ΣΤΟΙΧΕΙΑ ΘΕΩΡΙΑΣ ΠΑΙΓΝΙΩΝ ΚΑΙ ΛΗΨΗΣ ΑΠΟΦΑΣΕΩΝ

ΔΙΑΛΕΞΗ 7: ΣΥΝΕΧΗ ΠΑΙΓΝΙΑ

Παναγιώτης Μερτικόπουλος

Εθνικό και Καποδιστριακό Πανεπιστήμιο Αθηνών

Τμήμα Μαθηματικών

Χειμερινό Εξάμηνο, 2023–2024

Outline

1. A modern game
2. Definitions and further examples
3. Nash equilibrium and characterizations
4. Concave games

Example 1: Spot the fake

Which person is real?

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Which person is real?

Spoiler: https://thispersondoesnotexist.com

Deep learning 101

The "hello world" of deep learning: how to recognize a hand-written digit?

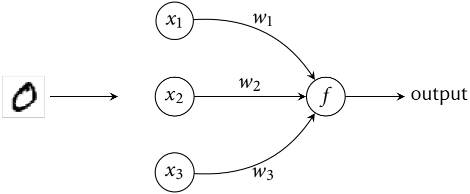
Figure: A sample from the MNIST database

Neural networks: use labeled data to infer hidden structures (“learn”)

The perceptron

A digit recognition ***perceptron:***

McCulloch & Pitts (1943)



1. x_1: is image intensity above 10%?
2. x_2: does image contain ?
3. x_3: does image contain a loop?

Output: $y = f(\sum_i w_i x_i)$: `if` $y = 1$, image depicts a 0; `else` image does not depict a 0

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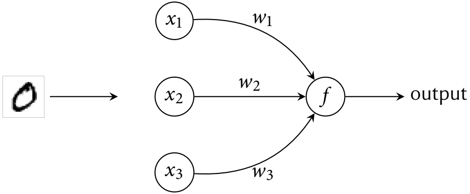

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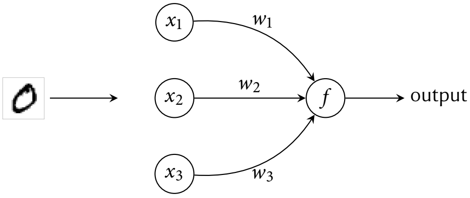

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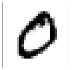
✓ ***Simple, but not simplistic:*** *much* better than guessing at random!

✗ What is f?

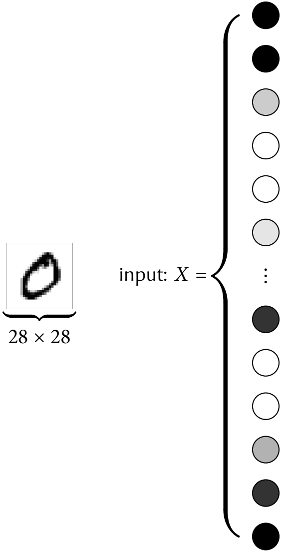
✗ How do we extract x_i?

How would this work in practice?



$\underbrace{}_{28 \times 28}$

How would this work in practice?



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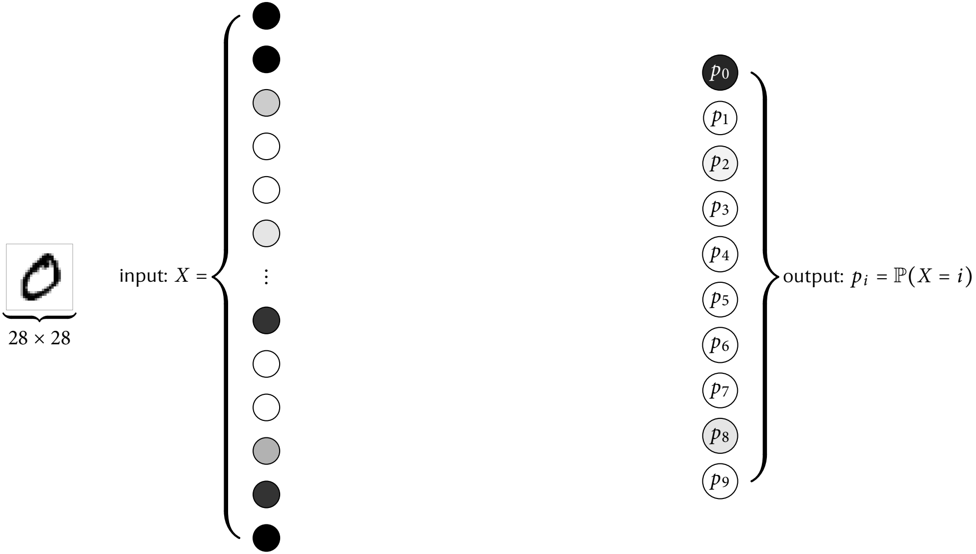

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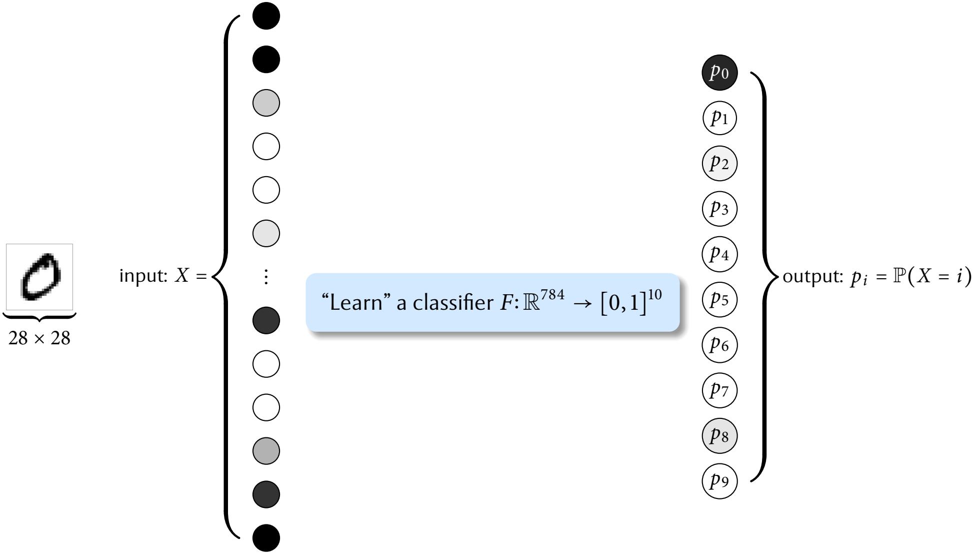

