On the notion of Index of a curve

Introduction [1] If $\gamma : [0,1] \to \mathbb{C}$ is any (piecewise) continuously differentiable function with $\gamma(0) = \gamma(1)$, it defines a closed curve $[\gamma] := \{\gamma(t) : t \in [0,1]\} \subseteq \mathbb{C}$. If a point $\lambda \in \mathbb{C}$ is not on the curve, i.e. $\lambda \notin [\gamma]$ then we define its *index or winding number with respect to* γ by

$$\mathrm{wind}(\gamma;\lambda) = n(\gamma;\lambda) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-\lambda}dz = \frac{1}{2\pi i}\int_0^1 \frac{1}{\gamma(t)-\lambda}\gamma'(t)dt\,.$$

We wish to extend this notion to more general continuous curves $\gamma \in C([0,1])$.

Clearly if $\lambda \notin [\gamma]$ and we put $\gamma_\lambda(t) = \gamma(t) - \lambda$ then $0 \notin [\gamma_\lambda]$ and

$$n(\gamma_\lambda;0) = n(\gamma;\lambda)\,.$$

So it suffices to consider $n(\gamma;0)$ for nowhere-vanishing $\gamma \in C([0,1])$.

Every $f \in C(\mathbb{T})$ which vanishes nowhere (so that $0 \notin f(\mathbb{T})$) defines a continuous function

$$\gamma_f : [0,1] \to \mathbb{C} : t \mapsto f(e^{2\pi i t})$$

with $\gamma_f(0) = \gamma_f(1)$ and $0 \notin [\gamma_f] = f(\mathbb{T})$.

In order to define the index of such a more general closed curve γ_f, there are two methods. One method [Rud87] approximates f by a sequence p_i of trigonometric polynomials, and shows that the sequence $(n(\gamma_{p_i},0))_i$ is eventually well defined and eventually constant, so that one may define $n(\gamma_f;o) := \lim_i n(\gamma_{p_i},0)$. The other method [Arv02] directly shows that a never vanishing $\gamma \in C([0,1])$ is an exponential and uses periodicity of the exponential function to construct the index.

Παρατήρηση 1. *Καθε μη μηδενικος μιγαδικος αριθμος z μπορει να γραφτει $z = |z|e^{2\pi i\theta}$ για καποιο $\theta \in \mathbb{R}$. Το θεμα ειναι οτι αν το z εξαρταται κατα συνεχη τροπο απο μια παραμετρο $t \in [0,1]$, τοτε (οπως θα δειξουμε) το θ* μπορει να επιλεγει *να εξαρταται κατα συνεχη τροπο απο το t (παρολο που η απεικονιση $[0,1] \to \mathbb{T} : \theta \mapsto e^{2\pi i\theta}$ δεν εχει συνεχη αντιστροφο).*

Παρατήρηση 2. *Οριζουμε*

$$\log z := -\sum_{n=1}^{\infty}\frac{1}{n}(1-z)^n, \quad |1-z| < 1\,.$$

Η σειρα συγκλινει απολυτα στον ανοικτο δισκο $D(1,1) = \{z \in \mathbb{C} : |1-z| < 1\}$ και οριζει ολομορφη συναρτηση $\log$ *που ικανοποιει τη σχεση* $\log 1 = 0$ *και* $e^{\log z} = z$ *στον* $D(1,1)$. [2]

[1] Ind24, modified 24 Δεκεμβρίου 2024

[2] Verify that $\frac{d}{dz}\left(\frac{e^{\log z}}{z}\right) = 0$ on the open and connected set $D(1,1)$, so the quotient $\frac{e^{\log z}}{z}$ is the constant function $\frac{e^{\log 1}}{1} = 1$.

Πρόταση 3 (απο το βιβλιο του Arveson [Arv02]). *Αν η* $\gamma \in C([0,1])$ *δεν μηδενιζεται πουθενα (ισοδυναμα, αν ειναι αντιστρεψιμο στοιχειο της αλγεβρας* $C([0,1])$*), τοτε υπαρχει (μη μοναδικη)* $g \in C([0,1])$ *ωστε* $\gamma = e^g$.

Απόδειξη. Observe that since $\gamma([0,1])$ is a compact set and $0 \notin \gamma([0,1])$, the quantity $\epsilon := \min\{|\gamma(t)| : t \in [0,1]\}$ is strictly positive.

