Mathematical Commentaries in the Ancient World

A Global Perspective

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2 Philosophical Commentaries on Mathematical Texts

The Case of Proclus' Commentary on the First Book of Euclid's *Elements*

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2.1 Introduction

Our understanding of Greco-Roman philosophical commentaries has significantly changed in the past decades. Formerly they were regarded as secondary texts whose worth lay primarily in the light they shed on the texts being commented on and in the information about lost works found therein, whereas now they are regarded as philosophical works that merit study in their own right.[1] Undoubtedly, the current approach does better justice to the ancient commentary tradition but, much like the approach that it replaced, it is prone to be one-sided. It is liable to abstract the commentators' philosophical views from the commentaries' pedagogical aims, their structure, and their dependence on a base text. Proclus' (412–485 CE) commentary on the first book of Euclid's *Elements* exemplifies well the limitation of the one-sided approach that regards commentaries either as secondary texts or as philosophical works that stand in their own right. This commentary is considered to be unusual because, as John Vallance says, it forces its base text into second place, lacks the exegetical focus of the genre, and differs from didactic commentaries whose aim is advocating and explicating a fixed doctrine.[2] This understanding rests on the anachronistic assumption that commenting on texts for pedagogical purposes and developing philosophical views are mutually exclusive activities, thereby leaving Proclus' choice to use the commentary genre for propounding his own philosophical views puzzling. Jettisoning this

[1] The literature on the philosophy of the commentators is extensive. The major works that represent this turn of approach are: Sorabji, R. (ed.), *Aristotle Transformed: The Ancient Commentators and Their Influence* [*Aristotle Transformed*] (Ithaca 1990); Sorabji, R., *The Philosophy of the Commentators, 200–600 AD*, vols. 1–3 (London 2004); and more recently, Sorabji, R. (ed.), *Aristotle Re-interpreted: New Findings on Seven Hundred Years of the Ancient Commentators* (London and New York 2016).

[2] Vallance, J. T, 'Galen, Proclus and the Non-submissive Commentary', in G. W. Most (ed.), *Commentaries – Kommentare* (Göttingen 1999), 223–244, at 223.

assumption has consequences not only for understanding Proclus' commentary on the *Elements*. It is crucial for understanding a central feature of Greek philosophical commentaries, that is, their being exegetical and pedagogical works as well as philosophical works.

In view of this wider implication, in the first two sections of this chapter, I address two general questions: (1) How are commentaries' pedagogical aims related to their philosophical aims? and (2) What is the role of studying canonical texts in the Greek philosophical commentary tradition? This discussion leads to the conclusion that commentaries can serve both as exegetical and philosophical works for two reasons. First, their pedagogical aims are not distinct from their theoretical or philosophical aims, and second, commentators did not regard canonical texts as necessarily authoritative. Next, I examine Proclus' commentary on the *Elements* in light of this conclusion, arguing that, its unique features notwithstanding, it does not differ from other philosophical commentaries in its pedagogical approach and its treatment of the base text. Like other commentaries, its pedagogical aim is not distinct from its philosophical aim, and its base text and other canonical texts serve therein as a means of further philosophical inquiry.

2.2 Teaching and Philosophy

The vast majority of the late-antique philosophical commentaries were written in the fifth and sixth centuries CE by thinkers who taught in the Platonic schools in Athens and Alexandria, and some of them were based on lecture notes, as indicated by the expression 'from the voice of' (ἀπὸ φωνῆς / *apo phônês*), which appears in their titles.[3] Still, these commentaries reveal very little information about classroom practices in these schools.[4] The division found in some commentaries between a general discussion of a part of the text, called *protheôria*, and a detailed discussion of the

[3] This expression appears in the titles of almost all of Ammonius' commentaries; sometimes the name of the person, Asclepius or Philoponus, who took the notes is mentioned (e.g., the commentaries on the *Prior Analytics*, the *Posterior Analytics*, *On the Soul*, and *Generation and Corruption*; the titles of the last three indicate that the notes contain Philoponus' additions). On this subject, see Richard, M., 'Apo phōnēs', *Byzantion* (20) 1950, 191–222.

[4] A study of classroom practices in an earlier period is found in Snyder, H. G., *Teachers and Texts in the Ancient World: Philosophers, Jews and Christians* [*Teachers and Texts*] (London 2000). Unlike the sources that Snyder studied, late-antique commentaries do not facilitate an examination of questions such as whether the base text was read aloud by the teacher, the students, or not at all; or whether the problems addressed in commentaries arise from classroom discussions or from the commentators' engagement with earlier commentaries and other philosophical texts.

lemmata, called *lexis*, may reasonably reflect the order of discussion in classrooms, but generally it is difficult to learn from these commentaries about the teaching practices and the way they shaped the commentaries' content, mode of presentation, and style.[5] Moreover, although a sizable number of philosophical commentaries have come down to us, they do not facilitate secure generalizations about the institutional and pedagogical context in which they were written. In most cases we have only one or two commentaries on the same work; therefore, we cannot distinguish with certainty which features attest to common practices and a common tradition and which features reflect the commentators' personal style, interests, and aims.

Nevertheless, these commentaries contain valuable information about the teaching curriculum in the Platonic schools in Athens and Alexandria.[6] Studies of this subject show that students in these schools went through well-ordered stages of learning. After propaedeutic studies of the *trivium*, the *quadrivium*, and ethics, students studied Aristotle as the 'lesser mysteries' of philosophy and then proceeded to the study of Plato as the 'greater mysteries'. The study of Aristotle began with the *Categories* and his logical writings, and then advanced to his natural philosophy as found in the *Physics*, *On the Heavens*, and *On Generation and Corruption*, and finally culminated in his theology, that is, his *Metaphysics*. This order reflects the theoretical assumption that logic is a tool for philosophical inquiry and the Platonic ontological distinction between phenomenal and intelligible realities. Similarly, the order of studying Plato's dialogues was based on philosophical distinctions. The propaedeutic stage was devoted to the political, cathartic, and theoretical virtues. It began with the study of *Alcibiades* I, as an introduction, and proceeded to the *Gorgias* and the *Phaedo*, through which political and cathartic virtues were respectively taught. The theoretical virtues were taught in three ways: through the study of the *Cratylus* and the *Theaetetus*, which teach these virtues on the level of names and notions; through the study of the *Sophist* and the *Statesman*, which teach them on the level of physical realities; and the study of the *Phaedrus* and the *Symposium*, which teach theoretical virtues on the theological level. The study of the theoretical part of philosophy

[5] This division is found in Philoponus' commentaries on Aristotle's *Physics* and *On the Soul* and was systematically used in Olympiodorus', Elias', David's, and Stephanus' commentaries.

[6] Summaries of studies of the Neoplatonic curriculum and references are found in Sorabji, *Aristotle Transformed*, 5–10 and in Hoffmann, Ph., The Example of the Neoplatonic Commentators' [What Was Commentary in Late Antiquity?], in Gill, M. L., and Pellegrin, P. (eds.), *A Companion to Ancient Philosophy* (Oxford 2006), 597–622, at 605–606.

corresponds to the Platonic distinction between physical realities, taught through the *Timaeus*, and intelligible realities, taught through the *Parmenides*. The *Philebus* served as preparatory to the study of the latter because it acquaints students with the Good.[7]

The correspondence between this curriculum and Platonic ontology indicates that studies in the Neoplatonic schools were not aimed at teaching in the narrow sense of transmitting knowledge or training scholars and specialists in certain disciplines. Rather, these studies served as a means of edifying the students by leading them, or more accurately their souls, from the phenomenal realm to the intelligible realm. A major consequence of this observation is that from the Neoplatonic commentators' viewpoint, content and context, or in other words philosophical and didactic motivations, are not distinct. Therefore, discussions that from our perspective go beyond didactic and pedagogical aims are not atypical but accord with the commentators' view of the role and aim of teaching.

Neoplatonic commentaries have other common characteristics. The introductions to the extant Neoplatonic commentaries on the *Categories* show that, in teaching Aristotle, commentators addressed two sets of questions: a set of ten questions that served as a general introduction to Aristotle's philosophy and a set of six questions about a particular text, the *Categories* in this case.[8] Plato's dialogues were treated in a similar way. From the anonymous *Prolegomena to the Philosophy of Plato* dated to the second half of the sixth century CE, we learn that the study of Plato was guided by a set of questions that introduce readers to his philosophy in

7 In view of the correspondence between this curriculum and Plato's ontology, it is reasonable to infer that mathematical texts were studied at the intermediary level, either before Aristotle's *Metaphysics* or before the study of Plato. Geometry and at least a part of Euclid's *Elements* were included in the *quadrivium*.

8 The former was laid down by Proclus in his now lost *Sunanagnōsis*, i.e., reading a text with a teacher (Elias, *In Cat.* 107.24–26 Busse). A description of these questions is found in Hoffmann, 'What Was Commentary in Late Antiquity?', 608–613. For a comparison between the commentators' treatments of these questions, see Hadot, I., Hoffmann, Ph., Hadot, P., and Mahé, J.-P., *Simplicius. Commentaire sur les 'Catégories': Traduction sous la direction de I. Hadot* (Leiden 1990), 168–182. The common questions addressed in the introductions to philosophical and mathematical texts were extensively studied in Mansfeld, J., *Prolegomena: Questions to Be Settled before the Study of an Author, or a Text* (Leiden, New York, Köln 1994) and Mansfeld, J., *Prolegomena Mathematica: From Apollonius of Perga to Late Neoplatonism with an Appendix on Pappus and the History of Platonism* (Leiden, Boston, Köln 1998). Some of these common questions feature also in Proclus' prologues to his commentary on the *Elements*: the utility of mathematics (20.8–25.15); the utility of geometry (61.25–64.7); the qualities required from a mathematician (32.21–35.16); the aim of the *Elements* (70.19–71.24); the aim of the first book of the *Elements* (81.24–83.6); and the meaning of the title, *Elements* (71.24–75.4). The references to Proclus' commentary on the *Elements* here and throughout are to Friedlein's edition: *Procli Diadochi in primum Euclidis elementorum librum commentarii*, ed. G. Friedlein (Leipzig 1873).

general; and from the introductions to the extant commentaries on Plato's dialogues, we can reconstruct a set of questions that commentators addressed before the reading of each dialogue.[9] From the introductions to Aristotle's *Categories* we also learn that in addition to these sets of questions, Neoplatonic commentators shared a common exegetical aim, that is, to show that Plato and Aristotle are in agreement.[10]

These common features may give the impression that commentators had little leeway to express their own views, that they were constrained not only by the content and structure of the base text but also by a tradition that determined the questions that they should address and the aim of their interpretations. In fact, on the basis of these common features, Ilsetraut Hadot argued against Karl Praechter that there are no significant doctrinal differences between commentaries written in the Athenian school and those written in the Alexandrian school.[11] I do not venture to challenge this view; but a comparison between commentators' treatments of specific lemmata, rather than between introductions to commentaries, does reveal major doctrinal differences. For example, Simplicius' account of relational attributes in his commentary on the *Categories* is significantly different from his predecessors' accounts. Here he addresses the question of whether relational attributes exist over and above their subjects, which his predecessors left open, thereby paving the way to an examination of the unique feature of relational attributes, that is, their being attributes of two subjects.[12] This example highlights the limitation of the focus on common traditions and introductions to commentaries. It shows that this approach conceals another characteristic of commentaries, whose importance was

[9] On these questions see Hoffmann, 'What Was Commentary in Late Antiquity?', 613–614.

[10] Commentators adhered to this exegetical assumption in different ways, e.g., Simplicius holds that the difference between Aristotle and Plato is mainly terminological (*In Cat.* 7.29–32 Kalbfleich), whereas Syrianus admits that Aristotle disagrees with Plato on the Forms and the ontological status of mathematical objects (*In Met.*80.4–81.14 Kroll).

[11] This is the main thesis of her book, *Le problème du néoplatonisme alexandrin: Hiéroclès et Simplicius* (Paris 1978). Cf. Hadot, I., *Athenian and Alexandrian Neoplatonism and the Harmonization of Aristotle and Plato*, trans. M. Chase (Leiden and Boston 2015), and Hadot, I. 'The Role of the Commentaries on Aristotle in the Teaching of Philosophy According to the Prefaces of the Neoplatonic Commentaries on the *Categories*' *Oxford Studies in Ancient Philosophy*, suppl. 1991, Blumenthal, H., and Robinson, H. (eds.), *Aristotle and After*, 175–189.

[12] See my 'Simplicius on the Reality of Relations and Relational Change', *Oxford Studies in Ancient Philosophy* 37 (2009), 245–274. I have chosen this particular example because commentaries on the *Categories* serve as the main evidence for the doctrinal uniformity of the Athenian and Alexandrian schools. This example does not necessarily cast doubt on Hadot's thesis. The difference between Simplicius' commentary and the other commentaries on the *Categories* admits other explanations; for example, it may be due to the fact that this was Simplicius' last commentary and was written after the closure of the Athenian school in 529 CE.

stressed in Heinrich von Staden's study of medical and scientific commentaries, that is, that they are shaped by an overarching theme that gives them a direction.[13] The possibility of shaping commentaries in this way, despite the constraints that the base text and the tradition impose, is related to the role that the study of canonical texts played in the ancient philosophical and scientific traditions. The following section clarifies this point.

