Magnetic Vortex Lattices

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Ginzburg-Landau Equations

Equilibrium states of superconductors (macroscopically) and of the $U(1)$ Higgs model of particle physics are described by the Ginzburg-Landau equations (GLE):

$$-\Delta_A \Psi = \kappa^2 (1 - |\Psi|^2)\Psi$$
$$\operatorname{curl}^2 A = \operatorname{Im}(\bar{\Psi} \nabla_A \Psi)$$

where $(\Psi, A) : \mathbb{R}^d \to \mathbb{C} \times \mathbb{R}^d$, $d = 2, 3$, $\nabla_A = \nabla - iA$, $\Delta_A = \nabla_A^2$, the covariant derivative and covariant Laplacian, respectively, and κ is the Ginzburg-Landau material constant.

The GLE are the Euler-Lagrange equations for the Ginzburg-Landau energy functional

$$E_Q(\Psi, A) = \frac{1}{2} \int_\Sigma \left\{ |\nabla_A \Psi|^2 + |\operatorname{curl} A|^2 + \frac{\kappa^2}{2} (|\Psi|^2 - 1)^2 \right\}, \qquad (1)$$

with apprpriate b.c. . Here Σ is any domain in $\mathbb{R}^2$.

Origin of Ginzburg-Landau Equations

Superconductivity. Ψ is called the *order parameter*;
$|\Psi|^2$ is the density of (Cooper pairs of) superconducting electrons;
A is the magnetic potential;
$\text{Im}(\bar{\Psi}\nabla_A\Psi)$ is the superconducting current.

Particle physics. Ψ and A are the Higgs and $U(1)$ gauge (electro-magnetic) fields, respectively. (Part of Weinberg - Salam model of electro-weak interactions/ a standard model.)

Similar equations appear in other areas of physics and material sciences.

Extensions: Yang-Mills-Higgs and Seiberg-Witten equations

Ginzburg-Landau equations on surfaces

To model superconducting thin membranes, or quantum engines (nano-devises), one considers the GLE on 2D surfaces, Σ,

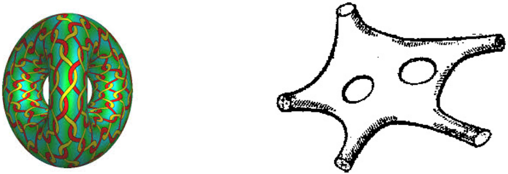


Figure: Compact and non-compact Riemann surfaces.

If the magnetic field $\neq 0$, then, instead of functions, Ψ, and vector-fields, A, we have to consider *sections* Ψ and *connection* one-forms, A, on a $U(1)$ line bundle L over Σ.

Ginzburg-Landau Equations

The Ginzburg-Landau equations on a $U(1)$ line bundle $L \to \Sigma$ over a manifold Σ are written as

$$\Delta_A \Psi = \kappa^2(|\Psi|^2 - 1)\Psi, \qquad (2a)$$

$$d^* dA = \mathrm{Im}(\bar{\Psi}\nabla_A \Psi). \qquad (2b)$$

Here Ψ is a section and A, a connection one-form on $L \to \Sigma$, $\Delta_A = \nabla_A^* \nabla_A$, ∇_A and ∇_A^* are the covariant derivative and its adjoint, and d and d^* are the exterior derivative and its adjoint.

The GLE are the Euler-Lagrange equations for the Ginzburg-Landau energy functional

$$E_Q(\Psi, A) = \frac{1}{2}\int_\Sigma \left\{ |\nabla_A \Psi|^2 + |dA|^2 + \frac{\kappa^2}{2}(|\Psi|^2 - 1)^2 \right\}. \qquad (3)$$

Let Σ be a Riemann surface (i.e. 2D complex Riem manif.) of finite vol.

