The spectrum of a Toeplitz operator with continuous symbol

Παρατήρηση 1. *Στο αρχειο (Ind24) θα δειξουμε οτι ο δεικτης στροφης επεκτεινεται σε ολες τις συνεχεις κλειστες καμπυλες* $\gamma : [0,1] \to \mathbb{C}$. *Αν* $\phi \in C(\mathbb{T})$, *γραφουμε* $n(\phi;\lambda)$ *για τον δεικτη στροφης της* $\gamma_\phi(t) := \phi(e^{2\pi i t})$, $t \in [0,1]$.

Θεώρημα 2. *Αν* $\phi \in C(\mathbb{T})$, *τοτε*

$$\sigma(T_\phi) = \operatorname{ran}(\phi) \ \cup \{\lambda \in \mathbb{C} \setminus \operatorname{ran}(\phi) : n(\phi;\lambda) \neq 0\}\,.$$

Απόδειξη. Observe that since $\phi \in C(\mathbb{T})$ we have $\operatorname{ran}(\phi) = \operatorname{essran}(\phi)$.

Hence by spectral inclusion,

$$\operatorname{ran}(\phi) = \sigma(M_\phi) \subseteq \sigma(T_\phi)\,.$$

Thus we have to show that if $\lambda \in \mathbb{C} \setminus \operatorname{ran}(\phi)$, then

$$\lambda \notin \sigma(T_\phi) \iff n(\phi;\lambda) = 0$$

$$\text{Equivalently: } T_{\phi-\lambda} \text{ invertible} \iff n(\phi-\lambda;0) = 0.$$

Thus replacing ϕ with $\phi - \lambda$ we have to show that if $0 \notin \operatorname{ran}(\phi)$ (i.e. if ϕ vanishes nowhere in $\mathbb{T}$, i.e. if $\min|\phi(e^{it})| > 0$), then

$$T_\phi \text{ invertible} \iff n(\phi;0) = 0.$$

• *Case (1)* Suppose $\phi = f_k$ for some $k \in \mathbb{Z}_+$:

Here $T_\phi = T_1^k$ is invertible iff $k = 0$, while $n(\phi;0) = k$ (στροφες γυρω απ το 0) vanishes iff $k = 0$. □

• *Case (2)* Suppose ϕ is an (analytic) polynomial $\phi = \sum_{j=0}^{n} c_j f_j$:

Extending ϕ to $\mathbb{C}$, consider

$$q(z) = \sum_{j=0}^{n} c_j z^j\,.$$

The polynomial q factors:

$$q(z) = cz^m \prod_{j=1}^{k}(z - z_j) \prod_{j=1}^{\ell}(z - w_j) \quad 0 < |z_j| < 1 < |w_j|$$

for some $c \in \mathbb{C}$, where $m \geq 0$ is the multiplicity of the root of q at 0, $\{z_j\}$ are the non-zero roots of q in $\mathbb{D}$ and $\{w_j\}$ are the roots of q outside $\overline{\mathbb{D}}$ (observe that q has no roots in $\mathbb{T}$ because ϕ never vanishes on $\mathbb{T}$!).

Then we have

$$\phi(e^{it}) = q(e^{it}) = ce^{imt}\prod_{j=1}^{k}(e^{it}-z_j)\prod_{j=1}^{\ell}(e^{it}-w_j)$$

$$\text{so}\quad \phi = cf_{m+k}\prod_{j=1}^{k}(1-z_j\bar{f}_1)\prod_{j=1}^{\ell}(f_1-w_j)$$

$$= c\bar{u}vf_{m+k}$$

where $u := \prod_{j=1}^{k}(1-\bar{z}_j f_1)$ and $v := \prod_{j=1}^{\ell}(f_1-w_j)$.

Since the functions v and f_{m+k} are in $\widetilde{H}^\infty$ we have

$$T_\phi = T_{c\bar{u}vf_{m+k}} = cT_{\bar{u}vf_{m+k}} = cT_{\bar{u}}T_{vf_{m+k}} = cT_{\bar{u}}T_vT_{f_{m+k}}\,.$$

But the function u (extended to $\mathbb{C}$) vanishes nowhere in $\overline{\mathbb{D}}$ (u has roots $1/\bar{z}_j$ which lie outside $\overline{\mathbb{D}}$). Hence the Toeplitz operator $T_{\bar{u}} = T_u^*$ is invertible because T_u is invertible. [1] Also the function v vanishes nowhere in $\overline{\mathbb{D}}$ and hence T_v is invertible.

