ΤΕΛΕΣΤΕΣ ΣΤΟΝ H^2 (Θ.23α-711)

ΑΣΚΗΣΕΙΣ ΙΙ

Μερικες ενδιαφερουσες/ωραιες/πρωτοτυπες λυσεις και σχολια που προταθηκαν στην ταξη:

Άσκηση 1. *Υπενθυμιζουμε τη γραμμικη επι ισομετρια* $f \mapsto \widetilde{f} : H^2 \to \widetilde{H}^2$.
Αν $f, g \in H^2$ *ειναι τετοιες ωστε* $fg \in H^2$, *δειξτε οτι* $\widetilde{fg} = \widetilde{f}\widetilde{g}$.

Απόδειξη. Αφου $f \in H^2$, αν $f_r(e^{it}) := f(re^{it})$, εχουμε $\lim_{r\nearrow 1} \|f_r - \tilde{f}\|_{L^2} = 0$ και συνεπως για καθε ακολουθια (r_n) με $0 \leqslant r_n \nearrow 1$ υπαρχει υπακολουθια (r_{k_n}) της (r_n) ώστε $\lim_n f_{r_{k_n}}(e^{it}) = \tilde{f}(e^{it})$ σχεδον για καθε $e^{it} \in \mathbb{T}$, δηλαδη για καθε $e^{it} \notin A$ οπου $A \subseteq \mathbb{T}$ εχει μετρο Lebesgue 0. Για τον ιδιο λογο, υπαρχει *υπακολουθια* (r'_n) *της* (r_{k_n}) ωστε $\lim_n g_{r'_n}(e^{it}) = \tilde{g}(e^{it})$ για καθε $e^{it} \notin B$ οπου $B \subseteq \mathbb{T}$ εχει μετρο 0. Εφοσον επισης $fg \in H^2$, υπαρχει *υπακολουθια* (r''_n) *της* (r'_n) ωστε $\lim_n (fg)_{r''_n}(e^{it}) = \widetilde{(fg)}(e^{it})$ για καθε $e^{it} \notin C$ οπου $C \subseteq \mathbb{T}$ εχει μετρο 0.

Αν ονομασουμε $X = A \cup B \cup C$, το συνολο X εχει μετρο 0 και για καθε $e^{it} \notin X$ εχουμε λοιπον

$$\begin{aligned} \lim_n f_{r''_n}(e^{it}) &= \tilde{f}(e^{it}) \\ \lim_n g_{r''_n}(e^{it}) &= \tilde{g}(e^{it}) \\ \lim_n (fg)_{r''_n}(e^{it}) &= \widetilde{(fg)}(e^{it})\,. \end{aligned}$$

Ομως απο τις πρωτες δυο σχεσεις εχουμε $\lim_n (fg)_{r''_n}(e^{it}) = \tilde{f}(e^{it})\tilde{g}(e^{it})$, οποτε συνδυαζοντας με την τριτη βρισκουμε οτι $\widetilde{(fg)}(e^{it}) = \tilde{f}(e^{it})\tilde{g}(e^{it})$ για καθε $e^{it} \notin X$, δηλαδη, σχεδον παντου.

Δειξαμε λοιπον οτι $\widetilde{fg} = \widetilde{f}\widetilde{g}$ ως στοιχεια του $\widetilde{H}^2$. □

Παρατηρηση Η στοιχειωδης αυτη αποδειξη αποφευγει τη χρηση του Θεωρηματος Fatou, που δεν εχουμε αποδειξει.

Άσκηση 2. Εστω $\phi \in L^\infty(\mathbb{T})$ και $M_\phi \in \mathcal{B}(L^2(\mathbb{T}))$ ο αντιστοιχος πολλαπλασιαστικος τελεστης.
Δειξτε οτι ο υποχωρος $\widetilde{H}^2 \subseteq L^2$ είναι M_ϕ-αναλλοιωτος αν και μονον αν $\phi \in \widetilde{H}^2$,
αν και μονον αν υπαρχει $\psi \in H^\infty$ ωστε $\widetilde{\psi} = \phi$.

(Παραλειπεται)

Άσκηση 3. Δειξτε οτι αν μια $\widetilde{f} \in \widetilde{H}^2$ παιρνει πραγματικες τιμες σχεδον παντου στο $\mathbb{T}$, τοτε ειναι σχεδον παντου ιση με την cf_0, οπου c μια (πραγματικη) σταθερα.

