On the Relationship between Nash–Cournot and Wardrop Equilibria*

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A noncooperative game is formulated on a transportation network with congestion. The players are associated with origin–destination pairs, and are facing demand functions at their respective destination nodes. A Nash–Cournot equilibrium is defined and conditions for existence and uniqueness of this solution are provided. The asymptotic behavior of the Nash–Cournot equilibrium is then shown to yield (under appropriate assumptions) a total flow vector corresponding to a Wardrop equilibrium.

1. INTRODUCTION

The aim of this article is to characterize noncooperative equilibria for a class of games defined on a transportation network subject to congestion effects, and to obtain convergence results showing that Nash equilibria can approximate Wardrop equilibria [26].

The consideration of a game defined on a transportation network occurs naturally if one imagines a finite number of players (economic agents or firms) sending flows from various origins toward different destinations, on a common transportation network. We will assume two sources of interactions between players: (a) through the demand laws of their selling market, and (b) via the congestion effect which modifies the unit transportation cost on each arc depending on the total flow circulating on the network. The two extreme cases of a one-player game and of a game with infinitely many "infinitesimal" players have already received considerable attention in the transportation literature. In the one player case, the problem reduces to the classical convex min-cost transportation problem. The limiting case of an infinite number of infinitesimal players corresponds, as will be shown later on in this paper, to the Wardrop equilibrium concept and, more generally, to competitive equilibria on a network [1, 2, 3, 5].

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The case of a finite number of players competing for the utilization of a common transportation network would correspond, for example, to a situation where different gas utility companies supplying different markets are sharing a common pipeline and main distribution network. In such a situation, one can imagine that, if they do not cooperate, the companies will tend to react to the transportation program of their competitors on the network. An equilibrium will be reached when each transportation program is the optimal reaction from the company concerned to the competitors programs.

The game considered in this article is closely related to the Cournot oligopoly model in the economic theory of imperfect competition. In the Cournot model, a finite number of firms compete on a given market characterized by a known demand function. The interaction between the firms comes from the price determination mechanism which is dependent on the total supply on the market.

In the model studied in this article, the structure is very similar, with a little more generality since the interaction between players stems from the unit transportation cost as well as from the demand laws.

Also, as there are many arcs, with a flow conservation constraint at each node, the network game model appears as a generalized Cournot model with many products and production constraints.

The characterization of equilibria will include (i) sufficient conditions for existence of an equilibrium; (ii) sufficient conditions for uniqueness of equilibrium; (iii) analysis of the asymptotic behaviour of equilibria when the number of players becomes very large, each player becoming infinitesimal.

2. EQUILIBRIUM ON A TRANSPORTATION NETWORK AND RELATIONSHIP WITH COURNOT OLIGOPOLY THEORY

Let $G \triangleq (N, A)$ be a directed network where N is a finite set of nodes and A is a set of arcs, i.e., one-way links between pairs of nodes.

Let $M \triangleq \{1, 2, \ldots, m\}$ be the set of players competing on the network. Associated with Player i of M is the flow vector

$$v^i \triangleq (v_a^i)_{a \in A}$$

where v_a^i is the flow sent by Player i and circulating on arc a.
Define

$$v_a \triangleq (v_a^i)_{i \in M}$$

and

$$V \triangleq (v_a^i)_{\substack{i \in M \\ a \in A}}.$$

V will thus represent the multiflow generated on the various arcs by the players.

We assume that, on a given arc a of A, the unit transportation cost for Player i, called also link traversal cost for i, is given as a function of the flow variables

$$S_a^i(V) \triangleq \text{link traversal cost on } a \in A \text{ for player } i \in M.$$

We finally assume that each player i is associated with a unique pair origin-destination $(x^i, y^i) \in N \times N$, where x^i corresponds to the production situs, and y^i corresponds to the selling market for this player. The selling market of Player i is characterized by the (inverse) demand function

$$p^i = f^i(q^1, q^2, \ldots, q^m)$$

which determines the unit selling price p^i of the product shipped by i from x^i to y^i, as a function of the total flows $q^1, \ldots, q^m$ sent by the different players to their respective markets.

