Σχολια σε μερικες ασκησεις

(Απο τη συζητηση στην ταξη για το φυλλαδιο III)

Άσκηση 4. Εστω $\phi \in L^\infty$.
(α) Δειξτε οτι ο τελεστης $T_\phi := PM_\phi|_{\widetilde{H}^2}$ ειναι ισομετρια αν-ν η ϕ ειναι εσωτερικη συναρτηση.
(β) Δειξτε οτι ο T_ϕ ειναι unitary (= ισομετρια και επι) αν-ν η ϕ ειναι (σ.π.) σταθερη.
(γ) Αν $\phi \in \widetilde{H}^\infty$, δειξτε οτι η $\{1, \phi, \phi^2, \dots\}$ ειναι ορθοκανονικη βαση του $\widetilde{H}^2$ αν-ν $\phi(z) = \lambda z$ οπου $\lambda \in \mathbb{T}$ σταθερα.

Απόδειξη. (α) If ϕ is inner, clearly T_ϕ is isometric:

$$\int |\phi(e^{it})|^2 |f(e^{it})|^2 dt = \int |f(e^{it})|^2 dt \quad \forall f \in \widetilde{H}^2 .$$

If conversely T_ϕ is isometric, then we must have $\|P(\phi f)\|_2 = \|f\|_2$ for every $f \in \widetilde{H}^2$ and in particular $\|P(\phi \mathbf{1})\|_2 = \|\mathbf{1}\|_2 = 1$. This forces $\phi \in \widetilde{H}^\infty$: for if there were $k > 0$ with $\hat{\phi}(-k) \neq 0$ then

$$\|P(\phi f_k)\|_2^2 = \sum_{n \geq 0} |\hat{\phi}(n-k)|^2 > \sum_{n \geq k} |\hat{\phi}(n-k)|^2 = \|P(\phi)\|_2^2 = 1$$

whereas $\|P(\phi f_k)\|_2^2 = \|f_k\|_2^2 = 1$.

It follows that $\|\phi\|_2 = 1$ and $\|\phi^2\|_2 = \|T_\phi(\phi)\|_2 = \|\phi\|_2 = 1$ and inductively $\|\phi^n\|_2 = 1$ for every $n \in \mathbb{N}$.

We claim that $|\phi| \leq 1$ a.e.: Indeed, for $\epsilon > 0$ put $A_\epsilon : \{t \in [0, 2\pi] : |\phi(e^{it}|| \geq 1 + \epsilon\}$ and observe that

$$1 = \|\phi^n\|_2^2 = \frac{1}{2\pi} \int |\phi(e^{it})|^2 dt \geq \frac{1}{2\pi} \int_{A_\epsilon} |\phi(e^{it})|^2 dt \geq \frac{1}{2\pi} m(A_\epsilon)(1+\epsilon)^n$$

which forces $m(A_\epsilon) = 0$ since $\lim_n (1+\epsilon)^n = \infty$. Thus the set $\{t \in [0, 2\pi] : |\phi(e^{it}|| > 1\} = \cup_{n \in \mathbb{N}} A_{1/n}$ must have measure zero.

Finally, since $|\phi| \leq 1$ a.e and $\|\phi\|_2 = 1$, we must have $|\phi| = 1$ a.e, for if $|\phi| < 1$ on a set of positive measure it would follow that $\frac{1}{2\pi} \int |\phi(e^{it})|^2 dt < 1$. Thus ϕ is inner.

Alternatively: Having shown that $\phi \in \widetilde{H}^\infty$ we now see that if the map T_ϕ is isometric then ti must map the orthonormal sequence $\{f_n : n \geq 0\}$ to an orthonormal sequence. Thus $\{\phi f_n : n \geq 0\}$ is ortonormal and in particular

$$0 = \langle \phi f_n, \phi f_0 \rangle = \frac{1}{2\pi} \int \phi(e^{it}) e^{int} \bar{\phi}(e^{it}) dt = \frac{1}{2\pi} \int |\phi(e^{it})|^2 e^{int} dt \qquad \text{for all } n > 0 .$$

Taking complex conjugates, we also have

$$\frac{1}{2\pi} \int |\phi(e^{it})|^2 e^{-int} dt \qquad \text{for all } n > 0 .$$

