

SPHERICAL AVERAGES OF FOURIER TRANSFORMS OF MEASURES WITH FINITE ENERGY; DIMENSION OF INTERSECTIONS AND DISTANCE SETS

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§1. *Introduction.* Let μ be a positive Radon measure with compact support in the euclidean n -space $\mathbb{R}^n$. Introducing the Fourier transform

$$\hat{\mu}(x) = \int e^{-ix \cdot y} d\mu y,$$

and the averages over the spheres $S(r) = \{x: |x| = r\}$,

$$\sigma(\mu)(r) = \int_{S(1)} |\hat{\mu}(r\xi)|^2 d\xi,$$

we can write the α -energy, $0 < \alpha < n$, of μ as

$$\begin{aligned} I_\alpha(\mu) &= \iint |x - y|^{-\alpha} d\mu x d\mu y \\ &= c_1 \int |x|^{\alpha-n} |\hat{\mu}(x)|^2 dx \\ &= c_2 \int_0^\infty r^{\alpha-1} \sigma(\mu)(r) dr, \end{aligned}$$

where the positive constants c_1 and c_2 depend only on n and α . The second equality is based on the Plancherel formula and the fact that $\hat{k}_{n-\alpha} = ck_\alpha$ where $k_\alpha(x) = |x|^{\alpha-n}$. Thus if $I_\alpha(\mu) < \infty$, we have for *many* large x and r ,

$$|\hat{\mu}(x)|^2 \leq c(\mu) |x|^{-\alpha}, \quad (1)$$

$$\sigma(\mu)(r) \leq c(\mu) r^{-\alpha}. \quad (2)$$

In general these inequalities need not hold for *all* large x and r . As for (1), for no α , $0 < \alpha < n$, does the finiteness of $I_\alpha(\mu)$ even force $\hat{\mu}(x)$ to tend to zero as $|x| \rightarrow \infty$. But the averages behave better. In Section 3 we shall prove that

$$\sigma(\mu)(r) \leq cI_\alpha(\mu) r^{-\alpha} \quad \text{for all} \quad 0 < r < \infty, \quad 0 < \alpha \leq \frac{1}{2}(n-1).$$

We shall also show that this is no longer true for $\alpha > \frac{1}{2}(n-1)$, and we derive a weaker estimate in this case.

In Section 4 we study the influence of the above averages on what we call the distance measure $\delta(\mu, \nu)$ of two measures μ and ν :

$$\delta(\mu, \nu)(A) = \mu \times \nu\{(x, y): |x - y| \in A\}, \quad A \subset \mathbb{R}.$$

We shall give bounds for the L^∞ - and L^2 -norms of the Radon-Nikodým-derivative of $\delta(\mu, \nu)$ and for the energy-integrals of $\delta(\mu, \nu)$ in terms of the energies of μ and ν and the behavior of the averages $\sigma(\mu)$ and $\sigma(\nu)$. By approximation, these problems are reduced to smooth functions f and g , and we are led to derive *a priori* inequalities for

$$\delta(f, g)(r) = \int \left(\int_{| \zeta - x | = r} f(\zeta) d\zeta \right) g(x) dx.$$

In Section 5 we apply the results of the earlier sections to distance sets. For Borel subsets A and B of $\mathbb{R}^n$, their distance set is

$$D(A, B) = \{|x - y|: x \in A, y \in B\},$$

with $D(A, A) = D(A)$. In [F2] Falconer proved that the Lebesgue measure $\mathcal{L}^1 D(A) > 0$ if the Hausdorff dimension $\dim A > \frac{1}{2}(n+1)$, and that $\dim D(A) \geq \dim A - \frac{1}{2}(n-1)$ if $\dim A \leq \frac{1}{2}(n+1)$. We improve this by showing that if $\dim B \leq \frac{1}{2}(n-1)$, then

$$\dim A + \dim B > n \quad \text{implies} \quad \mathcal{L}^1 D(A, B) > 0, \quad \text{and} \quad (3)$$

$$\dim A + \dim B \leq n \quad \text{implies} \quad \dim D(A, B) \geq \dim A + \dim B - n + 1. \quad (4)$$

It is not known whether the condition $\dim B \leq \frac{1}{2}(n-1)$ is really needed in (3) and (4). But we can replace it by the condition that B supports a measure ν for which the averages $\sigma(\nu)(r)$ decay at nearly optimal rate as $r \rightarrow \infty$. To explain this, recall that $\dim B$ can be defined as the supremum of the numbers α such that B supports a Radon probability ν with $I_\alpha(\nu) < \infty$. B is called a *Salem set* if the stronger condition holds that for any $\alpha < \dim B$, B supports a Radon probability ν with $|\hat{\nu}(x)|^2 = O(|x|^{-\alpha})$. Our condition is weaker than this, namely that for all $\alpha < \dim B$, B supports a Radon probability ν with $\sigma(\nu)(r) = O(r^{-\alpha})$. We call such sets B *generalized Salem sets*. We shall prove that (3) and (4) are valid if B is a generalized Salem set.

Although deterministic Salem sets are in some sense scarce, they abound as random sets. Kahane has shown that if $F \subset \mathbb{R}$ is compact, its images under n -dimensional Brownian motion and many other stochastic processes, are almost surely Salem sets, see [K1, Chapter XV]. In [KR] Kaufman found deterministic number-theoretic Salem sets. As for generalized Salem sets, the results of Section 2 show that all Borel sets in $\mathbb{R}^n$ of dimension at most $\frac{1}{2}(n-1)$ are such, and that there are also Borel sets which are not generalized Salem sets.

For intersections it was proved in [M2] that if A and B are Borel sets in $\mathbb{R}^n$, $\gamma < \dim A + \dim B - n$ and if $\dim B > \frac{1}{2}(n+1)$, then $\dim A \cap fB \geq \gamma$ for many isometries f . It is not known whether the condition $\dim B > \frac{1}{2}(n+1)$ is redundant, but we shall show that it is in case B is a generalized Salem set. Earlier it has been known that this assumption is not needed if B is sufficiently

smooth, e.g., a C^1 submanifold, [M1], or if one uses larger transformation groups, e.g., similarities, [K2], [M1], in place of isometries.

§2. *Preliminaries.* Throughout the whole paper n will be a positive integer, and α and β will be positive numbers with $0 < \alpha < n$, $0 < \beta < n$. Always $c_1, c_2, \dots$ will denote constants depending only on n, α and β . By $\mathcal{D}$ or $\mathcal{D}(\mathbb{R}^n)$ we mean the space of *non-negative*, compactly supported, infinitely differentiable functions on the euclidean n -space $\mathbb{R}^n$. Mostly the results stated for functions in $\mathcal{D}$ hold in much greater generality. As usual, L^p stands for the Lebesgue space of the p th power integrable real-valued functions on $\mathbb{R}^n$.

A measure in this paper will always mean a Radon outer measure on $\mathbb{R}^n$. The set of all such measures μ with $\mu\mathbb{R}^n < \infty$ and with the support of μ , $\text{spt } \mu$, compact is denoted by $\mathcal{M}$ or $\mathcal{M}(\mathbb{R}^n)$. The Lebesgue measure on $\mathbb{R}^n$ is $\mathcal{L}^n$.

Absolutely continuous measures will be identified with functions, and vice versa, any locally integrable non-negative function f on $\mathbb{R}^n$ with the measure $A \mapsto \int_A f d\mathcal{L}^n$. Thus for example the notation $\mu \in L^p$ means that μ is absolutely continuous with Radon-Nikodým derivative in L^p . The Fourier transform of $\mu \in \mathcal{M}$ is

$$\hat{\mu}(x) = \int e^{-ix \cdot y} d\mu y.$$

Let A be a subset of $\mathbb{R}^n$. The α -capacity of A is

$$C_\alpha(A) = \sup \{ I_\alpha(\mu)^{-1} : \mu \in \mathcal{M}, \text{ spt } \mu \subset A, \mu A = 1 \},$$

where

$$I_\alpha(\mu) = \iint |x - y|^{-\alpha} d\mu x d\mu y$$

is the α -energy of μ . We define the *capacitary dimension* of A by

$$\dim A = \sup \{ \alpha \in (0, n) : C_\alpha(A) > 0 \},$$

with $\sup \emptyset = 0$. It agrees with the *Hausdorff dimension* if A is a Suslin set. Recall that all Borel sets are Suslin sets.

