Σημειωσεις στους τελεστες Toeplitz

Υπενθυμιση[1]

Θεώρημα 1. *Το συνολο των τελεστων στον $L^2(\mathbb{T})$ που μετατιθενται με τον M_1 ειναι το*

$$\{M_\phi : \phi \in L^\infty(\mathbb{T})\}.$$

Πρόταση 2. *Ο μεταθετης του unilateral shift $T_1 = M_1|_{\widetilde{H}^2}$ στον $\widetilde{H}^2$ ειναι το συνολο των αναλυτικων τελεστων Toeplitz: Αν $A \in \mathcal{B}(\widetilde{H}^2)$, τοτε*

$$AT_1 = T_1 A \quad \Longleftrightarrow \quad A = T_\phi \text{ για καποιο } \phi \in \widetilde{H}^\infty.$$

Σχεδιο Αποδειξης. Let $A \in \mathcal{B}(\widetilde{H}^2)$ be such that $AT_1 = T_1 A$.

Define $\phi := A(f_0) \in \widetilde{H}^2$.

- The hypothesis $AT_1 = T_1 A$ implies that $A(f_n) = \phi f_n \ \forall n \in \mathbb{Z}_+$, hence $A(p) = \phi p$ for any polynomial p.

- Given $\tilde{f} \in \widetilde{H}^2$ let (p_n) be a sequence of polynomials such that $\|\tilde{f} - p_n\|_{L^2} \to 0$.
Hence $\|A(\tilde{f}) - A(p_n)\|_{L^2} \to 0$.
Therefore there is a subsequence (p_{k_n}) such that $|\tilde{f} - p_{k_n}| \to 0$ a.e. AND $|A(\tilde{f}) - A(p_{k_n})| \to 0$ a.e.

It follows from $A(p) = \phi p$ that

$$A(\tilde{f}) = \lim_n A(p_{k_n}) = \lim_n \phi p_{k_n} = \phi \tilde{f} \ \text{ a.e..}$$

- It remains to show that $\phi \in \widetilde{H}^\infty$. For this, observe that if $\psi \in H^2$ is such that $\tilde{\psi} = \phi$ then for all $z \in \mathbb{D}$ the equality

$$\langle \tilde{\psi}\tilde{f}, k_z \rangle = \langle \phi \tilde{f}, k_z \rangle = \langle A(\tilde{f}), k_z \rangle$$

(where $k_z(e^{it}) = \sum_{k \geq 0} (\bar{z} e^{it})^k$ is Szegő's kernel) gives

$$\begin{aligned} \langle \tilde{f}, A^*(k_z) \rangle &= \langle A(\tilde{f}), k_z \rangle = \langle \tilde{\psi}\tilde{f}, k_z \rangle = \langle \widetilde{\psi f}, k_z \rangle \\ &= (\psi f)(z) = \psi(z) f(z) = \psi(z) \langle \tilde{f}, k_z \rangle = \langle \tilde{f}, \overline{\psi(z)} k_z \rangle \end{aligned}$$

for all $\tilde{f} \in \widetilde{H}^2$ and so $A^*(k_z) = \overline{\psi(z)} k_z$ which shows that $\|A^*\| \geq |\overline{\psi(z)}|$ hence $|\psi(z)| \leq \|A\|$ for all $z \in \mathbb{D}$; thus $\psi \in H^\infty$ hence $\phi \in \widetilde{H}^\infty$. ☐

Παρατήρηση 3. *Αν E, F ειναι χωροι Banach και $D \subseteq E$ ειναι πυκνος γραμμικος υποχωρος, τοτε καθε συνεχης γραμμικη απεικονιση $A : D \to F$ εχει μοναδικη συνεχη γραμμικη επεκταση $\tilde{A} : E \to F$. Η αποδειξη στηριζεται στο γεγονος οτι μια συνεχης γραμμικη απεικονιση ειναι ομοιομορφα συνεχης.*

Τι μπορουμε να πουμε για μια συνεχη διγραμμικη *απεικονιση $\psi : D \times D \to F$; Η ψ δεν ειναι εν γενει ομοιομορφα συνεχης. Παρολα αυτα ομως:*

Λήμμα 4. *Εστω E, F χωροι Banach, $D \subseteq E$ πυκνος γραμμικος υποχωρος, και $\psi : D \times D \to F$ μια συνεχης διγραμμικη απεικονιση. Τοτε υπαρχει μοναδικη συνεχης διγραμμικη απεικονιση $\tilde{\psi} : E \times E \to F$ που επεκτεινει την ψ.*

Ομοιως για sesquilinear απεικονισεις.

