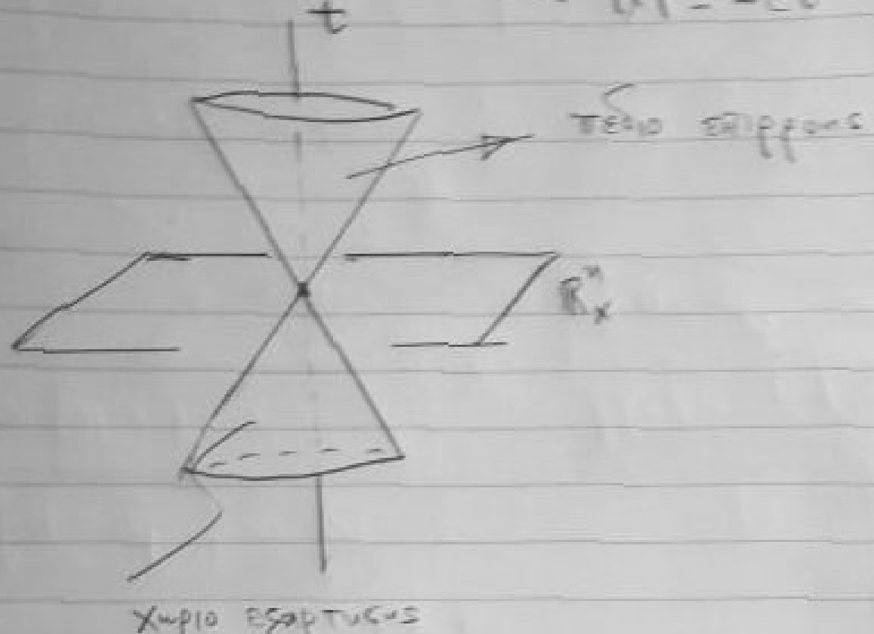
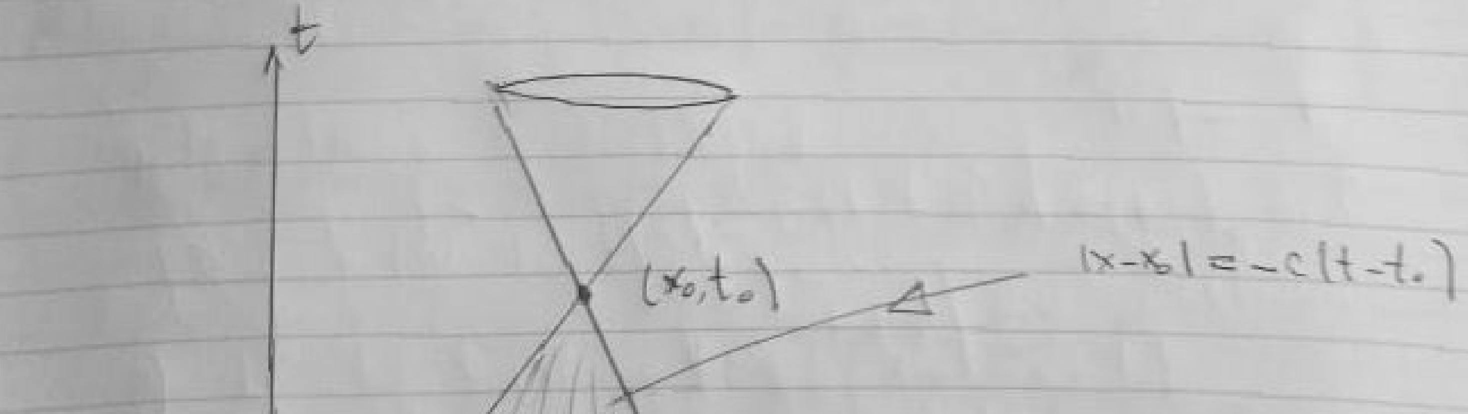
Διάλεξη 22

Σχόλια (Κώνοι, οπισθοδρομικοί – εμπροσθοδρομικοί, $d\tilde{S}_{\tilde{y}}$

1) $$\big||x|-ct\big| = 0 \quad \begin{cases} |x| = ct \\ |x| = -ct \end{cases}$$



2) $$\big||x-x_0|-c(t-t_0)\big| = 0$$



$\partial B(x_0, ct_0) = \{|x-x_0| = ct_0\}$

$$u(x,t) = \frac{1}{4\pi t^2 c^2} \int\limits_{|y-x|=ct} \left[t\,h(y) + g(y) + \nabla g(y)\cdot(y-x) \right] dS_y$$

$u(x,0) = g(x)$, $u_t(x,0) = h(x)$

Τύπος Kirchhoff, $n=3$.