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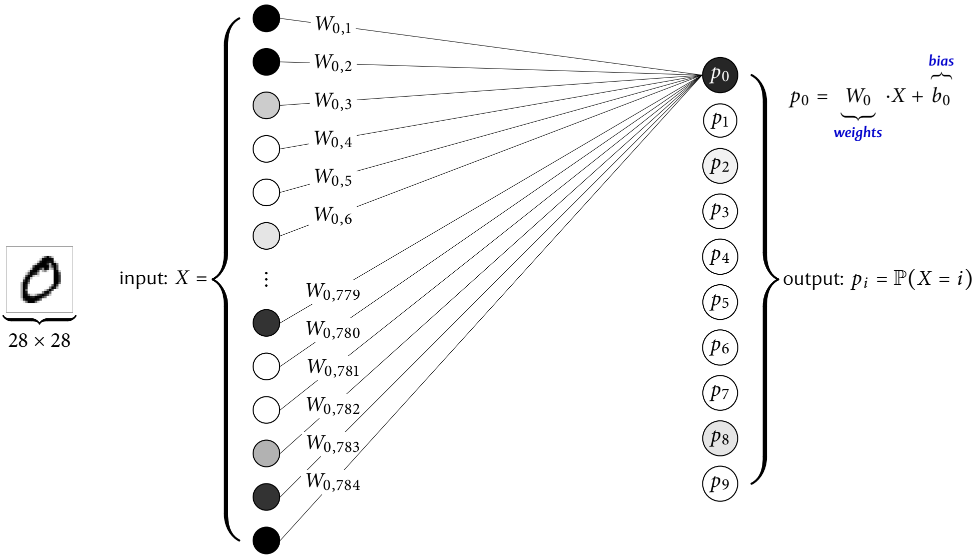

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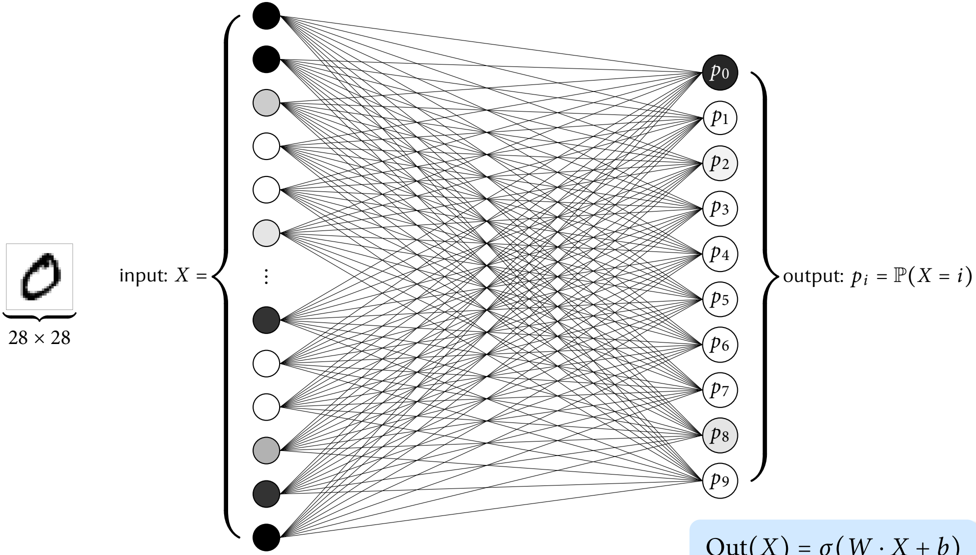


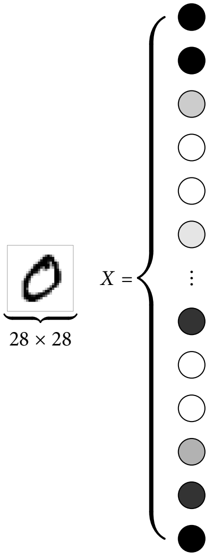
$$\mathrm{Out}(X) = \sigma(W \cdot X + b)$$

How would this work in practice?

Not expressive enough!

$$\mathrm{Out}(X) = \sigma(W \cdot X + b)$$

Deep neural networks



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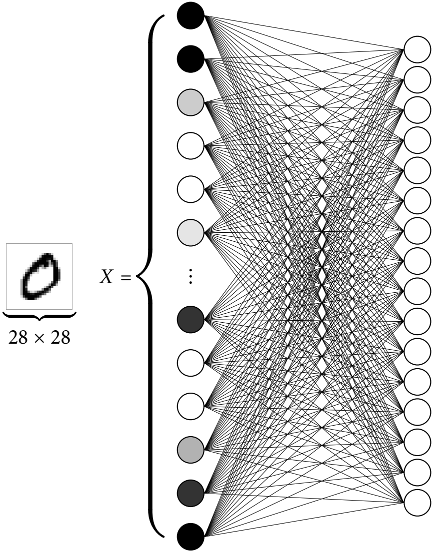