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Απόδειξη. Observe that if $x, x', y, y' \in D$, we have

$$\begin{aligned}\|\psi(x,y)-\psi(x',y')\| &= \|\psi(x-x',y)+\psi(x',y-y')\| \le \|\psi(x-x',y)\|+\|\psi(x',y-y')\| \\ &\le \|\psi\|\|x-x'\|\|y\|+\|\psi\|\|x'\|\|y-y'\| \qquad (1)\end{aligned}$$

where $\|\psi\| = \sup\{\|\psi(x,y)\| : x, y \in D, \|x\| = 1 = \|y\|\}$, which is finite by the assumed continuity of ψ.

Now given $\xi, \eta \in E$, we wish to define $\tilde{\psi}(\xi,\eta) \in F$. Choosing sequences $(x_n), (y_n)$ in D with $\|x_n - \xi\| \to 0$ and $\|y_n - \eta\| \to 0$, by the previous inequality we have for all $m, n \in \mathbb{N}$

$$\|\psi(x_n,y_n)-\psi(x_m,y_m)\| \le \|\psi\|\|x_n-x_m\|\|y_n\|+\|\psi\|\|x_n\|\|y_n-y_m\|\,.$$

Since the sequences (x_n) and (y_n) are convergent, they are Cauchy and bounded (!), say by M. Thus the last inequality gives

$$\|\psi(x_n,y_n)-\psi(x_m,y_m)\| \le \|\psi\|M(\|x_n-x_m\|+\|y_n-y_m\|)\,.$$

This shows that the sequence $(\psi(x_n,y_n))_n$ is Cauchy in F; since the latter is complete, the sequence converges, and we may define

$$\tilde{\psi}(\xi,\eta) := \lim_n \psi(x_n,y_n)\,.$$

Observe that this is well defined, i.e. does not depend on the sequences used to approximate ξ and η. Indeed, if also $(x_n'), (y_n')$ are sequences in D with $\|x_n' - \xi\| \to 0$ and $\|y_n' - \eta\| \to 0$ then by (1) we have

$$\|\psi(x_n,y_n)-\psi(x_n',y_n')\| \le \|\psi\|\|x_n-x_n'\|\|y_n\|+\|\psi\|\|x_n'\|\|y_n-y_n'\| \to 0$$

since $(x_n'), (y_n')$ are bounded. Therefore $\psi(x_n', y_n') \to \tilde{\psi}(\xi,\eta)$.

It is clear that $\tilde{\psi}$ extends ψ (use constant sequences to approximate elements of D).

To show that $\tilde{\psi}$ is linear in the first variable, let $\xi = \lim x_n \in E$ and $\xi' = \lim x_n' \in E$ with $x_n, x_n' \in D$ and let $\lambda \in \mathbb{C}$. Then for all $\eta = \lim y_n \in E$ with $y_n \in D$, since $x_n + \lambda x_n' \to \xi + \lambda\xi'$ by the definition of $\tilde{\psi}$ we have

$$\tilde{\psi}(\xi+\lambda\xi',\eta) = \lim \psi(x_n+\lambda x_n', y_n) = \lim(\psi(x_n,y_n)+\lambda\psi(x_n',y_n)) = \tilde{\psi}(\xi,\eta)+\lambda\tilde{\psi}(\xi',\eta)\,.$$

Similarly we show linearity in the second variable.

Finally to show that $\tilde{\psi}$ is continuous, for all $\xi = \lim x_n \in E$ and $\eta = \lim y_n \in E$ with $x_n, y_n \in D$ we have

$$\|\tilde{\psi}(\xi,\eta)\| = \lim_n \|\psi(x_n,y_n)\| \le \limsup_n \|\psi\|\|x_n\|\|y_n\| = \|\psi\|\|\xi\|\|\eta\|$$

which shows that $\|\tilde{\psi}\| \le \|\psi\|$ (but in fact equality holds since $\tilde{\psi}$ extends ψ). ☐

Παρατήρηση 5. *Ενας φραγμενος τελεστης* $A \in \mathcal{B}(\widetilde{H}^2)$ *εχει πινακα Toeplitz ως προς την ο/κ βαση* $\{f_n : n \in \mathbb{Z}_+\}$ *αν-ν* $T_1^* A T_1 = T$.

