ΣΤΟΙΧΕΙΑ ΘΕΩΡΙΑΣ ΠΑΙΓΝΙΩΝ ΚΑΙ ΛΗΨΗΣ ΑΠΟΦΑΣΕΩΝ

ΕΞΕΛΙΚΤΙΚΕΣ ΔΥΝΑΜΙΚΕΣ

Παναγιώτης Μερτικόπουλος

Εθνικό και Καποδιστριακό Πανεπιστήμιο Αθηνών

Τμήμα Μαθηματικών

Χειμερινό Εξάμηνο, 2023–2024

Outline

Population games, I: Symmetric models

Definition (Single-population games)

A ***single-population game*** is a collection of the following primitives:

- A continuous ***population of players*** modeled by $\mathcal{N} = [0,1]$
- A finite set of ***actions / pure strategies*** $\mathcal{A} = \{1, \dots, m\}$, common for all players in the population
- An ensemble of ***payoff functions*** $v_\alpha \colon \mathcal{X} \equiv \Delta(\mathcal{A}) \to \mathbb{R}$, one per $\alpha \in \mathcal{A}$

A population game with primitives as above will be denoted by $\mathcal{G} \equiv \mathcal{G}(\mathcal{A}, v)$.

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Setup of the game:

- ***Action selection*** given by some $i \mapsto \chi(i) \in \mathcal{A}$ — # $\chi \colon \mathcal{N} \to \mathcal{A}$ assumed measurable
- ***Population state*** $x \in \mathcal{X} \equiv \Delta(\mathcal{A})$ defined as — # as a measure: $x = \lambda \circ \chi^{-1}$

$$x_\alpha = \lambda(\chi^{-1}(\alpha)) = \text{mass of players playing } \alpha \in \mathcal{A}$$

- ***Anonymity:*** payoffs determined by the *state* of the population, not *individual* player choices

$$v_\alpha(x) = \text{payoff to } \alpha\text{-players when the population is at state } x \in \mathcal{X}$$

Example I: Symmetric random matching

Example (Symmetric / Single-population random matching)

- ***Given:*** $m \times m$ payoff matrix M — # symmetric two-player finite game
- ***Matching:*** Two players are drawn randomly to play M — # independent draws from $x \in \mathcal{X}$
- If the population is at state $x \in \mathcal{X}$:

$$\mathbb{P}(\text{matching } \alpha \text{ against } \beta) = x_\alpha x_\beta$$

- Mean payoff to an α-strategist:

$$v_\alpha(x) = \mathbb{E}_{\beta \sim x}[M_{\alpha\beta}] = \sum_{\beta \in \mathcal{A}} M_{\alpha\beta} x_\beta = (Mx)_\alpha$$

- Mean population payoff:

$$u(x) = \mathbb{E}_{\alpha,\beta \sim x}[M_{\alpha\beta}] = \sum_{\alpha,\beta \in \mathcal{A}} M_{\alpha\beta} x_\alpha x_\beta = x^\top M x$$

NB:

- Mean population payoff is **quadratic** in x — # symmetric matching

Population games, II: Asymmetric models

Definition (Multi-population games)

A ***multi-population game*** is a collection of the following primitives:

- N distinct ***populations of players:*** $\mathcal{N} = \coprod_{i=1}^{N} [0, \rho_i]$ — # ρ_i = total mass of i-th population
- A finite set of ***actions / pure strategies*** $\mathcal{A}_i = \{1, \dots, m_i\}$ per population
- An ensemble of ***payoff functions*** $v_{i\alpha_i} \colon \mathcal{X} \equiv \prod_j \Delta(\mathcal{A}_j) \to \mathbb{R}$, one per $\alpha_i \in \mathcal{A}_i$, $i = 1, \dots, N$

A population game with primitives as above will be denoted by $\mathcal{G} \equiv \mathcal{G}(\mathcal{N}, \mathcal{A}, v)$.

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A population game with primitives as above will be denoted by $\mathcal{G} \equiv \mathcal{G}(\mathcal{N}, \mathcal{A}, v)$.