Let J_p be the Bessel function of order p , see [W]. The basic estimates we need are: for $0 < t < \infty$,

$$|J_p(t)| \leq ct^{-1/2}, \quad (2.1)$$

$$|J_p(t)| \leq ct^p. \quad (2.2)$$

We shall frequently work with $J_{\frac{1}{2}(n-2)}$, when $n \geq 2$, and fix the notation

$$J = J_{\frac{1}{2}(n-2)}.$$

§3. *Spherical averages.* For $0 < r < \infty$ and $x \in \mathbb{R}^n$, let

$$S(x, r) = \{y \in \mathbb{R}^n : |x - y| = r\} \quad \text{and} \quad S(r) = S(0, r).$$

For μ and ν in $\mathcal{M}$, we define the averages of $\hat{\mu}\bar{\hat{\nu}}$ and $|\hat{\mu}|^2$ over $S(r)$ by

$$\sigma(\mu, \nu)(r) = \int_{S(1)} \hat{\mu}(r\zeta) \overline{\hat{\nu}(r\zeta)} d\zeta,$$

$$\sigma(\mu)(r) = \sigma(\mu, \mu)(r) = \int_{S(1)} |\hat{\mu}(r\zeta)|^2 d\zeta.$$

By Hölder's inequality

$$|\sigma(\mu, \nu)(r)|^2 \leq \sigma(\mu)(r) \sigma(\nu)(r). \quad (3.1)$$

Sometimes the weighted averages

$$\Sigma(\mu, \nu)(r) = r^{\frac{1}{2}(n-1)} \sigma(\mu, \nu)(r), \quad (3.2)$$

and $\Sigma(\mu) = \Sigma(\mu, \mu)$ will be more convenient.

Let $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be an approximate identity of the form $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon)$ such that $\varphi_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ with

$$\text{spt } \varphi_\varepsilon = \{x: |x| \leq \varepsilon\}, \quad \varphi_\varepsilon(x) = \varphi_\varepsilon(-x), \quad \int \varphi_\varepsilon d\mathcal{L}^n = 1 \quad \text{and} \quad 0 \leq \hat{\varphi}_\varepsilon \leq 1.$$

For any $\mu \in \mathcal{M}$ the convolutions $\mu * \varphi_\varepsilon$ belong to $\mathcal{D}$ and converge weakly to μ , $\mu * \varphi_\varepsilon \rightarrow^w \mu$, as $\varepsilon \downarrow 0$. Also

$$(\mu * \varphi_\varepsilon)^\wedge \rightarrow \hat{\mu}, \quad |(\mu * \varphi_\varepsilon)^\wedge| = |\hat{\mu} \hat{\varphi}_\varepsilon| \leq |\hat{\mu}|,$$

and for $\mu, \nu \in \mathcal{M}$ and $0 < r < \infty$,

$$\lim_{\varepsilon \downarrow 0} \sigma(\mu * \varphi_\varepsilon, \nu * \varphi_\varepsilon)(r) = \sigma(\mu, \nu)(r). \quad (3.3)$$

The mutual α -energy of μ and ν in $\mathcal{M}$ is

$$I_\alpha(\mu, \nu) = \iint |x - y|^{-\alpha} d\mu x d\nu y.$$

Suppose $I_\alpha(\mu) < \infty$ and $I_\alpha(\nu) < \infty$. Then, cf. the proof of [L, Lemma 1.2, p. 82],

$$\lim_{\varepsilon \downarrow 0} I_\alpha(\mu * \varphi_\varepsilon, \nu * \varphi_\varepsilon) = I_\alpha(\mu, \nu). \quad (3.4)$$

Using this, Plancherel's formula and the fact $\hat{k}_{n-\alpha} = ck_\alpha$, where $k_\alpha(x) = |x|^{\alpha-n}$, [L, Section I.1], one easily verifies

$$I_\alpha(\mu, \nu) = c_1 \int |x|^{\alpha-n} \hat{\mu}(x) \overline{\hat{\nu}(x)} dx = c_2 \int_0^\infty r^{\alpha-1} \sigma(\mu, \nu)(r) dr. \quad (3.5)$$

By Hölder's inequality,

$$I_{\frac{1}{2}(\alpha+\beta)}(\mu, \nu) \leq I_\alpha(\mu)^{1/2} U_\beta(\nu)^{1/2}. \quad (3.6)$$

From now on we shall assume, unless otherwise explicitly stated, that $n \geq 2$.

3.7. LEMMA. For $\mu, \nu \in \mathcal{M}$ and $0 < r < \infty$,

$$\sigma(\mu, \nu)(r) = c \iint (r|x-y|)^{(2-n)/2} J(r|x-y|) d\mu x d\nu y.$$

Proof. Since the integrand is bounded and continuous by (2.2), we may, approximating as above, assume that $\mu = f, \nu = g \in \mathcal{D}$. The normalized surface measure τ_r on $S(r)$;

$$\int \varphi d\tau_r = \int_{S(1)} \varphi(r\xi) d\xi,$$

has the Fourier transform

$$\hat{\tau}_r(x) = c_1(r|x|)^{\frac{1}{2}(2-n)} J(r|x|),$$

and the measure $\hat{f} d\tau_r$ is the transform of $c_2 f * \hat{\tau}_r$ in the sense of distributions. Hence by Parseval's formula

$$\begin{aligned} \sigma(f, g)(r) &= \int \hat{f} \bar{\hat{g}} d\tau_r = c_3 \int (f * \hat{\tau}_r) g d\mathcal{L}^n \\ &= c \iint (r|x-y|)^{\frac{1}{2}(2n-n)} J(r|x-y|) f(x) g(y) dx dy. \end{aligned}$$

For $\alpha \leq \frac{1}{2}(n-1)$ we have the following sharp inequality.

3.8. THEOREM. For $0 < \alpha \leq (n-1)/2$, $\mu, \nu \in \mathcal{M}$ and $0 < r < \infty$,

$$|\sigma(\mu, \nu)(r)| \leq cr^{-\alpha} I_\alpha(\mu, \nu).$$

Proof. For $\alpha = \frac{1}{2}(n-1)$ this follows immediately from Lemma 3.7 and (2.1). If $\alpha < \frac{1}{2}(n-1)$, we assume, as we may, that $\mu = f, \nu = g \in \mathcal{D}$, and define γ by $2\gamma = \frac{1}{2}(n-1) - \alpha$, and let k_γ and φ_ε be as before. Set $f_\varepsilon = (f * k_\gamma) \hat{\varphi}_\varepsilon$ and $g_\varepsilon = (g * k_\gamma) \hat{\varphi}_\varepsilon$. Then $\hat{f}_\varepsilon = (\hat{f} \hat{k}_\gamma) * \varphi_\varepsilon$ and $\hat{g}_\varepsilon = (\hat{g} \hat{k}_\gamma) * \varphi_\varepsilon$. Recalling that $k_\gamma(x) = c_1|x|^{-\gamma}$, we obtain

$$\lim_{\varepsilon \downarrow 0} r^{2\gamma} \sigma(f_\varepsilon, g_\varepsilon)(r) = c_1^2 \sigma(f, g)(r).$$

We shall apply the case $\alpha = \frac{1}{2}(n-1)$ and the formula (3.5) to f_ε and g_ε although they are not compactly supported. But this is easily seen to be legitimate as they are smooth and rapidly decreasing. Hence

$$\begin{aligned} |r^{2\gamma} \sigma(f_\varepsilon, g_\varepsilon)(r)| &\leq c_2 r^{2\gamma - \frac{1}{2}(n-1)} I_{\frac{1}{2}(n-1)}(f_\varepsilon, g_\varepsilon) \\ &= c_3 r^{-\alpha} \int |x|^{\frac{1}{2}(n-1)-n} ((\hat{f} \hat{k}_\gamma) * \varphi_\varepsilon)(x) ((\hat{g} \hat{k}_\gamma) * \varphi_\varepsilon)(x) dx \\ &\rightarrow c_4 r^{-\alpha} \int |x|^{\alpha-n} \hat{f}(x) \bar{\hat{g}}(x) dx = c_5 r^{-\alpha} I_\alpha(f, g), \end{aligned}$$

as $\varepsilon \downarrow 0$, and the theorem follows.

3.9. Remarks. Theorem 3.8 gives the best possible asymptotic estimate. If $\alpha \leq \frac{1}{2}(n-1)$ and $I_\alpha(\mu) < \infty$, then

$$\sigma(\mu)(r) = O(r^{-\alpha}) \quad \text{as} \quad r \rightarrow \infty.$$

This and Theorem 3.8 fail for $\alpha > \frac{1}{2}(n-1)$, see Example 3.12. However, see Theorem 3.10.

In particular, $\sigma(\mu)(r) \rightarrow 0$ as $r \rightarrow \infty$ if $I_\alpha(\mu) < \infty$ for some $\alpha > 0$. But this is easily seen to hold much more generally.