Thus T_ϕ is invertible iff $T_{f_{m+k}} = T_1^{m+k}$ is invertible which happens iff $m+k=0$. ($\iff$ q has no roots in $\mathbb{D}$.)

On the other hand, since the roots of $\bar{u}$ and v lie outside $\overline{\mathbb{D}}$, we have $n(\bar{u};0)=0$ and $n(v;0)=0$. By (Ind24) [2]

$$n(\phi;0) = n(\bar{u};0) + n(v;0) + n(f_{m+k};0) = m+k\,.$$

so $n(\phi;0)=0$ iff $m+k=0$ iff T_ϕ is invertible. □

- *Case (3)* Suppose ϕ is a trigonometric polynomial $\phi = \sum_{j=-n}^{n} c_j f_j$:

In this case we shift n places to the right to obtain an analytic polynomial: define

$$q(z) = \sum_{j=-n}^{n} c_j z^{n+j}\,.$$

Thus $q(e^{it}) = \sum_{j=-n}^{n} c_j e^{it(n+j)} = e^{int}\phi(e^{it})$ for $e^{it}\in\mathbb{T}$, i.e. $\phi = \tilde{q}f_{-n}$ (note $\tilde{q} = q|_{\mathbb{T}}$). Using the argument and notation of Case (2), we may write

$$\phi = c\bar{u}vf_{m+k-n}\,.$$

Suppose that $m+k-n\geq 0$. Then f_{m+k-n} is in $\widetilde{H}^\infty$ and the same argument as in Case (2) yields that T_ϕ is invertible iff $m+k-n=0$ and that $n(\phi;0)=m+k-n$ which vanishes iff $m+k-n=0$, completing the proof.

Suppose that $m+k-n\leq 0$. Then $\bar{f}_{m+k-n}$ is in $\widetilde{H}^\infty$ and if we write $\phi = cf_{m+k-n}\bar{u}v$ we see that the function $\bar{f}_{m+k-n}u$ is in $\widetilde{H}^\infty$ hence we have the factorization

$$T_\phi = cT_{f_{m+k-n}\bar{u}v} = cT_{f_{m+k-n}\bar{u}}T_v = cT_{f_{m+k-n}}T_{\bar{u}}T_v\,.$$

Thus, as before we see that T_ϕ is invertible iff $T_{f_{m+k-n}}$ is invertible which happens iff $m+k-n=0$, iff $n(\phi;0)$ vanishes. □

[1] $T_uT_{u^{-1}} = T_{uu^{-1}} = I = T_{u^{-1}}T_u$ since both u and u^{-1} are analytic.

[2] αποδειξη στο αρχειο (Ind24)

- *Case (4)* (general) Choose a trigonometric polynomial $p \in C(\mathbb{T})$ such that

$$\|\phi - p\| < \frac{1}{3} \min |\phi|$$

and note that p never vanishes ($|p| > \|\phi\| - \frac{1}{3} \min |\phi| \geq \frac{2}{3} \min |\phi|$) and

$$\psi := \frac{\phi - p}{p} \quad \text{satisfies} \quad \|\psi\| < \frac{\frac{1}{3} \min |\phi|}{\frac{2}{3} \min |\phi|} = \frac{1}{2}\,.$$

It follows that $u = \log(1 + \psi)$ is a well defined continuous function and

$$1 + \psi = e^u\,.$$

(indeed, the series $\log(1+\psi) = -\sum_{n=1}^{\infty} \frac{1}{n}(-\psi)^n$, converges because $\|1 - (1+\psi)\| < 1$.) [3]
Therefore (from (`Ind`))

$$n(1 + \psi; 0) = 0$$

and hence, since $\phi = p(1 + \psi)$,

$$n(\phi; 0) = n(p; 0) + n(1 + \psi; 0) = n(p; 0)\,.$$

Since $1+\psi$ is analytic and invertible the relation $T_\phi = T_p T_{(1+\psi)}$ shows that T_ϕ is invertible if and only if T_p is invertible. By Case (3), this happens iff $n(p; 0) = 0$, equivalently iff $n(\phi; 0) = 0$ and we are done. □

[3]Λεπτομερειες στο αρχειο (`Ind24`)