Setup of the game:

- ***Population state*** $x \in \mathcal{X} \equiv \prod_j \Delta(\mathcal{A}_j)$: # state of i-th population: $x_i \in \mathcal{X}_i \equiv \Delta(\mathcal{A}_i)$

$$x_{i\alpha_i} = \text{mass of players of population } i \text{ playing } \alpha_i \in \mathcal{A}_i$$

- ***Anonymity:*** payoffs determined by the *state* of the population, not *individual* player choices

$$v_{i\alpha_i}(x) = \text{payoff to players of population } i \text{ playing } \alpha_i \in \mathcal{A}_i \text{ when the population is at state } x \in \mathcal{X}$$

Example II: Asymmetric random matching

Example (Asymmetric / Multi-population random matching)

- ***Given:*** finite game $\Gamma \equiv \Gamma(\mathcal{N}, \mathcal{A}, u)$; N unit mass populations
- ***Matching:*** N players are drawn randomly to play Γ, one per population # independent draws from $x \in \mathcal{X}$
- If the population is at state $x \in \mathcal{X}$:

$$\mathbb{P}(\text{matching } \alpha_i \text{ against } \alpha_{-i}) = x_{i\alpha_i} \cdot x_{-i,\alpha_{-i}}$$

- Mean payoff to an α-strategist of population i:

$$v_{i\alpha_i}(x) = \mathbb{E}_{\alpha_{-i} \sim x_{-i}}[u_\alpha(\alpha_i; \alpha_{-i})] = u_i(\alpha_i; x_{-i})$$

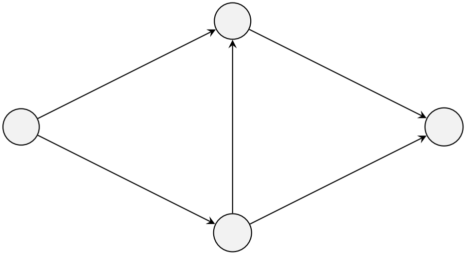
- Mean payoff of population i:

$$u_i(x) = \mathbb{E}_{\alpha \sim x}[u_i(\alpha)] = \sum_{\alpha_1 \in \mathcal{A}_1} \cdots \sum_{\alpha_N \in \mathcal{A}_N} x_{1,\alpha_1} \cdots x_{N,\alpha_N} \, u_i(\alpha_1, \dots, \alpha_N)$$

NB:

- Mean population payoff is **multilinear** in x # asymmetric matching

Example III: Nonatomic congestion games



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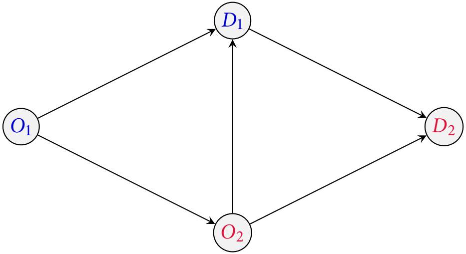

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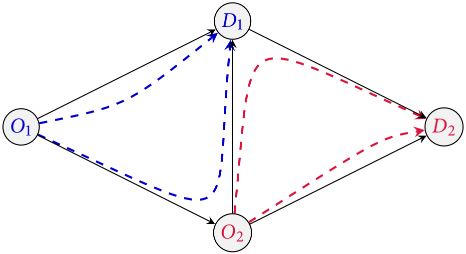

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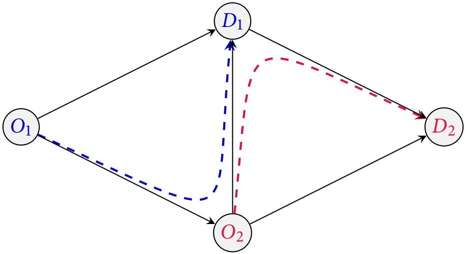

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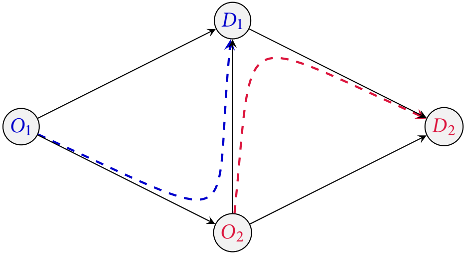

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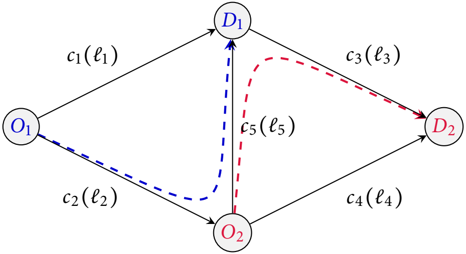

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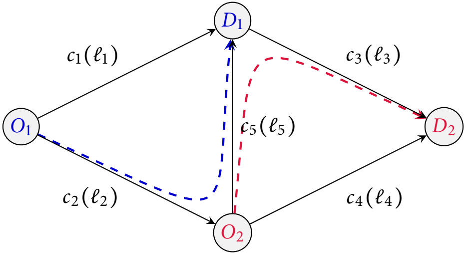

