Al-Nayrīzī's Own Proof of Euclid's Parallel Postulate

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In the beginning of Book 1 of the *Elements*, Euclid (ca. 300 BC) lists a number of basic assumptions. The most controversial of these is his fifth postulate, which reads as follows: "that if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles" [7, vol. 1 p. 155]. Compare Figure 1: the straight line n falls on the two straight lines ℓ and m, $\alpha + \beta$ is less than 180°, and the postulate says that ℓ and m intersect at a point X at the same side as the angles α and β.

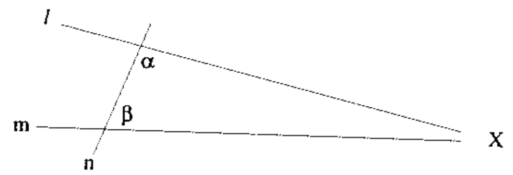


Fig. 1

The fifth postulate is called the parallel postulate since it underlies most of Euclid's theory of parallels. Euclid does not use the parallel postulate before proposition 29 of the first Book of his *Elements*, although he could have used it to simplify the proofs of earlier propositions. This may be an indication that Euclid did not really like to assume the parallel postulate. To get rid of this postulate, he would have had to prove it by means of his other assumptions and postulates and the first 28 propositions of the *Elements*. This was an ideal which many later geometers from ancient times until the nineteenth century sought to realize. Many attempts to prove the postulate were made, but some of these proofs were false and in other proofs a new postulate was assumed (which turned out to be equivalent to the parallel postulate) [10], [12]. In the early nineteenth century Nikolai Ivanovich Lobachevskii (1792–1856) and János Bólyai (1802–1860) invented a so-called Non-Euclidean geometry in which Euclid's other postulates and axioms hold but not the parallel postulate, and around 1870 it was proved that if Euclid's geometry does not contain contradictions, this non-Euclidean geometry cannot contain contradictions either. Hence it turned out that the parallel postulate cannot be proved from Euclid's other postulates and assumptions. Thus in plane Euclidean geometry one always needs either the parallel postulate or some equivalent to it.

The parallel postulate was intensively studied in the medieval Islamic tradition from the ninth century on, and more than ten "proofs" have come down to us [12, pp. 42–90]. This paper is about an early Islamic proof of the parallel postulate by Abu'l-ʿAbbās al-Faḍl ibn Ḥātim al-Nayrīzī,[1] a mathematician and astronomer who originated from the city of Nayrīz in Iran and who worked in Baghdād. Since he wrote a treatise on meteorology for Caliph al-Muʿtaḍid, who reigned from 892 to 902, he must have lived around 900. Following Suter [17, p. 45], many authors have stated that al-Nayrīzī died in or around the year 310 of the Hijra (922/3 CE), but I have not found evidence for this date in medieval sources.[2] As we will see below, al-Nayrīzī probably influenced Thābit ibn Qurra (836–901),[3] so we can assume that the main activity of al-Nayrīzī was in the second half of the ninth century CE.

Al-Nayrīzī wrote two texts relating to the parallel postulate. The first is a short treatise, entitled *On the Proof of the Famous Postulate of Euclid*. This treatise has not been published previously, except in two Russian translations, one by Grigorian in 1971 [6],[4] and one by Abdurakhmanov and Rosenfeld in 1982 [1]. The treatise has been briefly summarized by Sabra [14, p. 6, col. 2] and by Rosenfeld [12, pp. 56–57], a more extensive Russian summary with figures is in [13, pp. 42–45]. An edited Arabic text of the treatise with English translation can be found in this paper. The treatise is not found in the collections of texts on the parallel postulate which were published by Jaouiche [8], [9]

The second text is al-Nayrīzī's commentary to Euclid's *Elements* which is extant in a medieval Latin translation by Gerard of Cremona [5], [2]; a slightly different version has been preserved in part in an Arabic manuscript together with an Arabic translation of the *Elements* attributed to al-Ḥajjāj (ca. 830 CE) [3].[5] In this commentary to the *Elements*, al-Nayrīzī renders a proof of the parallel postulate by a late Greek geometer who is known under the arabicized name Aghānis and who was a contemporary of Simplicius (fifth century AD). The proof by Aghānis is not otherwise extant, but it is well-known in the modern historical literature since its publication by Curtze in 1899.

