Το γινόμενο Blaschke

Θεώρημα 1. *Εστω*[1] (z_k) *ακολουθια στο* $\mathbb{D}\backslash\{0\}$ *με* $\sum_{i=1}^{\infty}(1-|z_i|)<\infty$ *και* $s\in\mathbb{Z}_+$. *Τοτε υπαρχει μια εσωτερικη συναρτηση* B *που εχει συνολο ριζων* $Z(B)=\{0\}\cup\{z_k : k\in\mathbb{N}\}$ *και η ριζα* $z=0$ *εχει πολλαπλοτητα* s. *Για καθε* $z\in\mathbb{D}$, *εχουμε*

$$B(z)=z^s\prod_{i=1}^{\infty}\frac{\bar{z}_i}{|z_i|}\frac{z_i-z}{1-\bar{z}_i z}\ .$$

οπου το απειρογινομενο συγκλινει ομοιομορφα στα συμπαγη του $\mathbb{D}$.

Επομενως, η συνθηκη $z_n\in\mathbb{D}\backslash\{0\}$ με $\sum_{i=1}^{\infty}(1-|z_i|)<\infty$ ειναι ικανη και αναγκαια για την υπαρξη $f\in H^2$ (μαλιστα, εσωτερικης) με ακριβως αυτες τις ριζες.

Σχολιο. Η συγκλιση του απειρογινομενου σημαινει οτι, για καθε συμπαγες υποσυνολο K του $\mathbb{D}$, μονον πεπερασμενο πληθος ορων του απειρογινομενου εχει ριζες στο K και οτι το απειρογινομενο που αποτελειται απο τους υπολοιπους ορους συγκλινει ομοιομορφα στο συμπαγες K σε μια ολομορφη συναρτηση που δεν εχει καμμια ριζα στο K.

Απόδειξη. Αρκει να δειξουμε την υπαρξη εσωτερικης συναρτησης B_0 με συνολο ριζων $Z(B_0)=\{z_k : k\in\mathbb{N}\}$ και να θεσουμε $B(z):=z^sB_0(z)$.

Let ϕ_i be the inner function

$$\phi_i(z):=\frac{\bar{z}_i}{|z_i|}\frac{z_i-z}{1-\bar{z}_i z}$$

and let

$$B_n(z):=\prod_{k=1}^{n}\phi_k(z)$$

which is an inner function with roots $Z(B_n)=\{z_1,\dots,z_n\}$.

Claim 1 The sequence (B_n) is Cauchy in the Banach space H^2.

Proof of Claim 1 For $n>m$, we have

$$\begin{aligned}\|B_n-B_m\|^2=\left\|\tilde{B}_n-\tilde{B}_m\right\|^2&=\frac{1}{2\pi}\int_{-\pi}^{\pi}|\tilde{B}_n(e^{it})-\tilde{B}_m(e^{it})|^2\,dt\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}(\tilde{B}_n(e^{it})-\tilde{B}_m(e^{it}))\overline{(\tilde{B}_n(e^{it})-\tilde{B}_m(e^{it}))}\,dt\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(|\tilde{B}_n(e^{it})|^2+|\tilde{B}_m(e^{it})|^2-2\operatorname{Re}(\tilde{B}_n(e^{it})\overline{\tilde{B}_m(e^{it})})\right)\,dt\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1+1-2\operatorname{Re}\left(\frac{\tilde{B}_n(e^{it})}{\tilde{B}_m(e^{it})}\right)\right)\,dt\end{aligned}$$

[1]blaschke, 25 Νοεμβρίου 2024

since $|\tilde{B}_n(e^{it})| = 1$ a.e., and so $\overline{\tilde{B}_n(e^{it})} = \frac{1}{\tilde{B}_n(e^{it})}$ a.e. Notice now that $B_n(z) = B_m(z)\prod_{k=m+1}^n \phi_k(z)$ and so $\frac{B_n}{B_m}$ is a well defined inner function. Hence by the Poisson integral formula (at $r = 0$) we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\tilde{B}_n(e^{it})}{\tilde{B}_m(e^{it})}\,dt = \frac{B_n(0)}{B_m(0)}\,.$$

Now

$$\frac{B_n(0)}{B_m(0)} = \prod_{k=m+1}^{n} \phi_k(0) = \prod_{k=m+1}^{n} \frac{\bar{z}_k}{|z_k|}\frac{z_k}{1-0} = \prod_{k=m+1}^{n} |z_k|$$

and so

$$\|B_n - B_m\|^2 = 2 - 2\prod_{k=m+1}^{n} |z_k|\,.$$

Since $\sum\limits_{i=1}^{\infty}(1-|z_i|) < \infty$, we know that the infinite product $\prod_{k=1}^{\infty}|z_k|$ converges to a nonzero number p. It is easy to see that this implies that the quotient $\prod_{k=m+1}^{n}|z_k| = \dfrac{\prod_{k=1}^{n}|z_k|}{\prod_{k=1}^{m}|z_k|}$ can be made arbitrarily close to 1, making $\|B_n - B_m\|$ arbitrarily small. [2]

This proves Claim 1.

Therefore there exists $B_0 \in H^2$ such that $\lim_n \|B_n - B_0\|_{H^2} = 0$.