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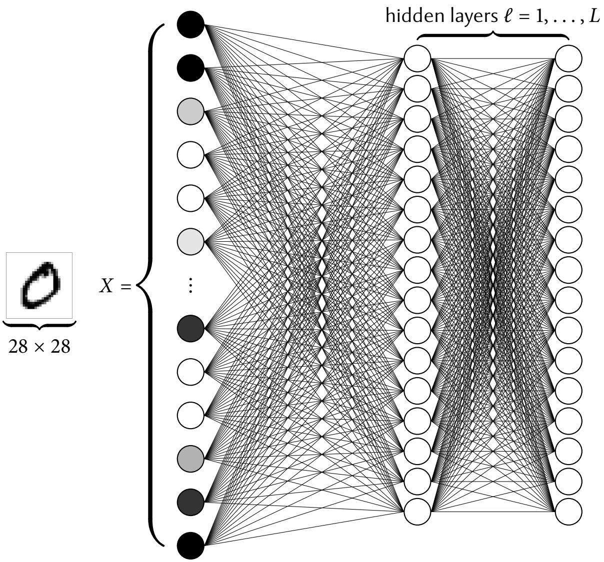

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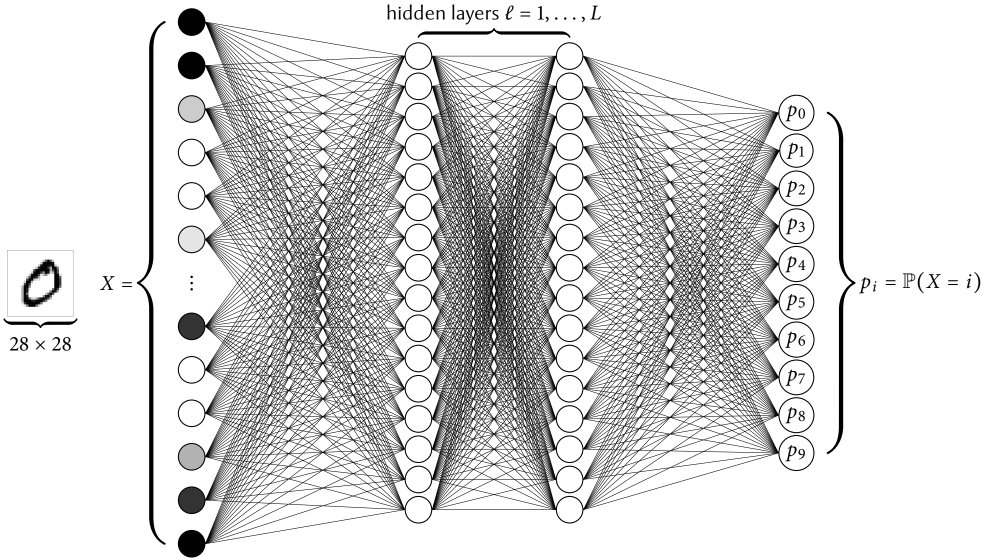

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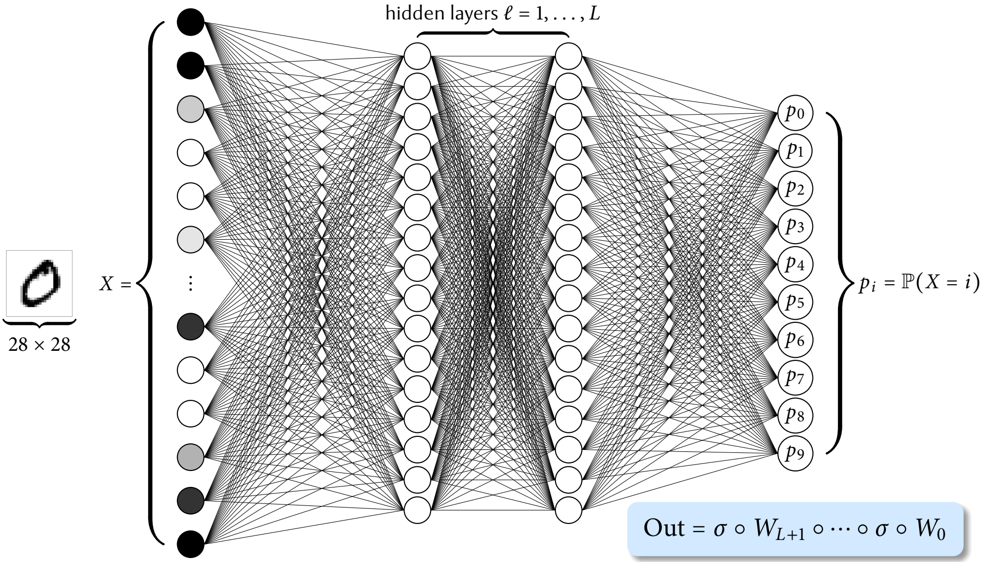
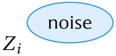


$$\text{Out} = \sigma \circ W_{L+1} \circ \cdots \circ \sigma \circ W_0$$

The schematics of GANs



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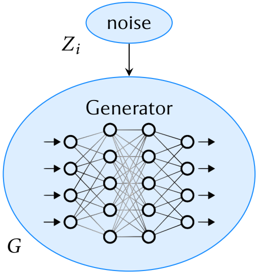

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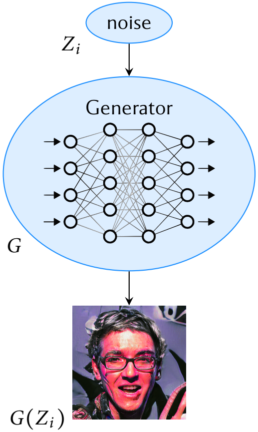