2.3 Canonical and Authoritative Texts

In a passage from his *Geography*, the geographer and historian Strabo (64 BCE–c. 24 CE) associates the decline and revival of the Peripatetic school of thought with the loss and rediscovery of Aristotle's writings. He says that in the absence of books, the early Peripatetics who succeeded Theophrastus (c. 371–287 BCE) could not philosophize seriously (πραγματικῶς / *pragmatikôs*) but made hollow statements (ληκυθίζειν / *lêkuthizein*), whereas later Peripatetics fared better than their predecessors, owing to the publication of Aristotle's works (*Geography* 13.1.54). The adequacy of Strabo's explanation aside, it sheds light on Peripatetic thinkers' view of the role played by the study of texts, showing that they regarded it as indispensable for serious philosophical inquiry.[14] The few descriptions we have of commentators' own exegetical undertakings show that later thinkers, working in different disciplines and of different theoretical leanings, adopted the Peripatetic approach and regarded the study of canonical texts as a means of inquiry. To examine the consequences of this approach, I discuss two passages: one from the commentary on Hippocrates' *On Fractures* written by the second-century CE physician Galen (c. 129–200 CE), and the other from a commentary on Aristotle's *Categories* written by the sixth-century CE Neoplatonic philosopher Simplicius (c. 490 – c. 560 CE):

> Before I begin the detailed interpretation, it is worth saying in general about any interpretation that its purpose is to render clear what is unclear in the writings, to prove that what is written in the text is true or to refute it as false, and, if someone sophistically attacks it, to defend it. Although [scil. the latter] is distinct from interpretation, it is customarily done by

[13] Heinrich von Staden calls this characteristic 'plot'. See von Staden, H., 'A Woman Does Not Become Ambidextrous: Galen and the Culture of Scientific Commentary', in Gibson, R. K., and Shuttleworth Kraus, C. (eds.), *The Classical Commentary: Histories, Practices, Theories*, 109–139, at 118.

[14] Cf. Snyder, *Teachers and Texts*, 66.

> almost all writers of commentaries. Indeed, nothing prevents an interpreter from moderately treating this matter, but contending with full authority in defense of the writer's doctrines is beyond the boundary of an interpretation.[15] (Galen, *On Hippocrates' On Fractures* xviiiB 318–319 K.)
>
> The worthy interpreter of Aristotle's writings should not significantly fall short of the latter's greatness of intellect. Further, he should be acquainted with the philosopher's entire writings and be knowledgeable about the customary Aristotelian linguistic style. He should also be impartial in judgment, so that he should neither seek to show, by taking the argument lightheartedly, that well-said things are unconvincing, nor, if something needs attention, should he insist to prove that [scil. Aristotle] is always and in every way infallible, as if he enrolled himself in the philosopher's school of thought.[16] (Simplicius, *On Aristotle's Categories* 7.23–29 Kalbfleisch.)

These passages call into question the assumption that canonical texts are necessarily authoritative. They show that although Galen and Simplicius regarded Hippocrates' and Aristotle's writings as canonical texts that merit commentaries, they did not think that they expressed *the* truth. In their views, commentators should not act as followers of the base texts' authors and defend their views at all costs. Rather, they should be impartial, as Simplicius says, and distinguish the matters about which the author of the base text is right from those about which he is wrong.[17] Thus although commentators should clarify the base text, as Galen says, they should not slavishly adhere to the views found therein but should use the text as a means of inquiry that would lead to the right medical theory or the true philosophical stance. This approach to the study of texts clarifies why the commentary genre leaves room for doctrinal divergences and for the development of new ideas and interpretative stances.

Proclus' commentary on the *Elements* differs from Galen's, Simplicius', and his own other commentaries in engaging with a mathematical text,

[15] All translations are mine.

[16] Parallel passages are found in other introductions to commentaries on the *Categories*: Ammonius, 8.11–19 Busse; Elias, 122.25–123.1 Busse, Philoponus 6.30–35 Busse, and Olympiodorus, *Proleg.* 10.24–33 Busse. I quote and translate these passages in Appendix 2 at A3–6.

[17] The fact that Galen wrote polemical commentaries also indicates that ancient thinkers did not necessarily regard canonical texts as authoritative. On these commentaries see, Fleming, R., 'Commentary' in Hankinson, R. J. (ed.), *The Cambridge Companion to Galen* (Cambridge 2008), 323–354, at 325.

which does not give rise to questions regarding the truth value of its content. Still, adopting the above approach to the study of texts, he occasionally uses the commentary to make mathematical contributions of his own. The best-known and most significant contribution is his refutation of Ptolemy's and other attempts to prove the parallel postulate (365.7–371.10), and his own proof of this postulate (371.10–373.2). Further, in keeping with the impartial exegetical approach advocated by Galen and Simplicius, Proclus at times criticizes Euclid and at other times approves and praises him. For example, he finds Euclid's definition of an angle in terms of inclination (κλίσις / *klisis*) problematic on the grounds that it only captures the angles' relational aspect and disregards their qualitative and quantitative aspects (122.22–123.23); and he points out that in the proof of proposition I.36 (parallelograms which are on equal bases and in the same parallels are equal to one another), Euclid ignores the possibility that the bases of two parallelograms might have a common segment, and he offers a proof that takes this possibility into account (401.4–402.2). He treats Euclid positively, for example, in his discussion of proposition I.23 (to construct a rectilineal angel equal to a given rectilineal angle on a given straight line and at a point on it), where he shows that Euclid's proof is superior to Apollonius of Perga's alternative one because the latter disrupts the axiomatic-deductive order in basing his proof on theorems proved in the third book of the *Elements* (335.16–19),[18] or when he defends Euclid's reasoning against the Epicurean contention that proposition I.20 (any two sides of a triangle taken together are greater than the remaining side), needs no proof because it is evident (322.4–323.3).

Another feature of the study of texts can be inferred from the description of philosophical inquiry in Plotinus (204/5–270 CE), the founder of Neoplatonism. In his discussion of eternity and time, he describes the method of tackling philosophical difficulties as a study of the different views held by ancient philosophers on the matter at stake but stresses that this inquiry does not end here. He says that although some of the ancient philosophers have discovered the truth, a proper philosophical inquiry should address the following questions: which of them attained the truth to the highest degree, and how we should understand it (*Enn.*III.7.1, 7–16 Henry and Schwyzer). From this description we see that authority admits

[18] Theorems III.28 and 29.

degrees and that the relation between a commentary and its base text can be more complex than it may seem. Plotinus' claim that several ancient philosophers have discovered the truth and later Neoplatonists' exegetical aim of harmonizing Plato and Aristotle suggest that a commentary on a given text can aim to establish the authority not only of its base text but also of other texts and other authors.

This observation is crucial for understanding Proclus' commentary on Euclid's *Elements*. In his study of the revival of Pythagoreanism in late antiquity, Dominic O'Meara convincingly shows that Proclus' choice to write a commentary on the *Elements* should be understood in light of his tendency to downplay the scientific claims of the Pythagoreans and reinstate Plato's authority on mathematical matters by stressing the latter's use of the demonstrative method as opposed to the Pythagorean, inspired, symbolic, and revelatory approach.[19] To this end, Proclus describes Euclid as a devout Platonist (68.20–21); he identifies the *Elements*' aim (σκοπός / *skopos*) with the construction of the five Platonic solids (70.24–25, 71.22–24); he interprets proposition I.46 (on a given straight line to construct a square) as paving the way for the construction of the cosmic solids (423.13–14); and he relates the origin of the word 'mathematics' to Plato's theory of recollection (45.14–22).

Another significant consequence of O'Meara's study is that Proclus' departure from his predecessor, Iamblichus (c.245–c.325 CE), and the tradition that regarded Nicomachus of Gerasa's (c.60–c.120 CE) *Introduction to Arithmetic* as the canonical text for studying mathematics do not result from a sheer adherence to Plato's authority but have the pedagogical and philosophical aim of teaching and examining scientific demonstrative method on the basis of Euclid's axiomatic-deductive model. In this respect, Proclus' commentary on the *Elements* is not significantly different from other commentaries. As I show in the following two sections, here too the pedagogical aim is not separate from the theoretical aim, and a certain overarching theme related to Proclus' stress on demonstrative method gives the commentary a direction.

[19] O'Meara, D. J., *Pythagoras Revived: Mathematics and Philosophy in Late Antiquity* [*Pythagoras Revived*] (Oxford 1989), 148. The clearest expression of Proclus' attitude to Pythagorean mathematics is found in his comments on *Elements* I.47, i.e., Pythagoras' theorem, where he attributes its discovery to Pythagoras and then says that he admires Euclid more because he proved it and because in *Elements* VI.31 he proved a more general theorem (426.6–14).

2.4 Proclus' Pedagogical Aim

At the end of the second prologue to the commentary on the *Elements*, Proclus describes its aim as follows:[20]

> Beginning the investigation of particular matters, we inform in advance those who will come across [scil. this book] that they should not demand of us the things that our predecessors overused, namely lemmata, cases, and the like. We are surfeited with these matters and we will seldom treat them. But we will devote the commentary primarily to those matters that are theoretically more important and contribute to the whole of philosophy, thereby emulating the Pythagoreans whose customary proverb was: 'figure and a stepping-stone and not figure and three obols'. By this they indicate that we should pursue the type of geometry that makes each theorem a basis for a step upward and carries the soul to the heights and does not let it descend to the realm of perceptibles, by satisfying the needs of mortals and in aiming at these, neglects to turn around from there. (84.8–23)

At first glance, this passage confirms Vallance's view that Proclus' commentary on the *Elements* forces its base text into second place. Here he quotes a Pythagorean proverb which suggests that the study of figures, namely, geometry, is merely a stepping-stone, and treats with disdain mathematical issues, such as cases (i.e., variant constructions) and lemmata (i.e., propositions that require proof).[21] The latter point led Peter Riedlberger to the conclusion that on the basis of this passage 'we may imagine how Proclus knocked out any idea of mathematical rigor from the heads of his pupils'.[22] However, closer examination calls this understanding into question. It shows that by ascribing lesser importance to cases and lemmata, Proclus does not favor philosophical speculations over the *Elements*' mathematical content, but favors instruction in mathematical

[20] Proclus' commentary on the *Elements* has two prologues, one on mathematics and the other on geometry. The rest of the commentary is devoted to Euclid's definitions, the axioms and postulates, and the propositions of book I.

[21] For Proclus' use of these terms, see 211.4–212.11.

[22] Riedelberger, P. (ed., trans., and comm.), *Domninus of Larissa, Encheiridion and Spurious Works* (Pisa and Rome 2013), 41. Regarding this passage, it is important to note that in light of O'Meara's study, it is better to understand Proclus' mention of the Pythagorean proverb as a way of informing his readers that his choice to write a commentary on the *Elements* and not on Nicomachus' *Introduction to Arithmetic* is in line with Iamblichus and the Pythagorean tradition. Further, Proclus' interpretation of this proverb fully accords with his stress on scientific method. He does not regard the study of geometry as a stepping-stone but describes by this term its axiomatic-deductive structure, i.e., its use of previously proven theorems as a basis for proving other theorems, and he ascribes to this structure the role of elevating the soul to higher realms of reality.

method over training in mathematical skills. The following similar passage, where Proclus describes his commentary's aim, supports this understanding:

> Having briefly examined the account of theorems and problems, their difference, their parts, and the divisions [scil. found] therein, let us turn to the interpretation of [scil. the propositions] proven by the author of the *Elements*. In so doing, we select the more exact comments written about them by the ancients but we cut short their endless verbosity and teach the more technical matters related to scientific methods, thereby paying greater attention to the investigation of important things than to the variety of cases and lemmata, which, we see, attracts for the most part the youth. (200.6–18)

Here again Proclus ascribes little importance to the examination of cases and lemmata, but contrary to the conclusion that Riedlberger draws from the above similar passage, he does not dismiss mathematical technicalities but stresses the greater importance of technical matters and scientific method. The difference between examination of cases and the technical and methodological matters that Proclus stresses becomes clear from his discussions of the former. In keeping with the above passages, Proclus mentions cases about 20 times in the 432 pages of his commentary. The majority of these discussions are devoted to methodological questions, such as why certain problems or theorems admit cases (222.22–223.6, 225.8–11, 418.10–14), whether Euclid's proofs hold for all cases (400.10–11, 406.16–17, 413.19–21, 417.2–3), or why Euclid chooses a specific case (232.11–233.2, 399.4–5); but on three occasions he explains why he presents cases, saying that he does so for the sake of exercise (224.15–16. 227.6–8, 290.14–15). These discussions bring to light the significance of Proclus' promise to pay less attention to cases and lemmata. They show that rather than favoring philosophical speculations over mathematical issues, he favors methodological matters over training in mathematical skills.[23]

Further, Proclus' claim in the above passage from the second prologue that he devotes his commentary to matters that contribute to the whole of philosophy (ἡ ὅλη φιλοσοφία / *hê holê philosophia*) may cast doubt on this conclusion, by suggesting that in Proclus' commentary philosophical matters override the study of geometry. However, the following survey of Proclus' philosophical interpretations shows that with the exception of the two philosophical prologues, the philosophical treatment that is more

[23] Proclus encourages his readers or students to practice or study on their own certain subjects on other occasions too, e.g., 113.6–8, 220.4–6, 345.9–13, 414.15–20.

prevalent in this commentary turns on logical and methodological issues closely related to Euclid's text, rather than on philosophical issues that go beyond its logical aspect and mathematical content.

Interpretations of the latter type are widespread in Proclus' comments on Euclid's definitions.[24] Here, alongside with and usually after discussions of the definitions themselves, alternative definitions, classifications of geometrical objects, and the like, Proclus presents Pythagorean and Neoplatonic interpretations. For example, in his comment on the definition of a surface he says that the Pythagoreans say that the surface is related to the triad (114.25–115.8),[25] in his comments on the definition of an angle he says that plane angles express the more immaterial and simpler modes of unification than solid angles do (129.10–15), and in his comments on the definition of the circle he refers to the Pythagorean table of opposites, saying that the circle is analogous to the things listed in the column of the limit (147.3–5).

Interpretations of this type are significantly scarcer in Proclus' comments on the propositions. In fact, the only lengthy symbolic interpretation found in this commentary appears in his comments on Euclid's first proof, to construct an equilateral triangle on a given finite straight line. I cite it at length, so that its difference from the more widespread methodologically oriented interpretations of the propositions becomes clear:

> It is evident to everyone that the equilateral triangle is the most beautiful triangle and most akin to the circle, which has all its lines [scil. drawn] from the center equal and one simple line that bounds it from without. And it seems that the encompassing by two circles [. . .] brings to light as in an image how the things that proceed from the principles receive from them perfection, identity, and equality. In this way too, things that move in a straight line turn round in a circle through their eternal generation and the souls having discursive thinking represent through their reversals and returnings to their starting-points the unalterable activity of the intellect. It is also said that the life-giving source of the souls is encompassed by two intellects. Accordingly, if the circle is an image of intelligible being and the triangle is an image of the first soul because of the equality and similarity of the angles and sides, it would be reasonable to demonstrate this [i.e., proposition], which takes an intermediate equilateral [scil. area] in them, through circles. And if every soul proceeds from intellect and reverts to intellect and participates in the intellect in a twofold

24 Cf. O'Meara, *Pythagoras Revived*, 174.

25 Proclus does not explain why a surface is related to the triad, but he probably has in mind the idea that the minimal number of straight lines needed for enclosing an area is three.

