Διαλεξη 8 (# 1)

$\|\varphi_n\| = 1$ $\quad -\Delta \varphi_n = \lambda_n \varphi_n, \quad \varphi_n\big|_{\partial U} = 0 \qquad f = \sum c_n \varphi_n$ (L^2)

$0 \leq \alpha < 1 \qquad X^\alpha = \{ f \in L^2 \mid \sum \lambda_n^{2\alpha} c_n^2 < +\infty \} \qquad U \subset \mathbb{R}^k$

$W_0^{2\alpha, 2}$

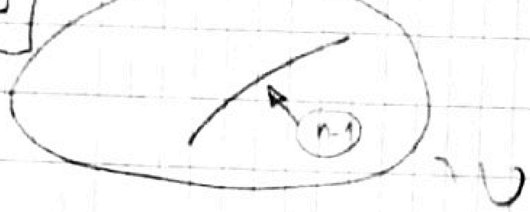
$\alpha = 1 \longrightarrow \sum \lambda_n^2 c_n^2 < +\infty \longrightarrow W^{2,2}$

$\alpha = 0 \longrightarrow \sum c_n^2 < +\infty \longrightarrow L^2$

$\alpha = \frac{1}{2} \longrightarrow \sum |\lambda_n| c_n^2 \longrightarrow W^{1,2}$

$\alpha = 1 \leftarrow \boxed{2\tau\varphi}$ $\quad \alpha = \frac{1}{4} \longrightarrow \sum \lambda_n^{\frac{1}{2}} c_n^2 < +\infty \qquad W$

$\alpha = \frac{1}{4} \leftarrow \boxed{\frac{1}{2}\tau\varphi}$



$W^{\frac{1}{2}}$

$[0,1] \subset \mathbb{R}^1$

$\varphi_n = e^{inx}, \quad \lambda_n = n^2$

$W^{\frac{1}{2},2} = \{ \sum |c_n|^2 n < +\infty \}$

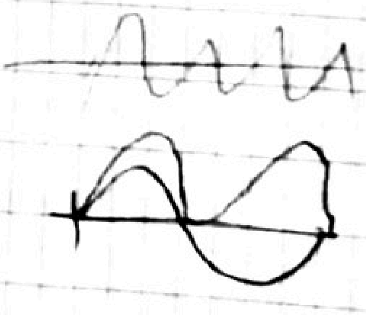
$f(x) = \sum c_n e^{inx}$

$f(0) = \sum c_n \qquad \sum |c_n| < +\infty$

$\sum |c_n| = \sum |c_n| \sqrt{n} \frac{1}{\sqrt{n}} \leq \left(\sum |c_n|^2 n \right)^{1/2} \left(\sum \frac{1}{n} \right)$

$\int A_n^2 \sin^2 nx = 1$

$A_n^2 n$



$W^{2\alpha,2}(U) \subset C^{\gamma}, \quad 4\alpha > n \qquad p > n$

$\gamma = 1 - \frac{n}{4\alpha} \qquad 1 > \frac{n}{4\alpha}$

Θ2

U φρ, $\partial U \in C^1$, $u \in W^{1,p}(U)$

$u \in W_0^{1,p}(U) \Leftrightarrow Tu = 0$ → $W_0^{1,p}(U)$ πυρήνας του T

$$W_0^{1,p}(U) = \{ u \in W^{1,p}(U) \mid Tu = 0 \} =: \mathcal{N}(T)$$

Σχ. Ορισμός $\overline{}^{W^{1,p}}$ νόρμα

$$W_0^{1,p}(U) = \overline{C_c^\infty(U)}$$

$W_0^{1,p}(U) \subseteq \mathcal{N}(T)$ εύκολο.

