. The space $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, if $1 \leq p < \infty$.

Proof. By Theorem A 2.2 we know that $C_c(\Omega)$ is dense in $L^p(\Omega)$ if $1 \leq p < \infty$. If $f \in C_c(\Omega)$, then arguing as in the last part of the proof of Lemma A 2.5 (iii), we may construct a sequence in $C_c^\infty(\Omega)$, namely f_n for $n \geq n_0$, such that $f_n \to f$ in $L^p(\Omega)$, $n \to \infty$. $\qquad\square$

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