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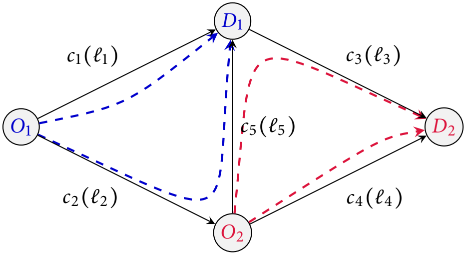

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- **Nonatomic congestion game:** $\mathcal{G} \equiv \mathcal{G}(\mathcal{N}, \mathcal{A}, -c)$

Mixing versus matching

⚠ Symmetric Random matching $\neq$ Mixed extension

Population matched against itself $\Longrightarrow$ ***symmetric interactions***

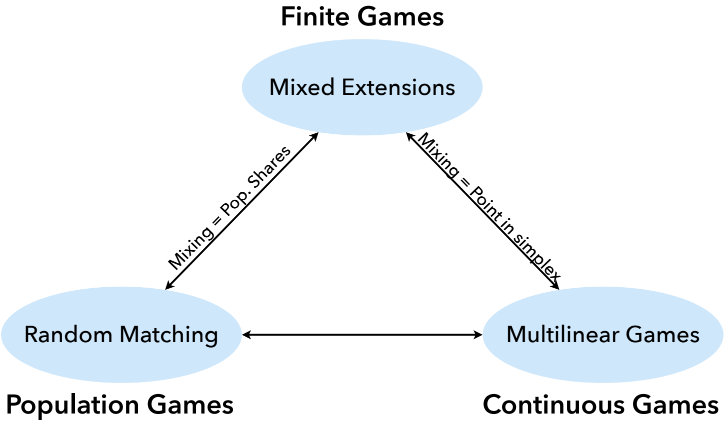
⚠ *Asymmetric* random matching $=$ Mixed Extension

Populations matched against each other $\Longrightarrow$ ***asymmetric interactions***

⚠ Multi-population games $\supsetneq$ Mixed Extensions

Nonatomic congestion games, ...

Relations between classes



Nash equilibrium

Nash equilibrium (Nash, 1950, 1951)

"No player has an incentive to deviate from their chosen strategy if other players don't"

- **In finite games** (mixed extension formulation):

$$u_i(x_i^*; x_{-i}^*) \geq u_i(x_i; x_{-i}^*) \quad \text{for all } x_i \in \mathcal{X}_i,\ i \in \mathcal{N}$$

- **In population games:**

$$v_{i\alpha_i}(x^*) \geq v_{i\beta_i}(x^*) \quad \text{whenever } \alpha_i \in \operatorname{supp}(x^*)$$

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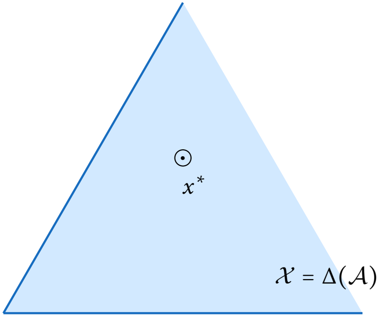
Variational formulation (Stampacchia, 1964)

$$\langle v(x^*), x - x^* \rangle \leq 0 \quad \text{for all } x \in \mathcal{X}$$

where $v(x) = (v_1(x), \dots, v_N(x))$ is the ***payoff field*** of the game

Geometric characterization

At Nash equilibrium, payoff vectors are outward-pointing



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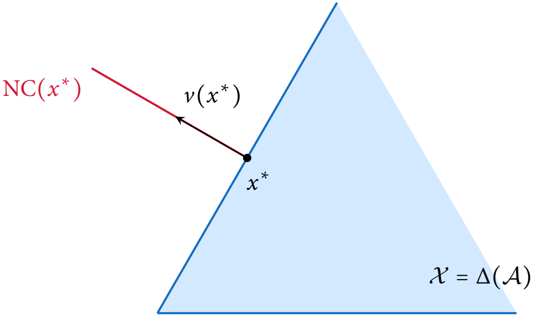

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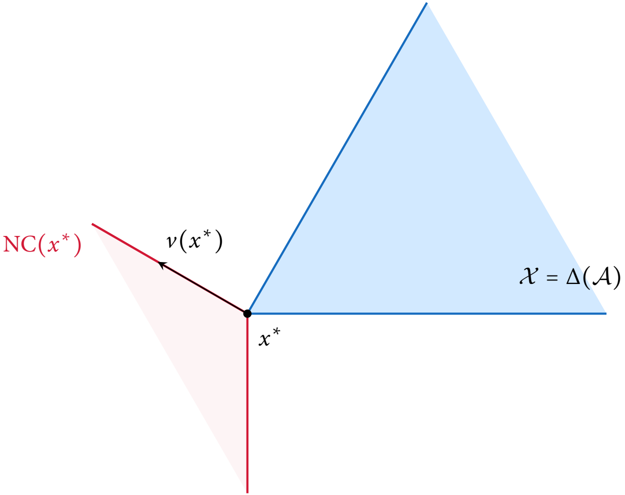