It is possible to describe the system as a network with a multiflow circulation by introducing a return arc a^i from each market y^i to each corresponding production situs x^i, $i \in M$, with an associated traversal cost

$$S^i_{a^i}(v_{a^1}, \ldots, v_{a^m}) \triangleq -f^i(v_{a^1}, \ldots, v_{a^M})$$

Definition 2.1. The set of feasible flows for Player i is the set Φ^i of vectors $(v^i, v^i_{a^i})$ satisfying the flow conservation equations at each node of N. By a slight abuse of notation, these vectors will still be denoted v^i. ■

We notice that, according to the definition of the return arcs a^i one has

$$v^i_{a^j} = 0 \qquad \text{if } i \neq j.$$

Definition 2.2. A multiflow vector V^* constitutes a Nash–Cournot equilibrium if, for each player i the following holds:

(i) $v^{i*} \in \Phi^i$

(ii) $$\sum_{a \in A} v^{i*}_a S^i_a(V^*) + v^{i*}_{a^i} S^i_{a^i}(V^*) = \underset{v^i \in \Phi^i}{\text{Min}} \sum_{a \in A} v^i_a S^i_a(V^{*(i)}) + v^i_{a^i} S^i_{a^i}(V^{*(i)}) \qquad (1)$$

where we have denoted $V^{*(i)}$ the flow vector defined by

$$V^{*(i)} \triangleq (v^{1*}, \ldots, v^{i-1*}, v^i, v^{i+1*}, \ldots, v^{m*}) \qquad (2). \ ■$$

According to this definition, at equilibrium, each player i minimizes a function equal to his transportation cost from his production situs to his market, minus the total value of sales on his market, and given the transportation and sales programs of his competitors.

Remark 2.1. Condition (1) expresses the usual property of noncooperative equilibria for games in normal form. If one defines the strategy set of Player i as Φ^i and the pay-off* of Player i as a function $J^i : \Phi^1 \times \cdots \times \Phi^m \to \mathrm{R}$ given by

$$J^i(v^1, v^2, \ldots, v^m) = v^i_{a^i} f^i(v_{a^1}, v_{a^2}, \ldots, v_{a^m}) - \sum_{a \in A} v^i_a S^i_a(V) \qquad (3)$$

*We follow the usual conventions according to which players maximize their pay-offs in game theory whereas users minimize their transportation costs in traffic equilibrium theory.

the equilibrium condition (1) reduces to

$$J^i(v^{1*}, \ldots, v^{i*}, \ldots, v^{m*}) = \underset{v^i \in \Phi^i}{\text{Max}}\ J^i(v^{1*}, \ldots, v^i, \ldots, v^{m*}) \tag{4}$$

i.e., the usual Nash equilibrium definition.

Remark 2.2. The Cournot oligopoly model [16, 22] describes m firms with cost functions $c_i(q_i)$ competing on a market described by an inverse demand function $p = f(Q)$, where Q is the total supply on the market. A Cournot equilibrium is a vector $q^* = (q_1^*, q_2^*, \ldots, q_m^*)$ such that

(i) $q_i^* \geq 0, \qquad i \in M$

(ii) $$q_i^*\, f\left(\sum_{j\in M} q_j^*\right) - c_i(q_i^*)$$

$$= \underset{q_i \geq 0}{\text{Max}}\ q_i\, f\left(q_i + \sum_{j\neq i} q_j^*\right) - c_i(q_i). \tag{5}$$

There is a clear relationship between condition (1) for the network equilibrium model and condition (4) for the Cournot oligopoly model. In the Cournot model, the coupling between the firms stems from the common market demand law. In the network equilibrium model, the coupling is also due to the congestion effect on unit transportation costs. In the latter model, however, one has for each player a vector-valued strategy, subject to polyhedral constraints ($v^i \in \Phi^i$) instead of a scalar-valued, nonnegative strategy for the Cournot model. Furthermore, in this network model, the congestion effects and the demand laws are given a very general form.

Although our model could be considered as a generalization of the oligopoly models presented in [14, 16, 24], the efficient algorithms proposed for the computation of Cournot equilibria in these references do not extend directly.

Classical results of game theory [9, 20] can be used for establishing conditions for existence and uniqueness of an equilibrium as defined by (1)–(2). Gabay and Moulin [10] considered Nash games in variational inequality form, extending existence, uniqueness, and stability results previously obtained by Lions and Stampacchia [12].