Al-Nayrīzī's proof in the short treatise is related to the proof by Aghānis but the two proofs are not the same. I now discuss the most important differences in modern notation and terminology.[6] The proofs by Aghānis and al-Nayrīzī are based on the concept of equidistant lines. Their definitions of this concept are unclear but amount to the following: Two straight lines ℓ and m are called equidistant if (1) the distance from any point P on ℓ to line m is the same, (2) the distance from any point Q on

1 On al-Nayrīzī see [16, vol. 5, pp. 283–285; vol. 6, pp. 191–192].
2 In his German translation of a chapter of the *Fihrist* [18, p. 67], Suter says: "Weitere Angaben über sein Leben habe ich nicht gefunden, da er aber für al-Muʿtaḍid ein Werk verfasst hat, so muss er ums Jahr 900 zur Zeit Ṯābits gelebt haben." I therefore suspect that the date 310/922–3 is an invention of Suter.
3 On Thābit see [16, vol. 5, pp. 264–272].
4 Grigorian reads the name of al-Nayrīzī as al-Tabrīzī.
5 The differences between the Arabic and the Latin versions are discussed in [4].
6 For the text of the proof of Aghānis see [3, vol. 1 pp. 118–131], [2, pp. 55–61], for a French translation see [9, pp. 129–136], and for a summary see e. g. [12, pp. 43–44].

m to line ℓ is the same, and (3) the distances in (1) and (2) are equal. Al-Nayrīzī provides a philosophical motivation for the existence of equidistant lines.

Aghānis and al-Nayrīzī begin with proofs of a few simple propositions on equidistant lines.[7] In these propositions they both assume that for any straight line ℓ and any point P not on ℓ there exists a straight line m through P equidistant to ℓ. They prove among other things that two straight lines are equidistant if and only if they have a common perpendicular. Aghānis uses the term "parallel lines"[8] but al-Nayrīzī uses the expression "lines which preserve the distance between them," which I have translated as "equidistant lines."

The main difference between the proofs by Aghānis and al-Nayrīzī appears in their final propositions (Figure 2, notation adapted to al-Nayrīzī, cf. Figure 5 below).

Assume that two straight lines AB and GD are intersected by a third line AG in such a way that $\angle BAG + \angle AGD < 180°$. Without loss of generality we can assume that $\angle AGD$ is acute. By *Elements* I:11–12 one can draw perpendiculars AE onto GD and AZ onto AE, now AZ and GD are equidistant according to the propositions which Aghānis and al-Nayrīzī proved before.[9] Choose an arbitrary point H on AB.

Aghānis drops the perpendicular HF onto AE. He then bisects AE, then bisects the half of AE, and so on, until he arrives at a point E' between A and F. (We have $AE = 2^k AE'$ for some number $k \geq 1$). He then draws perpendicular $E'P'$ to AF to intersect AH at point P' and he defines P on AB such that $AP = 2^k AP'$. He then proves that P is also on GD, so the point of intersection has been found.

Al-Nayrīzī draws a perpendicular THK to AZ and GD. He then defines point L on TK such that $TL = n \cdot TH$ is the smallest multiple of TH which is equal to or greater than TK. (Figure 2 is drawn for the case $L \neq K$). Draw SL perpendicular to HL, then SL is equidistant to GD and to AB. Al-Nayrīzī defines C on AB such that $AC = n \cdot AH$ and he proves that C is also on the straight line SL, so AB and SL intersect. He concludes that GD intersects AB between points C and A.

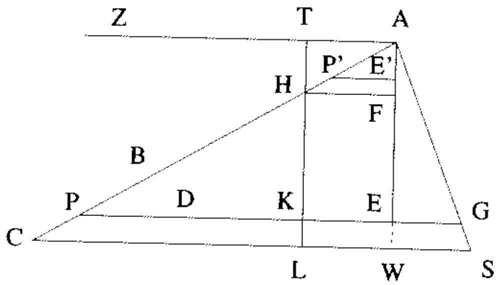


Fig. 2

7 See the footnotes to my translation below for the exact correspondence. The fourth proposition of Aghānis, which corresponds to *Elements* I:28, does not occur in al-Nayrīzī's treatise.

8 Note that Gerard of Cremona translated the Arabic word for parallel (*muwāzī*, *mutawāzī*) as "equidistans", which therefore appears in the Latin translation of the proof by Aghānis.

9 Clearly the only interesting case is where AE is between AG and AB. If this is not the case, AB and GD have a point of intersection between E and G.

