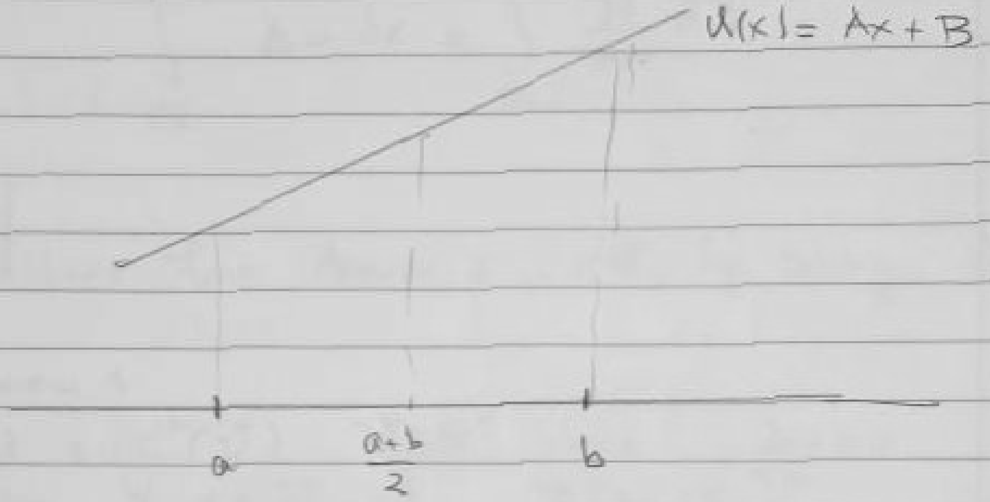
§4 lecture 4 : Mean Value Theorems For harmonic fns

$n=1$

(1) $$u''=0 \qquad u(x),\ x\in\mathbb{R}$$



(2) $$u\left(\frac{a+b}{2}\right)=\frac{1}{2}\left(u(a)+u(b)\right)$$

Notation

$$\frac{1}{|\Omega|}\int_\Omega f\,dx =: ⨍_\Omega f\,dx$$

$|\Omega|$ = Lebesgue measure of Ω

measure of ball in $\mathbb{R}^n$

Exs

(3) $$⨍_{B(x,r)} u\,dx = \frac{1}{|B(x,r)|}\int_{B(x,r)} u\,dx$$

(4) $$⨍_{\partial B(x,r)} u\,dS = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u\,dS$$

"area" of sphere $\partial B(x,r)$

$|B(x,r)| = \mathcal{L}^n(B(x,r))$

$|\partial B(x,r)| = \mathcal{L}^{n-1}(\partial B(x,r))$

<u>Lemma</u>. $\Omega \subset \mathbb{R}^n$, open, $u \in C^2(\Omega)$, then

(4) $$\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} \, dS$$

<u>Proof</u>

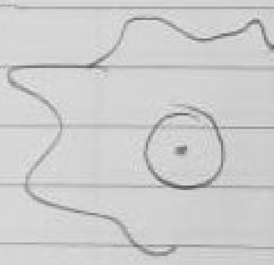
Follows from Assum 2, p18, by taking $v = 1$

$\square$

<u>Theorem 1</u>

Let $u \in C^2(U)$, $U \subset \mathbb{R}^n$ open, $\Delta u = 0$

Then $\forall x \in U$ s.t. $B(x,r) \subset U$



(5) $$u(x) = ⨍_{B(x,r)} u(y)\,dy = ⨍_{\partial B(x,r)} u \, dS_y$$

<u>Proof</u>

1. Set $y = x + rz$

$$\phi(r) := ⨍_{\partial B(x,r)} u(y)\,dS_y = ⨍_{\partial B(0,1)} u(x+rz)\,dS_z$$

$$\left(dS_y = r^{n-1} dS_z \, , \; |\partial B(x,r)| = |\partial B(0,1)| \, r^{n-1} \right)$$

$$\phi'(r) = ⨍_{\partial B(0,1)} \frac{\partial}{\partial r}\left(u(x+rz)\right) dS_z$$

$$\left(\frac{\partial}{\partial r} u(x+rz) = \sum_i \frac{\partial u}{\partial y_i} \frac{\partial y_i}{\partial r} \, , \; y = x+rz \, , \; y_i = x_i + r z_i \right.$$

