Διάλεξη 11

(I) Σχόλια επί της εκτίμησης Morrey ($n<p\le\infty$)

(M) $\boxed{\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\, \|\nabla u\|_{L^p(\mathbb{R}^n)}}$, $u\in C^1(\mathbb{R}^n)$

1) C^∞ **δεν** είναι πυκνό στο $C^{0,\gamma}$

Παράδειγμα

$f(x)=\sqrt{|x|}\in C^{0,1/2}$

Θεωρούμε $f_n\in C^2$, άρτιες, $\Rightarrow$ $f_n(x)\overset{\text{Mac}}{=}x^2 g_n(x)$ $\quad\begin{pmatrix} g_n(0)\neq 0 \\ g_n'(0)\neq 0 \\ g_n''(0)\neq 0.\end{pmatrix}$

$$|f-f_n|_{C^{0,\frac12}} \ge \sup_{x\neq 0}\frac{|f(x)-f_n(x)|}{\sqrt{x}}$$

$$\ge \sup_{x\neq 0}\frac{\sqrt{x}-x^2 g_n(x)}{\sqrt{x}}$$

$$= \sup_{x\neq 0}\left(1-x^{3/2}g_n(x)\right)$$

$$\ge 1$$

> Κλειστότητα των C^∞ στην $C^{0,\gamma}$ νόρμα $=: h^\gamma$ χώρος $\neq C^{0,\gamma}$ (μικροί Hölder)

2) Έστω $u\in W^{1,p}$. Επιλέγουμε $u_n\in C_c^\infty$,
$\nabla u_n \xrightarrow{L^p} \nabla u$

(M) $\Rightarrow$

$$\|u_n - u_m\|_{C^{0,\gamma}} \le C \|\nabla u_n - \nabla u_m\|_{L^p}$$

$\{\nabla u_n\}$ Cauchy στο L^p $\Longrightarrow$ $\{u_n\}$ Cauchy στο $C^{0,\gamma}$

$$\Rightarrow \quad u_n \xrightarrow{C^{0,\gamma}} \hat{u} \in C^{0,\gamma}$$

$\boxed{\text{συνεχής}}$

$$u_n \xrightarrow{W^{1,p}} u \quad \Rightarrow \quad u_n(x) \longrightarrow u(x) \quad \text{a.e.}$$

Επίσης

$$\underbrace{\sup |u_n(x) - \hat{u}(x)|}_{\downarrow \, 0} \ge |u_n(x) - \hat{u}(x)|$$

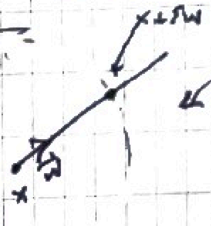
$\therefore \quad \hat{u}(x) = u(x) \quad$ a.e. $\quad \Rightarrow \boxed{u(x) = \hat{u}(x)}$

αντιπρόσωπος
<u>συνεχής</u>

3) <u>Επί της Απόδειξης της (M)</u>

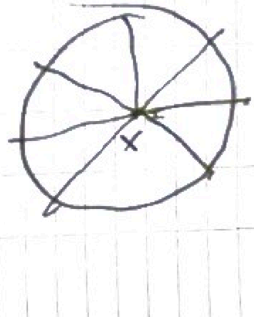
(A) • $$\fint_{B(x,r)} |u(x)-u(y)|\,dy \le C \int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\,dy$$

<u>Εξηγήσεις</u>



$\boxed{0<s<r}$ $$|u(x+sw)-u(x)| \le \int_0^s |\nabla u(x+tw)|\,dt$$

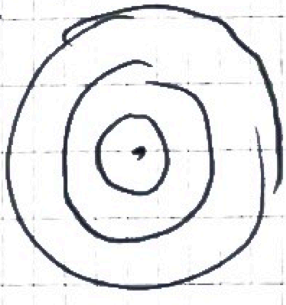
$$\int_{|w|=1} |u(x+sw)-u(x)|\,dS_w$$



$$\int_{|w|=1}\int_0^s |\nabla u(x+tw)|\,dt\,dS_w = \int_0^s\left(\int_{|w|=1} |\nabla u(x+tw)|\,dS_w\right)dt$$

Fubini

όλες τις $|w|=1$



$$= \int_0^s \left(\int_{\partial B_t(x)} | \quad | \frac{dS_w}{dS_t}\, dS_t\right)dt$$

$$= \int_0^s\left(\int_{\partial B_t(x)} | \quad | \frac{1}{t^{n-1}}\, dS_t\right)dt$$

$$= \int_{B(x,s)} \underline{\qquad}\, dt \le \int_{B(x,r)} \underline{\quad}\, dt$$

