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NOTES IN

MATHEMATICAL LOGIC

ATHENS 2026

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Chapter 1

Introduction

Let us begin by trying to describe what Logic is. It is often said that "Logic studies the laws of thought", but this is misleading. The study of the laws of thought is the subject of Psychology, while Logic is only interested in those thoughts that constitute proofs. In proofs we are not interested in the content, but in the form. To see what we mean by this, let us take a look at the following proofs:

(1) Every man is mortal.
Socrates is mortal.
Therefore, Socrates is a man.

(2) Every horse is an invertebrate animal.
Doris is a horse.
Therefore, Doris is an invertebrate animal.

We observe that both of them have the same form:

Every A is B.
C is A.
C is B.

In the first proof, all three propositions are true, while in the second proof, the first and third propositions are false and the second proposition is true (– let's assume that "Doris" is the name of a horse). What must hold for a proof is the following: If the premisses are true, then the conclusion is also true. Mathematical Logic, which will concern us here, studies mathematical proofs.

As far as we know, Aristotle was the first person to deal systematically with Logic and to formulate inference rules, which Euclid used. Serious logical studies, in a direction different from that of Aristotle and his

students, were carried out by members of the Stoic philosophical school, especially Chrysippus. For many centuries, philosophers dealt essentially only with Aristotelian Logic, which was considered to be a rival to Stoic Logic - after the Renaissance it became clear that these were complementary, not conflicting theories.

Already in antiquity it was seen that the use of natural languages, e.g. Greek, created problems in logical studies. Let us look at an example that comes from the Platonic dialogue *Euthydemus*:

(1) This is a horse.
This is white.
Therefore, this is a white horse.

(2) This dog is a father.
This dog is yours.
Therefore, this dog is your father.

The sentences in group (1) obviously constitute a valid argument, while those in (2) do not, although they appear to have (at first glance) a similar linguistic structure to those in (1).

So it became clear that it was advisable to use symbolic or "formal" languages in Logic. Until the Middle Ages, the use of symbols was very limited - some symbols were used as abbreviations. The use of symbols in Mathematics, which had contributed to its development, led to the introduction of symbols for the study of logical issues. It was found that the use of symbols could increase not only the precision in the formulation of arguments, thus avoiding problems such as the above, but also the generality of the application of methods and rules.

G. Leibniz (1646-1716) was the first to realize the importance of introducing a good symbolism in the context of developing a calculus of arguments. Also B. Bolzano (1781-1848) used a semi-formal language, namely German enriched with various types of constants and variables, to study basic concepts of Logic, such as the concept of logical inference. The founders of "Symbolic" or "Formal" or "Mathematical" Logic are considered to be A. De Morgan (1806-1871) and G. Boole (1815-1864). In 1847, the work *Formal Logic* of the former and the work *Mathematical Analysis of Logic* of the latter were published, with which we consider that modern Logic was born.

At the end of the 19th century, three overlapping schools of Logic emerged: the "algebraic", the "logicist" and the "formalist". To their astonishing flourishing, which lasted into the twentieth century, contributed the discovery of non-Euclidean geometries and paradoxes in Set Theory, which was significantly developed by G. Cantor (1845-1918).

The main representatives of the algebraic school were G. Boole, J. Venn (1834-1923), C. Peirce (1839-1914) and E. Schröder (1841-1902). This school focused its interest on the study of the relationship between numerical operations, such as addition and multiplication, and logical operations. The main goal was to develop calculi with application in various contexts, such as propositions, sets and probabilities. Starting with one or more algebraic systems, algebraists formulated a set of axioms that were true in each of these systems. The system that Boole developed was essentially what we today call "Boolean algebra".

The members of the logicist school believed that Logic was the basis for all kinds of precise argumentation. Its main representatives were G. Frege (1848-1925), perhaps the greatest logician since Aristotle, and B. Russell (1872-1970). Frege developed a rich formal language with mathematical rigor in his *Begriffsschrift* and attempted to show that various parts of Mathematics, such as Number Theory, were actually parts of Logic. Unfortunately, his system was contradictory, as Russell showed with his famous paradox, and had to be rejected. In an attempt to improve the situation, Russell and A. Whitehead (1861-1947) developed the "theory of types" in their *Principia Mathematica*.

The formalist school had as its most important representatives R. Dedekind (1831-1916), G. Peano (1858-1932) and D. Hilbert (1862-1943). The aim of the research of this school was the construction of axiomatic systems for individual branches of Mathematics, e.g. Geometry, Number Theory etc. The basic direction was the same as that of Euclid in the *Elements*, but with an emphasis on the precise formulation of the logical rules that are allowed to be used. Hilbert was the initiator of the so-called "Hilbert's program", the aim of which was to formalize Mathematics and prove that the usual methods of proof cannot lead to contradictions.

Other important logicians in the twentieth century were L. Löwenheim (1878-1957), T. Skolem (1887-1963), A. Tarski (1901-1983), A. Church (1903-1995), K. Gödel (1906-1978) and S. Kleene (1909-1994). Gödel, in particular, is considered the leading logician of this century, since he proved two of the most fundamental theorems of Logic, namely the "completeness theorem of predicate calculus" and the "incompleteness theorem of formal arithmetic".

The formal languages that we will use in the following are the following: the "language of propositional logic" and the "first-order languages". To the language of propositional logic we are led by making a rough abstraction from a natural language, while to first-order languages we are led by making a finer one to the language we use in Mathematics, that is,

to a natural language along with some other symbols. When speaking of formal languages, we are using natural languages. In our study, therefore, we must distinguish between the "object language", which is the formal language that we are studying, and the "meta-language", which is the natural language that we are using in our study. For us, the meta-language is English, along with some symbols and the object-language is one of the formal languages mentioned earlier, and the way of working is the one we use in Mathematics – this is one reason why the Logic we are interested in is called "Mathematical Logic".

A natural language has two sides:

- the "syntactical", i.e., a set of rules describing the permissible ways of constructing words and propositions, without any reference to interpretations of the words and propositions, and
- the "semantic", i.e., the attribution of meaning to words and propositions and then the assignment of truth value ("true" or "false") to propositions.

Exactly the same thing happens with formal languages. The main difference is that with such a language we avoid the ambiguities of a natural language, by strictly defining (in the meta-language) the syntactic and semantic concepts.

When we talk about a symbol (or combination of symbols) we give it a name, which is usually the symbol (or combination of symbols) itself in quotes. For example, "Heraklion" is the name of a Greek word, while Heraklion is the name of a Greek city. Here, where the distinction between the object language and the metalanguage is important, it is very often necessary to use names of symbols (or combinations of symbols) of the object language. For simplicity, we accept the following:

- A symbol (or combination of symbols) of the object language will also be used as its name in the metalanguage.
- As the name of a symbol (or combination of symbols) of the meta-language will be used
  - the symbol itself (or the same combination), if there is no possibility of confusion and
  - the symbol (or combination) within quotation marks, otherwise.

Some of the symbols of formal languages are "variables" that take values in some set of sentences, numbers etc. In the meta-language, we need something similar, and for this we use "syntactic variables" that take values in some set (of combinations) of symbols of the object language.

Chapter 2

Propositional Logic

2.1 The language of propositional logic

This language, denoted with $\mathcal{L}_0$, is poor in comparison with a natural language: it has no letters which can be used to make words and then propositions, but only symbols for basic propositions, which by the use of connecting symbols allow us to construct complex propositions. The set of its symbols has the following elements:

- $p_0, p_1, p_2, ...$, which are called "propositional variables"
- $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, which are called "connectives" – respectively, "negation", "conjunction", "disjunction", "implication", "equivalence"
- (,), which are called "parentheses" – respectively, "left parenthesis" and "right parenthesis".

The set of propositional variables is denoted with $PV(\mathcal{L}_0)$. We will use $p, q, r, ...$ as syntactic variables taking values in $PV(\mathcal{L}_0)$.

The connectives and parentheses are called *logical symbols*, because, as we will see, they have a definite meaning, while the propositional variables are called *non-logical symbols*, because they can be interpreted in various ways.

Having seen the symbols of $\mathcal{L}_0$, we now turn to the rules for constructing combinations of symbols that will be meaningful.

Definition 2.1.1 An *expression* of $\mathcal{L}_0$ is any finite sequence of symbols of $\mathcal{L}_0$, e.g. $\rightarrow p_0 \vee \neg$, $(p_1 \rightarrow p_2$.

$EX(\mathcal{L}_0)$ will denote the set of expressions of $\mathcal{L}_0$; we will be using the letters $\alpha, \beta, \gamma, ...$ as syntactic variables taking values in $EX(\mathcal{L}_0)$.

Of course, only some of the expressions of $\mathcal{L}_0$ will play the role played by propositions in a natural language, namely the ones described in the following definition.

Definition 2.1.2 An expression α is a *propositional formula* iff (that is, if and only if)

- it is a propositional variable or
- it is of the form $(\neg\beta)$, $(\beta \wedge \gamma)$, $(\beta \vee \gamma)$, $(\beta \to \gamma)$, $(\beta \leftrightarrow \gamma)$, where β, γ are already constructed propositional formulas.

Here are some propositional formulas: p_1, $(p_0 \wedge p_2)$, $(\neg p_1)$, $(p_3 \vee (p_5 \leftrightarrow p_0))$. The set of propositional formulas will be denoted with $PF(\mathcal{L}_0)$. The letters $\varphi, \psi, \chi, \ldots$ (attention, the letter χ is different from the letter x) will be used as syntactic variables taking values in $PF(\mathcal{L}_0)$.

The existence of the set of propositional formulas is based on the recursion theorem concerning the set of natural numbers:
If A is an arbitrary set, $a \in A$ and $g : A \to A$, then there exists a function $f : \mathbb{N} \to A$ such that

$$\begin{aligned} & f(0) = a \\ \text{and} \quad & f(n+1) = g(f(n)) \text{ for every } n \in \mathbb{N}. \end{aligned}$$

In this case, we consider the set $\mathcal{P}(EX(\mathcal{L}_0))$, i.e., the powerset of $EX(\mathcal{L}_0)$, its element $PV(\mathcal{L}_0)$ and the function $h : \mathcal{P}(EX(\mathcal{L}_0)) \to \mathcal{P}(EX(\mathcal{L}_0))$ defined as follows:

$$h(X) = \{\alpha : \ \alpha \text{ of the form } (\neg\beta), (\beta \wedge \gamma), (\beta \vee \gamma), (\beta \to \gamma), (\beta \leftrightarrow \gamma), \text{ where } \beta, \gamma \in X\},$$

for every $X \subseteq EX(\mathcal{L}_0)$.
By the recursion theorem, there exists a function $f : \mathbb{N} \to \mathcal{P}(EX(\mathcal{L}_0))$ such that

$$\begin{aligned} & f(0) = PV(\mathcal{L}_0) \\ \text{and} \quad & f(n+1) = f(n) \cup h(f(n)) \text{ for every } n \in \mathbb{N}, \end{aligned}$$

so that the propositional formulas are precisely the elements of the sets $PV(\mathcal{L}_0)$, $h(PV(\mathcal{L}_0))$, $h(PV(\mathcal{L}_0) \cup h(PV(\mathcal{L}_0)))$, ..., that is, the elements of $\bigcup_{n \in \mathbb{N}} f(n)$.

Remark. For convenience, sometimes we use [,] or { , } instead of (,) in propositional formulas.

Since the set $PF(\mathcal{L}_0)$ was defined "recursively", when we want to prove that all propositional formulas have a property, we work "inductively", i.e., we apply the following theorem.

Theorem 2.1.1 (Principle of induction for $PF(\mathcal{L}_0)$) If A is a set of propositional formulas such that

a) $PV(\mathcal{L}_0) \subseteq A$ and

b) for every $\varphi, \psi \in A$, the formulas $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, $(\varphi \leftrightarrow \psi)$ belong to A,

then $A = PF(\mathcal{L}_0)$.

Proof. Supposing that A is a set of propositional formulas for which a) and b) hold, we will show that $PF(\mathcal{L}_0) \subseteq A$. To prove this, by the discussion after Definition 2.1.2, it suffices to show that $B = \mathbb{N}$, where

$$B = \{n \in \mathbb{N} : \text{for every formula } \varphi \in f(n), \varphi \in A\}.$$

Now suppose $B \neq \mathbb{N}$ so that $\mathbb{N} - B$ is a non-empty subset of $\mathbb{N}$. By the least element principle, $\mathbb{N} - B$ has a least element, say n. Then there exists a formula χ such that $\chi \in f(n)$ but $\chi \notin A$. By a), χ is not a propositional variable; hence it is of the form $(\neg\varphi)$ or $(\varphi \wedge \psi)$ or ..., for some φ or φ, ψ belonging to some $f(m)$, for $m < n$. Suppose that it is of the form $(\neg\varphi)$. Then $\varphi \in A$, by the hypothesis about n, which, by b), implies that $(\neg\varphi) \in A$, i.e., $\chi \in A$, which is impossible. Working similarly, we can arrive at a contradiction, if we suppose that χ is of the form $(\varphi \wedge \psi)$ or .... Hence the claim holds. □

Example. We will prove that every propositional formula contains the same number of left and right parentheses. Let A be the set of propositional formulas having this property. Obviously, $PV(\mathcal{L}_0) \subseteq A$ (since every propositional formula contains 0 left parentheses and 0 right parentheses). Suppose now that $\varphi, \psi \in A$. The formula $(\neg\varphi)$ contains one more left and one more right parenthesis than φ, hence it also contains the same number of left and right parentheses; therefore, $(\neg\varphi) \in A$. Similarly, we can check that $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, $(\varphi \leftrightarrow \psi) \in A$ and hence $A = PF(\mathcal{L}_0)$, by the principle of induction.

As a consequence of the example above, we can prove that the way parentheses are used does not lead to ambiguities, in the sense that, for every propositional formula φ, there exists a unique procedure for constructing it from the propositional variables occurring in it. To prove this, we will need the following.

Lemma 2.1.1 For any propositional formula φ, no proper initial segment of φ is a propositional formula.

Proof. Exercise.

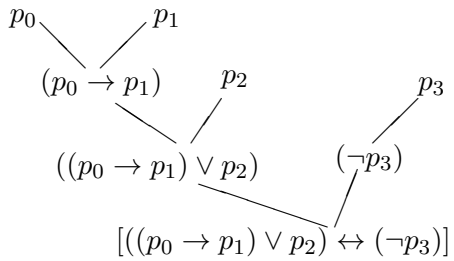
Theorem 2.1.2 (Unique Readability Theorem) For the functions $G_\neg : PF(\mathcal{L}_0) \to PF(\mathcal{L}_0)$, $G_\wedge, G_\vee, G_\to, G_\leftrightarrow : PF(\mathcal{L}_0) \times PF(\mathcal{L}_0) \to PF(\mathcal{L}_0)$ defined as $G_\neg(\varphi) = (\neg\varphi)$, $G_\wedge(\varphi, \psi) = (\varphi \wedge \psi)$ και όμοια για τις $G_\vee$, $G_\to$, $G_\leftrightarrow$, the following are true:

(a) the set $PV(\mathcal{L}_0)$ and the ranges of $G_\neg$, $G_\wedge$ etc. are pairwise disjoint

(b) $G_\neg$, $G_\wedge$ etc. are 1-1 functions.

Proof. (a) Suppose, for example, that for some propositional formulas φ, ψ, χ it is the case that $G_\neg(\varphi) = G_\wedge(\psi, \chi)$, that is, the formula $(\neg\varphi)$ is identical with $(\psi \wedge \chi)$. Then the sequence of symbols $\neg\varphi)$ is identical with the sequence $\psi \wedge \chi)$, hence the first symbol of the formula ψ must be $\neg$, which is impossible (– ψ could have only the form $(\neg\psi_1)$). Hence the intersection of the ranges of $G_\neg$, $G_\wedge$ is empty.
(b) Suppose, for example, that for some propositional formulas φ, ψ, χ, θ we have $G_\to(\varphi, \psi) = G_\to(\chi, \theta)$, that is, the formula $(\varphi \to \psi)$ is identical with $(\chi \to \theta)$. Then the sequence of symbols $\varphi \to \psi)$ is identical with the sequence $\chi \to \theta)$. Comparing now the sequences φ και χ, we distinguish three cases: φ is a proper initial segment of χ, χ is a proper initial segment of φ, φ is identical with χ. But the first two cases are not possible, because of the previous Lemma. Hence φ is identical with χ. Similarly, we can show that ψ is identical with θ, so our claim holds. □

As we mentioned, the previous result guarantees the uniqueness of the way of constructing an arbitrary propositional formula from propositional variables by means of the functions $G_\neg$, $G_\wedge$ etc. In other words, every propositional formula corresponds to a unique "tree diagram", that is, a diagram representing its construction from propositional variables using connectives. For example, the following tree diagram corresponds to the propositional formula $[((p_0 \to p_1) \vee p_2) \leftrightarrow (\neg p_3)]$:



In fact, we can give an algorithm which, given a propositional formula, produces the unique tree diagram corresponding to it. Tree diagrams will be useful for us when we discuss the semantics of $\mathcal{L}_0$.

Many times the parentheses contained in a propositional formula are so many that it is not easy to conceive its structure. For this reason, we omit some of them, according to the following rules:

- The external parentheses of a propositional formula are omitted.
- $\neg$ is considered to be stronger than the other connectives and $\wedge, \vee$ are considered to be stronger than $\rightarrow, \leftrightarrow$.
- $\wedge, \vee$ are considered to be equivalent and the same holds for $\rightarrow, \leftrightarrow$.

Hence, we may write $p_0 \wedge p_1$ instead of $(p_0 \wedge p_1)$, $\neg p_0 \vee p_1$ instead of $((\neg p_0) \vee p_1)$ and $p_2 \wedge p_1 \rightarrow p_0 \vee p_4$ instead of $((p_2 \wedge p_1) \rightarrow (p_0 \vee p_4))$.

Having said enough about the syntax of $\mathcal{L}_0$, we will now discuss its semantics. An interpretation of $\mathcal{L}_0$, which is called "intended interpretation", is the following.

- The propositional variables correspond to elementary (or atomic) propositions of a natural language, that is, minimal propositions which can be characterized as true or false.
- The connectives correspond to the expressions "not", "and", "or", "if ... then ...", "... if and only if ...".
- The parentheses correspond to punctuation marks indicating the beginning and the end of a proposition.

Under this interpretation, the propositional formulas of $\mathcal{L}_0$ correspond to propositions of a natural language that are constructed from elementary propositions by using the expressions "not", "and" etc. For example, if p_0 corresponds to the elementary proposition "2/3 is a rational number", p_3 to "-4 is an integral number" and p_1 to "$\sqrt{7}$ is a real number", then the propositional formula $p_0 \wedge p_3 \rightarrow \neg p_1$ corresponds to the proposition "if 2/3 is a rational number and -4 is an integral number, then $\sqrt{7}$ is not a real number", and $p_0 \rightarrow (p_3 \rightarrow \neg p_1)$ corresponds to the proposition "if 2/3 is an integral number, then $\sqrt{7}$ is not a real number, assuming that -4 is an integral number".

Let us now see how a *truth value* is assigned to a propositional formula.

Definition 2.1.3 *Truth assignment* (or *evaluation*) is a function $a : PV(\mathcal{L}_0) \rightarrow \{1, 0\}$, where 1, 0 are called *truth values* and correspond to the notions "true", "false".

Remark. As the previous definition shows, we focus on two-valued logic. Instead of two, we could consider 3 or 4 or infinitely many truth values, but we will not deal at all with these variations.

Given a truth assignment a, determining the truth value of an arbitrary propositional formula, i.e., finding a function $\overline{a} : PF(\mathcal{L}_0) \to \{1, 0\}$, involves the rules/tables given below. These tables are called *truth tables* and are suggested to us by the intended interpretation of $\mathcal{L}_0$.

$\overline{a}(\varphi)$	$\overline{a}(\neg\varphi)$
1	0
0	1

$\overline{a}(\varphi)$	$\overline{a}(\psi)$	$\overline{a}(\varphi \wedge \psi)$	$\overline{a}(\varphi \vee \psi)$	$\overline{a}(\varphi \to \psi)$	$\overline{a}(\varphi \leftrightarrow \psi)$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

Special attention should be paid to the column corresponding to implication, which is usually called "material implication" – for a discussion, see section 8.3 of [8].

Given a truth assignment a, since, as we saw, the set of propositional formulas was defined recursively by means of the function h, there exists a unique extension $\overline{a}$ of a constructed on the basis of the truth tables of the connectives. The proof is similar to that of the recursion theorem for $\mathbb{N}$ – first we extend a to the set $h(PV(\mathcal{L}_0))$, then to $h(PV(\mathcal{L}_0) \cup h(PV(\mathcal{L}_0)))$ etc., obtaining $\overline{a}$ as the limit of this sequence of functions.