Equivariant functions and vector fields

By the key uniformization theorem for Riemann surfaces, a Riem. surface Σ of genus 1 is torus and can be given as $\mathbb{T} = \mathbb{C}/\Lambda$, where Λ is a standard lattice, and of genus ≥ 2 can be given by

$$\Sigma = \mathbb{H}/\Gamma,$$

where $\mathbb{H}$ is the Poincaré half-plane

$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

and Γ, a Fuchsian group. (i.e. a discrete subgroup of the group of isometries $PSL(2, \mathbb{R})$ acting on $\mathbb{H}$ by Möbius transforms

$$\gamma\, z = \frac{az+b}{cz+d},\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Sections and *connections* of the line bundle over the Riem. surf. Σ $\Longleftrightarrow$ $\Gamma-$equivariant functions and vector fields,

$$(\Psi, A)(s^{-1}x) = T^{\text{gauge}}_{\chi_s}(\Psi, A),\ \forall s \in \Gamma, \tag{4}$$

where $T^{\text{gauge}}_{\chi} : (\Psi, A) \to (e^{i\chi}\Psi, A + d\chi)$, the *gauge transform*.

The GLE on line bundles on $\Sigma \Longleftrightarrow$ the GLE for complex functions and real vector-fields on $\mathbb{H}$ satisfying (4).

Examples

An important class of examples are the Riemann surfaces

$$\Sigma := \mathbb{H}/\Gamma(N), \quad N = 1, 2, \dots, \tag{5}$$

where $\Gamma(N)$ is the principal congruence subgroup of level N,

$$\Gamma(N) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : a \equiv d \equiv 1, b \equiv c \equiv 0 \mod N \right\}.$$

An explicit bundle $E \to \Sigma$ satisfying condition

$$\dim_{\mathbb{C}} \mathrm{Null}(-\Delta_{A^{b_r}} - b_r) = 1,$$

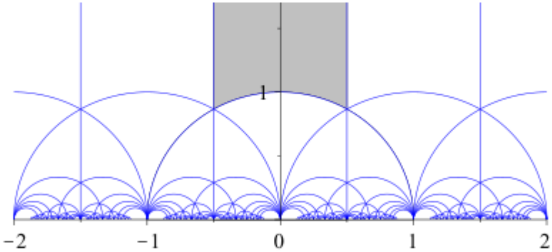
is

$$\Sigma = \mathbb{H}/\Gamma(6), \quad \deg E = 12. \tag{6}$$

Fundamental domains

Recall a fundamental domain, F, of a group Γ acting on a topological space X is a closed subset of X s.t.

$$X = \cup_{g \in \Gamma} gF \quad \text{and} \quad gF \cap g'F = \partial(gF) \cap \partial(g'F)\ \forall g \neq g'$$



Tiling of the Poincaré (hyperbolic) plane by fundamental domains

Fundamental domains 2

The GLE on line bundles on $\Sigma \Longleftrightarrow$ the GLE for complex functions and real vector-fields on a fundamental domain F of Γ satisfying appropriate (equivariant) boundary conditions.

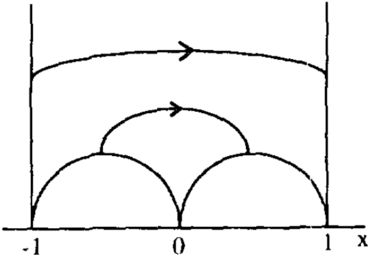
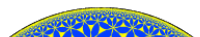
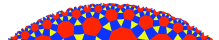


Figure: A fundamental domain of the principal congruence subgroup $\Gamma(2)$ of level 2.

$$\Gamma(N) := \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : a \equiv d \equiv 1, b \equiv c \equiv 0 \bmod N\right\}.$$

Examples of tiling of the Poincaré disk by fundamental domains:

General properties

Gauge symmetry: If (Ψ, A) is a solution of GLE, then for any $h \in C^\infty(\Sigma, U(1))$ the pair $(h\Psi, A - ih^{-1}dh)$ is also a soln of GLE

Quantization of flux: Let $F_A = dA$ be the curvature of a connection A on a line bundle L. Then

Theorem.(Chern-Weil correspondence) The flux of the curvature F_A is quantized: $\frac{1}{2\pi}\int_\Sigma F_A \ (= c_1(L)) = \deg(L) \in \mathbb{Z}$.