Theorem 2.1. Assume the following

A1. (i) For each player i there is a value $\bar{q}^i$ such that the demand function satisfies

$$f^i(q^1, \ldots, q^m) = 0 \qquad \text{if } q^i \geq \bar{q}^i \text{ for any } q^1, \ldots, q^{i-1}, q^{i+1}, \ldots, q^m.$$

(ii) There exists $q^j < \bar{q}^j$ for $j = 1, \ldots, m$ such that

$$f^i(q^1, \ldots, q^m) > 0 \qquad \text{for all } i \in M.$$

(iii) The functions f^i and S_a^i are positive-valued for all $i \in M$.

A2. For each player i, the payoff function (3) is continuous on $\Phi^1 \times \cdots \times \Phi^m$ and strictly quasi-concave with respect to $v^i \in \Phi^i$.

Then there exists an equilibrium.

Proof. Under A1, one can restrict the strategy set of each player i to the compact subset

$$\psi^i \triangleq \{v^i \in \Phi^i : v^i_{a'} \leq \tilde{q}^i\}$$

Then the game is defined with a finite set of players, compact strategy sets and continuous and strictly quasi-concave (with respect to v^i), payoff functions Φ^i. Then Theorem 7.1 of Ref. [9] applies to guarantee the existence of an equilibrium. ■

Remark 2.3. Assumption A2 of Theorem 2.1 will be satisfied if the revenue functions $q^i f^i(q^1, \ldots, q^i, \ldots, q^m)$ are concave w.r.t. q^i and the costs $v^i_a S^i_a(V)$ are convex w.r.t. v^i_a.

Sufficient conditions for uniqueness of equilibrium are much more restrictive. Rosen [20] has given a general condition which has been explicited further by Goodman [11]. In our framework, this will yield the following results.

Theorem 2.2. Assume the following hypotheses.

A3. Each payoff function J^i is C^2, and there exists $r > 0$ in R^m such that the symmetric matrix

$$[G(V, r) + G^T(V, r)] \tag{6}$$

is negative definite, where $G(V, r)$ is the Jacobian with respect to V of the mapping

$$g(V,r) = \begin{bmatrix} g^1(V,r) \\ g^2(V,r) \\ \vdots \\ g^m(V,r) \end{bmatrix} = \begin{bmatrix} r_1 \nabla_{v^1} J^1(V) \\ r_2 \nabla_{v^2} J^2(V) \\ \vdots \quad \vdots \\ r_m \nabla_{v^m} J^m(V) \end{bmatrix} \tag{7}$$

Then the equilibrium satisfying (4) is unique.

Proof. See Rosen [20]. ■

Theorem 2.3. Assume that each payoff function J^i is C^1 and negative semidefinite over Φ^i. Then V^* is an equilibrium solution if and only if it satisfies the variational inequality:

$$(V^* - V)^T g(V^*, r) \geq 0 \qquad \forall\, V \in \Phi^1 \times \cdots \times \Phi^m. \tag{8}$$

Proof. For a fixed player i, the first order necessary and sufficient optimality conditions (Luenberger [13]) state that no feasible ascent direction exist at the optimum, i.e.,

$$(v^* - v^i)^T g^i(V^*) \geq 0 \qquad \forall\, v^i \in \Phi^i. \tag{9}$$

Aggregating (9) for all players yields the desired result. ■

Remark 2.4. The assumption A3 of Theorem 2.2 could be replaced by the slightly more general condition of strict monotonicity of $g(V, r)$ for some $r > 0$ given in R^m.

This monotonicity condition is also the basic condition for convergence of iterative schemes (see [4, 5, 18, 19]) used for computation of the solution V^* to the variational inequality (9).

Therefore the uniqueness condition given by Rosen is also the condition permitting the computation of equilibrium by known iterative methods (e.g. projection and relaxation methods).

Corollary 2.1. Assume the following

A4. Each payoff function J^i is C^2, strictly concave in v^i and convex in $(v^1, \ldots, v^{i-1}, v^{i+1}, \ldots, v^m)$,

A5. There exists $r > 0$ in R^m such that $\sigma(V,r) \triangleq \sum_{i=1}^m r_i J^i(V)$ is concave in V.