Theorem 2.1.3 Let a be an arbitrary truth assignment. Then there exists a unique function $\overline{a} : PF(\mathcal{L}_0) \to \{1, 0\}$ such that

1) $\overline{a}$ is an extension of a and

2) for any propositional formulas φ, ψ

$$\begin{array}{l} \overline{a}(\neg\varphi) = F_{\neg}(\overline{a}(\varphi)), \\ \overline{a}(\varphi \wedge \psi) = F_{\wedge}(\overline{a}(\varphi), \overline{a}(\psi)), \\ \overline{a}(\varphi \vee \psi) = F_{\vee}(\overline{a}(\varphi), \overline{a}(\psi)), \\ \overline{a}(\varphi \to \psi) = F_{\to}(\overline{a}(\varphi), \overline{a}(\psi)), \\ \overline{a}(\varphi \leftrightarrow \psi) = F_{\leftrightarrow}(\overline{a}(\varphi), \overline{a}(\psi)), \end{array} \qquad (*)$$

where $F_{\neg} : \{1, 0\} \to \{1, 0\}$ and $F_{\wedge}, F_{\vee}, F_{\to}, F_{\leftrightarrow} : \{1, 0\}^2 \to \{1, 0\}$ are the functions corresponding to the truth tables above.

Proof. We call *admissible function* a function $b : T \to \{1, 0\}$ such that $PV(\mathcal{L}_0) \subseteq T \subseteq PF(\mathcal{L}_0)$, b extends a and the following holds

3) for any propositional formulas φ, ψ, if $\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi, \varphi \to \psi, \varphi \leftrightarrow \psi \in T$, then $\varphi, \psi \in T$ and (*) above are true.

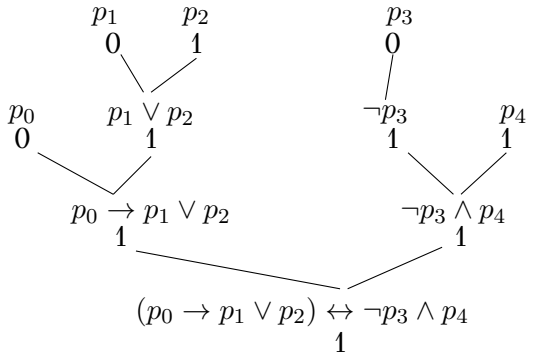
We now consider the set A of admissible functions and define $\overline{a} = \bigcup\{b : b \in A\}$. Using set-theoretical arguments and the unique readability theorem, we can prove that
a) $\overline{a}$ is a function
b) the domain of $\overline{a}$ is $PF(\mathcal{L}_0)$
c) $\overline{a}$ is an admissible function
d) $\overline{a}$ is the only admissible function having $PF(\mathcal{L}_0)$ as domain. □

Given a truth assignment a, to determine $\overline{a}(\varphi)$ we use the tree diagram of φ. Intuitively, it is obvious that $\overline{a}(\varphi)$ depends only on the truth values that a assigns to the finitely many propositional variables occurring in φ. This fact is provable, namely, the following holds.

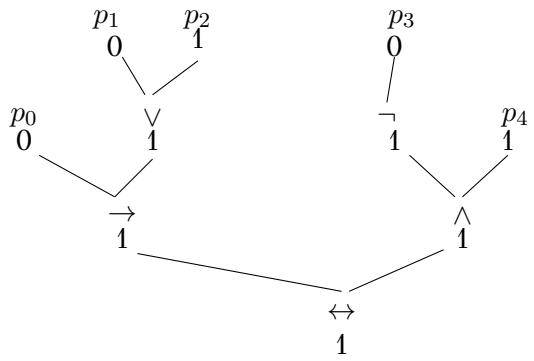
Proposition 2.1.1 For any propositional formula φ and truth assignments a_1, a_2, if a_1, a_2 agree on the propositional variables occurring in φ, then $\overline{a_1}(\varphi) = \overline{a_2}(\varphi)$.

Proof. Exercise 7.

Example. Let a be a truth assignment such that $a(p_2) = a(p_4) = 1$ and $a(p_0) = a(p_1) = a(p_3) = 0$. Then we determine $\overline{a}((p_0 \to p_1 \vee p_2) \leftrightarrow \neg p_3 \wedge p_4)$ as follows:



For simplicity, we mention only propositional variables and connectives:



We have said enough about $\mathcal{L}_0$. Let us see now how we can use it to study proofs.

2.2 Tautological implication

In this section we will use notions that we defined while discussing the semantical side of $\mathcal{L}_0$.

Definition 2.2.1 Let $T \subseteq PF(\mathcal{L}_0)$, $\varphi \in PF(\mathcal{L}_0)$ and a a truth assignment. We say that

- α) a *satisfies* φ iff $\overline{a}(\varphi)=1$ and a *satisfies* T iff a satisfies all elements of T.
- β) T is *satisfiable* iff there exists a truth assignment satisfying it.
- γ) φ is a *tautology* iff every truth assignment satisfies φ.
- δ) φ is a *contradiction* iff $\neg\varphi$ is a tautology.

Definition 2.2.2 Let $T \subseteq PF(\mathcal{L}_0)$ and $\varphi \in PF(\mathcal{L}_0)$. We say that T *tautologically implies* φ, and we write $T \models \varphi$, iff every truth assignment satisfying T satisfies also φ.

Remarks. 1) Obviously $T \models \varphi$ for all $\varphi \in T$.
2) $\emptyset \models \varphi$ iff φ is a tautology, since every truth assignment satisfies $\emptyset$. In other words, tautologies are the tautological consequences of $\emptyset$.
3) If T is not satisfiable, then clearly $T \models \varphi$ for all φ.
4) If $T = \{\psi\}$ and $T \models \varphi$, then we write simply $\psi \models \varphi$ and say that ψ *tautologically implies* φ.
5) If $\varphi \models \psi$ and $\psi \models \varphi$, then we write $\varphi \equiv \psi$ and say that φ, ψ are *tautologically equivalent*.

The next theorem shows that we can use the method of "reduction to the impossible".

Theorem 2.2.1 For any $T \subseteq PF(\mathcal{L}_0)$ and $\varphi \in PF(\mathcal{L}_0)$,

$$T \models \varphi \text{ iff } T \cup \{\neg\varphi\} \text{ is not satisfiable.}$$

Proof. ($\Rightarrow$) Suppose that $T \models \varphi$, but $T \cup \{\neg\varphi\}$ is satisfiable. Then there exists a truth assignment a that satisfies $T \cup \{\neg\varphi\}$. But now we have arrived at a contradiction, because it must be the case that $\overline{a}(\varphi) = 1$, since a satisfies T, and $T \models \varphi$, and, at the same time, $\overline{a}(\varphi) = 0$, since a satisfies $\neg\varphi$. Therefore, the implication holds.
($\Leftarrow$) Suppose that $T \cup \{\neg\varphi\}$ is not satisfiable, but $T \not\models \varphi$. Then there exists a truth assignment which satisfies T but does not satisfy φ. Hence there exists a truth assignment satisfying $T \cup \{\neg\varphi\}$, which is impossible. Therefore, the implication holds. □

The following theorem will be proved later.

Theorem 2.2.2 (Compactness Theorem) Let T be an infinite set of propositional formulas. If every finite subset of T is satisfiable, then T is satisfiable.

In what follows, we will describe an algorithm, on the basis of which we decide whether or not a finite set of propositional formulas tautologically implies a propositional formula. So suppose that we are given a finite $T \subseteq PF(\mathcal{L}_0)$ and a propositional formula φ. Because of Proposition 2.1.1, we are interested only in the truth values that a truth assignment assigns to the propositional variables occurring in elements of $T \cup \{\varphi\}$. Suppose that n different propositional variables, $n > 0$, occur in the elements of $T \cup \{\varphi\}$. We construct a truth table with 2^n rows (why?) and m columns, where $m > 0$ is the number of different propositional formulas involved in the tree diagrams of the elements of $T \cup \{\varphi\}$. After determining the truth values in all columns, we examine all rows to check if every truth assignment satisfying T also satisfies φ or not.

Examples. 1) We will show that $p \wedge q \to r \models p \to (q \to r)$.

p	q	r	$p \wedge q$	$q \to r$	$p \wedge q \to r$	$p \to (q \to r)$
1	1	1	1	1	1	1
1	1	0	1	0	0	0
1	0	1	0	1	1	1
1	0	0	0	1	1	1
0	1	1	0	1	1	1
0	1	0	0	0	1	1
0	0	1	0	1	1	1
0	0	0	0	1	1	1

As we see in the last two rows, the formula $p \to (q \to r)$ takes value 1 every time that $p \wedge q \to r$ takes value 1; hence our claim holds.
2) We will show that the propositional formula $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$ is a tautology (i.e., it is tautologically implied by $\emptyset$).

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$	$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$
1	1	0	0	1	0	0	1
1	0	0	1	0	1	1	1
0	1	1	0	0	1	1	1
0	0	1	1	0	1	1	1

The last column of this truth table shows that the propositional formula we are interested in always takes value 1, hence it is a tautology.

Remark. Sometimes we can attain our aim without using truth tables. Let us see two examples.
(a) To check that the propositional formula

$$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$$

is a tautology, we work as follows:

1) We start assuming p takes truth value 1. Then we see that the propositional formula takes value 1, without having the need to know the truth values of q, r.
2) Then we suppose that p has truth value 0. This is not enough to find the truth value of the whole propositional formula. For this reason, we suppose that also q has truth value 0, in which case the propositional formula takes value 1.
3) Finally, we suppose that p has value 0 and q has value 1. Given the symmetry in the occurrences of q, r, if we suppose that r takes value 0, we are again in case 2. For this reason, we suppose that r has value 1. Then the propositional formula takes again value 1. Schematically:

p	$\vee$	$(q$	$\wedge$	$r)$	$\leftrightarrow$	$(p$	$\vee$	$q)$	$\wedge$	$(p$	$\vee$	$r)$
1	1				1	1	1		1	1	1	
0	0	0	0		1	0	0	0	0	0		
0	1	1	1	1	1	1	1	1	1	0	1	1

(b) Suppose now that we want to check whether or not the propositional formula $(p \wedge q \to r) \to (p \vee q \to r)$ is a tautology. We check whether or not there exists a truth assignment which assigns value 0 to this formula.

Such an assignment would assign value 1 to the formula $p \wedge q \to r$ and value 0 to the formula $p \vee q \to r$. Continuing in the same way, we find a truth assignment a such that $a(p) = 1$ and $a(q) = a(r) = 0$; this assignment assigns value 0 to the propositional formula under examination and hence this formula is not a tautology.

A list of fundamental tautologies, that are called "laws of propositional logic", is the following:

Laws of commutativity:	$\varphi \wedge \psi \leftrightarrow \psi \wedge \varphi$
	$\varphi \vee \psi \leftrightarrow \psi \vee \varphi$
Laws of associativity:	$\varphi \wedge (\psi \wedge \chi) \leftrightarrow (\varphi \wedge \psi) \wedge \chi$
	$\varphi \vee (\psi \vee \chi) \leftrightarrow (\varphi \vee \psi) \vee \chi$
Laws of distributivity:	$\varphi \wedge (\psi \vee \chi) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
	$\varphi \vee (\psi \wedge \chi) \leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \chi)$
Law of double negation:	$\neg\neg\varphi \leftrightarrow \varphi$
Law of negation of implication:	$\neg(\varphi \to \psi) \leftrightarrow \varphi \wedge \neg\psi$
De Morgan's laws:	$\neg(\varphi \wedge \psi) \leftrightarrow \neg\varphi \vee \neg\psi$
	$\neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \wedge \neg\psi$
Law of excluded middle:	$\varphi \vee \neg\varphi$
Law of contraposition:	$(\varphi \to \psi) \leftrightarrow (\neg\psi \to \neg\varphi)$
Law of exportation:	$(\varphi \wedge \psi \to \chi) \leftrightarrow (\varphi \to (\psi \to \chi))$
Laws of replacement:	$(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$
	$(\varphi \leftrightarrow \psi) \leftrightarrow (\varphi \to \psi) \wedge (\psi \to \varphi)$
	$(\varphi \vee \psi) \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$
	$(\varphi \wedge \psi) \leftrightarrow \neg(\neg\varphi \vee \neg\psi)$.

2.3 Complete sets of connectives

Let us consider a language that has as symbols the propositional variables, parentheses and some of the five connectives, perhaps together with other connectives, and which is similar to $\mathcal{L}_0$. Is this new language semantically richer than $\mathcal{L}_0$, poorer than $\mathcal{L}_0$ or equivalent to $\mathcal{L}_0$; Our aim in this section is to answer this question.

First we need a definition.

Definition 2.3.1 Let $k \in \mathbb{N}$. A *Boolean function with* $k+1$ *variables* is a function $f : \{1, 0\}^{k+1} \to \{1, 0\}$.

It is obvious that to any propositional formula with $k+1$ different propositional variables corresponds a Boolean function with $k+1$ variables. Is the converse also true? The answer is "yes", as shown by the next theorem.

Theorem 2.3.1 If f is a Boolean function with $k+1$ variables, $k \in \mathbb{N}$, then there exists a propositional formula φ in which the variables occurring are $p_0, \ldots, p_k$ such that for any truth assignment a the following holds:

$$\overline{a}(\varphi) = f(a(p_0), \ldots, a(p_k)),$$

that is, the truth table of φ completely represents f.

Before proving this theorem, let us see an example that will make the proof more comprehensible. So let f be the Boolean function with three variables that is defined by:

$$\begin{array}{ll} f(0,0,0) = 0 & f(1,0,0) = 1 \\ f(0,0,1) = 1 & f(1,0,1) = 0 \\ f(0,1,0) = 1 & f(1,1,0) = 0 \\ f(0,1,1) = 0 & f(1,1,1) = 1. \end{array}$$

(for simplicity, we write $0,0,0$ instead of $\langle 0,0,0\rangle$ etc.)
First we separate the elements of $\{1,0\}^3$ to which f assigns the value 1. These are the following:

$$\langle 0,0,1\rangle,\ \langle 0,1,0\rangle,\ \langle 1,0,0\rangle,\ \langle 1,1,1\rangle \quad \textbf{(*)}.$$

To each of these ordered triples we correspond a propositional formula in which the only propositional variables occurring are p_0, p_1, p_2, as follows: If the first coordinate of the triple is 1, then we take p_0, while if it is 0, we take $\neg p_0$. Similarly for the other two coordinates of the triple and finally we take the conjunction of the three elementary propositional formulas obtained. So to $\langle 0,0,1\rangle$ corresponds the propositional formula $\neg p_0 \wedge \neg p_1 \wedge p_2$.
The disjunction of the propositional formulas corresponding to the ordered triples (*) is the propositional formula

$$(\neg p_0 \wedge \neg p_1 \wedge p_2) \vee (\neg p_0 \wedge p_1 \wedge \neg p_2) \vee (p_0 \wedge \neg p_1 \wedge \neg p_2) \vee (p_0 \wedge p_1 \wedge p_2),$$

whose truth table realizes completely the function f.

Let us now turn to the proof of the theorem.
Proof. We distinguish two cases.
First case. Suppose that the range of f is $\{0\}$. Then the truth table of the formula

$$(p_0 \wedge \neg p_0) \vee (p_1 \wedge \neg p_1) \vee ... \vee (p_k \wedge \neg p_k)$$

fully describes f.

Second case. Suppose that there are $n+1$ ordered $k+1$-ples, $n, k \in \mathbb{N}$, which f corresponds to 1, say the following

$$< \tau_{00}, \tau_{01}, ..., \tau_{0k} >$$
$$< \tau_{10}, \tau_{11}, ..., \tau_{1k} >$$
$$.....$$
$$< \tau_{n0}, \tau_{n1}, ..., \tau_{nk} >.$$

We claim that the propoitional formula φ that we are looking for is $\psi_0 \vee \psi_1 \vee ... \vee \psi_n$, where $\psi_i = \chi_{i0} \wedge \ldots \wedge \chi_{ik}$, $i = 0, \ldots, n$, and

$$\chi_{ij} = \begin{cases} p_j, & \text{if } \tau_{ij} = 1 \\ \neg p_j, & \text{if } \tau_{ij} = 0. \end{cases}$$

Indeed, let a be an arbitrary truth assignment. If $a(p_0) = \tau_{i0}, \ldots, a(p_k) = \tau_{ik}$ for some i, then

$$\overline{a}(\varphi) = 1 = f(a(p_0), \ldots, a(p_k)).$$

If, on the other hand, $\langle a(p_0), \ldots, a(p_k) \rangle \neq \langle \tau_{i0}, \ldots, \tau_{ik} \rangle$ for all $i = 0, \ldots, n$, then

$$f(a(p_0), \ldots, a(p_k)) = 0 = \overline{a}(\varphi).$$

□

Remarks. a) When the truth table of φ fully describes (or realizes) the Boolean function f, we will say that "φ represents f".
b) There are always more than one propositional formulas representing the same Boolean function f – if φ represents f, then any propositional formula tautologically equivalent to φ also represents f. In the example that we have just given, the formula $p_0 \leftrightarrow (p_1 \leftrightarrow p_2)$ also represents f.

Definition 2.3.2 We will say that a propositional formula is in *disjunctive normal form* iff it is of the form $\psi_0 \vee \psi_1 \vee ... \vee \psi_n$, where ψ_i is a propositional formula of the form $\chi_{i0} \wedge \ldots \wedge \chi_{ik}$, $i = 0, \ldots, n$, and χ_{ij} is a propositional variable or negation of a propositional variable.

An immediate consequence of Theorem 2.3.1 is

Corollary 2.3.1 Every propositional formula is tautologically equivalent to a propositional formula in disjunctive normal form.

Proof. Exercise. □

Now we are able to answer the question we posed at the beginning of this section, concerning two of all possible cases, namely the case in which we take all initial connectives plus some extra ones and the case in which we take only some of the initial connectives.

First case. So suppose that we take, in addition to the initial connectives, a new connective # and develop the new language $\mathcal{L}_0'$ as we did with $\mathcal{L}_0$ (that is, we define propositional formulas, determine the truth table of # etc.).

Let now $\varphi_{\#}$ be a propositional formula of $\mathcal{L}_0'$. Its truth table is fully described by a Boolean function, say f. By Theorem 2.3.1, there exists a propositional formula ψ of $\mathcal{L}_0$ that represents f. Then the formulas $\varphi_{\#}$, ψ have exactly the same truth table, hence they are tautologically equivalent. Therefore $\mathcal{L}_0'$ is semantically equivalent to $\mathcal{L}_0$. Of course, the same holds if we take more than one extra connectives.

Second case. According to Corollary 2.3.1, if we keep only $\neg, \wedge, \vee$, we get the same language. As we will see, it suffices to keep less. However, to proceed we need a definition.

Definition 2.3.3 Let C be a set of connectives. We say that C is *complete* (or *sufficient*) iff every propositional formula is tautologically equivalent to a propositional formula containing only connectives that belong to C.

As we have seen, $\{\neg, \wedge, \vee\}$ is a complete set. Omitting one of the connectives $\wedge, \vee$ we obtain again a complete set.

Proposition 2.3.1 $\{\neg, \wedge\}$, $\{\neg, \vee\}$ are complete sets of connectives.

Proof. We will show that the set $\{\neg, \wedge\}$ is complete – the proof for $\{\neg, \vee\}$ is similar.

So let φ be an arbitrary propositional formula. By Corollary 2.3.1, there exists a propositional formula φ^* in which only the connectives $\neg, \wedge, \vee$ occur and such that $\varphi \equiv \varphi^*$.

The third law of replacement gives $\psi \vee \chi \equiv \neg(\neg\psi \wedge \neg\chi)$. Therefore, starting from "inside out" in φ^*, that is, from the less complex subformulas in which $\vee$ appears and continuing to those of larger complexity, we can get rid of all its occurrences in φ^*. In this way we obtain a propositional formula φ^{**} in which only $\neg, \wedge$ occur and such that $\varphi \equiv \varphi^{**}$ (see Exercise 12). □

Remarks. i) In order to give a proper proof of the former proposition we have to use the principle of induction for $PF(\mathcal{L}_0)$, suitably modified

(that is, instead of $PF(\mathcal{L}_0)$ it holds for the set $PF'(\mathcal{L}_0)$ of all propositional formulas in which the only connectives occurring are $\neg, \wedge, \vee$).
ii) Because of Proposition 2.3.1, we could have taken only two connectives from the beginning and introduced the other three as abbreviations (this is exactly what we will do in predicate logic).

Having a complete set of connectives C and a propositional formula φ, in order to find a formula containing connectives only from C which is tautologically equivalent to φ we use the laws of replacement, following the procedure described in the proof of the previous proposition.
Example. Let us find a propositional formula in which only $\neg, \vee$ appear and which is tautologically equivalent to the formula $p \vee q \to (q \leftrightarrow r)$. Schematically, we proceed as follows:

$p \vee q \to (q \to r) \wedge (r \to q)$	2nd law of replacement
$p \vee q \to (\neg q \vee r) \wedge (\neg r \vee q)$	1st law of replacement
$p \vee q \to \neg(\neg(\neg q \vee r) \vee \neg(\neg r \vee q))$	4th law of replacement
$\neg(p \vee q) \vee \neg(\neg(\neg q \vee r) \vee \neg(\neg r \vee q))$	1st law of replacement.