If $\mu, \nu \in \mathcal{M}$ and $\mu\{y\} = 0$ for ν almost all $y \in \mathbb{R}^n$, then $\lim_{r \rightarrow \infty} \sigma(\mu, \nu)(r) = 0$.

In fact, given $\varepsilon > 0$ the continuity condition on μ yields $\delta > 0$ such that $\int \mu\{x: |x-y| \leq \delta\} d\nu y < \varepsilon$. Splitting the integration in the formula of Lemma 3.7 into $|x-y| \leq \delta$ and $|x-y| > \delta$, and using the estimates (2.1) and (2.2), we find

$$|\sigma(\mu, \nu)(r)| \leq c_1 \int \mu\{x: |x-y| \leq \delta\} d\nu y + c_2(r\delta)^{\frac{1}{2}(1-n)} \mu(\mathbb{R}^n) \nu(\mathbb{R}^n) < 2\varepsilon$$

for sufficiently large r .

3.10. THEOREM. Let $\mu \in \mathcal{M}$ and suppose that the diameter of $\text{spt } \mu$ is at most 1. Then for $0 < r < \infty$,

$$\sigma(\mu)(r) \leq c I_\alpha(\mu) r^{-\frac{2}{3}\alpha - \frac{1}{6}(n-5)}.$$

Proof. We assume $\alpha \geq \frac{1}{2}(n+1)$, since otherwise Theorem 3.8 gives a better estimate. Let $\gamma = \frac{2}{3}\alpha + \frac{1}{6}(n-5)$. Then $\alpha - \gamma \geq 1$. Again we may assume $\mu = f \in \mathcal{D}$. Let

$$\tau(r) = r^\gamma \sigma(f)(r).$$

Then

$$I_\alpha(f) = c_1 \int_0^\infty r^{\alpha-\gamma-1} \tau(r) dr, \quad (1)$$

and $\tau(r) \leq I_\alpha(f)/c_1$ for some r in every interval $[k, k+1]$, $k \geq 1$. Hence it suffices to verify the required estimate $\tau(r) \leq c I_\alpha(f)$ at a point $r_0 > 2$ where τ has a local maximum. If $\tau(r_0) > 2 I_\alpha(f)/c_1$, we find $r_1, r_0 < r_1 \leq 2r_0$, such that $\tau(r_1) = \tau(r_0)/2$ and $\tau(r_1) \leq \tau(r)$ for $r_0 \leq r \leq r_1$. Then as $\tau'(r_0) = 0$,

$$\frac{1}{2} \tau(r_0) = - \int_{r_0}^{r_1} \tau'(r) dr = - \int_{r_0}^{r_1} \int_{r_0}^r \tau''(t) dt dr. \quad (2)$$

By Lemma 3.7

$$\tau(r) = cr^\gamma \iint (r|x-y|)^{\frac{1}{2}(2-n)} J(r|x-y|) f(x) f(y) dx dy.$$

Differentiating twice, using the identities

$$\frac{d}{dt} (t^{-p} J_p(t)) = -t^{-p} J_{p+1}(t)$$

and

$$\frac{d^2}{dt^2}(t^{-p}J_p(t)) = t^{-p}J_{p+2}(t) - t^{-p-1}J_{p+1}(t),$$

cf. [W, Section 3.2], recalling the assumption $\text{diam}(\text{spt } \mu) \leq 1$, and applying the estimate (2.1), we see that for $t \geq 1$,

$$|\tau''(t)| \leq c_2 I_{\frac{1}{2}(n-1)}(f) t^{\gamma-\frac{1}{2}(n-1)} \leq c_2 I_\alpha(f) t^{\gamma-\frac{1}{2}(n-1)}.$$

Inserting this in (2) we get

$$\begin{aligned} \tau(r_0) &\leq c_2 I_\alpha(f) \int_{r_0}^{r_1} \int_{r_0}^r t^{\gamma-\frac{1}{2}(n-1)} dt dr \\ &\leq c_3 I_\alpha(f) r_0^{\gamma-\frac{1}{2}(n-1)} (r_1 - r_0)^2. \end{aligned}$$

Hence, by (1),

$$\begin{aligned} I_\alpha(f) &\geq c_1 \int_{r_0}^{r_1} r^{\alpha-\gamma-1} \tau(r) dr \\ &\geq \frac{1}{2} c_1 r_0^{\alpha-\gamma-1} (r_1 - r_0) \tau(r_0) \\ &\geq c_4 I_\alpha(f)^{-\frac{1}{2}} r_0^{\alpha-\gamma-1-\frac{1}{2}\gamma+\frac{1}{4}(n-1)} \tau(r_0)^{\frac{3}{2}} \\ &= c_4 I_\alpha(f)^{-\frac{1}{2}} \tau(r_0)^{\frac{3}{2}}, \end{aligned}$$

which gives the required estimate.

3.11. Remarks. The assumption $\text{diam}(\text{spt } \mu) \leq 1$ was used to guarantee $I_\gamma(\mu) \leq I_\alpha(\mu)$ for $\gamma < \alpha$. Without it a more complicated estimate involving lower order energies would follow.

As noted in the proof, Theorem 3.10 improves Theorem 3.8 only when $\alpha > \frac{1}{2}(n+1)$. I don't know what is the best possible exponent. The following example in the plane presents some limitations. It shows in particular that when $n = 2$, the asymptotic estimate given by Theorem 3.8 for μ with $I_\alpha(\mu) < \infty$ and $\alpha \geq \frac{1}{2}$,

$$\sigma(\mu)(r) = O(r^{-\frac{1}{2}}) \quad \text{as } r \rightarrow \infty,$$

is best possible for $\frac{1}{2} \leq \alpha < 1$.

3.12. EXAMPLE. For each α , $0 < \alpha < 1$, there is $\mu_\alpha \in \mathcal{M}(\mathbb{R}^2)$ such that $I_\alpha(\mu_\alpha) < \infty$ and

$$\limsup_{r \rightarrow \infty} r^{\frac{1}{2}} \sigma(\mu_\alpha)(r) > 0.$$

For each α, α' , $1 \leq \alpha < \alpha'$, there is $\mu_{\alpha, \alpha'} \in \mathcal{M}(\mathbb{R}^2)$ such that $I_\alpha(\mu_{\alpha, \alpha'}) < \infty$ and

$$\limsup_{r \rightarrow \infty} r^{\frac{1}{2}\alpha'} \sigma(\mu_{\alpha, \alpha'})(r) > 0.$$

Proof. Let $0 < \alpha < 1$. There is $\mu = \mu_\alpha \in \mathcal{M}(\mathbb{R}^2)$ such that

$$\text{spt } \mu \subset \{(x, y): y = 0\}, \quad I_\alpha(\mu) < \infty \quad \text{and} \quad \hat{\mu}(x, 0) \not\rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

In fact, if $\alpha < \beta < 1$ and $\beta = \log 2 / \log(1/d)$, we take as μ the natural probability measure on the Cantor set constructed by deleting the mid-interval of length $(1-2d)l$ from intervals of length l . The formula for the Fourier transform of this measure can be found in [KS, p. 15]. Since $\hat{\mu}$ is Lipschitz continuous and since $\hat{\mu}(x, y) = \hat{\mu}(x, 0)$ for $x, y \in \mathbb{R}$, we can find arbitrarily large numbers r for which $|\hat{\mu}(x, y)| > a$ for $x \in [r-a, r]$ and $y \in \mathbb{R}$, where a is a positive number independent of r . The length of the arc $S(r) \cap \{(x, y): r-a \leq x \leq r\}$ is at least $br^{\frac{1}{2}}$ for some $b > 0$ independent of r . It follows that

$$\sigma(\mu)(r) = (2\pi r)^{-1} \int_{S(r)} |\hat{\mu}(\zeta)|^2 d\zeta \geq (a^2 b / 2\pi) r^{-\frac{1}{2}},$$

and the first assertion is proved.