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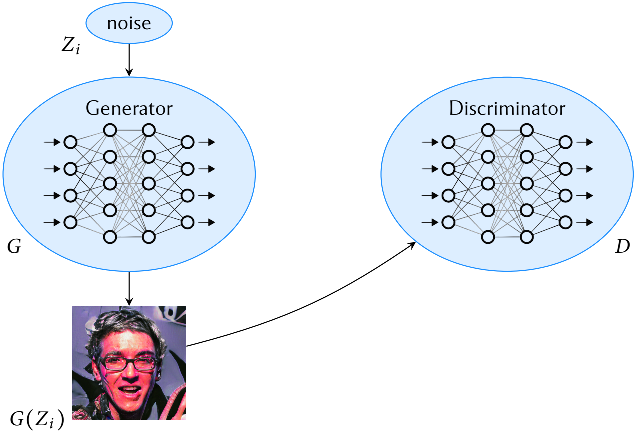

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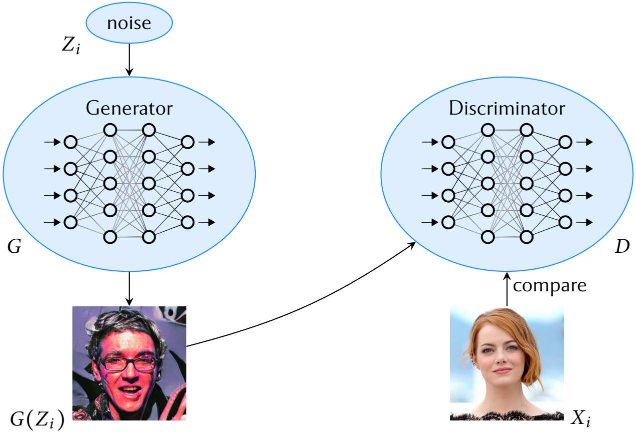

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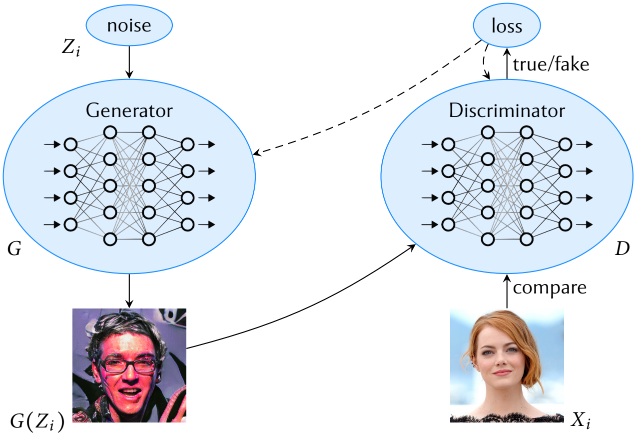


Model likelihood: $\ell(G, D) = \prod_{i=1}^{N} D(X_i) \times \prod_{i=1}^{N} (1 - D(G(Z_i)))$

GAN training

How to find good generators (G) and discriminators (D)?

Discriminator: maximize (log-)likelihood estimation

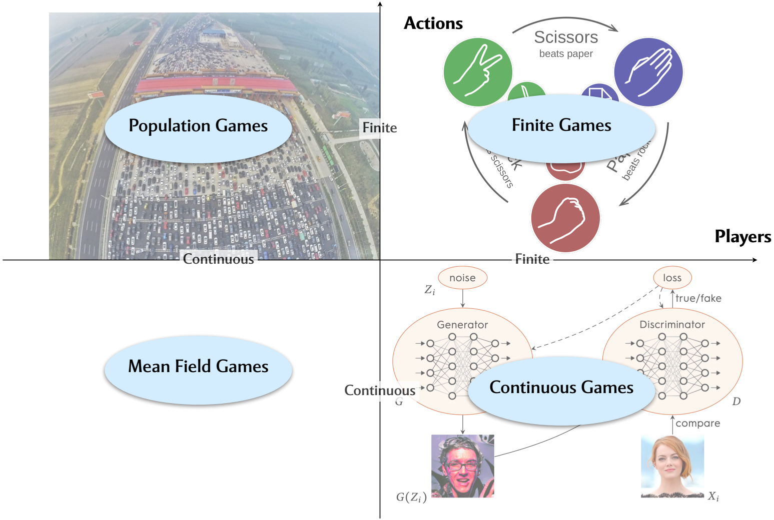
$$\max_{D \in \mathcal{D}} \log \ell(G, D)$$

Generator: minimize the resulting divergence

$$\min_{G \in \mathcal{G}} \max_{D \in \mathcal{D}} \log \ell(G, D)$$

A (very complex) zero-sum game!

Taxonomy


Actions
Players
Finite
Continuous
Finite
Continuous
Population Games
Finite Games
Mean Field Games
Continuous Games
Scissors
beats paper
noise
loss
true/fake
Generator
Discriminator
compare
Z_i
$G(Z_i)$
X_i
D

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Continuous games

Definition (Continuous games)

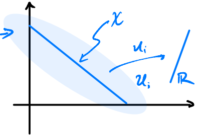
A ***continuous game*** is a collection of the following primitives:

- A finite set of ***players*** $\mathcal{N} = \{1, \ldots, N\}$
- A closed convex set of ***actions*** $\mathcal{X}_i \subseteq \mathbb{R}^{d_i}$ for each player $i \in \mathcal{N}$
- A ***payoff function*** $u_i \colon \mathcal{X} := \prod_j \mathcal{X}_j \to \mathbb{R}$ for each player $i \in \mathcal{N}$

(usually differentiable)

A continuous game with primitives as above will be denoted by $\mathcal{G} \equiv \mathcal{G}(\mathcal{N}, \mathcal{X}, u)$.