Special Case Suppose that

$$|\gamma(t) - \gamma(s)| < \epsilon \quad \text{for all } \ t, s \in [0,1].$$

Then

$$\begin{aligned} |\gamma(t)| \left|1 - \frac{\gamma(s)}{\gamma(t)}\right| &= |\gamma(t) - \gamma(s)| < \epsilon \leq |\gamma(t)| \\ \text{so} \quad \left|1 - \frac{\gamma(s)}{\gamma(t)}\right| &< 1 \quad \text{for all } \ t, s \in [0,1]\,. \end{aligned}$$

Thus $g(s) := \log\left(\frac{\gamma(s)}{\gamma(0)}\right)$ is defined (απο την Παρατηρηση 2) and continuous on $[0,1]$ and we see that

$$e^{g(s)} = \frac{\gamma(s)}{\gamma(0)}, \qquad s \in [0,1]\,.$$

General Case Since γ is uniformly continuous, we may find a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ such that

$$|\gamma(t) - \gamma(s)| < \epsilon \quad \text{for all } \ t, s \in [t_{k-1}, t_k] \ \text{ and all } k.$$

By the Special case applied to the interval $[t_{k-1}, t_k]$ for each $k = 1, \ldots, n$, there exists $g_k \in C([0,1])$ so that

$$\gamma(t_{k-1}) e^{g_k(s)} = \gamma(s), \qquad s \in [t_{k-1}, t_k]\,.$$

Note that $g_k(t_{k-1}) = \log 1 = 0$.

Now define $g_0 : [0,1] \to \mathbb{C}$ as follows:

$$\begin{aligned} &g_0(s) = g_1(s), \quad s \in [0, t_1] \\ &g_0(s) = g_1(t_1) + g_2(s), \quad s \in [t_1, t_2] \\ &\quad \vdots \\ &g_0(s) = g_1(t_1) + g_2(t_2) + \cdots + g_n(s), \quad s \in [t_{n-1}, 1]\,. \end{aligned}$$

Then g_0 is continuous and for each k,

$$\begin{aligned} e^{g_0(s)} &= e^{g_1(t_1)} e^{g_2(t_2)} \ldots e^{g_k(s)}, \quad s \in [t_{k-1}, t_k] \\ &= \frac{\gamma(t_1)}{\gamma(0)} \frac{\gamma(t_2)}{\gamma(t_1)} \cdots \frac{\gamma(s)}{\gamma(t_{k-1})} = \frac{\gamma(s)}{\gamma(0)} \end{aligned}$$

and so, choosing z_0 s.t. $\gamma(0) = e^{z_0}$ (since $\gamma(0) \neq 0$) and putting $g(s) := g_0(s) + z_0$ we obtain

$$e^{g(s)} = e^{g_0(s)} \gamma(0) = \gamma(s), \qquad s \in [0,1]\,.$$

□

Definition of the index Given $f \in C(\mathbb{T})$ which vanishes nowhere (so that $0 \notin f(\mathbb{T})$), define the continuous function

$$\gamma : [0,1] \to \mathbb{C} : t \mapsto f(e^{2\pi it})$$

Note that $\gamma(0) = \gamma(1)$ and $0 \notin [\gamma]$. By the Proposition, there is a continuous $g : [0,1] \to \mathbb{C}$ such that

$$f(e^{2\pi it}) = \gamma(t) = e^{2\pi i g(t)}, \quad t \in [0,1].$$

Notice that

$$e^{2\pi i g(0)} = f(e^{2\pi i 0}) = f(e^{2\pi i 1}) = e^{2\pi i g(1)}$$

and therefore (since exp is periodic with period $2\pi i$)

$$g(1) - g(0) \in \mathbb{Z}.$$

This difference depends only on f and not on g. For if $h \in C([0,1])$ is such that

$$f(e^{2\pi it}) = e^{2\pi i h(t)}, \quad t \in [0,1]$$

then

$$e^{2\pi i h(t)} = f(e^{2\pi it}) = e^{2\pi i g(t)}, \quad t \in [0,1]$$

and therefore $t \mapsto h(t) - g(t)$ is (continuous and integer-valued on $[0,1]$, hence) [3] constant. In particular, $h(1) - h(0) = g(1) - g(0)$.

Thus we may unambiguously define

Ορισμός 1. *Αν $f \in C(\mathbb{T})$ και $0 \notin f(\mathbb{T})$, οριζουμε*

$$\mathrm{wind}(f;0) = n(f;0) := g_f(1) - g_f(0)$$

οπου $g_f : [0,1] \to \mathbb{C}$ ειναι οποιαδηποτε συνεχης συναρτηση που ικανοποιει $f(e^{2\pi it}) = e^{2\pi i g_f(t)}$, $t \in [0,1]$.

Αν $f \in C(\mathbb{T})$ και $\lambda \in \mathbb{C} \backslash f(\mathbb{T})$, οριζουμε

$$\mathrm{wind}(f;\lambda) = n(f;\lambda) := n(f_\lambda;0)$$

οπου $f_\lambda = f - \lambda \mathbf{1}$.

