JOURNAL OF COMBINATORIAL THEORY 8, 134–145 (1970)

Probability of a Pure Equilibrium Point in n-Person Games*

MELVIN DRESHER

The RAND Corporation,
Santa Monica, California

ABSTRACT

A "random" n-person non-cooperative game—the game that prohibits communication and therefore coalitions among the n players—is shown to have with high probability a pure strategy solution. Such a solution is by definition an equilibrium point or a set of strategies, one for each player, such that if $n - 1$ players use their equilibrium strategies then the n-th player has no reason to deviate from his equilibrium strategy. It is shown that the probability of a solution in pure strategies for large random n-person games converges to $(1 - 1/e)$ for all $n \geqslant 2$.

1. INTRODUCTION

The concept of a solution frequently used for an n-person non-cooperative game is the equilibrium point [1]. In order to assure the existence of a solution it is necessary to introduce mixed strategies (probabilistic mixtures of ordinary or "pure" strategies). Except for the 2-person game, however, it is generally very difficult to compute a mixed strategy solution. Further, many decision makers may be reluctant to accept the operational notion of a mixed strategy.

These limitations of mixed strategies lead naturally to the hope that mixed strategy solutions are rarely required, i.e., a game chosen at random will in fact possess a pure strategy solution. For a 2-person zero-sum game this hope is not fulfilled; Goldman [2] showed that for such a game with many strategies it is almost certain that all solutions will require mixed strategies, the chance of a pure strategy solution being almost negligible.

It was conjectured that 2-person non-zero-sum games would have a similar property. But Goldberg, Goldman, and Newman [4] showed that for the 2-person game the probability of a pure strategy solution is quite

* Presented at the Yale University Conference on Combinatorial Theory in honor of Professor Oystein Ore (May, 1968).

large when the players have many strategies to choose among, in fact converging to $1 - e^{-1}$.

The present paper extends the results to n-person games ($n > 2$). It is shown that the probability that an n-person game ($n \geqslant 2$) has a pure strategy solution converges to $1 - e^{-1}$ as the number of strategies of each of the n players increases. Further, this result is also valid if only two of the n sets of player strategies increase without bound.

2. Games and Truncations

In the normal form of an n-person noncooperative game the i-th player ($i \leqslant n$) has m_i strategies which we label u_i ($1 \leqslant u_i \leqslant m_i$). A *play* of a game can be represented by an n-vector $U = (u_1, u_2, \ldots, u_n)$, giving us $\prod_{i=1}^{n} m_i = \pi$ possible plays. For each play U and each player i there exists a *payoff* $M_i(U)$, representing the payoff to the i-th player for the play U. There are therefore $n\pi$ payoffs.

We now define a *truncation* of a play with respect to the i-th player to be an $n - 1$ vector:

$$U_i = (u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n).$$

A truncation of a play leaves out the i-th player's strategy, a fact our notation expresses as

$$U = (U_i, u_i).$$

A game is called *zero-sum* if $\sum_1^n M_i(U) = 0$ for every play U. Despite a few allusions to properties of such games for purposes of contrast, the games treated in this paper are not constrained to be zero-sum.

3. Equilibrium Points

Nash [1] first introduced the notion of an equilibrium point, and he showed that every game possesses such a point in mixed strategies. An n-vector of pure strategies $U^* = (u_1^*, u_2^*, \ldots, u_n^*)$ is an *equilibrium point in pure strategies* if for each $i \leqslant n$ and $u_i \leqslant m_i$,

$$M_i(U^*) \geqslant M_i(U_i^*, u_i). \tag{1}$$

Equivalently, we have, for each $i \leqslant n$,

$$M_i(U^*) = \max_{u_i \leqslant m_i} M_i(U_i^*, u_i). \tag{2}$$

If the above condition is satisfied, U^* will be referred to as a pure equilibrium point or PE solution or just PE. For a 2-person zero-sum game a PE solution is the same as a saddle-point. We also call a PE point a *solution* of the n-person game.

4. Random Games

It is wellknown that PE solutions are rare for 2-person zero-sum games. For example, the probability that a "random" 2-person zero-sum game has a PE solution is

$$\frac{m_1!\, m_2!}{(m_1 + m_2 - 1)!}\,.$$

This result, proved in [2] and [3], exhibits the need for mixed strategies, even if the number of strategies for each player is not very large in the 2-person zero-sum game.