Βασική Εκτίμηση: $u \in C^1(\bar{U})$

$$\int_{\partial U} |u|^p \, dS \le C \int_U (|Du|^p + |u|^p) \, dx$$

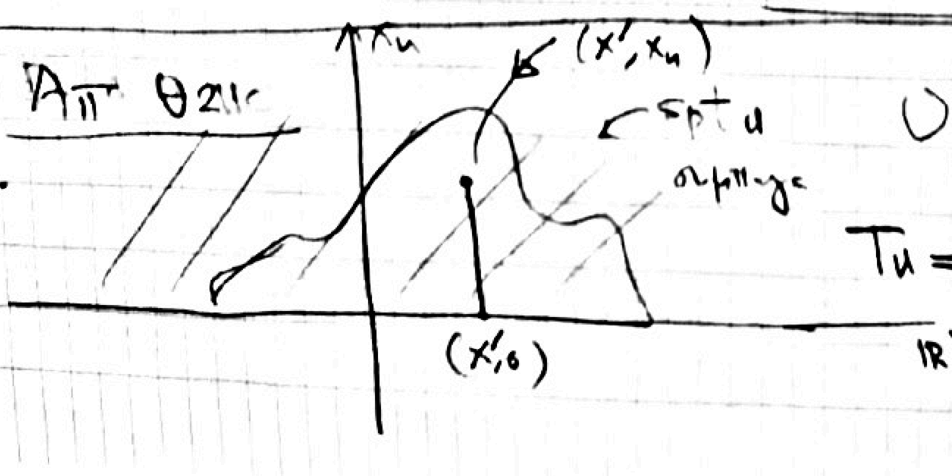
Θ1 T $W^{1,p}(U) \to$

⇓ Θ1

$$\|Tu\|_{L^p(\partial U)} \le C \|u\|_{W^{1,p}(U)}$$

Απ Θ2 ←

A.



$U = \mathbb{R}^n_+ = \{x_n > 0\}$

$Tu = 0$

*3

$\exists \; u_m \in C^1(\overline{\mathbb{R}^n_+})$

$u_m \longrightarrow u \quad W^{1,p}(\mathbb{R}^n_+)$

(Θ. πυκνότητας, sψT u συμπαγές $\partial U \in C^1$)

Σημ $p=2$ $W^{1/2,q}$ … $\int_{\mathbb{R}^{n-1}} \left| \frac{u(x', x_n)}{\sqrt{x_n}} \right| \leq \int_0^{x_n} \int_{\mathbb{R}^{n-1}}$

Θα δείξουμε την Εκτίμηση — Ήδη συνεχώς

(*) $\displaystyle\int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p dx' \leq x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |\nabla u|^p dx' dt$ $\quad 1 \leq p < \infty$

Σημ: (*) $\Rightarrow \lim_{x_n \to +\infty} \displaystyle\int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p dx' = 0$

$\left(\int |u(x', 0)|^p dx' = \int |Tu|^p dx' \right)$ $\quad \frac{\partial u}{\partial \nu}$

Απ (*)

$$u_m(x', x_n) = u_m(x', 0) + \int_0^{x_n} \frac{\partial u_m}{\partial x_n}(x', t)\, dt$$

$$|u_m(x', x_n)| \leq |u_m(x', 0)| + \int_0^{x_n} \left| \frac{\partial u_m}{\partial x_n}(x', t) \right| dt$$

$$|u_m(x', x_n)|^p \leq C \left[|u_m(x', 0)|^p + \left(\int_0^{x_n} \left| \frac{\partial u_m}{\partial x_n}(x', t) \right| dt \right)^p \right]$$

$$\left(\int_0^{x_n} f\, dt\right)^p \overset{\text{Holder}}{\leq} \left(\int_0^{x_n} f^p dt\right)^{1/p} \left(\int_0^{x_n} 1^q dt\right)^{1/q} \qquad \frac{1}{p} + \frac{1}{q} = 1$$

$$= \left(\int_0^{x_n} f^p dt\right)^{1/p} x_n^{\frac{p-1}{p}} \qquad f = \frac{\partial u_m}{\partial x_n} \qquad \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}, \quad q = \frac{p}{p-1}$$

$$\left(\int_0^{x_n} \left| \frac{\partial u_m}{\partial x_n} \right| dt\right)^p \leq \left(\int_0^{x_n} \left| \frac{\partial u_m}{\partial x_n} \right|^p dt\right) x_n^{p-1}$$