To show that a set of connectives C is not complete is harder than showing that it is complete. Usually we do the following: We show that the truth table of any propositional formula containg connectives only from C has some specific property. Then we find a propositional formula containing a connective containing a connective not belonging to C and whose truth table does not satisfy this specific property.
Example. We will show that $\{\wedge, \to\}$ is not complete.
We observe that any propositional formula containing only the connectives $\wedge, \to$ takes value 1 when all propositional variables occurring in it take value 1 (– for a proper proof, we use the principle of induction). However, the propositional formula $\neg p_0$ does not satisfy this property.

We will finish this section with a discussion concerning all n-ary connectives, for $n \leq 2$, that we could have chosen. Obviously the truth table of an n-ary connective is fully described by a Boolean function with n variables. But the number of such functions is 2^{2^n} and hence there are 2^{2^n} many possible n-ary connectives.
$\underline{n=0}$. There exist two 0-ary connectives, namely $\top$ and $\bot$, which form by themselves propositional formulas such that, for any truth assignment a, we have that $\overline{a}(\top) = 1$ and $\overline{a}(\bot) = 0$.
$\underline{n=1}$. There exist four 1-ary connectives; two of them correspond to the constant functions with value 1, 0 correspondingly, one corresponds to the identity function on $\{1, 0\}$ and the fourth one is $\neg$, which we have

already seen.
$\underline{n=2}$. There exist sixteen possible 2-ary connectives, out of which only 10 are essentially binary – the remaining ones are essentially 0-ary or 1-ary. Apart from the ones we have already seen, we have to mention $+$, $\downarrow$, $|$, which correspond to the expressions "... or ... but not both" (exclusive disjunction), "neither ... nor ...", "not ... or not ..." and have the following truth tables:

$\overline{a}(\varphi)$	$\overline{a}(\psi)$	$\overline{a}(\varphi + \psi)$	$\overline{a}(\varphi \downarrow \psi)$	$\overline{a}(\varphi \mid \psi)$
1	1	0	0	0
1	0	1	0	1
0	1	1	0	1
0	0	0	1	1

Remarks. i) The connective $\downarrow$ was introduced in 1940 by W. V. Quine (see [25]).
ii) The connective $|$ is called "Sheffer's stroke" and was used for the first time in 1913 by H. M. Sheffer (see [28]). We note that, however, this connective had been studied before by E. Stamm (see [29]).

Example. $\{|\}$ is complete (we suppose that we have modified the definition of the set of propositional formulas etc.).
Obviously, for any propositional formulas φ, ψ:

$$\neg\varphi \equiv \varphi|\varphi \quad \text{and} \quad \varphi \vee \psi \equiv (\varphi|\varphi)|(\psi|\psi).$$

Since $\{\neg, \vee\}$ is complete, this is true also for $\{|\}$.
Similarly, we can show that $\{\downarrow\}$ is complete.

Proposition 2.3.2 For any binary connective $*$ different from $|$, $\downarrow$, the set $\{*\}$ is not complete.

Proof. Suppose that $*$ is a binary connective such that $\{*\}$ is complete, but $*$ differs from $|$ and $\downarrow$.
If $*(1,1) = 1$, then every propositional formula in which only $*$ occurs would take value 1 when all its propositional variables took value 1. But $\neg p_0$ does not satisfy this property. Hence $*(1,1) = 0$.
Similarly we see that it must be the case that $*(0,0) = 1$. Concerning the values $*(1,0)$, $*(0,1)$ there are four cases:

$$\text{a) } 1,\ 1 \quad \text{b) } 1,\ 0 \quad \text{c) } 0,\ 1 \quad \text{d) } 0,\ 0.$$

Since $*$ differs from both $|$ and $\downarrow$, cases a) and d) are excluded. If case c) holds, then $*(\varphi, \psi) \equiv \neg\varphi$. If case b) holds, then $*(\varphi, \psi) \equiv \neg\psi$. So, in all

cases, $*$ would be definable from $\neg$. But $\{\neg\}$ is not complete, and hence $\{*\}$ is not complete. $\square$

Remark. If from the ten essentially binary connectives we exclude $\leftrightarrow$, $+$, anyone of the remaining eight connectives forms together with $\neg$ a complete set.

2.4 Propositional calculus

The following question arises naturally:
Is it possible to systematize tautologies by means of an axiomatic system? In other words, is there a set of axioms and rules, on the basis of which we can produce all tautologies and only these?
Here, of course, the notion of "axiom system" will be defined precisely in the context of the language of propositional logic – usually, axiomatic systems are not presented through formal languages.

Definition 2.4.1 An *axiomatic* (or *formal* or *deductive*) *system* for propositional logic, which we call *propositional calculus*, is composed of the following:

- a set of propositional formulas A and
- a set K whose elements are (binary) relations between sets of propositional formulas and propositional formulas.

The elements of A are called *axioms* and those of K are called *inference rules* (or *deductive rules*). If $\kappa \in K$ and $\tau_1, \ldots, \tau_n, \tau$ are propositional formulas such that $\langle \{\tau_1, \ldots, \tau_n\}, \tau \rangle \in \kappa$, then we say that τ *is a direct consequence of* $\tau_1, ..., \tau_n$ *by* κ.

Definition 2.4.2 Let $\mathcal{A} = \langle A, K \rangle$ be an axiomatic system for propositional calculus, $B \subseteq PF(\mathcal{L}_0)$ and $\tau \in PF(\mathcal{L}_0)$.

- *A formal proof in* $\mathcal{A}$ *from* B is a finite sequence of propositional formulas $\tau_1, \ldots, \tau_n$ such that, for any $i = 1, \ldots, n$, we have
  1) $\tau_i \in A \cup B$ or
  2) τ_i is a direct consequence of previous propositional formulas by some inference rule.

  When $B = \emptyset$, instead of *formal proof in* $\mathcal{A}$ *from* $\emptyset$ we say simply *formal proof in* $\mathcal{A}$.

- We say that τ *is formally provable in* $\mathcal{A}$ *from* B, and denote this by $B \vdash_{\mathcal{A}} \tau$, iff there exists a formal proof $\tau_1, \ldots, \tau_n$ in $\mathcal{A}$ from B such that τ_n is τ.
  When $B \vdash_{\mathcal{A}} \tau$, the elements of B are called *hypotheses* and τ is called *conclusion*.
  If $B = \emptyset$, then we write simply $\vdash_{\mathcal{A}} \tau$ and say that τ *is a formal theorem of* $\mathcal{A}$.

Remarks. 1) The formal proofs that we can construct in $\mathcal{A}$ correspond to proofs we can construct with the propositions of a natural language that we mentioned at the end of the description of the intended interpretation of the formal language $\mathcal{L}_0$. This is the reason for using the term "propositional calculus".
2) It is obvious that if $B \vdash_{\mathcal{A}} \tau$, then $B' \vdash_{\mathcal{A}} \tau$ for some finite set $B' \subseteq B$.
3) It is also obvious that $B \vdash_{\mathcal{A}} \tau$ for all $\tau \in B$.
4) If $B \vdash_{\mathcal{A}} \tau$, then clearly $B' \vdash_{\mathcal{A}} \tau$ for any $B \subseteq B' \subseteq T(\Gamma_0)$.

After these generalities, let us see what is the particular axiomatic system $\mathcal{A}_0 = \langle A_0, K_0 \rangle$ which we will study. A_0 is the set whose elements are all propositional formulas produced by the following *axiom schemas*

$AS1.$ $\varphi \to (\psi \to \varphi)$

$AS2.$ $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

$AS3.$ $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$

and K_0 is the set whose only element is the rule of inference *Modus Ponens* which (we will denote by MP and) is described as follows:

$$\frac{\varphi \to \psi, \ \ \varphi}{\psi}$$

A natural question that arises is the following: Why do only the connectives $\neg, \to$ occur in the axiom schemas of $\mathcal{A}_0$? The answer is that this is not a problem, since, as we have seen, the set $\{\neg, \to\}$ is complete.

Example. The following formal proof in $\mathcal{A}_0$ shows that $\vdash_{\mathcal{A}_0} \varphi \to \varphi$. Note that, for simplicity, when we have a formal proof $\tau_1, \ldots, \tau_n$ in $\mathcal{A}_0$ (from some set B), we write the propositional formulas one under the other, with explanations on the right.

1.	$[\varphi \to ((\varphi \to \varphi) \to \varphi)] \to [(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)]$	$AS2$
2.	$\varphi \to ((\varphi \to \varphi) \to \varphi)$	$AS1$
3.	$(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$	1, 2, MP
4.	$(\varphi \to (\varphi \to \varphi))$	$AS1$
5.	$\varphi \to \varphi$	3, 4, MP.

In the sequel, we will deal with some theorems concerning the axiomatic system $\mathcal{A}_0$. Here we should stress the difference between "formal theorems" and "theorems". A formal theorem is an expression of a formal language, while a theorem is an expression of the meta-language. The difference between a "formal proof" and a "proof" is analogous.

Theorem 2.4.1 (Deduction Theorem) For every $T \subseteq PF(\mathcal{L}_0)$, $\varphi, \psi \in PF(\mathcal{L}_0)$,

$$\text{if } T \cup \{\varphi\} \vdash_{\mathcal{A}_0} \psi, \text{ then } T \vdash_{\mathcal{A}_0} \varphi \to \psi.$$

Proof. Suppose that $T \cup \{\varphi\} \vdash_{\mathcal{A}_0} \psi$, where $T \subseteq PF(\mathcal{L}_0)$ and $\varphi, \psi \in PF(\mathcal{L}_0)$. Then there exists a formal proof $\psi_1, \ldots, \psi_n = \psi$ in $\mathcal{A}_0$ from $T \cup \{\varphi\}$. Using the principle of induction, we will show that for all $i = 1, ..., n$: $T \vdash_{\mathcal{A}_0} \varphi \to \psi_i$.
First step. $i = 1$. We distinguish two cases.
a) $\psi_1 \in A_0 \cup T$.
Then $T \vdash_{\mathcal{A}_0} \varphi \to \psi_1$, as seen from the following formal proof in $\mathcal{A}_0$ from T:

1.	$\psi_1 \to (\varphi \to \psi_1)$	$AS1$
2.	ψ_1	element of $A_0 \cup T$
3.	$\varphi \to \psi_1$	1, 2, MP.

b) ψ_1 is identical with φ.
By the previous example, we know that $\vdash_{\mathcal{A}_0} \varphi \to \varphi$. Hence $T \vdash_{\mathcal{A}_0} \varphi \to \varphi$, that is, $T \vdash_{\mathcal{A}_0} \varphi \to \psi_1$.
Second step. Suppose that for all k, $1 < k < i \leq n$, we have $T \vdash_{\mathcal{A}_0} \varphi \to \psi_k$. We will show that $T \vdash_{\mathcal{A}_0} \varphi \to \psi_i$. Again, we distinguish two cases.
i) $\psi_i \in A_0 \cup T \cup \{\varphi\}$.
Then we obtain what is required as before.
ii) ψ_i is a direct consequence by MP, from ψ_j, ψ_l, $j, l < i$, where, without loss of generality, ψ_l is of the form $\psi_j \to \psi_i$.
Because of the inductive hypothesis, we have that

$$T \vdash_{\mathcal{A}_0} \varphi \to \psi_j \quad \text{and} \quad T \vdash_{\mathcal{A}_0} \varphi \to (\psi_j \to \psi_i).$$

We now construct a formal proof in $\mathcal{A}_0$ from T as follows:

$$\left.\begin{array}{ll} 1. & ..... \\ .. & ..... \\ \lambda. & \varphi \to \psi_j \end{array}\right\} \text{formal proof from } T \text{ in } \mathcal{A}_0$$

$$\left.\begin{array}{ll} .. & ..... \\ .. & ..... \\ \mu. & \varphi \to (\psi_j \to \psi_i) \end{array}\right\} \text{formal proof from } T \text{ in } \mathcal{A}_0$$

μ+1.	$(\varphi \to (\psi_j \to \psi_i)) \to ((\varphi \to \psi_j) \to (\varphi \to \psi_i))$	$AS2$
μ+2.	$(\varphi \to \psi_j) \to (\varphi \to \psi_i)$	μ, μ+1, MP
μ+3.	$\varphi \to \psi_i$	λ, μ+2, MP.

Therefore, $T \vdash_{\mathcal{A}_0} \varphi \to \psi_i$ and the proof of the Theorem is complete. □

Remarks. (i) The deduction theorem (through the intended interpretation of $\mathcal{L}_0$) expresses the following method of proof:
When from hypotheses H we are trying to prove that $A \Rightarrow B$, it suffices to prove B from hypotheses $H \cup \{A\}$.
(ii) It is easy to see that the converse of the Deduction Theorem holds; indeed, suppose that $T \vdash_{\mathcal{A}_0} \varphi \to \psi$. Now observe that the formal proof

$$\left.\begin{array}{ll} 1. & ..... \\ .. & ..... \\ \lambda. & \varphi \to \psi \end{array}\right\} \text{formal proof from } T \text{ in } \mathcal{A}_0$$

λ+1.	φ	element of $T \cup \{\varphi\}$
λ+2.	ψ	λ, λ+1, MP.

shows that $T \cup \{\varphi\} \vdash_{\mathcal{A}_0} \psi$.

Corollary 2.4.1 1) $\{\varphi \to \psi, \psi \to \chi\} \vdash_{\mathcal{A}_0} \varphi \to \chi$.
2) $\{\varphi \to (\psi \to \chi), \psi\} \vdash_{\mathcal{A}_0} \varphi \to \chi$.

Proof.
1) It suffices, by the deduction theorem, to show that

$$\{\varphi \to \psi, \psi \to \chi, \varphi\} \vdash_{\mathcal{A}_0} \chi.$$

But this follows from the next formal proof:

1.	$\varphi \to \psi$	hypothesis
2.	$\psi \to \chi$	hypothesis
3.	φ	hypothesis
4.	ψ	1, 3, MP
5.	χ	2, 4, MP.

2) Similarly. □

In what follows, we will use the theorems that we have proved to shorten formal proofs in $\mathcal{A}_0$. For example, if, in the process of constructing

a formal proof, we have already written the propositional formulas $\varphi \to \psi$, $\psi \to \chi$, then we can add $\varphi \to \chi$ and justify this by saying "Corollary 2.4.1", instead of repeating the corresponding formal proof. By the way, we note that, in systems of *natural deduction*, that are an alternative view of the approach we are following for propositional calculus, there is a rule called *Hypothetical Syllogism* and is depicted as follows:

$$\frac{\begin{array}{c}\varphi \to \psi \\ \psi \to \chi\end{array}}{\varphi \to \chi}$$

We continue with a result that will be very useful later.

Theorem 2.4.2 The following propositional formulas are formal theorems of $\mathcal{A}_0$ (for any propositional formulas φ, ψ):

1) $\neg\neg\varphi \to \varphi$
2) $\varphi \to \neg\neg\varphi$
3) $\neg\varphi \to (\varphi \to \psi)$
4) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$
5) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
6) $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$
7) $\varphi \to (\neg\psi \to \neg(\varphi \to \psi))$
8) $(\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi)$
9) $(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$.

Proof: Exercise 18.

One more consequence of the Deduction Theorem is the following

Theorem 2.4.3 (Contraposition Theorem) For any set of propositional formulas T and any propositional formulas φ, ψ:

$$T \cup \{\varphi\} \vdash_{\mathcal{A}_0} \neg\psi \quad \text{iff} \quad T \cup \{\psi\} \vdash_{\mathcal{A}_0} \neg\varphi.$$

Proof. ($\Rightarrow$) Suppose that $T \cup \{\varphi\} \vdash_{\mathcal{A}_0} \neg\psi$. Then, by the Deduction Theorem, $T \vdash_{\mathcal{A}_0} \varphi \to \neg\psi$. The following formal proof shows that $T \vdash_{\mathcal{A}_0} \psi \to \neg\varphi$:

$$\left.\begin{array}{ll} 1. & \ldots\ldots \\ .. & \ldots\ldots \\ \lambda. & \varphi \to \neg\psi \end{array}\right\} \text{formal proof in } \mathcal{A}_0$$

$\lambda{+}1.$ $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$ Theorem 2.4.2, 6)
$\lambda{+}2.$ $\psi \to \neg\varphi$ $\lambda, \lambda{+}1$, MP.

Using the Deduction Theorem again, we deduce that $T \cup \{\psi\} \vdash_{\mathcal{A}_0} \neg\varphi$.
($\Leftarrow$) Similarly. □

We continue with a result corrsponding to the method of "reduction to the impossible", which is used very often towards constructing proofs. Before we see the theorem, we will define a new notion.

Definition 2.4.3 Let T be a set of propositional formulas. We say that
a) T is *consistent* iff there is no propositional formula φ such that $T \vdash_{\mathcal{A}_0} \varphi$ and $T \vdash_{\mathcal{A}_0} \neg\varphi$
b) T is *contradictory* iff T is not consistent.

We should note that, if T is a contradictory set of propositional formulas, then $T \vdash_{\mathcal{A}_0} \psi$ for any propositional formula ψ. Indeed, suppose that for some propositional formula φ we have that $T \vdash_{\mathcal{A}_0} \varphi$ and $T \vdash_{\mathcal{A}_0} \neg\varphi$. Then, using the formal theorem $\neg\varphi \to (\varphi \to \psi)$ (Theorem 2.4.2, 3)), we can construct a formal proof of ψ from T, for any propositionl formula ψ.

Theorem 2.4.4 (Reduction to the Impossible) For any set of propositional formulas T and any propositional formula φ:

$$\text{if } T \cup \{\varphi\} \text{ is contradictory, then } T \vdash_{\mathcal{A}_0} \neg\varphi.$$

Proof. Suppose that $T \cup \{\varphi\}$ is contradictory. Then for some propositional formula ψ:

$$T \cup \{\varphi\} \vdash_{\mathcal{A}_0} \psi \text{ and } T \cup \{\varphi\} \vdash_{\mathcal{A}_0} \neg\psi.$$

Hence, by the Deduction Theorem,

$$T \vdash_{\mathcal{A}_0} \varphi \to \psi \text{ and } T \vdash_{\mathcal{A}_0} \varphi \to \neg\psi.$$

Using now the formal theorem $(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$ (Theorem 2.4.2, part 9)), we can construct a formal proof in $\mathcal{A}_0$ which shows that $T \vdash_{\mathcal{A}_0} \neg\varphi$. □

We will not continue the study of $\mathcal{A}_0$, for the following reasons:

- $\mathcal{A}_0$ is not the most suitable framework for the study of mathematical proofs, since the language $\mathcal{L}_0$ has limited expressive capability.
- There is an easier way than constructing formal proofs in $\mathcal{A}_0$ to check if a propositional formula is a formal theorem of $\mathcal{A}_0$, namely, checking that this propositional formula is a tautology or not – this fact follows from theorems we will see in the next section.

We will end this section mentioning two sets of axioms A_0', A_0'', each one of which we can take in place of A_0 without changing anything, i.e., the systems $\mathcal{A}_0$, $\mathcal{A}_0' = \langle A_0', K_0\rangle$ and $\mathcal{A}_0'' = \langle A_0'', K_0\rangle$ have the same formal theorems.
The elements of A_0' are all propositional formulas obtained from the following axiom schemas:

$$\begin{array}{l}\varphi \vee \varphi \to \varphi\\ \varphi \to \varphi \vee \psi\\ \varphi \vee \psi \to \psi \vee \varphi\\ (\psi \to \chi) \to (\varphi \vee \psi \to \varphi \vee \chi).\end{array}$$

A_0'' has as elements all propositional formulas of the form

$$(\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\} \to \tau) \to [(\tau \to \varphi) \to (\theta \to \varphi)].$$

Remarks. i) The system A_0' comes from [17].
ii) The axiom schema of the system A_0'' was invented by C. A. Meredith (see [23]). Much later, a shorter axiom schema for propositional calculus was invented (see [21]).

2.5 Soundness and completeness

In this section we will see that the axiomatic system $\mathcal{A}_0$ is *sound*, that is, all its formal theorems are tautologies, and *complete*, that is, all tautologies are formal theorems of the system. In other words, inferring propositional formulas from others syntactically is the same thing as inferring them semantically.

On the basis of the results in section 2.3, we may assume that the only connectives occurring in propositional formulas are $\neg, \to$ (and the induction principle holds with the appropriate modification).

To prove the first theorem, we need the next lemma.

Lemma 2.5.1 Given a truth assignment a, we define the function $^{-a} : PF(\mathcal{L}_0) \to PF(\mathcal{L}_0)$ as follows:

$$\overline{\varphi}^a = \begin{cases} \varphi, & \text{if } \overline{a}(\varphi) = 1\\ \neg\varphi, & \text{otherwise.}\end{cases}$$

Then for any propositional formula φ and any truth assignment a the following holds

$$\{\overline{p_0}^a, ..., \overline{p_k}^a\} \vdash_{\mathcal{A}_0} \overline{\varphi}^a,$$

where $k \in \mathbb{N}$ is such that all propositional variables occurring in φ are among $p_0, \ldots, p_k$.