> fashion, it would be well stated that the triangle, being a symbol of the threefold subsistence of the souls, is generated by being enclosed by two circles. But let these [scil. remarks] remind (ἀναμιμνησκέτω / *anamimnêsketô*) us of the nature of things from likeness.[26] (213.14–214.15)

In this passage Proclus goes beyond Euclid's proof, offering a symbolic interpretation which shows that this proof elusively represents Neoplatonic philosophy. He states that Euclid's first proof is derived from principles, not from the principles of geometry (i.e., definitions, axioms, and postulates), but from the metaphysical principles of Neoplatonic ontology. In so doing, he conceives of the equilateral triangle as an image of the circle and explains Euclid's construction of an equilateral triangle by means of two circles in the light of the Neoplatonic assumption that higher realms of reality endow the lower realms with their essential properties. In the same vein, he tells his readers that the circle and the equilateral triangle symbolize the relation between a paradigm and its image, on which Neoplatonic metaphysics rests, and says that it is found also in the phenomenal and the intelligible realms. In the former this relation finds expression in the eternal and circular generation of things that move in a straight line, that is, the four elements (earth, water, air, and fire) that are generated one from another, and in the latter it finds expression in the soul's discursive mode of thinking that proceeds from consequences to premises and back from premises to consequences. Concluding this account, Proclus justifies Euclid's procedure through a symbolic interpretation that likens the circle to the intellect and the triangle to the soul and shows that the construction of a triangle by means of two circles reflects the mode of generation found in the higher realms of reality.

Apart from this passage, only a few shorter interpretations of this type are found in the commentary on the propositions,[27] while the rest of the Platonic interpretations are more mathematically oriented. For example, in his comments on proposition I.32 (in any triangle, if one of the sides is produced, the exterior angle of the triangle is equal to the two interior and opposite angles, and the interior angles of the triangle are equal to two right angles), Proclus prepares his students for reading Plato's account of the constitution of the four elements from triangles found in the *Timaeus*

[26] Proclus' use of the verb *anamimnêskein* (to remind) is an allusion to Plato's theory of recollection.

[27] E.g., 294.2–12, 314.19–315.4, 353.3–11. Similarly, in the discussion of the propositions, references to the Pythagoreans are confined to attributions to them of certain theorems, proofs, and discoveries (e.g., 305.3, 379.17–18, 419.15–16, 426.6–9).

(53d–54b), by drawing the consequences that each of the angles of an equilateral triangle is equal to two-thirds of a right angle; the two angles of an isosceles triangle whose vertical angle is right are equal to half a right angle; and a scalene right angle triangle has one of its two other angles equal to two-thirds of a right angle and the other equal to one-third of a right angle (383.17–384.4).[28]

With the exception of these and a few other examples, the rest of Proclus' comments on the propositions aim primarily to clarify logical, methodological, and mathematical matters. For example, he explains mathematical terms, such as 'given' (271.14–15), 'porism' (301.21–302.11), and 'locus theorem' (394.16–395.2), he describes logical procedures, such as conversion of propositions and reduction to impossibility (252.5–253.15 and 254.22–256.8), he clarifies the difference between transitive and non-transitive relations (373.5–23), and he discusses alternative proofs (e.g., 227.9–228.3, 249.20–250.19, 266.16–269–25, 323.5–326.5, 334.3–335.14). To illustrate how Euclid's *Elements* serves Proclus in addressing logical and methodological matters, I quote at length two representative lemmata, proposition I.7 (Figure 2.1) and proposition I.10 (Figure 2.2):

> *Elements* I.7: *Given two straight lines constructed on a straight line, there cannot be constructed on the same straight line and on the same side of it two other straight lines equal respectively to the former two and having the same extremities but meeting at a different point.*
>
> This theorem has a certain rare quality and it is not at all customary in scientific propositions. For formulating [scil. propositions] negatively and not affirmatively is exceedingly unsuitable to them. At any rate, the propositions of geometrical and arithmetical theorems are for the most part affirmative. The reason is, as Aristotle says, that the universal affirmative [scil. conclusion] is most befitting the sciences because it is more self-sufficient and needs no negative [scil. premise], whereas the universal negative [scil. conclusion] needs also the affirmative, if one intends to make a proof. For without an affirmative [scil. premise] there is neither demonstration nor syllogism, and for this reason, those sciences that are demonstrative prove mostly affirmative [scil. propositions] and rarely use negative conclusions.
>
> And the enunciation of the theorem is complete and of admirable precision and is safeguarded by all the additions needed for rendering it irrefutable and indisputable against those who try to quibble. First, he [i.e., Euclid] states 'on the same straight line', lest we prove that two straight lines equal respectively to the first two [scil. can be constructed] upon another

[28] See also 427.24–428.2 where Proclus explains how Euclid's proof of proposition I.47 is useful for understanding Plato's *Republic*.

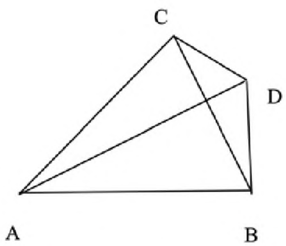

Figure 2.1 The diagram of Proposition I.7 of Euclid's *Elements*.

> line and by fallacious reasoning mislead those who use this theorem. Second, given one straight line, he does not say that the two lines constructed on it are simply equal to the two straight lines – for this is possible – but that they are equal 'respectively'. Would it be extraordinary to assume that either of the constructed lines is equal to either of the other lines, by lengthening one line and shortening the other? But he says that it is impossible that the lines are equal respectively. Third, he adds 'meeting at another point'. For what if one makes the two other lines equal respectively to the already constructed lines congruent to one another and so construct them on the same given lines and meeting at the same point, i.e., the apex? For if these straight lines are equal, certainly their extremities coincide. Fourth, [scil. he says] 'on the same side', for given one straight line, could we not construct two of the straight lines on one side of the given line and the other two lines on its other side, so that the [scil. given] straight line is a common base of two triangles having opposite apexes? He adds 'on the same side', in order to prevent us from attributing our mistake, if we make it, to the author of the *Elements*. Fifth, he asserts 'having the same extremities as the initially [scil. constructed] straight lines'. For it would be possible to construct upon the same straight line two lines equal respectively to the [scil. other] two lines that meet at different points on the same side [scil. of the given line], using the whole straight line and constructing the two lines upon it, while the constructed lines do not have the same extremities as the other [scil. two lines] but different ones. If we conceive of two diagonals in a square [scil. constructed] on one of the square's sides, there will be two lines equal to other [scil. two lines] – a side and a diagonal, the former being equal to the parallel side and the latter to the other diagonal – but the equal lines will not have the same extremities. For neither the parallel [scil. sides] nor the diagonals, though equal, will have the same extremities. If all these qualifications are maintained, the enunciation is proven to be true and the deduction indisputable. (259.15–262.3)

Proclus' focus on the methodological and logical aspects of geometrical proofs is evident from this lemma. Here he does not explain Euclid's proof by going through its stages and pointing to the axioms, postulates, and previously proven theorems on which it is based. Rather, he uses this theorem for discussing a methodological topic, that is, the suitability of negative conclusions to scientific demonstrations. This discussion takes its cue from *Posterior Analytics* I.14, where Aristotle claims that the first syllogistic figure, which deduces a universal affirmative conclusion from two affirmative universal premises (namely, a deduction of the form: every A is B, every B is C, therefore every A is C) is the most scientific syllogistic figure. His explicit mention of Aristotle notwithstanding, Proclus alters Aristotle's argument. Whereas Aristotle focuses on the syllogistic figure or form, arguing that the first figure is self-sufficient because it is irreducible to the other syllogistic figures (79a30–33), Proclus focuses on the deduced conclusions, arguing that universal affirmative conclusions are self-sufficient because they can be deduced without negative premises. Further, Proclus uses Aristotle's discussion selectively. He does not mention Aristotle's claims that first figure syllogisms are as a rule explanatory (79a22–25) and that knowledge of the essence can be attained only through this syllogistic figure (79a25–30). By treating *Posterior Analytics* I.14 in this way, Proclus adapts Aristotle's discussion to the *Elements*. He shifts the focus from syllogistic figures to universal negative conclusions because none of Euclid's proofs has a syllogistic form, and he omits Aristotle's arguments about explanation and knowledge of the essence because Euclid's proofs are neither explanatory in the Aristotelian sense nor do they lead to knowledge of the essence. Thus although Proclus interprets Euclid in light of Aristotle's methodological considerations, he does not read them into the *Elements* but adapts them to Euclid's form of reasoning.

In the second paragraph of this lemma, Proclus meticulously discusses Euclid's phrasing of the enunciation of proposition I.7, but here too he focuses on methodological matters rather than on clarifying Euclid's words. He comments on five qualifications that appear in the enunciation and shows how each qualification excludes possible misunderstandings that would make the theorem false. This exegetical approach is primarily pedagogical. As the first and last sentences of this paragraph indicate, Proclus' aim is to highlight Euclid's precision and show that seemingly minute details are crucial for formulating true theorems and irrefutable proofs. Further, his detailed explanation of each qualification calls to mind the notion of elaboration (ἐπεξεργασία / *epexergasia*) that Alexandrian scholars used in their scholia to Homer in characterizing and often justifying extensive literary descriptions that have a function in a certain context.[29] This similarity and Proclus' admiring tone imply that he treats the *Elements* as a classical text comparable in perfection to the *Iliad* and the *Odyssey*, thereby suggesting that just as Homer is a model of exemplary literary style, so too Euclid is a model of scientific precision.[30]

> *Elements* I.10: *To bisect a given finite straight line.*
>
> This problem too posits a finite straight line, since it is by no means possible to bisect [scil. a line] that is infinite on both sides, and if it is infinite only on one side the division is to unequal parts, wherever the point of section is taken; for the part that extends to infinity is greater than the remaining part since the latter is finite. So the alternative left to one who intends to bisect [scil. a line] is to assume that it is bounded on both sides.
>
> From this problem, some people may perhaps be led to suppose that geometers take in advance the hypothesis that a line does not consist of indivisible parts. For if it did, a bounded line would consist of either an odd number of parts or an even number of parts. But if it consists of an odd number of parts, it seems that when the line is bisected, an indivisible part is also bisected since [scil. otherwise] one of its parts would consist of more indivisible parts and would be greater than the other. Consequently, it will be impossible to bisect the given straight line, if indeed its magnitude consists of indivisible parts. But if it does not consist of indivisible parts, it is divisible to infinity. So it seems, they say, that this assumption too, namely that a magnitude consists of parts divisible to infinity, is an agreed geometrical principle. But we at any rate will say what Geminus [scil. has said] about these matters, i.e., that geometers assume, in accordance with common notions, that continuous [scil. magnitudes] are

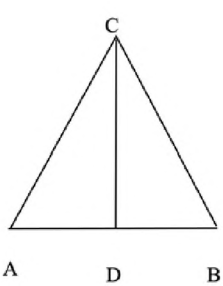

Figure 2.2 The diagram of Proposition I.10 of Euclid's *Elements*.

[29] See Nünlist, R., *The Ancient Critic at Work: Terms and Concepts of Literary Criticism in Greek Scholia* (Cambridge 2009), 204–208.

[30] For similar discussions of Euclid's phrasing see 235.15–236.1; 292.12–23; 295.7–296.14; 305.21–306.8; 326.22–327.8.

> divisible; for we say that continuous [scil. magnitudes] consist of parts that touch one another, and these can certainly be divided. However, geometers do not assume in advance that continuous [scil. magnitudes] are divided to infinity, but they prove it from proper principles. For when they prove that among magnitudes there is an incommensurable magnitude and that not all [scil. magnitudes] are commensurable with one another, what else could one say they are proving but that any magnitude is always divided into parts and that we will never reach an indivisible part, which is the minimal measure of magnitudes? This [scil. statement] is demonstrable, whereas the other one, that states that any continuous [scil. entity] is divisible, hence a bounded continuous line is also divisible, is an axiom. The author of the *Elements* too bisects the bounded straight line on the basis of this notion, but not on the assumption that it is divisible to infinity. For being divisible and being divisible to infinity are not identical [. . .].
>
> The geometer bisects the bounded line by using the first and the ninth propositions in the construction and only the fourth one in the demonstration. For he proves that the bases are equal through the [scil. bisection] of the angles. But Apollonius of Perga bisects the given bounded straight line in this way [. . .]. His proof too assumes an equilateral triangle, but instead of proceeding from the bisection of the angle at [scil. the apex] C, it proves that [scil. the line] is bisected on the basis of the equality of the bases. So, the author of the *Elements*' demonstration is much better, simpler, and he proceeds from the principles. (277.16–280.12)

Like the lemma quoted above, this lemma focuses on the enunciation of the theorem and clarifies a methodological point. It explains why Euclid assumes a finite straight line, by eliminating the two other possible alternatives, that is, that the given line is infinite on both sides and that it is infinite on one side, on the ground that these infinite lines cannot be bisected. In the rest of the lemma Proclus addresses the question whether geometers assume that magnitudes are infinitely divisible, and discusses Apollonius' alternative proof. Seemingly, these discussions are unrelated. The first one bears on the philosophical debate about the existence of indivisible magnitudes and the second one is mathematical. However, on closer examination, these discussions address the same methodological subject, that is, the axiomatic-deductive structure of the *Elements*. In the first discussion, Proclus counters the contention that the problem of bisecting a finite straight line presupposes an assumption that does not feature in the *Elements*' list of axioms, that is, that a line does not consists of indivisible parts. This contention casts doubt on the axiomatic-deductive structure of the

Elements. It implies not only that Euclid's axiomatic-deductive system is incomplete, in being based on a tacit assumption; it also implies that this assumption cannot serve as an axiom because, being one of the points of contention between the Epicureans, who denied it, and the Aristotelians, Platonists, and Stoics, who endorsed it, this assumption is not generally agreed.[31] Addressing this objection, Proclus appeals to Geminus' solution and offers three arguments:[32] First, geometers assume that continuous magnitudes are divisible, not that they are *infinitely* divisible; second, the former assumption is a common notion, that is, an evident and generally agreed principle;[33] and third, geometers prove the latter assumption when they show that certain magnitudes are incommensurable (e.g., the diagonal and the side of a square). In this discussion, then, Proclus raises a philosophical topic, but he confines his discussion to its methodological consequences. He defends Euclid's axiomatic-deductive system against the charges of the Atomists, and he highlights the distinction between indemonstrable and indisputable assumptions that serve as axioms and the demonstrable theorems derived therefrom.