Let $\alpha = \beta + \gamma$, $\alpha' = \beta + \gamma'$, $0 < \beta < 1$, $0 < \gamma < \delta < \varepsilon < \eta < \gamma' < 1$ with $1 + \eta = \alpha'$. From [KS, pp. 14-15] we see that we can choose $\mu \in \mathcal{M}(\mathbb{R})$ with the following properties.

$$\mu[x - \rho, x + \rho] \leq \rho^\beta \quad \text{for} \quad x \in \mathbb{R}, \quad 0 < \rho < \infty, \quad (1)$$

and there are $a > 0$ and $t > 1$ such that for every $k = 1, 2, \dots$ there is $r_k \in [t^k, t^{k+1}]$ with

$$|\hat{\mu}(x)| > a \quad \text{for} \quad x \in [r_k - a, r_k]. \quad (2)$$

We also find $\nu \in \mathcal{M}(\mathbb{R})$ such that

$$\nu[y - \rho, y + \rho] \leq \rho^\delta \quad \text{for} \quad y \in \mathbb{R}, \quad 0 < \rho < \infty, \quad (3)$$

and $I_\varepsilon(\nu) = \infty$. As $\varepsilon < \eta$ and

$$\int |y|^{\varepsilon-1} |\hat{\nu}(y)|^2 dy = c I_\varepsilon(\nu) = \infty,$$

we must have, by a geometric series argument, for arbitrarily large k

$$\int_0^{bt^{k/2}} |\hat{\nu}(y)|^2 dy \geq t^{\frac{1}{2}(1-\eta)k}, \quad (4)$$

where $b > 0$ as in the first part of the proof.

We show that $\mu_{\alpha, \alpha'} = \mu \times \nu$ has the required properties. First for any disc $B((x, y), \rho) \subset \mathbb{R}^2$,

$$(\mu \times \nu)B((x, y), \rho) \leq \mu[x - \rho, x + \rho] \nu[y - \rho, y + \rho] \leq \rho^{\beta+\delta}$$

by (1) and (3). As $\alpha < \beta + \delta$, an easy calculation, like that in the proof of Theorem 6.3 in [F1], shows then that $I_\alpha(\mu \times \nu) < \infty$. Secondly, note that

$(\mu \times \nu)^\wedge(x, y) = \hat{\mu}(x)\hat{\nu}(y)$. If k is as in (4) and $r = r_k$, we have by (2) and (4),

$$\begin{aligned}\sigma(\mu \times \nu)(r) &= (2\pi r)^{-1} \int_{S(r)} |(\mu \times \nu)^\wedge(\xi)|^2 d\xi \\ &\geq a^2(2\pi r)^{-1} \int_0^{br^{1/2}} |\hat{\nu}(y)|^2 dy \geq a^2(2\pi r)^{-1} t^{\frac{1}{2}(1-\eta)k} \\ &\geq a^2(2\pi)^{-1} t^{\frac{1}{2}(\eta-1)} r^{-\frac{1}{2}(1+\eta)} = a^2(2\pi)^{-1} t^{\frac{1}{2}(\eta-1)} r^{-\frac{1}{2}\alpha'},\end{aligned}$$

and the second assertion follows.

§4. *Distance measures.* For $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$ we define the *distance measure* $\delta(\mu, \nu) \in \mathcal{M}(\mathbb{R})$ of μ and ν by

$$\int \varphi d\delta(\mu, \nu) = \iint \varphi(|x - y|) d\mu x d\nu y$$

for continuous functions φ on $\mathbb{R}$. That is, $\delta(\mu, \nu)$ is the image of $\mu \times \nu$ under the distance map. Sometimes it will be convenient to use the weighted measure $\Delta(\mu, \nu)$ determined by

$$\int \varphi d\Delta(\mu, \nu) = \int s^{\frac{1}{2}(1-n)} \varphi(s) d\delta(\mu, \nu)s. \quad (4.1)$$

Note that

$$\delta(\mu \nu)(R) = \mu(R^n) \nu(R^n), \quad \delta(\mu_i, \nu_i) \xrightarrow{w} \delta(\mu, \nu),$$

whenever $\mu_i \xrightarrow{w} \mu$, $\nu_i \xrightarrow{w} \nu$ and the supports of μ_i and ν_i are contained in a fixed compact set, and that

$$\int s^{-\alpha} d\delta(\mu, \nu)s = I_\alpha(\mu, \nu). \quad (4.2)$$

4.3. LEMMA. If $f, g \in \mathcal{D}$, then $\delta(f, g)$ is continuous with compact support, $\delta(f, g)(s) = 0$ for $s \leq 0$, and for $0 < s < \infty$,

$$\begin{aligned}\delta(f, g)(s) &= \int \left(\int_{S(x, s)} f(\xi) d\xi \right) g(x) dx \\ &= c_1 s^{\frac{1}{2}n} \int |x|^{\frac{1}{2}(2-n)} J(s|x|) \hat{f}(x) \overline{\hat{g}(x)} dx \\ &= c_2 s^{\frac{1}{2}n} \int_0^\infty r^{\frac{1}{2}n} J(sr) \sigma(f, g)(r) dr.\end{aligned}$$

Proof. It is clear that $\delta(f, g)$ is continuous with compact support and that it vanishes for $s \leq 0$. Also the first and third identities are immediate. To prove that the first member equals the third, let φ be continuous with compact support, and let ψ be the radial function $\psi(x) = \varphi(|x|)$ on $\mathbb{R}^n$. Then

$$\begin{aligned} \int \varphi d\delta(f, g) &= \iint \varphi(|x-y|)f(x)g(y)dx dy \\ &= \int (\psi * f)gd\mathcal{L}^n = c_3 \int \hat{\psi}\hat{f}\hat{g}d\mathcal{L}^n. \end{aligned}$$

By the formula for the Fourier transform of a radial function, see [SW, p. 155],

$$\hat{\psi}(x) = c_4|x|^{\frac{1}{2}(2-n)} \int_0^\infty \varphi(s)J(s|x|)s^{\frac{1}{2}n}ds,$$

whence

$$\int \varphi d\delta(f, g) = c_5 \int \varphi(s)s^{\frac{1}{2}n} \int |x|^{\frac{1}{2}(2-n)}J(s|x|)\hat{f}(x)\overline{\hat{g}(x)}dx ds,$$

and the lemma follows.

4.4. THEOREM. Suppose $\mu, \nu \in \mathcal{M}$, $I_\alpha(\mu) < \infty$, $I_\beta(\nu) < \infty$, and $\alpha + \beta \geq n + 1$. Then $\delta(\mu, \nu) \in L^\infty$ and for $0 < s < \infty$,

$$\delta(\mu, \nu)(s) \leq cs^{\frac{1}{2}(\alpha+\beta)-1}I_\alpha(\mu)^{\frac{1}{2}}I_\beta(\nu)^{\frac{1}{2}}.$$

Proof. Approximating by $\mu * \psi_\varepsilon$ and $\nu * \varphi_\varepsilon$ we reduce to the case $\mu = f$, $\nu = g \in \mathcal{D}$. Then by Lemma 4.3 for $0 < s < \infty$

$$\delta(f, g)(s) = c_1 s^{\frac{1}{2}n} \int_0^\infty r^{\frac{1}{2}n}J(sr)\sigma(f, g)(r)dr = \delta_1(s) + \delta_2(s)$$

where $\delta_1(s)$ is the integral from 0 to $1/s$ and $\delta_2(s)$ from $1/s$ to ∞ . We use (2.2), (3.1), the fact $\frac{1}{2}(\alpha + \beta) < n$, Hölder's inequality and (3.5) to estimate

$$\begin{aligned} \delta_1(s) &\leq c_2 s^{n-1} \int_0^{1/s} r^{n-1}|\sigma(f, g)(r)|dr \\ &\leq c_2 s^{n-1} \int_0^{1/s} r^{n-1}\sigma(f)(r)^{\frac{1}{2}}\sigma(g)(r)^{\frac{1}{2}}dt \\ &= c_2 s^{\frac{1}{2}(\alpha+\beta)-1} \int_0^{1/s} (sr)^{n-\frac{1}{2}(\alpha+\beta)}(r^{\alpha-1}\sigma(f)(r))^{\frac{1}{2}}(r^{\beta-1}\sigma(g)(r))^{\frac{1}{2}}dr \\ &\leq c_2 s^{\frac{1}{2}(\alpha+\beta)-1} \int_0^{1/s} (r^{\alpha-1}\sigma(f)(r))^{\frac{1}{2}}(r^{\beta-1}\sigma(g)(r))^{\frac{1}{2}}dr \end{aligned}$$

$$\begin{aligned} &\leq c_2^{\frac{1}{2}(\alpha+\beta)-1} \left(\int_0^\infty r^{\alpha-1} \sigma(f)(r) dr \right)^{\frac{1}{2}} \left(\int_0^\infty r^{\beta-1} \sigma(g)(r) dr \right)^{\frac{1}{2}} \\ &= c_3 s^{\frac{1}{2}(\alpha+\beta)-1} I_\alpha(f)^{\frac{1}{2}} I_\beta(g)^{\frac{1}{2}}. \end{aligned}$$

Similarly using (2.1), (3.1), the fact $n+1 \leq \alpha + \beta$, Hölder's inequality and (3.5),

$$\begin{aligned} \delta_2(s) &\leq c_4 s^{\frac{1}{2}(n-1)} \int_{1/s}^\infty r^{\frac{1}{2}(n-1)} \sigma(f)(r)^{\frac{1}{2}} \sigma(g)(r)^{\frac{1}{2}} dr \\ &= c_4 s^{\frac{1}{2}(\alpha+\beta)-1} \int_{1/s}^\infty (sr)^{\frac{1}{2}(n+1-\alpha-\beta)} (r^{\alpha-1} \sigma(f)(r))^{\frac{1}{2}} (r^{\beta-1} \sigma(g)(r))^{\frac{1}{2}} dr \\ &\leq c_5 s^{\frac{1}{2}(\alpha+\beta)-1} I_\alpha(f) I_\beta(g). \end{aligned}$$

The combination of these estimates completes the proof. Note that $\sigma(\mu, \nu)$ is bounded since it has compact support.

