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Correlated Equilibrium and Potential Games[1]

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Abstract: Any correlated equilibrium of a strategic game with bounded payoffs and convex strategy sets which has a smooth concave potential, is a mixture of pure strategy profiles which maximize the potential. If moreover, the strategy sets are compact and the potential is strictly concave, then the game has a unique correlated equilibrium.

1 Introduction

It is well known that games arising from Cournot competition models with linear demand and linear cost functions have a unique strategic equilibrium. A surprising result due to Luchuan Liu asserts that these Cournot games also have a unique correlated equilibrium. This paper generalizes Liu's result. We consider the class of strategic games that have convex strategy sets, a smooth concave potential and bounded payoffs. It is shown for this class that the set of correlated equilibrium coincides with the set of mixtures of pure strategies which maximize the potential. As a corollary we obtain uniqueness of correlated equilibrium if in addition the potential is strictly concave.

2 Preliminaries

1. Strategic Equilibrium. A *strategic game* is a triple $(N; (S^i)_{i\in N}; (h^i)_{i\in N})$ where N is the set of players and for every player i in N, S^i is the set of strategies of player i, and the payoff function h^i is a real valued function defined on S, where S denotes the cartezian product $S = \times_{i\in N} S^i$. For s in S, $h^i(s)$ represents the payoff to player i. For every s in S, i in N, and every strategy t^i in S^i we denote by $(s|{}^it^i)$ or $(s|t^i)$ for short the N-tuple of strategies in S whose ith component is t^i and for $j \neq i$, the jth component equals s^j. A point s in S is a *pure strategy equilibrium* if $h^i(s) \geq h^i(s|t^i)$ for every t^i in S^i, and every i in N.

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2. *Potential Games.* A *potential* for a game $G=(N;(S^i)_{i\in N};(h^i)_{i\in N})$ is a real valued function P on S ($P:S\to\mathbb{R}$) such that for every player i, every strategy profile s in S and every t^i in S^i, $P(s|t^i)-P(s)=h^i(s|t^i)-h^i(s)$. A *potential game* is a game that has a potential. Obviously, any point s in S which maximizes a potential for a game G is a pure strategy Nash equilibrium of G.

Potential games were introduced in Monderer and Shapley (1994). The strategic game $(N;\ (S^i)_{i\in N};(h^i)_{i\in N})$, where $S^i=[0,a/b]$, $a>0$ and $b>0$, and $h^i(q_1,\dots,q_n)=(a-b\sum_{j=1}^n q_j)q_i-c_i(q_i)$, $c_i:[0,a/b]\to\mathbb{R}$ a bounded measurable function, is called the auxiliary game arising from Cournot competition with a linear price function and arbitrary cost functions $c_i(q_i)$, $1\le i\le n$. Monderer and Shapley (1994) show that this auxiliary game is a potential game. The function $P((q_1,\dots,q_n))=a\sum_{j=1}^n q_j-b\sum_{j=1}^n q_j^2-b\sum_{1\le i<j\le n}q_iq_j-\sum_{j=1}^n c_j(q_j)$ is a potential for this game. Note that this potential function P is smooth and concave whenever the cost functions c_i are smooth and convex, and that the convexity assumption here on the cost functions c_i, could be replaced by assuming that the cost functions have second derivatives and $b+c_i''(q_i)\ge 0$.

3. *Correlated Equilibria.* A correlated equilibrium of a game in strategic form is defined by means of a probability distributions on $S=S^1\times\cdots\times S^n$. Thus, we assume a given measurable structure on each S^i and assume that the payoffs are measurable.

A *correlated equilibrium* of a game $G=(N;(S^i)_{i\in N};(h^i)_{i\in N})$ is a probability distribution μ on $S=\times_{i\in N}S^i$ such that for every i in N and every measurable function $\xi^i:S^i\to S^i$,

$$\int_S h^i(x)d\mu(x)\ge\int_S h^i(x|\xi^i(x^i))d\mu(x).$$

3 The Results

Theorem 1: Let $G=(N;(S^i)_{i\in N})$ be a game in strategic form (normal form) with bounded payoffs, such that:

(1) $S^i, i\in N$, are convex.
(2) G has a smooth (C^1) concave potential.

Then any correlated equilibrium of G is a mixture of pure strategy profiles which maximize the potential.

Proof: Let P be a smooth concave potential for G, and let μ be a probability measure on $S=S^1\times\cdots\times S^n$. First note that any potential P for a game G with bounded payoffs is bounded. We will prove that μ is a correlated equilibrium iff

$\int_S P(x)d\mu(x) \geq P(y)$ for every y in S. First assume that $\int_S P(x)d\mu(x) \geq P(y)$ for every y in S. Then for any measurable $f:S \to S$,

$$\int_S (P(x) - P(f(x))d\mu(x) \geq 0$$

which implies in particular that for any $i \in N$ and any measurable $f:S^i \to S^i$

$$\int_S (h^i(x) - h^i(x|f^i(x^i)))d\mu(x) = \int_S (P(x) - P(x|f^i(x^i)))d\mu(x) \geq 0$$

which proves that μ is a correlated equilibrium.