It is natural to inquire about the need for mixed strategies in arbitrary n-person games. Is it likely that we can get by with pure strategies? To answer to this inquiry we analyze "random games."

We define a *random n-person game* by the following properties:

(i) The $n\pi$ payoffs $M_i(U)$, are independent random variables.

(ii) For each i, the payoffs $M_i(U)$ have the same (independent of U) continuous probability distribution.

From the above definition of a random game it follows that, with probability one, the $n\pi$ payoffs are distinct in such a game. From now on, the zero-probability set of games not having distinct payoffs will be ruled out of the analysis. Further, the probability that a random n-person game has a PE solution is now welldefined.

Let $E(U)$ be the event that play U is a PE solution of the game. More generally, for any family F of plays, let $E(F)$ be the event that every U in F is a PE solution. Now let F_t denote the set of all F with cardinality t, and set

$$S_t = \sum \{\Pr(E(F)) \mid F \text{ in } F_t\}.$$

Let $P_n(m_1, m_2, ..., m_n)$ be the probability that a random n-person game, where the n players have $m_1, m_2, ..., m_n$ strategies, respectively, has at least one PE solution. Then

$$P_n(m_1, m_2, ..., m_n) = \Pr\left\{\sum_U E(U)\right\}.$$

Then, by the so-called method of inclusion and exclusion,

$$P_n(m_1, m_2, \ldots, m_n) = \sum_{t=1} (-1)^{t+1} S_t .$$

For any play U, the n events

$$M_i(U) = \max_{u_i \leqslant m_i} M_i(U_i, u_i)$$

are independent since they involve disjoint sets of independent random variables. Since the i-th of these events has probability $1/m_i$, we have

$$\Pr\{E(U)\} = \frac{1}{\pi} .$$

5. Possible Sets of t Equilibrium Points

In order to determine S_t we shall derive a condition that t given plays of a random game have non-zero probability of being simultaneously among its equilibrium points. Our definition of equilibrium point and random game yields the following:

THEOREM 1. *A necessary and sufficient condition that $U^1, U^2, \ldots, U^t$ are, with non-zero probability, t equilibrium points of an n-person random game is that*

$$U_i^1, U_i^2, \ldots, U_i^t \quad \textit{are distinct for each } i \leqslant n.$$

Proof. Suppose

$$U_i^1 = U_i^2.$$

Then, since U^1 and U^2 are equilibrium points,

$$\begin{aligned} M_i(U^1) &= \max_{u_i \leqslant m_i} M_i(U_i^1, u_i) \\ &= \max_{u_i \leqslant m_i} M_i(U_i^2, u_i) = M_i(U^2), \end{aligned}$$

contradicting the stipulation that, with probability one, all $n\pi$ payoffs are distinct.

The sufficiency follows from the continuity assumption on the payoff distribution and from the fact that the nt events

$$M_i(U^j) = \max_{u_i \leqslant m_i} M_i(U_i^j, u_i)$$

involve disjoint sets of independent random variables.

Since the U's are n-vectors and the U_i's are $(n - 1)$-vectors, the theorem states that each pair of U's must differ in at least two of their n components in order for all to be PE solutions.

Using Theorem 1, we can give an explicit formula for S_t. Let $F_t{}^*$ consist of those F in F_t for which $\Pr\{E(F)\} > 0$, so that

$$S_t = \sum \{\Pr(E(F)) \mid F \text{ in } F_t{}^*\}.$$

Now the members $F = \{U^1, U^2, \ldots, U^t\}$ of $F_t{}^*$ are characterized in Theorem 1, and for any such F in $F_t{}^*$ the t events $E(U^j)$ refer to disjoint sets of independent random variables and so are independent. Since $\Pr\{E(U^j)\} = 1/\pi$, it follows that