(#4)

$$\Rightarrow \int_{\mathbb{R}^{n-1}} |u_m(x',x_n)|^p dx' \le C\Big[\int_{\mathbb{R}^{n-1}} |u_m(x',0)|^p dx' + x_n^{p-1} \int_{\mathbb{R}^{n-1}} \Big(\int_0^{x_n} \Big|\frac{\partial u_m}{\partial x_n}\Big|^p dt\Big)$$

$u_m(x',x_n) \to u(x',x_n)$

Δηλ.

$$\le C\Big[\int_{\mathbb{R}^{n-1}} |u_m(x',0)|^p dx' + x_n^{p-1} \int_{\mathbb{R}^{n-1}}\int_0^{x_n} |\nabla u_m|$$

$$= \int_{\mathbb{R}^{n-1}} |Tu_m|^p dx' + x_n^{p-1} \int_{\mathbb{R}^{n-1}}\int |\nabla u_m|$$

$m \to +\infty$

$Tu_m \to Tu = 0$

$\Downarrow$ $m \to +\infty$

$$x_n^{p-1} \int_{\mathbb{R}^{n-1}}\int_0^{x_n} |\nabla u|^p dx' dt.$$

B. <u>Παρατήρηση</u>

Θέλουμε να δείξουμε ότι $W^{1,p}(\mathbb{R}^+_n) = W_0^{1,p}(\mathbb{R}^+_n)$

$u \in C_c^1(\mathbb{R}^+_n)$

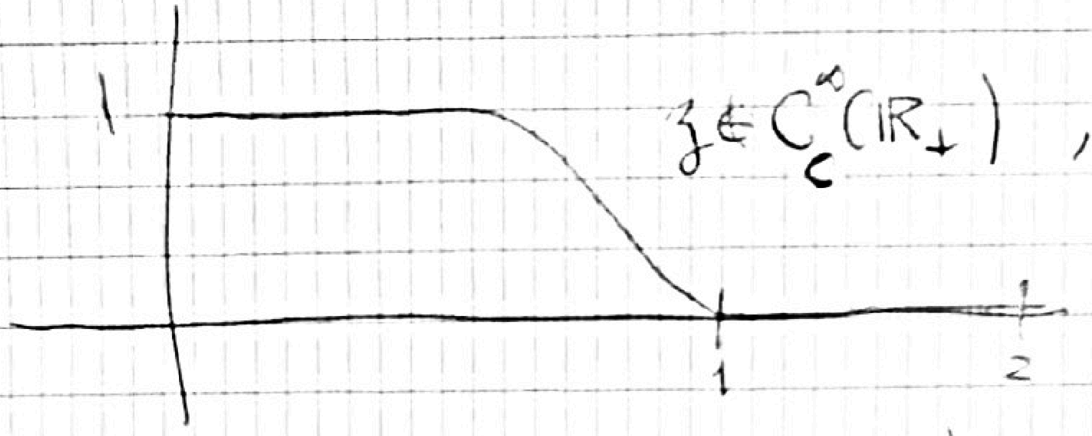
δείξαμε ότι $\exists \{\hat{W}_m\} \subset C_c^1(\mathbb{R}^n)$

$\hat{W}_n \to u$, $W^{1,p}(\mathbb{R}^n)$ ενώ α.