Απόδειξη. We have learnt in [605] that if a function g in $L^2(\mathbb{T})$ is a.e. real valued, then $\hat{g}(-n) = \overline{\hat{g}(n)}$. Indeed,

$$2\pi\hat{g}(-n) = \int g(e^{it})e^{int}dt = \int g(e^{it})\overline{e^{-int}}dt = \overline{\left(\int g(e^{it})e^{-int}dt\right)} = 2\pi\overline{\hat{g}(n)}.$$

But our function $g = \widetilde{f}$ is in $\widetilde{H}^2$, so necessarily $\hat{g}(-n) = 0$ when $n \geqslant 1$. By the previous calculation we must have $\hat{g}(n) = 0$ when $n \neq 0$ and so the Fourier series reduces to $\widetilde{f} = g = \hat{g}(0)f_0$. □

Άσκηση 4. Δειξτε οτι αν ενας κλειστος υποχωρος $E \subseteq L^2(\mathbb{T})$ ειναι M_1-αναλλοιωτος, τοτε
ή $M_1(E) = E$ ή αλλιως $\bigcap_{n\in\mathbb{N}} M_1^n(E) = \{0\}$.

Απόδειξη. By assumption $M_1(E) \subseteq E$. If $M_1(E) \neq E$ then E is simply invariant (not reducing) [1] so by Beurling there exists $\phi \in L^2(\mathbb{T})$ with $|\phi| = 1$ a.e. on $\mathbb{T}$ such that $E = \phi\widetilde{H}^2 = M_\phi(\widetilde{H}^2)$. Observe that

[1] if it were reducing it would also be invariant under $M_1^* = M_1^{-1}$ and so $E \subseteq M_1(E) \subseteq E$.

$M_\phi M_1 = M_1 M_\phi$. Hence, for $n \geqslant 1$,

$$M_1^n(E) = M_1^n M_\phi(\widetilde{H}^2) = M_\phi M_1^n(\widetilde{H}^2)\,.$$

But $\widetilde{H}^2 = \overline{\text{span}}\{f_0, f_1, \dots\}$ and so (since M_1^n is isometric) $M_1^n(\widetilde{H}^2) = \overline{\text{span}}\{f_n, f_{n+1}, \dots\}$. Επεται οτι

$$\begin{aligned}\bigcap_{n\geqslant 0} M_1^n(E) &= \bigcap_{n\geqslant 0} M_\phi M_1^n(\widetilde{H}^2) = M_\phi \bigcap_{n\geqslant 0} M_1^n(\widetilde{H}^2) \qquad (M_\phi \text{ is 1-1})\\ &= M_\phi \bigcap_{n\geqslant 0} \overline{\text{span}}\{f_n, f_{n+1}, \dots\}\,.\end{aligned}$$

But clearly $\bigcap_{n\geqslant 0} \overline{\text{span}}\{f_n, f_{n+1}, \dots\} = \{0\}$ (every f_k for $k \geqslant 0$ is orthogonal to it). Thus

$$\bigcap_{n\geqslant 0} M_1^n(E) = \{0\}\,.$$

Άσκηση 5. Δειξτε οτι ενα πολυωνυμο $f(z) = \sum_{k=0}^{N} a_k z^k$ ειναι εξωτερικη συναρτηση (ως στοιχειο του χωρου H^2) αν και μονον αν δεν εχει καμμια ριζα στο $\mathbb{D}$. Μπορειτε να γενικευσετε για την περιπτωση που η f ειναι ολομορφη σε μια περιοχη του $\overline{\mathbb{D}}$;

Απόδειξη. Θα χρειασθει μια ενδιαφερουσα παρατηρηση:

Λήμμα 1. *If f, g are in H^∞ and are outer, then fg is outer.*

Απόδειξη. Note that f outer means that f is a cyclic vector for T_1, i.e. that $\text{span}\{T_1^n(f) : n \geqslant 0\}$ is dense in H^2. But

$$\text{span}\{T_1^n(f) : n \geqslant 0\} = \text{span}\{\zeta^n f : n \geqslant 0\} = \text{span}\{T_f(\zeta^n) : n \geqslant 0\}$$

and since $f \in H^\infty$, the operator $T_f : H^2 \to H^2 : h \mapsto fh$ is bounded, hence the above is equivalent to showing that the closure of $T_f(H^2)$ is H^2.

Since $fg \in H^\infty$, it thus suffices to show that the closure of $T_{fg}(H^2)$ is H^2.

Of course $fgH^2 \subseteq H^2$, hence $\overline{fgH^2} \subseteq H^2$.

To show equality, it is enough to show that $gH^2 \subseteq \overline{fgH^2}$ (because then $\overline{gH^2} \subseteq \overline{fgH^2}$ and since g is bounded and outer, as remarked above we have $\overline{gH^2} = H^2$).