Thus the Foureir coefficients $\hat{g}(k)$ of the function $g := |\phi|^2$ vanish for all $k \neq 0$, and hence g is a constant ($= \hat{g}(0)$) a.e., showing that ϕ is inner. □

(β) If T_ϕ is an isometry and onto, then ϕ is inner, as just shown; also T_ϕ is onto, so there must be $f \in \widetilde{H}^2$ so that $T_\phi(f) = \mathbf{1}$, i.e. $\phi f = \mathbf{1}$. Thus $\frac{1}{\phi}$ must be in $\widetilde{H}^2$. But since $|\phi| = 1$ a.e., we have $\frac{1}{\phi} = \overline{\phi}$. Thus the function ϕ and its complex conjugate must both be in $\widetilde{H}^2$, hence ϕ must be constant.

Το αντιστροφο ειναι βεβαια προφανες. □

(γ) If the sequence $\{\phi^n : n \geq 0\}$ is orthonormal, then $\|\phi^n\|_2 = 1$ for all $n \geq 0$, which, as we saw in the proof of (α) shows that ϕ must be inner and so T_ϕ must be an isometry.

Now since $\{\mathbf{1}, \phi, \phi^2, \dots\}$ is an orthonormal basis of $\widetilde{H}^2$, the sequence $\{\phi, \phi^2, \dots\}$ is an orthonormal basis of the space $\{\mathbf{1}\}^\perp$ which is the closed linear span of $\{f_1, f_2, \dots\}$, i.e. the space $f_1\widetilde{H}^2$. But $\{\phi, \phi^2, \dots\} = T_\phi(\{\mathbf{1}, \phi, \phi^2, \dots\})$ and so

$$f_1\widetilde{H}^2 = \overline{\text{span}}(\{\phi, \phi^2, \dots\}) = T_\phi(\overline{\text{span}}(\{\mathbf{1}, \phi, \phi^2, \dots\}) = T_\phi(\widetilde{H}^2) = \phi\widetilde{H}^2$$

which, by the uniqueness in Beurling's Theorem, shows that $\phi = \lambda f_1$, for some constant λ.

Το αντιστροφο ειναι παλι προφανες: η $\{\mathbf{1}, (\lambda f_1), (\lambda f_1)^2, \dots\} = \{\mathbf{1}, \lambda f_1, \lambda^2 f_2, \dots\}$ ειναι ορθοκανονικη βαση του $\widetilde{H}^2$. □

Σχολιο Σε εναν χωρο πιθανοτητας (οπως στο $[0,1]$ με το μετρο Lebesgue) οι νορμες $\|\cdot\|_p$ για $p = 1, 2, \dots$ αποτελουν αυξουσα ακολουθια, και μια μετρησιμη συναρτηση f ειναι ουσιωδως φραγμενη αν-ν $\sup_p \|f\|_p < \infty$, οποτε $\sup_p \|f\|_p = \|f\|_\infty$. Η αποδειξη γινεται οπως στο (α) της Ασκησης.
Δεν αρκει απλως να εχουμε $\|f\|_p < \infty$ για καθε p. Η τομη ολων των L^p ειναι ενας ενδιαφερων γραμμικος χωρος, γνησιως μεγαλυτερος απο τον L^∞.[1]

Άσκηση 7. Αν ενας φραγμενος τελεστης $X : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ ικανοποιει την ιδιοτητα, καθε S-αναλλοιωτος κλειστος υποχωρος του $\ell^2(\mathbb{Z}_+)$ να ειναι X-αναλλοιωτος, τοτε $SX = XS$.

[Υποδειξη: Εξεταστε τους συζυγεις τελεστες.]

Πρωτη Αποδειξη. Δουλευουμε στον $\ell^2(\mathbb{Z}_+)$.

Recall that a bounded operator leaves a closed subspace E invariant if-f its adjoint leaves $E^\perp$ invariant.

Also recall that for every $z \in \mathbb{D}$, if $x_z := (1, z, z^2, \dots) \in \ell^2(\mathbb{Z}_+)$ we have $S^* x_z = z x_z$. Thus the closed subspace $E_z = \text{span}\{x_z\}$ is S^* invariant, and hence $E_z^\perp$ is S-invariant.