4.5. Remark. If $\mu = \nu$, Theorem 4.4 tells us that $\delta(\mu) \in L^\infty$ provided $I_{\frac{1}{2}(n+1)}(\mu) < \infty$. The example 2.2 in [M2] shows that for no $\alpha < \frac{1}{2}(n+1)$ does $I_\alpha(\mu) < \infty$ imply $\delta(\mu) \in L^\infty$.

Next we study L^2 -relations between Σ and Δ ; for the definitions recall (3.2) and (4.1).

4.6. THEOREM. For $f, g, h, k \in \mathcal{D}$,

$$\int_0^\infty \Delta(f, g)(s) \Delta(h, k)(s) ds = c \int_0^\infty \Sigma(f, g)(r) \Sigma(h, k)(r) dr.$$

Proof. Let $S = \Sigma(f, g)$ and $T = \Sigma(h, k)$. Using Lemma 4.3 and Hankel's Theorem, see [W, p. 456], we compute

$$\begin{aligned} \int_0^\infty \Delta(f, g)(s) \Delta(h, k)(s) ds &= c \int_0^\infty \int_0^\infty \int_0^\infty s p^{\frac{1}{2}} J(sp) S(p) r^{\frac{1}{2}} J(sr) T(r) dp dr ds \\ &= c \int_0^\infty r^{\frac{1}{2}} T(r) \int_0^\infty s J(sr) \int_0^\infty p J(sp) p^{-\frac{1}{2}} S(p) dp ds dr \\ &= c \int_0^\infty r^{\frac{1}{2}} T(r) r^{-\frac{1}{2}} S(r) dr \\ &= c \int_0^\infty S(r) T(r) dr. \end{aligned}$$

4.7. COROLLARY. For $f, g \in \mathcal{D}$,

$$\int_0^\infty \Delta(f, g)(s)^2 ds = c \int \Sigma(f, g)(r)^2 dr,$$

$$\int_0^\infty \Delta(f)(s)\Delta(g)(s)ds = c \int \Sigma(f)(r)\Sigma(g)(r)dr.$$

Introducing the notation

$$\sigma_\alpha(\mu) = \sup_{0 < r < \infty} r^\alpha \sigma(\mu)(r) \quad \text{for} \quad \mu \in \mathcal{M}, \quad (4.8)$$

we get a corollary for measures.

4.9. COROLLARY. Suppose $\mu, \nu \in \mathcal{M}$, $I_\alpha(\mu) < \infty$ and $\sigma_{n-\alpha}(\nu) < \infty$. Then $\Delta(\mu, \nu) \in L^2$ and

$$\int_0^\infty \Delta(\mu, \nu)(s)^2 ds \leq c I_\alpha(\mu) \sigma_{n-\alpha}(\nu).$$

Proof. We may again assume $\mu = f$, $\nu = g \in \mathcal{D}$. By Corollary 4.7, (3.2), (3.1) and (3.5), we have

$$\begin{aligned} \int_0^\infty \Delta(f, g)(s)^2 ds &= c_1 \int_0^\infty \Sigma(f, g)(r)^2 dr \\ &\leq c_1 \int_0^\infty r^{n-1} \sigma(f)(r) \sigma(g)(r) dr \\ &\leq c_1 \sigma_{n-\alpha}(g) \int_0^\infty r^{\alpha-1} \sigma(f)(r) dr = c_2 \sigma_{n-\alpha}(g) I_\alpha(f). \end{aligned}$$

4.10. Remark. In the case when $\mu = \nu$, we get $\Delta(\mu) \in L^2$, and hence $\delta(\mu) \in L^2$, provided $I_{\frac{1}{2}n}(\mu) < \infty$ and $\sigma_{\frac{1}{2}n}(\mu) < \infty$. I don't know if $I_\alpha(\mu) < \infty$ implies $\delta(\mu) \in L^2$ for some $\alpha, \frac{1}{2}n \leq \alpha < \frac{1}{2}(n+1)$. That it does not for $\alpha < \frac{1}{2}n$ follows from the examples in [F2].

Our next goal is to estimate energy-integrals of $\Delta(\mu, \nu)$. For this we need to have a closer look at the Bessel function J .

4.11. LEMMA. For $0 < t < \infty$

$$J(t) = at^{-\frac{1}{2}} \cos t + bt^{-\frac{1}{2}} \sin t + R(t),$$

where the constants a and b are not both zero, and

$$|R(t)| \leq c \min \{t^{-\frac{1}{2}}, t^{-\frac{3}{2}}\}.$$

This follows immediately for example from [SW, p. 158].

Recall that if $n=3$, then $J(t) = bt^{-\frac{1}{2}} \sin t$, and some of the calculations below would simplify.

4.12. Notation. We define the sign-function sgn by $\operatorname{sgn}(t) = 1$ for $t > 0$ and $\operatorname{sgn}(t) = -1$ for $t \leq 0$, and the even and odd extensions of Σ to $\mathbb{R}$ by

$$\Sigma_1(\mu, \nu)(r) = \Sigma(\mu, \nu)(|r|) \quad \text{and} \quad \Sigma_2(\mu, \nu)(r) = \operatorname{sgn}(r)\Sigma(\mu, \nu)(|r|) \quad \text{for } r \in \mathbb{R}.$$

We consider the operator

$$\mathcal{H} = I - iH: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}),$$

where I is the identity and H the Hilbert transform, that is $Hf = i \operatorname{sgn} \hat{f}$ for $f \in L^2$. Since the functions $r \mapsto |r|^{-\gamma}$, $0 \leq \gamma < 1$, satisfy Muckenhoupt's A_2 condition, $\mathcal{H}$ is continuous between the corresponding weighted L^2 -spaces, that is, see [M],

$$\int_{-\infty}^{\infty} |r|^{-\gamma} |\mathcal{H}\varphi(r)|^2 dr \leq c(\gamma) \int_{-\infty}^{\infty} |r|^{-\gamma} |\varphi(r)|^2 dr. \quad (4.13)$$

4.14. LEMMA. For $f, g \in \mathcal{D}$, $\Delta(f, g)$ has the decomposition

$$\Delta(f, g)(s) = F(f, g)(s) + K(f, g)(s)$$

where $F(f, g)(s) = K(f, g)(s) = 0$ for $s \leq 0$, and for $r \in \mathbb{R}$ and $0 < s < \infty$

$$F(f, g)(s) = c_1 \mathcal{H}(\Sigma_1(f, g))(r) + c_2 \mathcal{H}(\Sigma_2(f, g))(r),$$

$$K(f, g)(s) = c_3 s^{\frac{1}{2}} \int_0^{\infty} r^{\frac{1}{2}} R(sr) \Sigma(f, g)(r) dr.$$

Proof. Let $\Sigma = \Sigma(f, g)$, $\Sigma_i = \Sigma_i(f, g)$ for $i = 1, 2$. Using Lemmas 4.3 and 4.11 we obtain this decomposition with $K(f, g)$ as required and, for $0 < s < \infty$,

$$\begin{aligned} F(f, g)(s) &= c_4 \int_0^{\infty} (a \cos(sr) + b \sin(sr)) \Sigma(r) dr \\ &= \frac{1}{2} ac_4 \int \cos(sr) \Sigma_1(r) dr + \frac{1}{2} bc_4 \int \sin(sr) \Sigma_2(r) dr \\ &= 2a_1 \int e^{isr} \Sigma_1(r) dr + 2b_1 \int e^{isr} \Sigma_2(r) dr \end{aligned}$$

with $a_1 = \frac{1}{4} ac_4$ and $b_1 = \frac{1}{4} bc_4/i$. Then we can write for all $s \in \mathbb{R}$

$$\begin{aligned} F(f, g)(s) &= a_1 \int e^{isr} \Sigma_1(r) dr + a_1 \operatorname{sgn}(s) \int e^{isr} \Sigma_1(r) dr \\ &\quad + b_1 \int e^{isr} \Sigma_2(r) dr + b_1 \operatorname{sgn}(s) \int e^{isr} \Sigma_2(r) dr. \end{aligned}$$

Since $\widehat{Hf} = i \operatorname{sgn} \hat{f}$, we obtain by the inversion formula

$$F(f, g)^{\wedge} = 2\pi a_1 \mathcal{H}\Sigma_1 + 2\pi b_1 \mathcal{H}\Sigma_2.$$

4.15. LEMMA. If $f, g \in \mathcal{D}$ and $n-1 \leq \alpha + \beta \leq n+1$, then for $0 < s < \infty$,

$$|K(f, g)(s)| \leq cs^{\frac{1}{2}(\alpha+\beta-n-1)} I_{\alpha}(f)^{\frac{1}{2}} I_{\beta}(g)^{\frac{1}{2}}.$$

In particular, $K(f, g)$ and $sK(f, g)(s)$ are bounded.