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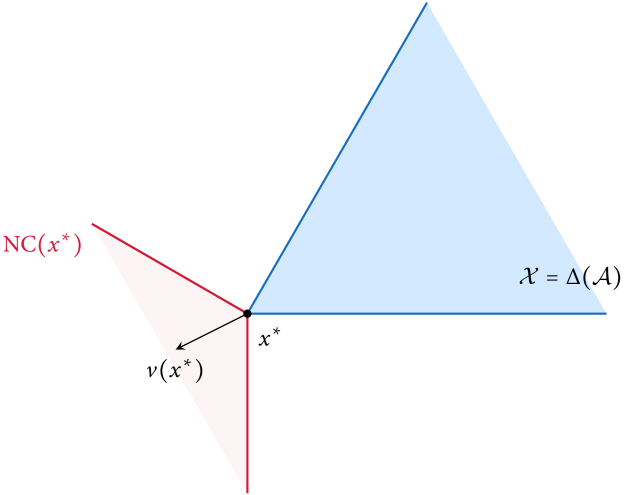

Outline

Basic questions

How do players learn from the history of play?

Do players end up playing a Nash equilibrium?

Learning, evolution and dynamics

What is "learning" in games?

Learning, evolution and dynamics

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The basic process:

- Players choose strategies and receive corresponding payoffs
- Depending on outcome and information revealed, they choose new strategies and they play again
- Rinse, repeat

Learning, evolution and dynamics

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The basic questions:

- *How do populations evolve over time?* # Population biology
- *How do people learn in a game?* # Economics
- *What algorithms should we use to learn in a game?* # Computer science
- Given a dynamical system on $\mathcal{X}$, what is its long-term behavior? # Mathematics

Age the First (1970's-1990's): Population Biology

- Strategies are ***phenotypes*** in a given species

$$z_\alpha = \text{absolute population mass of type } \alpha \in \mathcal{A}$$
$$z = \sum_\alpha z_\alpha = \text{absolute population mass}$$

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$$\dot{z}_\alpha = z_\alpha v_\alpha$$

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- Evolution of population shares ($x_\alpha = z_\alpha / z$):

$$\dot{x}_\alpha = \frac{d}{dt}\frac{z_\alpha}{z} = \frac{\dot{z}_\alpha z - z_\alpha \sum_\beta \dot{z}_\beta}{z^2} = \frac{z_\alpha}{z} v_\alpha - \frac{z_\alpha}{z}\sum_\beta \frac{z_\beta}{z} v_\beta$$

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Replicator dynamics (Taylor & Jonker, 1978)

$$\dot{x}_\alpha = x_\alpha [v_\alpha(x) - u(x)] \tag{RD}$$

Age the Second (1990's-2010's): Economics

- Agents receive **revision opportunities** to switch strategies

$$\rho_{\alpha\beta}(x) = \text{conditional switch rate from } \alpha \text{ to } \beta$$

NB: dropping player index for simplicity

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Imitate with probability proportional to excess payoff (Helbing, 1992; Schlag, 1998)

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- Inflow/outflow:

$$\textit{Incoming toward } \alpha = \sum_\beta \operatorname{mass}(\beta \rightsquigarrow \alpha) = \sum_{\beta\in\mathcal{A}} x_\beta \rho_{\beta\alpha}(x)$$

$$\textit{Outgoing from } \alpha = \sum_\beta \operatorname{mass}(\alpha \rightsquigarrow \beta) = x_\alpha \sum_{\beta\in\mathcal{A}} \rho_{\alpha\beta}(x)$$

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$$\textbf{\textit{Outgoing from }} \alpha = \sum_\beta \text{mass}(\alpha \rightsquigarrow \beta) = x_\alpha \sum_{\beta\in\mathcal{A}} \rho_{\alpha\beta}(x)$$

- Detailed balance:

$$\dot{x}_\alpha = \text{inflow}_\alpha(x) - \text{outflow}_\alpha(x) = \cdots = x_\alpha [v_\alpha(x) - u(x)] \tag{RD}$$

Age the Third (2000's-present): Computer Science

Learning in finite games

Require: finite game $\Gamma \equiv \Gamma(\mathcal{N}, \mathcal{A}, u)$

repeat

At each epoch $t \geq 0$ **do simultaneously** for all players $i \in \mathcal{N}$ — # continuous time

Choose ***mixed strategy*** $x_i(t) \in \mathcal{X}_i := \Delta(\mathcal{A}_i)$ — # mixing