So let $h \in H^2$. We show that gh is in $\overline{fgH^2}$: Since $H^2 = \overline{fH^2}$, there is a sequence (p_n) in H^2 with $\|fp_n - h\| \to 0$ and hence $\|gfp_n - gh\| \leqslant \|g\|_\infty \|fp_n - h\|$ tends to 0 as well. But each gfp_n is in fgH^2 and so the limit gh is in $\overline{fgH^2}$. □

Πρόταση 2 (Η γενικευση)**.** *If f is holomorphic in an open disc V containing $\overline{\mathbb{D}}$ and has no roots in $\mathbb{D}$, then it is outer.*

Απόδειξη. The function f may have roots in $\mathbb{T}$, but they must be finitely many, otherwise they would have an accumulation point in $\mathbb{T} \subseteq V$, hence f would vanish identically by the identity principle.

Thus we may factorize

$$f(z) = (z - c_1)\dots(z - c_n)h(z), \quad z \in V$$

where the c_i are the roots of f in $\mathbb{T}$ and h is holomorphic and has no roots in $\overline{\mathbb{D}}$. Since $\overline{\mathbb{D}}$ is compact, $\inf\{|h(z)| : z \in \overline{\mathbb{D}}\} > 0$. Thus $1/h$ is defined and bounded on $\overline{\mathbb{D}}$, which means that the operator T_h is *invertible* on H^2 (with inverse $T_{1/h}$) and so $T_h(H^2) = H^2$.

It thus remains to show that $z \mapsto (z - c_1)\dots(z - c_n)$ is an outer function. By Claim 1, this will follow if we prove that $u(z) := z - c$ is an outer function when $c \in \mathbb{T}$.

For this, let $g \in H^2$ be orthogonal to $T_1^n(u)$ for all $n \in \mathbb{Z}_+$. Thus we have, for all $n \in \mathbb{Z}_+$,

$$0 = \langle g, T_1^n(u)\rangle = \langle g, \zeta^n(\zeta - c)\rangle = \langle g, \zeta^{n+1}\rangle - \bar{c}\langle g, \zeta^n\rangle$$
$$\text{hence} \quad |\langle g, \zeta^{n+1}\rangle| = |\bar{c}||\langle g, \zeta^n\rangle| = |\langle g, \zeta^n\rangle| \,.$$

This means that the (Fourier) coefficients $\langle g, \zeta^n\rangle$ of g are constant in modulus. But g is in H^2, so the sequence $(\langle g, \zeta^n\rangle)$ must be square summable. This can only happen if all $\langle g, \zeta^n\rangle$ vanish, i.e. if $g = 0$.

Thus the linear span of $\{T_1^n(u) : n \geqslant 0\}$ has trivial orthogonal compelement, so it must be dense in H^2, οπως θελαμε. □

Άσκηση 6. Στο μαθημα χρησιμοποιηθηκε οτι, αν ενα αριθμησιμο συνολο $\{c_n : n \in \mathbb{N}\}$ ειναι πυκνο στον κυκλο $\mathbb{T}$ και θεωρησουμε τα σημεια $z_n = c_n(1 - \frac{1}{n^2})$, τοτε καθε σημειο του κυκλου $\mathbb{T}$ ειναι σημειο συσσωρευσης του $\{z_n : n \in \mathbb{N}\}$. Αποδειξη;

Παραλλαγη: Εστω X υποσυνολο του κυκλου (κλειστο, αν θελετε), και $\{c_n : n \in \mathbb{N}\}$ πυκνο στο κυκλο X. Ειναι αληθεια ότι καθε σημειο του X ειναι σημειο συσσωρευσης του $\{c_n(1 - \frac{1}{n^2}) : n \in \mathbb{N}\}$; Οχι παντα! Παρτε για παράδειγμα $X = \{e^{\pi i/n}, n \in \mathbb{N}\} \cup \{1\}$ και για c_n τα σημεια $e^{\pi i/n}$. Το σημειο $e^{\pi i} = 1$ ειναι αρκετα μακρια απ ολα τα z_n - γιατι ειναι μεμονωμενο σημειο του X.

Ενω ο κυκλος δεν εχει μεμονωμενα σημεια!

