2022-1-10

Άσκηση 4.5 (Εργασία 2)

Logistic regression $\quad y = 0, 1, \quad x \in \mathbb{R}$

$$\hat{f}(x) = \begin{cases} 0 & b_0 + b_1 x < 0 \\ 1 & b_0 + b_1 x > 0 \end{cases}$$

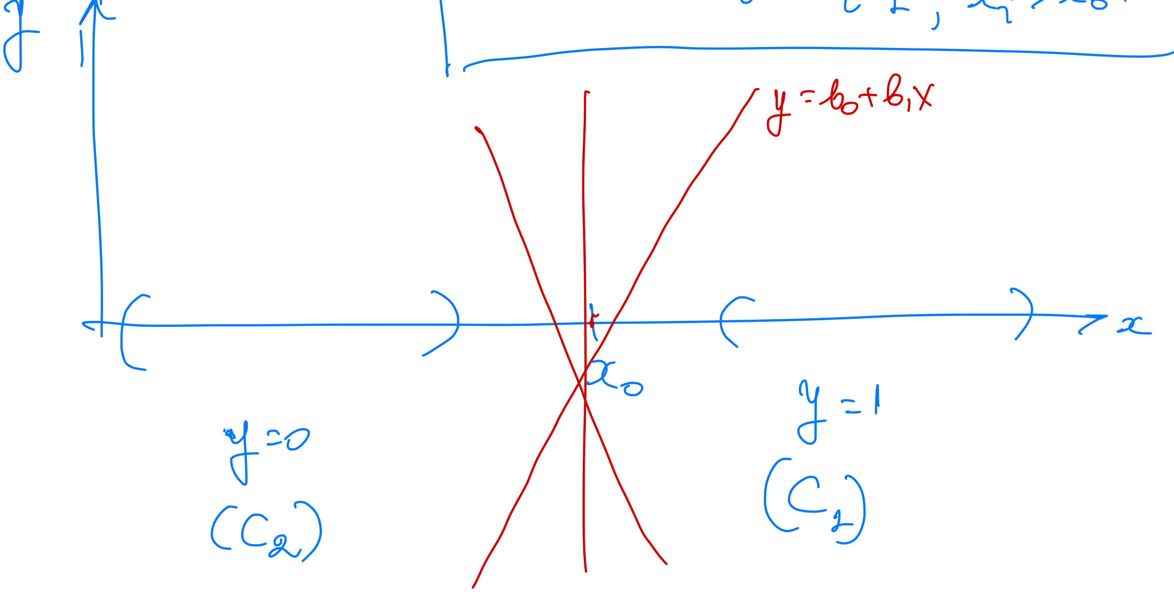
log-likelihood $\quad \ell(b) = \sum_{i=1}^{N} \left[y_i \log p(x_i; b) + (1 - y_i) \log(1 - p(x_i; b)) \right]$

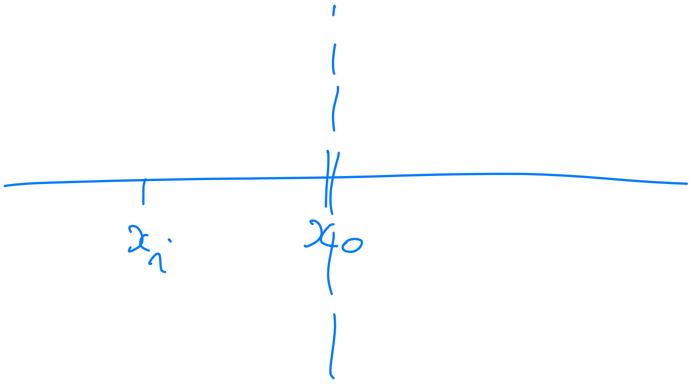
$b = (b_0, b_1)$

$$p(x_i; b) = P(y = 1 \mid x_i, b) = \frac{e^{b_0 + b_1 x_i}}{1 + e^{b_0 + b_1 x_i}}$$

$$\ell(b) = \sum_{i=1}^{N} \left\{ y_i (b_0 + b_1 x_i) - \log\left(1 + e^{b_0 + b_1 x_i}\right) \right\} \leftarrow \underset{b_0, b_1}{\max}$$

Ειδική περίπτωση : $\boxed{\exists\, x_0 \in \mathbb{R} : y_i = \begin{cases} 0, & x_i < x_0 \\ 1, & x_i > x_0 . \end{cases}}$