Proof. Let $\delta = \frac{1}{2}(n+1-\alpha-\beta)$. Then $0 \leq \delta \leq 1$. By Lemmas 4.14, 4.11, (3.1) and Hölder's inequality

$$\begin{aligned} |K(f, g)(s)| &= \left| c_1 s^{\frac{1}{2}} \int_0^{\infty} r^{\frac{1}{2}n} R(sr) \sigma(f, g)(r) dr \right| \\ &\leq c_2 \int_0^{1/s} r^{\frac{1}{2}(n-1)} |\sigma(f, g)(r)| dr + c_2 s^{-1} \int_{1/s}^{\infty} r^{\frac{1}{2}(n-3)} |\sigma(f, g)(r)| dr \\ &= c_2 s^{-\delta} \int_0^{1/s} (sr)^{\delta} r^{\frac{1}{2}(n-1)-\delta} |\sigma(f, g)(r)| dr \\ &\quad + c_2 s^{-\gamma} \int_{1/s}^{\infty} (sr)^{\delta-1} r^{\frac{1}{2}(n-1)-\delta} |\sigma(f, g)(r)| dr \\ &\leq c_2 s^{-\delta} \int_0^{\infty} r^{\frac{1}{2}(\alpha+\beta)-1} |\sigma(f, g)(r)| dr \\ &\leq c_2 s^{-\delta} \int_0^{\infty} (r^{\alpha-1} \sigma(f)(r))^{\frac{1}{2}} (r^{\beta-1} \sigma(g)(r))^{\frac{1}{2}} dr \\ &\leq c_2 s^{-\delta} \left(\int_0^{\infty} r^{\alpha-1} \sigma(f)(r) dr \right)^{\frac{1}{2}} \left(\int_0^{\infty} r^{\beta-1} \sigma(g)(r) dr \right)^{\frac{1}{2}} \\ &= c_3 s^{-\delta} I_{\alpha}(f)^{\frac{1}{2}} I_{\beta}(g)^{\frac{1}{2}}. \end{aligned}$$

The last statement follows with $\alpha = \beta = \frac{1}{2}(n-1)$ and $\alpha = \beta = \frac{1}{2}(n+1)$.

4.16. THEOREM. If $\beta \leq \frac{1}{2}(n-1)$ and $0 < \gamma = \alpha + \beta - n + 1 < 1$, then for $\mu, \nu \in \mathcal{M}$,

$$I_{\gamma}(\Delta(\mu, \nu)) \leq c I_{\alpha}(\mu) I_{\beta}(\nu).$$

Proof. We may assume $\mu = f, \nu = g \in \mathcal{D}$. Using Lemma 4.14 we write

$$\Delta = \Delta(f, g) = F(f, g) + K(f, g) = F + K.$$

We set

$$I_\gamma(G, H) = \iint |s-t|^{-\gamma} G(s)H(t)dsdt$$

for all functions G and H on $\mathbb{R}$ for which the right-hand side is defined. Then

$$I_\gamma(G) = I_\gamma(G, G) = c_1 \int |r|^{\gamma-1} |\hat{G}(r)|^2 dr$$

provided $I_\gamma(|G|) < \infty$. From the identity

$$\Delta(s)\Delta(t) = F(s)F(t) + \Delta(s)K(t) + \Delta(t)K(s) - K(s)K(t),$$

we see that

$$I_\gamma(\Delta) = I_\gamma(F) + 2I_\gamma(\Delta, K) - I_\gamma(K) \leq I_\gamma(F) + 2I_\gamma(\Delta, K),$$

since $I_\gamma(K) \geq 0$. As Δ is bounded with compact support, Lemmas 4.14 and 4.15 show that F and $sF(s)$ are bounded. Elementary calculations then reveal that $I_\gamma(|F|) < \infty$. Thus using Lemma 4.14, (4.13), (3.1) and Theorem 3.8, we have

$$\begin{aligned} I_\gamma(F) &= c_1 \int |r|^{\gamma-1} |\hat{F}(r)|^2 dr \\ &\leq c_2 \int_0^\infty r^{\gamma-1} \Sigma(f, g)(r)^2 dr = c_2 \int_0^\infty r^{\gamma+n-2} \sigma(f, g)(r)^2 dr \\ &\leq c_2 \int_0^\infty r^{\gamma+n-2} \sigma(f)(r) \sigma(g)(r) dr \\ &\leq c_3 I_\beta(g) \int_0^\infty r^{\alpha-1} \sigma(f)(r) dr = c_4 I_\alpha(f) I_\beta(g). \end{aligned}$$

To estimate $I_\gamma(\Delta, K)$, we use Lemma 4.15, the elementary fact

$$\int_0^\infty t^{\frac{1}{2}\gamma-1} |s-t|^{-\gamma} dt \leq c_5 s^{-\frac{1}{2}\gamma},$$

(4.2) and (3.6) to obtain

$$\begin{aligned} |I_\gamma(\Delta, K)| &= \left| \int_0^\infty \int_0^\infty \Delta(s)K(t) |s-t|^{-\gamma} dt ds \right| \\ &\leq c_6 I_\alpha(f)^{\frac{1}{2}} I_\beta(g)^{\frac{1}{2}} \int_0^\infty \Delta(s) \int_0^\infty t^{\frac{1}{2}\gamma-1} |s-t|^{-\gamma} dt ds \end{aligned}$$

$$\begin{aligned}
&\leq c_7 I_\alpha(f)^{\frac{1}{2}} I_\beta(g)^{\frac{1}{2}} \int_0^\infty \Delta(s) s^{-\frac{1}{2}\gamma} ds \\
&= c_7 I_\alpha(f)^{\frac{1}{2}} I_\beta(g)^{\frac{1}{2}} \int s^{-\frac{1}{2}(\alpha+\beta)} \delta(f, g)(s) ds \\
&= c_7 I_\alpha(f)^{\frac{1}{2}} I_\beta(g)^{\frac{1}{2}} I_{\frac{1}{2}(\alpha+\beta)}(f, g) \\
&\leq c_7 I_\alpha(f) I_\beta(g).
\end{aligned}$$

The theorem follows from these estimates.

4.17. THEOREM. If $0 < \gamma = \alpha + \beta - n + 1 < n$, then for $\mu, \nu \in \mathcal{M}$,

$$I_\gamma(\Delta(\mu, \nu)) \leq c I_\alpha(\mu)(I_\beta(\nu) + \sigma_\beta(\nu)).$$

Recalling the definition of $\sigma_\beta(\nu)$ from (4.8), the proof is almost identical with that of Theorem 4.16.

4.18. Remark. I don't know if the inequality of Theorem 4.17 holds without the term $\sigma_\beta(\nu)$ unless $\beta \leq \frac{1}{2}(n-1)$, which is the case of Theorem 4.16.

5. DISTANCE SETS. The distance sets of sets $A, B \subset \mathbb{R}^n$ is

$$D(A, B) = \{|x - y| : x \in A, y \in B\}.$$

For $\mu, \nu \in \mathcal{M}$ we clearly have

$$\text{spt } \delta(\mu, \nu) = D(\text{spt } \mu, \text{spt } \nu).$$

The following contains part of Falconer's result in [F2, Theorem 2.2].

5.1. THEOREM. Let A and B be subsets of $\mathbb{R}^n$. If $\alpha + \beta \geq n + 1$, $C_\alpha(A) > 0$ and $C_\beta(B) > 0$, then $\mathcal{L}^1 D(A, B) > 0$. In particular, $\dim A + \dim B > n + 1$ implies $\mathcal{L}^1 D(A, B) > 0$.