The main difference between the proofs of Aghānis and al-Nayrīzī is related to their constructions of E' and L respectively. Aghānis obtains point E' is by successive bisection of AE, and the existence of E' follows from *Elements* X:1 [7, vol. 3, pp. 14–15]. Euclid's proof of this proposition is based on the assumption that AF and AE are magnitudes of the same kind according to *Elements* V, definition 4. The existence of the multiple of TH greater than TK also follows from the assumption that $TH = AF$ and $TK = AE$ are magnitudes of the same kind. Thus one can say that al-Nayrīzī eliminated the continuous bisection process of *Elements* X:1. This may make the construction a bit more elegant, but I cannot see a fundamental advantage in al-Nayrīzī's approach. Hence I think it is fair to say that al-Nayrīzī's proof is a variation on a theme of Aghānis, but nevertheless a variation which deserves to be recorded, because it is certainly more than a textual modification.

I now turn to the possible influence of al-Nayrīzī's proof on other Islamic proofs of the parallel postulate. A strong connection exists between the proof of al-Nayrīzī and a proof in one of the two treatises by Thābit ibn Qurra on the parallel postulate, namely the *Treatise on the Fact That Two Lines Drawn at Angles Less Than Two Right Angles Meet.*[10] In this treatise Thābit also begins with a philosophical demonstration of equidistant lines, but he introduces motion in geometry. His mathematical discussion is more profound than that of al-Nayrīzī and Aghānis. Thābit begins with a straight line ℓ and he then obtains (by a philosophical argument) a straight line m which is equidistant with respect to ℓ in the sense that the distance of any point P of M to ℓ is the same. He then ***proves*** that the distance of any point Q on ℓ to m is the same as the distance of any point P on m to ℓ. Al-Nayrīzī and Aghānis simply assume this property in their definition of equidistant lines. The final proposition of Thābit is so close to the final proposition of al-Nayrīzī that they cannot be independent. Thābit does not draw perpendicular THK in the notation of Figure 2 but he finds a multiple $n \cdot AF = AW \geq AE$ (dotted line in Figure 2) and then defines SL as the line through W. The proof of al-Nayrīzī is somewhat defective in its wording (see the footnotes below) but Thābit's proof is clear. I therefore think that Thābit's proof is a revision of the proof of al-Nayrīzī, not the other way around. Directly or through Thābit, the proof by al-Nayrīzī influenced other Islamic geometers as well. Al-Nayrīzī's proof was known to ʿUmar al-Khayyāmī, who stated that al-Nayrīzī based himself on an assumption (on equidistant lines) which was not simpler than the thing to be proved (the parallel postulate) [9, pp. 185–186], [12, p. 57].

The following edition and translation of al-Nayrīzī's short treatise on the proof of the parallel postulate is based on two Arabic manuscripts: Tehran, Sepahsālār 597, 11b–12a (facsimile in [11, pp. 86–87]), and Paris, Bibliothèque Nationale, Fonds Arabe 2467, f. 89a–90a [20, p. 593]. The Tehran manuscript is dated 784 H./1380 CE and provided with figures. The Paris manuscript is undated, poorly legible and the figures have not been drawn. Jaouiche only had access to the Paris manuscript and decided not to add the treatise by al-Nayrīzī to his collections [9, p. 16], [8]. Numerous scribal

10 See for an English translation of this treatise [15, pp. 19–27], for a French translation [9, pp. 151–160] and for a discussion [12, pp. 52–56]. This treatise is called Thābit's first treatise by Sabra and Thābit's second treatise by Rosenfeld and Jaouiche. Manuscripts of the two treatises are listed in [16, vol. 5, p. 268 no. 6].

errors in the Paris manuscript have been noticed in the apparatus at the end of the edition. Sezgin mentions two other manuscripts of al-Nayrīzī's treatise [16, vol. 5, p. 284 no. 2]. According to Rosenfeld [12, p. 56] and Jaouiche [9, p. 16], the Berlin manuscript was lost during the second world war. The Hyderabad manuscript was not available to me.

In the edition I have adapted the orthography to modern usage but I have not corrected grammatical errors. My own explanatory additions to the translation are in parentheses. The Arabic letters in the geometrical figures have been transcribed according to the following system (I render the letters in the order of increasing numerical values):

ا = A, ب = B, جـ = G, د = D, ه = E, و = W, ز = Z, ح = H, ط = T, ك = K, ل = L, م = M, ن = N, س = S, ع = O, ف = F, ص = C, ق = Q, ر = R, ش = X, ت = t, ض = d.