Proclus' discussion of Apollonius' alternative proof has a similar aim. He compares this proof with Euclid's proof, showing that the former does not follow the proper order of an axiomatic system.[34] Specifically, he points out that Euclid bases his proof on the previous problem (*Elements* I.9), 'to bisect a given rectilinear angle,' and deduces the equality of the two parts of the given line from the equality of the two parts of the angle, whereas Apollonius does not take the bisection of an angle as proven, but bisects an angle and then proves the equality of the two parts of the given line. Thus the aim of this discussion is not mathematical but methodological. It emphasizes the proper axiomatic order, which requires that the bisection of an angle should logically precede the bisection of a line.

These examples do not capture the richness of Proclus' commentary on the *Elements*, but they suffice to show that it is more didactic than it might

[31] This and other passages in which Proclus addresses the Epicurean criticism of geometry, although there were no proponents of this school of thought at that time, attest to commentaries' role in preserving tradition.

[32] Geminus was an astronomer and a mathematician who flourished in the first century BCE.

[33] Proclus bases his contention that this assumption is evident on *Physics* V.3, where Aristotle defines the continuous as consisting of parts whose extremities touch and become one (227a10–12). By this definition, continuous entities are evidently divisible because they consist of parts.

[34] For this understanding of the expression 'from the principles' see Morrow, G. R. (trans.), *Proclus, a Commentary on the First Book of Euclid's* Elements: *Translation with Introduction and Notes* (Princeton 1970), 218, note 95.

seem on the basis of the concluding passage of its second prologue, in aiming to instruct its readers in the methodological and logical aspects of mathematical reasoning. As in other Neoplatonic commentaries, this didactic aim is not distinct from the philosophical aim. Proclus' focus on logic and methodology accords with his view that mathematical objects are intermediate between intelligible and perceptible realities and pre-exist in discursive reason (διάνοια / *dianoia*) (3.1–4.8). In keeping with this view, he understands the *Elements* as having two aims: one in respect of its subject matter, which is the construction of the Platonic solids, and the other in relation to the learner, which is the perfection (τελείωσις / *teleiôsis*) of discursive reason (71.5–9).[35] Having the latter aim, mathematics and geometry are stepping-stones not only because their objects elevate us from the perceptible realm and symbolize higher realms of reality, but also because in Proclus' view the rigorous deductive method found in the *Elements* should be employed in philosophy. True to this view, Proclus himself used Euclid as a model in his *Elements of Theology* and *Elements of Physics*, where he presents his metaphysics and natural philosophy in an axiomatic-deductive form.

The aim of Proclus' emphasis on logic and scientific methodology is not confined to perfecting his audience's discursive reason, that is, to teaching methods of reasoning, argumentation, and proofs; his commentary has also a more specific methodological and philosophical aim. In the concluding paragraph of this commentary, Proclus once again contrasts his commentary on the *Elements* with other commentaries circulating at that time and encourages his successors to write commentaries on the subsequent books of the *Elements* that, like his own commentary, contribute to the account of causes, to dialectical judgment, and to philosophical understanding (432.9–19). As I now show, the first subject on this list, that is, the account of causes, is the overarching theme of Proclus' commentary on the *Elements*.

2.5 The Overarching Theme of Proclus' Commentary

In his commentary on the *Elements*, Proclus departs not only from Iamblichus but also from the prevailing view of the ontological status of geometrical objects. Their Platonic commitments notwithstanding,

[35] See also Proclus' claim in the first prologue that one should use mathematics as a model that leads to dialectical reasoning (21.22–24).

Neoplatonic philosophers followed Aristotle and Alexander of Aphrodisias (fl. 200 CE) in regarding mathematical objects as abstractions that exist in our minds.[36] In contrast to this tradition, in the second prologue to his Euclid commentary Proclus develops the view of his teacher Syrianus (fourth and early fifth centuries CE) that geometrical objects are projections onto the imagination of objects that pre-exist in discursive reason. This view, Proclus holds, accords both with Plato, who identified mathematical objects with the objects of discursive reason (56.23–57.8), and with the geometers' use of constructions:[37]

> But if the objects of geometry are outside matter, pure *logoi*, and separate from sensibles [...], how then do we still bisect the straight line, the triangle, and the circle? And how do we speak about differences of angles and of the increases and decreases of figures, such as triangular and quadrangular? And how [scil. do we speak] of contacts of circles or lines? (49.24–50.6)

This attempt to offer a philosophy of geometry that accounts for geometrical constructions has to confront the difficulty that the use of geometrical constructions does not meet the requirements of explanatory Aristotelian demonstrations, which Proclus formulates in his comments on Euclid's first proof:

> We shall find sometimes that what is called 'proof' has the properties of a demonstration, in proving the sought through definitions as middle terms – and this is a perfect demonstration – but sometimes it attempts to prove from signs.[38] This should not be overlooked. For, although geometrical arguments always have their necessity through the underlying matter, they do not always draw their conclusions by means of demonstrative methods. For when it is proved that the interior angles of a triangle are equal to two right angles from the fact that the exterior angle of a triangle is equal to the two opposite interior angles, how can this demonstration be from the cause? How can the middle term be other than a sign? For the interior angles are equal to two right angles even if there are no exterior angles, for there is a triangle even if its side is not extended. (206.12–26)

[36] On the commentators' approach to mathematical objects, see Mueller, I., 'Aristotle's Doctrine of Abstraction in the Commentators', in R. Sorabji (ed.) *Aristotle Transformed*, 463–480.

[37] The following discussion is based on my articles: '*Methexis* and Geometrical Reasoning in Proclus' Commentary on Euclid's *Elements, Oxford Studies in Ancient Philosophy* 30 (2006), 361–389, at 383–388 and 'Proclus' Account of Explanatory Demonstrations in Mathematics and Its Context' ['Explanatory Demonstrations'], *Archiv für Geschichte der Philosophie*, 90 (2008), 137–164, at 138–139.

[38] A middle term is the term that appears in the two premises of a syllogistic deduction. For example, B is the middle term of the syllogistic deduction: every A is B and every B is C, therefore every A is C. Greek philosophers use the term sign in different ways; here Proclus uses it in describing deductions that base their conclusions on a necessary but not essential attribute.

Here Proclus highlights the tension between the aim of accounting for the use of geometrical constructions and his assumption that geometrical proofs should have the form of an Aristotelian explanatory demonstration. Explanatory Aristotelian demonstrations ground the conclusion in the definition or essence of the subject of demonstration, whereas from an Aristotelian viewpoint, as Proclus points out in the above passage, constructions are accidental. Euclid's proof of the equality of the sum of the interior angles of a triangle to two right angles does not ground this property in the definition of the triangle but instead deduces it from a construction step, whereby the triangle's base is extended and a line parallel to one of the triangle's sides is drawn. This problem is not only philosophical but also has indirect consequences for Euclid's authority inasmuch as it is crucial for countering two criticisms that call into question the status of mathematics as a demonstrative science: in one, which according to Proclus was raised by other Platonists, mathematics is not a science because it does not account for its axioms (29.14–32.20); and in the other, which he traces to Aristotle, geometry does not study causes (202.9–19).[39]

In his commentary on the *Elements* Proclus does not directly address this question or explain extensively how his philosophy of geometry accounts for both explanatory demonstrations and geometrical constructions. Instead he discusses it in a piecemeal fashion in his comments on three related propositions: I.16, the exterior angle of a triangle is greater than either of the interior and opposite angles; I.17, any two angles of a triangle are less than two right angles; and the second part of proposition I.32, the sum of the interior angles of a triangle is equal to two right angles (Figure 2.3). I discuss the relevant passages in the order of their appearance in the commentary:

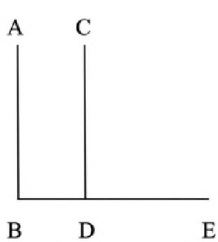

Figure 2.3 A diagram illustrating Proclus' construction.

[39] The association of this view with Aristotle goes back to the Stoic philosopher Posidonius (c.135–c.51 BCE). On this, see my 'Explanatory Demonstrations', 144–146.

> By means of this theorem we can also prove [scil. a theorem that states] that if a straight line that falls on two straight lines makes the exterior angle equal to the interior and opposite angle, the lines do not make a triangle nor do they intersect, since the same angle will be greater and equal, which is impossible. Let AB and CD be straight lines and let BE falling on them make the equal angles ABD and CDE. AB and CD do not intersect. For if while the angles remain equal, they intersect, angle CDE will be equal to angle ABD although being an exterior angle it is also greater than the interior and opposite angle. Necessarily then if [scil. the lines] intersect, the angles no longer remain equal but in every case the angle at D increases. For if while line AB remains unmoved, you think of line CD as moving toward it, so that they intersect, you make the interval at angle CDE greater, for the more CD moves toward AB, the more it moves away from DE. And if while line CD remains unmoved, you think of line AB as moving toward it, you make angle ABD lesser, for it moves simultaneously toward CD and toward BD. And if you make both [scil. lines] move toward each other, you will find that in moving toward BD, AB also decreases the angle while, in moving away from DE as a result of the motion toward AB, CD also increases angle CDE. Therefore, if there will be a triangle and lines AB and CD will intersect, of necessity the exterior angle will be greater than the interior and opposite angle [. . .]. The motion of the lines is the cause of these [scil. facts] [. . .] and you can infer from this how the generations of things bring to our view the true causes of the conclusion. (308.14–310.8)

Here Proclus uses proposition I.16 in an indirect proof that shows that two parallel lines do not intersect and make a triangle, arguing that, if they made a triangle, the exterior angle would be both equal to and greater than the interior and opposite angle. Although this proof is based on proposition I.16, Proclus devotes the rest of this passage to proving this proposition. The remarkable feature of this proof is the mode of construction that he employs. Instead of following Euclid's construction by means of ruler and compass, he uses a kinematic construction. First he moves line CD towards line AB and shows that angle CDE increases, then he moves line AB towards line CD and shows that angle ABD decreases, and finally he moves both lines towards each other and shows that angle ABD decreases while angle CDE increases. At the end of this passage he explains why this construction is methodologically preferable, saying that it reveals the true cause of the conclusion. He does not explain here why Euclid's construction is deficient in this respect, but he does do so in his comments on proposition I.17:

> And the proof of the author of the *Elements* takes a clear path. For it uses the preceding theorem. But, as in the previous theorem, one must discover the cause of the present property by looking at the generation of triangles. So again let lines AB and CD be at a right angle to BD. If then there is to be a triangle, AB and CD must incline toward each other. And the inclination decreases their interior angles, so that they become less than two right angles. For they were right [scil. angles] before the inclination [. . .]. This then is the cause, and not that the exterior angle is greater than each of the interior and opposite angles; for it is not necessary that the side be produced, nor that a certain exterior angle be constructed. But it is necessary that any of the interior angles be less than two right angles. And how can what is not necessary be the cause of what is necessary? Rather, as I said, the cause is the stated [scil. reason], i.e., the inclination of the lines toward the base, which decreases the angles. (310.17–311.23)

Here Proclus argues that Euclid's mode of construction is not explanatory because it derives a necessary attribute of triangles from a contingent attribute, that is, the fact that the base of triangles can be extended so as to make an exterior angle. Furthermore, this passage clarifies his appeal to the mode of generation in his comments on proposition I.16. It shows that by using this kinematic construction, Proclus establishes a necessary relation between the triangle's attributes and its essence. The inclination of lines AB and CD towards each other makes the two interior angles less than two right angles; hence, by the parallel postulate, they necessarily meet and make a triangle. In his comments on proposition I.32, Proclus directly relates this alternative construction to the problem that he raises in his comments on *Elements* I.1, showing that having the sum of its interior angles equal to two right angles is an essential attribute of a triangle:

> And we should state also this point, i.e., that having its interior angles equal to two right angles holds for a triangle essentially and by virtue of itself. For this reason, in his work on demonstrations where he examines essential predication Aristotle readily uses it as an example [. . .]. The truth of this theorem seems to be in line with our common notions. For if we think of a straight line and of lines standing at right angles at its extremities, then, if they incline so that they generate a triangle, we will see that they reduce the right angles that they made with the straight line in proportion to their inclination; the same amount that they have subtracted from these [scil. angles] is added through

the inclination to the angle at the vertex, so of necessity they make the three angles equal to two right angles. (384.5–21)

The above five passages are scattered throughout Proclus' commentary, but placed side by side they form a coherent argument. In the first passage Proclus states that one of the aims of his account of the ontological status of geometrical objects is to explain geometrical constructions. In the second one he presents a difficulty about the explanatory value of geometrical proofs that arises from the use of geometrical constructions. And in the last three passages he solves this problem by deriving the three discussed theorems from an alternative construction that ultimately furnishes an explanatory proof for the second part of proposition I.32. From these passages we see that the overarching theme of Proclus' commentary on the *Elements* is not merely scientific methodology in general but specifically the question of mathematical explanation that bears on the status of mathematics as a science. This specific theme, which is Proclus' unique and historically influential contribution to the philosophy of mathematics, runs through the commentary from the prologues to the detailed comments on the propositions and thereby unifies it.[40]

Further, Proclus' formulation of and answer to the question of mathematical explanation illustrates the complex engagement with several canonical texts in one commentary. The above passage is an implicit commentary on the *Posterior Analytics* I.4, 73b33-74a3, where Aristotle exemplifies his account of essential predication through the proposition that the sum of triangles' interior angles is equal to two right angles.[41] It justifies Aristotle's use of the second part of proposition I.32 as an example of essential predication, but at the same time it modifies his account of this type of predication and makes room for geometrical constructions by identifying the explanatory element of demonstrations with the mode of generation of geometrical objects and not with their essence or definition, as Aristotle does.