Proof. Assuming $C_\alpha(A) > 0$ and $C_\beta(B) > 0$, we find measures $\mu, \nu \in \mathcal{M}$ with $\text{spt } \mu \subset A$, $\text{spt } \nu \subset B$, $\mu A = \nu B = 1$, $I_\alpha(\mu) < \infty$ and $I_\beta(\nu) < \infty$. Then $\text{spt } \delta(\mu, \nu) \subset D(A, B)$ and $\delta(\mu, \nu)(D(A, B)) = 1$, whence by Theorem 4.4, $\mathcal{L}^1 D(A, B) > 0$.

Although capacity formulations follow as easily, we state the remaining results on distance sets only in term of dimensions.

5.2. Definitions. In order to formulate further results, we shall introduce some new concepts. Let $A \subset \mathbb{R}^n$. Recall that the Fourier dimension of A , $\dim_F A$, is the supremum of the numbers α such that A supports a $\mu \in \mathcal{M}$ with $\mu A = 1$ and for some $C < \infty$

$$|\hat{\mu}(x)|^2 \leq C|x|^{-\alpha} \quad \text{for } x \in \mathbb{R}^n.$$

A is called a Salem set if $\dim_F A = \dim A$. (For these see [K1].) Analogously we define (as a convenient working term and not a proposal for a new concept

of dimension) the *spherical Fourier dimension*, $\dim_{SF} A$, to be the supremum of the numbers α such that A supports a $\mu \in \mathcal{M}$ with $\mu A = 1$ and $\sigma_\alpha(\mu) < \infty$, that is,

$$\sigma(\mu)(r) \leq Cr^{-\alpha} \quad \text{for } 0 < r < \infty.$$

We call A a *generalized Salem set* if $\dim_{SF} A = \dim A$. Note that always

$$\dim_F A \leq \dim_{SF} A \leq \dim A.$$

Thus Salem sets are generalized Salem sets. By Theorem 3.8 all sets A in $\mathbb{R}^n$ with $\dim A \leq \frac{1}{2}(n-1)$ are generalized Salem sets, while Example 3.12 indicates that this is no longer true beyond the borderline $\frac{1}{2}(n-1)$. However, when $\dim A \geq \frac{1}{2}(n-1)$, Theorems 3.8 and 3.10 give

$$\dim_{SF} A \geq \max \left\{ \frac{1}{2}(n-1), \frac{2}{3} \dim A + \frac{1}{6}(n-5) \right\}.$$

5.3. THEOREM. *Let A and B be subsets of $\mathbb{R}^n$. If $\dim A + \dim_{SF} B > n$, then $\mathcal{L}^1 D(A, B) > 0$. In particular if B is a generalized Salem set and $\dim A + \dim B > n$, then $\mathcal{L}^1 D(A, B) > 0$.*

Proof. This is an immediate consequence of Corollary 4.9.

5.4. COROLLARY. *If A and B are subsets of $\mathbb{R}^n$ such that $\dim A + \dim B > n$ and $\dim B \leq \frac{1}{2}(n-1)$, then $\mathcal{L}^1 D(A, B) > 0$.*

The following result contains the remaining part of Falconer's Theorem 2.2 in [F2].

5.5. THEOREM. *If A and B are subsets of $\mathbb{R}^n$ such that $\dim A + \dim B \leq n$ and $\dim B \leq \frac{1}{2}(n-1)$, then*

$$\dim D(A, B) \geq \dim A + \dim B - n + 1.$$

Proof. Apply Theorem 4.16.

I don't know whether the condition $\dim B \leq \frac{1}{2}(n-1)$ in Theorems 5.4 and 5.5 is redundant. It is if B is a generalized Salem set.

5.6. THEOREM. *Let A and B be subsets of $\mathbb{R}^n$. If $\dim A + \dim_{SF} B \leq n$, then*

$$\dim D(A, B) \geq \dim A + \dim_{SF} B - n + 1.$$

In particular, if B is a generalized Salem set and $\dim A + \dim B \leq n$, then

$$\dim D(A, B) \geq \dim A + \dim B - n + 1.$$

This follows from Theorem 4.17.

5.7. Remark. Theorems 5.3 and 5.6 hold also in $\mathbb{R}^1$. Then $\dim_{SF} B = \dim_F B$, and generalized Salem sets are Salem sets. In fact, in $\mathbb{R}^1$ it suffices to prove the conclusions for the difference set

$$\{x - y : x \in A, y \in B\}$$

in place of the distance set $D(A, B)$. The corresponding difference measure of two measures μ and ν is $\mu * \tilde{\nu}$ where $\tilde{\nu}(E) = \nu\{-x: x \in E\}$. Then

$$|(\mu * \tilde{\nu})^\wedge(x)|^2 = |\hat{\mu}(x)|^2 |\hat{\nu}(-x)|^2.$$

Consequently, if $I_\alpha(\mu) < \infty$ and $|\hat{\nu}(x)|^2 \leq C|x|^{-\beta}$, $\mu * \tilde{\nu} \in L^2$ in the case $\alpha + \beta \geq 1$, and $I_{\alpha+\beta}(\mu * \tilde{\nu}) < \infty$ in the case $\alpha + \beta < 1$.

§6. Intersections. To derive results for intersections, we are led to estimate integrals like

$$\int_0^\infty s^{-\gamma} \Delta(f)(s) \Delta(g)(s) ds, \quad 0 \leq \gamma < 1.$$

For $\gamma = 0$ this was done in Section 4. The kernels

$$k_\gamma(s) = |s|^{\gamma-1} \quad \text{for} \quad s \in \mathbb{R}, \quad 0 < \gamma < 1$$

satisfy the well-known identities

$$k_\gamma * k_\delta = c(\gamma, \delta) k_{\gamma+\delta} \quad \text{for} \quad \gamma, \delta \in (0, 1), \quad \gamma + \delta < 1. \quad (6.1)$$

6.2. LEMMA. Let $\varphi \in \mathcal{D}(\mathbb{R})$ be such that $\int \varphi \leq 1$ and

$$\varphi * k_\delta \leq c(\delta) k_\delta \quad \text{for} \quad 0 < \delta < 1.$$

If $\alpha, \beta \in (\frac{1}{2}(n-1), \frac{1}{2}(n+1))$ and $0 < \gamma = \alpha + \beta - n < 1$, then for $f, g \in \mathcal{D}(\mathbb{R}^n)$

$$\int_0^\infty s^{-\gamma} \Delta(f)(s) (\varphi * \Delta(g))(s) ds \leq c I_\alpha(f) (I_\beta(g) + \sigma_\beta(g)).$$

Proof. Using the decomposition of Lemma 4.14 for $\Delta(f)$ and $\Delta(g)$, we write

$$D = \Delta(f) = F(f) + K(f) = F + K,$$

$$E = \Delta(g) = F(g) + K(g) = G + L.$$

Then

$$D(s)E(t) = F(s)G(t) + K(s)E(t) + D(s)L(t) - K(s)L(t),$$

and the left hand side of the asserted inequality is dominated by $|I_1| + |I_2| + |I_3| + |I_4|$, where

$$I_1 = \int_0^\infty s^{-\gamma} F(s) (\varphi * G)(s) ds,$$

$$I_2 = \int_0^\infty s^{-\gamma} K(s) (\varphi * E)(s) ds,$$

$$I_3 = \int_0^{\infty} s^{-\gamma} D(s) (\varphi * L)(s) ds,$$

$$I_4 = \int_0^{\infty} s^{-\gamma} K(s) (\varphi * L)(s) ds.$$

Writing $S = \Sigma(f)$ and $T = \Sigma(g)$, we see from the proof of Lemma 4.14 that

$$F(s) = c_1 \int_0^{\infty} \cos(sr) S(r) dr + c_2 \int_0^{\infty} \sin(sr) S(r) dr,$$

$$G(s) = c_1 \int_0^{\infty} \cos(sr) T(r) dr + c_2 \int_0^{\infty} \sin(sr) T(r) dr.$$

Inserting these into the formula

$$I_1 = \int_0^{\infty} s^{-\gamma} F(s) \int G(s-t) \varphi(t) dt ds$$

and using the addition formulae for trigonometric functions, we find that I_1 is a sum of constant multiples of terms like

$$\int_0^{\infty} \int_0^{\infty} S(p) T(r) \int \varphi(t) \sin tr dt \int_0^{\infty} s^{-\gamma} \cos sp \sin sr ds dp dr$$

with other combinations of sines and cosines. The absolute value of the t -integral is bounded by 1 and of the s -integral by $c_3 |p-r|^{\gamma-1}$, as the use of product formulae for trigonometric functions shows. Hence

$$|I_1| \leq c_4 \int_0^{\infty} \int_0^{\infty} S(p) T(r) |p-r|^{\gamma-1} dp dr.$$