Then the equilibrium is unique.

Proof. See Goodman [11]. ■

Example 2.1. We consider a network where the link traversal cost on any arc a of A depends only on the total flow

$$w_a = \sum_{i=1}^{m} v_a^i$$

and is the same for all players: i.e.,

$$S_a^i(V) \equiv S_a(w_a). \tag{10}$$

On the arc a^i the total flow circulating is $v_{a^i}^i$ and the link traversal cost is a function of the total flow circulating on the arcs a^j, $j \in M$.

$$S_{a^i}(V) = -f\left(\sum_{j=1}^{m} v_{a^j}^j\right) \tag{11}$$

which is also assumed to be the same for all players.

The payoff function (3) is now given by

$$J^i(v^1, v^2, \ldots, v^m) = v_a^i f\left(\sum_{j=1}^{m} v_{a^j}^j\right) - \sum_{a \in A} V_a^i S_a(w_a) \tag{12}$$

Goodman's conditions will be verified with $r_1 = r_2 = \cdots = r_m = 1$ if the following holds.

P1. For each arc $a \in A$ the total cost function $w_a S_a(w_a)$ is convex in w_a

P2. The total revenue function $q f(q)$ is concave w.r.t. the variable $q = \sum_{j=1}^m v_{a^j}^j$

P3. For each arc $a \in A$ the link traversal cost $S_a(w_a)$ is a concave function of w_a

P4. The (inverse) demand law $f(q)$ is convex w.r.t. q.

3. CONVERGENCE OF NASH–COURNOT EQUILIBRIA TO A WARDROP EQUILIBRIUM

In the traffic equilibrium problem one considers the network $G \triangleq (N, A)$ with the total link-flow vector

$$w \triangleq (w_a)_{a \in A}$$

which is obtained as the summation of the flow vectors generated by all the users of the network. Let us assume that for any user the link traversal cost depends only on w and is independent of the particular player considered,

$$S_a^i(V) \equiv S_a(w) \qquad \forall\, a \in A, \text{ for any user } i.$$

Each user i is associated with a pair origin destination $(x^i, y^i) \in N \times N$, and uses a unique path for sending his flow.

The Wardrop equilibrium principle is related to the situation where the flow w is generated by a very large number of infinitesimal users. By infinitesimal it is meant that, when a particular user i unilaterally switches his flow from a particular path from x^i to y^i to another path from x^i to y^i then there is no sensible modification on the link traversal costs. The traffic flow is a Wardrop equilibrium if no user can reduce his cost by switching from his current path to another one connecting the same origin-destination pair.

This definition can be made more precise.

Definition 3.1. A flow vector w^* is a Wardrop equilibrium if, for any pair (x^1, y^1) the following holds

$$\sum_{a \in k^*} S_a(w^*) \leq \sum_{a \in k} S_a(w^*) \tag{13}$$

for any $k^* \in P_i^*$ and $k \in P_i$, where P_i is the set of all paths from x^i to y^i and P_i^* is the subset of P_i consisting of all paths which are actually used (i.e., such that a positive flow is circulating on them).

It is shown in [1, 5, 23] that a Wardrop equilibrium flow vector w^* is characterized as being a solution of the variational inequality

$$(w^* - w)^T S(w^*) \leq 0 \qquad \forall\, w \in \varphi \tag{14}$$

where φ is the set of all feasible flows w.

We are interested in the following question: is it possible to approximate a Wardrop equilibrium flow w^* by the total flow $w^*(n)$ resulting from a Nash equilibrium for the game played on the network G by a set of nm players partitioned into m classes I_i, $i \in M$, each class corresponding to a particular origin–destination pair (x^i, y^i)? Such result would allow a new interpretation of a flow satisfying condition (14), and also it would establish a similarity with classical results in economic theory dealing with the convergence of Cournot equilibria to competitive equilibria.

A relationship between traffic equilibrium and Nash equilibrium has already been established by Rosenthal [21], Devarajan [8], and Dafermos and Sparrow [3]. Rosenthal has considered a discrete version of the traffic equilibrium problem with a particular link traversal cost function. Devarajan found an equivalence between the Wardrop equilibrium flow and the one resulting from a Nash equilibrium for an N-player game, each player corresponding to a particular origin–destination pair.