finite: $\Gamma \equiv \Gamma(N, A, u)$



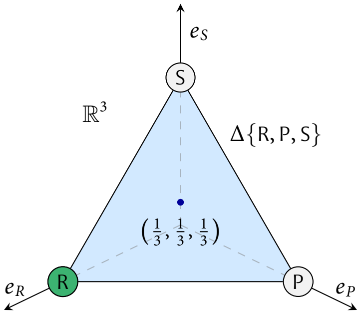
Notes:

- **Generality:** $\mathcal{X}_i$ could be more general, but almost always closed and convex in practice
- **Differentiability:** convenient to assume u_i differentiable in an open neighborhood of $\mathcal{X}$ in $\mathbb{R}^d$ $\# d = \sum_i d_i$

Example: Mixed extensions

Playing with mixed strategies:

- Players: $\mathcal{N} = \{1, \ldots, N\}$
- Pure strategies: $a_i \in \mathcal{A}_i$
- Mixed strategies: $x_i \in \mathcal{X}_i \equiv \Delta(\mathcal{A}_i)$
- Choose mixed strategy $x_i \in \mathcal{X}_i$
- Choose action $a_i \sim x_i$
- Get payoff $u_i(a_i; a_{-i})$



Example

The ***mixed extension*** $\Delta(\Gamma)$ of a finite game $\Gamma \equiv \Gamma(\mathcal{N}, \mathcal{A}, u)$ can be seen as a continuous game with

- Players: $\mathcal{N} = \{1, \ldots, N\}$
- Action sets: $\mathcal{X}_i = \Delta(\mathcal{A}_i)$
- Payoff functions: $u_i(x) = \mathbb{E}_{a \sim x}[u_i(a)] = \sum_{a_1 \in \mathcal{A}_1} \cdots \sum_{a_N \in \mathcal{A}_N} x_{1,a_1} \cdots x_{N,a_N} u_i(a_1, \ldots, a_N)$

Example: Cournot competition

A ***Cournot oligopoly*** consists of the following elements:

- N firms compete in a market for a given product — # no product differentiation
- Each firm $i = 1, \ldots, N$ can produce up to C_i of the good in question — # C_i = production capacity
- Production has a cost of $c_i > 0$ per unit
- The good is priced as a function $P(x_{\text{tot}})$ of the total production $x_{\text{tot}} = \sum_j x_j$ — # usually $P(x) = a + bx$
- The utility of each firm is given by $u_i(x) = x_i P(x_{\text{tot}}) - c_i x_i$

Example (Cournot competition)

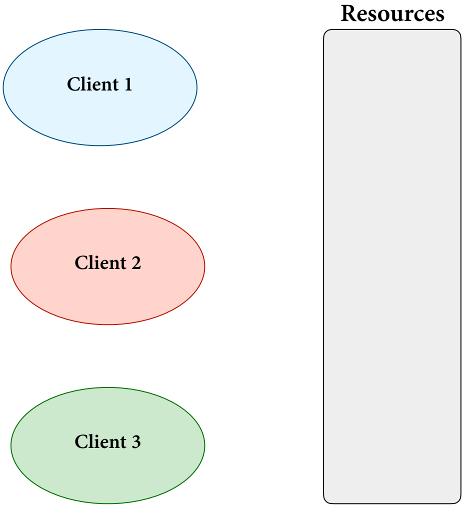
A ***Cournot oligopoly*** can be seen as a continuous game with

- Players: $\mathcal{N} = \{1, \ldots, N\}$
- Action sets: $\mathcal{X}_i = [0, C_i]$
- Payoff functions: $u_i(x) = x_i P(x_{\text{tot}}) - c_i x_i$ — # utility = revenue - cost

Example: Resource allocation / Kelly auctions

Proportionally fair resource allocation

➻ Tullock (1967, 1980); Kelly et al. (1998):



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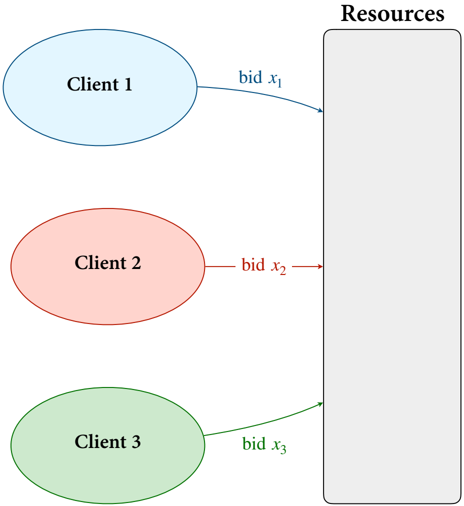

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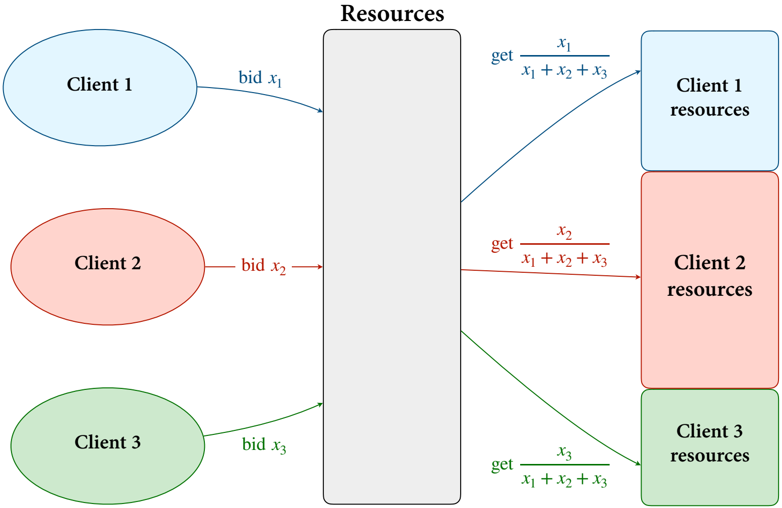