Using (4.8) and (6.1) we obtain

$$\begin{aligned} |I_1| &\leq c_4 \sigma_{\beta}(g) \int_0^{\infty} S(p) (k_{\frac{1}{2}(n+1)-\beta} * k_{\gamma})(p) dp \\ &= c_5 \sigma_{\beta}(g) \int_0^{\infty} S(p) k_{\alpha-\frac{1}{2}(n-1)}(p) dp \\ &= c_5 \sigma_{\beta}(g) \int_0^{\infty} p^{\alpha-1} \sigma(f)(p) dp = c_6 \sigma_{\beta}(g) I_{\alpha}(f). \end{aligned}$$

To estimate I_2 we use Lemma 4.15, the assumption on φ , (4.1) and (4.2) to get

$$\begin{aligned}
 |I_2| &= \left| \int_0^\infty s^{-\gamma} K(s) (\varphi * E)(s) ds \right| \\
 &\leq c_7 I_\alpha(f) \int_0^\infty s^{-\gamma+\alpha-\frac{1}{2}(n+1)} (\varphi * E)(s) ds \\
 &\leq c_7 I_\alpha(f) \int (\varphi * k_{\alpha-\gamma-\frac{1}{2}(n-1)})(s) E(s) ds \\
 &\leq c_8 I_\alpha(f) \int k_{\frac{1}{2}(n+1)-\beta}(s) E(s) ds \\
 &= c_8 I_\alpha(f) \int_0^\infty s^{-\beta} \delta(g)(s) ds = c_9 I_\alpha(f) I_\beta(g).
 \end{aligned}$$

Similarly,

$$|I_3| \leq c_{10} I_\alpha(f) I_\beta(g).$$

Finally, estimating as above, using Lemma 4.14 and Lemma 4.11,

$$\begin{aligned}
 |I_4| &\leq c_7 I_\alpha(f) \int k_{\frac{1}{2}(n+1)-\beta}(s) |L(s)| ds \\
 &\leq c_{11} I_\alpha(f) \int s^{\frac{1}{2}n-\beta} \int_0^\infty r^{\frac{1}{2}} |R(sr)| T(r) dr ds \\
 &= c_{11} I_\alpha(f) \int_0^\infty r^{\frac{1}{2}} T(r) \int_0^\infty s^{\frac{1}{2}n-\beta} |R(sr)| ds dr \\
 &\leq c_{12} I_\alpha(f) \int_0^\infty r^{\frac{1}{2}} T(r) \left(\int_0^{1/r} r^{-\frac{1}{2}} s^{\frac{1}{2}n-\beta-\frac{1}{2}} ds + \int_{1/r}^\infty r^{-\frac{3}{2}} s^{\frac{1}{2}n-\beta-\frac{3}{2}} ds \right) dr \\
 &= c_{13} I_\alpha(f) \int_0^\infty r^{\beta-\frac{1}{2}(n+1)} T(r) dr = c_{14} I_\alpha(f) I_\beta(g).
 \end{aligned}$$

6.3. LEMMA. For $\alpha + \beta \geq n$, $0 < \delta \leq 1$ and $\mu, \nu \in \mathcal{M}$,

$$\begin{aligned}
 &\int \int \int_{|x-u| \geq 2\delta} |x-u|^{1-\alpha-\beta} \nu\{v: ||x-u|-|y-v|| \leq \delta\} d\nu dy d\mu d\mu x \\
 &\leq c \delta I_\alpha(\mu) (I_\beta(\nu) + \sigma_\beta(\nu)).
 \end{aligned}$$

Proof. We may assume $\mu = f$, $\nu = g \in \mathcal{D}$. Letting $\kappa = \delta(f)$ and $\lambda = \delta(g)$, the left-hand side divided by δ equals

$$I = \delta^{-1} \int_{2\delta}^{\infty} s^{1-\alpha-\beta} \int_{s-\delta}^{s+\delta} \lambda(t) dt \kappa(s) ds.$$

If $\alpha \geq \frac{1}{2}(n+1)$, we obtain by Theorem 4.4

$$\begin{aligned} I &\leq c_1 I_\alpha(f) \delta^{-1} \int_{2\delta}^{\infty} s^{-\beta} \int_{s-\delta}^{s+\delta} \lambda(t) dt ds \\ &= c_1 I_\alpha(f) \delta^{-1} \int_{\delta}^{\infty} \lambda(t) \int_{\max\{2\delta, t-\delta\}}^{t+\delta} s^{-\beta} ds dt \\ &\leq c_2 I_\alpha(f) \int_0^{\infty} t^{-\beta} \lambda(t) dt = c_3 I_\alpha(f) I_\beta(g), \end{aligned}$$

and similarly if $\beta \geq \frac{1}{2}(n+1)$. If α and β are both less than $\frac{1}{2}(n+1)$, they are also greater than $\frac{1}{2}(n-1)$, as $\alpha + \beta \geq n$, and we can apply Lemma 6.2. Choose $\varphi \in \mathcal{D}$ as in Lemma 6.2 such that $\varphi(s) \geq 1/(4\delta)$ for $|s| \leq \delta$. Then I is dominated by c_4 times the integral of Lemma 6.2, and the desired inequality follows.

With the aid of Lemma 6.3 the arguments of [M1] and [M2] may be used to obtain intersection results as in those papers. Letting θ_n be the Haar measure on the orthogonal group $O(n)$, they yield for example:

6.4. THEOREM. *Let A and B be subsets of $\mathbb{R}^n$ and let $\varepsilon > 0$. If $\gamma = \dim A + \dim_{SF} B - n$, then for θ_n almost all rotations $g \in O(n)$,*

$$\mathcal{L}^n\{z \in \mathbb{R}^n : \dim A \cap (g(B) + z) \geq \gamma - \varepsilon\} > 0.$$

In particular, if B is a generalized Salem set, this holds with $\gamma = \dim A + \dim B - n$.

6.5. THEOREM. *Let $B \subset \mathbb{R}^n$ and $0 < \beta < \dim_{SF} B$, and suppose $\alpha + \beta - n > 0$. Then there is $c(B)$, $0 < c(B) < \infty$, such that for any $A \subset \mathbb{R}^n$*

$$c(B) C_\alpha(A) \leq \int_* \int_* C_{\alpha+\beta-n}(A \cap (g(B) + z)) d\theta_n g d\mathcal{L}^n z.$$

where $\int_$ denotes lower integral. In particular, if B is a generalized Salem, this holds for $0 < \beta < \dim B$.*

In Theorem 6.5 we can take $c(B) = c(I_\beta(\nu) + \sigma_\beta(\nu))$ for a probability measure ν with $\text{spt } \nu \subset B$ and $I_\beta(\nu) + \sigma_\beta(\nu) < \infty$.

6.6. Remarks. The analogues of Theorems 6.4 and 6.5 hold also in $\mathbb{R}^1$. Recalling 5.7 one easily derives the required estimate for the integral I appear-

ing in the proof of Lemma 6.3. Of course, in $\mathbb{R}^1$ we can restrict to translations and for example the statement corresponding to Theorem 6.4 is as follows.

If $A, B \subset \mathbb{R}^1$, $\varepsilon > 0$ and $\gamma = \dim A + \dim_F B - 1$, then

$$\mathcal{L}^1\{z \in \mathbb{R}^1: \dim A \cap (B+z) \geq \gamma - \varepsilon\} > 0.$$

I don't know whether Theorems 6.4 and 6.5 hold with $\dim_{SF} B$ replaced by $\dim B$ when $n \geq 2$. For $n = 1$, they don't hold, see [K2] or [M1]. There are some, but perhaps not so promising, hopes to sharpen the present Fourier-analytic method to solve this problem as well as the earlier-mentioned problems on distance sets. These hopes are based on the observation that in all estimates above the uniform bound $\sigma_\beta(\nu)$ could be replaced by suitable L^p -norms of the function $r^\beta \sigma(\nu)(r)$. The question is whether such L^p -norms could be dominated by appropriate energy-integrals of ν . The discouragement comes from the measures μ_α in Example 3.12. Evidently $r^{\frac{1}{2}} \sigma(\mu_\alpha)(r)$, being Lipschitz (by the proof of Theorem 3.10) and not vanishing at infinity, cannot belong to any L^p for $p < \infty$. But could some weighted inequalities work?

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