Encounter ***mixed payoff vector*** $v_i(x(t))$ and get **mixed payoff** $u_i(x(t)) = \langle v_i(t), x(t) \rangle$ — # feedback phase

until end

Defining elements

- ***Time:*** continuous
- ***Players:*** finite
- ***Actions:*** finite
- ***Mixing:*** yes
- ***Feedback:*** mixed payoff vectors

Exponential weights

Exponential reinforcement mechanism:

- Score each action based on its cumulative payoff over time:

$$y_{i\alpha_i}(t) = \int_0^t v_{i\alpha_i}(x(s))\,ds$$

- Play an action with probability exponentially proportional to its score

$$x_{i\alpha_i}(t) \propto \exp(y_{i\alpha_i}(t))$$

Exponential weights in continuous time

$$\begin{aligned} \dot{y}_{i\alpha_i} &= v_{i\alpha_i}(x) \\ x_{i\alpha_i} &= \frac{\exp(y_{i\alpha_i})}{\sum_{\beta_i \in \mathcal{A}_i} \exp(y_{i\beta_i})} \end{aligned} \tag{EW}$$

Replicator dynamics

How do mixed strategies evolve under (EW)?

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$$\begin{aligned}\dot{x}_{i\alpha_i} &= x_{i\alpha_i}\left[v_{i\alpha_i}(x) - \sum\nolimits_{\beta_i \in \mathcal{A}_i} x_{i\beta_i} v_{i\beta_i}(x)\right] \\ &= x_{i\alpha_i}\left[u_i(\alpha_i; x_{-i}) - u_i(x)\right]\end{aligned} \tag{RD}$$

"The per capita growth rate of a strategy is proportional to its payoff excess"

➥ Hofbauer & Sigmund (1998); Weibull (1995); Hofbauer & Sigmund (2003); Sandholm (2010)

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Proposition

Solution orbits of (EW) $\iff$ *Interior orbits of* (RD)

Basic properties

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Structural properties

➾ Weibull, 1995; Hofbauer & Sigmund, 1998

- **Well-posed:** every initial condition $x \in \mathcal{X}$ admits unique solution trajectory $x(t)$ that exists for all time

  # Assuming v Lipschitz

- **Consistent:** $x(t) \in \mathcal{X}$ for all $t \geq 0$

  # Assuming $x(0) \in \mathcal{X}$

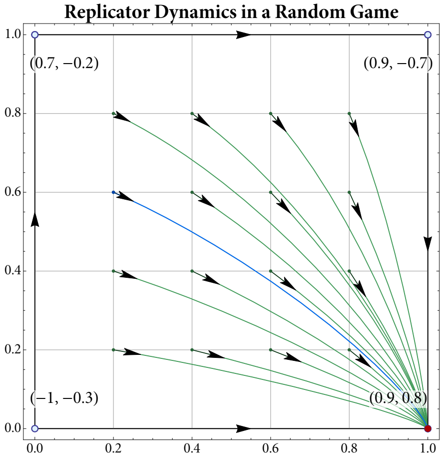
- **Faces are forward invariant** ("strategies breed true"):

$$x_{i\alpha_i}(0) > 0 \iff x_{i\alpha_i}(t) > 0 \quad \text{for all } t \geq 0$$
$$x_{i\alpha_i}(0) = 0 \iff x_{i\alpha_i}(t) = 0 \quad \text{for all } t \geq 0$$

Evolution of mixed strategies I: 2×2 *games*

What do the dynamics look like?

phase portraits



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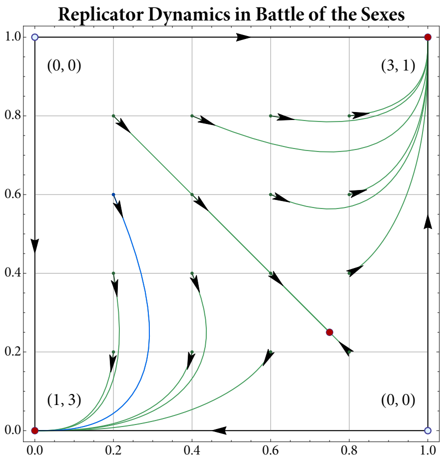

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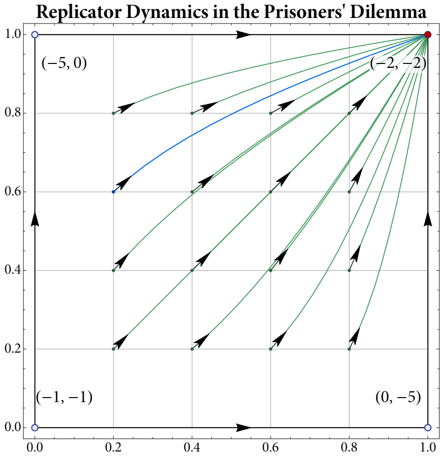