$$\Pr\{E(F)\} = \frac{1}{\pi^t}$$

for each F in F_t. Let N_t represent the cardinality of $F_t{}^*$; then

$$S_t = N_t/\pi^t,$$

and

$$P_n(m_1, m_2, \ldots, m_n) = \sum_{t=1} (-1)^{t+1} N_t \pi^{-t}. \tag{3}$$

6. Equilibrium Points in Two-Person Games

If $n = 2$, a play of the game can be represented by a 2-vector $U = (\alpha, \beta)$. It follows from Theorem 1 that, in order for $(\alpha^1, \beta^1), \ldots, (\alpha^t, \beta^t)$ to be a possible set of t equilibrium points,

$$\alpha^1, \alpha^2, \ldots, \alpha^t \quad \text{are distinct}$$

and

$$\beta^1, \beta^2, \ldots, \beta^t \quad \text{are distinct.}$$

To compute N_t, we observe that t distinct α's can be chosen in $\binom{m_1}{t}$ ways and t distinct β's can be chosen in $\binom{m_2}{t}$ ways, and then the two sets can be paired off in $t!$ ways. Thus

$$N_t = \binom{m_1}{t}\binom{m_2}{t} t!,$$

and

$$P_2(m_1, m_2) = \sum_{t=1} (-1)^{t+1} \binom{m_1}{t}\binom{m_2}{t} t!(m_1 m_2)^{-t}. \tag{4}$$

This result was first obtained by Goldberg, Goldman, and Newman [4]. They also obtained the asymptotic value of $P_2(m_1, m_2)$.

7. Equilibrium Points in Three-Person-Games

If $n = 3$, it is convenient to decompose the set of $\pi = m_1m_2m_3$ plays into m_1m_2 sets of the form S_{ij}. Each member $U = (u_1, u_2, u_3)$ of S_{ij} is such that $u_1 = i$, $u_2 = j$, $u_3 \leqslant m_3$. Thus each set S_{ij} contains m_3 plays. Now each S_{ij} can contain at most one equilibrium point. Therefore N_t is the number of ways of carrying out the following process:

(i) Choose a family S_1^t of t sets $S_{i_1j_1}, S_{i_2j_2}, \dots, S_{i_tj_t}$ from the m_1m_2 sets S_{ij}.

(ii) Choose one member from each of these t sets so that the resulting t plays obey the condition of Theorem 1.

Let $\mu(t \mid S_1^t)$ be the number of ways of making the t choices in (ii) above. Thus $\mu(t \mid S_1^t)$ is the number of ways of choosing t equilibrium points from the t given sets $S_{i_1j_1}, S_{i_2j_2}, \dots, S_{i_tj_t}$, and we have

$$N_t = \sum \mu(t \mid S_1^t),$$

where the sum is over all choices of S_1^t. If we consider the choice of S_1^t as made at random, then $\mu(t \mid S_1^t)$ is a random variable whose mean value will be denoted by $\mu(t)$. Since each S_1^t has probability $1/\binom{m_1m_2}{t}$ of being chosen it follows that

$$N_t = \binom{m_1m_2}{t}\mu(t). \tag{5}$$

From the above definition of $\mu(t \mid S_1^t)$ we have the following inequalities:

$$m_3(m_3 - 1) \cdots (m_3 - t + 1) \leqslant \mu(t \mid S_1^t) \leqslant m_3^t. \tag{6}$$

Therefore, its mean value, $\mu(t)$, also satisfies the inequality

$$\binom{m_3}{t} t! \leqslant \mu(t) \leqslant m_3^t. \tag{7}$$

For example, if $t = 1$, $\mu(1) = m_3$ and

$$N_1 = (m_1m_2)m_3 = \pi.$$

If $t = 2$, we have

$$\mu(2 \mid S_1^2) = m_3^2, \qquad \text{if} \quad i_1 \neq i_2, \quad j_1 \neq j_2,$$

$$\mu(2 \mid S_1^2) = (m_3 - 1)m_3, \qquad \text{if} \quad i_1 = i_2 \quad \text{or} \quad j_1 = j_2.$$

We can now compute

$$\mu(2) = \frac{m_3^2(m_1 - 1)(m_2 - 1) + (m_3 - 1)\, m_3(m_1 + m_2 - 2)}{(m_1 - 1)(m_2 - 1) + m_1 + m_2 - 2}$$

$$= m_3 \left(\frac{\pi - S + 2}{m_1 m_2 - 1} \right),$$

where $S = m_1 + m_2 + m_3$.