3) $d\tilde{S}_{\tilde{y}}$

α)

$$E(R) = \iint\limits_{Q} \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

β) $\partial \tilde{B}(\tilde{x},t) = \left\{ (y_1-x_1)^2 + (y_2-x_2)^2 + (y_3-0)^2 = t^2 \right\}$

$$y_3 = \left(t^2 - \left[(y_1-x_1)^2 + (y_2-x_2)^2 \right] \right)^{1/2} = \gamma(y_1, y_2)$$

$$\iint\limits_{B(x,t)} \tilde{g}\, d\tilde{S}_{\tilde{y}} = \iint\limits_{B(x,t)} g(y) \left(1 + |\nabla \gamma|^2 \right)^{1/2} dy \qquad y = (y_1, y_2)$$

$$\left(1 + |\nabla \gamma|^2 \right)^{1/2} = t \left(t^2 - |y-x|^2 \right)^{1/2}.$$

Ασκήσεις 18, 19, 20 Evans.

Η μη ομογενής Εξίσωση

$$(1) \quad u_{tt} - \Delta u = f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,\infty)$$

Θεωρούμε την κάτωθι ομογενή

$$(2) \quad \begin{cases} u_{tt}(x,t;s) - \Delta u(x,t;s) = 0 \\ u(x,t;s) = 0 \quad , \; x \in \mathbb{R}^n, \; t = s \\ u_t(x,t;s) = f(x,s) \quad , \; \text{—} '' \text{—} \end{cases}$$

Ορίζουμε

$$(3) \quad u(x,t) := \int_0^t u(x,t;s)\,ds \qquad (x \in \mathbb{R}^n, t \geq 0)$$

Έχουμε

$$u_{tt}(x,t) = \frac{\partial^2}{\partial t^2}\left(\int_0^t u(x,t;s)\,ds\right)$$

$$= \frac{\partial}{\partial t}\Big[\cancel{u(x,t;t)}^{\overset{(2)}{0}} + \int_0^t u_t(x,t;s)\,ds\Big]$$

$$= u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\,ds$$

$$\overset{(2)}{=} f(x,t) + \int_0^t \Delta u(x,t;s)\,ds$$

$$= f(x,t) + \Delta \int_0^t u(x,t;s)\,ds = \Delta u + f.$$

$\square$

Για Παράδειγμα

$n=1$

(4) $\begin{cases} u_{tt}(x,t;s) - u_{xx}(x,t;s) = 0 \\ u(x,s;s) = 0 \quad , \quad u_t(x,s;s) = f(x,s) \end{cases}$

D'Alembert $\Longrightarrow$

$$u(x,t;s) = \frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy$$

(5) $$u(x,t) = \frac{1}{2}\int_0^t \left(\int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy \right) ds \,.$$

$n=3$

$t-s=\tau$

$u(2) \Rightarrow \quad u(x,s+\tau;s) = \hat{u}(x,t-s)$

ικανοποιεί την

$$\begin{cases} \hat{u}_{\tau\tau} - \Delta\hat{u} = 0 \\ \hat{u}(x,0) = 0 \\ \hat{u}_\tau(x,0) = f(x,s) \end{cases}$$

Kirchhoff $\Longrightarrow$

$$u(x,t;s) = (t-s) \fint_{\partial B(x,t-s)} f(y,s)\,dS_y$$

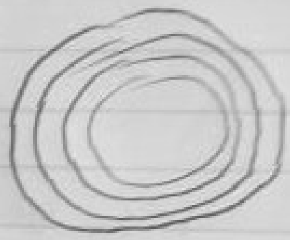
$\Rightarrow$

$$u(x,t) = \int_0^t (t-s)\left(\fint_{\partial B(x,t-s)} f(y,s)\, dS_y \right) ds$$

$$= \frac{1}{4\pi} \int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{(t-s)}\, dS_y \, ds$$

$r = t-s$

$$= \frac{1}{4\pi} \int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\, dS_y \, dr$$



$$= \frac{1}{4\pi} \int_{B(x,t)} \frac{f(y, t-|y-x|)}{|y-x|}\, dy .$$

$\square$

Μέθοδος Ενέργειας

(1) $u_{tt} = \Delta u$,

(2) $E(t) := \frac{1}{2} \int_{\mathbb{R}^n} \left(u_t^2 + |\nabla u|^2\right) dx$

$$\frac{dE}{dt} = \int_{\mathbb{R}^n} \left(u_t u_{tt} + \nabla u \, \nabla u_t \right) dx$$