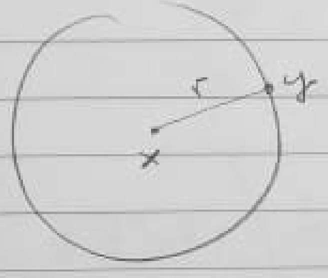
$$\frac{\partial y_i}{\partial r} = z_i$$

$$\sum \frac{\partial u}{\partial y_i} \frac{\partial y_i}{\partial r} = \nabla u\,(x+rz)\cdot z$$

$$\therefore \quad f'(r) = \fint_{\partial B(0,1)} \nabla u(x+rz)\cdot z \, dS_z$$

$$\left(= \fint_{\partial B(0,1)} \nabla u\,(x+rz)\cdot \frac{y-x}{r}\, dS_z \right)$$

$$= \fint_{\partial B(x,r)} \nabla u(y)\cdot \underbrace{\frac{y-x}{r}}_{\vec{n}(y)} \, dS_y$$



$$= \fint_{\partial B(x,r)} \frac{\partial u}{\partial n(y)} \, dS_y$$

$$= \frac{1}{|\partial B(x,r)|} \int_{B(x,r)} \Delta u \, dy$$

$$= 0$$

$$\therefore \quad \phi'(r) = 0 \quad \Rightarrow \quad \phi(r) = \text{const}$$

$$\Rightarrow \quad \phi(r) = \lim_{t \to 0} \phi(t) = u(x)$$

2. **Remark** (Appendix Evans § C.3) [see p. 26 present notes]

$$(6) \quad \int_{B(x,r)} u \, dy = \int_0^r \left(\int_{\partial B(x,\sigma)} u \, dS \right) d\sigma$$

$$\left(n=2 \quad \iint_{B(x,r)} u \, dx_1 dx_2 = \int_0^{2\pi} \left(\int_0^r u \, \sigma \, d\sigma \right) d\theta \right.$$

$$= \int_0^r \left(\int_0^{2\pi} u \, \underbrace{\sigma \, d\theta} \right) d\sigma$$

$$\sigma \, d\theta = dS(\sigma)$$

$$\left. = \int_0^r \left(\int_{\partial B(x,\sigma)} u \, dS(\sigma) \right) d\sigma \right)$$

$$(6) \overset{(5)(i)}{\Longrightarrow} \int_{B(x,r)} u \, dy = \int_0^r \overbrace{|\partial B(x,\sigma)|}^{n\alpha(n)\sigma^{n-1}} u(x) \, d\sigma$$

$$= u(x) \, \alpha(n) r^n$$

$$= u(x) \, |B(x,r)| \, .$$

$\square$

Special Case of the Coarea formula (Appendix C3)

Lemma

Let $f : \mathbb{R}^n \to \mathbb{R}$ continuous, $\int_{\mathbb{R}^n} |f| dx < \infty$

Then

(7) $$\frac{d}{dr} \int_{B(x_0,r)} f dx = \int_{\partial B(x_0,r)} f dS \quad , \forall B(x_0,r) \subset \mathbb{R}^n$$

$\square$

The Mean Value Property characterizes harmonicity

Theorem 2 : If $u \in C^2(U)$ satisfies (U open, $\subset \mathbb{R}^n$)

(8) $$u(x) = ⨍_{\partial B(x,r)} u \, dS_y$$

$\forall B(x,r) \subset U$, then u is harmonic.

Proof

Recall that if $\phi(r) := ⨍_{\partial B(x,r)} u(y) dS_y$ then

we have calculated

(9) $$\phi'(r) = \frac{1}{|\partial B(x,r)|} \int_{B(x,r)} \Delta u(y) dy$$

Since $\phi(r) = u(x) \quad \forall r$ s.t. $B(x,r) \subset U$

$\Rightarrow \phi'(r) = 0 \quad , \quad r \leq r_0 \quad ,$ i.e.

(10) $$\frac{1}{|\partial B(x,r)|} \int_{B(x,r)} \Delta u(y) dy = 0 \quad , \quad r \leq r_0$$

Taking $r \to 0$ by the Mean Value Theorem, since $\Delta u \in C^2(U)$ $\Longrightarrow$

(11) $$\Delta u(x) = 0 \quad .$$

$\square$

Άσκηση 3^* (Δ. Τσαγούλης)
Mean Value Property for Poisson's Equation

Let $u \in C^2(\bar{U})$ solution to

(12) $$\Delta u = f(x)$$

Then $\forall$ $B(\xi, r) \subset U$ we have

(13) $$\fint_{B(\xi,r)} u \, dS_y = u(\xi) + \frac{1}{|S^{n-1}|} \int_0^r \frac{a(\sigma)}{\sigma^{n-1}} d\sigma$$

where

(14) $$a(\sigma) = \int_{B(\xi,\sigma)} f \, dx \quad .$$

$\square$

Άσκηση 4 (Δ. Τσαγούλης)

(a) Consider the equation

(15) $$\Delta u = cu \quad , \quad c = \text{constant} \, ,$$

$u = u(x)$, $x \in \mathbb{R}^n$.

Set

(16) $$h(\rho) = \int_{B(\xi,\rho)} u(y) \, dy$$

Utilizing Askuon 3 show that h satisfies

(17) $$h''(\rho) - \frac{2}{\rho} h'(\rho) + c h(\rho) = 0$$

(b) Solve (17) for $c > 0$ and $c < 0$ explicitly.