Substituting in (5) we have

$$N_2 = \binom{m_1 m_2}{2} \mu(2) = \frac{\pi(\pi - S + 2)}{2}.$$

To compute $\mu(3)$ we need to examine four cases:

$$\mu(3 \mid S_1^3) = \begin{cases} m_3(m_3 - 1)(m_3 - 2), & \text{if } i_1 = i_2 = i_3 \text{ and } j_1 \neq j_2 \neq j_3 \\ & \text{or if } j_1 = j_3 = j_3 \text{ and } i_1 \neq i_2 \neq i_3\,; \\ m_3(m_3 - 1)^2, & \text{if } i_1 = i_2 \neq i_3\,, \quad j_1 \neq j_2 \neq j_3 \\ & \text{or if } j_1 = j_2 \neq j_3\,, \quad i_1 \neq i_2 \neq i_3\,; \\ m_3^3, & \text{if } i_1 \neq i_2 \neq i_3\,, \quad j_1 \neq j_2 \neq j_3\,; \\ m_3(m_3 - 1)^2, & \text{if } i_1 = i_2 \neq i_3 \text{ and } j_1 = j_3 \neq j_2 \\ & \text{or if } j_1 = j_2 \neq j_3 \text{ and } i_1 = i_3 \neq i_2\,. \end{cases}$$

The frequencies associated with each of the four above values of $\mu(3 \mid S_1^3)$ are proportional to, respectively,

$$\begin{aligned} &(m_1 - 1)(m_1 - 2) + (m_2 - 1)(m_2 - 2), \\ &2(m_1 - 1)(m_2 - 1)^2 + 2(m_1 - 1)^2(m_2 - 1), \\ &(m_1 - 1)(m_2 - 1)(m_1 m_2 - m_1 - m_2), \\ &2(m_1 - 1)(m_2 - 1). \end{aligned}$$

The sum of the above frequencies is $(m_1 m_2 - 1)(m_1 m_2 - 2)$.

Using the above frequencies and values of $\mu(3 \mid S_1^3)$ we obtain the value of $\mu(3)$ as a function of m_1, m_2, m_3. In particular, if $m_1 = m_2 = m_3 = m$, we have

$$\mu(3) = \frac{m(m - 1)(m^4 + 2m^3 - 8m^2 + 6)}{(m + 1)(m^2 - 2)}$$

and

$$N_3 = \binom{m^2}{3} \mu(3) = \frac{m^3(m - 1)^2(m^4 + 2m^3 - 8m^2 + 6)}{6}.$$

In a similar manner we can compute the values of N_t, where

$$t \leqslant \min(m_1 m_2, m_3),$$

and then compute the required probability

$$\begin{aligned} P_3(m_1, m_2, m_3) &= \sum_{t=1} (-1)^{t+1} N_t \pi^{-t} \\ &= \sum_{t=1} (-1)^{t+1} \binom{m_1 m_2}{t} \mu(t) \pi^{-t}. \end{aligned} \tag{8}$$

It is of interest to determine the asymptotic value of $P_3(m_1, m_2, m_3)$ as the number of strategies increases for each player. We note that the absolute value of the t-th term of the series for P_3 is

$$\begin{aligned} N_t \pi^{-t} &= \binom{m_1 m_2}{t} \mu(t) \pi^{-t} \\ &= \frac{1}{t!} \mu(t) \prod_{k=1}^{t} \frac{(m_1 m_2 - k + 1)}{m_1 m_2 m_3}. \end{aligned}$$

From (7) it follows that the absolute value of the t-th term satisfies the inequality

$$\binom{m_3}{t} \prod_{k=1}^{t} \left(\frac{m_1 m_2 - k + 1}{m_1 m_2 m_3} \right) \leqslant N_t \pi^{-t} \leqslant \frac{1}{t!} \prod_{k=1}^{t} \left(\frac{m_1 m_2 - k + 1}{m_1 m_2} \right)$$

or

$$\frac{1}{t!} \prod_{k=1}^{t} \left(1 - \frac{k-1}{m_3} \right) \left(1 - \frac{k-1}{m_1 m_2} \right) \leqslant N_t \pi^{-t} \leqslant \frac{1}{t!} \prod_{k=1}^{t} \left(1 - \frac{k-1}{m_1 m_2} \right).$$

Hence we obtain

$$\lim_{m_1, m_2, m_3 \to \infty} N_t \pi^{-t} = \frac{1}{t!},$$

which by (8) suggests

$$\lim_{m_1, m_2, m_3 \to \infty} P_3(m_1, m_2, m_3) = \sum_{t=1} \frac{(-1)^{t+1}}{t!} = 1 - e^{-1}.$$

Detailed proof of this will be given in the proof of Theorem 2.