Constant curvature connections on L: A with F_A of the form $F_A = b\omega$, where b is a constant and ω is the symplectic *volume form* on Σ. Then Chern-Weil corresp. thm implies

$$b = \frac{1}{\mathrm{vol}(X)}\int_\Sigma F_A = \frac{2\pi n}{\mathrm{vol}(\Sigma)}. \tag{7}$$

Proposition. $(0, A)$ *solves GLEs* $\Longleftrightarrow$ A *is a c. c. connect. on* L.

Result 1: Existence and expansion

Let (Σ, h_r), $r > 0$, be a compact or non-compact Riemann surface equipped with the finite area hyperbolic metric

$$h_r = \frac{r}{(\operatorname{Im} z)^2} dz \otimes d\bar{z} \quad (r > 0). \tag{8}$$

Let L be a unitary line bundle over Σ and $\deg L$, the topol. degree of L.

Theorem 1 (Existence and expansion). Suppose $r > 0$ satisfies $0 < \left|\kappa^2 r - b\right| \ll 1$, with $b := 2\pi \deg L / |\Sigma| > 0$.

Then $\exists\ \epsilon > 0$ s.th. GLE with metric (8) has a C^2 branch of solns (Ψ_s, A_s, r_s), $s \in \mathbb{C}$, $|s| \leq \epsilon$, satisfying $r_s = b/\kappa^2 + O(|s|^2)$ and

$$\Psi_s = s\xi + O_{\mathcal{H}^k}(|s|^3), \tag{9}$$

$$A_s = A^{b_{r_s}} + |s|^2 \alpha + O_{\vec{H}^k}(|s|^4), \tag{10}$$

where $\xi = O_{\mathcal{H}^k}(1)$ is gauge-equivalent to a holom. section of L, $b_r := b/r$ and $\alpha = O_{\vec{\mathcal{H}}^k}(1)$ is a co-closed 1-form satisfying

$$d\alpha = \frac{1}{2} * (1 - |\xi|^2) \qquad * = \text{Hodge operator}. \tag{11}$$

Result 2: Uniqueness

Recall, $b_r := b/r$, with $b := 2\pi \deg L/|\Sigma| > 0$.

Theorem 2 (Uniqueness). Under the conditions of Theorem 1, $\mathrm{Null}(-\Delta_{A^{b_r}} - b_r)$ is finite dimensional and, if

$$\dim_{\mathbb{C}} \mathrm{Null}(-\Delta_{A^{b_r}} - b_r) = 1, \tag{12}$$

then we can take $s \in \mathbb{R}_{\geq 0}$, and the solution (Ψ_s, A_s, r_s), $s \in \mathbb{R}_{\geq 0}$, $|s| \leq \epsilon$, is unique in $U \subset X^k$, up to a gauge symmetry, and equation $r = r_s$ can be solved for s giving $s = s(r)$ leading to the solution

$$(\Psi(r), A(r)) = (\Psi_{s(r)}, A_{s(r)}), \qquad \forall r > 0.$$

Result 3: Energy

Theorem 3 (Energy asymptotics). For the solution $(\Psi_{s(r)}, A_{s(r)})$ constructed above and the constant curvature solution $(0, A^{b_r})$,

$$\mathcal{E}(\Psi_{s(r)}, A_{s(r)}) = \mathcal{E}(0, a^{b_r}) - \frac{|\Sigma|_r}{4} \frac{\left|\kappa^2 - b_r\right|^2}{(\kappa^2 - \frac{1}{2})\beta(r) + \frac{1}{2}} + O(\left|\kappa^2 - b_r\right|^3). \tag{13}$$

where, recall, $b_r = b/r$, with $b := 2\pi \deg E/(|\Sigma|\, r)$ and

$$\beta(r) := \min\{\frac{\langle|\xi|^4\rangle}{\langle|\xi|^2\rangle^2} : \xi \in \mathrm{Null}(-\Delta_{A^{b_r}} - b_r)\},\ \langle f\rangle := \frac{1}{|\Sigma|_r}\int f.$$

$\Longrightarrow$ $\mathcal{E}(\Psi_{s(r)}, A_{s(r)}) < \mathcal{E}(0, A^{b_r})$, provided $\kappa > \kappa_c(r)$, where

$$\kappa_c(r) := \sqrt{\frac{1}{2}\left(1 - \frac{1}{\beta(r)}\right)}. \tag{14}$$

Hence, if $\kappa > \kappa_c(r)$, then the solutions constructed in Thm 1 are energetically favourable compared to the constant curvature one.