[40] For a comparison between Proclus' and Philoponus' approaches to the question of mathematical explanation, see my 'John Philoponus and the Conformity of Mathematical Proofs to Aristotelian Demonstrations,' in Chemla, K. (ed.), *The History of Mathematical Proof in Ancient Traditions* (Cambridge 2012), 206–227. For Proclus' influence on the Renaissance debate about the certainty of mathematics, see Helbing, M. O., 'La fortune des Commentaires de Proclus sur le premier livre des *Éléments* d'Euclide à l'époque de Galilée', in Bechtle, G., and O'Meara, D. J. (eds.), *La philosophie des mathématiques de l'Antiquité tardive* (Fribourg 2000), 173–193, at 180–183.

[41] Certain passages from the first prologue to this commentary can be understood as an implicit commentary on Iamblichus' *On Pythagoreanism* III; for examples see O'Meara, *Pythagoras Revived*, 160–166.

Finally, these passages shed light on the difference between discursive interpretations, for example, Themistius' (317–c.390 CE) paraphrases of Aristotle's works or Galen's *The Elements According to Hippocrates*, and running commentaries. The former presume knowledge of the base text but do not present it in lemmata, whereas running commentaries, which proceed from one quoted lemma to another, focus on the authors' exact words. This mode of presentation is not only related to issues of authority and the canonical status of certain works, but it also facilitates a different mode of philosophical or theoretical inquiry. It enables commentators to treat the lemmata as data that call for explanation. In propounding his philosophy of geometry and his account of mathematical explanation, Proclus treats Euclid's text in this way. In the first passage quoted above he presents Euclid's use of geometrical constructions as counter-evidence that casts doubt on the Platonic view that geometrical objects are separate from matter, and later he states that he seeks to offer an alternative account that accords with the facts themselves (τοῖς πράγμασιν αὐτοῖς / *tois pragmasin autois*, 50.16). Similarly, he appeals to Euclid's proof of the second part of proposition I.32 in raising a question about the adequacy of Aristotle's model of demonstrative proof, and he presents his answer by examining Euclid's proofs of this and of two related propositions. In so doing, he does not slavishly follow Aristotle or Euclid. He modifies Aristotle's account of explanatory demonstrations by basing them on the mode of generation of objects and not on their essence, and he uses kinematic constructions instead of Euclid's constructions by ruler and compass. Consequently, in this commentary Proclus does not force the *Elements* into second place nor does he use them as a pretext for expounding his own philosophy of mathematics. Instead, his philosophy of mathematics takes its cue from and finds support in Euclid's geometrical reasoning but, in keeping with the late-antique commentary tradition, it also uses it as a basis for further philosophical inquiry.

In conclusion, Proclus' commentary on the first book of Euclid's *Elements* defies the dichotomous distinction between secondary texts and philosophical works that stand in their own right. In this commentary, Proclus goes beyond the base text, emphasizes its methodological aspects instead of its mathematical content, and approaches it from a philosophical viewpoint. Nevertheless, in so doing, he does not regard the *Elements* as a dispensable background but uses it as a model for teaching rigorous reasoning, and, when he approaches it from a philosophical perspective, he adapts the philosophical views to the base text. He develops his philosophical views in the commentary, but even in this case the *Elements* serves as a basis. The use of constructions therein enables Proclus to formulate the question of mathematical explanation, which did not arise in commentaries on Aristotle's *Posterior Analytics* – the natural place to discuss methodological topics.[42] These features of Proclus' commentary on the *Elements* conform to the view of Greek thinkers about the role of the study of canonical texts. These texts were usually commented on for teaching purposes, but they served as a basis for further philosophical inquiry, and the pedagogical purposes were not regarded as distinct from the philosophical or theoretical purposes. Understanding Proclus' commentary on the *Elements* in light of this approach to canonical texts does better justice to his commentary and also explains a central feature of Greek philosophical commentaries: their being *both* secondary texts and works that stand in their own right.

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[42] On this see my 'John Philoponus and the Conformity of Mathematical Proofs to Aristotelian Demonstrations,' in Chemla, K. (ed.), *The History of Mathematical Proof in Ancient Traditions* (Cambridge 2012), 206–227.

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Appendix 2 Translation, Orna Harari

A Commentators' Objectives

A1 Galen, On Hippocrates' On Fractures *xviiiB 318–319 K*

Before I begin the detailed interpretation, it is worth saying in general about any interpretation that its purpose is to render clear what is unclear in the writings, to prove that what is written in the text is true or to refute it as false, and, if someone sophistically attacks it, to defend it. Although [scil. the latter] is distinct from interpretation, it is customarily done by almost all writers of commentaries. Indeed, nothing prevents an interpreter from moderately treating this matter, but contending with full authority in defense of the writer's doctrines is beyond the boundary of an interpretation.

Πρὸ τῆς τῶν κατὰ μέρος ἐξηγήσεως ἄμεινον ἀκηκοέναι καθόλου περὶ πάσης ἐξηγήσεως, ὡς ἔστιν ἡ δύναμις αὐτῆς, ὅσα τῶν ἐν τοῖς συγγράμμασίν ἐστιν ἀσαφῆ, ταῦτ' ἐργάσασθαι σαφῆ. τὸ δ' ἀποδεῖξαί τι τῶν γεγραμμένων ὡς ἀληθὲς ἢ ὡς ψεῦδος ἐλέγξαι, καὶ εἰ κατηγόρησέ τις σοφιστικῶς ἀπολογήσασθαι, κεχώρισται μὲν ἐξηγήσεως, εἴθισται δὲ γίγνεσθαι πρὸς ἁπάντων ὡς εἰπεῖν τῶν γραφόντων ὑπομνήματα. καὶ νὴ Δία οὐδὲν κωλύει καὶ τούτου μετρίως ἅπτεσθαι τὸν ἐξηγητήν. τὸ δ' ἀγωνίζεσθαι τελέως ὑπὲρ τῶν τοῦ γράφοντος δογμάτων ἐκπέπτωκε τὸν ὅρον τῆς ἐξηγήσεως.

A2 Simplicius, On Aristotle's Categories *7.23–29 Kalbfleisch*

The worthy interpreter of Aristotle's writings should not fall significantly short of the latter's greatness of intellect. Further, he should be acquainted with the philosopher's entire writings and be knowledgeable about the customary Aristotelian linguistic style. He should also be impartial in judgment, so that he should neither seek to show, by taking the argument lightheartedly, that well-said things are unconvincing, nor, if something needs attention, should he insist to prove that [scil. Aristotle] is always and in every way infallible, as if he enrolled himself in the philosopher's school of thought.

Τὸν δὲ ἄξιον τῶν Ἀριστοτελικῶν συγγραμμάτων ἐξηγητὴν δεῖ μὴ πάντῃ τῆς ἐκείνου μεγαλονοίας ἀπολείπεσθαι. δεῖ δὲ καὶ τῶν πανταχοῦ τῷ φιλοσόφῳ γεγραμμένων ἔμπειρον εἶναι καὶ τῆς Ἀριστοτελικῆς συνηθείας ἐπιστήμονα. δεῖ δὲ καὶ κρίσιν ἀδέκαστον ἔχειν, ὡς μηδὲ τὰ καλῶς λεγόμενα κακοσχόλως ἐκδεχόμενον ἀδόκιμα δεικνύναι μηδὲ εἴ τι δέοιτο ἐπιστάσεως, πάντῃ πάντως ἄπταιστον φιλονεικεῖν ἀποδεῖξαι, ὡς εἰς τὴν αἵρεσιν ἑαυτὸν ἐγγράψαντα τοῦ φιλοσόφου.

A3 Ammonius, On Aristotle's Categories *8.11–19 Busse*

In the tenth place, one should ask of what quality the interpreter of Aristotle's writings should be. And we say that he should know very well the [scil. writings] that he is about to interpret and be also intelligent, so that he could on the one hand present the thought of the philosopher and on the other hand examine closely the truth of his sayings. For he should not sell himself completely, so to speak, and sustain whatever [scil. the philosopher] may say and be always eager to confirm the sayings that he interprets as entirely true, even if it is not the case. But when he judges, he should closely examine each of them, and if it should turn out this way, set the truth before Aristotle. So this is how an interpreter should interpret.

Δέκατον ἐπὶ πᾶσι δεῖ ζητεῖν ὁποῖον δεῖ εἶναι τὸν ἐξηγούμενον τὰ Ἀριστοτέλους συγγράμματα. καὶ λέγομεν ὅτι δεῖ καὶ ἄριστα εἰδέναι αὐτὸν ἃ μέλλει ἐξηγεῖσθαι, εἶναι μέντοι καὶ ἄνδρα ἔμφρονα, ὡς τὸ μὲν παριστᾶν τὴν τοῦ φιλοσόφου διάνοιαν τὸ δὲ τὴν ἐν τοῖς λεγομένοις ἀλήθειαν ἐξετάζειν· οὐδὲ γὰρ δεῖ ὥσπερ ἐκμεμισθωκέναι πάντως ἑαυτὸν καὶ ἀνέχεσθαι ὅ τι ἂν λέγηται καὶ σπουδάζειν πάντως ἐκεῖνα κρατῦναι ἃ ἐξηγεῖται ὡς ἀληθῆ πάντα, κἂν μὴ οὕτως ἔχῃ, ἀλλὰ δεῖ ἕκαστον κρίνοντα βασανίζειν ἐπίπροσθεν Ἀριστοτέλους θέμενον, εἰ τύχοι, τὴν ἀλήθειαν. οὕτως οὖν ἐξηγεῖσθαι χρὴ τὸν ἐξηγούμενον.

A4 Elias, On Aristotle's Categories *122.25–123.1 Busse*

And the interpreter should be both an interpreter and a man of knowledge. And the explanation of the unclarities in the text is the task of an interpreter, whereas the judgment of truth and falsehood, or of the futile and fruitful [scil. is the task] of a man of knowledge. He should not side with the [scil. authors] that he may interpret after the manner of actors who put on different masks due to the habit of imitating different characters. He should become neither an Aristotelian when he interprets Aristotle's works, saying that a philosopher of his qualities has not been born, nor a Platonist when he interprets Plato's works, saying that no philosopher on a par with Plato has been born. He should not argue vehemently in all manners that the ancient philosopher whom he interprets is always right but firmly state 'this man is a friend and the truth is a friend; while both are dear, the truth is dearer'.[43]

Ὁ δὲ ἐξηγητὴς ἔστω ἅμα ἐξηγητὴς καὶ ἐπιστήμων. ἔστι δὲ ἐξηγητοῦ μὲν ἔργον ἡ ἀνάπτυξις τῶν ἀσαφῶν ἐν τῇ λέξει, ἐπιστήμονος δὲ ἡ κρίσις τοῦ ἀληθοῦς καὶ

[43] This saying is a paraphrase of Aristotle's *Nicomachean Ethics* I.6, 1096a16–17.

τοῦ ψεύδους, ἤτοι ἀνεμίων καὶ γονίμων. δεῖ αὐτὸν μὴ συμμεταβάλλεσθαι οἷς ἂν ἐξηγῆται δίκην τῶν ἐν σκηνῇ ὄντων καὶ διάφορα πρόσωπα ὑποδυομένων διὰ τὸ μιμεῖσθαι διάφορα ἤθη, καὶ Ἀριστοτελικὸν μὲν γίνεσθαι τὰ τοῦ Ἀριστοτέλους ἐξηγούμενον καὶ λέγειν ὅτι οὐκ ἐγένετο φιλόσοφος τοιοῦτος, Πλατωνικὰ δὲ ἐξηγούμενον Πλατωνικὸν γίνεσθαι καὶ λέγειν ὅτι οὐκ ἐγένετο κατὰ Πλάτωνα φιλόσοφος. δεῖ αὐτὸν μὴ ἐκ παντὸς τρόπου βιάζεσθαι καὶ λέγειν ὅτι πάντως ἀληθεύει ὁ ἀρχαῖος ὃν ἐξηγεῖται, ἀλλὰ πανταχοῦ ἐπιλέγειν 'φίλος ὁ ἀνήρ, φίλη δὲ καὶ ἡ ἀλήθεια, ἀμφοῖν δὲ φίλοιν προκειμένοιν φιλαιτέρα ἡ ἀλήθεια'.

A5 Philoponus on Aristotle's Categories *6.30–35 Busse*

And the interpreter of this [scil. philosopher] should neither attempt to prove badly stated things in a biased way, like oracles received from a tripod, nor maliciously receive well-stated things, out of hostility. But he ought to be a dispassionate judge of the things said, and first clarify the intention of the ancient [scil. philosopher] and convey his opinion, and then state his own judgment.

ὁ δὲ τοῦτον ἐξηγούμενος ὀφείλει μήτε κατ' εὔνοιαν ἐπιχειρεῖν τὰ κακῶς λεγόμενα συνιστᾶν καὶ ὡς ἀπὸ τρίποδος ταῦτα δέχεσθαι μήτε τὰ καλὰ κακοτρόπως δέχεσθαι κατὰ ἀπέχθειαν, ἀλλὰ κριτὴς ἀπαθὴς τῶν λεγομένων ὑπάρχειν, καὶ πρῶτα μὲν τὴν διάνοιαν τοῦ ἀρχαίου σαφηνίζειν καὶ ἑρμηνεύειν τὰ αὐτῷ δοκοῦντα, ἔπειτα τὴν παρ' ἑαυτοῦ ἐπιφέρειν κρίσιν.

A6 Olympiodorus, Introduction to Aristotle's Logic and Commentary on the Categories *10.24–33 Busse*

Of what quality should an interpreter of Aristotle's writings be? And we say that such [scil. a person] should be a skilled interpreter and a man of knowledge: skilled interpreter, because of the clear interpretation of the truth; a man of knowledge, because he discerns truth from falsehood. He should not be a slave of a school of thought and let it hire, so to speak, his ears and tongue, and as a result do everything, [scil. I mean] do, say, and understand, in accordance with its opinions. And he should not side with the schools of thought, so that he would not be like those who put on different masks on stage, and thus on this account, he himself too would be in this state. But he must have true impressions and criteria, through which he can discern what is right from what is not.