Arabic Text

بسم الله الرحمان الرحيم (١)

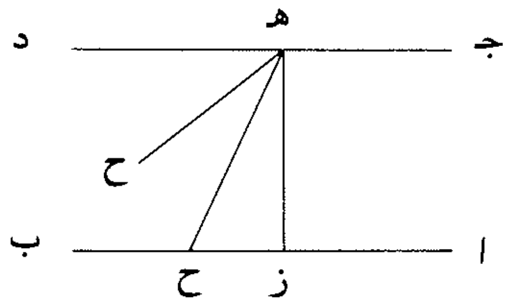
رسالة لفضل بن حاتم النيريزى (٢) فى بيان المصادرة المشهورة لاقليدس

قال علمنا بالامور المختلفة بعد علمنا بالمتساوية لان المساواة متقدّمة فى الطبع على الاختلاف لانها من الامور الجارية مجرى (٣) الطبع والاختلاف خارج عنه.

فالخطوط التى تحفظ الابعاد بينهما لكونه اعتدالاً ما يتقدّم على التى لا تحفظ لان الامور الطبيعية اولى بان تكون موجودة من الخارج (٤) عنها لان الخارجة عنها انّما تسير بها والّا فكيف يكون وجود غير الطبيعية ان لم تكن الطبيعية متقدّمة عليها. فيجب باضطرار ان تكون خطوط تحفظ الابعاد بينها

< آ > فمن القضايا الاوّل المقرّ بها ان كل مستقيمين فى سطح يحفظان (٥) البعد بينهما ولا يلتقيان وان اخرجا بلا نهاية كـاب جد فى هذه الاشكال الاربعة فان البعد بينهما عمود عليهما ولانه يمكن ان يخرج من كل نقطة من احدهما الى الآخر خطوط لا نهاية لكثرتها يجب ان يكون البعد بينهما اقصرها وهو هز (٦) مثلاً فهو البعد بينهما وكذا الخط الاقصر من الخطوط الخارجة من (٧) اى نقطة لان البعد محفوظ فيكون الابعاد متساوية فاقول ان هذه الابعاد

اعمدة عليها (٨) والّا فليكن هح عموداً على اب فـح قائمة و ز اعظم منها لان هح اطول من هز فـ ز ح اعظم من قائمتين خلف او لان كل زاويتين من المثلّث اصغر منهما و ح قائمة فـز اصغر منها فـهح اقصر من هز وفرضنا اطول خلف (٩) فـهز عمود عليها وهو المراد.

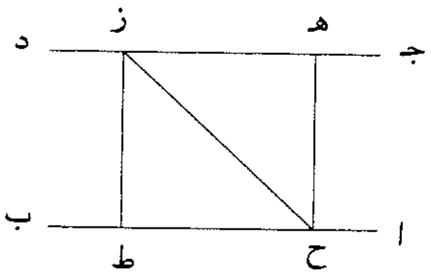


< ب > الاعمدة الواقعة بينهما كـهز فهى البعد بينهما (١٠) اى انها اقصر الخطوط والّا فليكن البعد هح فهو اقصر الخطوط وعمود عليهما فـزح من مثلث هزح (١١) قائمتان خلف (٩) فـهز هو البعد بينهما وهو المراد.

< جـ > كل عمود على احدها كـهز (١٢) على اب فهو عمود على الآخر والّا فليكن هح عموداً على جد فهو البعد بينهما لشكل ب فهو كـهز لانه (١٣) ايضاً البعد بينهما لشكل ب فـح قائمة كـز خلف (٩) فـهز (١٤) عمود عليها وهو المطلوب (١٥).

< د > اذا اقام عمود على خطين مستقيمين فهما يحفظان البعد بينهما والّا فليكن هح يحفظ البعد بينه وبين اب و هز عمود على اب بل على جد فهو عمود على (١٦) هح لشكل جـ فـزهح الجزء قائمة ومساوية لـزهد الكل (١٧) هذا محال (١٨) فلا يحفظ البعد مع اب غير (١٩) جد وهو المطلوب (١٥).

