Lecture 1 XEIMEPINO - 2020

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1. nalikako@math.uoa.gr (315 Μαθηματικό)
Office Hours (via skype) Monday 1-2
PDE Seminar (via Webex) Friday 3-4

2. $\boxed{\text{Textbook : Evans PDE, 2nd Edition AMS}}$
Related Textbooks
W. Strauss (Advanced Undergraduate) Parabolic
F. John (classic, not easy) Conservation laws
J. Smoller (Reaction-Diffusion + Shock waves)
{ Gilbarg - Trudinger (Elliptic, Advanced)
{ Evans (handwritten elliptic regularity, eclass)
{ Han - Lin (Elliptic - Modern Advanced)

P. Germain (Current lecture Notes, NLS eclass)
C. Dafermos (Conservation laws Vols I, II)
Protter - Weinberger (Maximum Principles)
Αλικάκος - Αλικάκος (Προπτυχιακό, Σύγχρονη Επιλογή)
Δάσιος - Κυριάκη (— " —)

3. Structure of the course - Determination of Grade
(a) Assignments - HWK problems
(b) Oral Exam
(c) Take Home Written Exam

§ 1 Introduction

PDE Physics + Geometry → Curvature, Gravitation

Conservation laws (Evolution Equations)

2^{nd} order Eqs Importance derives from $\begin{cases} \text{Newton } F = ma \\ \text{Curvature } K, H \end{cases}$

(1) Laplace's Equation

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , u = u(x,y)$$

harmonic function

- Complex Analysis: $f(z) = u(x,y) + i v(x,y)$

  u, v harmonic conjugates

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \qquad z = x + iy$$

- Harmonic maps on the plane $\quad V : \mathbb{R}^2 \to \mathbb{R}^2 , \quad V = (u, v)$

  $\Delta u = \Delta v = 0$

- Newtonian Theory of Gravitation

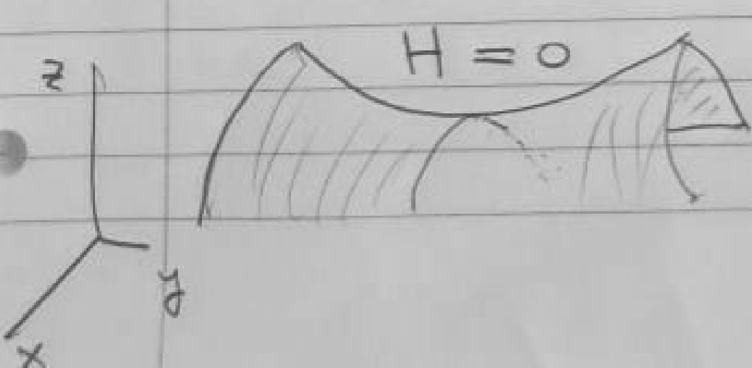
$$\Delta u = 0 \quad \text{on } \mathbb{R}^n \setminus 0$$

$$\Delta u = f \quad (\text{Poisson's Equation})$$

(2) Minimal Surface Equation

$$H = \frac{k_1 + k_2}{2} = \text{Mean Curvature}$$

$H = 0$



k_i = principal curvatures

$k_1 > 0, \; k_2 < 0$

$k_1 + k_2 = 0$

Catenoid

$z = u(x,y)$ (graph)

$$H = \frac{(1+u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1+u_y^2)u_{xx}}{2(1+u_x^2+u_y^2)^{3/2}}$$

Note: linearization of MSE is Laplace's Eq.

Phase Transitions – Allen–Cahn equation

(3) $\Delta u - (u^3 - u) = 0$, $-1 \le u \le 1$

$u = -1$, $u = 1$ pure phase

Evolution Equations

(4) Diffusion Equation

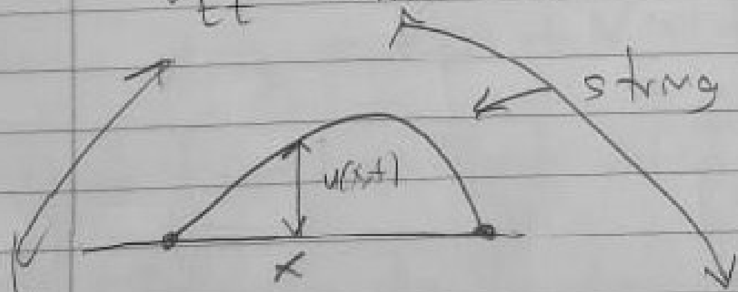
$$\frac{\partial u}{\partial t} = \Delta u \qquad (u_t = u_{xx})$$