$\delta_i = x_i - x_0$

$$y_i = \begin{cases} 0 & \delta_i < 0 \\ 1 & \delta_i > 0. \end{cases}$$

$$\ell(b) = \sum_{i=1}^{N} \left[y_i (b_0 + b_1 x_0 + b_1 \delta_i) - \log\left(1 + e^{\overbrace{b_0 + b_1 x_0}^{\gamma_0} + \overbrace{b_1}^{\gamma_1} \delta_i}\right) \right]$$

$$\left.\begin{aligned} \frac{\partial \ell}{\partial \gamma_0} &= 0 \\ \frac{\partial \ell}{\partial \gamma_1} &= 0 \end{aligned}\right| \Rightarrow \left.\begin{aligned} &\sum_{C_1} \delta_i - \sum_{i=1}^{N} \delta_i \varphi_i = 0 \\ &|C_1| - \sum_{i=1}^{N} \varphi_i = 0 \end{aligned}\right] \quad \varphi_i = \varphi_i(\gamma_0, \gamma_1, x_i)$$

λύση (γ_0, γ_1)

Αν υπάρχουν (γ_0, γ_1) τ.ω. $\varphi_i = \begin{cases} 0 & i \notin C_1 \\ 1, & i \in C_1 \end{cases}$

δείξτε το

5.4 Γενικό truncated power series (cubic splines)

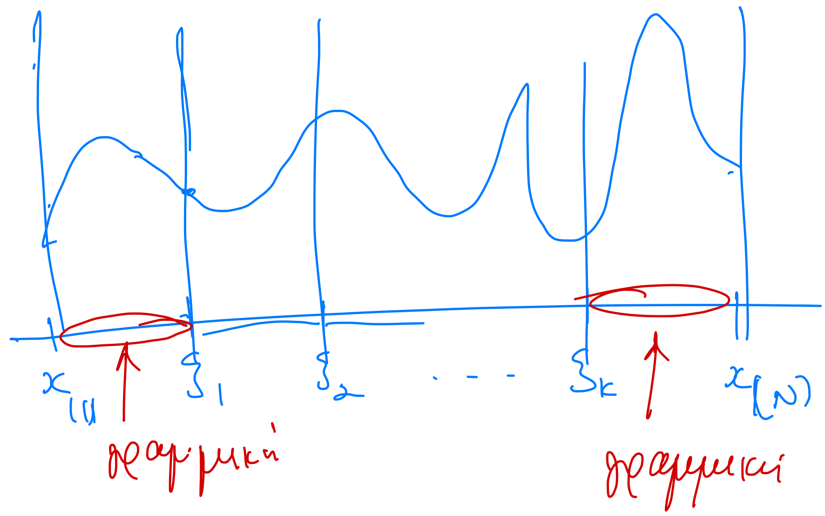
$$f(x) = \sum_{j=0}^{3} b_j x^j + \sum_{k=1}^{K} \theta_k (x - \xi_k)_+^3$$

$$\left[\text{βάση} \quad 1,\ x,\ x^2,\ x^3,\ (x-\xi_1)_+^3, \dots, (x-\xi_K)_+^3 \right]$$

training set $\quad x_1, \dots, x_N$

$$x_{(1)} = \min x_i\ , \quad x_{(N)} = \max x_i$$

$$x_{(1)} < \xi_1\ , \quad x_{(N)} > \xi_K .$$



$x < \xi_1$

(a) $x < \xi_1 \Rightarrow (x - \xi_k)_+ = 0 \Rightarrow f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$

$$\Longrightarrow \quad b_2 = 0,\ b_3 = 0.$$

(b) $x > \xi_K \Rightarrow (x - \xi_k)_+ = x - \xi_k \quad \forall k = 1, \dots, K$

$$f(x) = b_0 + b_1 x + \sum_{k=1}^{N} \theta_k \left(x^3 - 3x^2 \xi_k + 3x\xi_k^2 - \xi_k^3 \right)$$

$$= \left(b_0 - \sum_{k=1}^{K} \theta_k \xi_k^3 \right) + \left(b_1 + 3\sum_{k=1}^{K} \theta_k \xi_k^2 \right) x - 3\cancel{\left(\sum_{k=1}^{K} \theta_k \xi_k \right)}^{=0} x^2 + \cancel{\left(\sum_{k=1}^{K} \theta_k \right)}^{=0} \cdot x^3$$

$$\Rightarrow \sum_{k=1}^{K} \theta_k = 0 \quad \Rightarrow \theta_K = -\sum_{k=1}^{K-1} \theta_k$$

$$\boxed{\sum_{k=1}^{K} \theta_k \xi_k = 0.} \quad \Rightarrow \theta_K \xi_K = -\sum_{\ell=1}^{K-1} \theta_\ell \xi_\ell$$

$$f(x) = b_0 + b_1 x + \sum_{\ell=1}^{K-1} \theta_\ell (x-\xi_\ell)_+^3 - \left(\sum_{\ell=1}^{K-1} \theta_\ell\right)(x-\xi_K)_+^3$$

$$= b_0 + b_1 x + \sum_{\ell=1}^{K-1} \theta_\ell \cdot \underbrace{\left[(x-\xi_\ell)_+^3 - (x-\xi_K)_+^3\right]}_{e_\ell(x)}$$