8. Pure Equilibrium Points in n-Person Games

We now evaluate the probability of a PE solution in a random n-person game, where the i-th player has m_i strategies. In such a game the set of

$\pi = m_1 m_2 \cdots m_n$ plays can be decomposed into $m_1 m_2 \cdots m_{n-1} = M$ sets of the form $S_{i_1 i_2 \cdots i_{n-1}}$ where each set contains m_n plays. Each member $U = (u_1, u_2, \ldots, u_n)$ of $S_{i_1 i_2 \cdots i_{n-1}}$ has the property that $u_1 = i_1$, $u_2 = i_2, \ldots, u_{n-1} = i_{n-1}$, and $u_n \leqslant m_n = m$. Thus each of the M sets contains m plays.

From Theorem 1 it follows that each set $S_{i_1 i_2 \cdots i_{n-1}}$ can contain at most one PE point. Therefore choosing t plays which can simultaneously be equilibrium points from the π plays is equivalent to choosing t of the M sets and then choosing one play from each of these t chosen sets. Again, let $\mu(t \mid S_1{}^t)$ be the number of ways of choosing t plays which can simultaneously be equilibrium points from the t given sets (we emphasize that only one point may be chosen from each set) and let $\mu(t)$ represent its mean value. Then, we have

$$N_t = \binom{M}{t} \mu(t).$$

From our definition of the random variable $\mu(t \mid S_1{}^t)$,

$$m(m-1) \cdots (m-t+1) \leqslant \mu(t \mid S_1{}^t) \leqslant m^t.$$

Therefore the mean value $\mu(t)$ satisfies the same inequality, or

$$\binom{m}{t} t! \leqslant \mu(t) \leqslant m^t. \tag{10}$$

The required probability of a PE point in a random game is given by

$$P_n(m_1, m_2, \ldots, m_n) = \sum_{t=1} (-1)^{t+1} \binom{M}{t} (Mm)^{-t} \mu(t). \tag{11}$$

For each M and m one can compute the probability P_n by first computing $\mu(t)$ when $t \leqslant \min(m, M)$. Now from the definition of $\mu(t)$ we have $\mu(1) = m$. In order to compute $\mu(2)$ we pick two sets, $S_{i_1 i_2 \cdots i_{n-1}}$ and $S_{j_1 j_2 \cdots j_{n-1}}$, from the M sets. We have then

$$\mu(2 \mid S_1{}^2) = \begin{cases} m^2, & \text{if } i_1 \neq j_1, i_2 \neq j_2, \ldots, i_{n-1} \neq j_{n-1}, \\ m(m-1), & \text{if } i_1 = j_1 \text{ or } i_2 = j_2, \ldots, \text{ or } i_{n-1} = j_{n-1}. \end{cases}$$

Now the frequency associated with $\mu(2 \mid S_1{}^2) = m^2$ is proportional to D, where

$$D = (m_1 - 1)(m_2 - 1) \cdots (m_{n-1} - 1).$$

The frequency associated with $\mu(2 \mid S_1{}^2) = m(m-1)$ is proportional to DE, where

$$E = \sum_{i=1}^{n-1} \frac{1}{m_i - 1}.$$

Hence we have

$$\mu(2) = \frac{m^2 D + m(m-1)\,DE}{D + DE}$$
$$= m\left(m - \frac{E}{1+E}\right).$$

In a similar manner we can compute $\mu(3)$, $\mu(4)$,..., $\mu(\bar{m})$, where

$$\bar{m} = \min(m, M),$$

and then obtain P_n. Of course, the computation of $\mu(t)$ becomes more cumbersome with each value of t. However, P_n has an asymptotic value given by