(B) Ιδιομορφία

$$\int_{B(x,r)} \frac{|\nabla u|}{|x-y|^{n-1}}\,dy \le \left(\int_{B(x,r)} \frac{|\nabla u|^p}{1}\, dy\right)^{1/p}\underbrace{\left(\int_{B(x,r)} \frac{dy}{|x-y|^{(n-1)\frac{p}{p-1}}}\right)^{\frac{p-1}{p}}}_{I}$$

$$I = \left(c_n\int_0^r \frac{ds}{s^{(n-1)\frac{p}{p-1}}}\, s^{n-1}\, ds\right)^{\frac{p-1}{p}} = c_n\left(r^{(n-1)\left(-\frac{1}{p-1}\right)+1}\right)^{\frac{p-1}{p}}$$

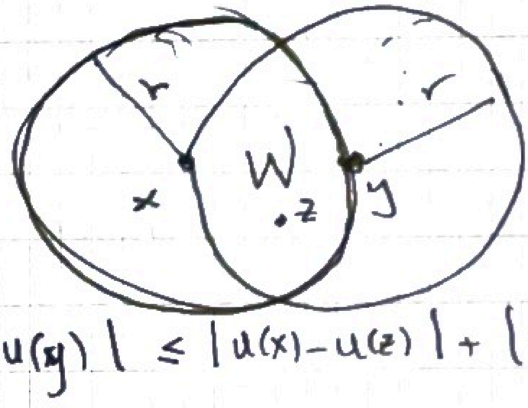
$$= \left(r^{\frac{p-n}{p-1}}\right)^{\frac{p-1}{p}} = r^{\frac{p-n}{p}} = r^{1-\frac{n}{p}}.$$

c_n = σταθερά

(A): $\fint_{B(x,r)} |u(x)-u(z)|\,dz \le c\int_{B(x,r)} \frac{|\nabla u(z)|}{|z-x|^{n-1}}$

(C)

$|x-y|=r$



, $|u(x)-u(y)| \le |u(x)-u(z)| + |u(z)-u(y)|$

$$\fint_W |u(x)-u(y)|\,dz \overset{!}{\le} \fint_W |u(x)-u(z)|\,dz + \fint_W |u(z)-u(y)|\,dz$$

$$\mathcal{L}^n(W)\,|u(x)-u(y)| \le \rule{2em}{0.4pt} + \rule{2em}{0.4pt} \qquad |A| = \mathcal{L}^n(A)$$

$$\frac{1}{|W|}\int_W |u(x)-u(z)|\,dz \le \frac{1}{|W|}\int_{B(x,r)} |u(x)-u(z)|\,dz = \boxed{\frac{|B(x,r)|}{|W|}} \fint_{B(x,r)} |u(x)-u(z)|$$

ανεξάρτητο του r!

#

II Λύσεις Ασκήσεων

1) $C^{0,\alpha}(U)$, $\alpha > 1$, αποτελείται από σταθερές συναρτήσεις, U συνεκτικό

Λύση (Αλέξανδρος)

$$|u(x)-u(y)| \le c|x-y|^{\alpha} \Rightarrow \frac{|u(x)-u(y)|}{|x-y|} \le c|x-y|^{\alpha-1} \xrightarrow[x\to y]{} 0$$

$\Rightarrow \nabla u = 0$. u Lip. Θα δείξουμε ότι Lip $= W^{1,\infty}$

$\Rightarrow$ u Sobolev, $\nabla u = 0 \Rightarrow u \equiv$ σταθερά (Ασκ 11) □

2) $u = \ln\left(\ln\left(1+\frac{1}{|x|}\right)\right) \in W^{1,n}(B_1)$, $n>1$

(για $n=1$ δεν ισχύει διότι $W^{1,1}([-1,1]) \hookrightarrow L^\infty([-1,1])$)

$$u(r) = \ln\left(\ln\left(1+\frac{1}{r}\right)\right)$$

$$u'(r) = \frac{1}{\ln\left(1+\frac{1}{r}\right)} \, \frac{1}{1+\frac{1}{r}} \, \frac{1}{r^2}$$

Κοντά στο 0 έχουμε:

$$\frac{1}{1+\frac{1}{r}} \, \frac{1}{r^2} = \frac{r}{r+1} \, \frac{1}{r^2} = \frac{1}{r(r+1)} \simeq \frac{1}{r}$$

$$\int_0^1 (u'(r))^n r^{n-1}\,dr \sim \int_0^1 \frac{1}{\left(\ln\left(1+\frac{1}{r}\right)\right)^n} \, \frac{1}{r^n} \, r^{n-1}\,dr$$

$$\sim \int_0^1 \frac{1}{(\ln r)^n} \, \frac{1}{r}\,dr = -\int_0^1 \frac{1}{(\ln r)^n}\,d(\ln r)$$

$$= \int_1^\infty \frac{1}{s^n}\,ds < +\infty.$$

$=\!\!\!/\!\!\!=$