< ه > اذا اخرج من احد مستقيمين يحفظان البعد بينهما كجد عمودان (٢٠) الى الآخر كهح زط فما بين موقعيهما من الآخر كحط مساو لما بينهما من الاول كهز لان العمودين يحفظان البعد بينهما لشكل د وهو المطلوب (١٥) ·



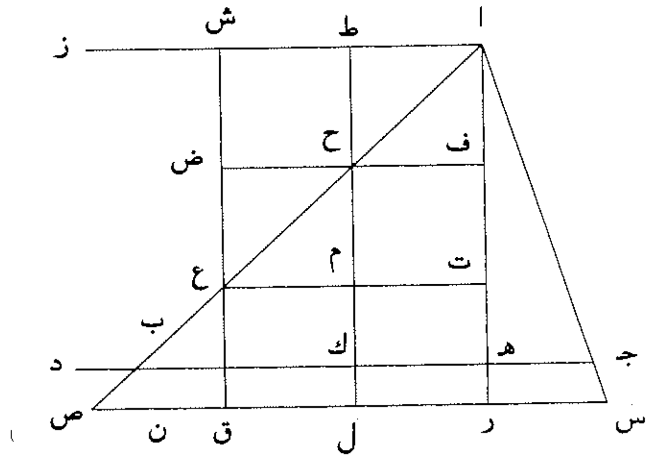
< و > اذا كان اب جد يحفظان البعد بينهما ووقع عليهما وح فان الزاويتين اللتين فى جهة واحدة معادلتان لقائمتين لان عمودى وط (٢١) هح متساويان لانّهما البعد بينهما فحط كهو و حو (٢٢) مشترك فزاوية هوح (٢٣) كوحط (٢٤) لكن هوح مع حود كقائمتين فكذا دوح وبحو كقائمتين وهو المراد·

< ز > اذا وقع مستقيم كاج على مستقيمين كاب جد وصيّر مجموع الداخلتين فى جهة كجاب اجد اصغر من قائمتين فانهما يلتقيان فى تلك الجهة فلان الزاويتين اصغر من قائمتين فلا يحفظ اب جد البعد بينهما فنخرج عمود اه على جد واز (٢٥) على اه فهو يحفظ البعد مع جد ونأخذ من اب مقداراً ما كاح ونخرج عمود حط على از والى لك من جد فحط اما ان يقدر طك او (٢٦) يفضل عليه بفضلة كلك فحط يقدر طل وليكن بثلاث مرات على ح وم ونخرج عمود لسن (٢٧) على لط واه الى ر واج الى (٢٨) س فسن از يحفظان البعد بينهما وكذا سن جد فلو بينا ان اب يلقى سن لظهر انه يلقى جد قبله·

فنقسم اب مع الذى على استقامة كبص بامثال اح كحع عص ونصل سقص فهو خط مستقيم يلقى اب على ص لانا نجيز على ع البعد الذى بين (٢٩) از سن وليكن (٣٠) فشق ار طل شق هى (٣١) الابعاد بين از سن فيحفظ الابعاد التى بينهما ولا يلتقى ونجيز (٣٢)

على ح البعد الذى بين ار شق وليكن لحض وكذا على ع وليكن عمت فلتساوى اح حع (٣٣) وقائمتين ض ف ومتقاطعتين احف عحض يكون عض كاف اى شض لما (٣٤) تبين فى شكل ه لكن شض كطح وضع (٣٥) كحم فشع كطم فيبقى عق كمل بل مح بل عض (٣٦) وزاوية صعق كحعض فقص كضح وعقص قائمة كحضع وكانت عقل قائمة فسقص (٣٧) خط مستقيم لقى اب على ص فاب لقى جد قبل ان يلقى سص وهو المطلوب وذلك ما اردناه·

تمت الرسالة (٣٨) ·



Apparatus

Paris, Bibliothèque Nationale, Fonds Arabe 2467, 89a–90a : (ب)

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١ بسم الله الرحمان الرحيم : ناقص فى مخطوط (ب)· ٢ النيريزى: التبريزى (ب)· ٣ مجرى : محرى (ب)· ٤ الخارج : الخارجة (ب)· ٥ يحفظان: الحمطان (ب)· ٦ هز : قار (ب)· ٧ من: فى (ب)· ٨ عليها: كذا فى مخطوطين (ت) (ب)· ٩ خلف: هذا خلف (ب)· ١٠ كهز فهى