Proof. We use the principle of induction (on the set of propositional formulas).
1) Let φ be a propositional variable, say p_i. Then what is required holds obviously, since the propositional formula $\overline{\varphi}^a$ is identical with the propositional formula $\overline{p_i}^a$.
2) Now suppose that the property holds for the propositional formulas φ, ψ. We will show that it holds also for the formulas $\neg\varphi$, $\varphi \to \psi$.

a) Obviously, the propositional variables occurring in the propositional formulas φ, $\neg\varphi$ are exactly the same, say they are among p_0, ..., p_k. Now we consider a truth assignment a. By the inductive hypothesis, we have $\{\overline{p_0}^a, \ldots, \overline{p_k}^a\} \vdash_{\mathcal{A}_0} \overline{\varphi}^a$.
If $\overline{a}(\varphi) = 1$, then $\overline{a}(\neg\varphi) = 0$ and hence $\overline{\neg\varphi}^a = \neg\neg\varphi$. But $\vdash_{\mathcal{A}_0} \varphi \to \neg\neg\varphi$, by Theorem 2.4.2. Therefore, $\{\overline{p_0}^a, \ldots, \overline{p_k}^a\} \vdash_{\mathcal{A}_0} \overline{\neg\varphi}^a$.
If $\overline{a}(\varphi) = 0$, then $\overline{a}(\neg\varphi) = 1$ and hence $\overline{\varphi}^a = \neg\varphi$ and $\overline{\neg\varphi}^a = \neg\varphi$. Therefore, $\{\overline{p_0}^a, \ldots, \overline{p_k}^a\} \vdash_{\mathcal{A}_0} \overline{\neg\varphi}^a$.
So in both cases the formula $\neg\varphi$ satisfies the required property.

b) Suppose that the propositional variables occurring in φ are among $p_0, \ldots, p_k$ and those occurring in ψ are among $p_0, \ldots, p_l$, where, without loss of generality, $k \leq l$. Clearly, the propositional variables occurring in $\varphi \to \psi$ are among $p_0, ..., p_l$. We consider a truth assignment a. By the inductive hypothesis, we have

$$\{\overline{p_0}^a, \ldots, \overline{p_k}^a\} \vdash_{\mathcal{A}_0} \overline{\varphi}^a \quad (1)$$
$$\{\overline{p_0}^a, \ldots, \overline{p_l}^a\} \vdash_{\mathcal{A}_0} \overline{\psi}^a \quad (2).$$

By Remark 4) on page 24, (1) implies

$$\{\overline{p_0}^a, \ldots, \overline{p_l}^a\} \vdash_{\mathcal{A}_0} \overline{\varphi}^a \quad (3).$$

If $\overline{a}(\varphi \to \psi) = 0$, in which case $\overline{a}(\varphi) = 1$ and $\overline{a}(\psi) = 0$, we obtain

$$\overline{\varphi \to \psi}^a = \neg(\varphi \to \psi),\ \overline{\varphi}^a = \varphi \text{ and } \overline{\psi}^a = \neg\psi \quad (4).$$

But by part 7) of Theorem 2.4.2, we have that

$$\vdash_{\mathcal{A}_0} \varphi \to (\neg\psi \to \neg(\varphi \to \psi)) \quad (5).$$

From (2), (3), (4) and (5) we deduce that

$$\{\overline{p_0}^a, \ldots, \overline{p_l}^a\} \vdash_{\mathcal{A}_0} \overline{\varphi \to \psi}^a,$$

and so what is required holds.
If $\overline{a}(\varphi \to \psi) = 1$, in which case $\overline{a}(\varphi) = 0$ or $\overline{a}(\psi) = 1$, we will have that

$$\overline{\varphi \to \psi}^a = \varphi \to \psi,\ \overline{\varphi}^a = \neg\varphi \text{ or } \overline{\psi}^a = \psi \quad (6).$$

If $\overline{a}(\varphi) = 0$, then what is required holds by (3), (6) and the fact that $\vdash_{\mathcal{A}_0} \neg\varphi \to (\varphi \to \psi)$ (see 3) of Theorem 2.4.2).
If $\overline{a}(\psi) = 1$, then what is required follows by (2), (6) and the fact that $\vdash_{\mathcal{A}_0} \psi \to (\varphi \to \psi)$ (see $AS1$).

So in all cases what is required holds. □

Remark. Note that, when we apply Lemma 2.5.1 for a given propositional formula φ, we essentially have to consider only assignments for the propositional variables occurring in φ.

Now we have available everything we need to prove the following result.

Theorem 2.5.1 For any propositional formula φ, if $\models \varphi$, then $\vdash_{\mathcal{A}_0} \varphi$.

Proof. Let φ be an arbitrary propositional formula such that $\models \varphi$ and the propositional variables occurring in φ are among $p_0, \ldots, p_k$. Without loss of generality, we may assume that $k>0$.
Because of the previous Lemma, for any truth assignment a we have:

$$\{\overline{p_0}^a, \ldots, \overline{p_k}^a\} \vdash_{\mathcal{A}_0} \varphi \quad (\text{since } \overline{a}(\varphi) = 1).$$

Now consider truth assignments a_1, a_2, concerning $p_0, \ldots, p_k$, such that $a_1(p_k) = 1$, $a_2(p_k) = 0$ and $a_1(p_i) = a_2(p_i)$, for all $0 \leq i \leq k-1$. Then

$$\{\overline{p_0}^{a_1}, \ldots, \overline{p_{k-1}}^{a_1}, p_k\} \vdash_{\mathcal{A}_0} \varphi$$
$$\{\overline{p_0}^{a_2}, \ldots, \overline{p_{k-1}}^{a_2}, \neg p_k\} \vdash_{\mathcal{A}_0} \varphi.$$

Hence, by the Deduction Theorem,

$$\{\overline{p_0}^{a_1}, \ldots, \overline{p_{k-1}}^{a_1}\} \vdash_{\mathcal{A}_0} p_k \to \varphi$$
$$\{\overline{p_0}^{a_2}, \ldots, \overline{p_{k-1}}^{a_2}\} \vdash_{\mathcal{A}_0} \neg p_k \to \varphi.$$

But by Theorem 2.4.2, part 8), $(p_k \to \varphi) \to ((\neg p_k \to \varphi) \to \varphi)$ is a formal theorem of $\mathcal{A}_0$.
Hence for every truth assignment a concerning $p_0, \ldots, p_{k-1}$

$$\{\overline{p_0}^a, \ldots, \overline{p_{k-1}}^a\} \vdash_{\mathcal{A}_0} \varphi.$$

Continuing in the same way, we can eliminate all propositional formulas $\overline{p_0}^a, \ \ldots, \overline{p_{k-1}}^a$ and deduce that $\vdash_{\mathcal{A}_0} \varphi$. □

The following more general result holds (we will sketch the proof in the next chapter).

Theorem 2.5.2 (Completeness of Propositional Calculus) For any set of propositional formulas T and any propositional formula φ,

$$\text{if } T \models \varphi, \text{ then } T \vdash_{\mathcal{A}_0} \varphi.$$

As we mentioned before, the converses of the results above also hold.

Theorem 2.5.3 For any propositional formula φ, if $\vdash_{\mathcal{A}_0} \varphi$, then $\models \varphi$.

Theorem 2.5.4 (Soundness of Propositional Calculus) For any set of propositional formulas T and any propositional formula φ,

$$\text{if } T \vdash_{\mathcal{A}_0} \varphi, \text{ then } T \models \varphi.$$

Proof. Suppose that $T \vdash_{\mathcal{A}_0} \varphi$. Then there exists a formal proof $\varphi_1, ..., \varphi_n$ in $\mathcal{A}_0$, where φ_n is identical with φ. As we will see in Exercise 19, every axiom of $\mathcal{A}_0$ is a tautology and the rule MP preserves tautological implication. Using these facts, we can show by induction that $T \models \varphi_i$, for all $i = 1, \ldots, n$. Hence $T \models \varphi_n$, that is, $T \models \varphi$. □

The previous theorem guarantees that A_0 is not contradictory. Indeed, suppose that there exists a propositional formula φ such that $\vdash_{\mathcal{A}_0} \varphi$ and $\vdash_{\mathcal{A}_0} \neg\varphi$. Then we would have that $\models \varphi$ and $\models \neg\varphi$, which is impossible.

As a corollary of the Completeness and Soundness Theorems we prove the Compactness Theorem that we mentioned in section 2.2.

Proof. Suppose that T is an infinite set of propositional formulas such that every finite subset is satisfiable, but T itself is not satisfiable. Then $T \models p_0 \wedge \neg p_0$ (see Remark after Definition 2.2.2). Hence $T \vdash_{\mathcal{A}_0} p_0 \wedge \neg p_0$ (by the Completeness Theorem), which implies that there exists some finite set $T' \subseteq T$ such that $T' \vdash_{\mathcal{A}_0} p_0 \wedge \neg p_0$. By the Soundness Theorem, it follows that $T' \models p_0 \wedge \neg p_0$ holds. Since T' is satisfiable, there exists a truth assignment a satisfying T'. Then a must satisfy $p_0 \wedge \neg p_0$, which is impossible. □

A consequence of the Completeness and SoundnessTheorems is that there is an algorithm for deciding whether or not φ is formally provable or not in $\mathcal{A}_0$ from some finite set $T \subseteq PF(\mathcal{L}_0)$, namely, the algorithm that decides whether or not $T \models \varphi$. As we will see in the next chapter,

things are not so simple in predicate logic, although in that context, too, a completeness theorem is proved.

EXERCISES

1. Show that there are no propositional formulas with 2, 3 or 6 symbols, but for any other natural number $k > 0$ there exists a propositional formula with k symbols.
2. Let φ be a propositional formula, n the number of instances of binary connectives and m the number of instances of propositional variables occurring in φ. Prove that $m = n+1$.
3. Show that the expression $(p_1 \rightarrow \wedge p_0)$ is not a propositional formula.
4. Show that no proper initial segment of any propositional formula is a propositional formula.
5. Construct the tree diagrams of the propositional formulas

$$[(p_1 \wedge p_0) \rightarrow ((\neg p_2) \vee (p_3 \leftrightarrow p_1))],\ (p_1 \vee (p_2 \leftrightarrow p_0)),$$
$$[((p_0 \wedge p_1) \rightarrow p_2) \rightarrow p_0].$$

6. Let $f, g : PF(\mathcal{L}_0) \rightarrow \mathbb{N}$ be the functions defined as follows:

$f(p) = 0$	$g(p) = 0$
$f(\neg\varphi) = f(\wedge)+1$	$g(\neg\varphi) = g(\varphi)+1$
$f(\varphi \wedge \psi) = f(\varphi)+f(\psi)+1$	$g(\varphi \wedge \psi) = max(g(\varphi), g(\psi))+1$

   and similarly with $\vee, \rightarrow, \leftrightarrow$ in place of $\wedge$.
   a) Find the numbers

$$f((p_1 \rightarrow p_2) \wedge (\neg p_2 \rightarrow p_0)),\ g((p_1 \vee (p_2 \rightarrow p_3)) \wedge (\neg p_3 \vee \neg p_4))).$$

   b) Prove that for any propositional formula φ the following holds: $g(\varphi) \leq f(\varphi)$.
7. Prove Proposition 2.1.1.
8. Check whether or not each of the following propositional formulas tautologically implies the other:

$$p_0 \leftrightarrow (p_1 \leftrightarrow p_2),\ (p_0 \wedge (p_1 \wedge p_2)) \vee (\neg p_0 \wedge (\neg p_1 \wedge \neg p_2)).$$

9. Is the propositional formula $((p_0 \rightarrow p_1) \rightarrow p_0) \rightarrow p_0$ a tautology?

10. Prove that for any $\varphi, \psi \in PF(\mathcal{L}_0)$, $\Sigma \subset PF(\mathcal{L}_0)$ it is true that
    α) $\Sigma \cup \{\varphi\} \models \psi$ iff $\Sigma \models \varphi \to \psi$
    β) $\varphi \equiv \psi$ iff $\models \varphi \leftrightarrow \psi$.

11. Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be a sequence of propositional formulas. For an arbitrary propositional formula φ, φ^* denotes the propositional formula obtained by replacing p_0 with φ_0, p_1 with φ_1, p_2 with φ_2 etc.

    a) Let a be a truth assignment. We consider a truth assignment b such that $b(p_i) = \overline{a}(\varphi_i)$ for $i = 0, 1, 2, \ldots$. Prove that $\overline{b}(\varphi) = \overline{a}(\varphi^*)$.

    b) Prove that if φ is a tautology, then φ^* is a tautology.

12. For an arbitrary propositional formula φ we define recursively the set of *propositional subformulas of* φ, denoted by Sub(φ), as follows:

$$\begin{aligned} &\mathsf{Sub}(p) = \{p\} \\ &\mathsf{Sub}(\neg\varphi) = \mathsf{Sub}(\varphi) \cup \{\neg\varphi\} \\ &\mathsf{Sub}(\varphi \wedge \psi) = \mathsf{Sub}(\varphi) \cup \mathsf{Sub}(\psi) \cup \{\varphi \wedge \psi\} \end{aligned}$$

    and similarly for $\vee, \to, \leftrightarrow$ in place of $\wedge$.
    Let now φ, ψ, χ be propositional formulas such that $\chi \in$ Sub(φ). We define the propositional formula $\varphi[\chi/\psi]$ to be the formula obtained from φ, if we replace every occurrence of χ in φ with ψ. Prove that if $\chi \equiv \psi$, then $\varphi \equiv \varphi[\chi/\psi]$.

13. There are three suspects for a murder, say A, B, C. During interrogation, they say:

    A. I did not commit the murder. The victim was an old acquaintance of B. Furthermore, C hated the victim.
    B. I did not commit the murder. I hardly knew the victim. In any case, I was not in town on the day of the murder.
    C. I did not commit the murder. I saw A and B with the victim on the day of the murder. One of them must have committed the murder.

    Knowing that one of the three committed the murder and that the innocent ones tell the truth, determine the murderer.

14. Prove the following *Duality Principle*:
    Suppose that in φ there are occurrences of only the connectives $\neg, \vee, \wedge$. Let φ^* be the propositional formula obtained from φ by alternating $\wedge, \vee$ and replacing every occurrence of a propositional variable by its negation. Prove that $\neg\varphi \equiv \varphi^*$.

15. Using laws of propositional logic and Exercise 12, simplify the following propositional formulas:

$$[p_1 \wedge ((p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2))] \vee \neg(p_3 \vee \neg p_1),$$
$$(\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi),$$
$$[(\varphi \vee \psi) \wedge (\psi \to \varphi)] \vee [(\psi \wedge \chi) \vee \neg(\psi \to \chi)].$$

16. A prisoner who has been condemned to death is obliged to make a statement. If the statement is true, he will be hanged; if the statement is false, he will be shot. What statement should he make so that to confuse his executors?

17. Which of the following sets are satisfiable and why?

$$PV(\mathcal{L}_0),\ PF(\mathcal{L}_0),\ \{p_0, p_0 \to p_1, \neg p_1\},\ \{p_i \to p_{i+1} : i \in \mathbb{N}\}.$$

18. Construct formal proofs to show that all propositional formulas mentioned in Theorem 2.4.2 are formal theorems of $\mathcal{A}_0$.

19. Check that the axioms of $\mathcal{A}_0$ are tautologies and that the rule MP preserves tautological implication.

20. Prove that the propositional formulas $[\varphi \to (\psi \to \chi)] \leftrightarrow (\varphi \wedge \psi \to \chi)$ and $(\varphi \vee \psi) \wedge (\varphi \to \chi) \wedge (\psi \to \chi) \to \chi$ are formal theorems of $\mathcal{A}_0$.

21. Let f be the Boolean function defined by:

$$\begin{array}{llll} f(0,0,0)=1 & f(0,1,1)=0 & f(1,1,0)=0 & f(1,0,1)=0 \\ f(0,0,1)=1 & f(1,0,0)=1 & f(1,1,1)=0 & f(0,1,0)=1. \end{array}$$

    Find a propositional formula in disjunctive normal form that realizes f. Is there a shorter such formula?

22. Determine the truth tables of the 9 possible binary connectives that we have not mentioned.

23. Prove that $\{\downarrow\}$, $\{\neg, \to\}$ are complete sets of connectives.

24. Prove that the set $\{+, \leftrightarrow\}$ is not complete.

25. Rewrite the following propositional formulas in disjunctive normal form:

$$\neg(p_1 \to p_2) \vee (\neg p_1 \wedge p_3),\ \ p_0 \to ((p_2 \to \neg p_0) \vee p_3).$$

26. a) Prove that for any set of propositional formulas T:

$$T \text{ is consistent iff } T \text{ is satisfiable.}$$

b) Is $\{\neg p_2 \wedge p_3 \rightarrow p_1, p_2 \rightarrow (\neg p_1 \rightarrow p_3), p_1 \leftrightarrow \neg p_3\}$ consistent and why?

27. If $T \subseteq PF(\mathcal{L}_0)$, $\overline{T}$ denotes the set $\{\varphi \in PF(\mathcal{L}_0) : T \vdash_{\mathcal{A}_0} \varphi\}$. Prove that for any sets T, $\Sigma \subseteq PF(\mathcal{L}_0)$ the following hold:

a) $\overline{\overline{T}} = \overline{T}$ b) $\overline{T \cap \Sigma} \subseteq \overline{T} \cap \overline{\Sigma}$
c) $\overline{T} \cup \overline{\Sigma} \subseteq \overline{T \cup \Sigma}$ d) $T(\mathcal{L}_0) - \overline{T} \subseteq \overline{T(\mathcal{L}_0) - T}$.

Find examples for T, Σ for which $\subset$ (proper subset) holds in b) – d).

Chapter 3

Predicate Logic

3.1 First-order languages

As we mentioned before, the language $\mathcal{L}_0$ has very limited expressive abilities and that is why it is reasonable to look for other, more expressive, formal languages. With the symbols of $\mathcal{L}_0$ we were not able to penetrate the interior of propositions of the English language, which is imperative, if we want, for example, to speak in a unified way about sets of similar cases. Suppose we want to express in a formal language that "the number a has the property A" and that "the number b has the property A". In the context of $\mathcal{L}_0$ this can be done through two distinct propositional variables, let say p_0 and p_1, which leads to the loss of the information that we are referring to the numbers a, b with respect to the same property A. However, if there is a symbol in the formal language that represents the property A, we could express the two propositions in a way that highlights their common part. For example, if the symbol P_A corresponds to A, the propositions can be transcribed as $P_A(a)$ and $P_A(b)$ respectively. The so-called *first-order languages* are formal languages that offer these features. Such languages are used to construct axiomatic systems for studying parts of Mathematics, e.g. of Set Theory, Number Theory etc.

The symbols of a first-order language $\mathcal{L}_1$ are the following:

a) $v_0, v_1, v_2, \ldots$, which are called "atomic variables"

b) $\neg, \rightarrow$, which are called "connectives" (respectively, "negation", "implication")

c) (,), which are called "parentheses" (respectively, "left parenthesis", "right parenthesis")

d) $\approx$, which is called "equality"

e) $\forall$, which is called "universal quantifier"

f) for every $n \in \mathbb{N} - \{0\}$, a set (maybe empty) $\{P_{ni} | i \in I_n\}$, whose elements are called "n-ary predicate symbols"

g) for every $n \in \mathbb{N} - \{0\}$, a set (maybe empty) $\{f_{nj} | j \in J_n\}$, the elements of which are called "n-ary function symbols"

h) a set (maybe empty) $\{c_k | k \in K\}$, the elements of which are called "atomic constants".

We denote the set of the atomic variables with $V(\mathcal{L}_1)$ and the set of atomic constants with $C(\mathcal{L}_1)$.
We will be using the following syntactic variables:

$x, y, z, \ldots, x_1, x_2, \ldots$	for atomic variables
$P, Q, R, \ldots$	for predicate symbols
$f, g, h, \ldots$	for function symbols
$c, d, e, \ldots$	for atomic constants.

Remarks. 1) The symbols of categories b) - e) are called "logical symbols", because they will always have the same meaning, based on intuitive logic, while those of categories f) - h) are called "non-logical symbols", because they can be interpreted in several ways. The symbols of categories a) - e) are common to all first-order languages.
2) Some authors include $\approx$ in the non-logical symbols, but this does not cause significant differences in the development of predicate logic. Furthermore, some authors consider constant symbols as 0-ary function symbols. Finally, some authors consider function symbols as a special category of predicate symbols, studying first-order languages without the use of symbols of category g) (such is the approach in [7]).
3) We could allow the inclusion of 0-ary predicate symbols which would correspond to the propositional variables $\mathcal{L}_0$. In this way, we would attain the incorporation of propositional logic in predicate logic – in such a case, of course, we should make all necessary modifications in what will follow.
4) As for $\mathcal{L}_0$, we will use [,] and $\{,\}$ instead of (,).
5) We suppose that every first-order language has at least one predicate or function symbol.
6) Later we will introduce the symbols $\wedge, \vee, \leftrightarrow, \exists$. The reason that their presence is not necessary from the beginning is that the set $\{\neg, \rightarrow, \forall\}$ is complete (we have assumed that we have extended the notion of completeness to apply to sets with connectives and quantifiers as elements).