οἷον δεῖ εἶναι τὸν ἐξηγούμενον τὰ συγγράμματα Ἀριστοτέλους. καί φαμεν ὅτι τοιόνδε <δεῖ> εἶναι ἐξηγηματικὸν καὶ ἐπιστημονικόν· ἐξηγηματικὸν μὲν διὰ τὸ ἐξηγεῖσθαι σαφῶς τὴν ἀλήθειαν, ἐπιστημονικὸν δὲ ὡς διακρίνοντα τὴν ἀλήθειαν ἀπὸ τοῦ ψεύδους. δεῖ δὲ αὐτὸν μὴ δουλεύειν αἱρέσει καὶ ὥσπερ ἐκμεμισθωκέναι αὐτῇ τὰ ὦτα καὶ τὴν γλῶτταν, ὡς ἅπαντα ποιεῖν πρὸς τὰ ἐκείνῃ δοκοῦντα, καὶ ποιεῖν καὶ λέγειν καὶ ἀκούειν. δεῖ δὲ αὐτὸν μὴ συμμεταβάλλεσθαι ταῖς αἱρέσεσιν, ἵνα μὴ ὥσπερ οἱ ἐν σκηνῇ διάφορα ὑποδύοντες πρόσωπα, οὕτω καὶ αὐτὸς ταύτῃ τι πάθῃ, ἀλλὰ τύπους ἔχειν χρὴ καὶ κανόνας ἀληθεῖς, δι᾽ ὧν δύναται διακρῖναι τὸ ὀρθῶς ἔχον καὶ τὸ μὴ τοιοῦτον.

B The Aim of Proclus' Commentary

B1 Proclus, Commentary on the First Book of Euclid's *Elements 84.8–23*

Beginning the investigation of particular matters, we inform in advance those who will come across [scil. this book] that they should not demand of us the things that our predecessors overused, namely lemmata, cases, and the like. We are surfeited with these matters and we will seldom treat them. But we will devote the commentary primarily to those matters that are theoretically more important and contribute to the whole of philosophy, thereby emulating the Pythagoreans whose customary proverb was: 'figure and a stepping-stone and not figure and three obols'. By this they indicate that we should pursue the type of geometry that makes each theorem a basis for a step upward and carries the soul to the heights and does not let it descend to the realm of perceptibles, by satisfying the needs of mortals and in aiming at these, neglects to turn around from there.

Ἀρχὴν δὲ ποιούμενοι τῆς τῶν καθ᾽ ἕκαστα ζητήσεως προαγορεύομεν τοῖς ἐντευξομένοις, μὴ ταῦτα παρ᾽ ἡμῶν ἀπαιτεῖν ὅσα διατεθρύληται τοῖς πρὸ ἡμῶν λημμάτια καὶ πτώσεις καὶ εἴ τι τοιοῦτο. τούτων μὲν γὰρ διακορεῖς ἐσμὲν καὶ σπανίως αὐτῶν ἐφαψόμεθα. ὅσα δὲ πραγματειωδεστέραν ἔχει θεωρίαν καὶ συντελεῖ πρὸς τὴν ὅλην φιλοσοφίαν, τούτων προηγουμένην ποιησόμεθα τὴν ὑπόμνησιν, ζηλοῦντες τοὺς Πυθαγορείους, οἷς πρόχειρον ἦν καὶ τοῦτο σύμβολον "σχᾶμα καὶ βᾶμα, ἀλλ᾽ οὐ σχᾶμα καὶ τριώβολον" ἐνδεικνυμένων, ὡς ἄρα δεῖ τὴν γεωμετρίαν ἐκείνην μεταδιώκειν, ἣ καθ᾽ ἕκαστον θεώρημα βῆμα τίθησιν εἰς ἄνοδον καὶ ἀπαίρει τὴν ψυχὴν εἰς ὕψος, ἀλλ᾽ οὐκ ἐν τοῖς αἰσθητοῖς καταβαίνειν ἀφίησιν καὶ τὴν σύνοικον τοῖς θνητοῖς χρείαν ἀποπληροῦν καὶ ταύτης στοχαζομένην τῆς ἐντεῦθεν περιαγωγῆς καταμελεῖν.

B2 Ibid. *200.6–18*

Having briefly examined the account of theorems and problems, their difference, their parts, and the divisions [scil. found] therein, let us turn to the interpretation of [scil. the propositions] proven by the author of the *Elements*. In so doing, we select the more exact comments written about them by the ancients but we cut short their endless verbosity and teach the more technical matters related to scientific methods, thereby paying greater attention to the investigation of important things than to the variety of cases and lemmata, which, we see, attracts for the most part the youth.

νυνὶ δὲ ἀναλαβόντες ἐπὶ βραχὺ τὸν τῶν θεωρημάτων καὶ προβλημάτων λόγον καὶ περὶ τῆς διαφορᾶς αὐτῶν καὶ τῶν ἑκατέρου μερῶν καὶ τῶν ἐν αὐτοῖς διαιρέσεων ἐπὶ τὴν ἐξήγησιν τραπώμεθα τῶν δεικνυμένων ὑπὸ τοῦ στοιχειωτοῦ, τὰ μὲν γλαφυρώτερα τῶν εἰς αὐτὰ γεγραμμένων τοῖς παλαιοῖς ἀναλεγόμενοι καὶ τὴν ἀπέραντον αὐτῶν πολυλογίαν συντέμνοντες, τὰ δὲ τεχνικώτερα καὶ μεθόδων ἐπιστημονικῶν ἐχόμενα παραδιδόντες, τῇ τῶν πραγμάτων ἐπεξεργασίᾳ πλέον ἀπονέμοντες ἢ τῇ ποικιλίᾳ τῶν πτώσεων καὶ τῶν λημμάτων, οἷς ὡς τὸ πολὺ τοὺς νεαροπρεπεῖς ἐπιτρέχοντας ὁρῶμεν.

C Proclus' Exegetical Strategies

C1 Ibid. *213.14–214.15*

It is evident to everyone that the equilateral triangle is the most beautiful triangle and most akin to the circle, which has all its lines [scil. drawn] from the center as equal and one simple line bounds it from without. And it seems that the encompassing by two circles [...] brings to light as in an image how the things that proceed from the principles receive from them perfection, identity, and equality. In this way too, things that move in a straight line turn round in a circle through their eternal generation and the souls having discursive thinking represent through their reversals and returnings to their starting-points the unalterable activity of the intellect. It is also said that the life-giving source of the souls is encompassed by two intellects. Accordingly, if the circle is an image of intelligible being and the triangle is an image of the first soul because of the equality and similarity of the angles and sides, it would be reasonable to demonstrate this [i.e. proposition], which takes an intermediate equilateral [scil. area] in them, through circles. And if every soul proceeds from intellect and reverts to intellect and participates in the intellect in a twofold fashion, it would be well stated that the triangle, being a symbol of the threefold subsistence of the souls, is generated by being enclosed by two circles. But let these [scil. remarks] remind us of the nature of things from likeness.

Τὸ μὲν οὖν ἰσόπλευρον τρίγωνον ὅτι κάλλιστον ἐν τοῖς τριγώνοις καὶ τῷ κύκλῳ συγγενέστατον τῷ πάσας ἴσας ἔχοντι τὰς ἐκ τοῦ κέντρου καὶ μίαν καὶ ἁπλῆν τὴν ἔξωθεν αὐτὸ ὁρίζουσαν γραμμὴν παντὶ καταφανές. ἔοικεν δὲ ἡ τῶν δύο κύκλων περίληψις [...] δηλοῦν ὡς ἐν εἰκόσιν, ὅπως καὶ τὰ προελθόντα ἀπὸ τῶν ἀρχῶν τὸ τέλειον καὶ τὸ ταὐτὸν καὶ τὸ ἴσον ἀπ᾿ ἐκείνων καταδέχεται. κατὰ γὰρ τοῦτον τὸν τρόπον καὶ τὰ ἐπ᾿ εὐθείας κινούμενα κύκλῳ περιάγεται διὰ τῆς ἀεὶ γενεσίας, καὶ αἱ ψυχαὶ μεταβατικὰς ἔχουσαι νοήσεις διὰ τῶν ἀποκαταστάσεων καὶ τῶν περιόδων ἀπεικονίζουσι τὴν ἀμετάβατον ἐνέργειαν τοῦ νοῦ. λέγεται δὲ καὶ ὑπὸ δύο νοῶν ἡ ζωογόνος πηγὴ περιέχεσθαι τῶν ψυχῶν. εἰ τοίνυν ὁ μὲν κύκλος εἰκών ἐστι τῆς νοερᾶς οὐσίας, τὸ δὲ τρίγωνον τῆς πρωτίστης ψυχῆς διά τε τὴν ἰσότητα καὶ τὴν ὁμοιότητα τῶν γωνιῶν καὶ πλευρῶν, εἰκότως ἂν καὶ τοῦτο διὰ τῶν κύκλων ἐν αὐτοῖς μέσον ἀπολαμβανόμενον ἰσόπλευρον ἀποδεικνύοιτο. εἰ δὲ καὶ πᾶσα ψυχὴ πρόεισιν ἀπὸ νοῦ καὶ ἐπιστρέφει πρὸς νοῦν καὶ μετέχει τοῦ νοῦ δυαδικῶς, εὖ ἂν ἔχοι καὶ ταύτῃ τὸ τρίγωνον τῆς τριφυοῦς τῶν ψυχῶν ὑποστάσεως σύμβολον ὂν ὑπὸ δυεῖν κύκλων περιληφθὲν λαμβάνειν τὴν γένεσιν. Ἀλλὰ ταῦτα μὲν ὡς ἐξ εἰκόνων ἡμᾶς ἀναμιμνησκέτω τῆς τῶν πραγμάτων φύσεως.

C2 Ibid. *259.15–262.3*

Lemma (*Elements* I.7): *Given two straight lines constructed on a straight line, there cannot be constructed on the same straight line and on the same side of it two other straight lines equal respectively to the former two and having the same extremities but meeting at a different point.*

Commentary: This theorem has a certain rare quality and it is not at all customary in scientific propositions. For formulating [scil. propositions] negatively and not affirmatively is exceedingly unsuitable to them. At any rate, the propositions of geometrical and arithmetical theorems are for the most part affirmative. The reason is, as Aristotle says, that the universal affirmative [scil. conclusion] is most befitting the sciences because it is more self-sufficient and needs no negative [scil. premise], whereas the universal negative [scil. conclusion] needs also the affirmative, if one intends to make a proof. For without an affirmative [scil. premise] there is neither demonstration nor syllogism, and for this reason, those sciences that are demonstrative prove mostly affirmative [scil. propositions] and rarely use negative conclusions.

And the enunciation of the theorem is complete and of admirable precision and is safeguarded by all the additions needed for rendering it irrefutable and indisputable against those who try to quibble. First, he [i.e., Euclid] states 'on the same straight line', lest we prove that two straight lines equal respectively to the first two [scil. can be constructed] upon another line and by fallacious reasoning mislead those who use this theorem. Second, given one straight line, he does not say that the two lines constructed on it are simply equal to the two straight lines – for this is possible – but that they are equal 'respectively'. Would it be extraordinary to assume that either of the constructed lines is equal to either of the other lines, by lengthening one line and shortening the other? But he says that it is impossible that the lines are equal respectively. Third, he adds 'meeting at another point'. For what if one makes the two other lines equal respectively to the already constructed lines congruent to one another and so construct them on the same given lines and meeting at the same point, i.e., the apex? For if these straight lines are equal, certainly their extremities coincide. Fourth, [scil. he says] 'on the same side', for given one straight line, could we not construct two of the straight lines on one side of the given line and the other two lines on its other side, so that the [scil. given] straight line is a common base of two triangles having opposite apexes? He adds 'on the same side', in order to prevent us from attributing our mistake, if we make it, to the author of the *Elements*. Fifth, he asserts 'having the same extremities as the initially [scil. constructed] straight lines'. For it would be possible to construct upon the same straight line two lines equal respectively to the [scil. other] two lines that meet at different points on the same side [scil. of the given line], using the whole straight line and constructing the two lines upon it, while the constructed lines do not have the same extremities as the other [scil. two lines] but different ones. If we conceive of two diagonals in a square [scil. constructed] on one of the square's sides, there will be two lines equal to other [scil. two lines] – a side and a diagonal, the former being equal to the parallel side and the latter to the other diagonal – but the equal lines will not have the same extremities. For neither the parallel [scil. sides] nor the diagonals, though equal, will have the same extremities. If all these qualifications are maintained, the enunciation is proven to be true and the deduction indisputable.

Prop. VII: *Ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι οὐ σταθήσονται ἑκατέρα ἑκατέρᾳ πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις.*

Ἐστὶ μὲν τὸ θεώρημα τοῦτο σπάνιόν τι πεπονθὸς καὶ οὐ πάνυ ταῖς ἐπιστημονικαῖς προτάσεσιν εἰωθός. τὸ γὰρ ἀποφατικῶς σχηματίζεσθαι καὶ μὴ καταφατικῶς οὐ σφόδρα αὐταῖς οἰκεῖον. τὰ γοῦν πολλὰ καταφάσεις εἰσὶν αἱ προτάσεις τῶν τε γεωμετρικῶν καὶ τῶν ἀριθμητικῶν θεωρημάτων. αἴτιον δέ, ὡς φησὶν Ἀριστοτέλης, ὅτι τὸ καθόλου καταφατικὸν ταῖς ἐπιστήμαις ἐστὶ μάλιστα προσῆκον ὡς αὐταρκέστερον καὶ μηδὲν τῆς ἀποφάσεως προσδεόμενον, τὸ δὲ καθόλου ἀποφατικὸν δεῖται καὶ τῆς καταφάσεως, εἰ μέλλοι δείκνυσθαι. ἄνευ γὰρ καταφάσεως οὔτε ἀπόδειξίς ἐστιν, οὔτε συλλογισμὸς οὐδείς, καὶ διὰ τοῦτο αἱ ἀποδεικτικαὶ τῶν ἐπιστημῶν τὰ μὲν πλεῖστα καταφατικὰ δεικνύουσι, σπανίως δὲ χρῶνται καὶ τοῖς ἀποφατικοῖς συμπεράσμασι.