Key step: Linearized GLE

Linearize GLE around the constant curvature solution $(0, A^b) \Longrightarrow$

$$(-\Delta_{A^b} - \kappa^2 r)\xi = 0, \;\; d^* d\alpha = 0.$$

Let $\mathcal{S}(\Sigma) \equiv \mathcal{S}_k(\Sigma)$ denote the space of cusp forms on Σ with weight $k = 2b = 4\pi n / |\Sigma|$. We have the following.

Theorem. Let $\Sigma = \mathbb{H}/\Gamma$ be a non-compact Riemann surface with elliptic points. Then $-\Delta_{A^b}$ is self-adjoint and satisfies

(a) $-\Delta_{A^b} \geq b$ and b is an eigenvalue of $-\Delta_{A^b}$ if and only if $\mathcal{S}(\Sigma) \neq \emptyset$, and the multiplicity of b equals to $\dim \mathcal{S}(\Sigma)$;

(b) The essential spectrum of $-\Delta_{A^b}$ consists of m branches each of which filling in the semi-axis $[1/4 + b^2, \infty)$, where $m = \#$ cusps (defined later). Hence,

$$\sigma_{\mathrm{ess}}(-\Delta_{A^b}) = [1/4 + b^2, \infty).$$

Cusps

Definition (Cusp) Let Γ be a Fuchsian group. A point $c \in \mathbb{R} \cup \{\infty\}$ is called a *cusp* of $\Gamma \iff \exists\, \gamma \in \Gamma$ that is conjugate-equivalent to some horizontal translation $z \mapsto z + h$, $h \in \mathbb{R}$, s.th. $\gamma c = c$.

For example, $\Gamma = SL(2, \mathbb{Z})$ has the only cusp $c = \infty$, as every integral translation $z \mapsto z + n$, $n \in \mathbb{Z}$ fixes c.

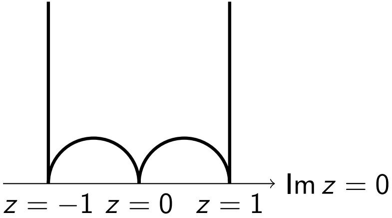


Figure: A fundamental domain of $\Gamma(2)$ in $\mathbb{H}$ with three cusps $c_1 = 1, c_2 = 0, c_3 = \infty$. ($-1$ is equivalent to c_2 thru transl. $z \mapsto z + 2$.)

The principal congruence subgroup $\Gamma = \Gamma(N)$ has 3 cusps for $N = 2$ and $\frac{1}{2} N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$ cusps for $N > 2$.

Ideas of the proof 1: Decomposition of Σ

(a) Let $\bar\partial_{A^b}$ = proj of ∇_{A^b} on $(0,1)$-forms. By the Weitzenböck -type formula, $-\Delta_a = \partial_a''^*\partial_a'' + *F_a$, we have

$$-\Delta_{A^b} \geq b \quad \text{and} \quad \mathrm{Null}(-\Delta_{A^b} - b) = \mathrm{Null}\,\bar\partial_{A^b}$$

$\Longrightarrow$ b is an eigenvalue of $-\Delta_{A^b}$ iff Null $\bar\partial_{A^b}$ is non-empty.

(b) We identify Σ with a fundamental domain $F_\Sigma \subset \mathbb{H}$ of Γ and decompose F_Σ into a compact connected set, U_0, and neighbourhoods U_i of the cusps c_i, in such a way that

$U_i \cap U_j = \emptyset$ for $1 \leq i \neq j \leq m$, and $U_0 := F_\Sigma \setminus \bigcup_{i=1}^m U_i$ is compact.