Obviously a first-order language is defined when its non-logical symbols are defined. Usually the first-order languages we are interested in have a

finite number of symbols of the categories f) - h).
Examples. i) The language $\mathcal{L}_1^{pl}$ of predicate logic has only predicate symbols P_{nm}, with $n, m \in \mathbb{N}-\{0\}$, and atomic constants $c_1, c_2, c_3, \ldots$
ii) The language $\mathcal{L}_1^{st}$ of set theory has only one binary predicate symbol, namely $\in$. Some authors prefer $\mathcal{L}_1^{st}$ to have also the atomic constant $\emptyset$.
iii) The language $\mathcal{L}_1^{nt}$ of number theory has a unary finction symbol $'$, two binary function symbols $\oplus, \odot$ and one atomic constant $\mathbf{0}$. Some authors prefer $\mathcal{L}_1^{nt}$ to have also one binary predicate symbol $\prec$.

In the following we will generally speak of a first-order language $\mathcal{L}_1$, unless we are referring to a specific one.

Definition 3.1.1 *Expression* of $\mathcal{L}_1$ is a finite sequence of its symbols, e.g. $\forall x_0 \approx x_0 x_1$, $(x_1 \rightarrow \neg$.

We denote the set of expressions of $\mathcal{L}_1$ with $EX(\mathcal{L}_1)$ and we use $\alpha, \beta, \gamma, \ldots$ as syntactic variables taking values in the set of expressions.

Let us now see which expressions of $\mathcal{L}_1$ will be meaningful. The relevant definitions will be made recursively, as for $\mathcal{L}_0$.

Definition 3.1.2 We say that the expression α is a *term* iff

1) $\alpha \in V(\mathcal{L}_1) \cup C(\mathcal{L}_1)$ or
2) α is of the form $f\beta_1 \ldots \beta_n$, where f is an n-ary function symbol and $\beta_1, \ldots, \beta_n$ are already constructed terms.

Here are some terms of $\mathcal{L}_1$: c_k, v_3, $f_{2i}v_1v_5$, $f_{2j}f_{2i}v_1v_1c_k$. We denote the set of terms of $\mathcal{L}_1$ with $T(\mathcal{L}_1)$. The letters $t, r, s, \ldots$ are used as syntactic variables with values in $T(\mathcal{L}_1)$. To make the terms more readable, we will write $f(t_1, \ldots, t_n)$ instead of $ft_1 \ldots t_n$.

Definition 3.1.3 We will say that the expression α is a *formula* iff α is of the form

1) $\approx t_1 t_2$, where t_1, t_2 are terms, or
2) $Pt_1 \ldots, t_n$, where P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, or
3) $(\neg\beta)$ or $(\beta \rightarrow \gamma)$ or $\forall x\beta$, where β, γ are already constructed formulas and x is an atomic variable.

A formula is called *atomic* iff it is of the form 1) or 2).

Here are some formulas of $\mathcal{L}_1$: $\approx c_k f_{2j} v_1 v_5$, $(\neg(\approx c_k v_1 \rightarrow P_{2i} c_k v_5))$. We denote the set of atomic formulas of $\mathcal{L}_1$ with $AF(\mathcal{L}_1)$ and the set of formulas with $F(\mathcal{L}_1)$. The letters $\varphi, \psi, \chi, \ldots$ are used as syntactic variables with values in $F(\mathcal{L}_1)$. For the formulas to be more readable, we will write $P(t_1, \ldots, t_n)$ instead of $Pt_1 \ldots t_n$.

Some of the formulas of $\mathcal{L}_1$ will have a specific meaning, while the rest will have meaning under certain conditions. This distinction is based on the following definition.

Definition 3.1.4 Let x be an atomic variable and φ be a formula. We say that

α) x *occurs free in* φ iff

1) φ is atomic and x occurs in φ or
2) φ is of the form $(\neg\psi)$ and x occurs free in ψ or
3) φ is of the form $(\psi \rightarrow \chi)$ and x occurs free in ψ or in χ or
4) φ is of the form $\forall y\psi$, where x is different from y and x occurs free in ψ.

β) x *is bound in* φ iff x does not occur free in φ.

γ) φ is a *sentence* iff all variables are bound in φ.

For example, x occurs free in the formula $(\forall x Q(x) \rightarrow P(x))$, y is bound in the formula $\forall y(P(y) \rightarrow \forall x Q(x))$ and the formula $\forall x \forall y(\approx xy \rightarrow \approx yx)$ is a sentence.
The set of all sentences of $\mathcal{L}_1$ is denoted with $S(\mathcal{L}_1)$.

Remark. According to the previous definition, a variable can occur free in a formula and, at the same time, be bound in a subformula of it.

In analogy with the principle of induction of Chapter 2 we have the

Theorem 3.1.1 (Principle of Induction) a) If $A \subseteq T(\mathcal{L}_1)$ such that

1) $V(\mathcal{L}_1) \cup C(\mathcal{L}_1) \subseteq A$ and

2) $f(t_1, ..., t_n) \in A$, for every n-ary function symbol f and every $t_1, \ldots, t_n \in A$,

then $A = T(\mathcal{L}_1)$.
b) If $A \subseteq F(\mathcal{L}_1)$ such that

1) $AF(\mathcal{L}_1) \subseteq A$ and

2) $(\neg\psi) \in A$, $(\psi \rightarrow \chi) \in A$, $\forall x\psi \in A$, for every $\psi, \chi \in A$ and $x \in V(\mathcal{L}_1)$,

then $A = F(\mathcal{L}_1)$.

As in Chapter 2, we can prove that every formula is uniquely readable, i.e., it is constructed based on a unique tree diagram of atomic formulas using the symbols $\neg, \rightarrow, \forall$.

For convenience, we introduce the symbols $\wedge, \vee, \leftrightarrow, \exists$ as follows:

We will write	$(\varphi \wedge \psi)$	instead of	$(\neg(\varphi \rightarrow (\neg\psi)))$,
	$(\varphi \vee \psi)$	instead of	$((\neg\varphi) \rightarrow \psi)$,
	$(\varphi \leftrightarrow \psi)$	instead of	$((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$, that is
		instead of	$(\neg((\varphi \rightarrow \psi) \rightarrow (\neg(\psi \rightarrow \varphi))))$,
	$\exists x\varphi$	instead of	$(\neg\forall x(\neg\varphi))$.

Also for convenience and for reasons of tradition, we will write

$t_1 \approx t_2$ instead of $\approx t_1 t_2$,	$t_1 \not\approx t_2$ instead of $(\neg\approx t_1 t_2)$
$t_1 \in t_2$ instead of $\in t_1 t_2$,	$t_1 \notin t_2$ instead of $(\neg\in t_1 t_2)$
t_1' instead of $'t_1$,	$(t_1 \oplus t_2)$ instead of $\oplus t_1 t_2$
$(t_1 \odot t_2)$ instead of $\odot t_1 t_2$.	

As we did with propositional formulas, we omit parentheses, according to the following rules:

1) Outer parentheses are omitted.

2) The symbols $\neg, \forall, \exists$ are considered stronger than all the rest and $\wedge, \vee$ stronger than $\rightarrow, \leftrightarrow$.

3) The symbols $\wedge, \vee$ are considered to be equivalent and the same holds for $\rightarrow, \leftrightarrow$.

We also omit the outer parentheses of terms of $\mathcal{L}_1^{nt}$, when there is no possibility of misunderstanding.

Let us now turn to the semantics of a first-order language. Different first-order languages have different "main" (or "intended") interpretations, but they agree on logical symbols. Thus, the main interpretation of logical symbols and the symbols defined by them (i.e. $\wedge, \vee, \leftrightarrow, \exists$) is as follows:

a) $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ correspond to the expressions "not", "and", "or" etc., as for $\mathcal{L}_0$.

b) (,) correspond to punctuation marks, as for $\mathcal{L}_0$.

c) $\approx$ corresponds to the expression "is identical to".

d) $\forall, \exists$ correspond to the expressions "for all", "there exists (at least one)".

Let us see now two examples of intended interpretations of the non-logical symbols of a first-order language.
Example 1. The intended interpretation of the non-logical symbols of $\mathcal{L}_1^{st}$ is the following:

- the variables correspond to sets and
- $\in$ corresponds to the expression "belongs to".

According to this interpretation, the formulas of $\mathcal{L}_1^{st}$ correspond to propositions concerning set theory.
Example 2. The intended interpretation of the non-logical symbols of $\mathcal{L}_1^{nt}$ is the following:

- The variables correspond to natural numbers,
- $'$ corresponds to the expression "successor of",
- $\oplus, \odot$ correspond, respectively, to the expressions "plus", "times",
- $\mathbf{0}$ corresponds to the expression "zero".

According to this interpretation, the formulas of $\mathcal{L}_1^{nt}$ correspond to propositions concerning number theory. For example, if the variables x_1, x_2, x_3, x_4 correspond, repectively, to the natural numbers 4, 1, 2, 3, then the formula $(x_1 \oplus x_2)' \approx x_3 \odot x_4$ corresponds to the proposition "the successor of 4 plus 1 is equal to 2 times 3" and the sentence $\forall x_1 \exists x_2 x_2 \approx x_1'$ to the proposition "for every natural number there exists a natutal number that is equal to the successor of the first".

We continue with some examples of translations of sentences in English in suitable first-order languages.
Examples. a) Let's see which formula of $\mathcal{L}_1^{st}$ corresponds to the proposition "For any two sets, the first is equal to the second if they have the same elements".
We observe that this proposition has the same meaning as "For any two sets, the first is equal to the second if every element of the first is also an element of the second and vice versa".
Finally, ignoring some grammatical elements of the proposition, we see that the formula we are looking for is the following:

$$\forall x_1 \forall x_2 [x_1 \approx x_2 \leftrightarrow \forall x_3 (x_3 \in x_1 \leftrightarrow x_3 \in x_2)].$$

b) The formula of $\mathcal{L}_1^{nt}$ corresponding to the proposition "Every nonzero natural number is equal to the successor of some natural number" is $\forall x_1 [x_1 \not\approx \mathbf{0} \rightarrow \exists x_2 \, x_1 \approx x_2']$.

To see how a truth value is assigned to each formula of $\mathcal{L}_1$ we need some definitions.

Definition 3.1.5 A *structure* (or *interpretation*) $\mathfrak{A}$ for $\mathcal{L}_1$ consists of the following:

a) a non-empty set which is called *universe* or *field* of $\mathfrak{A}$ and is denoted with $|\mathfrak{A}|$ (– in which the variables take values)

b) for every $n \in \mathbb{N}-\{0\}$, a set (maybe empty) $\{P_{ni}^{\mathfrak{A}} | i \in I_n\}$ of n-ary relations on $|\mathfrak{A}|$ – to the symbol P_{ni} corresponds the relation $P_{ni}^{\mathfrak{A}} \subseteq |\mathfrak{A}|^n$

c) for every $n \in \mathbb{N}-\{0\}$, a set (maybe empty) $\{f_{nj}^{\mathfrak{A}} | j \in J_n\}$ of functions with domain the set $|\mathfrak{A}|^n$ and range a subset of the set $|\mathfrak{A}|$ – to the symbol f_{nj} corresponds the function $f_{nj}^{\mathfrak{A}} : |\mathfrak{A}|^n \to |\mathfrak{A}|$

d) a set (maybe empty) $\{c_k^{\mathfrak{A}} | k \in K\}$ of elements of $|\mathfrak{A}|$ – to the symbol c_k corresponds the element $c_k^{\mathfrak{A}}$ of $|\mathfrak{A}|$.

Remark. The languages we study are called *first-order* because they contain variables whose values are elements of a set. Second-order languages have variables whose values are subsets of some set etc.

Examples. 1) The structure $\mathfrak{N}_s$ for $\mathcal{L}_1^{st}$ is defined as follows:

- $|\mathfrak{N}_s| = \mathbb{N}$ and
- to $\in$ corresponds the relation $\in^{\mathfrak{N}_s} = \{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} | m < n\}$.

2) The structure $\mathfrak{N}_s^*$ for $\mathcal{L}_1^{st}$ is defined as follows:

- $|\mathfrak{N}_s^*| = \mathbb{N}$ and
- to $\in$ corresponds the relation $\in^{\mathfrak{N}_s^*} = \{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} | n < m\}$.

3) The structure $\mathfrak{N}$ for $\mathcal{L}_1^{nt}$ is defined as follows:

- $|\mathfrak{N}| = \mathbb{N}$
- to $', \oplus, \odot$ correspond the usual functions of successor, addition and multiplication on $\mathbb{N}$ (denoted by $'^{\mathbb{N}}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}$)
- to $\mathbf{0}$ correponds $0 \in \mathbb{N}$.

4) The structure $\mathfrak{N}^*$ for $\mathcal{L}_1^{nt}$ is defined as follows:

- $|\mathfrak{N}^*| = \{-n | n \in \mathbb{N}\} = \mathbb{N}^*$
- the functions $'^{\mathfrak{N}^*}, \oplus^{\mathfrak{N}^*}, \odot^{\mathfrak{N}^*}$ are defined as follows:

$$'^{\mathfrak{N}^*}(-n) = -(n +^{\mathbb{N}} 1)$$
$$\oplus^{\mathfrak{N}^*}(-m, -n) = -(m +^{\mathbb{N}} n)$$
$$\odot^{\mathfrak{N}^*}(-m, -n) = -(m \cdot^{\mathbb{N}} n)$$

- to $\mathbf{0}$ corresponds $0 \in \mathbb{N}^*$.

A structure for $\mathcal{L}_1$ is therefore a world for which the formulas of $\mathcal{L}_1$ express something. The assignment of truth values to formulas must therefore be made within the framework of a structure.

Definition 3.1.6 Let $\mathfrak{A}$ be a structure for $\mathcal{L}_1$.

1) A *valuation in* $\mathfrak{A}$ is a function $v : V(\Gamma_1) \to |\mathfrak{A}|$. Since the set of terms was defined recursively, we can prove that for any valuation v in $\mathfrak{A}$ there exists a unique function $\overline{v} : T(\mathcal{L}_1) \to |\mathfrak{A}|$ such that
   - $\overline{v}(x) = v(x)$ for every variable x
   - $\overline{v}(c) = c^{\mathfrak{A}}$ for every constant symbol c
   - $\overline{v}(f(t_1, ..., t_n)) = f^{\mathfrak{A}}(\overline{v}(t_1), ..., \overline{v}(t_n))$ for every n-ary function symbol f and arbitrary terms $t_1, ..., t_n$.
2) If v is a valuation in $\mathfrak{A}$, the *value of t for v in $\mathfrak{A}$* is the element $\overline{v}(t)$ of $|\mathfrak{A}|$.
3) If v is a valuation in $\mathfrak{A}$ and $a \in |\mathfrak{A}|$, then $v(x|a)$ denotes the valuation that gives value a to x and agrees with v for all other variables.

Examples.
i) Let us consider the structure $\mathfrak{N}$ for $\mathcal{L}_1^{nt}$ and a valuation v in $\mathfrak{N}$ such that $v(x_1) = 2$, $v(x_2) = 0$, $v(x_3) = 1$. Then $\overline{v}(x_1 \oplus (x_2' \odot x_3)) = \overline{v}(x_1) +^{\mathfrak{N}} \overline{v}(x_2' \odot x_3) = v(x_1) +^{\mathfrak{N}} (\overline{v}(x_2') \cdot^{\mathfrak{N}} \overline{v}(x_3)) = v(x_1) +^{\mathfrak{N}} (\overline{v}(x_2) +^{\mathfrak{N}} 1) \cdot^{\mathfrak{N}} v(x_3)) = 2 +^{\mathbb{N}} (0 +^{\mathbb{N}} 1) \cdot^{\mathbb{N}} 1 = 3$.
ii) Let us consider now the structure $\mathfrak{N}^*$ for $\mathcal{L}_1^{nt}$ and a valuation v in $\mathfrak{N}^*$ such that $v(x_1) = -1$, $v(x_2) = 0$, $v(x_3) = -4$. Then $\overline{v}(x_2'' \odot (x_3 \oplus x_1)) = \overline{v}(x_2'') \odot^{\mathfrak{N}^*} \overline{v}(x_3 \oplus x_1) = (\overline{v}(x_2'))'^{\mathfrak{N}^*} \odot^{\mathfrak{N}^*} (\overline{v}(x_3) \oplus^{\mathfrak{N}^*} \overline{v}(x_1)) = ((\overline{v}(x_2))'^{\mathfrak{N}^*})'^{\mathfrak{N}^*} \odot^{\mathfrak{N}^*} (v(x_3) \oplus^{\mathfrak{N}^*} v(x_1)) = (0'^{\mathfrak{N}^*})'^{\mathfrak{N}^*} \odot^{\mathfrak{N}^*} ((-4) \oplus^{\mathfrak{N}^*} (-1)) = (-1)'^{\mathfrak{N}^*} \odot^{\mathfrak{N}^*} (-5) = (-2) \odot^{\mathfrak{N}^*} (-5) = -10$.

Definition 3.1.7 *(Tarski)* Let $\mathfrak{A}$ be a structure for $\mathcal{L}_1$, φ a formula, T a set of formulas and v a valuation in $\mathfrak{A}$. We say that

1) *v satisfies φ in $\mathfrak{A}$* or *φ is true for v in $\mathfrak{A}$*, denoted by $\mathfrak{A} \models \varphi[v]$, iff
   - $\overline{v}(t_1) = \overline{v}(t_2)$, if φ is of the form $t_1 \approx t_2$
   - $\langle \overline{v}(t_1), ..., \overline{v}(t_n) \rangle \in P^{\mathfrak{A}}$, if φ is of the form $P(t_1, ..., t_n)$
   - it is false that $\mathfrak{A} \models \psi[v]$, if φ is of the form $\neg\psi$
   - if $\mathfrak{A} \models \psi[v]$, then $\mathfrak{A} \models \chi[v]$, if φ is of the form $\psi \to \chi$
   - for every $a \in |\mathfrak{A}|$: $\mathfrak{A} \models \psi[v(x|a)]$, if φ is of the form $\forall x \psi$.

   If φ is not true for v in $\mathfrak{A}$, then we write $\mathfrak{A} \not\models \varphi[v]$.

2) *T is true for v in $\mathfrak{A}$* or *v satisfies T in $\mathfrak{A}$* iff every element of T is true for v in $\mathfrak{A}$.

Remark. From the way we defined $\wedge, \vee, \leftrightarrow, \exists$, it follows that for every structure $\mathfrak{A}$ for $\mathcal{L}_1$ and every valuation v in $\mathfrak{A}$:

$$\begin{array}{lll} \mathfrak{A} \models \psi \wedge \chi[v] & \text{iff} & (\mathfrak{A} \models \psi[v] \text{ and } \mathfrak{A} \models \chi[v]) \\ \mathfrak{A} \models \psi \vee \chi[v] & \text{iff} & (\mathfrak{A} \models \psi[v] \text{ or } \mathfrak{A} \models \chi[v]) \\ \mathfrak{A} \models \psi \leftrightarrow \chi[v] & \text{iff} & (\mathfrak{A} \models \psi[v] \text{ and } \mathfrak{A} \models \chi[v]) \text{ or} \\ & & (\mathfrak{A} \not\models \psi[v] \text{ and } \mathfrak{A} \not\models \chi[v]) \\ \mathfrak{A} \models \exists x \psi[v] & \text{iff} & (\text{there exists } a \in |\mathfrak{A}| : \mathfrak{A} \models \psi[v(x|a)]). \end{array}$$

Examples. a) Let us consider the structure $\mathfrak{N}_s$ for $\mathcal{L}_1^{st}$ that we defined in the first example before Definition 3.1.6, and an evaluation v in $\mathfrak{N}_s$ such that $v(x_1) = 3$, $v(x_2) = 5$. Then $\mathfrak{N}_\sigma \models x_1 \in x_2[v]$, since $\langle v(x_1), v(x_2)\rangle \in \in^{\mathfrak{N}_\sigma}$ (that is, $\langle 3, 5\rangle \in \in^{\mathfrak{N}_\sigma}$).
b) Let us consider now the structure $\mathfrak{N}^*$ for $\mathcal{L}_1^{nt}$ and an evaluation v in $\mathfrak{N}^*$ such that $v(x_2) = 0$. Then $\mathfrak{N}^* \models \forall x_1 x_1 \oplus x_2 \approx x_1[v]$, since for every $n \in \mathbb{N}$ it holds that $(-n) \oplus^{\mathfrak{N}^*} 0 = (-n)$, i.e., $\mathfrak{N}^* \models x_1 \oplus x_2 \approx x_1[v(x_1|-n)]$. If, however, v' is a valuation in $\mathfrak{N}^*$ such that $v'(x_2) = -1$, then $\mathfrak{N}^* \not\models \forall x_1 x_1 \oplus x_2 \approx x_1[v]$, since it is not true that $(-n) \oplus^{\mathfrak{N}^*} (-1) = -n$ for every $n \in \mathbb{N}$.