Απόδειξη. For all $n, m \in \mathbb{Z}_+$ we have

$$\langle T_1^* A T_1 f_n, f_m\rangle = \langle A T_1 f_n, T_1 f_m\rangle = \langle A f_{n+1}, f_{m+1}\rangle\,.$$

Thus $T_1^* A T_1 = T$ iff $\langle A f_n, f_m\rangle = \langle A f_{n+1}, f_{m+1}\rangle$ $\forall n, m \in \mathbb{Z}_+$ iff the matrix of T has constant diagonals, i.e iff the matrix of T is Toeplitz. ☐

Πρόταση 6. *Ενας φραγμενος τελεστης* $A \in \mathcal{B}(\widetilde{H}^2)$ *ειναι τελεστης Toeplitz αν-ν εχει πινακα Toeplitz ως προς την ο/κ βαση* $\{f_n : n \in \mathbb{Z}_+\}$*, αν-ν* $T_1^* A T_1 = T$.

Απόδειξη. Suppose the matrix of A is Toeplitz, so that $T_1^* A T_1 = T$.

We 'lift' the setting to $L^2(\mathbb{T})$ in order to apply Proposition 2.
Define an increasing family of closed subspaces $E_n := \text{span}\{f_k,\, k \geq -n\},\ n \in \mathbb{Z}_+$:

$$\widetilde{H}^2 := E_0 \subseteq E_1 := \text{span}\{f_{-1}, f_0, f_1, \dots\} \subseteq \dots \subseteq L^2(\mathbb{T})$$

whose union $E_\infty := \bigcup_n E_n$ is dense in $L^2(\mathbb{T})$. Note that for all $n \in \mathbb{N}$ we have $M_1(E_n) \subseteq E_{n-1}$, hence $M_1^n(E_n) \subseteq E_0 = \widetilde{H}^2$. Thus for each $n \in \mathbb{N}$ we may define

$$A_n := M_1^{-n} A M_1^n \in \mathcal{B}(E_n)\,.$$

Note that $\|A_n\| \leq \|A\|$ for all $n \in \mathbb{N}$.

- *Claim 1.* For all $\xi, \eta \in E_n$, $(n \in \mathbb{N})$ we have

$$\langle A_{n+1}\xi, \eta\rangle = \langle A_n \xi, \eta\rangle\,.$$

Proof of the Claim.

$$\begin{aligned}\langle A_{n+1}\xi, \eta\rangle &= \langle M_1^{-(n+1)} A M_1^{n+1}\xi, \eta\rangle = \langle A M_1 M_1^n \xi, M_1 M_1^n \eta\rangle \\ &= \langle A T_1 M_1^n \xi, T_1 M_1^n \eta\rangle \quad \text{since } M_1^n\xi \text{ and } M_1^n\eta \text{ are in } \widetilde{H}^2 \\ &= \langle (T_1^* A T_1) M_1^n \xi, T_1 M_1^n \eta\rangle = \langle A M_1^n \xi, T_1 M_1^n \eta\rangle \quad \text{since } T_1^* A T_1 = A \\ &= \langle A_n \xi, \eta\rangle\,.\end{aligned}$$

- It follows or all $\xi, \eta \in E_\infty$ the sequence of complex numbers

$$(\langle A_n \xi, \eta\rangle)_n$$

is eventually constant, hence convergent.

Hence we may define

$$\psi : E_\infty \times E_\infty \to \mathbb{C} : (\xi, \eta) \mapsto \lim_n \langle A_n \xi, \eta\rangle\,.$$

Since every $(\xi, \eta) \mapsto \langle A_n \xi, \eta\rangle$ is a sesquilinear form, bounded by $\|A\|$, the same holds for ψ.

- *Using Lemma 4, we may now extend* ψ *to a bounded sesquilinear form* $\tilde{\psi} : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to \mathbb{C}$. Since $L^2(\mathbb{T})$ is a Hilbert space, there exists a unique bounded operator $A_0 \in \mathcal{B}(L^2(\mathbb{T}))$, which satisfies

$$\langle A_0(\xi), \eta\rangle = \psi(\xi, \eta) = \lim_n \langle A_n \xi, \eta\rangle \quad \forall\, \xi, \eta \in E_\infty\,.$$

- *Claim.* The operator A_0 commutes with the bilateral shift M_1.