§5 The Maximum Principle

1. Theorem 1 ($n=1$)
Let $u'' \geq 0$ on (a,b), $u \in C^2(a,b) \cap C^1[a,b]$
Then

(i) $\max\limits_{[a,b]} u = \max\{u(a), u(b)\}$

(ii) If $u \neq$ constant $\Longrightarrow$ $u(x) < \max\{u(a), u(b)\}$, $x \in (a,b)$

(i.e. u can not attain its maximum in the interior of (a,b))

(iii) If $u \neq 0$ and $u(c) = \max u$
$\Longrightarrow$

$c = a$ or b

and

$u'(c) < 0$ if $c = a$

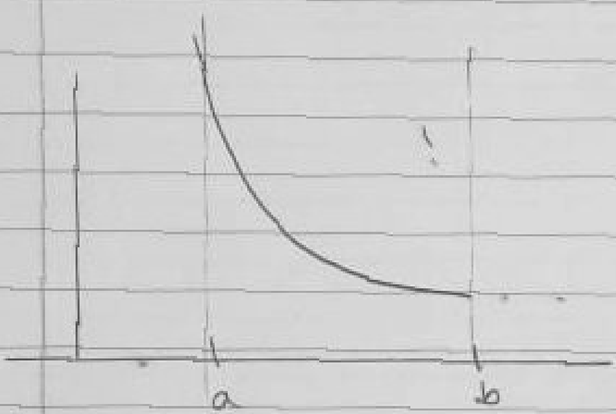
$u'(c) > 0$ if $c = b$.

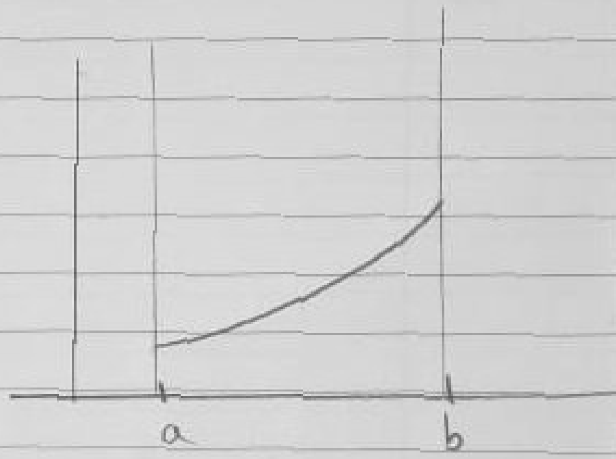
$\square$

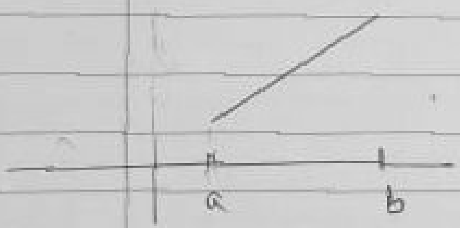
Observations

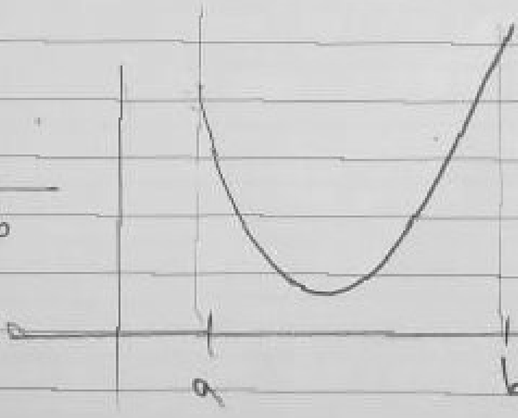
α) If u is subharmonic, $u \not\equiv$ const, then u attains its maximum on the boundary

β) The derivative at the max is different from zero









Convexity

Proof

u is convex, hence its graph is above every tangent. Thus u can not have an interior (local) maximum. (unless is a constant)

(i) Suppose $u(b) = \max\limits_{[a,b]} u$.

$$\Longrightarrow \quad u'(b) \geq 0.$$

Suppose $u'(b) = 0$. From $u'' \geq 0$ it follows that $u'(x) \leq 0$, $x \leq b$, hence u is nonincreasing,

$$\Longrightarrow \quad u(x) \equiv u(b), \; x \in [a,b].$$

(ii) Suppose $u(a) = \max_{[a,b]} u$.

$\Rightarrow \quad u'(a) \leq 0.$

Suppose $u'(a) = 0$, From $u'' \geqslant 0$ it follows that $u'(x) \geqslant 0$, hence u is nondecreasing hence $u(x) \equiv u(a)$, $x \in [a,b]$.

$\square$