Πρόταση 4. *The index or winding number satisfies, for $f, h \in C(\mathbb{T})$ invertible:*

1. *$n(fh;0) = n(f;0) + n(h;0)$;*
2. *$n(f;0) = n \in \mathbb{Z} \iff \exists\, u \in C(\mathbb{T})$ s.t. $f = f_n e^u$ (where $f_n(e^{it}) = e^{int}$, $e^{it} \in \mathbb{T}$).*

Απόδειξη. For (1), let $g_f, g_h \in C([0,1])$ be s.t.

$$f(e^{2\pi it}) = e^{2\pi i g_f(t)}, \;\; h(e^{2\pi it}) = e^{2\pi i g_h(t)}, \quad t \in [0,1].$$

Then

$$(fh)(e^{2\pi it}) = e^{2\pi i (g_f(t) + g_h(t))},$$

[3] for each $n \in \mathbb{Z}$ the inverse image $\{t \in [0,1] : h(t) - g(t) = n\}$ is a clopen nonempty subset of $[0,1]$, hence equal to $[0,1]$

so

$$\begin{aligned} n(fh;0) &= (g_f(1)+g_h(1))-(g_f(0)+g_h(0)) \\ &= (g_f(1)-g_f(0))+(g_h(1))-g_h(0)) \\ &= n(f;0)+n(h;0)\,. \end{aligned}$$

We prove (2):

• Suppose first that $f=e^u$ for some $u\in C(\mathbb{T})$. Then f never vanishes and so by the Proposition we already have $f(e^{2\pi it})=e^{2\pi i g_f(t)}$ where $g_f(t):=\frac{1}{2\pi i}u(e^{2\pi it})$, $t\in[0,1]$ and so

$$n(f;0)=g_f(1)-g_f(0)=0$$

(since $e^{2\pi i1}=e^{2\pi i0}$).

If conversely $n(f;0)=0$ then by the definition of $n(f;0)=0$ there exists $g_f\in C([0,1])$ with $f(e^{2\pi it})=e^{2\pi i g_f(t)}$ for all $t\in[0,1]$ and $g_f(1)-g_f(0)=0$. Thus g_f is periodic with period 1 so the function $u:\mathbb{T}\to\mathbb{C}$ given by $u(e^{2\pi it})=2\pi i g_f(t)$ is well defined and continuous on $\mathbb{T}$ and $f=e^u$.

We have shown that (2) holds for $n=0$.

• It follows (using (1)) that if $f\in C(\mathbb{T})$ is invertible then $n(1/f;0)+n(f;0)=n(\mathbf{1};0)=0$ since $\mathbf{1}=e^0$. Hence $n(1/f;0)=-n(f;0)$.

• Note that $n(f_1;0)=1$.

Indeed, $f_1(e^{2\pi it})=e^{2\pi it}$ so $g_{f_1}(t)=t$ gives $n(f_1;0)=g_{f_1}(1)-g_{f_1}(0)=1$.

• It now follows inductively that for all $n\in\mathbb{N}$ we have $n(f_n;0)=n$, and then, since $f_{-n}=1/f_n$, the same holds for all $n\in\mathbb{Z}$.

• Thus if $f=f_ne^u$ then $n(f;0)=n(f_n;0)+n(e^u;0)=n+0=n$.

Conversely if $n(f;0)=n$ then letting $g=f_{-n}f$ we see that $n(g;0)=-n+n(f;0)=0$. It follows that g is an exponential, $g=e^u$, and so $f=f_ng=f_ue^u$. □

Πρόταση 5. *If $f\in C(\mathbb{T})$ is invertible and continuously differentiable, then*

$$n(f;0)=\frac{1}{2\pi i}\int_{\gamma_f}\frac{1}{z}dz$$

where $\gamma_f(t)=f(e^{2\pi it}),\ t\in[0,1]$.

Απόδειξη. Observe that since γ_f is continuously diferentiable and never vanishes, the integral

$$\int_0^1\frac{1}{\gamma_f(t)}\gamma_f'(t)dt$$

exists. Since $0\notin[\gamma_f]$ there is a $z_0\in\mathbb{C}$ such that $\gamma_f(0)=e^{z_0}$.