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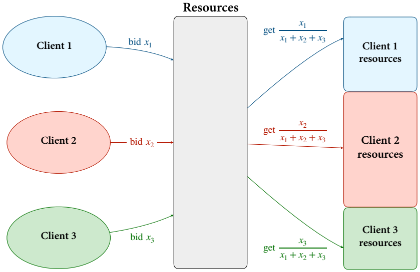

Example (Resource allocation)

A ***Kelly auction*** is a continuous game with

- Players: $\mathcal{N} = \{1, \ldots, N\}$
- Action sets: $\mathcal{X}_i = [0, B_i]$ # B_i = maximum budget of player i
- Payoff functions: $u_i(x) = \dfrac{g_i x_i}{x_1 + \cdots + x_N} - x_i$ # g_i = marginal profit from the good

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Nash equilibrium

Nash equilibrium

Let $\mathcal{G} \equiv \mathcal{G}(\mathcal{N}, \mathcal{X}, u)$ be a continuous game. An action profile $x^* = (x_1^*, \dots, x_N^*)$ is a ***Nash equilibrium*** of $\mathcal{G}$ if

$$u_i(x_i^*; x_{-i}^*) \geq u_i(x_i; x_{-i}^*) \quad \text{for all } x_i \in \mathcal{X}_i \text{ and all } i \in \mathcal{N}. \tag{NE}$$

Intuition:

- ***Stability:*** no player has an incentive to deviate
- ***Unilateral resilience:*** stable against *individual* player deviations, not multi-player ones

Characterization: best responses

Definition (Best responses)

The action $x_i^* \in \mathcal{X}_i$ is a *best response* to $x_{-i} \in \mathcal{X}_{-i}$ if

$$u_i(x_i^*; x_{-i}) \geq u_i(x_i; x_{-i}) \quad \text{for all } x_i \in \mathcal{X}_i.$$

or, equivalently, if

$$x_i^* \in \arg\max_{x_i \in \mathcal{X}_i} u_i(x_i; x_{-i})$$

The set-valued function $\mathrm{BR}_i \colon \mathcal{X}_{-i} \rightrightarrows \mathcal{X}_i$ given by

$$\mathrm{BR}_i(x_i) := \arg\max_{x_i \in \mathcal{X}_i} u_i(x_i; x_{-i})$$

is called the ***best response correspondence*** of player i. Collectively, we also let

$$\mathrm{BR}(x) = \prod_{i \in i} \mathrm{BR}_i(x_i)$$

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Nash equilibrium as fixed points

$$x^* \text{ is a Nash equilibrium of } \mathcal{G} \iff x_i^* \in \mathrm{BR}_i(x_{-i}^*) \text{ for all } i \in \mathcal{N} \iff x^* \in \mathrm{BR}(x^*)$$

Παράδειγμα: Υπολογισμός ΣΣΙ Nash σε ένα δυοπώλιο Cournot (Cournot duopoly)

Μας δίνονται $N=2$ εταιρείες με το ίδιο κόστος παραγωγής c ανά μονάδα προϊόντος κ' γραμμική εξάρτηση ανάμεσα στη συνολική παραγωγή κ' την τιμή του προϊόντος, δηλ. $P(x_{tot}) = a - b\,x_{tot}$ $\quad (x_{tot} = x_1 + x_2)$

Να υπολογισθούν τα ΣΣΙ Nash αυτού του μοντέλου Cournot.

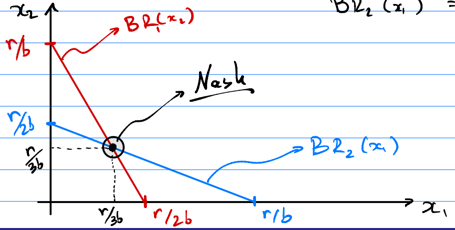
1ο Βήμα: Υπολογίζουμε τις συναρτήσεις ωφέλειας:

$$\begin{cases} u_1(x) = x_1\,[P(x_{tot}) - c] = [a - b(x_1+x_2) - c]\,x_1 \\ \quad\quad\;\; = r x_1 - b x_1 x_2 - b x_1^2 \\ u_2(x) = r x_2 - b x_1 x_2 - b x_2^2 \end{cases} \qquad (r = a - b)$$

2ο Βήμα: Υπολογίζουμε τις βέλτιστες αποκρίσεις:

$$BR_1(x_2) = \arg\max_{x_1}\{r x_1 - b x_1 x_2 - b x_1^2\} = \frac{r - b x_2}{2b}$$

$$BR_2(x_1) = \cdots\cdots = \frac{r - b x_1}{2b}$$



Συμπεραίνουμε ότι το ΣΣΙ Nash του παιγνίου είναι $x_1^* = x_2^* = r/3b$

Characterization: payoff gradients

Definition (Individual payoff gradients)

The ***individual payoff gradient*** of player $i \in \mathcal{N}$ is the vector field

$$v_i(x) = \nabla_{x_i} u_i(x_i; x_{-i})$$

and, collectively, the game's ***individual gradient field*** is

$$v(x) = (v_1(x), \dots, v_N(x))$$

Notes:

- In finite games: $\partial_{a_i} u_i(x) = u_i(a_i; x_{-i}) \implies$ ***individual gradients = mixed payoff vectors***
- In general: convenient to assume u_i differentiable in an open neighborhood of $\mathcal{X}$

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$$u_i(x) = \mathbb{E}_{a \sim x}[u_i(a)] = \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} x_{ia_i} x_{-i,a_{-i}} u_i(a_i; a_{-i})$$