البعد بينهما: ناقص فى مخطوط (ب). ١١ هزح : زح (ب). ١٢ كهز: لهز(ب). ١٣ فهو كهز
لانه : لهز ولانه (ب). ١٤ فهز: فهو(ب). ١٥ المطلوب: المطه (ب). ١٦ جد فهو عمود
على: ناقص فى مخطوط (ب). ١٧: الكل: الشكل (ب). ١٨: محال: مح (ت). ١٩ مع اب غير:
مع اب عن (ب). ٢٠ عمودان: عمودا (ب). ٢١ وط : زط (ب). ٢٢ كهو وحو: كه وجح
و (ب). ٢٣ هوح : وح (ب). ٢٤ وحط : محط (ت) (ب). ٢٥ از : اد (ب). ٢٦ او : و
(ب). ٢٧ لمن : ل س ىد (ب). ٢٨ الى: ناقص فى مخطوط (ب). ٢٩ بين: عن (ب). ٣٠
وليكن : ولكن (ب). ٣١ هى: كمن (ب). ٣٢ ونجيز: و محتر (ب). ٣٣ حع : ع ح (ب).
٣٤ لما: كما (ب). ٣٥ و ضع : و ض (ب). ٣٦ بل عض : ل ع ض (ب). ٣٧ فسقص:
فسقمه (ب). ٣٨ تمت الرسالة : تمت الرسالة والحمد لله على كل حال فى يوم الجمعة ثامن
ذى الحجة لسنة احدى وثمانين وسبع مائة والحمد لله ... وصلى الله على سيدنا محمد النبى
وآله (ت).

Translation

In the name of God, the Merciful, the Compassionate.

Treatise by Faḍl ibn Ḥātim al-Nayrīzī[11] *on the Proof of the Famous Postulate of Euclid.*

He said: Our knowledge of unequal things comes after our knowledge of equal things, because equality is by (its) nature prior to inequality. For they (equal things) belong to the things which follow the course of nature, and inequality is outside nature. Thus lines which maintain the distance between them, since it[12] is evenness, are the ones who are prior to lines which do not maintain the distance, since it is more appropriate for natural things to be found than for things outside nature. For the things outside nature go (only) with them (i.e. the natural things), and if this were not so, how could the unnatural exist if the natural were not prior to it? Thus it is necessary that there are lines which maintain the distances between them.[13]

11 The Paris manuscript reads "al-Tabrīzī."
12 The reference is presumably to the "nature" (*ṭab*ʿ) of these equidistant lines.
13 Henceforth I will use the word equidistant in my translation. Al-Nayrīzī's assumption of the existence of equidistant straight lines is equivalent to Euclid's parallel postulate.

<1>[14] One of the first theorems which are established is the following: For every pair of equidistant lines in one plane which do not meet even if they are produced indefinitely,[15] such as AB, GD in these four propositions, the distance[16] between them is perpendicular to them.

(Figure 3) Since infinitely many lines can be drawn from any point on one of the two lines to the other line, it is necessary that the distance between them is the shortest of these, for example EZ. Then it is the distance between them and similarly the shortest line among the lines drawn from any (other) point, since the distance is preserved; so the distances are equal. I say that these distances are perpendicular to them.[17] If this is not the case, let EH be perpendicular to AB, then H is (a) right (angle), and Z is greater than it, since EH is longer than EZ, so Z and H are (together) greater than two right angles, contradiction.[18]

Or, since every two angles of the triangle are (together) less than two of them (two right angles), and H is a right angle, Z is less than it, so EH is shorter than EZ,[19] but we assumed it to be longer, contradiction. Thus EZ is perpendicular to both of them, and that is what was desired.

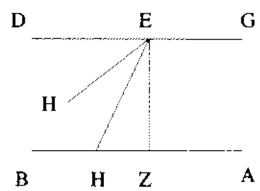

Fig. 3

<2> (Figure 3) Perpendiculars falling between them such as EZ are the distance between them, that is to say that they are the shortest lines.[20] If this is not the case, let the distance be EH, then it is the shortest line, and (hence) perpendicular to both of them. So (angles) Z, H in triangle EZH are two right angles, contradiction.[21] So EZ is the distance between them, and this is what was desired.