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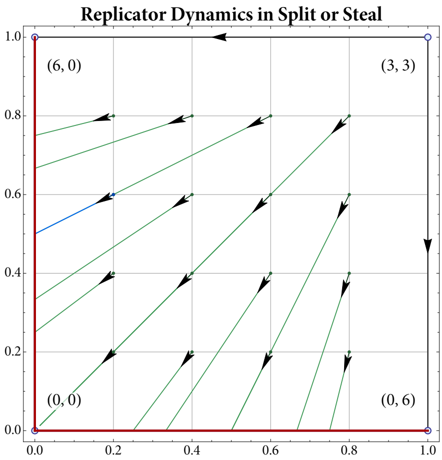

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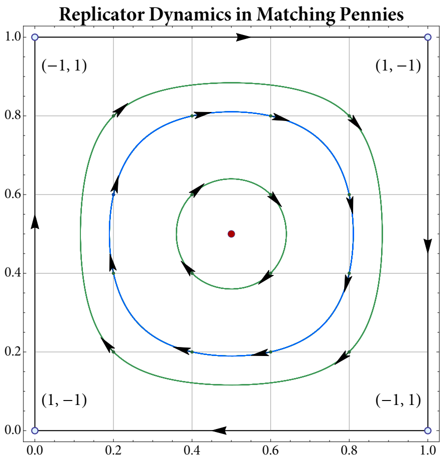

Evolution of mixed strategies II: $2 \times 2 \times 2$ games

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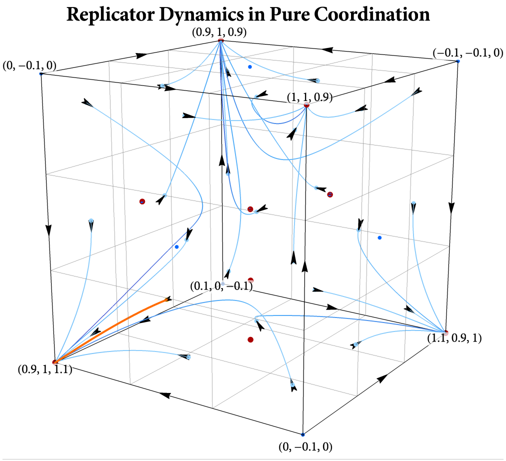

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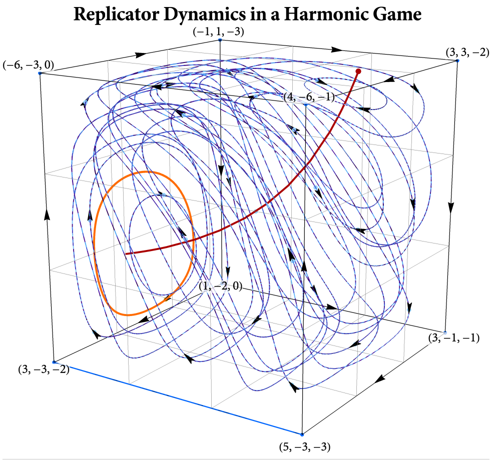

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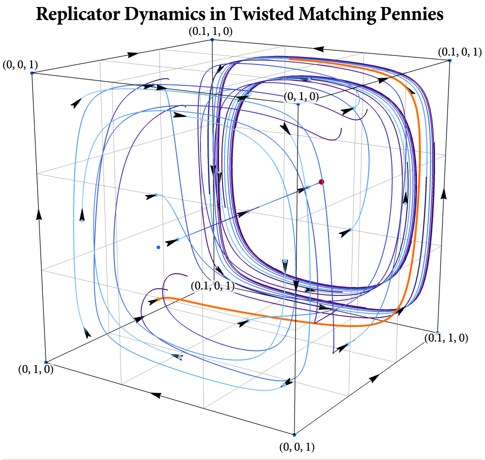

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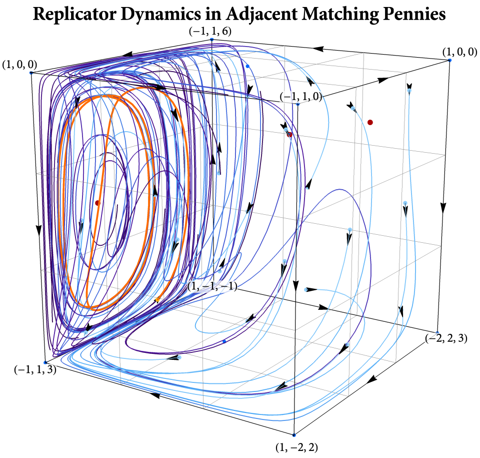