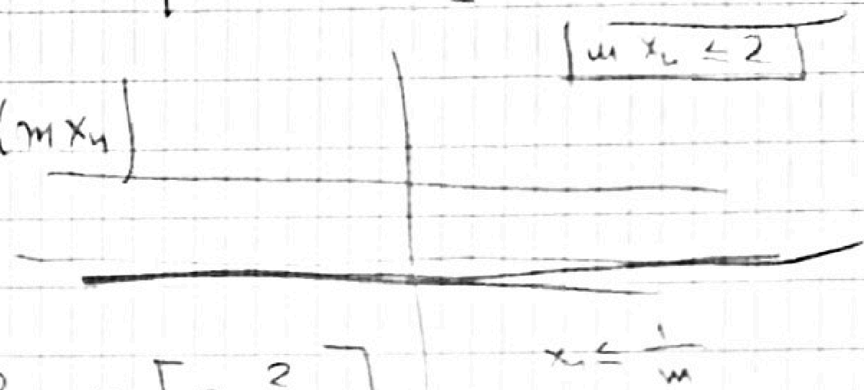
Θα δείξουμε αρχικά των $\exists \{W_n\} \subset W^{1,p}(\mathbb{R}^n_+)$

supp $W_m \subset\subset \mathbb{R}^n_+$

(*5)



$\zeta \in C_c^\infty(\mathbb{R}_+)$, $0 \le \zeta \le 1$



$\zeta_m(x) := \zeta(m x_n)$

$x = (x', x_n)$

$\operatorname{supp} \zeta_m \subset \mathbb{R}_{n-1} \times \left[0, \frac{2}{m}\right]$

$w_m(x) =: u(x)\left(1 - \zeta_m(x)\right)$

$\operatorname{supp} w_m \subset \{ |x_n| \ge \frac{1}{m} \} \cap \ldots \subset\subset \mathbb{R}^n_+$

$$\begin{cases} \dfrac{\partial w_m}{\partial x_n} = \dfrac{\partial u}{\partial x_n}\left(1 - \zeta_m(x)\right) - m\,\zeta'(m x_n)\, u(x) \\ \nabla_{x'} w_m = \left(\nabla_{x'} u\right)\left(1 - \zeta_m(x)\right) \end{cases}$$

$1 - \zeta_m \to 1$

$$\int_{\mathbb{R}^n_+} |\nabla w_m - \nabla u|^p dx \le \int_{\mathbb{R}^n_+} |\nabla_{x'} w_m - \nabla_{x'} u|^p dx$$

$$+ \int_{\mathbb{R}^n_+} \left| \frac{\partial w_m}{\partial x_n} - \frac{\partial u}{\partial x_n} \right|^p dx$$

$= I + II$

(#6)

I

$$\int_{\mathbb{R}^n_+} |\nabla_{x'} u\,(1-\zeta_m(x)) - \nabla_{x'} u|^p\,dx = \int_{\mathbb{R}^n_+} |\zeta_m(x)|^p\,|\nabla_{x'} u|^p\,dx$$

$$= \int_0^{2/m} |\nabla_{x'} u|^p\,dx \xrightarrow[m\to+\infty]{} 0$$

II

$$\int_{\mathbb{R}^n_+} \left|\frac{\partial w_m}{\partial x_n} - \frac{\partial u}{\partial x_n}\right|^p dx = \int \left|\frac{\partial u}{\partial x_n}\,(1-\zeta_m(x)) - m\zeta'(m x_n)\,u(x) - \frac{\partial u}{\partial x_n}\right|^p dx$$

$$\leq C\int \left|\frac{\partial u}{\partial x_n}\right|^p \zeta_m^p(x)\,dx + m^p \int |\zeta'(m x_n)\,u(x)|^p\,dx$$

$$\downarrow \quad C\int_0^{2/m} \quad \downarrow 0 \qquad\qquad m^p \int_{|x_n|\leq \frac{2}{m}} |u(x)|^p\,dx$$

$$m^p \int_{|x_n|\leq \frac{2}{m}} |u|^p\,dx = m^p \int^{2/m} \left(\int_{\mathbb{R}^{n-1}} |u|^p\,dx' \right) dx_n$$

$$\leq \left(m^p \int_0^{2/m} x_n^{p-1}\,dx_n \right) \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |\nabla u|^p\,dx'\,dt$$

$$< m^p \left(\frac{2}{m}\right)^p \frac{1}{p} \qquad \underbrace{\qquad\qquad}_{\to 0}$$

$\square$