Υποθετοντας οτι η $x \longrightarrow u(x,t)$ εχει συμπαγη φορεα

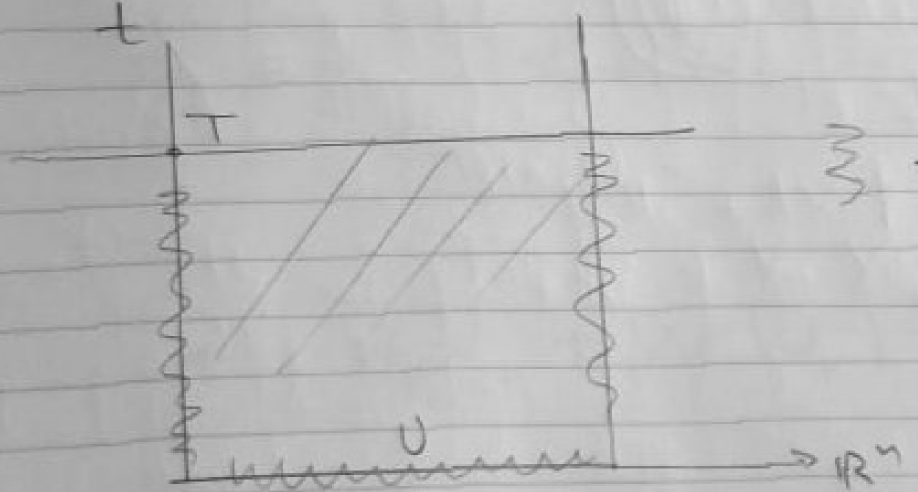
$$= \int_{\mathbb{R}^n} \left(u_t u_{tt} - \Delta u \; u_t \right) dx$$

$$\overset{(1)}{=} 0$$

$\therefore \quad E(t) \equiv$ σταθερα (διατηρηση ενεργειας).

<u>Θεωρημα 1</u> (Μοναδικοτητα)

Θεωρουμε $U \subset \mathbb{R}^n$, ανοικτο, φραγμενο, $\partial U \in C^1$



$\wr = \Gamma_T = \partial U \times [0,T)$

$u_{tt} - \Delta u$

$$(3) \begin{cases} u_{tt} - \Delta u = f(x,t) , & U \times (0,T) \\ u = g , & \partial U \times [0,T) \\ u_t(x,0) = h , & x \in U . \end{cases}$$

Υπαρχει το πολυ μια λυση $u(x,t) \in C^2(\bar{U}_T)$.

<u>Αποδειξη</u>

Εστω u_1, u_2 λ-σεις $\Rightarrow$ $w = u_1 - u_2$
επιλυει την

$$
(4)\quad \begin{cases} w_{tt} - \Delta w = 0 \\ w = 0 \quad \text{στο} \quad \partial U \times [0,T) \\ w_t(x,0) = 0, \quad x \in U \end{cases}
$$

$$
\frac{d}{dt}\left[\frac{1}{2} \int_U \left(w_t^2 + |\nabla w|^2 \right) dx \right]
$$

$$
= \int_U \left(w_t w_{tt} + \nabla w \, \nabla w_t \right) dx
$$

$$
= \int_U w_t w_{tt} \, dx - \int_U (\Delta w \, w_t) \, dx + \int_{\partial U} \frac{\partial w}{\partial \nu} \cancel{w}^{\,0} \, dS
$$

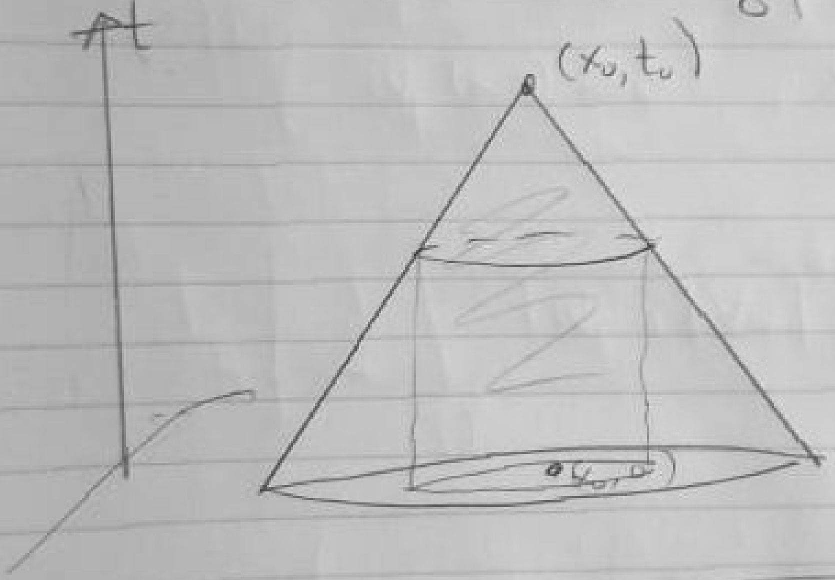
$= 0$

$$\therefore \int_U (w_t^2 + |\nabla w|^2)\,dx = \int_U w_t^2(x,0) + |\nabla w(x,0)|^2$$

$$= 0.$$

$\Box$

<u>Θεωρημα 2</u> (Πεπερασμενη ταχυτητα μεταδοσης σηματος)
Χωριο εξαρτησης



$|x - x_0| \le -(t - t_0)$

Στερεος οπισθοδρομικος κωνος.

$\mathbb{R}^n$

Εστω $u = u_t = 0$, $|x - x_0| \le t_0$, $x \in \mathbb{R}^n$

$\Longrightarrow u(x,t) \equiv 0$ στον Στερεο Κωνο

<u>Αποδειξη.</u>

$$e(t) := \frac{1}{2} \int_{B(x_0, t_0 - t)} \left(u_t^2(x,t) + |\nabla u(x,t)|^2\right) dx, \quad 0 \le t \le t_0$$