$$\boxed{= b_0 + b_1 x + \sum_{\ell=1}^{K-1} \theta_\ell e_\ell(x)} \quad , \quad e_\ell(x) = (x-\xi_\ell)_+^3 - (x-\xi_K)_+^3$$

$$\sum_{\ell=1}^{K} \theta_\ell \xi_\ell = 0 \Rightarrow \theta_K \xi_K = -\sum_{\ell=1}^{K-1} \theta_\ell \xi_\ell \Rightarrow$$

$$\Rightarrow -\xi_K \left(\sum_{\ell=1}^{K-1} \theta_\ell\right) = -\sum_{\ell=1}^{K-1} \theta_\ell \xi_\ell \Rightarrow \boxed{\sum_{\ell=1}^{K-1} \overbrace{\theta_\ell (\xi_K - \xi_\ell)}^{J_\ell} = 0.}$$

$$\boxed{\theta_{K-1}(\xi_K - \xi_{K-1})} = -\sum_{\ell=1}^{K-2} \theta_\ell (\xi_K - \xi_\ell)$$

$$f(x) = b_0 + b_1 x + \sum_{\ell=1}^{K-1} \theta_\ell \, e_\ell(x)$$

$$= b_0 + b_1 x + \sum_{\ell=1}^{K-1} \frac{\theta_\ell}{\xi_K - \xi_\ell} \cdot (\xi_K - \xi_\ell) \, e_\ell(x) =$$

$$= b_0 + b_1 x + \sum_{\ell=1}^{K-1} \underbrace{\theta_\ell (\xi_K - \xi_\ell)}_{\gamma_\ell} \cdot \underbrace{\boxed{\frac{e_\ell(x)}{\xi_K - \xi_\ell}}}_{d_\ell(x)}$$

$$\Rightarrow f(x) = b_0 + b_1 x + \sum_{\ell=1}^{K-1} \gamma_\ell \, d_\ell(x)$$

όπου $\sum_{\ell=1}^{K-1} \gamma_\ell = 0 \Rightarrow \gamma_{K-1} = - \sum_{\ell=1}^{K-2} \gamma_\ell$

$$\Rightarrow f(x) = b_0 + b_1 x + \sum_{\ell=1}^{K-2} \gamma_\ell \, d_\ell(x) - \sum_{\ell=1}^{K-2} \gamma_\ell \, d_{K-1}(x)$$

$$\Rightarrow f(x) = b_0 + b_1 x + \sum_{\ell=1}^{K-2} \gamma_\ell \underbrace{\left(d_\ell(x) - d_{K-1}(x)\right)}_{N_{\ell+2}(x)}$$

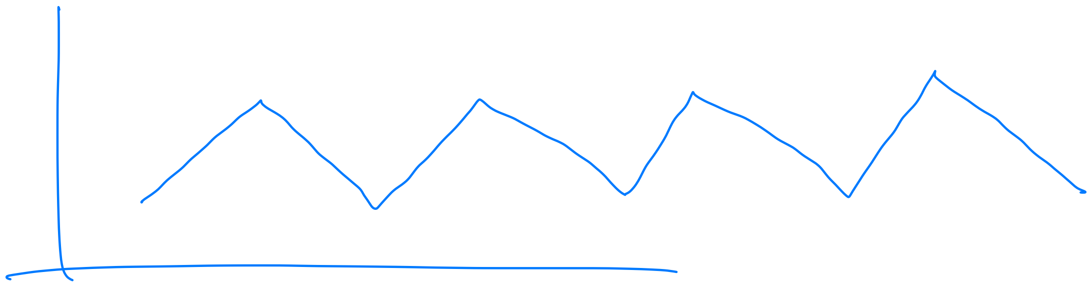
Συναρτήσεις βάσης

$N_1(x) = 1$, $N_2(x) = x$, $N_3(x) = d_1(x) - d_{K-1}(x)$, ..., $N_K(x) = d_{K-2}(x) - d_{K-1}(x)$

Άσκηση 5.6 (ειδική περίπτωση)

Έστω ότι σε ένα training set θέλουμε να προσαρμόσουμε μια περιοδική συνάρτηση με γνωστή περίοδο T, της μορφής spline τάξης 2 (τμηματικά γραμμική).