Both these results are very different from the one we proceed to prove. First we consider "players" instead of "users," which means that we allow each player to allocate his flow to several paths. Secondly, the payoff functions of our game do not depend on integrability conditions of the cost functions involved. Such a condition is required in Devarajan [8].

To prove the next results, we will use a replication strategy which is similar, in spirit, to that of Debreu and Scarf [6] and Debreu [7].

Let us call $\Gamma(n)$ the game defined on the network G in the following way. There are m origin–destination pairs (x^i, y^i), $i = 1, \ldots, m$. For each i there is also a set M_i of n identical players who share the same link traversal cost functions $S_a(w)$ and the same demand law $f^i(q^1, \ldots, q^m)$, where q^j is the total flow arriving on market, j, i.e.,

$$q^j = \sum_{l \in M_j} v^l_{a^j} \qquad j = 1, \ldots, m. \tag{15}$$

All the results of Section 2 concern a game $\Gamma(1)$. These results extend directly to any game $\Gamma(n)$. Adapting the variational inequality formulation derived in theorem 2.3 to this situation, we obtain:

Lemma 3.1. Consider the game $\Gamma\,(n)$ with payoffs.

$$J^l(v^1, v^2, \ldots, v^{mn}) = v^l_{a^i} f^i(q^1, \ldots, q^m) - \sum_{a \in A} v^l_a S_a(w) \tag{16}$$

for any $l \in M_i$, $i = 1, \ldots, m$, and where w and q^j have been defined previously.

Assume that each function J^l is C^1. Then a Nash equilibrium solution $V^*(n)$ to the game $\Gamma(n)$ necessarily satisfied the variational inequality

$$(v^{l*}(n) - v^l)^T \nabla J^l(V^*(n)) \gtrsim 0 \qquad \forall\, V^l \in \Phi^l, \forall\, l \in M_i,\ i = 1, \ldots, m. \tag{17}^*$$

Remark 3.1. By using the extended set of arcs $\bar{A} \triangleq A \cup \{a^i : i \in M\}$ with traversal cost on each arc a^i defined as

$$S_{a^i}(w) \triangleq -f^i(q^1, \ldots, q^m) \tag{18}$$

the payoff (16) can be rewritten under the more symmetric form

$$J^l(v^1, v^2, \ldots, v^{mn}) = \sum_{a \in A} v^l_a S_a(w) \tag{19}$$

*For the explanation of the $\gtrsim$ sign, see footnote in Section 2.

and therefore the inequalities (17) can be rewritten

$$(v^{l*}(n) - v^l)^T (S(w^*(n)) + (\nabla S(w^*(n)))^T v^{l*}(n)) \leq 0$$
$$\forall\, v^l \in \Phi^l \qquad \forall\, l \in M_i,\ i = 1, \ldots, m. \tag{20}$$

We are now ready to prove the main result of this section.

Theorem 3.1. Under the assumptions of Theorem 2.1 there exists a Wardrop equilibrium on the network G, with flow vector w^* and a sequence $\{\Gamma(n_k)\}_{k \in \mathrm{N}}$ of games on G admitting Nash equilibria multiflow $V^*(n_k)$ such that the resulting total flow vectors $w^*(n_k)$ verify

$$\lim_{k \to \infty} w^*(n_k) = w^*$$

Proof. Consider any game $\Gamma(n)$. By a direct adaptation of Theorem 2.1, we can prove that there exists a Nash equilibrium multiflow $V^*(n)$. Furthermore, the players in M_i being all identical, there exists a Nash equilibrium for which all players of M_i have the same strategy. This common Nash-equilibrium strategy can be written

$$v^{l*}(n) = \frac{1}{n}\, \omega^{i*}(n), \qquad \forall\, l \in M_i$$

where $\omega^{i*}(n)$ is a flow which is uniformly bounded for all n.

The associated total link-flow vector

$$w^*(n) = \sum_{i=1}^{m} \sum_{l \in M_i} v^{l*}(n) = \sum_{i=1}^{m} \omega^{i*}(n)$$

is also uniformly bounded. Hence there exists a subsequence $(n_k)_{k \in N}$ such that $w^*(n_k)$ converges to a limit w^*.