Μετα απ αυτην την παρατηρηση, η λυση ειναι αμεση:

If $z \in \mathbb{T}$ and $\epsilon > 0$ the disk $D(z, \epsilon/2)$ contains *infinitely many terms of* $\{c_n : n \in \mathbb{N}\}$. So there exists a n_ϵ *as large as I like, and I choose* $n_\epsilon > \sqrt{2/\epsilon}$ *(εχω το δικαιωμα!)* so that $c_n \in D(z, \epsilon/2)$ for all $n \geqslant n_\epsilon$. But then

$$|z - z_n| \leqslant |z - c_n| + |c_n - z_n| < \frac{\epsilon}{2} + \frac{1}{n^2} < \epsilon\,.$$

□

Άσκηση 7 (Προαιρετικα)**.** Μια συναρτηση $h : \mathbb{D} \to \mathbb{C}$ λεγεται *πολλαπλασιαστης (multiplier)* του χωρου H^2 αν ικανοποιει $hf \in H^2$ για καθε $f \in H^2$. Δειξτε οτι τα ακολουθα ειναι ισοδυναμα:
(α) Η h ειναι πολλαπλασιαστης του χωρου H^2.
(β) Η απεικονιση $f \mapsto hf$ οριζει *φραγμενο* τελεστη $H^2 \to H^2$.
(γ) $h \in H^\infty$.

Απόδειξη. The implications (γ)⇒(β)⇒(α) are immediate.

We show that (β)⇒(γ):

First, since $\mathbf{1} \in H^2$, we have $h = h\mathbf{1} \in H^2$ by hypothesis. To show that h is in fact bounded, recall that for each $z \in \mathbb{D}$,

$$h(z) = \langle h, k_z\rangle$$

where $k_z(w) = \dfrac{1}{1 - \bar{z}w}$ is the Szegő kernel . Thus for all $f \in H^2$ we have

$$\langle f, T_h^* k_z\rangle = \langle T_h f, k_z\rangle = \langle hf, k_z\rangle = (hf)(z) = h(z)f(z)$$
$$= h(z)\langle f, k_z\rangle = \langle f, \overline{h(z)}k_z\rangle$$

and therefore

$$T_h^* k_z = \overline{h(z)}k_z$$

which shows that the complex number $\overline{h(z)}$ is an eigenvalue of the operator T_h^*. It follows that

$$|h(z)| = |\overline{h(z)}| \leqslant \|T_h^*\| = \|T_h\|$$

for all $z \in \mathbb{D}$, and so $\|h\|_\infty = \sup\{|h(z)| : z \in \mathbb{D}\} \leqslant \|T_h\|$, οπως θελαμε.

Now we show that (α)⇒(β):

First proof. The hypothesis means that we have a well defined map

$$T_h : f \mapsto hf : H^2 \to H^2$$

which is obviously linear. Since its domain and range are complete, to show that this mapping is bounded it suffices to prove that its graph $\{(f, hf) : f \in H^2\}$ is closed in $H^2 \times H^2$ in the product topology

For this, by linearity of T_h, it is enough to prove that if a sequence (f_n) in H^2 satisfies $\|f_n\| \to 0$ *and* $\|hf_n - g\| \to 0$ for some $g \in H^2$, then necessarily $g = 0$ (εξηγειστε γιατι).

Now for each $z \in \mathbb{D}$, we have

$$g(z) = \langle g, k_z\rangle = \lim_n \langle hf_n, k_z\rangle = \lim_n (hf_n)(z) = h(z) \lim_n f_n(z) = h(z) \lim_n \langle f_n, k_z\rangle = 0$$

This shows that $g = 0$ as claimed.

Second proof . We work in $L^2(\mathbb{T})$: note that the hypothesis gives that for all $\tilde{f} \in \widetilde{H}^2$ we have $\widetilde{hf} \in \widetilde{H}^2$, i.e. $\tilde{h}\tilde{f} \in \widetilde{H}^2$ (Exercise 1). We show that this implies that $\tilde{h}$ is in $L^\infty(\mathbb{T})$. It will follow (Exercise 2) that h is bounded in $\mathbb{D}$, i.e. that $h \in H^\infty$.

Write $g = \tilde{h}$ for brevity. Suppose that g is not essentially bounded. We will prove that there is a $v \in \widetilde{H}^2$ for which gv is not in $L^2(\mathbb{T})$.

(a) For this, we will first prove that there is a $u \in L^2(\mathbb{T})$ for which gu is not in $L^2(\mathbb{T})$.

Decompose $\mathbb{R}_+$ as a disjoint union $\bigcup_{n\geqslant 0}[n, n+1)$ of intervals and define

$$A_n := \{e^{it} \in \mathbb{T} : n \leqslant |g(e^{it})| < n+1\} = |g|^{-1}([n, n+1)) \,.$$

These are disjoint measurable subsets of $\mathbb{T}$ and their union is $\{e^{it} \in \mathbb{T} : |g(e^{it})| < \infty\}$ whose complement $\{e^{it} \in \mathbb{T} : |g(e^{it})| = \infty\}$ has measure zero, since $|g| \in L^2(\mathbb{T})$.