The fact that $\lim_n \left\|\tilde{B}_n - \tilde{B}_0\right\|_{\tilde{H}^2} = 0$ (i.e. in the norm of $L^2(\mathbb{T})$) implies in particular that there is a subsequence (k_n) so that $\lim_n \tilde{B}_{k_n}(e^{it}) = \tilde{B}_0(e^{it})$ a.e., and so $|\tilde{B}_0(e^{it})| = \lim_n |\tilde{B}_{k_n}n(e^{it})| = 1$ a.e. since the B_n are inner functions.

Thus B_0 is an inner function.

Claim 2 The infinite product $\prod_{k=1}^{\infty}\phi_k$ converges (to B_0) uniformly on compact subsets of $\mathbb{D}$.

Proof of Claim 2 Take any $r \in (0,1)$, so that $\bar{B}(0,r) \subset \mathbb{D}$.

Firstly, the condition $\sum\limits_{i=1}^{\infty}(1-|z_i|) < \infty$ implies that $|z_n| \to 1$, hence at most finitely many of the z_n can be in $\bar{B}(0,r)$.

• Suppose first that no z_n is in the closed ball $\bar{B}(0,r)$. Then no $\phi_n(z)$ has any roots in $B(0,r)$, and the same holds for the B_n. Since $B_n(z) \to B_0(z)$ uniformly on compact sets

[2] Writing $p_n := \prod_{k=1}^{n}|z_k|$ we have $\lim \frac{p_n}{p} = 1$ hence $\lim \frac{p}{p_n} = 1$. Thus given $\epsilon \in (0,1)$ there exists n_0 such that for $n > m > n_0$ we have $1-\epsilon < \frac{p_n}{p} < 1+\epsilon$ and also $1-\epsilon < \frac{p}{p_m} < 1+\epsilon$ hence $(1-\epsilon)^2 < \frac{p_n}{p_m} < (1+\epsilon)^2$ and so $1 - \frac{p_n}{p_m} < 2\epsilon - \epsilon^2 < 2\epsilon$. Therefore

$$\|B_n - B_m\|^2 = 2 - 2\prod_{k=m+1}^{n}|z_k| < 4\epsilon\,,$$

showing that (B_n) is Cauchy.

of B(0,r) and B_0 is not identically zero on $B(0,r)$, [3] by Hurwitz's Theorem B_0 has no roots in $B(0,r)$. In other words, for any $z \in B(0,r)$, the partial products $B_n(z)$ converge to the *nonzero* complex number $B_0(z)$, which means exactly that $\prod_{k=1}^{\infty} \phi_k(z) = B_0(z)$. The convergence is uniform on compact sets of $B(0,r)$, as observed earlier.

• Suppose now that the roots $\{z_1, \dots, z_N\}$ are in $\bar{B}(0,r)$ and $|z_m| > r$ for $m > N$. We repeat the same argument for the sequence $(\phi_n)_{n>N}$: the partial products $B_n'(z) := \prod_{k=N+1}^{n} \phi_k(z)$ have no roots in $B(0,r)$ hence their limit $B_0'(z) := \lim_n B_n'(z)$ (which, as before, is not identically zero, being the limit of inner functions) has no roots in $B(0,r)$. This shows that $B_0'(z) = \prod_{k=N+1}^{\infty} \phi_k(z)$ exists and is a nonzero complex number for any $z \in B(0,r)$.

It follows that

$$B_0(z) = \left(\prod_{k=1}^{N} \phi_k(z) \right) B_0'(z) = \left(\prod_{k=1}^{N} \phi_k(z) \right) \left(\prod_{k=N+1}^{\infty} \phi_k(z) \right) = \prod_{k=1}^{\infty} \phi_k(z)$$

converges for all $z \in B(0,r)$, uniformly on compact sets, and vanishes exactly at the points $\{z_1, \dots, z_N\}$.

Finally, let K be a compact subset of $\mathbb{D}$. There exists $r \in (0,1)$ so that $K \subset B(0,r) \subseteq \bar{B}(0,r) \subseteq \mathbb{D}$. By the previous arguments, the infinite product converges to $B_0(z)$ uniformly on K.

The Claim is proved.

It follows that B_0 cannot vanish at any $z_0 \in \mathbb{D} \setminus \{z_1, \dots, z_n, \dots\}$; for such a z_0 would lie in some ball $B(0,r)$, and we have just seen that the roots of B_0 in $B(0,r)$ must belong to $\{z_1, \dots, z_n, \dots\}$. Since conversely $B_0(z_k) = 0$ for any k (because $B_n(z_k) = 0$ for all $n \geq k$, we have shown that

$$Z(B_0) = \{z_1, \dots, z_n, \dots\}.$$

Finally we show that the multiplicity of each $\hat{z} \in Z(B)$ is exactly the number of factors $s_{\hat{z}}$ in which $\hat{z}$ occurs. (This number is of course finite: any root of a nonzero holomorphic function must have finite multiplicity.) Writing

$$B_0(z) = \left(\prod_{z_k = \hat{z}} \phi_k(z) \right) \left(\prod_{z_k \neq \hat{z}} \phi_k(z) \right)$$

it is clear that the multiplicity of $\hat{z}$ is at least the number $s_{\hat{z}}$ of factors appearing in the first term. To show that it cannot exceed that number, note that the second term of the product does not vanish at $\hat{z}$, by the earlier argument applied to the sequence $\{z_n\}$ with the (finitely many) terms for which $z_k = \hat{z}$ removed. Thus the multiplicity of $\hat{z}$ as a root of B_0 is exactly the number of terms occuring in the first term. □

[3] otherwise it would be identically zero on $\mathbb{D}$ (by the identity principle) which would contradict the fact its boundary function is nonzero