Replacing $v^{l*}(n)$ by ω^i/n_k in (20) and summing over all players, there comes

$$(w^*(n_k) - w)^T S(w^*(n_k)) + Q_{n_k} \leq 0 \qquad \forall\, w \in \varphi \tag{21}$$

with

$$Q_{n_k} = \sum_{i=1}^{m} \sum_{l \in M_i} \left(\frac{\omega^{i*}(n_k)}{n_k} - v^l \right) (\nabla S(w^*(n_k)))^T \frac{\omega^{i*}(n_k)}{n_k}. \tag{22}$$

It is readily seen that, φ being compact, one has $\lim_{k \to \infty} Q_{n_k} = 0$, therefore, w^* satisfies

$$(w^* - w)^T S(w^*) \leq 0 \qquad \forall\, w \in \varphi$$

and is, by (14), a Wardrop equilibrium. ∎

Remark 3.2. The quasi-concavity of the payoff function is not essential in the proof of Theorem 3.1. It suffices to guarantee that a Nash equilibrium solution exists for any n.

The next result strengthens the convergence result.

Theorem 3.2. If for each game $\Gamma(n)$ there exists a Nash equilibrium solution and if $S(w)$ is strictly monotone, then there exists a unique Wardrop equilibrium w^*, and for any sequence $(w^*(n))_{n\in N}$ of Nash equilibria one has

$$\lim_{n\to\infty} w^*(n) = w^*.$$

Proof. It is well known that, when S is strictly monotone, there exists a unique solution w^* to (14) (Ref. [4]).

Also it is easily proved that, for any Nash equilibrium for the game $\Gamma(n)$, all the players in a set M_i have the same strategy.

Now assume that some sequence $\{w^*(n)\}_{n\in N}$ does not converge, and let $\{w^*(n_k)\}_{k\in N}$ be a subsequence such that

$$\lim_{k\to\infty} w^*(n_k) = \bar{w} \neq w^*.$$

Taking the limit in (21) there comes:

$$\begin{aligned}\lim_{k\to\infty}\ & (w^*(n_k) - w^*)^T S(w^*(n_k)) + Q_{n_k} \\ &= (\bar{w} - w^*)^T S(\bar{w}) \\ &\leq 0\end{aligned}$$

This last inequality implies, by strict monotomicity of S, that we must have $\bar{w} = w^*$, a contradiction.

Consequently:

$$\lim_{n\to\infty} w^*(n) = w^*. \qquad \blacksquare$$

Remark 3.3. Most of the algorithms [4, 18, 19] recently proposed for computing Wardrop equilibria assume that $S(w)$ is uniformly monotone, which means that there exists a positive constant β such that

$$(w - w')^T(S(w) - S(w')) \geq \beta \|w - w'\|^2 \qquad \forall\, w, w' \in \varphi. \tag{23}$$

Under this hypothesis it is possible to derive a sharper estimate of the convergence rate of the oligopoly model towards competitive equilibrium. More precisely we have:

Theorem 3.3 If for each game $\Gamma(n)$ there exists a Nash equilibrium solution and if $S(w)$ is uniformly monotone on φ, then there exists a positive constant α such that

$$\|w^*(n) - w^*\|^2 \leq \frac{\alpha}{n}$$

for any $n \in N$ and any Nash-equilibrium of $\Gamma(n)$.

Proof. By (14), (21), (22), and (23) one has

$$\begin{aligned}\beta\|w^*(n) - w^*\|^2 &\le (w^*(n) - w^*)^T(S(w^*(n)) - S(w^*))\\ &\le (w^*(n) - w^*)^T S(w^*(n))\\ &\le -Q_n\end{aligned}$$

where $\lim_{n\to\infty} Q_n = 0$. Furthermore, from (22) it appears that there exists a constant γ such that

$$|Q_n| \le \frac{\gamma}{n}.$$

Thus

$$\|w^*(n) - w^*\|^2 \le \frac{\alpha}{n}$$

with $\alpha = \gamma/\beta$. ∎

Remark 3.4. When the demand for a product at the destination nodes is constant, return arcs are useless, and the definition of the games $\Gamma(n)$ differs slightly. If g_i denotes the flow requirement between origin x^i and destination y^i, we set the flow requirement of each of the n identical players as equal to g_i/n. We denote this new game $\tilde{\Gamma}(n)$ its solution in total link flows $\tilde{w}(n)$.