By the assumption, $E_z^\perp$ is X-invariant, and hence $E_z = E_z^{\perp\perp}$ is X^*-invariant.

Hence $X^* x_z \in \text{span}\{x_z\}$, i.e. there exists $w_X \in \mathbb{C}$ s.t. $X^* x_z = w_X x_z$.

Thus we have

$$\begin{aligned} X^*S^*(x_z) &= X^*(z x_z) = w_X z x_z \\ \text{and} \quad S^*X^*(x_z) &= S^*(w_X x_z) = z w_X x_z\,. \end{aligned}$$

It follows that

$$(X^*S^* - S^*X^*)(x_z) = 0$$

for all $z \in \mathbb{D}$. But we know that the closed linear span of $\{x_z : z \in \mathbb{D}\}$ is dense in $\ell^2(\mathbb{Z}_+)$. Therefore we have $X^*S^* - S^*X^* = 0$ and so $SX - XS = 0$ οπως θελαμε. □

Δευτερη Αποδειξη. Δουλευουμε στον H^2.

For every $z \in \mathbb{D}$ the subspace $F_z := \{f \in H^2 : f(z) = 0\}$ is S-invariant (i.e. T_1-invariant).

By the assumption, F_z is X-invariant, and hence $F_z^\perp$ is X^*-invariant. But recall that $F_z = \{k_z\}^\perp$ where k_z is the Szegő kernel, $k_z(w) = \sum_{k=0}^\infty \bar{z}^k w^k$ (indeed for every $g \in H^2$ we have $g(z) = \langle g, k_z \rangle$), so $F_z^\perp = \text{span}\{k_z\}$.

Hence $X^* k_z \in \text{span}\{k_z\}$, i.e. there exists $u_X \in \mathbb{C}$ s.t. $X^* k_z = u_X k_z$.

Also note that $S^* k_z = \bar{z} k_z$.

[1] Ενα παραδειγμα ειναι η συναρτηση $f(x) = \log x$, $x \in (0,1]$, ενα αλλο η συναρτηση $g : (0,1] \to \mathbb{R}$ οπου για καθε $n \in \mathbb{N}$ οταν $x \in (2^{-(n+1)}, 2^{-n}]$ οριζουμε $g(x) = n$: και οι δυο ανηκουν σε ολους τους L^p, $p \in [1, \infty)$ αλλα δεν ειναι φραγμενες.

Thus we have

$$\begin{aligned} X^*S^*(k_z) &= X^*(\bar{z}k_z) = u_X\bar{z}k_z \\ \text{and} \quad S^*X^*(k_z) &= S^*(u_Xk_z) = \bar{z}u_Xk_z\,. \end{aligned}$$

It follows that

$$(X^*S^* - S^*X^*)(k_z) = 0$$

for all $z \in \mathbb{D}$. Hence for all $f \in H^2$ we have

$$\langle (SX - XS)f, k_z \rangle = \langle f, (X^*S^* - S^*X^*)(k_z) \rangle = 0$$

for all $z \in D$.
Thus the function $g = (SX - XS)f \in H^2$ satisfies $g(z) = \langle g, k_z \rangle = 0$ for all $z \in \mathbb{D}$, i.e. $g = 0$.

We have shown that $(SX - XS)f = 0$ for all $f \in H^2$, i.e. that $SX - XS = 0$, οπως θελαμε. □

Για το αντιστροφο: εχουμε δειξει οτι αν $SX = XS$, δηλαδη $T_1X = XT_1$, τοτε $X = T_\phi$ για καποια $\phi \in H^\infty$. Ομως καθε μη μηδενικος T_1-αναλλοιωτος υποχωρος του H^2 ειναι της μορφης ψH^2 οπου ψ εσωτερικη. Και προφανως καθε ψH^2 ειναι T_ϕ-αναλλοιωτος, αφου για καθε $\psi f \in \psi H^2$ εχουμε $T_\phi(\psi f) = \phi(\psi f) = \psi(\phi f) \in \psi H^2$.

Αρα, αν $SX = XS$ τοτε καθε S-αναλλοιωτος υποχωρος ειναι X-αναλλοιωτος.