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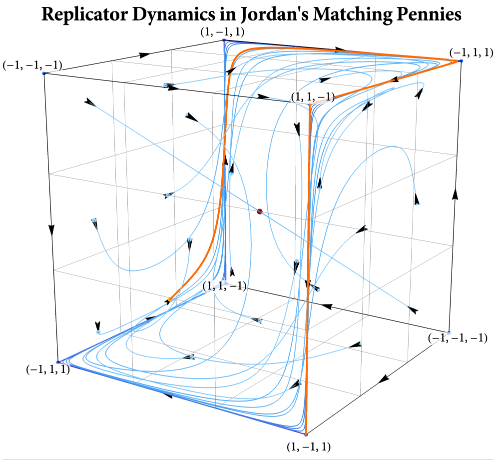

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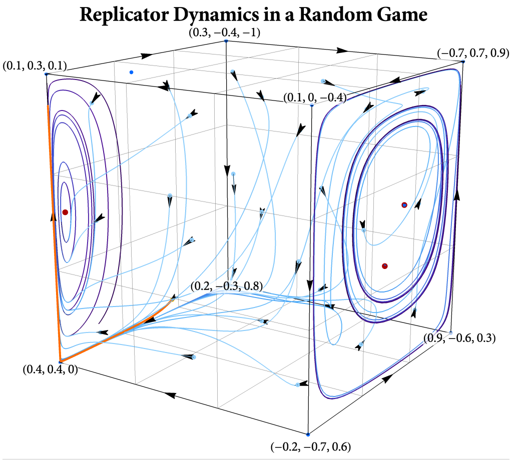

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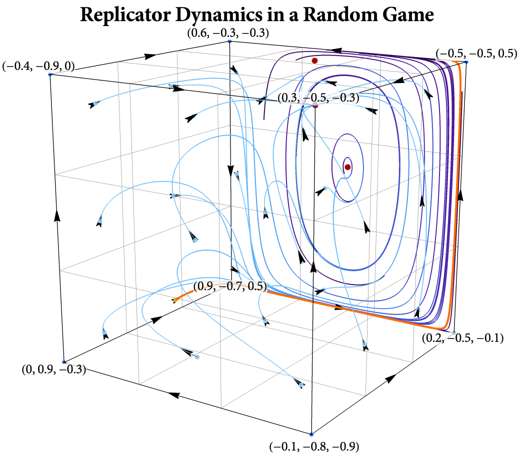

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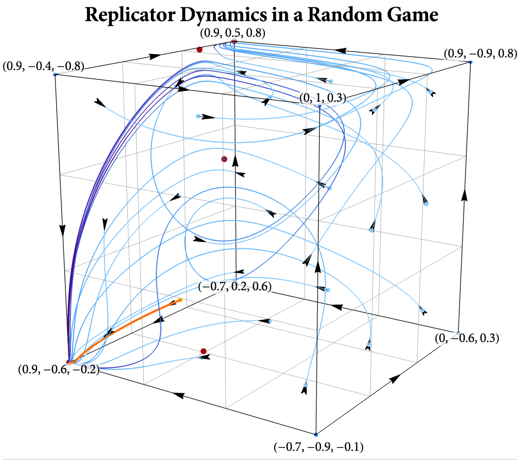

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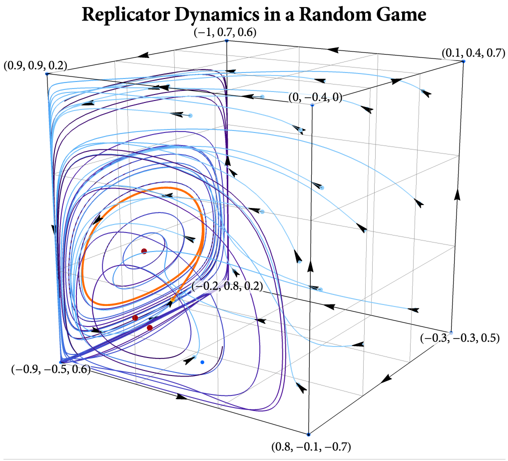

Outline

Dynamics and rationality

Are game-theoretic solution concepts consistent with the players' dynamics?