Corollary 3.1. For a network G with fixed demand, there exists a Wardrop equilibrium w^* and a subsequence of games $\tilde{\Gamma}(n_k)$ such that

$$w^* = \lim_{k\to\infty} \tilde{w}^*(n_k).$$

If w^* is unique, then: $w^* = \lim_{n\to\infty} \tilde{w}^*(n)$. If the function S is uniformly monotone, then:

$$\|\tilde{w}^*(n) - w^*\|^2 \le \frac{\alpha}{n}$$

for some positive number α. ∎

Remark 3.5. When the number of players tends to infinity, each infinitesimal player may still use several distinct paths. This is illustrated by the following example, using the two-link network of Fig. 1, where the flow requirement between origin A and destination B is 2 units.

Straightforward computations for the n-player game $\tilde{\Gamma}(n)$ defined on the network of Fig. 1 yield:

$$v_1^i = \frac{1}{n}\left(1 - \frac{3}{n+1}\right)$$

and

$$v_2^i = \frac{1}{n}\left(1 + \frac{3}{n+1}\right) \qquad i = 1, 2, \ldots, n$$

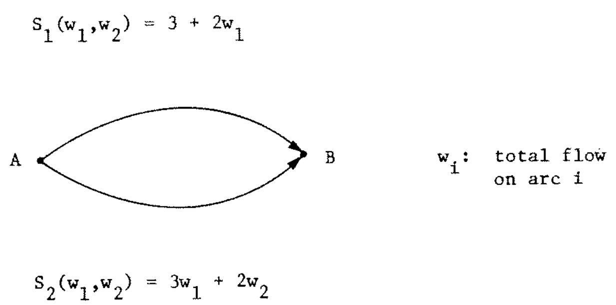

FIG. 1. A 2-link example.

Convergence of the above flows towards the Wardrop equilibrium $w_1 = w_2 = 1$ according to the statement of theorem 3.3 can readily be observed.

Also, a Nash strategy for an infinitesimal player must include both paths. Otherwise, at least one player sends all of his flow on the upper path; this cannot result in a Nash equilibrium, since this player can reduce his cost (increase his payoff) by sending a fraction of flow on the bottom path. Actually, for each $\bar{\Gamma}_n$, a Nash equilibrium is achieved when all players send an equal amount of flow on both paths. Although the asymptotic Nash equilibrium is equivalent to a Wardrop equilibrium as far as total link flows are concerned, it is conceptually quite different.

Remark 3.6. The theorems of this section provide also new convergence results for oligopoly models. If we incorporate supply curves into the transportation costs, the Wardrop equilibrium of Section 2 corresponds to the spatial price equilibrium model studied by Takayama and Judge [25] (see Fig. 2). The Wardrop equilibrium (competitive equilibrium) is then reached when the following equation is satisfied:

Supply price (production situs i) + transportation cost (production situs i to consumption market j) = price of market j.

Another particular case is the standard oligopoly model consisting of a single consumption market. Although convergence of the Nash solution of the oligopoly model to a competitive equilibrium when the number of agents increases is not always guar-

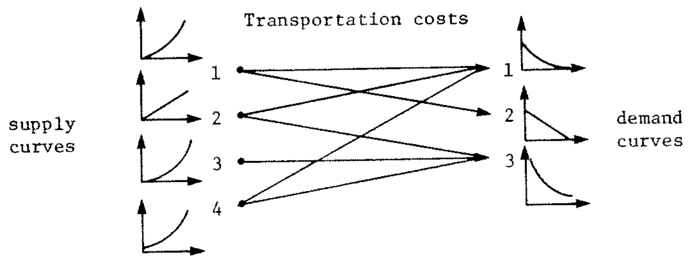

FIG. 2. Takayama-Judge model.

anteed (see [22], for instance), this is true under our assumptions about the cost and demand functions.

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