Next, assume that there exists y in S with

$$\int_S P(x)d\mu(x) < P(y).$$

As P is concave, for any x in S, the function $\varepsilon \to P((1-\varepsilon)x + \varepsilon y)$ is concave. Therefore, the following limit and inequality exist for every x in S,

$$\lim_{\varepsilon \to 0+} \frac{P((1-\varepsilon)x + \varepsilon y) - P(x)}{\varepsilon} \geq P(y) - P(x). \tag{1}$$

Let $P_{(y^i - x^i)}$ denote the directional derivation of P in the direction $y^i - x^i$. As $y - x = \sum_{i=1}^n y^i - x^i$ and P is smooth,

$$\lim_{\varepsilon \to 0+} \frac{P((1-\varepsilon)x + \varepsilon y) - P(x)}{\varepsilon} = \sum_{i=1}^n P_{(y^i - x^i)}(x). \tag{2}$$

Therefore it follows from (1) and (2) that $\sum_{i=1}^n P_{(y^i - x^i)}(x) \geq P(y) - P(x)$, implying that $\int_S (\sum_{i=1}^n P_{(y^i - x^i)}(x))d\mu(x) \geq \int_S (P(y) - P(x))d\mu(x) > 0$. Therefore, there exists i in N with

$$\int_S P_{(y^i - x^i)}(x)d\mu(x) > 0. \tag{3}$$

For $0 < \varepsilon \leq 1$, let $x^i_\varepsilon = (1-\varepsilon)x^i + \varepsilon y^i$. The concavity of P implies the concavity of $h^i(x|x^i_\varepsilon) - h^i(x)$ as a function of ε. Therefore, $[h^i(x|x^i_\varepsilon) - h^i(x)]/\varepsilon$ increases to $P_{(y^i - x^i)}(x)$ as $\varepsilon \to 0+$, and thus by the monotone convergence theorem and (3),

$$\lim_{\varepsilon \to 0} \int_S \frac{h^i(x|x^i_\varepsilon) - h^i(x)}{\varepsilon} d\mu(x) = \int_S P_{(y^i - x^i)}(x)d\mu(x) > 0.$$

Therefore, there exists $\varepsilon > 0$ such that

$$\int_S h^i(x \mid x^i_\varepsilon) d\mu(x) > \int_S h^i(x) d\mu(x)$$

proving that μ is not a correlated equilibrium.

If $\int_S P(x)d\mu(x) \geq P(y)$ for every y in S, then $P(x) \geq \sup\{P(y): y \in S\}$ for $\mu - a.e.\ x$, thus $\max\{P(y): y \in S\}$ exists and for $\mu - a.e.\ x$ $P(x) = \max\{P(y): y \in S\}$ and x is a (pure strategy) Nash equilibrium. ■

The following is a straight forward corollary of Theorem 1.

Corollary: Let $G = (N; (S^i)_{i \in N}; (h^i)_{i \in N})$ be a game in strategic form (normal form) with bounded payoffs, such that:

(1) $S^i, i \in N$, are convex.
(2) G has a smooth (C^1) concave potential P.

Then the set of pure strategy equilibria coincides with $\arg\max P$ which is a convex subset of S.

Theorem 2: Let $G = (N; (S^i)_{i \in N}; (h^i)_{i \in N})$ be a game in normal form, such that:

(1) S^i are convex and compact.
(2) G has a (smooth) C^1 strictly concave potential.

Then G has a unique correlated equilibria.

Proof: Let $\bar{x}$ be the unique maximizer of a potential function P for the game G. Then $\bar{x}$ is a pure strategy Nash equilibrium, and by the previous theorem it follows that $\bar{x}$ is the unique correlated equlibrium. ■

Remarks: The results assert that if G has a concave smooth potential, then the set of pure strategy equilibria is convex and any correlated equilibrium is a convex combination of pure strategy Nash equilibria, and that any pure strategy Nash equilibrium x, maximizes the potential. It is easy to construct a two person game G with strategy sets $[0, 1]$ having a concave potential, and a pure strategy Nash equilibrium x that does not maximize the potential. For instance, let P be the smallest nonnegative concave function on $[0, 1]^2$ with $P(1/2,\ 1/2) = 1$, i.e., $P(x, y) = 1 - 2\max(|x - 1/2|,\ |y - 1/2|)$. Then any point (x, y) with $|x - 1/2| = |y - 1/2|$ is a pure strategy Nash equilibrium. Thus the conclusion of Theorem 1 no longer holds if one drops the smoothness assumption. Similarly, one could construct a strictly convex potential on $[0,1]^2$ with the corresponding game having multiple equilibria. Indeed, set

$$P(x,y) = 1 - 2|y-x| - (x-1/2)^2 - (y-1/2)^2.$$

Then the two player game $(\{1,2\},[0,1]^2,(P,P))$ has a strictly concave potential P which is maximized at the point (1/2, 1/2), and any point (x,y) with $x=y$ is a pure strategy equilibrium. However, all these equilibrium points (x,x) with $x \neq 1/2$ lacks some form of stability. It is therefore of interest to look for a dynamic type theorem which addresses the correlated equilibrium points of strategic games with a concave potential.

References

Aumann RJ (1974) Subjectivity and correlation in randomized strategies. Journal of Mathematical Economics 1: 67–96

Liu L (1993) Correlated equilibrium of cournot oligopoly competition. To appear in Journal of Economic Theory

Monderer D, Shapley LS (1994) Potential games. To appear in Games and Economic Behavior

Rosenthal RW (1973) A class of games possessing pure-strategy equilibria. International Journal of Game Theory 2: 65–67

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