Μια Ασκηση

Άσκηση 1. Αν ενας φραγμενος τελεστης $A : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ ικανοποιει την ιδιοτητα, καθε S-αναλλοιωτος κλειστος υποχωρος του $\ell^2(\mathbb{Z}_+)$ να ειναι A-αναλλοιωτος, τοτε $SA = AS$.
Ισχυει και το αντιστροφο.

[Υποδειξη: Εξεταστε τους συζυγεις τελεστες.]

Πρωτη Αποδειξη. Δουλευουμε στον $\ell^2(\mathbb{Z}_+)$.

Recall that a bounded operator leaves a closed subspace E invariant if-f its adjoint leaves $E^\perp$ invariant.

Also recall that for every $z \in \mathbb{D}$, if $x_z := (1, z, z^2, \dots) \in \ell^2(\mathbb{Z}_+)$ we have $S^* x_z = z x_z$. Thus the closed subspace $E_z = \mathrm{span}\{x_z\}$ is S^* invariant, and hence $E_z^\perp$ is S-invariant.

By the assumption, $E_z^\perp$ is A-invariant, and hence $E_z = E_z^{\perp\perp}$ is A^*-invariant.

Hence $A^* x_z \in \mathrm{span}\{x_z\}$, i.e. there exists $w_A \in \mathbb{C}$ s.t. $A^* x_z = w_A x_z$.

Thus we have

$$\begin{aligned} A^*S^*(x_z) &= A^*(zx_z) = w_A z x_z \\ \text{and} \quad S^*A^*(x_z) &= S^*(w_A x_z) = z w_A x_z \,. \end{aligned}$$

It follows that

$$(A^*S^* - S^*A^*)(x_z) = 0$$

for all $z \in \mathbb{D}$. But we know that the closed linear span of $\{x_z : z \in \mathbb{D}\}$ is dense in $\ell^2(\mathbb{Z}_+)$. Therefore we have $A^*S^* - S^*A^* = 0$ and so $SA - AS = 0$ οπως θελαμε. □

Δευτερη Αποδειξη. Δουλευουμε στον H^2.

For every $z \in \mathbb{D}$ the subspace $F_z := \{f \in H^2 : f(z) = 0\}$ is S-invariant (i.e. T_1-invariant).

By the assumption, F_z is A-invariant, and hence $F_z^\perp$ is A^*-invariant. But recall that $F_z = \{k_z\}^\perp$ where k_z is the Szegő kernel, $k_z(w) = \sum_{k=0}^{\infty} \bar{z}^k w^k$ (indeed for every $g \in H^2$ we have $g(z) = \langle g, k_z \rangle$), so $F_z^\perp = \mathrm{span}\{k_z\}$).

Hence $A^* k_z \in \mathrm{span}\{k_z\}$, i.e. there exists $u_A \in \mathbb{C}$ s.t. $A^* k_z = u_A k_z$.

Also note that $S^* k_z = \bar{z} k_z$.

Thus we have

$$\begin{aligned} A^*S^*(k_z) &= A^*(\bar{z}k_z) = u_A \bar{z} k_z \\ \text{and} \quad S^*A^*(k_z) &= S^*(u_A k_z) = \bar{z} u_A k_z \,. \end{aligned}$$

It follows that

$$(A^*S^* - S^*A^*)(k_z) = 0$$

for all $z \in \mathbb{D}$. Hence for all $f \in H^2$ we have

$$\langle (SA - AS)f, k_z \rangle = \langle f, (A^*S^* - S^*A^*)(k_z) \rangle = 0$$

for all $z \in D$.
Thus the function $g = (SA - AS)f \in H^2$ satisfies $g(z) = \langle g, k_z \rangle = 0$ for all $z \in \mathbb{D}$, i.e. $g = 0$.

We have shown that $(SA - AS)f = 0$ for all $f \in H^2$, i.e. that $SA - AS = 0$, οπως θελαμε.

Για το αντιστροφο: εχουμε δειξει οτι αν $SA = AS$, δηλαδη $T_1 A = A T_1$, τοτε $A = T_\phi$ για καποια $\phi \in H^\infty$. Ομως καθε μη μηδενικος T_1-αναλλοιωτος υποχωρος του H^2 ειναι της μορφης ψH^2 οπου ψ εσωτερικη. Και προφανως καθε ψH^2 ειναι T_ϕ-αναλλοιωτος, αφου για καθε $\psi f \in \psi H^2$ εχουμε $T_\phi(\psi f) = \phi(\psi f) = \psi(\phi f) \in \psi H^2$.

Αρα, αν $SA = AS$ τοτε καθε S-αναλλοιωτος υποχωρος ειναι A-αναλλοιωτος. □