It is intuitively obvious that whether or not a formula is true for a valuation v in a structure $\mathfrak{A}$ depends on the values that v gives only to the variables that appear free in this formula. This can be proven, as the following result shows.

Proposition 3.1.1 For any formula φ, any structure $\mathfrak{A}$ and any valuations v_1, v_2 in $\mathfrak{A}$, if v_1, v_2 agree on the variables that appear free in φ, then

$$\mathfrak{A} \models \varphi[v_1] \text{ iff } \mathfrak{A} \models \varphi[v_2].$$

An immediate conclusion is that a sentence of $\mathcal{L}_1$ is true for all valuations or none of them (indeed, if φ is any sentence, then there are no variables that occur free in φ, so all valuations agree on the variables that occur free in φ). This fact leads us to the following definition.

Definition 3.1.8 Let φ be a sentence of $\mathcal{L}_1$, S be a set of sentences of $\mathcal{L}_1$ and $\mathfrak{A}$ a structure for $\mathcal{L}_1$. We say that

- φ is *true in* $\mathfrak{A}$ or $\mathfrak{A}$ is a *model of* φ, denoted by $\mathfrak{A} \models \varphi$, iff φ is true for all evaluations in $\mathfrak{A}$.
- φ is *false in* $\mathfrak{A}$, denoted by $\mathfrak{A} \not\models \varphi$, iff φ is not true in $\mathfrak{A}$.
- $\mathfrak{A}$ is *model of* S iff $\mathfrak{A}$ is a model of every element of S.

Example. We will show that the sentence $\exists x_0 \forall x_1 x_0 \odot x_1 \approx x_1$ is true in the structure $\mathfrak{N}$ for $\mathcal{L}_1^{nt}$. Let v be an arbitrary valuation in $\mathfrak{N}$. Then
$\mathfrak{N} \models \exists x_0 \forall x_1 x_0 \odot x_1 \approx x_1[v]$ iff
there exists $m \in \mathbb{N} : \mathfrak{N} \models \forall x_1 x_0 \odot x_1 \approx x_1[v(x_0|m)]$ iff
there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$:

$$\mathfrak{N} \models x_0 \odot x_1 \approx x_1[v(x_0|m, x_1|n)] \text{ iff}$$

there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$:

$$\overline{v(x_0|m, x_1|n)}(x_0 \odot x_1) = \overline{v(x_0|m, x_1|n)}(x_1) \text{ iff}$$

there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N} : m \odot^{\mathfrak{N}} n = n$, which is true.

Let us not forget, however, that our purpose is to use first-order languages for the study of mathematical proofs. Here, as in propositional logic, there are two equivalent ways to work: the semantic and the syntactic. As in propositional logic, we will begin with the semantic approach.

3.2 Logical implications

Definition 3.2.1 Let T be a set of formulas and φ be a formula of $\mathcal{L}_1$. We say that

1) T *logically implies* φ, denoted by $T \models \varphi$, iff for every structure $\mathfrak{A}$ and every valuation v in $\mathfrak{A}$, if T is true for v, then φ is also true for v.
2) φ is *valid* or φ is *logically true* iff for every structure $\mathfrak{A}$ and every valuation v in $\mathfrak{A}$, φ is true for v.

Remarks. a) For every φ it holds that $\emptyset \models \varphi$ iff φ is valid, that is, the valid formulas are exactly the logical consequences of the $\emptyset$ (they correspond, that is, to tautologies).
b) The notions φ *logically implies* ψ and φ, ψ *are logically equivalent* are defined as their counterparts in propositional logic.
c) Combining Definitions 3.1.8 and 3.2.1, we see that for every set S of sentences and sentence φ of $\mathcal{L}_1$:

$$S \models \varphi \text{ iff every model of } S \text{ is a model of } \varphi.$$

Now let us see some examples of logical implications.
Examples. Let Q be a unary predicate symbol.
1) We will show that $\forall x_1 Q(x_1) \models Q(x_2)$. So let $\mathfrak{A}$ be a structure and v a valuation in $\mathfrak{A}$ such that $\mathfrak{A} \models \forall x_1 Q(x_1)[v]$. Then for every $a \in |\mathfrak{A}| : \mathfrak{A} \models Q(x_1)[v(x_1|a)]$, that is, for every $a \in |\mathfrak{A}| : a \in Q^{\mathfrak{A}}$. Obviously $v(x_2) \in Q^{\mathfrak{A}}$

and so $\mathfrak{A} \models Q(x_2)[v]$, that is, the claim holds.
2) We will show that $Q(x_1) \not\models \forall x_1 Q(x_1)$. This time our task is a little different, namely to find a structure $\mathfrak{A}$ and a valuation v in $\mathfrak{A}$ such that $Q(x_1)$ is true for v in $\mathfrak{A}$, but $\forall x_1 Q(x_1)$ is not true for v in $\mathfrak{A}$. So let $\mathfrak{A}$ be a structure with universe the set $\{a_0, a_1\}$ (where $a_0 \neq a_1$) and $Q^{\mathfrak{A}} = \{a_0\}$. Now let v be a valuation $\mathfrak{A}$ such that $v(x_1) = a_0$. Then $\mathfrak{A} \models Q(x_1)[v]$, since $v(x_1) \in Q^{\mathfrak{A}}$, but $\mathfrak{A} \not\models \forall x_1 Q(x_1)[v]$, since $a_1 \notin Q^{\mathfrak{A}}$. We should note that, instead of $\mathfrak{A}$ we can take any structure $\mathfrak{B}$ such that the set $|\mathfrak{B}|$ has at least two elements and $Q^{\mathfrak{B}} \subset |\mathfrak{B}|$.
3) We will show that $\models \forall x_1 Q(x_1) \to \exists x_2 Q(x_2)$. So let $\mathfrak{A}$ be a structure and v a valuation in $\mathfrak{A}$ such that $\mathfrak{A} \models \forall x_1 Q(x_1)[v]$. Then for every $a \in |\mathfrak{A}| : \mathfrak{A} \models Q(x_1)[v(x_1|a)]$, that is, for every $a \in |\mathfrak{A}| : a \in Q^{\mathfrak{A}}$. Let a_0 be an arbitrary element of $|\mathfrak{A}|$. Then $a_0 \in Q^{\mathfrak{A}}$ and hence $\mathfrak{A} \models Q(x_2)[v(x_2|a_0)]$. Therefore, there exists $a \in |\mathfrak{A}| : \mathfrak{A} \models Q(x_2)[v(x_2|a)]$ and hence $\mathfrak{A} \models \exists x_2 Q(x_2)$, that is, the claim holds.
4) We consider the following propositions in English:

No regular number is transcendental.
Some peculiar numbers are transcendental.
Some regular numbers are peculiar.

Using corresponding sentences of a first-order language, we will show that the first two do not logically entail the third. Let $\mathcal{L}_1$ be a first-order language that has at least three unary predicate symbols P, Q, R. The following sentences of $\mathcal{L}_1$ correspond to the sentences in English above:

$$\forall x_1(P(x_1) \to \neg Q(x_1)),\ \exists x_1(R(x_1) \wedge Q(x_1)),\ \exists x_1(P(x_1) \wedge R(x_1)).$$

We will show that

$$\{\forall x_1(P(x_1) \to \neg Q(x_1)), \exists x_1(R(x_1 \wedge Q(x_1))\} \not\models \exists x_1(P(x_1) \wedge R(x_1)). \quad (*)$$

It suffices to find a structure $\mathfrak{A}$ such that the first two sentences are true in $\mathfrak{A}$, while the third is not true in $\mathfrak{A}$. So we consider a structure $\mathfrak{A}$ with $|\mathfrak{A}| = \{a_1, a_2, a_3\}$ (where a_1, a_2, a_3 are different individuals) and $P^{\mathfrak{A}} = \{a_1\}$, $Q^{\mathfrak{A}} = \{a_2\}$, $R^{\mathfrak{A}} = \{a_2, a_3\}$. Using the definitions, we see that
$\mathfrak{A} \models \forall x_1(P(x_1) \to \neg Q(x_1))$ iff
(for every valuation v in $\mathfrak{A}$ and every $a \in |\mathfrak{A}|$:

$$\mathfrak{A} \models P(x_1) \to \neg Q(x_1)[v(x_1|a)]) \text{ iff}$$

(for every valuation v in $\mathfrak{A}$ and every $a \in |\mathfrak{A}|$:

$$\text{if } \mathfrak{A} \models P(x_1)[v(x_1|a)], \text{ then } \mathfrak{A} \not\models Q(x_1)[v(x_1|a)]) \text{ iff}$$

(for every valuation v in $\mathfrak{A}$ and every $a \in |\mathfrak{A}|$:

$$\text{if } a \in P^{\mathfrak{A}}, \text{ then } a \notin Q^{\mathfrak{A}}),$$

which is true. Therefore, $\forall x_1(P(x_1) \rightarrow \neg Q(x_1))$ is true in $\mathfrak{A}$. In a similar manner, we can show that the sentence $\exists x_1(R(x_1) \wedge Q(x_1))$ is true in $\mathfrak{A}$, while $\exists x_1(P(x_1 \wedge R(x_1))$ is not. Therefore, (*) holds.

As we saw in Chapter 2, there is an algorithm which, given a finite set T of propositional formulas and any propositional formula φ, can decide whether $T \models \varphi$ or not and, therefore, due to the completeness theorem, can decide whether $T \vdash_{\mathcal{A}_0} \varphi$ or not. Unfortunately, the same is not true for first-order logic, since the cases that need to be considered are infinite, because there are infinite possible structures for $\mathcal{L}_1$. Thus, we focus our attention on the construction and study of axiom systems for first-order logic.

3.3 Predicate calculus

We will begin with a definition.

Definition 3.3.1 For arbitrary formulas φ, ψ, we say that φ is a *generalization* of ψ iff φ is identical with ψ or φ is of the form $\forall y_1 ... \forall y_n \psi$, for some variables $y_1, \ldots, y_n$.

Foe example, the formula $\forall x_1(P(x_1, x_2) \rightarrow P(x_2, x_1))$ is a generalization of $P(x_1, x_2) \rightarrow P(x_2, x_2)$ and $\forall x_1 \forall x_3(Q(x_1) \vee R(x_1))$ is a generalization of $Q(x_1) \vee R(x_1)$.

An axiom system for predicate logic

$$\mathcal{A}_1 = \langle A_1, K_1 \rangle$$

is defined as follows:

- The set K_1 is the same as K_0 that we saw in propositional calculus, that is, it has only one element, namely the rule MP.
- The set of axioms A_1 is equal to the union of two disjoint sets, i.e., Λ_1 and M_1. The elements of Λ_1 are called *logical axioms*, they describe the rôle of the logical symbols of $\mathcal{L}_1$ and they are all generalizations of formulas of the following form:

  $AS1$. φ,
  where φ is the formula obtained if in a tautology we replace all propositional variables with formulas of $\mathcal{L}_1$, e.g., $P(x_1) \wedge Q(x_1, x_3) \rightarrow Q(x_1, x_3) \wedge P(x_1)$

$AS2.$ $\forall x\varphi \to \varphi^x_t$,
where the atomic variable x is substitutable by the term t in φ (see Definition 3.3.2 below) and φ^x_t is obtained from φ by replacing x, in all its free occurrences in φ, with t.

$AS3.$ $\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$

$AS4.$ $\varphi \to \forall x\varphi$, where x does not occur free in φ.

$AS5.$ $x \approx x$

$AS6.$ $x \approx y \to (\varphi \to \varphi^*)$,
where φ is an atomic formula and φ^* is the formula obtained from φ by replacing the variable x (in some or all of its occurrences) with the variable y.

The elements of M_1 are called *non-logical axioms* and describe the rôle of the non-logical symbols of $\mathcal{L}_1$. Obviously, then, the non-logical axioms of an axiom system for set theory will be different from those of one for number theory.

The most common axiom system for number theory $\mathcal{A}_1^{nt}$ has the following non-logical axioms (of Peano):

$P1.$ $\forall x_1\, x_1' \not\approx \mathbf{0}$

$P2.$ $\forall x_1 \forall x_2 (x_1' \approx x_2' \to x_1 \approx x_2)$

$P3.$ $\forall x_1\, x_1 \oplus \mathbf{0} \approx x_1$

$P4.$ $\forall x_1 \forall x_2\, x_1 \oplus x_2' \approx (x_1 \oplus x_2)'$

$P5.$ $\forall x_1\, x_1 \odot \mathbf{0} \approx \mathbf{0}$

$P6.$ $\forall x_1 \forall x_2\, x_1 \odot x_2' \approx (x_1 \odot x_2) \oplus x_1$

$P7.$ $\forall x_1 ... \forall x_n [\varphi(\mathbf{0}, \vec{x}) \wedge \forall x_0 (\varphi(x_0, \vec{x}) \to \varphi(x_0', \vec{x})) \to \forall x_0 \varphi(x_0, \vec{x})]$,
where $\varphi(x_0, \ldots, x_n)$ is a formula of $\mathcal{L}_1^{nt}$ such that the variables occurring free in it are among $x_0, \ldots, x_n$.

Axiom schema $P7$ corresponds to the principle of induction for the set of natural numbers.

The most common axiomatic system for set theory $\mathcal{A}_1^{st}$ has as non-logical axioms the Zermelo-Fraenkel axioms, some of which are:

$ZF1.$ $\forall x_1 \forall x_2 [x_1 \approx x_2 \leftrightarrow \forall x_3 (x_3 \in x_1 \leftrightarrow x_3 \in x_2)]$
$ZF2.$ $\forall x_1 \exists x_2 \forall x_3 [x_3 \in x_2 \leftrightarrow \forall x_4 (x_4 \in x_3 \to x_4 \in x_1)]$
$ZF3.$ $\exists x_1 \forall x_2\, x_2 \notin x_1$

For the remaining Zermelo-Fraenkel axioms, one may consult [11].

Remarks. a) It should be emphasized that it is not excluded that $M_1=\emptyset$, that is, that the axiom system does not have any non-logical axioms.
b) Instead of the sets Λ_1, K_1 we saw before, we can take the sets Λ_1', K_1', where

- K_1' is the set having two elements, namely the rule MP and the rule of generalization

$$\frac{\varphi}{\forall x\varphi}$$

- Λ_1' is the set having as elements all generalizations of the formulas produced by $AS1$, $AS2$, $AS5$, $AS6$ and the schema

$AS3'$. $\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$, where x is bound in φ.

c) As we will see later, $AS5$ and $AS6$ imply that $\approx$ is a symmetric and transitive relation. Also, these axioms ensure the possibility of replacing any variable x with any variable y in any formula φ, with the assumption that x is substitutable by y in φ (see Exercise 53).
d) The formal proofs in $\mathcal{A}_1$ correspond to proofs that we make with propositions of the English language in the sense of the main interpretation of $\mathcal{L}_1$. Thus, for example, formal proofs in

$$\mathcal{A}_1^{nt} = \langle \Lambda_1 \cup P, K_1 \rangle$$

correspond to proofs in number theory. Since every first-order language has at least one predicate or function symbol, the term *predicate calculus* (function symbols can be considered as predicate symbols) has been established.
e) Let T be a set of formulas of $\mathcal{L}_1$ and $\mathcal{A}_1^T = \langle \Lambda_1 \cup T, K_1 \rangle$, $\mathcal{A}_1^\emptyset = \langle \Lambda_1, K_1 \rangle$. From the definitions it follows that

$$\vdash_{\mathcal{A}_1^T} \varphi \text{ iff } T \vdash_{\mathcal{A}_1^\emptyset} \varphi.$$

Therefore, studying the axiom system $\mathcal{A}_1^\emptyset$ we essentially study all axiom systems, which of course have as background the same first-order language. In what follows, we will deal exclusively with $\mathcal{A}_1^\emptyset$, actually writing, for simplicity, $\mathcal{A}_1$ instead of $\mathcal{A}_1^\emptyset$.

Before proceeding, we need to provide some clarifications regarding the axiom schema $AS2$. The idea is "if every object has the property (expressed by the formula) φ, then the particular object t also has φ", but we must be careful, lest some part of t be incompatible with some part of φ; no variable occurring in t should be bound by any quantifier of φ.

Let us see with an example what we want to avoid. Let φ be the formula $\exists y\, x \not\approx y$ and t be the term y. Then φ_t^x is the formula $\exists y\, y \not\approx y$, so $\forall x\varphi \to \varphi_t^x$ is the sentence

$$\forall x \exists y\, x \not\approx y \to \exists y\, y \not\approx y,$$

which is false in almost all structures for $\mathcal{L}_1$. Indeed, the sentence $\forall x \exists y\, x \not\approx y$ is true in every structure whose universe has at least two elements, while $\exists y\, y \not\approx y$ is false in every structure. So if we do not put any restriction on the substitutions of variables by terms, we will have as an axiom the sentence $\forall x \exists y\, x \not\approx y \to \exists y\, y \not\approx y$, which is incompatible with our reasonable wish to take valid formulas as logical axioms.

Definition 3.3.2 For every formula φ, variable x and term t, we say that x is *substitutable by t in φ* iff

- φ is atomic or
- φ is of the form $\neg\psi$ and x is substitutable by t in ψ or
- φ is of the form $\psi \to \chi$ and x is substitutable by t in ψ and in χ or
- φ is of the form $\forall y\psi$ and
  - i) x does not occur free in $\forall y\psi$ or
  - ii) y does not occur in t and x is substitutable by t in ψ.

For example, x is substitutable by itself in every formula φ and x is not substitutable by y in the formula $P(x) \to \exists y Q(x, y)$.
Clearly, if no variable occurring in t occurs in φ, then x is substitutable by t in φ.

Now that we have fully described the axiomatic system $\mathcal{A}_1$, let's see an example of a formal proof, showing that the formula $\forall x(P(x) \to \exists y P(y))$ is a formal theorem of it.

1.	$\forall x[(\forall y \neg P(y) \to \neg P(x)) \to (P(x) \to \neg\forall y \neg P(y))]$	$AS1$
2.	$\forall x[(\forall y \neg P(y) \to \neg P(x)) \to (P(x) \to \neg\forall y \neg P(y))] \to$ $[\forall x(\forall y \neg P(y) \to \neg P(x)) \to \forall x(P(x) \to \neg\forall y \neg P(y))]$	$AS3$
3.	$\forall x(\forall y \neg P(y) \to \neg P(x)) \to \forall x(P(x) \to \neg\forall y \neg P(y))$	1, 2, MP
4.	$\forall x(\forall y \neg P(y) \to \neg P(x))$	$AS2$
5.	$\forall x(P(x) \to \neg\forall y \neg P(y))$	3, 4, MP.

The first of the theorems we will prove for $\mathcal{A}_1$ talks about the following proof method we use in Mathematics: If we want to show that every x has a property, we show that any x has this property.

Theorem 3.3.1 (Generalization Theorem) Let T be a set of formulas of $\mathcal{L}_1$, φ a formula of $\mathcal{L}_1$ and x a variable. If $T \vdash_{\mathcal{A}_1} \varphi$ and x does not occur free in any element of T, then $T \vdash_{\mathcal{A}_1} \forall x\varphi$.

Proof. Suppose that x does not occur free in any element of T and $T \vdash_{\mathcal{A}_1} \varphi$. Then there exists a formal proof $\varphi_1, \ldots, \varphi_n = \varphi$ in $\mathcal{A}_1$ from T. Using induction on i, we will show that $T \vdash_{\mathcal{A}_1} \forall x\varphi_i$, for every $1 \leq i \leq n$.
First step. $i = 1$. We distinguish two cases.
1) $\varphi_1 \in \Lambda_1$. Then $\forall x\varphi_1 \in \Lambda_1$ and hence obviously $T \vdash_{\mathcal{A}_1} \forall x\varphi_1$.
2) $\varphi_1 \in T$. Then the following formal proof shows that $T \vdash_{\mathcal{A}_1} \forall x\varphi_1$.

1.	φ_1	hypothesis
2.	$\varphi_1 \to \forall x\varphi_1$	$AS4$
3.	$\forall x\varphi_1$	1, 2, MP.