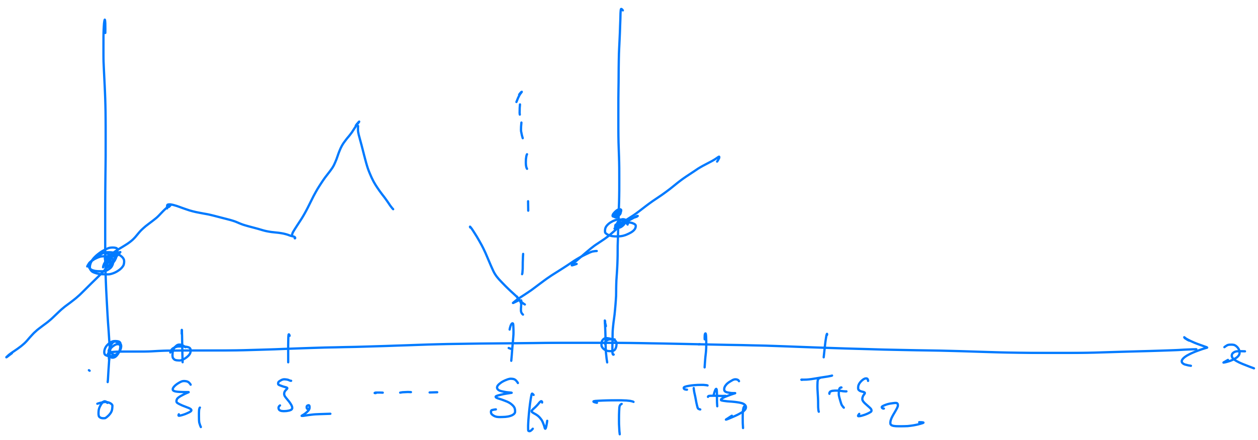
π.χ.:



Γενική μορφή truncated power series $(M=2)$

$$f(x) = b_0 + b_1 x + \sum_{\ell=1}^{K} \theta_\ell (x-\xi_k)_+$$

Επίσης $f(x+T)=f(x) \ \forall x$



Θεωρούμε $f(x)$: $x \in [0, T]$.

Θέτουμε κόμβους $\xi_1, \dots, \xi_K \in [0, T)$

$f(0) = f(T)$

$f'(0) = f'(T)$

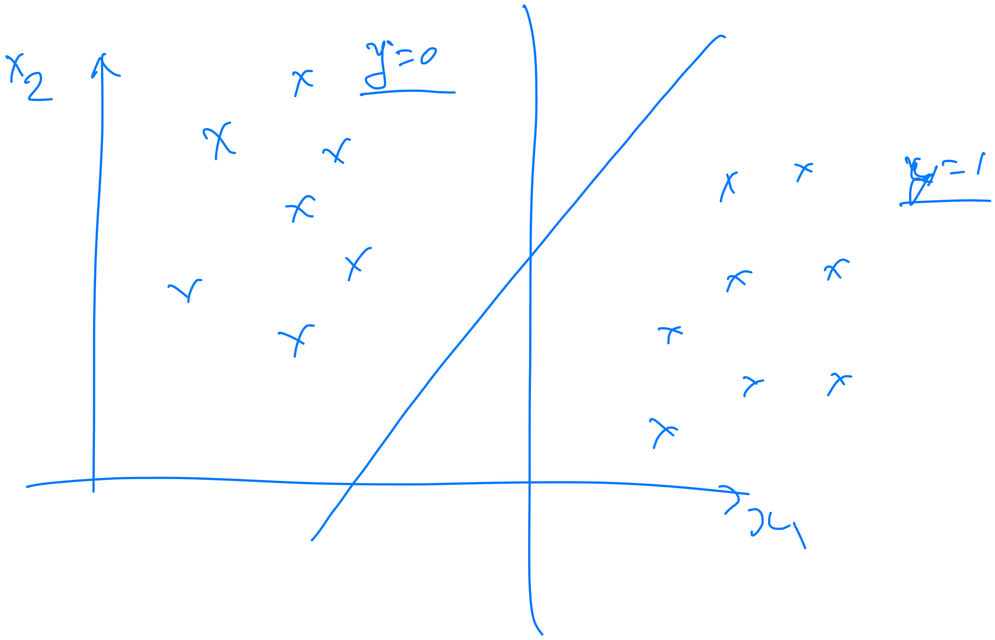
$$f(0) = f(T) \Rightarrow$$

$$\Rightarrow b_0 = b_0 + b_1 T + \sum_{\ell=1}^{K} \theta_\ell (T - \xi_\ell) \Rightarrow$$

$$\Rightarrow b_1 = -\frac{1}{T} \sum_{\ell=1}^{K} \theta_\ell (T - \xi_\ell)$$

$$\Rightarrow f(x) = b_0 - \frac{x}{T} \sum_{\ell=1}^{K} \theta_\ell (T - \xi_\ell) + \sum_{\ell=1}^{K} \theta_\ell \underline{(x - \xi_\ell)_+}$$

$$\Rightarrow f(x) = b_0 + \sum_{\ell=1}^{K} \frac{\theta_\ell}{T} \left[T (x - \xi_\ell)_+ - (T - \xi_\ell) x \right]$$



x_2
$y=0$
$y=1$
x_1