Since |g| is not essentially bounded, for all n the set $\{e^{it} \in \mathbb{T} : |g(e^{it})| \geqslant n\}$, which equals $\bigcup_{k\geqslant n} A_k$, must have positive measure; therefore for all n there exists $k_n \geqslant n$ so that A_{k_n} has positive measure.

Now for each $n \in \mathbb{N}$ let χ_n be the characteristic (aka indicator) function of A_{k_n}. Note that $\chi_n \in L^2$ and $\|\chi_n\|_2^2 = m(A_{k_n}) > 0$. The sum

$$\sum_{n=0}^{\infty} \frac{1}{k_n} \frac{\chi_n}{\|\chi_n\|_2}$$

defines a function $u \in L^2$ because[2] (it is measurable and)

$$\|u\|_2^2 = \int \left(\sum_{n=0}^{\infty} \frac{1}{k_n^2} \frac{\chi_n^2}{\|\chi_n\|_2^2} dm \right) = \sum_{n=0}^{\infty} \int \left(\frac{1}{k_n^2} \frac{\chi_n^2}{\|\chi_n\|_2^2} dm \right) = \sum_{n=0}^{\infty} \frac{1}{k_n^2} \frac{m(A_{k_n})}{m(A_{k_n})} < \infty$$

(Beppo Levi). On the other hand

$$|gu|^2 = \sum_{n=0}^{\infty} \frac{1}{k_n^2 \|\chi_n\|_2^2} |g\chi_n|^2 = \sum_{n=0}^{\infty} \frac{1}{k_n^2 m(A_{k_n})} |g|^2 \chi_n$$

hence

$$\begin{aligned} \|gu\|_2^2 = \int |gu|^2 dm &\overset{B.L.}{=} \sum_{n=0}^{\infty} \frac{1}{k_n^2 m(A_{k_n})} \int |g|^2 \chi_n dm \\ &= \sum_{n=0}^{\infty} \frac{1}{k_n^2 m(A_{k_n})} \int_{A_{k_n}} |g|^2 dm \geqslant \sum_{n=0}^{\infty} \frac{1}{k_n^2 m(A_{k_n})} m(A_{k_n}) k_n^2 \end{aligned}$$

[2]alternatively: the terms are pairwise orthogonal and square summable elements of the Hilbert space L^2

(because for $e^{it} \in A_{k_n}$ we have $|g(e^{it})| \geqslant k_n$).

Hence $\|gu\|_2^2 = \infty$: the function gu is *not in* $L^2(\mathbb{T})$.

(b) Now we modify u to obtain a function in $\widetilde{H}^2$. If

$$u = \sum_{n\in\mathbb{Z}} \hat{u}(n) f_n = \sum_{n=0}^{\infty} \hat{u}(n) f_n + \sum_{k=1}^{\infty} \hat{u}(-k) f_{-k}$$

is the Fourier series of u (all three series converge in $\|\cdot\|_2$) define

$$v := \sum_{n=0}^{\infty} \hat{u}(n) f_n, \quad w := \sum_{k=1}^{\infty} \overline{\hat{u}(-k)} f_k \,.$$

These both converge in $\|\cdot\|_2$ and hence define elements of $\widetilde{H}^2$. Note that

$$u = v + \bar{w}$$

hence $gu = gv + g\bar{w}$. If both gv and gw were in L^2, we would have (since $(|gv| + |g\bar{w}|)^2 \leqslant 2(|gv|^2 + |g\bar{w}|^2)$ and $|g\bar{w}| = |g|.|\bar{w}| = |g|.|w| = |gw|$)

$$\|gu\|_2^2 = \int (|gv| + |g\bar{w}|)^2 dm \leqslant 2\int |gv|^2 dm + 2\int |g\bar{w}|^2 dm = 2\int |gv|^2 dm + 2\int |gw|^2 dm < \infty$$

contrary to the choice of u.

Conclusion: there are $v, w \in \widetilde{H}^2$ such that either gv or gw is not in $\widetilde{H}^2$. □

Παρατήρηση 3. *Το ενδιαφερον της δευτερης αποδειξης ειναι οτι ειναι «κατασκευαστικη» και στοιχειωδης, και δεν στηριζεται στο θεωρημα κλειστου γραφηματος.*