$u(x,t) = \text{density} = \frac{m}{V}$, $\int u(x,t)\,dx = \text{Mass}$

Alternatively: $u(x,t) = \text{Temperature}$

(5) Wave Equation

$$u_{tt} - u_{xx} = 0 \qquad (u_{tt} - \Delta u = 0)$$



$$\kappa(x,t) = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

curvature

F=ma

(linearization of curvature)

|| Difference between linear and nonlinear wave propagation:

If velocity depends on the amplitude $\Rightarrow$ Nonlinear

Ex: Burgers' Equation

(5) $$u_t + u u_x = 0$$

$u(x,t)$

$x(t)$ 1d medium

(5) expresses the requirement that the velocity of a particle $x(t)$ at position x, at time t is CONSTANT:

$$\frac{dx}{dt} = u(x(t),t) \Rightarrow \frac{d^2x}{dt^2} = u_x \overset{=u}{x'} + u_t$$

Constant velocity $\Leftrightarrow \frac{d^2x}{dt^2} = 0 \Leftrightarrow u_t + u u_x = 0$

Schrödinger's Equation (Dispersion Equation) Quantum Mech.

(6) $$i\psi_t - \psi_{xx} + V(x)\psi = 0$$ (one particle)

$(x,t) \to \psi(x,t) = \underset{\psi_1}{\mathrm{Re}\,\psi} + i\,\underset{\psi_2}{\mathrm{Im}\,\psi} \in \mathbb{C}$

(System in ψ_1, ψ_2), $x \to |\psi(x,t)|^2$ probability density.

If the frequency depends the velocity then the equation becomes nonlinear (focusing / defocusing)

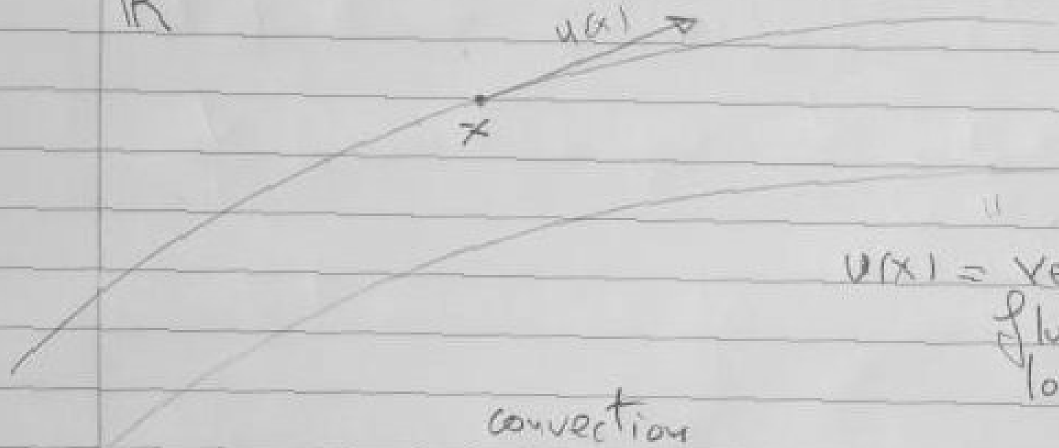
(7) $$i\psi_t - \psi_{xx} = |\psi|^2\psi$$ (many particles) mean field theory

(Real) Fluids (that is with viscosity)

(8) Navier-Stokes System - fluids

Eulerian description

$\mathbb{R}^3$



$u(x)$ = velocity of fluid at the location x

$u(x) = (u_1(x), u_2(x), u_3(x))$

convection

$$\left\{\begin{array}{l} \underbrace{\frac{\partial u_i}{\partial t}}_{\text{acceleration}} + (u\cdot\nabla)u_i = \underbrace{\nu\Delta u_i - \nabla p(x,t)}_{\text{Forces}} \\ \\ \operatorname{div} u = 0 \qquad \text{(incompressibility)} \end{array}\right.$$

Unknowns: (u_1, u_2, u_3), p (4 unknowns)

in 4 equations.

6

N-S : Existence/Uniqueness OPEN

Clay Institute Problem # 4
(21st Century 10^6 \$ prize)

Perelman (Hamilton) solved the Poincare Conjecture Problem # 6

(9) General Relativity (Einstein's Equation in vacum)

metric

$$Ric(g) = 0$$

$\square$