- Do dominated strategies die out in the long run?
- Are Nash equilibria stationary?
- Are they ***stable?*** Are they ***attracting?***
- Do the replicator dynamics always converge?
- What other behaviors can we observe?
- …

Dominated strategies

Suppose $\alpha_i \in \mathcal{A}_i$ is ***dominated*** by $\beta_i \in \mathcal{A}_i$

- Consistent payoff gap:

$$v_{i\alpha_i}(x) \leq v_{i\beta_i}(x) - \varepsilon \quad \text{for some } \varepsilon > 0$$

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- Consistent difference in choice probabilities

$$\frac{x_{i\alpha_i}(t)}{x_{i\beta_i}(t)} = \frac{\exp(y_{i\alpha_i}(t))}{\exp(y_{i\beta_i}(t))} \leq \exp(-\varepsilon t)$$

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Let $x(t)$ be a solution orbit of (EW)/(RD). *If $\alpha_i \in \mathcal{A}_i$ is dominated, then*

$$x_{i\alpha_i}(t) = \exp(-\Theta(t)) \quad \textit{as } t \to \infty$$

In words: under (EW)/(RD)*, dominated strategies become extinct at an exponential rate.*

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Self-check: extend to *iteratively* dominated strategies

Stationarity of equilibria

Nash equilibrium: $v_{i\alpha_i}(x^*) \geq v_{i\beta_i}(x^*)$ for all $\alpha_i, \beta_i \in \mathcal{A}_i$ with $x^*_{i\alpha_i} > 0$

- Supported strategies have equal payoffs:

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✗ **The converse does not hold!**

➬ **Self-check:** All vertices of $\mathcal{X}$ are stationary. General statement?

Stability

Are all stationary points created equal?

Definition (Lyapunov stability)

x^* is ***(Lyapunov) stable*** if, for every neighborhood $\mathcal{U}$ of x^* in $\mathcal{X}$, there exists a neighborhood $\mathcal{U}'$ of x^* such that

$$x(0) \in \mathcal{U}' \implies x(t) \in \mathcal{U} \quad \text{for all } t \geq 0$$

➠ Trajectories that start close to x^* remain close for all time

Stability and equilibrium

Proposition (Folk)

Suppose that x^* is Lyapunov stable under (EW) / (RD). *Then x^* is a Nash equilibrium.*

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Proof. Argue by contradiction:

- **Suppose that x^* is not Nash.** Then

$$v_{i\alpha_i^*}(x^*) = u_i(\alpha_i^*; x_{-i}^*) < u_i(\alpha_i; x_{-i}^*) = v_{i\alpha_i}(x^*)$$

  for some $\alpha_i^* \in \operatorname{supp}(x_i^*)$, $\alpha_i \in \mathcal{A}_i$, $i \in \mathcal{N}$

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- There exist $\varepsilon > 0$ and neighborhood $\mathcal{U}$ of x^* such that $v_{i\alpha_i}(x) - v_{i\alpha_i^*}(x) > \varepsilon$ for $x \in \mathcal{U}$

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- There exist $\varepsilon > 0$ and neighborhood $\mathcal{U}$ of x^* such that $v_{i\alpha_i}(x) - v_{i\alpha_i^*}(x) > \varepsilon$ for $x \in \mathcal{U}$
- If $x(t)$ is contained in $\mathcal{U}$ for all $t \geq 0$ (Lyapunov property), then:

$$y_{i\alpha_i^*}(t) - y_{i\alpha_i}(t) = c + \int_0^t [v_{i\alpha_i^*}(x(s)) - v_{i\alpha_i}(x(s))]\, ds < c - \varepsilon t$$

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- We conclude that $x_{i\alpha_i^*}(t) \to 0$, contradicting the Lyapunov stability of x^*. □

Asymptotic stability

Are Nash equilibria attracting?

Definition

- x^* is ***attracting*** if $\lim_{t\to\infty} x(t) = x^*$ whenever $x(0)$ is close enough to x^*
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i.e., $\lim_{t\to\infty} x_{i\alpha_i}(t) = 0$

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- Proof complete by showing Lyapunov stability ➻ Left as self-check exercise □

The "folk theorem" of evolutionary game theory

Theorem ("folk"; Hofbauer & Sigmund, 2003)

Let Γ be a finite game. Then, under (RD), we have:

1. x^* *is a Nash equilibrium* $\implies$ x^* *is stationary*
2. x^* *is the limit of an interior trajectory* $\implies$ x^* *is a Nash equilibrium*
3. x^* *is stable* $\implies$ x^* *is a Nash equilibrium*
4. x^* *is asymptotically stable* $\iff$ x^* *is a strict Nash equilibrium*

Notes:

- Single-population case similar ***except*** $\implies$ of (4)
- ✗ **Converse to (1), (2) and (3) does not hold!**
- ✓ Proof of (2) similar to (3) — Do as self-check
- Proof of " $\impliedby$ " in (4): requires different techniques

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