Second step. Suppose that $i \geq 2$ and $T \vdash_{\mathcal{A}_1} \forall x\varphi_k$ for every k such that $k < i$. We will show that $T \vdash_{\mathcal{A}_1} \forall x\varphi_i$. We again distinguish two cases.
1) $\varphi_i \in \Lambda_1 \cup T$. Then the claim holds as before.
2) φ_i is an immediate consequence, on the basis of MP, of φ_j, φ_j, $j, l < i$, where, without loss of generality, φ_l is of the form $\varphi_j \to \varphi_i$. By the inductive hypothesis, we have that

$$T \vdash_{\mathcal{A}_1} \forall x\varphi_j \quad \text{and} \quad T \vdash_{\mathcal{A}_1} \forall x(\varphi_j \to \varphi_i).$$

We now construct the following formal proof in $\mathcal{A}_1$:

1.	...	
..	...	formal proof of $\forall x\varphi_j$ from T
λ.	$\forall x\varphi_j$	
λ+1.	...	
..	...	formal proof of $\forall x(\varphi_j \to \varphi_i)$ from T
μ.	$\forall x(\varphi_j \to \varphi_i)$	
μ+1.	$\forall x(\varphi_j \to \varphi_i) \to (\forall x\varphi_j \to \forall x\varphi_i)$	$AS3$
μ+2.	$\forall x\varphi_j \to \forall x\varphi_i$	μ, μ+1, MP
μ+3.	$\forall x\varphi_i$	λ, μ+2, MP.

Therefore, $T \vdash_{\mathcal{A}_1} \forall x\varphi_i$, and so the proof of the inductive step is complete. It follows that $T \vdash_{\mathcal{A}_1} \forall x\varphi_n$, that is, $T \vdash_{\mathcal{A}_1} \forall x\varphi$. □

Remark. The restriction we set for x in the previous theorem is necessary, as shown by the following example: Let $T = \{P(x)\}$ and φ be the formula $P(x)$, where P is a unary predicate symbol. Then it is obvious that $T \vdash_{\mathcal{A}_1} \varphi$, but it is not true that $T \vdash_{\mathcal{A}_1} \forall x\varphi$, because $P(x) \not\models \forall xP(x)$ (see Example

2 after Definition 3.2.1) and, as we will see later, for any set T of formulas and any formula φ, if $T \vdash_{\mathcal{A}_1} \psi$, then $T \models \psi$.

As for propositional calculus, we can prove the following theorems.

Theorem 3.3.2 (Deduction Theorem) For every set of formulas T and any formulas φ, ψ,

$$T \cup \{\varphi\} \vdash_{\mathcal{A}_1} \psi \text{ iff } T \vdash_{\mathcal{A}_1} \varphi \to \psi.$$

Theorem 3.3.3 (Contraposition Theorem) For every set of formulas T and any formulas φ, ψ,

$$T \cup \{\varphi\} \vdash_{\mathcal{A}_1} \neg\psi \text{ iff } T \cup \{\psi\} \vdash_{\mathcal{A}_1} \neg\varphi.$$

Theorem 3.3.4 (Reduction to the Impossible) For any set of formulas T and any formula φ,

$$T \cup \{\varphi\} \text{ is inconsistent iff } T \vdash_{\mathcal{A}_1} \neg\varphi.$$

Let us now look at an example, where we use the previous theorems to show that a formula is a formal theorem of $\mathcal{A}_1$.
Example. We will show that for every formula φ it holds that

$$\vdash_{\mathcal{A}_1} \exists x \forall y \varphi \to \forall y \exists x \varphi.$$

Because of the Deduction Theorem, it suffices to show that

$$\exists x \forall y \varphi \vdash_{\mathcal{A}_1} \forall y \exists x \varphi.$$

Because of the Generalization Theorem, since y is bound in $\exists x \forall y \varphi$, it suffices to show that

$$\exists x \forall y \varphi \vdash_{\mathcal{A}_1} \exists x \varphi, \text{ that is } \neg\forall x \neg\forall y \varphi \vdash_{\mathcal{A}_1} \neg\forall x \neg\varphi.$$

Because of the Contraposition Theorem, it suffices to show that

$$\forall x \neg\varphi \vdash_{\mathcal{A}_1} \neg\neg\forall x \neg\forall y \varphi.$$

But $\neg\neg\forall x \neg\forall y \varphi \leftrightarrow \forall x \neg\forall y \varphi$ is an axiom, so it suffices to show that

$$\forall x \neg\varphi \vdash_{\mathcal{A}_1} \forall x \neg\forall y \varphi.$$

Using again the Generalization Theorem, we deduce that is is sufficient to show that $\forall x \neg\varphi \vdash_{\mathcal{A}_1} \neg\forall y \varphi$. Because of Reduction to the Impossible, it suffices to show that the set $\{\forall x \neg\varphi, \forall y \varphi\}$ is inconsistent. The following formal proof in $\mathcal{A}_1$ guarantees that $\{\forall x \neg\varphi, \forall y \varphi\} \vdash_{\mathcal{A}_1} \neg\varphi$.

1.	$\forall x \neg\varphi$	hypothesis
2.	$\forall x \neg\varphi \to \neg\varphi$	$AS2$
3.	$\neg\varphi$	1, 2, MP.

Similarly we can show that $\{\forall x\neg\varphi, \forall y\varphi\} \vdash_{\mathcal{A}_1} \varphi$, and hence our claim follows.

In general, when we want to show that $T \vdash_{\mathcal{A}_1} \varphi$, we follow the following strategy:

- If φ is of the form $\psi \to \chi$, then it suffices, using the Deduction Theorem, to show that $T \cup \{\psi\} \vdash_{\mathcal{A}_1} \chi$.
- If φ is of the form $\forall x\psi$, where x does not occur free in any element of T, then it suffices, using the Generalization Theorem, to show that $T \vdash_{\mathcal{A}_1} \psi$.
- If φ is of the form $\neg\psi$, then we distinguish subcases:
  - i) If ψ is of the form $\psi_1 \to \psi_2$, then it suffices to show that $T \vdash_{\mathcal{A}_1} \psi_1$ and $T \vdash_{\mathcal{A}_1} \neg\psi_2$. Indeed, having these, using (on the basis of $AS1$) the axiom $\psi_1 \to (\neg\psi_2 \to \neg(\psi_1 \to \psi_2))$, we can show that $T \vdash_{\mathcal{A}_1} \varphi$.
  - ii) If ψ is of the form $\neg\chi$, then it suffices to show that $T \vdash_{\mathcal{A}_1} \chi$, since the formula $\chi \to \neg\neg\chi$ is an axiom.
  - iii) If ψ is of the form $\forall x\chi$, then it suffices to show that $T \vdash_{\mathcal{A}_1} \neg\chi_t^x$, for some term t such that x is substitutable by t in χ. Indeed, having shown that $T \vdash_{\mathcal{A}_1} \neg\chi_t^x$, using the axioms $\forall x\chi \to \chi_t^x$ και $(\forall x\chi \to \chi_t^x) \to (\neg\chi_t^x \to \neg\forall x\chi)$, we can show that $T \vdash_{\mathcal{A}_1} \varphi$.

We continue with some more examples.

Examples. a) We will show that if x does not occur free in φ, then

$$\vdash_{\mathcal{A}_1} (\varphi \to \forall x\psi) \leftrightarrow \forall x(\varphi \to \psi).$$

Before we proceed, we make the following observation: When we want to show that $T \vdash_{\mathcal{A}_1} \chi_1 \leftrightarrow \chi_2$, it suffices to show that $T \vdash_{\mathcal{A}_1} \chi_1 \to \chi_2$ and $T \vdash_{\mathcal{A}_1} \chi_2 \to \chi_1$, since the formula

$$(\chi_1 \to \chi_2) \to ((\chi_2 \to \chi_1) \to (\chi_1 \leftrightarrow \chi_2))$$

is an axiom (on the basis of $AS1$).
So here it suffices to show that

$$\vdash_{\mathcal{A}_1} (\varphi \to \forall x\psi) \to \forall x(\varphi \to \psi) \text{ and}$$
$$\vdash_{\mathcal{A}_1} \forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi).$$

For the first part, because of the Deduction Theorem and the Generalization Theorem, it suffices to show that

$$\{\varphi \to \forall x\psi, \varphi\} \vdash_{\mathcal{A}_1} \psi.$$

But this is a consequence of the following formal proof:

1.	$\varphi \to \forall x\psi$	hypothesis
2.	φ	hypothesis
3.	$\forall x\psi$	1, 2, MP
4.	$\forall x\psi \to \psi$	$AS2$
5.	ψ	3, 4, MP.

For the second part, again because of the Deduction and the Generalization Theorems, it suffices to show that

$$\{\forall x(\varphi \to \psi), \varphi\} \vdash_{\mathcal{A}_1} \psi,$$

which is justified by the following formal proof:

1.	$\forall x(\varphi \to \psi)$	hypothesis
2.	φ	hypothesis
3.	$\forall x(\varphi \to \psi) \to (\varphi \to \psi)$	$AS2$
4.	$\varphi \to \psi$	1, 3, MP
5.	ψ	2, 4, MP.

b) We will show that, if x does not occur free in φ, then

$$\vdash_{\mathcal{A}_1} (\exists x\psi \to \varphi) \leftrightarrow \forall x(\psi \to \varphi).$$

It suffices to show that

$$\begin{array}{rl} & \vdash_{\mathcal{A}_1} (\exists x\psi \to \varphi) \to \forall x(\psi \to \varphi) \text{ and} \\ & \vdash_{\mathcal{A}_1} \forall x(\psi \to \varphi) \to (\exists x\psi \to \varphi), \\ \text{i.e.,} & \vdash_{\mathcal{A}_1} (\neg\forall x\neg\psi \to \varphi) \to \forall x(\psi \to \varphi) \text{ and} \\ & \vdash_{\mathcal{A}_1} \forall x(\psi \to \varphi) \to (\neg\forall x\neg\psi \to \varphi). \end{array}$$

For the first part, it suffices to show that

$$\{\neg\forall x\neg\psi \to \varphi, \psi\} \vdash_{\mathcal{A}_1} \varphi.$$

But this is evident from the following formal proof:

1.	$\neg\forall x\neg\psi \to \varphi$	hypothesis
2.	ψ	hypothesis
3.	$(\forall x\neg\psi \to \neg\psi) \to (\psi \to \neg\forall x\neg\psi)$	$AS1$
4.	$\forall x\neg\psi \to \neg\psi$	$AS2$
5.	$\psi \to \neg\forall x\neg\psi$	3, 4, MP
6.	$\neg\forall x\neg\psi$	2, 5, MP
7.	φ	1, 6, MP.

We work similarly for the second part.
c) We will show that $\vdash_{\mathcal{A}_1} \forall x \forall y (x \approx y \to y \approx x)$.
Because of the Generalization and the Deduction Theorems, it suffices to show that $x \approx y \vdash_{\mathcal{A}_1} y \approx x$. But this is justified by the following formal proof:

1.	$x \approx y$	hypothesis
2.	$x \approx y \to (x \approx x \to y \approx x)$	$AS6$
3.	$x \approx x \to y \approx x$	1, 3, MP
4.	$x \approx x$	$AS5$
5.	$y \approx x$	3, 4, MP.

d) We will show that for every binary function symbol f:

$$\vdash_{\mathcal{A}_1} \forall z_1 \forall z_2 \forall y_1 \forall y_2 [z_1 \approx y_1 \to (z_2 \approx y_2 \to f(z_1, z_2) \approx f(y_1, y_2))].$$

It suffices to show that

$$\{z_1 \approx y_1, z_2 \approx y_2\} \vdash_{\mathcal{A}_1} f(z_1, z_2) \approx f(y_1, y_2).$$

So we construct the following formal proof:

1.	$z_1 \approx y_1$	hypothesis
2.	$z_2 \approx y_2$	hypothesis
3.	$z_1 \approx y_1 \to (f(z_1, z_2) \approx f(z_1, z_2) \to f(z_1, z_2) \approx f(y_1, z_2))$	$AS6$
4.	$f(z_1, z_2) \approx f(z_1, z_2) \to f(z_1, z_2) \approx f(y_1, z_2)$	1, 3, MP
5.	$\forall x_1 x_1 \approx x_1$	$AS5$
6.	$\forall x_1 x_1 \approx x_1 \to f(z_1, z_2) \approx f(z_1, z_2)$	$AS2$
7.	$f(z_1, z_2) \approx f(z_1, z_2)$	5, 6, MP
8.	$f(z_1, z_2) \approx f(y_1, z_2)$	4, 7, MP
9.	$z_2 \approx y_2 \to (f(z_1, z_2) \approx f(y_1, z_2) \to f(z_1, z_2) \approx f(y_1, y_2)$	$AS6$
10.	$f(z_1, z_2) \approx f(y_1, z_2) \to f(z_1, z_2) \approx f(y_1, y_2)$	2, 9, MP
11.	$f(z_1, z_2) \approx f(y_1, y_2)$	8, 10, MP.

e) We will show that for any unary predicate symbols P, Q and any constant symbol c, it holds that

$$\{\forall x (P(x) \to Q(x)), \forall z P(z)\} \vdash_{\mathcal{A}_1} Q(c).$$

We construct the following formal proof:

1.	$\forall x (P(x) \to Q(x))$	hypothesis
2.	$\forall z P(z)$	hypothesis
3.	$\forall z P(z) \to P(c)$	$AS2$
4.	$P(c)$	2, 3, MP
4.	$\forall x (P(x) \to Q(x)) \to (P(c) \to Q(c))$	$AS2$
6.	$P(c) \to Q(c)$	1, 5, MP
7.	$Q(c)$	4, 6, MP.

If in the formal proof in e) above we replace c everywhere with y, then we will get a formal proof showing that

$$\{\forall x(P(x) \to Q(x)), \forall z P(z)\} \vdash_{\mathcal{A}_1} Q(y).$$

The fact that a constant in many cases plays the same rôle as a free variable is the basis for the following theorem, which we state without proof.

Theorem 3.3.5 (Generalization on Constants) Let T be a set of formulas, φ be a formula and c be an atomic constant. If $T \vdash_{\mathcal{A}_1} \varphi$ and c does not occur in any element of T, then there exists a variable y not occurring in φ such that $T \vdash_{\mathcal{A}_1} \forall y \varphi_y^c$. (with φ_y^c we denote the formula obtained from φ by replacing c by y everywhere)

A corollary of the previous theorem is the following theorem, which corresponds to the following method we use for constructing mathematical proofs: Suppose we want to prove B, assuming that there is an x with the property A. We call c an object that has property A and prove B.

Theorem 3.3.6 (Existential Instantiation) For every set T of formulas, formulas φ, ψ, variable y and atomic constant c, if $T \cup \{\varphi_c^y\} \vdash_{\mathcal{A}_1} \psi$ and c does not occur in any element of $T \cup \{\varphi, \psi\}$, then $T \cup \{\exists y \varphi\} \vdash_{\mathcal{A}_1} \psi$.

Let's revisit the example immediately following Theorem 3.3.4, this time using the existential instantiation theorem. So we want to show that

$$\vdash_{\mathcal{A}_1} \exists x \forall y \varphi \to \forall y \exists x \varphi.$$

Because of the Deduction Theorem, it suffices to prove that

$$\exists x \forall y \varphi \vdash_{\mathcal{A}_1} \forall y \exists x \varphi.$$

Using now Existential Instantiation, it suffices to show that

$\forall y \varphi_c^x \vdash_{\mathcal{A}_1} \forall y \exists x \varphi$, where c is a constant that does not occur in φ.

Finally, because of the Generalization Theorem, it suffices to show that

$$\forall y \varphi_c^x \vdash_{\mathcal{A}_1} \exists x \varphi, \text{ i.e., } \forall y \varphi_c^x \vdash_{\mathcal{A}_1} \neg \forall x \neg \varphi.$$

But this holds, as is evident from the following formal proof:

1.	$\forall y \varphi_c^x$	hypothesis
2.	$\forall y \varphi_c^x \to \varphi_c^x$	$AS2$
3.	φ_c^x	1, 2, MP
4.	$\forall x \neg \varphi \to \neg \varphi_c^x$	$AS2$
5.	$(\forall x \neg \varphi \to \neg \varphi_c^x) \to (\varphi_c^x \to \neg \forall x \neg \varphi)$	$AS1$
6.	$\varphi_c^x \to \neg \forall x \neg \varphi$	4, 5, MP
7.	$\neg \forall x \neg \varphi$	3, 6, MP.

So fa, we have dealt with the axiomatic system $\mathcal{A}_1$ at length. In what follows, we will show that the syntactic and semantic approaches to a (mathematical) proof are equivalent.

3.4 Soundness and completeness

We begin with a definition.

Definition 3.4.1 Let T be a set of formulas. We say that T is *satisfiable* iff there is a structure $\mathfrak{A}$ and a valuation v in $\mathfrak{A}$ such that T is true for v in $\mathfrak{A}$.

First we give the counterpart of Theorem 2.5.3.

Theorem 3.4.1 (Soundness Theorem for Predicate Calculus) For every set T of formulas and every formula φ:
a) If $T \vdash_{\mathcal{A}_1} \varphi$, then $T \models \varphi$.
b) If T is satisfiable, then T is consistent.

Proof. a) We take for granted that the logical axioms are valid formulas and that Modus Ponens preserves logical implication (see Exercise 61). Now let us suppose that $T \vdash_{\mathcal{A}_1} \varphi$. Then there exists a formal proof $\varphi_1, ..., \varphi_n$ in $\mathcal{A}_1$ from T and φ_n is identical with φ. By induction on i we will show that $T \models \varphi_i$ for every $i = 1, ..., n$.
1) $i = 1$. Obviously $\varphi_1 \in \Lambda_1 \cup T$. If $\varphi_1 \in \Lambda_1$, then φ_1 is valid and hence $T \models \varphi_1$. If $\varphi_1 \in T$, then obviously $T \models \varphi_1$.
2) Suppose now that $i \leq n$ such that $T \models \varphi_k$ for every $k < i$. We will show that $T \models \varphi_i$. If $\varphi_i \in \Lambda_1 \cup T$, then the claim holds as in 1). If φ_i is an immediate consequence, on the basis of MP, of φ_j, φ_l, for some $j, l < i$, where φ_l is identical with $\varphi_j \to \varphi_i$, then, by the inductive hypothesis, it will hold that $T \models \varphi_j$ and $T \models \varphi_j \to \varphi_i$. Hence $T \models \varphi_i$, since MP preserves logical implication.
The inductive proof is complete and hence $T \models \varphi$.
b) Suppose that T is satisfiable, but inconsistent. Then, for some formula φ, $T \vdash_{\mathfrak{A}_1} \varphi$ and $T \vdash_{\mathfrak{A}_1} \neg\varphi$. Hence, because of a), for some formula φ it holds that $T \models \varphi$ and $T \models \neg\varphi$. Let now $\mathfrak{A}$ be a structure and v a valuation in $\mathfrak{A}$ such that T is true for v in $\mathcal{A}$. Then $\mathfrak{A} \models \varphi[v]$ and $\mathfrak{A} \models \neg\varphi[v]$, which is impossible.
The claim follows. □

Remark. From the previous proof we see that a) ⇒ b). In fact, the converse also holds (see Exercise 62).

The following very important theorem was proved by K. Gödel and was published in 1930 (see [13]).

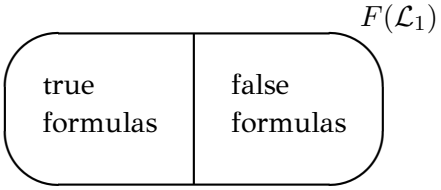
Theorem 3.4.2 (Completeness Theorem for Predicate Calculus) For every set T of formulas and every formula φ it holds that

a) If $T \models \varphi$, then $T \vdash_{\mathcal{A}_1} \varphi$.

b) If T is conistent, then T is satisfiable.

We note that a), b) are equivalent - see Exercise 63.

Proof. We will sketch the proof of b) given by L. Henkin in 1949 (see [16]). Suppose T is a consistent set of formulas of a first-order language $\mathcal{L}_1$. The idea of the proof comes from the following analysis: Suppose that we have an arbitrary structure $\mathfrak{A}$ for $\mathcal{L}_1$ and an arbitrary valuation v in $\mathfrak{A}$. Then a partition of the set of formulas of $\mathcal{L}_1$ into two "equal" parts results: the set of formulas satisfied by v in $\mathfrak{A}$ and the set of formulas not satisfied by v in $\mathfrak{A}$:



Conversely, if we are given a partition as above, it is possible to extract from it the definition of a structure $\mathfrak{A}$ and a valuation v such that the formulas that are true for v in $\mathfrak{A}$ are exactly the formulas of the part containing the logically true formulas. Therefore, the search for a valuation and a structure for which the set T is satisfied is equivalent to the search for a set $\Sigma \subseteq F(\mathcal{L}_1)$ which satisfies the following:

a) $T \subseteq \Sigma$ and Σ is consistent,

b) Σ is *complete*, i.e., for every $\varphi \in F(\mathcal{L}_1)$: it holds that $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$,

c) Σ indeed represents truth.

To ensure the validity of c), we need to use the so-called *Henkin witnesses*, i.e., constants which ensure that if the formula $\exists y\, \varphi$ has been included in Σ, then the formula φ_c^y has also been included for an appropriate such constant. We note that for technical reasons (which will become clear later) these constants must differ from those already available in $\mathcal{L}_1$. We also note that special care must be taken for $\approx$, since this must be interpreted as identity in the universe of the structure.

