On the notion of Index of a curve

Introduction [1] If $\gamma : [0,1] \to \mathbb{C}$ is any (piecewise) continuously differentiable function with $\gamma(0) = \gamma(1)$, it defines a closed curve $[\gamma] := \{\gamma(t) : t \in [0,1]\} \subseteq \mathbb{C}$. If a point $\lambda \in \mathbb{C}$ is not on the curve, i.e. $\lambda \notin [\gamma]$ then we define its *index or winding number with respect to* γ by

$$\operatorname{wind}(\gamma;\lambda) = n(\gamma;\lambda) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-\lambda}dz = \frac{1}{2\pi i}\int_0^1 \frac{1}{\gamma(t)-\lambda}\gamma'(t)dt\,.$$

We wish to extend this notion to general continuous curves $\gamma \in C([0,1])$.

Clearly if $\lambda \notin [\gamma]$ and we put $\gamma_\lambda(t) = \gamma(t) - \lambda$ then $0 \notin [\gamma_\lambda]$ and

$$n(\gamma_\lambda; 0) = n(\gamma;\lambda)\,.$$

So it suffices to consider $n(\gamma;0)$ for nowhere-vanishing $\gamma \in C([0,1])$.

Every $f \in C(\mathbb{T})$ which vanishes nowhere (so that $0 \notin f(\mathbb{T})$) defines a continuous function

$$\gamma_f : [0,1] \to \mathbb{C} : t \mapsto f(e^{2\pi i t})$$

with $\gamma_f(0) = \gamma_f(1)$ and $0 \notin [\gamma_f] = f(\mathbb{T})$.

In order to define the index of such a more general closed curve γ_f, there are two methods. One method [Rud87] approximates f by a sequence p_i of trigonometric polynomials, and shows that the sequence $(n(\gamma_{p_i}, 0))_i$ is eventually well defined and eventually constant, so that one may define $n(\gamma_f; o) := \lim_i n(\gamma_{p_i}, 0)$. The other method [Arv02] directly shows that a never vanishing $\gamma \in C([0,1])$ is an exponential and uses periodicity of the exponential function to construct the index.

Παρατήρηση 1. *Καθε μη μηδενικος μιγαδικος αριθμος* z *μπορει να γραφτει* $z = |z|e^{2\pi i\theta}$ *για καποιο* $\theta \in \mathbb{R}$*. Το θεμα ειναι οτι αν το* z *εξαρταται κατα συνεχη τροπο απο μια παραμετρο* $t \in [0,1]$*, τοτε (οπως θα δειξουμε με την Προταση 3) το* θ μπορει να επιλεγει *να εξαρταται κατα συνεχη τροπο απο το* t *(παρολο που η απεικονιση* $[0,1] \to \mathbb{T} : \theta \mapsto e^{2\pi i\theta}$ *δεν εχει συνεχη αντιστροφο).*

Παρατήρηση 2. *Οριζουμε*

$$\log z := -\sum_{n=1}^{\infty} \frac{1}{n}(1-z)^n, \quad |1-z| < 1\,.$$

Η σειρα συγκλινει απολυτα στον ανοικτο δισκο $D(1,1) = \{z \in \mathbb{C} : |1-z| < 1\}$ *και οριζει ολομορφη συναρτηση* $\log$ *που ικανοποιει τη σχεση* $\log 1 = 0$ *και* $e^{\log z} = z$ *στον* $D(1,1)$*.* [2]

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[1] Ind26, modified 16 Μαΐου 2026

[2] Verify that $\frac{d}{dz}\left(\frac{e^{\log z}}{z}\right) = 0$ on the open and connected set $D(1,1)$, so the quotient $\frac{e^{\log z}}{z}$ is the constant function $\frac{e^{\log 1}}{1} = 1$.

Πρόταση 3 (απο το βιβλιο του Arveson [Arv02]). *Αν η* $\gamma \in C([0,1])$ *δεν μηδενιζεται πουθενα (ισοδυναμα, αν ειναι αντιστρεψιμο στοιχειο της αλγεβρας* $C([0,1])$*), τοτε υπαρχει (μη μοναδικη)* $g \in C([0,1])$ *ωστε* $\gamma = e^g$.

Απόδειξη. Observe that since $\gamma([0,1])$ is a compact set and $0 \notin \gamma([0,1])$, the quantity $\epsilon := \min\{|\gamma(t)| : t \in [0,1]\}$ is strictly positive.

Special Case Suppose that

$$|\gamma(t) - \gamma(s)| < \epsilon \quad \text{for all } \ t, s \in [0,1].$$

Then

$$\begin{aligned} |\gamma(t)| \left| 1 - \frac{\gamma(s)}{\gamma(t)} \right| &= |\gamma(t) - \gamma(s)| < \epsilon \leq |\gamma(t)| \\ \text{so} \qquad \left| 1 - \frac{\gamma(s)}{\gamma(t)} \right| &< 1 \quad \text{for all } \ t, s \in [0,1]\,. \end{aligned}$$

Thus $g(s) := \log\left(\frac{\gamma(s)}{\gamma(0)}\right)$ is defined (απο την Παρατηρηση 2) and continuous on $[0,1]$ and we see that

$$e^{g(s)} = \frac{\gamma(s)}{\gamma(0)}, \qquad s \in [0,1]\,.$$

General Case Since γ is uniformly continuous, we may find a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ such that

$$|\gamma(t) - \gamma(s)| < \epsilon \quad \text{for all } \ t, s \in [t_{k-1}, t_k] \ \text{ and all } k.$$

By the Special case applied to the interval $[t_{k-1}, t_k]$ for each $k = 1, \dots, n$, there exists $g_k \in C([0,1])$ so that

$$\gamma(t_{k-1}) e^{g_k(s)} = \gamma(s), \qquad s \in [t_{k-1}, t_k]\,.$$

Note that $g_k(t_{k-1}) = \log 1 = 0$.

Now define $g_0 : [0,1] \to \mathbb{C}$ as follows:

$$g_0(s) = \begin{cases} g_1(s), & s \in [0, t_1] \\ g_1(t_1) + g_2(s), & s \in (t_1, t_2] \\ \vdots & \\ g_1(t_1) + g_2(t_2) + \cdots + g_n(s), & s \in (t_{n-1}, 1]\,. \end{cases}$$

Then g_0 is continuous and for each k, if $s \in (t_{k-1}, t_k]$,

$$\begin{aligned} e^{g_0(s)} &= e^{g_1(t_1)} e^{g_2(t_2)} \dots e^{g_k(s)}, \\ &= \frac{\gamma(t_1)}{\gamma(0)} \frac{\gamma(t_2)}{\gamma(t_1)} \cdots \frac{\gamma(s)}{\gamma(t_{k-1})} = \frac{\gamma(s)}{\gamma(0)} \end{aligned}$$

and so, choosing z_0 s.t. $\gamma(0) = e^{z_0}$ (since $\gamma(0) \neq 0$) and putting $g(s) := g_0(s) + z_0$ we obtain

$$e^{g(s)} = e^{g_0(s)} \gamma(0) = \gamma(s), \qquad s \in [0,1]\,.$$

□

Αναφορές

[Arv02] William Arveson, *A short course on spectral theory*, Graduate Texts in Mathematics, vol. 209, Springer-Verlag, New York, 2002. MR 1865513

[Rud87] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR 924157