THEOREM 2. *For all n-person games* $(n \geqslant 2)$

$$\lim_{\substack{m_1 m_2 \ldots m_{n-1} \to \infty \\ m_n \to \infty}} P_n(m_1, m_2, \ldots, m_n) = 1 - e^{-1}.$$

Proof. Equation (11) may be written as

$$P_n(M, m) = \sum_{t=1} \frac{(-1)^{t+1}}{t!} \mu(t)\, m^{-t} \prod_{i=1}^{t-1} \left(1 - \frac{i}{M}\right).$$

Hence we have

$$P_n(M, m) - (1 - e^{-1}) = \sum_{t=1} \frac{(-1)^t}{t!} \left[1 - \mu(t)\, m^{-t} \prod_{i=1}^{t-1} \left(1 - \frac{i}{M}\right)\right]. \tag{12}$$

Now let

$$\lambda_t(M, m) = \mu(t)\, m^{-t} \prod_{i=1}^{t-1} \left(1 - \frac{i}{M}\right).$$

From (10) it follows that for all t

$$\prod_{i=1}^{t-1} \left(1 - \frac{i}{m}\right)\left(1 - \frac{i}{M}\right) \leqslant \lambda_t(M, m) \leqslant \prod_{i=1}^{t-1} \left(1 - \frac{i}{M}\right) \leqslant 1. \tag{13}$$

Now for all $i \leqslant T \leqslant M$ we have

$$1 \geqslant 1 - \frac{i}{M} \geqslant 1 - \frac{T}{M}.$$

Hence

$$\prod_{i=1}^{t-1} \left(1 - \frac{i}{M}\right) > \left(1 - \frac{T}{M}\right)^{t-1} > \left(1 - \frac{T}{M}\right)^{T} \qquad \text{for } t \leqslant T \leqslant M.$$

Similarly we have that

$$\prod_{i=1}^{t-1}\left(1-\frac{i}{m}\right) > \left(1-\frac{T}{m}\right)^T \qquad \text{for } t \leqslant T \leqslant m.$$

Substituting the above inequalities in (13) we have

$$\begin{aligned}\lambda_t(M, m) &\geqslant \left(1-\frac{T}{M}\right)^T\left(1-\frac{T}{m}\right)^T \qquad \text{for } t \leqslant T \leqslant \min(m, M)\\ &\geqslant \left(1-\frac{T^2}{M}\right)\left(1-\frac{T^2}{m}\right)\\ &\geqslant \left(1-\frac{T^2}{M}-\frac{T^2}{m}\right).\end{aligned}$$

Now T is arbitrary but $T < M$ and $T < m$. Suppose we restrict T so that $T^3 < M$, and $T^3 < m$, then $T^2/M < 1/T$ and $T^2/m < 1/T$, and we obtain the inequality

$$\lambda_t(M, m) \geqslant 1 - \frac{2}{T} \qquad \text{for } t \leqslant T < T^3 < \min(m, M).$$

Therefore

$$0 \leqslant 1 - \lambda_t(M, m) < \frac{2}{T} \qquad \text{for } t \leqslant T < T^3 < \min(m, M).$$

Returning to (12) we have for $t \leqslant T < T^3 < \min(m, M)$

$$\begin{aligned}| P_n(M, m) - (1 - e^{-1})| &\leqslant \sum_{t=1}^{T} \frac{1}{t!}\left(\frac{2}{T}\right) + \left| \sum_{t>T} \frac{(-1)^t}{t!}[1-\lambda_t(M, m)]\right|\\ &\leqslant \frac{2}{T}e + \left| \sum_{t>T} \frac{(-1)^t}{t!}[1-\lambda_t(M, m)]\right|.\end{aligned}$$

The second term represents the "tail" of a converging alternating series. Thus, given any $\delta > 0$, we can choose T sufficiently large that

$$\left| \sum_{t>T} \frac{(-1)^t}{t!}[1-\lambda_t(M, m)]\right| < \delta, \tag{14}$$

and

$$| P_n(M, m) - (1 - e^{-1})| < \frac{2}{T}e + \delta.$$

Now by choosing $T > 2e/\delta$, we have

$$| P_n(M, m) - (1 - e^{-1})| < 2\delta,$$

which proves the theorem.

It is of interest to note that Theorem 2 requires only that two of the n sets of player strategies grow without bound.

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Printed in Belgium