< 3 > (Figure 3) Every perpendicular to one of them, such as EZ (perpendicular) to AB, is perpendicular to the other. If this is not the case, let EH be perpendicular to GD, then it is the distance between them by proposition 2,[22] so it is equal to EZ

14 The proposition numbers, which are missing in the manuscript, are analogous to the numbers in the Russian translation in [1].
15 The fact that the lines do not meet is a consequence of their equidistance.
16 For al-Nayrīzī, who did not have the modern concept of a real number, a distance is always a line segment.
17 This is the first theorem of Aghānis [3, pp. 120–123] [2, pp. 56–57] [9, p. 131].
18 According to *Elements* I:17 "in any triangle two angles taken together in any manner are less than two right angles" [7, vol. 1, p. 281].
19 According to *Elements* I:19 "in any triangle the greater angle is subtended by the greater side" [7, vol. 1, p. 284].
20 This is part of the second theorem of Aghānis [3, pp. 122–123] [2, p. 57] [9, pp. 131–132].
21 The contradiction is with *Elements* I:17.
22 This argument rests on the following assumption: the distance to GD of any point H on line AB is equal to the distance to AB of any point E on line GD.

since this is also the distance between them by proposition 2. Therefore H is a right angle and equal to Z, contradiction. Thus EZ is perpendicular to both of them, and this is what was required.

<4> (Figure 3) If a line is perpendicular to two straight lines, the two (lines) are equidistant.[23] If this is not the case, let EH and AB be equidistant[24] and let EZ be perpendicular to AB and to GD rather (than GH). Then it is perpendicular to EH by proposition 3. Therefore the part ZEH is (a) right angle and equal to the whole ZED. This is impossible.[25] Thus only GD is equidistant to AB, and this is what was required.

<5> (Figure 4) If one draws from one of two straight equidistant lines, such as GD, two perpendiculars EH, ZT to the other (straight line), the (segment) HT between their meeting points with the other (straight line) is equal to the segment EZ of the first line between them.[26] For the two perpendiculars are equidistant by proposition 4,[27] and that is what was required.

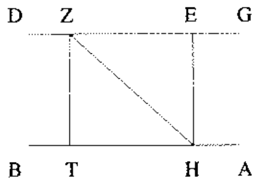

Fig. 4

<6> (Figure 4) If AB and GD are equidistant and WH[28] falls on them, the two angles on one side are (together) equal to two right angles.[29] Since the two perpendiculars WT, EH are equal because they are the distance between them (AB, GD), HT is equal to EW, and HW is common, therefore[30] angle EWH is equal to angle WHT,[31] but EWH together with HWD is equal to two right angles,[32] thus in the

23 This is part of the second theorem of Aghānis [3, pp. 122–123] [2, p. 57] [9, pp. 131–132].

24 In Figure 3 the reference is to the line through point E and the point H which is not on line AB. Here al-Nayrīzī assumes that for any given straight line AB and any point E not on AB there exists a straight line through E which is equidistant to AB. In non-Euclidean (hyperbolic) geometry two non-intersecting straight lines have a unique common perpendicular, and equidistant straight lines do not exist.

25 According to Common Notion 5, "The whole is greater than the part" [7, vol. 1, p. 155].

26 Aghānis also proves this fact in the course of his third proposition [3, pp. 122–125] [2, pp. 57–58] [9, pp. 132–133].

27 More precisely: Lines ZT and EH are equidistant since they have a common perpendicular EZ, and the two common perpendiculars EZ, TH are equal because they are the distance between ZT and EH by proposition 2.

28 The point W corresponds to the point Z in Figure 4. The letters Z and W are similar in Arabic, but the manuscripts have clearly Z in proposition 5 and W in proposition 6.

29 The same theorem is proved in the third proposition of Aghānis [3, pp. 122–125] [2, pp. 57–58] [9, pp. 132–133].

30 According to *Elements* I:8 "if two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have equal the angles which are contained by the equal straight lines" [7, vol. 1, p. 261].

31 Both manuscripts have EHT here.

32 *Elements* I:13 [7, vol. 1, p. 275].

same way DWH and BHW are (together) equal to two right angles, and that is what was desired.

<7> (Figure 5) If a straight line AG falls on two straight lines AB, GD and makes the sum of the two interior angles on one side, such as GAB, AGD less than two right angles, then the two (lines) meet on that side.[33]

For since the two angles are less than two right angles, AB and GD are not equidistant.[34] So we drop perpendicular AE onto GD and AZ onto AE,[35] then it is equidistant to GD.[36] We take from AB some magnitude AH and we drop the perpendicular HT onto AZ and (we extend it) to (point) K on GD. Then either HT measures TK[37] or it[38] exceeds it by an excess such as LK, then HT measures TL, let it be three times, at H and M.[39] We draw perpendicular LSN[40] to LT and we extend AE to R and AG to S.[41]

Then SN and AZ are equidistant, and similarly SN and GD. If we prove that AB meets SN it will be evident that it meets GD before it.