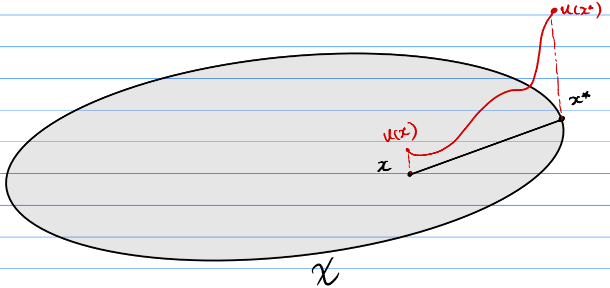
$$\frac{\partial u_i}{\partial x_{ia_i}} = \sum_{a_{-i} \in A_{-i}} x_{-i,a_{-i}} u_i(a_i; a_{-i}) = u_i(a_i; x_{-i}) = v_{ia_i}(x)$$

= a_i-th coordinate of mixed payoff vector $v_i(x)$ of player i

Ρόλος μονομερών παραγώγων στη μεγιστοποίηση

→ Αν x_i^* μεγιστοποιεί τη συνάρτηση $u_i(x_i; x_{-i}^*)$ για κάποιο $x_{-i}^* \in \mathcal{X}_{-i}$, τι μπορούμε να πούμε για τη μ. π/γο $v_i(x^*)$?

→ Ξεχνώντας προσωρινά τους άλλους παίκτες, αν x^* μεγιστοποιεί τη συνάρτηση $u(x)$, τι μπορούμε να πούμε για την π/γο $v(x^*) = \nabla u(x^*)$?



x^* μέγιστο $\Rightarrow u(x^*) \geq u(x) \quad \forall x$

$\Rightarrow$ Η συν/ση $\phi(t) = u(x^* + t(x - x^*))$ θα έχει $\phi'(0) \leq 0 \quad \forall x$

Όμως $\phi'(0) = \langle \underbrace{\nabla u(x^*)}_{v(x^*)}, x - x^* \rangle$

$$\Rightarrow \boxed{\langle v(x^*), x - x^* \rangle \leq 0 \quad \forall x \in \mathcal{X}}$$

ΣΤΑΣΙΜΟΤΗΤΑ ΠΡΩΤΗΣ ΤΑΞΗΣ

⇒ ΑΠΑΡΑΙΤΗΤΗ ΣΥΝΘΗΚΗ ΓΙΑ ΜΕΓΙΣΤΟΠΟΙΗΣΗ

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Nash equilibrium as first-order stationary points

$$u_i(x_i^*; x_{-i}^*) \geq u_i(x_i; x_i^*) \implies \begin{cases} \langle v_i(x^*), x_i - x_i^* \rangle \leq 0 & \text{for all } x_i \in \mathcal{X}_i \text{ and all } i \in \mathcal{N} \\ \langle v(x^*), x - x^* \rangle \leq 0 & \text{for all } x \in \mathcal{X} \end{cases} \quad \text{(FOS)}$$

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$$v(x) = (v_1(x), \ldots, v_N(x))$$

Nash equilibrium as first-order stationary points

$$u_i(x_i^*; x_{-i}^*) \geq u_i(x_i; x_i^*) \implies \begin{cases} \langle v_i(x^*), x_i - x_i^* \rangle \leq 0 & \text{for all } x_i \in \mathcal{X}_i \text{ and all } i \in \mathcal{N} \\ \langle v(x^*), x - x^* \rangle \leq 0 & \text{for all } x \in \mathcal{X} \end{cases} \quad \text{(FOS)}$$

When do we have (??) $\implies$ (NE)?

Outline

1. A modern game
2. Definitions and further examples
3. Nash equilibrium and characterizations
4. Concave games

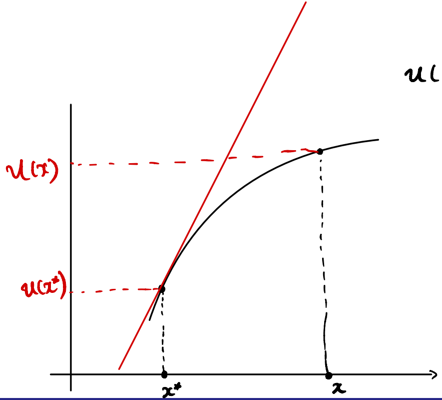
Concave games

Definition (Concave games)

A ***concave game*** is a continuous game with ***individually concave*** payoff functions, i.e.,

$$u_i(x_i; x_{-i}) \text{ is concave in } x_i$$

for all $x_{-i} \in \mathcal{X}_{-i}$ and all $i \in \mathcal{N}$.



$$u(x^*) + \langle v(x^*), x - x^* \rangle \geq u(x)$$

$$\Rightarrow \quad u(x) - u(x^*) \leq \langle v(x^*), x - x^* \rangle$$

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Notes:

- **Gradient dominance:**
$$u_i(x_i; x_{-i}) \leq u_i(x_i^*; x_{-i}) + \langle v_i(x_i^*; x_{-i}), x_i - x_i^* \rangle$$
- **Stationarity implies optimality:**
$$\langle v_i(x_i^*; x_{-i}), x_i - x_i^* \rangle \leq 0 \implies u_i(x_i; x_{-i}) \leq u_i(x_i^*; x_{-i})$$
- Closed convex $\arg\max_{x_i \in \mathcal{X}_i} u_i(x_i; x_{-i})$

Concave games

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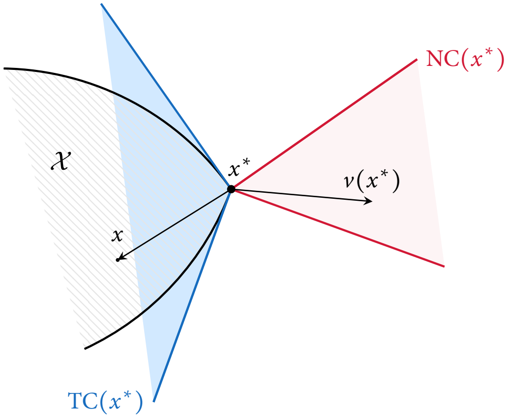
Proposition (Variational characterization of Nash equilibria)

Let $\mathcal{G} \equiv \mathcal{G}(i, \mathcal{X}, u)$ be a concave game. Then

$$x^* \text{ is a Nash equilibrium} \iff \langle v(x^*), x - x^* \rangle \leq 0 \quad \text{for all } x \in \mathcal{X}$$

Stampacchia Variational Inequality (SVI)

Geometric characterization



At Nash equilibrium, individual payoff gradients are outward-pointing

Existence of Nash equilibria

Theorem (Debreu, 1952)

Every concave game with compact action spaces admits a Nash equilibrium.