Proof of the Claim. For all $\xi, \eta \in E_\infty$, since $M_1^* = M_1^{-1}$, we have

$$\begin{aligned}\langle M_1^{-1} A_0 M_1 \xi, \eta\rangle &= \langle A_0 M_1 \xi, M_1 \eta\rangle = \lim_n \langle A_n M_1 \xi, M_1 \eta\rangle = \lim_n \langle M_1^{-n} A M_1^n (M_1\xi), (M_1\eta)\rangle \\ &= \lim_n \langle M_1^{-1} M_1^{-n} A M_1^n M_1 \xi, \eta\rangle = \lim_n \langle M_1^{-(n+1)} A M_1^{n+1} \xi, \eta\rangle = \langle A_0 \xi, \eta\rangle\,.\end{aligned}$$

Since E_∞ is dense in $L^2(\mathbb{T})$, we have $A_0 = M_1^{-1} A_0 M_1$ or $M_1 A_0 = A_0 M_1$, proving the Claim.

- It follows from the previous Theorem that there exists a $\phi \in L^\infty(\mathbb{T})$ such that $A_0 = M_\phi : L^2(\mathbb{T}) \to L^2(\mathbb{T})$.

To complete the proof, we show the

- *Claim.* $A = T_\phi : \widetilde{H}^2 \to \widetilde{H}^2$.

Proof of the Claim. If $\xi, \eta \in \widetilde{H}^2$ we have, for all $n \in \mathbb{N}$,

$$\begin{aligned}\langle PA_n\xi, \eta\rangle &= \langle A_n\xi, P\eta\rangle = \langle A_n\xi, \eta\rangle \\ &= \langle M_1^{-n} A M_1^n \xi, \eta\rangle = \langle A M_1^n \xi, M_1^n \eta\rangle = \langle A T_1^n \xi, T_1^n \eta\rangle \quad (\xi, \eta \in \widetilde{H}^2) \\ &= \langle (T_1^*)^n A T_1^n \xi, \eta\rangle = \langle A\xi, \eta\rangle\end{aligned}$$

since $(T_1^*)^n A T_1^n = A$ from the hypothesis $T_1^* A T_1 = A$ and induction.
Thus $\langle A\xi, \eta\rangle = \langle PA_n\xi, \eta\rangle$ for all n and so

$$\langle A\xi, \eta\rangle = \lim_n \langle PA_n\xi, \eta\rangle = \lim_n \langle A_n\xi, P\eta\rangle = \langle A_0\xi, P\eta\rangle = \langle M_\phi\xi, P\eta\rangle = \langle PM_\phi\xi, \eta\rangle$$

and therefore $PM_\phi|_{\widetilde{H}^2} = A$, as required. □

Εναλλακτικη αποδειξη της Προτασης 2. If $A \in \mathcal{B}(\widetilde{H}^2)$ satisfies $AT_1 = T_1A$, then $T_1^* A T_1 = T_1^* T_1 A = A$ since T_1 is an isometry. It follows from Proposition 6 that there exists $\phi \in L^\infty(\mathbb{T})$ such that $A = T_\phi$.

To show that in fact $\phi \in \widetilde{H}^\infty$, we need to prove that $\hat{\phi}(-m) = 0$ for all $m > 0$. But recall that the matrix $[\langle T_\phi f_m, f_n\rangle]$ of T_ϕ is given by $\langle T_\phi f_m, f_n\rangle = \hat{\phi}(n-m)$. Thus, for all $m > 0$,

$$\begin{aligned}\hat{\phi}(-m) &= \langle T_\phi f_m, f_0\rangle = \langle T_\phi T_1 f_{m-1}, f_0\rangle \quad (\text{since } m > 0) \\ &= \langle T_1 T_\phi f_{m-1}, f_0\rangle \quad (\text{by hypothesis}) \\ &= \langle T_\phi f_m, T_1^* f_0\rangle = 0\end{aligned}$$

as required. □

Συνεπεια για $T \in \mathcal{B}(\ell^2(\mathbb{Z}_+))$:

Πόρισμα 7. *Ενας φραγμενος τελεστης* $T \in \mathcal{B}(\ell^2(\mathbb{Z}_+))$ *ειναι τελεστης Toeplitz, δηλαδη υπαρχει* $\phi \in L^\infty(\mathbb{T})$ *ωστε* $\langle Te_m, e_n\rangle = \hat{\phi}(n-m)$, *αν-ν*

$$S^*TS = T.$$