Μια Ασκηση απο το βιβλιο των Rosenthal & Martínez-Avendaño

Exercise 2.4 (i) *Let $\phi \in \widetilde{H}^\infty$. If M_ϕ is isometric on $\widetilde{H}^2$, show that ϕ is inner.*

• If $\phi \in \widetilde{H}^\infty$ and M_ϕ is isometric then it sends the O.N. family $\{f_n : n \geqslant 0\}$ to an O.N. family. Thus $\langle \phi f_n, \phi f_m \rangle = 0$ for $n, m \in \mathbb{Z}_+,\ n \neq m$.

In particular $\langle \phi f_n, \phi f_0 \rangle = 0$ for all $n \geqslant 1$, δηλαδη $\int \phi f_n \overline{\phi} dm = 0$ for all $n \geqslant 1$, δηλαδη $\int |\phi|^2 f_n dm = 0$ for all $n \geqslant 1$.

Now, taking complex conjugates, you get $\int |\phi|^2 \overline{f_n} dm = 0$ for all $n \geqslant 1$, δηλαδη $\int |\phi|^2 f_{-n} dm = 0$ for all $n \geqslant 1$ δηλαδη $\int |\phi|^2 f_n dm = 0$ for all $n \in \mathbb{Z},\ n \neq 0$.

Thus the function $|\phi|^2$ has all Fourier coeff. $= 0$ except the zeroth, so $|\phi|^2$ is (a.e.) a constant multiple of f_0, δηλαδη a constant. (Πιο αναλυτικα, if $g := |\phi|^2$ then $\hat{g}(n) = 0$ for all $n \in \mathbb{Z}, n \neq 0$, hence the function $g - \hat{g}(0) f_0$ has all Fourier coeff. $= 0$ hence $g - \hat{g}(0) f_0 = 0$ a.e.) Δηλαδη η ϕ ειναι εσωτερικη.

• Το αντιστροφο ειναι ευκολο, αλλα και το εχουμε κανει.

Exercise 2.4 (ii) *Let $\phi \in \widetilde{H}^\infty$. Δειξτε οτι η $\{\mathbf{1}, \phi, \phi^2, \dots\}$ ειναι OK βαση του $\widetilde{H}^2$ αν-ν $\phi(z) = \lambda z,\ \lambda \in \mathbb{T}$.*

• The family $\{f_n : n \geqslant 0\}$ where $f_n(z) = z^n,\ z \in \mathbb{T}$ is an ON basis of $\widetilde{H}^2$; this means that they are O.N and their CLOSED linear span is $\widetilde{H}^2$, equivalently no nonzero element of $\widetilde{H}^2$ is orthogonal to the whole family. The same properties hold if you replace z^n by $\lambda^n z^n$ (since $|\lambda| = 1$). So if $\phi(z) = \lambda z$ then $\{\mathbf{1}, \phi, \phi^2, \dots\}$ is an ON basis of $\widetilde{H}^2$. Αυτη ειναι η τετριμμενη κατευθυνση.

• Αντιστροφα, υποθετουμε οτι η $\{\mathbf{1}, \phi, \phi^2, \dots\}$ is an ON basis of $\widetilde{H}^2$. Since $\phi \in \widetilde{H}^\infty$, M_ϕ is a bounded operator.
Now $M_\phi(\{\mathbf{1}, \phi, \phi^2, \dots\}) = \{\phi, \phi^2, \dots\}$ which is an ON family by hypothesis. It follows that M_ϕ is isometric, [1] so $\overline{\text{span}\{\phi, \phi^2, \dots\}} = M_\phi(\overline{\text{span}\{\mathbf{1}, \phi, \phi^2, \dots\}}) = M_\phi \widetilde{H}^2$.
But since $\{\mathbf{1}, \phi, \phi^2, \dots\}$ is an ON basis of $\widetilde{H}^2$, the closed subspace spanned by $\{\phi, \phi^2, \dots\}$ is the orthogonal complement of $\mathbf{1}$. Similarly, the closed subspace spanned by $\{f_1, f_2, \dots\} = \{f_1, f_1^2, \dots\}$ is also the orthogonal complement of $\mathbf{1}$. So these spaces are equal. Thus $\phi \widetilde{H}^2 = f_1 \widetilde{H}^2$. By uniquness in Beurling's theorem, there is a constant $\lambda \in \mathbb{T}$ s.t. $\phi = \lambda f_1$ i.e $\phi(z) = \lambda z,\ z \in \mathbb{T}$, οπως θελαμε.

Σχολιο: Η ακολουθη γενικοτερη ασκηση ηταν η 4η ασκηση στο 3ο φυλλαδιο ασκησεων:

Άσκηση 1. Εστω $\phi \in L^\infty$. (α) Δειξτε οτι ο τελεστης $T_\phi := PM_\phi|_{\widetilde{H}^2}$ ειναι ισομετρια αν-ν η ϕ ειναι εσωτερικη συναρτηση.
(β) Δειξτε οτι ο T_ϕ ειναι unitary (= ισομετρια και επι) αν-ν η ϕ ειναι (σ.π.) σταθερη.
(γ) Αν $\phi \in \widetilde{H}^\infty$, δειξτε οτι η $\{1, \phi, \phi^2, \dots\}$ ειναι ορθοκανονικη βαση του $\widetilde{H}^2$ αν-ν $\phi(z) = \lambda z$ οπου $\lambda \in \mathbb{T}$ σταθερα.

[1] preserves norm and sc. product on an ON basis hence (linearity and continuity) on the whole space