Existence of Nash equilibria

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Every concave game with compact action spaces admits a Nash equilibrium.

Proof idea: # as in finite games

- Fixed point characterization of Nash equilibria

$$x^* \text{ is a Nash equilibrium} \iff x^* \in \mathrm{BR}(x^*)$$

Existence of Nash equilibria

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Every concave game with compact action spaces admits a Nash equilibrium.

Proof idea: # as in finite games

- Fixed point characterization of Nash equilibria

$$x^* \text{ is a Nash equilibrium} \iff x^* \in \mathrm{BR}(x^*)$$

- If the game is concave, $\mathrm{BR}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is ~~nonempty~~, closed and convex

If $\mathcal{X}$ is compact $\Rightarrow$ $BR(x)$ is nonempty

Existence of Nash equilibria

Theorem (Debreu, 1952)

Every concave game with compact action spaces admits a Nash equilibrium.

Proof idea: # as in finite games

- Fixed point characterization of Nash equilibria

$$x^* \text{ is a Nash equilibrium} \iff x^* \in \mathrm{BR}(x^*)$$

- If the game is concave, $\mathrm{BR}: \mathcal{X} \rightrightarrows \mathcal{X}$ is nonempty, closed and convex
- Invoke Kakutani's fixed-point theorem for set-valued functions

Theorem (Kakutani, 1941)

Let $\mathcal{C}$ be a nonempty compact convex subset of $\mathbb{R}^d$, and let $F: \mathcal{C} \rightrightarrows \mathcal{C}$ be a set-valued function such that:

(P1) *$F(x)$ is nonempty, closed and convex for all $x \in \mathcal{C}$*

(P2) *F is **upper hemicontinuous** at all $x \in \mathcal{C}$, i.e., $\tilde{x} \in F(x)$ whenever $x_t \to x$ and $\tilde{x}_t \to \tilde{x}$ for sequences $x_t \in \mathcal{C}$ and $\tilde{x}_t \in F(x_t)$.*

Then there exists some $x^* \in \mathcal{C}$ such that $x^* \in F(x^*)$.

Proof of Debreu's theorem

Verify the conditions of Kakutani's theorem for $\mathcal{C} \leftarrow \mathcal{X}$ and $F \leftarrow \mathrm{BR}$:

(P1) $\mathrm{BR}(x)$ is a face of $\mathcal{X}$, so it is nonempty, closed and convex ➧ Why?

(P2) Argue by contradiction

- Suppose there exist sequences $x_t, \tilde{x}_t \in \mathcal{X}$, $t = 1, 2, \ldots$, such that $x_t \to x$, $\tilde{x}_t \to \tilde{x}$ and $\tilde{x}_t \in \mathrm{BR}(x_t)$, but $\tilde{x} \notin \mathrm{BR}(x)$.
- Then there exists a player $i \in \mathcal{N}$ and a deviation $x_i' \in \mathcal{X}_i$ such that

$$u_i(x_i'; x_{-i}) > u_i(\tilde{x}_i; x_{-i})$$

- But since $\tilde{x}_{i,t} \in \mathrm{BR}(x_{-i,t})$ by assumption, we also have:

$$u_i(x_i'; x_{-i,t}) \leq u_i(\tilde{x}_{i,t}; x_{-i,t})$$

- Since $x_t \to x$, $\tilde{x}_t \to \tilde{x}$ and u_i is continuous, taking limits gives

$$u_i(x_i'; x_{-i}) \leq u_i(\tilde{x}_i; x_{-i})$$

which contradicts our original assumption. □

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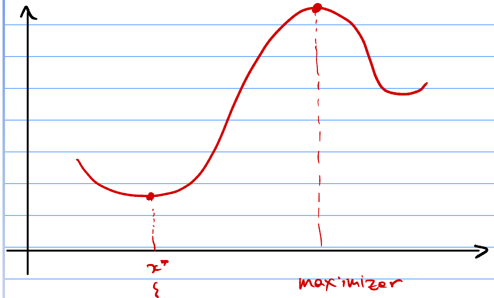
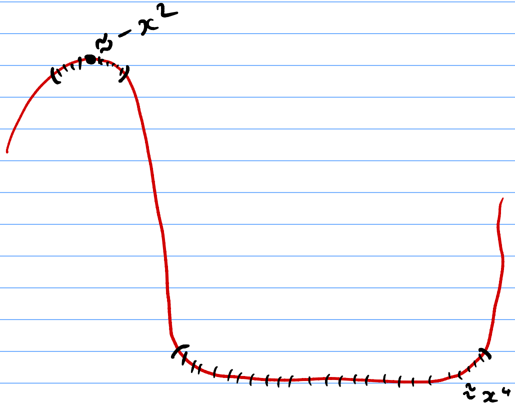
Ρόλος μονομερών παραγώγων στη μεγιστοποίηση

$$\langle v(x^*), x - x^* \rangle \le 0 \quad \forall x \in \mathcal{X}$$

⇓

ΕΙΝΑΙ ΤΟ x^* NASH?





ελάχιστο $\Rightarrow v(x^*) = 0 \Rightarrow \langle v(x^*), x - x^* \rangle = 0 \quad \forall x$

⟹ ΣΙΓΟΥΡΑ ΟΧΙ NASH