Θαυμαστῆς δὲ ἀκριβείας ἐστὶν ἡ πρότασις τοῦ θεωρήματος πλήρης καὶ πάσαις ἠσφάλισται ταῖς προσθήκαις, δι᾽ ὧν ἀνέλεγκτος ἀποτετέλεσται καὶ ἀναμφισβήτητος τοῖς συκοφαντεῖν ἐπιχειροῦσι. πρῶτον μὲν γὰρ τὸ ἐπὶ τῆς αὐτῆς εὐθείας εἴληπται, ἵνα μὴ ἐπ᾽ ἄλλης δύο δυσὶν ἴσας δεικνύωμεν ἑκατέραν ἑκατέρᾳ καὶ παραλογιζώμεθα τοὺς τῇ προτάσει χρωμένους. ἔπειτα μιᾶς εὐθείας οὔσης οὐ φησὶν ἐπὶ ταύτην συσταθήσεσθαι τὰς δύο ταῖς δυσὶν ἴσας οὐχ ἁπλῶς – τοῦτο γὰρ δυνατόν – ἀλλ᾽ ἑκατέραν ἑκατέρᾳ. τί γὰρ θαυμαστὸν ἀμφοτέρας ἀμφοτέραις ἴσας λαβεῖν τῶν ἐπισυνισταμένων, τὴν μὲν ἐκτείναντα, τὴν δὲ συστείλαντα· ἀλλ᾽ ἑκατέραν ἑκατέρᾳ φησὶν ἀδύνατον. τρίτον προστίθησιν τὸ πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ. τί γάρ, εἴ τις ταῖς προϋφεστώσαις δύο ποιήσας ἴσας ἄλλας δύο καὶ ἑκατέραν ἑκατέρᾳ ἐφαρμόσειεν ταύτας ἐκείναις καὶ πρὸς τῷ αὐτῷ σημείῳ τῷ κορυφοῦντι τὰς ὑποκειμένας εὐθείας καὶ ταύτας συστήσαιτο; πάντως γὰρ ἴσων οὐσῶν τῶν εὐθειῶν καὶ τὰ πέρατα ἐφαρμόσει. τέταρτον τὸ ἐπὶ τὰ αὐτὰ μέρη. τί γάρ, εἰ μιᾶς εὐθείας ὑποκειμένης τὰς μὲν ἐπὶ τὸ ἕτερον αὐτῆς μέρος ποιήσαιμεν τῶν εὐθειῶν, τὰς δὲ ἐπὶ τὸ ἕτερον, ὥστε τὴν εὐθεῖαν κοινὴν εἶναι βάσιν τριγώνων δυεῖν τὰς κορυφὰς ἀντικειμένας ἐχόντων; ἵνα οὖν μὴ τοῦτο παθόντες τὴν αὐτῶν ἀπάτην ἐπὶ τὸν στοιχειωτὴν μεταγάγωμεν προσέθηκεν τὸ ἐπὶ τὰ αὐτὰ μέρη. πέμπτον ἐπήνεγκε τὸ τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις. καὶ γὰρ ἦν δυνατὸν ἐπὶ τῆς αὐτῆς εὐθείας δύο δυσὶν ἴσας ἑκατέραν ἑκατέρᾳ συστήσασθαι πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη ὅλῃ τῇ εὐθείᾳ χρησάμενον καὶ ἐπὶ ταύτης τὰς δύο συνιστάντα, τῶν συνισταμένων οὐ τὰ αὐτὰ πέρατα ἐκείναις ἐχουσῶν ἀλλὰ ἕτερα. ἐὰν γοῦν νοήσωμεν ἐν τετραγώνῳ δύο διαγωνίους ἐπὶ μιᾶς τῶν τοῦ τετραγώνου πλευρῶν, ἔσονται δύο δυσὶν ἴσαι, πλευρὰ καὶ διάμετρος τῇ παραλλήλῳ πλευρᾷ καὶ τῇ ἑτέρᾳ διαμέτρῳ, ἀλλ᾽ οὐχ αἱ ἴσαι τὰ αὐτὰ πέρατα ἑτέρᾳ διαμέτρῳ, ἀλλ᾽ οὐχ αἱ ἴσαι τὰ αὐτὰ πέρατα ἕξουσιν· οὔτε γὰρ αἱ παράλληλοι οὔτε αἱ διάμετροι τὰ αὐτὰ ἕξουσιν ἀλλήλαις, αὗται δὲ ἦσαν ἴσαι. Τούτων οὖν πάντων τῶν διορισμῶν φυλαττομένων ἥ τε πρότασις ἀληθὴς καὶ ὁ συλλογισμὸς ἀναμφισβήτητος ἀποδείκνυται.

C3 Ibid. *277.16–280.12*

Lemma (*Elements* I.10): *To bisect a given finite straight line.*

Commentary: This problem too posits a finite straight line, since it is by no means possible to bisect [scil. a line] that is infinite on both sides, and if it is infinite only on one side the division is to unequal parts, wherever the point of section is taken; for the part that extends to infinity is greater than the remaining part since the latter is finite. So the alternative left to one who intends to bisect [scil. a line] is to assume that it is bounded on both sides.

From this problem, some people may perhaps be led to suppose that geometers take in advance the hypothesis that a line does not consist of indivisible parts. For if it did, a bounded line would consist of either an odd number of parts or an even number of parts. But if it consists of an odd number of parts, it seems that when the line is bisected, an indivisible part is also bisected since [scil. otherwise] one of its parts would consist of more indivisible parts and would be greater than the other. Consequently, it will be impossible to bisect the given straight line, if indeed its magnitude consists of indivisible parts. But if it does not consist of indivisible parts, it is divisible to infinity. So it seems, they say, that this assumption too, namely that a magnitude consists of parts divisible to infinity, is an agreed geometrical principle. But we at any rate will say what Geminus [scil. has said] about these matters, i.e., that geometers assume, in accordance with common notions, that continuous [scil. magnitudes] are divisible; for we say that continuous [scil. magnitudes] consist of parts that touch one another, and these can certainly be divided. However, geometers do not assume in advance that continuous [scil. magnitudes] are divided to infinity, but they prove it from proper principles. For when they prove that among magnitudes there is an incommensurable magnitude and that not all [scil. magnitudes] are commensurable with one another, what else could one say they are proving but that any magnitude is always divided into parts and that we will never reach an indivisible part, which is the minimal measure of magnitudes? This [scil. statement] is demonstrable, whereas the other one, that states that any continuous [scil. entity] is divisible, hence a bounded continuous line is also divisible, is an axiom. The author of the *Elements* too bisects the bounded straight line on the basis of this notion, but not on the assumption that that it is divisible to infinity. For being divisible and being divisible to infinity are not identical [. . .].

The geometer bisects the bounded line by using the first and the ninth propositions in the construction and only the fourth one in the demonstration. For he proves that the bases are equal through the [scil. bisection] of the angles. But Apollonius of Perga bisects the given bounded straight line in this way [...]. His proof too assumes an equilateral triangle, but instead of proceeding from the bisection of the angle at [scil. the apex] C, it proves that [scil. the line] is bisected on the basis of the equality of the bases. So, the author of the *Elements*' demonstration is much better, simpler, and he proceeds from the principles.

Prop. X: *Τὴν δοθεῖσαν εὐθεῖαν πεπερασμένην δίχα τεμεῖν.*

Πρόβλημα καὶ τοῦτο πεπερασμένην μὲν εὐθεῖαν ὑποτιθέμενον, ἐπειδὴ κατ' ἀμφότερα ἄπειρον οὐδαμῶς ἐστιν ὁρίσαι, τῆς δὲ κατὰ θάτερα μόνον ἀπείρου, ὅπουπερ ἂν ληφθῇ σημεῖον, εἰς ἄνισα ἡ τομὴ γίνεται· μείζων γὰρ ἐξ ἀνάγκης ἡ ἐφ' ἃ ἡ ἄπειρος τῆς λοιπῆς πεπερασμένης οὔσης. λείπεται οὖν ἐπ' ἄμφω πεπερασμένην λαμβάνειν τὴν δίχα τέμνεσθαι μέλλουσαν. ἴσως δ' ἄν τινες ἐκ τούτου κινούμενοι τοῦ προβλήματος ὑπονοήσειαν ὅτι προείληπται παρὰ τοῖς γεωμέτραις ὡς ὑπόθεσις τὸ μὴ εἶναι τὴν γραμμὴν ἐξ ἀμερῶν. εἰ γὰρ εἴη, ἢ ἐκ περιττῶν ἐστιν ἡ πεπερασμένη ἢ ἐξ ἀρτίων. ἀλλ' εἰ ἐκ περιττῶν, ἔοικεν καὶ τὸ ἀμερὲς τέμνεσθαι δίχα τῆς εὐθείας τεμνομένης, ἐπεὶ θάτερον αὐτῆς μέρος ἐκ πλειόνων ἀμερῶν ὑπάρχον ἔσται μεῖζον τοῦ λοιποῦ. οὐκ ἄρα δυνατὸν ἔσται τὴν δοθεῖσαν εὐθεῖαν δίχα τεμεῖν, εἴπερ ἐξ ἀμερῶν τὸ μέγεθος. εἰ δὲ μὴ ἐξ ἀμερῶν, ἐπ' ἄπειρον διαιρεῖται. ἔοικεν οὖν, φασίν, ὡμολογῆσθαι τοῦτο καὶ εἶναι ἀρχὴ γεωμετρικὴ τὸ μέγεθος τῶν εἰς ἄπειρον εἶναι διαιρουμένων. ἡμεῖς δέ γε τὸ τοῦ Γεμίνου πρὸς ταῦτα ἐροῦμεν, ὅτι τὸ μὲν διαιρετὸν εἶναι τὸ συνεχὲς κατὰ κοινὴν ἔννοιαν οἱ γεωμέτραι προλαμβάνουσιν. τοῦτο γὰρ εἶναί φαμεν συνεχὲς τὸ ἐκ μερῶν συνημμένων ὑφεστώς. πάντως δὲ τοῦτο καὶ διαιρεῖσθαι δυνατόν. ὅτι δὲ καὶ ἐπ' ἄπειρον διαιρεῖται τὸ συνεχὲς οὐ προειλήφασιν ἀλλ' ἀποδεικνύουσιν ἐκ τῶν οἰκείων ἀρχῶν. ὅταν γὰρ δεικνύωσιν ὅτι ἔστιν τὸ ἀσύμμετρον ἐν τοῖς μεγέθεσι καὶ οὐ πάντα σύμμετρα ἀλλήλοις, τί ἄλλο δεικνύναι φήσει τις αὐτοὺς ἢ ὅτι πᾶν μέγεθος εἰς ἀεὶ διαιρεῖται καὶ οὐδέποτε ἥξομεν εἰς τὸ ἀμερές, ὅ ἐστι κοινὸν μέτρον τῶν μεγεθῶν ἐλάχιστον. τοῦτο οὖν ἀποδεικτόν ἐστιν, ἐκεῖνο δὲ ἀξίωμα, ὅτι πᾶν συνεχὲς διαιρετόν, ὥστε καὶ ἡ πεπερασμένη γραμμὴ συνεχὴς διαιρετή ἐστιν [...].

τέμνει δὲ δίχα τὴν πεπερασμένην εὐθεῖαν ὁ γεωμέτρης εἰς μὲν τὴν κατασκευὴν τῷ πρώτῳ καὶ τῷ ἐνάτῳ χρώμενος, εἰς δὲ τὴν ἀπόδειξιν τῷ τετάρτῳ μόνῳ. διὰ γὰρ τῶν γωνιῶν δείκνυσιν ἴσας τὰς βάσεις. Ἀπολλώνιος δὲ ὁ Περγαῖος τέμνει τὴν δοθεῖσαν εὐθεῖαν πεπερασμένην δίχα τοῦτον τὸν τρόπον [...] ἡ κατὰ Ἀπολλώνιον τοῦ προκειμένου προβλήματος ἀπόδειξις, ἀπὸ μὲν τοῦ ἰσοπλεύρου τριγώνου καὶ αὐτὴ ληφθεῖσα, ἀντὶ δὲ τοῦ λαβεῖν δίχα τεμνομένην τὴν πρὸς τῷ γ γωνίαν δεικνύουσα ὅτι δίχα τέτμηται διὰ τὴν ἰσότητα τῶν βάσεων. πολλῷ δὴ οὖν κρείττων ἡ τοῦ στοιχειωτοῦ ἀπόδειξις καὶ ἁπλουστέρα καὶ ἀπὸ τῶν ἀρχῶν.

D Commentaries and Philosophical Inquiry

D1 Ibid. *49.24–50.6*

But if the objects of geometry are outside matter, pure *logoi*, and separate from sensibles [...], how then do we still bisect the straight line, the triangle, and the circle? And how do we speak about differences of angles and of the increases and decreases of figures, such as triangular and quadrangular? And how [scil. do we speak] of contacts of circles or lines?

εἴτε ἔξω τῆς ὕλης ἐστὶ τὰ ὑποκείμενα τῇ γεωμετρίᾳ καὶ λόγοι καθαροὶ καὶ χωριστοὶ τῶν αἰσθητῶν [...] πῶς οὖν ἔτι τὴν εὐθεῖαν τέμνομεν καὶ τὸ τρίγωνον καὶ τὸν κύκλον; πῶς δὲ γωνιῶν διαφορὰς λέγομεν καὶ αὐξήσεις αὐτῶν καὶ μειώσεις σχημάτων, οἷον τριγωνικῶν ἢ τετραγωνικῶν; πῶς δὲ τὰς ἁφὰς τῶν κύκλων ἢ τῶν εὐθειῶν;

D2 Ibid. *206.12–26*

We shall find sometimes that what is called 'proof' has the properties of a demonstration, in proving the sought through definitions as middle terms – and this is a perfect demonstration – but sometimes it attempts to prove from signs. This should not be overlooked. For, although geometrical arguments always have their necessity through the underlying matter, they do not always draw their conclusions by means of demonstrative methods. For when it is proved that the interior angles of a triangle are equal to two right angles from the fact that the exterior angle of a triangle is equal to the two opposite interior angles, how can this demonstration be from the cause? How can the middle term be other than a sign? For the interior angles are equal to two right angles even if there are no exterior angles, for there is a triangle even if its side is not extended.

