Lecture 5: (Maximum Principle continued)

2. **Theorem 2** ($n \geq 2$, The Strong Maximum Principle)
Let $u \in C^2(U) \cap C(\bar{U})$, $U \subset \mathbb{R}^n$, open, bounded

(1) $$\Delta u = 0 \quad , \quad x \in U$$

Then

(i) $$\max_{\bar{U}} u = \max_{\partial U} u$$

(ii) Suppose U is connected and suppose $x_0 \in U$ s.t.

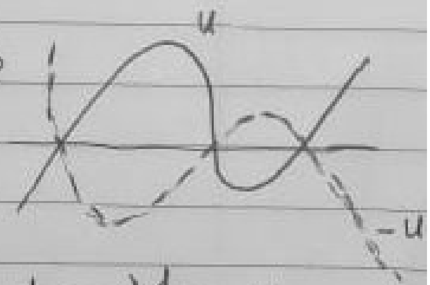
(2) $$u(x_0) = \max_{\bar{U}} u$$

Then

(3) $$u(x) \equiv u(x_0), \quad x \in U.$$

Note: The term "strong" refers to (ii).
(i) by itself is the "Maximum Principle".

Note (Strong Minimum Principle)
Replacing u with $-u$ we have:



$$\max(-u) = -\min u$$

and since $-u$ is harmonic

$$\max_{\bar{U}}(-u) = \max_{\partial U}(-u) \iff -\min_{\bar{U}} u = -\min_{\partial U} u$$

$$\iff \min_{\bar{U}} u = \min_{\partial U} u.$$

Also if U is connected, $x_0 \in U$

$$u(x_0) = \min_{\bar{U}} u$$

$\iff$

$$-u(x_0) = -\min_{\bar{U}} u = \max_{\bar{U}} (-u)$$

$\overset{(ii)}{\implies}$

$$-u(x_0) \equiv -u(x) \quad , x \in U$$

$$\therefore \quad u(x) \equiv u(x_0) \quad .$$

$\square$

<u>Note</u>

(i) follows from (ii) for connected U.
It also follows from (ii) for general U via the following argument:

Let U open in $\mathbb{R}^n$ and let

$$u(x_0) = \max_{\bar{U}} u =: M$$

We claim that

(4) $$\max_{\partial U} u = M.$$

If $x_0 \in \partial U$ we are set.
So suppose $x_0 \in U$. We now consider the maximal open connected component of U containing x_0. We denote it by U_{x_0}. We observe that

(5) $$\partial U_{x_0} \subset \partial U$$

We have
$$\partial U_{x_0} \subset \overline{U}_{x_0} \subset \overline{U} = U \cup \partial U$$

Let $\xi \in \partial U_{x_0}$, and suppose that $\xi \in U$. Then $\exists\ B(\xi;\varepsilon) \subset U$, $\varepsilon > 0$ small enough, by openess.

Also

$$(\alpha) \quad \underbrace{U_{x_0} \cup B(\xi;\varepsilon)}_{\text{open}} \subset U$$

$$(\beta) \quad U_{x_0} \cap B(\xi;\varepsilon) \neq \emptyset$$

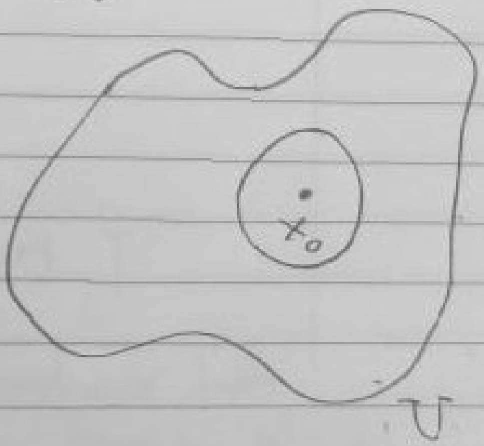
$(\alpha), (\beta) \implies U_{x_0} \cup B(\xi;\varepsilon)$ open, connected in U

$$U \supset U_{x_0} \cup B(\xi;\varepsilon) \supsetneq U_{x_0}$$

contradicting the maximality of U_{x_0}!

Thus (5) holds, and in turn (4) holds by (ii) applied to U_{x_0}.

<u>Proof of (ii) of Theorem 2</u>



1. Suppose $x_0 \in U$, $u(x_0) = \max\limits_{\overline{U}} u$

Choose $B(x_0;r) \subset U$.

$\Sigma_M \neq \emptyset$ by hypothesis.

$\therefore \quad \Sigma_M \equiv U$

The proof of Theorem 2 is complete.

□

Applications

A. Let U connected, $u \in C(U) \cap C(\bar{U})$ solution to

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U, \quad g \in C(\partial U) \end{cases}$$

Corollary (Positivity)

(i) $g \geq 0 \implies u \geq 0$ in U

(ii) $g \geq 0,\ g \not\equiv 0 \implies g > 0$ in U

Proof

1. By the minimum principle

$$\min_{\bar{U}} u = \min_{\partial U} u = \min_{\partial U} g \geq 0$$

2. By the strong minimum principle the minimum of u is not attained in U unless u is constant, equal to the minimum, $u \equiv 0$ impossible since $g \not\equiv 0$.

□

By the Mean Value Theorem

(6) $$M = u(x_0) = \frac{1}{|B(x_0;r)|} \int_{B(x_0;r)} u(y)\,dy$$

(7) $$M \geq u(y) \quad \forall y \in U$$

<u>Claim</u> : $u \equiv u(x_0)$ on $B(x_0;r)$

By contradiction. Suppose $\exists\ y^* \in B(x_0;r)$, $u(y^*) < M$

Then by continuity $u(z) < M$ in a nbd of z, hence

(8) $$\int_{B(x_0;r)} (M - u(y))\,dy > 0 .$$

y^*

$\Longleftrightarrow$

(9) $$\int_{B(x_0;r)} u(y)\,dy < M\,|B(x_0;r)| ,$$

in contradiction to (6).

Hence the claim is established

2. <u>Open-Closed Argument</u>

Consider the set

(10) $$\Sigma_M = \{ x \in U \mid u(x) = M \}$$

Σ_M is open since by 1. it contains balls.
Σ_M is also closed by the continuity of u.

B **Corollary** (Uniqueness)

Then there is at most one $C^2(U) \cap C(\bar{U})$ of

$$
(11) \quad \begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}
$$

Proof.

Let u_1, u_2 two solutions

$$
-\Delta u_1 = f \qquad\qquad -\Delta u_2 = f
$$

$$
u_1 = g \qquad\qquad u_2 = g
$$

$\Rightarrow$

$$
\begin{cases} \Delta(u_1 - u_2) = 0 & \text{in } U \\ u_1 - u_2 = 0 \end{cases}
$$

$$
u_1 - u_2 \in C^2(U) \cap C(\bar{U}).
$$

Then

$$
\max_{\bar{U}} (u_1 - u_2) = \max_{\partial U} (u_1 - u_2) = 0
$$

$$
\min_{\bar{U}} (u_1 - u_2) = \min_{\partial U} (u_1 - u_2) = 0
$$

$\square$