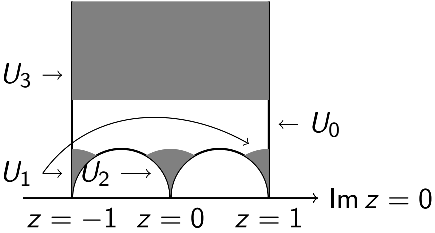


Figure: Schematic diagram for the decomposition of a fundamental domain of $\Gamma(2)$ in $\mathbb{H}$ with three cusps $c_1 = 1, c_2 = 0, c_3 = \infty$.

Ideas of the proof 2: Maps of cusps

We maps the domains U_i, $i = 1, \ldots, m$, isometrically onto the half-cylinders (for some $s_i \gg 1$)

$$Z_i := \{z \in \mathbb{C} : \operatorname{Im} z > s_i\}/\mathbb{Z}. \tag{15}$$

The corresponding maps φ_i are given by

$$\varphi_i : \begin{cases} z \mapsto -\dfrac{1}{z - c_i} & (c_i \neq \infty), \\ z \mapsto z & (c_i = \infty) \end{cases}$$

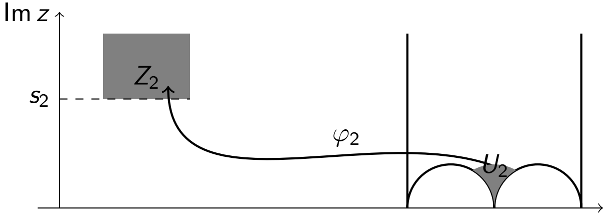


Figure: Schematic diagram illustrating map φ_2 associated to cusp $c_2 = 0$.

On Z_i's we solve the spectral problem explicitly and then patch different spectra using a partition of unity.

Idea pf the proof 3: Spectrum of a cusp

By the map φ_i, which maps U_i isometrically onto the half-cylinder Z_i

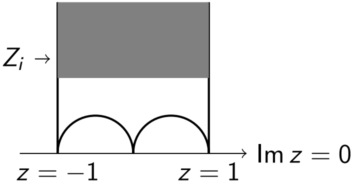


Figure: The half-cylinder $Z_i := \{z \in \mathbb{C} : \operatorname{Im} z > s_i\}/\mathbb{Z}$

the operator $-\Delta_{A^b}\big|_{U_i}$ is mapped unitarily to then operator

$$-\Delta_{A_0^b}, \quad \text{with } A_0^b := by^{-1}dx,$$

acting on $L^2(Z_i)$ with the Dirichlet b. c. $\Longrightarrow$ (easy estimate)

$$-\Delta_{A_0^b} \geq \frac{1}{4} + b^2.$$

Patching different spectra using a partition of unity, we conclude

$$\sigma_{\mathrm{ess}}(-\Delta_{A^b}) \subset [1/4 + b^2, \infty),$$

which concludes the essential part of (b). $\square$

Bifurcation from Constant Curvature Connection

Recall the linearized the GLE on the constant curvature solution $u^b = (0, A^b)$, where A^b is a c.c. connection on L:

$$(-\Delta_{A^b} - \kappa^2)\xi = 0, \; d^* d\alpha = 0$$

where ξ is a section on L and α one-form on Σ.

The first equation was investigated above to obtain that, if Null $\bar{\partial}_{A^b}$ (= the space of holomorphic sections of $L \to \Sigma$) is non-empty, then b is the smallest eigenvalue of $-\Delta_{A^b}$ and is isolated.

For the second equation we have
Proposition $d^* d \geq 0$ and the solution space to $d^* d\alpha = 0$ in $\vec{\mathcal{H}}^2$ is

$$\mathrm{Null}\, d^* d|_{\vec{\mathcal{H}}^2} = \{\text{harmonic 1-forms on } \Sigma\} = H^1_{\mathrm{DR}}(\Sigma, \mathbb{R}). \qquad (16)$$

$\Longrightarrow$ bifurcation of non-trivial energy minim. solns of the GLEs at $b = \kappa^2$

Summary

- We described the Ginzburg-Landau equations on general Riemann surfaces and its general properies.
- We presented our recent results on existence of energy minimizing solutions and gave some ideas of the proof.

Thank-you for your attention