Τὴν δὲ λεγομένην ἀπόδειξιν ὅτε μὲν καὶ τὰ ἴδια τῆς ἀποδείξεως ἔχουσαν εὑρήσομεν ἀπὸ τῶν ὁρισμῶν μέσων τὸ ζητούμενον δεικνύουσαν – αὕτη γὰρ ἀποδείξεως τελειότης – ὅτε δὲ ἐκ τεκμηρίων ἐπιχειροῦσαν. καὶ δεῖ μὴ λανθάνειν. πανταχοῦ μὲν γὰρ τὸ ἀναγκαῖον ἔχουσιν οἱ γεωμετρικοὶ λόγοι διὰ τὴν ὑποκειμένην ὕλην, οὐ πανταχοῦ δὲ περαίνονται διὰ τῶν ἀποδεικτικῶν μεθόδων. ὅταν γὰρ διὰ τοῦ τὴν ἐκτὸς τοῦ τριγώνου γωνίαν ἴσην εἶναι δύο ταῖς ἐντὸς καὶ ἀπεναντίας δεικνύηται τὸ τρίγωνον ἴσας ἔχον τὰς ἐντὸς τρεῖς γωνίας δυσὶν ὀρθαῖς, πῶς ἀπ᾽ αἰτίας ἡ ἀπόδειξις αὕτη, πῶς δὲ οὐχὶ τεκμήριόν ἐστι τὸ μέσον; καὶ γὰρ μήπω τῆς ἐκτὸς οὔσης γωνίας αἱ ἐντὸς οὖσαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. ἔστι γὰρ τὸ τρίγωνον καὶ τῆς πλευρᾶς μὴ ἐκβεβλημένης.

D3 **Ibid.** ***308.14–310.8 (see Figure 2.3, on p. 74)***

By means of this theorem we can also prove [scil. a theorem that states] that if a straight line that falls on two straight lines makes the exterior angle equal to the interior and opposite angle, the lines do not make a triangle nor do they intersect, since the same angle will be greater and equal, which is impossible. Let AB and CD be straight lines and let BE falling on them make the equal angles ABD and CDE. AB and CD do not intersect. For if while the angles remain equal, they intersect, angle CDE will be equal to angle ABD although being an exterior angle it is also greater than the interior and opposite angle. Necessarily then if [scil. the lines] intersect, the angles no longer remain equal but in every case the angle at D increases. For if while line AB remains unmoved, you think of line CD as moving toward it, so that they intersect, you make the interval at angle CDE greater, for the more CD moves toward AB, the more it moves away from DE. And if while line CD remains unmoved, you think of line AB as moving toward it, you make angle ABD lesser, for it moves simultaneously toward CD and toward BD. And if you make both [scil. lines] move toward each other, you will find that in moving toward BD, AB also decreases the angle while, in moving away from DE as a result of the motion toward AB, CD also increases angle CDE. Therefore, if there will be a triangle and lines AB and CD will intersect, of necessity the exterior angle will be greater than the interior and opposite angle [. . .]. The motion of the lines is the cause of these [scil. facts] [. . .] and you can infer from this how the generations of things bring to our view the true causes of the conclusion.

Διὰ τοῦδε δὲ τοῦ θεωρήματος κἀκεῖνο ἀποδείξομεν ὅτι, ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὴν ἐκτὸς γωνίαν ἴσην ποιῇ τῇ ἐντὸς καὶ ἀπεναντίον, οὐ ποιήσουσι τρίγωνον αἱ εὐθεῖαι οὐδὲ συμπεσοῦνται, ἐπεὶ ἔσται αὐτὴ καὶ μείζων καὶ ἴση ὅπερ ἀδύνατον. οἷον ἔστωσαν αἱ αβ γδ εὐθεῖαι καὶ εἰς αὐτὰς ἡ βε ἐμπεσοῦσα ποιείτω ἴσας τὰς ὑπὸ αβδ γδε. οὐ συμπεσοῦνται δὴ αἱ αβ γδ. εἰ γὰρ συμπεσοῦνται μενουσῶν τῶν ἴσων γωνιῶν ἔσται ἡ ὑπὸ γδε, τῇ ὑπὸ αβδ ἴση, ἐκτὸς οὖσα καὶ μείζων τῆς ἐντὸς καὶ ἀπεναντίον. ἀνάγκη ἄρα, εἰ συμπίπτουσιν, μηκέτι μένειν ἴσας τὰς γωνίας, ἀλλὰ ἐκ παντὸς αὔξεσθαι τὴν πρὸς τῷ δ. εἴτε γὰρ μενούσης ἀκινήτου τῆς αβ νοήσειας κινουμένην ἐπ᾽ αὐτὴν τὴν γδ, ἵνα συμπέσωσι, πλείονα ποιεῖς διάστασιν κατὰ τὴν γδε γωνίαν ὅσῳ γὰρ πρόσεισιν ἡ γδ τῇ αβ, τοσούτῳ μᾶλλον ἀφίσταται τῆς δε – εἴτε μενούσης τῆς γδ νοήσειας κινουμένην ἐπ᾽ αὐτὴν τὴν αβ, ἐλάττονα ποιήσεις τὴν αβδ γωνίαν – ἅμα γὰρ ἐπὶ τὴν γδ φέρεται καὶ ἐπὶ τὴν βδ – εἴτε καὶ ἄμφω κινουμένας ποιήσειας πρὸς ἀλλήλας, τὴν μὲν αβ εὑρήσεις ὡς ἐπὶ τὴν βδ φερομένην καὶ συνάγουσαν τὴν γωνίαν, τὴν δὲ γδ τῆς δε ἀφισταμένην διὰ τὴν ἐπὶ τὴν αβ κίνησιν καὶ αὔξουσαν τὴν ὑπὸ γδε γωνίαν. ἐξ ἀνάγκης ἄρα εἴ γε τρίγωνον ἔσται καὶ συμπεσοῦνται αἱ αβ γδ, καὶ μείζων ἡ ἐκτὸς ἔσται γωνία τῆς ἐντὸς καὶ ἀπεναντίον· [. . .] αἰτία δὲ τούτων ἡ κίνησις τῶν εὐθειῶν, τῆς μὲν ἐφ᾽ ἃ ποιεῖ τὴν ἐντὸς γωνίαν κινουμένης ἐπὶ ταῦτα, τῆς δὲ ἐφ᾽ ἃ ποιεῖ τὴν ἐκτὸς γωνίαν ἀπὸ τούτων φερομένης. καὶ ἔχεις ἐκ τούτων συλλογίζεσθαι, πῶς αἱ γενέσεις τῶν πραγμάτων ὑπ᾽ ὄψιν ἡμῖν τὰς ἀληθινὰς ἄγουσι τῶν ζητουμένων αἰτίας.

D4 Ibid. *310.17–311.23*

And the proof of the author of the *Elements* takes a clear path. For it uses the preceding theorem. But, as in the previous theorem, one must discover the cause of the present property by looking at the generation of triangles. So again let lines AB and CD be at a right angle to BD. If then there is to be a triangle, AB and CD must incline toward each other. And the inclination decreases their interior angles, so that they become less than two right angles. For they were right [scil. angles] before the inclination [. . .]. This then is the cause, and not that the exterior angle is greater than each of the interior and opposite angles; for it is not necessary that the side be produced, nor that a certain exterior angle be constructed. But it is necessary that any of the interior angles be less than two right angles. And how can what is not necessary be the cause of what is necessary? Rather, as I said, the cause is the stated [scil. reason], i.e., the inclination of the lines toward the base, which decreases the angles.

καὶ ἡ μὲν τοῦ στοιχειωτοῦ δεῖξις φανερὰν ἔχει τὴν ὁδόν. χρῆται γὰρ τῷ πρὸ τούτου θεωρήματι. δεῖ δὲ καθάπερ ἐν τῷ πρόσθεν εἰς τὴν γένεσιν ἀπιδόντα τῶν τριγώνων τὴν αἰτίαν εὑρεῖν τοῦ προκειμένου συμπτώματος. ἔστωσαν οὖν αἱ αβ πάλιν καὶ γδ τῇ βδ πρὸς ὀρθάς. εἰ οὖν μέλλοι τρίγωνον ἔσεσθαι, δεῖ συνεῦσαι τὰς αβ γδ πρὸς ἀλλήλας. ἡ δὲ σύνευσις αὐτῶν ἐλαττοῖ τὰς ἐντὸς γωνίας, ὥστε ἐλάττους γίνονται δυεῖν ὀρθῶν. εἰσὶ γὰρ ὀρθαὶ πρὸ τῆς συνεύσεως [. . .] Τοῦτο οὖν τὸ αἴτιόν ἐστιν, ἀλλ᾽ οὐχὶ τὸ μείζονα εἶναι τὴν ἐκτὸς ἑκατέρας τῶν ἐντὸς καὶ ἀπεναντίον γωνιῶν. ἐκβεβλῆσθαι μὲν γὰρ τὴν πλευρὰν οὐκ ἀναγκαῖον, οὐδὲ ἔξω τινὰ συνεστάναι γωνίαν, τῶν δὲ ἐντὸς γωνιῶν β ὁποιασοῦν εἶναι [β] ὀρθῶν ἐλάττους ἀναγκαῖον. τὸ δὲ μὴ ἀναγκαῖον τοῦ ἀναγκαίου πῶς ἂν αἴτιον εἴη; ἀλλ᾽ ὅπερ εἶπον τὸ μὲν αἴτιόν ἐστι τὸ ῥηθέν, ἡ σύνευσις τῶν εὐθειῶν ἐπὶ τὴν βάσιν ἐλαττοῦσα τὰς ὀρθάς.

D5 Ibid. *384.5–21*

And we should state also this point, i.e., that having its interior angles equal to two right angles holds for a triangle essentially and by virtue of itself. For this reason, in his work on demonstrations where he examines essential predication Aristotle readily uses it as an example [...]. The truth of this theorem seems to be in line with our common notions. For if we think of a straight line and of lines standing at right angles at its extremities, then, if they incline so that they generate a triangle, we will see that they reduce the right angles that they made with the straight line in proportion to their inclination; the same amount that they have subtracted from these [scil. angles] is added through the inclination to the angle at the vertex, so of necessity they make the three angles equal to two right angles.

Καὶ μὴν κἀκεῖνο ῥητέον, ὅτι τὸ τὰς ἐντὸς γωνίας δύο ὀρθαῖς ἴσας ἔχειν καθ' αὑτὸ καὶ ᾗ αὐτὸ ὑπάρχει τῷ τριγώνῳ· δι' ὃ καὶ Ἀριστοτέλης πρόχειρον ἔχει τὸ παράδειγμα τοῦτο ἐν ταῖς ἀποδεικτικαῖς πραγματείαις τὸ ᾗ αὐτὸ θεωρῶν [...] καὶ ἔοικεν καὶ κατὰ τὰς κοινὰς ἐννοίας προσπίπτειν ἡμῖν ἡ τοῦ θεωρήματος ἀλήθεια. ἐὰν γὰρ νοήσωμεν εὐθεῖαν καὶ ἐπὶ τῶν περάτων αὐτῆς τινας πρὸς ὀρθὰς ἑστώσας, εἶτα συνιούσας εἰς τριγώνου γένεσιν, ὁρῶμεν ὅτι, καθόσον συνεύουσιν, κατὰ τοσοῦτον ἐλαττοῦσι τὰς ὀρθάς, ἃς ἐποίουν πρὸς τῇ εὐθείᾳ, ὥστε ὅσον ἐκείνων ἀφεῖλον, τοσοῦτον προσλαβοῦσαι κατὰ τὴν πρὸς τῇ κορυφῇ σύνευσιν ἐξ ἀνάγκης τὰς τρεῖς ποιοῦσιν δυσὶν ὀρθαῖς ἴσας.

Chapter 2: Glossary of Greek Terms

ἀρχή *archê*: principle. An indemonstrable and fundamental proposition that serves as the basis for theorems and problems proved in an axiomatic-deductive system. In his commentary on the *Elements* Proclus uses this term in referring to the definitions, common notions, and postulates on which Euclid bases his proofs.

κοιναὶ ἔννοιαι *koinai ennoiai*: Common notions. In Euclid's *Elements* this term is used for the axioms: things that are equal to the same thing are equal to one another; if equals are added to equals, the wholes are equal; and if equals are subtracted from equals, the remainders are equal. Following the Hellenistic philosophical tradition, Proclus uses this term more generally in referring to any indemonstrable, evident, and indisputable proposition.

λέξις *lexis*: Literally, word, phrase, or text. In the commentary tradition this term (in the plural, *lexeis*) refers to the detailed discussion of single quotations from the base texts. This discussion follows the general discussion of the doctrine, called *protheôria.*

λῆμμα *lêmma*: lemma. In geometry this term refers to non-proven auxiliary propositions that are used in a proof. These propositions are not proved primarily because their proofs are simple. From Proclus' commentary on the *Elements* we can infer that commentators customarily supplied the proofs of these propositions.

προθεωρία *protheôria*: preliminary discussion. In its technical use in the context of commentaries it refers to a preliminary discussion of a large section of the base text, where commentators offer a general account of the doctrine propounded in this part. This discussion precedes the detailed interpretation of the lemmata, called *lexeis.* The distinction between a preliminary discussion and a detailed discussion does not feature in all commentaries. It does not appear in Proclus' commentary on the first book of Euclid's *Elements.*

πτῶσις *ptôsis*: case. In geometry this term refers to alternative constructions or different diagrams used in proofs of a given theorem or problem.

σκοπός *skopos*: aim. One of the subjects that commentators address in introductions to commentaries is the question of the aim or purpose of the base text. In his commentary on the *Elements* Proclus distinguishes the intrinsic aim of the text, which he identifies with the construction of the Platonic bodies, and its aim in relation to the learner, which is the perfection of discursive reason.