We divide AB and its rectilinear extension BC into parts (equal to) AH, namely HO, OC.[42] We join SQC;[43] it is a straight line which meets AB at C. For we let pass through O the distance between AZ, SN, let it be XQ. Then AR, TL, XQ are the distances between AZ, SN, so they (AR, TL, XQ) are equidistant and do not meet. We let pass through H the distance between AR, XQ, let it be FHd, and similarly through O, let it be OMt.[44] Then, since AH, HO are equal, (angles) d, F are right and AHF, OHd are (angles[45] between) two intersecting (lines), Od is equal to AF,[46]

33 This is Euclid's parallel postulate. Compare the final proposition in the proof by Aghānis in [3, pp. 126–131], [2, pp. 59–61], and [9, pp. 134–135] and my own discussion of the differences in the introduction to this paper.

34 This is a consequence of Proposition 6.

35 The constructions are explained in *Elements* I:12 [7, vol. 1, pp. 270–271].

36 Line AZ is equidistant to GD by Proposition 4.

37 The text means that TK is an integer multiple of HT.

38 The text is unclear, perhaps as a result of scribal error. Al-Nayrīzī must have meant: there is a multiple of HT which exceeds it (i.e. TK) ...

39 Al-Nayrīzī means that there exists an integer multiple of HT, namely $n \cdot HT = TL$, which is either equal to or greater than TK. This follows from the assumption that HT and TK are quantities of the same kind, as in *Elements* V definition 4. He assumes $n = 3$ in the rest of his exposition but the reasoning can be adapted to other values of n. The point M is defined by $HT = TM = ML$.

40 In the figure in the Tehran manuscript, point L is between points S and N, so it would be more correct to say SLN or NLS.

41 Al-Nayrīzī does not prove that AG and LN intersect. The existence of the point of intersection S is not essential in the rest of the argument.

42 Point C is defined as the point on AB or its rectilinear extension, such that $AH = HO = OC$. Here it is understood that the number of "parts" AH, HO, OC is n, that is 3 in this exposition.

43 The text is unclear. It would have been more correct to first define point Q on line SN, then join QC and prove that QC is on the same straight line as SQ.

44 The text is confusing. It is more correct to call the distance through O line Ot, since point M had already been defined above as the point on TK such that $TH = HM$. That line Ot passes through the point M has to be proved, see the next two footnotes.

45 These angles are equal because of *Elements* I:15.

46 Here al-Nayrīzī uses *Elements* I:26 [7, vol. 1, pp. 301–303]. He has now proved that the distance AF between AX and dF is equal to the distance Od between dF and Ot. It follows that Ot passes through the point M.

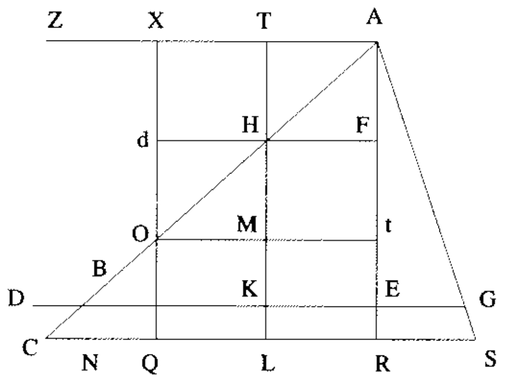

Fig. 5

that is Xd, because of what has been proved in proposition 5. But Xd is equal to TH and dO is equal to HM, so XO is equal to TM. Thus, by subtraction, OQ is equal to ML, or rather to MH, or rather to Od. Angle COQ is equal to angle HOd, so[47] QC is equal to dH and OQC is (a) right (angle), equal to HdO, but OQL was (a) right (angle). Therefore SQC is a straight line which meets AB at C. Therefore AB meets GD before it meets SC. That is what was desired, and that is what we wanted.

The treatise has ended.[48]

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47 Here al-Nayrīzī also uses $CO = OH$ and he applies *Elements* I:4 [7, vol. 1, pp. 247–248].

48 The Teheran manuscript adds: "Praise to God in all circumstances, on Friday 8 Dhū'l-Ḥijja of the year 781. Praise to God and may God bless our Lord Muḥammad the Prophet and his family." The date corresponds to March 16, 1380 CE.

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