Consider the differentiable function

$$g(s)=\frac{1}{2\pi i}\left(\int_0^s\frac{1}{\gamma_f(t)}\gamma_f'(t)dt+z_0\right),\quad s\in[0,1]\,.$$

We claim that

$$\gamma_f(t) = e^{2\pi i g(t)} \quad t \in [0,1]\,.$$

Indeed note that

$$2\pi i g'(t) = \frac{\gamma_f'(t)}{\gamma_f(t)}.$$

But this gives

$$\begin{aligned}\frac{d}{dt}\left(e^{-2\pi i g(t)}\gamma_f(t)\right) &= e^{-2\pi i g(t)}\gamma_f'(t) - 2\pi i g'(t) e^{-2\pi i g(t)}\gamma_f(t) \\ &= e^{-2\pi i g(t)}\left(\gamma_f'(t) - \frac{\gamma_f'(t)}{\gamma_f(t)}\gamma_f(t)\right) \\ &= 0\end{aligned}$$

hence the function $t \mapsto e^{-2\pi i g(t)}\gamma_f(t)$ is constant on $[0,1]$, equal to $e^{-2\pi i g(0)}\gamma_f(0) = e^{-z_0}e^{z_0}$ and so $e^{2\pi i g(t)} = \gamma_f(t)$ as claimed.

Thus $f(e^{2\pi i t}) = e^{2\pi i g(t)}$ and hence, by our defnition

$$n(f;0) = g(1) - g(0) = \frac{1}{2\pi i}\int_0^s \frac{1}{\gamma_f(t)}\gamma_f'(t)dt\,.$$

□

Ο δεικτης στροφης τριγωνομετρικου πολυωνυμου Εστω

$$p(e^{it}) = \sum_{k=-n}^{n} \hat{p}(k)e^{ikt}$$

τριγωνομετρικο πολυωνυμο με $0 \notin p(\mathbb{T})$. Τοτε το

$$q(e^{it}) = (f_n p)(e^{it}) = \sum_{m=0}^{2n} \hat{p}(m-n)e^{imt}$$

ειναι (αναλυτικο) πολυωνυμο, επομενως (οριζεται για καθε $z \in \mathbb{C}$ και) παραγοντοποιειται:

$$q(z) = cz^m \prod_{j=1}^{k}(z - z_j)\prod_{j'=1}^{l}(z - w_{j'}), \qquad 0 < |z_j| < 1 < |w_{j'}|$$

οπου $0, z_j$ are the roots of q in the open unit disc and $w_{j'}$ the roots of q outside the closed unit disc (notice that q has no roots on the circle because p never vanishes on $\mathbb{T}$).

Thus we have

$$p(e^{it}) = e^{-int}q(e^{it}) = ce^{i(m-n)t}\prod_{j=1}^{k}(e^{it} - z_j)\prod_{j'=1}^{l}(e^{it} - w_{j'})$$

$$\text{i.e.} \qquad p = cf_{m-n}\prod_{j=1}^{k} g_j \prod_{j'=1}^{l} h_{j'}$$

$$\text{where} \quad g_j(e^{it}) = e^{it} - z_j \quad \text{and} \quad h_{j'}(e^{it}) = e^{it} - w_{j'}\,,\ \ t \in [0, 2\pi]\,.$$

Thus from Proposition 4 (1) we have

$$n(p;0) = n(cf_{m-n};0) + \sum_{j=1}^{k} n(g_j;0) + \sum_{j'=1}^{l} n(h_{j'};0).$$

Now $n(cf_{m-n};0) = m - n$ of course. Also

$$\begin{aligned} n(g_j;0) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{g_j(t)} g_j'(t)dt = \frac{1}{2\pi i}\int_0^{2\pi} \frac{ie^{it}}{e^{it}-z_j}dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-e^{-it}z_j}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=0}^{\infty} (z_j e^{-it})^n dt = \frac{1}{2\pi}\sum_{n=0}^{\infty} z_j^n \int_0^{2\pi} e^{-int}dt \quad \text{(note } |z_j| < 1) \\ &= 1 \end{aligned}$$

for each $j = 1, \ldots, k$ and (since $|w_{j'}^{-1}| < 1$)

$$\begin{aligned} n(h_{j'};0) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{h_{j'}(t)} h_{j'}'(t)dt = \frac{1}{2\pi i}\int_0^{2\pi} \frac{ie^{it}}{e^{it}-w_{j'}}dt = \frac{1}{2\pi w_{j'}}\int_0^{2\pi} \frac{e^{it}}{e^{it}w_{j'}^{-1}-1}dt \\ &= \frac{-1}{2\pi w_{j'}}\int_0^{2\pi} e^{it}\sum_{n=0}^{\infty} (w_{j'}^{-1}e^{it})^n dt = \frac{-1}{2\pi w_{j'}}\sum_{n=0}^{\infty} w_{j'}^{-n} \int_0^{2\pi} e^{-i(n+1)t}dt \\ &= 0\,. \end{aligned}$$

Hence finally

$$n(p;0) = n(f_{m-n};0) + \sum_{j=1}^{k} n(g_j;0) + \sum_{j'=1}^{l} n(h_{j'};0) = (m-n) + k + 0.$$

Αναφορές

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