Let us now turn to the details. For convenience, we assume that $\mathcal{L}_1$ has a countable number of symbols, noting, however, that the theorem also holds for first-order languages with an uncountable number of symbols (see, e.g., [10], pp. 135-6).

Let $\mathcal{L}_1^*$ be the language obtained if we enrich $\mathcal{L}_1$ with a countable set of new constant symbols $\{c_0, c_1, c_2, ...\}$. On the basis of the analysis above, the proof has the following three main steps.

First step. We define a set Σ of formulas of $\mathcal{L}_1^*$ such that

i) $T \subseteq \Sigma$ and Σ is consistent

ii) Σ is complete in $\mathcal{L}_1^*$

iii) for every formula φ of $\mathcal{L}_1^*$ and every variable y there exists a new constant c such that $\exists y \varphi \rightarrow \varphi_c^y \in \Sigma$.

Second step. We define a structure $\mathfrak{A}$ and a valuation v in $\mathfrak{A}$ such that every element of Σ not containing $\approx$ is satisfied by v in $\mathfrak{A}$.

Third step. We define a structure $\mathfrak{A}'$ and a valuation v' in $\mathfrak{A}'$ such that every element of Σ is satisfied by v' in $\mathfrak{A}'$.

We will give a detailed proof only for the first step, but we will give sketches of the proofs of the other two steps (leaving details to the exercises).
Proof of first step. We first observe that T is consistent, if we consider it as a set of formulas of $\mathcal{L}_1^*$. Indeed, suppose that this is not the case, i.e., $T \vdash_{\mathcal{A}_1} \varphi$ and $T \vdash_{\mathcal{A}_1} \neg\varphi$, for some $\varphi \in F(\mathcal{L}_1^*)$. Based on Generalization on Constants, we can replace all the new constants that appear in the formal proofs showing that $T \vdash_{\mathcal{A}_1} \varphi$ and $T \vdash_{\mathcal{A}_1} \neg\varphi$ with variables and conclude that there are formal proofs using formulas of $\mathcal{L}_1$ and showing that $T \vdash_{\mathcal{A}_1} \varphi^*$ and $T \vdash_{\mathcal{A}_1} \neg\varphi^*$, for a suitable $\varphi^* \in F(\mathcal{L}_1)$. So T is contradictory as a subset of $F(\mathcal{L}_1)$, which is impossible.

Let $\langle \varphi_0, y_0 \rangle, \langle \varphi_1, y_1 \rangle, \ldots$ be an enumeration of all pairs that have as first coordinate a formula of $\mathcal{L}_1^*$ and as second a variable of $\mathcal{L}_1^*$ (there is a countable number of such pairs, because the set $F(\mathcal{L}_1^*)$ is countable – from Set Theory we know that if A is any countable set, then the set of all finite sequences of elements of A is countable). We define (recursively) a sequence of formulas $\theta_0, \theta_1, \ldots$ as follows:

a) θ_0 is the formula $\exists y_0 \varphi_0 \rightarrow (\varphi_0)_{c_{i_0}}^{y_0}$, where c_{i_0} is the first new constant that does not appear in φ_0, and

b) having defined $\theta_0, \ldots, \theta_{n-1}$, we define θ_n to be $\exists y_n \varphi_n \rightarrow (\varphi_n)_{c_{i_n}}^{y_n}$, where c_{i_n} is the first new constant that does not appear in $\theta_0, \ldots, \theta_{n-1}, \varphi_n$.

Claim. $T^* = T \cup \{\theta_0, \theta_1, \ldots\}$ is consistent.
Proof of claim. Suppose that T^* is contradictory. Then there exists a finite subset of T that is contradictory, hence there exists $m \in \mathbb{N}$ such that $T \cup \{\theta_0, \ldots, \theta_m\}$ is contradictory. Let k be the least such m and let us suppose that $k > 0$ (if $k = 0$, we work similarly).
Since $T \cup \{\theta_0, \ldots, \theta_k\}$ is contradictory, because of Reduction to the Impossible, it will hold that

$$T \cup \{\theta_0, \ldots, \theta_{k-1}\} \vdash_{\mathcal{A}_1} \neg\theta_k, \text{ that is,}$$
$$T \cup \{\theta_0, \ldots, \theta_{k-1}\} \vdash_{\mathcal{A}_1} \neg(\exists y_k \varphi_k \rightarrow (\varphi_k)^{y_k}_{c_{i_k}}).$$

But $\vdash_{\mathcal{A}_1} \neg(\varphi \rightarrow \psi) \rightarrow \varphi$ and $\vdash_{\mathcal{A}_1} \neg(\varphi \rightarrow \psi) \rightarrow \neg\psi$ for arbitrary formulas φ, ψ (on the basis of $AS1$). Hence

$$T \cup \{\theta_0, \ldots, \theta_{k-1}\} \vdash_{\mathcal{A}_1} \exists y_k \varphi_k \text{ και } T \cup \{\theta_0, ..., \theta_{k-1}\} \vdash_{\mathcal{A}_1} (\neg\varphi_k)^{y_k}_{c_{i_k}}.$$

Since c_{i_k} does not occur in any element of $T \cup \{\theta_0, \ldots, \theta_{k-1}, \varphi_k\}$, using Exercise 59, it holds that

$$T \cup \{\theta_0, \ldots, \theta_{k-1}\} \vdash_{\mathcal{A}_1} \forall y_k \neg\varphi_k.$$

Hence $T \cup \{\theta_0, \ldots, \theta_{k-1}\}$ is inconsistent, which is impossible because of the definition of k. Therefore, the claim holds.

Now let $\psi_0, \psi_1, \psi_2, \ldots$ be an enumeration of the formulas of $\mathcal{L}_1^*$. We define the sequence Σ_0, Σ_1, $\Sigma_2, \ldots$ as follows:

$$\Sigma_0 = T^*$$
$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\psi_n\}, & \text{if this is consistent} \\ \Sigma_n \cup \{\neg\psi_n\}, & \text{otherwise.} \end{cases}$$

Then Σ_n is consistent, for every $n \in \mathbb{N}$. Indeed, this holds for $n=0$, by the previous assertion. Now suppose that Σ_n is consistent, but Σ_{n+1} is contradictory. Then, by the definition of Σ_{n+1}, $\Sigma_n \cup \{\psi_n\}$ is contradictory and $\Sigma_n \cup \{\neg\psi_n\}$ is contradictory. Therefore $\Sigma_n \vdash_{\mathcal{A}_1} \neg\psi_n$ and $\Sigma_n \vdash_{\mathcal{A}_1} \neg\neg\psi_n$, so Σ_n is contradictory, which is absurd.
We set $\Sigma = \bigcup_{n \in \mathbf{N}} \Sigma_n$. Due to the definition of Σ_n, conditions a) - c) that we mentioned at the beginning hold. We observe that Σ has the following property: if $\Sigma \vdash_{\mathcal{A}_1} \varphi$, then $\varphi \in \Sigma$. Indeed, let $\varphi \in F(\mathcal{L}_1^*)$ be such that $\Sigma \vdash_{\mathcal{A}_1} \varphi$, but $\varphi \notin \Sigma$. Since Σ is complete, it will be true that $\neg\varphi \in \Sigma$, so obviously $\Sigma \vdash_{\mathcal{A}_1} \neg\varphi$. Consequently Σ is contradictory, which is not true.

Proof of second step. Let $\mathcal{L}_1^{*E}$ be the language we obtain from $\mathcal{L}_1^*$ by replacing the symbol $\approx$ with the binary predicate symbol E. We define the structure $\mathfrak{A}$ for $\mathcal{L}_1^{*E}$ as follows:

1) $|\mathfrak{A}| = T(\mathcal{L}_1^{*E}) = T(\mathcal{L}_1^*)$
2) $\langle s, t \rangle \in E^{\mathfrak{A}}$ iff $s \approx t \in \Sigma$

3) $\langle t_1, ..., t_n\rangle \in P^{\mathfrak{A}}$ iff $P(t_1, ..., t_n) \in \Sigma$

4) $f^{\mathfrak{A}}(t_1, ..., t_n) = f(t_1, ..., t_n)$

5) $c^{\mathfrak{A}} = c$.

Let v be the valuation in $\mathfrak{A}$ with $v(x) = x$ for each variable x. Using the induction principle for $T(\mathcal{L}_1^*)$ and $F(\mathcal{L}_1^*)$, we can show that

a) $\overline{v}(t) = t$, for every $t \in T(\mathcal{L}_1^*)$

b) $\mathfrak{A} \models \varphi_E^{\approx}[v]$ if $\varphi \in \Sigma$, for every $\varphi \in F(\mathcal{L}_1^*)$, where $\varphi_E^{\approx}$ is the formula we get from φ by replacing $\approx$ with E.

Proof of third step. We can show that $E^{\mathfrak{A}}$ is an equivalence relation on $|\mathfrak{A}|$ and that for every n-ary predicate symbol P and every n-place function symbol f the relation $P^{\mathfrak{A}}$ and the function $f^{\mathfrak{A}}$ are compatible with $E^{\mathfrak{A}}$, that is,

a) if $\langle t_1, \ldots, t_n\rangle \in P^{\mathfrak{A}}$ and $\langle t_1, s_1\rangle \in E^{\mathfrak{A}}, \ldots, \langle t_n, s_n\rangle \in E^{\mathfrak{A}}$, then $\langle s_1, \ldots, s_n\rangle \in P^{\mathfrak{A}}$,

b) if $\langle t_1, s_1\rangle \in E^{\mathfrak{A}}$ and $\ldots$ and $\langle t_n, s_n\rangle \in E^{\mathfrak{A}}$, then $\langle f^{\mathfrak{A}}(t_1, \ldots, t_n), f^{\mathfrak{A}}(s_1, \ldots, s_n)\rangle \in E^{\mathfrak{A}}$.

$[t]$ denotes the equivalence class of a term t of $\mathcal{L}_1^*$. We define the structure $\mathfrak{A}'$ for $\mathcal{L}_1^*$ as follows:

1) $|\mathfrak{A}'| = |\mathfrak{A}|/E^{\mathfrak{A}} = \{[t] : t \in |\mathfrak{A}|\}$

2) $\langle [t_1], \ldots, [t_n]\rangle \in P^{\mathfrak{A}'}$ iff $\langle t_1, \ldots, t_n\rangle \in P^{\mathfrak{A}}$

3) $f^{\mathfrak{A}'}([t_1], \ldots, [t_n]) = [f^{\mathfrak{A}}(t_1, \ldots, t_n)]$

4) $c^{\mathfrak{A}'} = [c]$.

Note that the definitions in 2), 3) above are good, because $P^{\mathfrak{A}}$, $f^{\mathfrak{A}}$ are compatible with $E^{\mathfrak{A}}$.
Let $h : |\mathfrak{A}| \to |\mathfrak{A}'|$ with $h(t) = [t]$ and v' the valuation in $\mathfrak{A}'$ that is defined by $v' = h \circ v$. By induction on $F(\mathcal{L}_1^*)$ we can prove that, for every $\varphi \in F(\mathcal{L}_1^*)$,

$$\mathfrak{A}' \models \varphi[v'] \text{ iff } \mathfrak{A} \models \varphi_E^{\approx}[v].$$

Hence for every $\varphi \in F(\mathcal{L}_1^*)$: $\varphi \in \Sigma$ iff $\mathfrak{A}' \models \varphi[v']$.
Since $T \subseteq \Sigma$, v' satisfies every element of T in $\mathfrak{A}'$. If we restrict now $\mathfrak{A}'$ to the original language $\mathcal{L}_1$, i.e., we consider the structure $\mathfrak{A}' \restriction \mathcal{L}_1$, nothing changes essentially, that is, v' satisfies T in the new structure $\mathfrak{A}' \restriction \mathcal{L}_1$ (see Exercise 64). Hence T is satisfiable. □

As in propositional logic, using the completeness and the soundness theorems, we can prove the following result.

Theorem 3.4.3 (Compactness Theorem) Let T be an infinite set of formulas. If every finite subset of T is satisfiable, then T is satisfiable.

We will end the section with an application of the compactness theorem. We will say that a structure is finite (or infinite or has n elements, where $n \in \mathbb{N}-\{0\}$) meaning that its universe is finite (or infinite or has n elements, respectively). We observe that for every natural number $n \geq 1$ there is a sentence φ_n expressing "there are at least n things", i.e., for every structure $\mathfrak{A}$:

$$\mathfrak{A} \models \varphi_n \quad \text{if} \quad \mathfrak{A} \text{ has at least } n \text{ elements.}$$

For example, φ_2 is the proposition $\exists x_1 \exists x_2\, x_1 \not\approx x_2$ and φ_3 is the proposition $\exists x_1 \exists x_2 \exists x_3 (x_1 \not\approx x_2 \wedge x_2 \not\approx x_3 \wedge x_3 \not\approx x_1)$.
Also for every natural number $n \geq 1$ there is a sentence ψ_n that expresses "there are exactly n things", i.e., for every structure $\mathfrak{A}$:

$$\mathfrak{A} \models \psi_n \quad \text{if} \quad \mathfrak{A} \text{ has exactly } n \text{ elements.}$$

So there are sets of sentences, e.g. $\{\psi_2\}$, whose every model is finite. According to the next theorem, if a set of sentences S has only finite models, then there is a bound on their size.

Theorem 3.4.4 Let S be a set of sentences. If, for any $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$, $m > n$, such that S has a model with m elements, then S has an infinite model.

Proof. We consider the set $S^* = S \cup \{\varphi_n : n \in \mathbb{N}, n \geq 1\}$. By the assumption, every finite subset of S^* is satisfiable. Therefore, by the compactness theorem, S^* is satisfiable, i.e., there exists a structure $\mathfrak{A}$ such that $\mathfrak{A} \models S^*$. Obviously then $\mathfrak{A} \models S$ and $\mathfrak{A}$ is infinite (since it has at least n elements, for every $n \in \mathbb{N}$). □

An immediate consequence of the previous theorem is

Theorem 3.4.5 There is no set of sentences S such that a structure is a model of S iff S is finite.

Proof. Suppose that S is a set of sentences such that for every structure $\mathfrak{A}$:

$$\mathfrak{A} \models S \quad \text{iff} \quad \mathfrak{A} \text{ is finite.} \qquad (*)$$

Then S has finite models as large as we wish, and hence, by Theorem 3.4.4, it has an infinite model, which contradicts (*). Therefore, the claim holds. □

3.5 Prenex normal form

We have seen that every propositional formula is tautologically equivalent to a formula in disjunctive normal form. We will show that something similar holds in predicate logic. For the proof we will need the following theorem.

Theorem 3.5.1 (Alphabetic Variants) For every formula φ, variable x and term t there exists a formula $\overline{\varphi}$ such that

a) $\vdash_{\mathcal{A}_1} \varphi \leftrightarrow \overline{\varphi}$ and

b) x is substitutable by t in $\overline{\varphi}$.

Proof. We use the principle of induction for $F(\mathcal{L}_1)$. We will give details only for the case in which φ is of the form $\forall y\psi$. The inductive hypothesis is that there exists a formula $\overline{\psi}$ such that $\vdash_{\mathcal{A}_1} \psi \leftrightarrow \overline{\psi}$ and x is substitutable by t in $\overline{\psi}$.
First case. If x is identical with y or y does not occur in t, then we take as $\overline{\varphi}$ the formula $\forall y\,\overline{\psi}$. It is easy to see that a), b) hold for $\overline{\varphi}$.
Second case. If the first case does not hold, we consider a variable z which (i) occurs neither in $\overline{\psi}$ nor in t and (ii) is different from x, and we take as $\overline{\varphi}$ the formula $\forall z\,\overline{\psi}^y_z$.
Since z does not occur in t and x is substitutable by t in $\overline{\psi}$, x is substitutable by t in $\overline{\varphi}$, and hence b) holds. It remains to show that a) is true.

1) By the inductive hypothesis, $\psi \vdash_{\mathcal{A}_1} \overline{\psi}$. Using Exercise 49, we see that $\forall y\psi \vdash_{\mathcal{A}_1} \forall y\,\overline{\psi}$. But $\forall y\,\overline{\psi} \vdash_{\mathcal{A}_1} \overline{\psi}^y_z$, since y is substitutable by z in $\overline{\psi}$. Hence, by the Generalization Theorem, $\forall y\,\overline{\psi} \vdash_{\mathcal{A}_1} \forall z\,\overline{\psi}^y_z$ and so $\forall y\psi \vdash_{\mathcal{A}_1} \forall z\,\overline{\psi}^y_z$.

2) Since z does not occur in $\overline{\psi}$, we have that

$$\forall z\,\overline{\psi}^y_z \vdash_{\mathcal{A}_1} \overline{\psi}^{y\ z}_{z\ y}.$$

But, by Exercise 58, $\overline{\psi}^{y\ z}_{z\ y}$ is obviously identical to $\overline{\psi}$. Note also that $\overline{\psi} \vdash_{\mathcal{A}_1} \psi$, by the inductive hypothesis. Therefore $\forall z\,\overline{\psi}^y_z \vdash_{\mathcal{A}_1} \psi$ and hence $\forall z\,\overline{\psi}^y_z \vdash_{\mathcal{A}_1} \forall y\,\psi$, because of the Generalization Theorem.

Definition 3.5.1 We say that the formula φ is in *prenex normal form* iff φ is *open*, that is, it contains no quantifiers, or φ is of the form $Q_1y_1 \ldots Q_ny_n\psi$, where $n \in \mathbb{N}-\{0\}$, ψ open and Q_i is either $\forall$ or $\exists$, $i = 1, \ldots, n$.

Theorem 3.5.2 Every formula is logically equivalent to a formula in prenex normal form.

Proof. We use the principle of induction for $F(\mathcal{L}_1)$.

1) Suppose that φ is atomic. Then φ is open, hence φ is in prenex normal form.

2) Suppose that φ is of the form $\forall x\psi$ and ψ is logically equivalent with ψ^* which is in prenex normal form. Then $\forall x\psi \equiv \forall x\psi^*$ (because of Exercise 49 and the completeness theorem. But $\forall x\psi^*$ is obviously in prenex normal form and so the claim holds.

3) Suppose that φ is of the form $\neg\psi$ and ψ is logically equivalent with ψ^* which is in prenex normal form. Then $\neg\psi \equiv \neg\psi^*$. But $\neg\psi^*$ is logically equivalent to a formula ψ^{**} in prenex normal form, since for any formula φ and any variable x:

$$\neg\forall x\psi \equiv \exists x\neg\psi \quad \text{and} \quad \neg\exists x\psi \equiv \forall x\neg\psi.$$

Hence φ is logically equivalent to the formula ψ^{**}, which is in prenex normal form.

4) Suppose that φ is of the form $\psi \to \chi$ and $\psi \equiv \psi^*$, $\chi \equiv \chi^*$, where ψ^*, χ^* are formulas in prenex normal form. Because of Theorem 3.5.1, we may suppose that the variables occurring in quantifiers in ψ do not occur in χ and conversely. Obviously $\psi \to \chi \equiv \psi^* \to \chi^*$. Using the Examples on pages 52, 53 and the Soundness Theorem, we deduce that for all φ_1, φ_2, x, if x is bound in φ_1, then

$$\varphi_1 \to \forall x\varphi_2 \equiv \forall x(\varphi_1 \to \varphi_2) \quad (1)$$
$$\exists x\varphi_2 \to \varphi_1 \equiv \forall x(\varphi_2 \to \varphi_1) \quad (2).$$

In a similar manner we show that if x is bound in φ_1, then

$$\varphi_1 \to \exists x\varphi_2 \equiv \exists x(\varphi_1 \to \varphi_2) \quad (3)$$
$$\forall x\varphi_2 \to \varphi_1 \equiv \exists x(\varphi_2 \to \varphi_1) \quad (4).$$

Using (1)–(4), we can find a formula φ^* in prenex normal form such that $\varphi^* \equiv \psi^* \to \chi^*$. Then $\varphi \equiv \varphi^*$ and the claim follows.

Example. Suppose that x, y do not occur in χ and z does not occur in ψ. There exist three formulas in prenex normal form which are logically equivalent to the formula $\forall x\exists y\psi \to \exists z\chi$, depending on the order according to which we apply (1)–(4).

$$
\begin{array}{lll}
\text{a)} \quad \forall x\exists y\psi \to \exists z\chi & \equiv & \exists z(\forall x\exists y\psi \to \chi) \\
& \equiv & \exists z\exists x(\exists y\psi \to \chi) \\
& \equiv & \exists z\exists x\forall y(\psi \to \chi) \\
\text{b)} \quad \forall x\exists y\psi \to \exists z\chi & \equiv & \exists x(\exists y\psi \to \exists z\chi) \\
& \equiv & \exists x\forall y(\psi \to \exists z\chi) \\
& \equiv & \exists x\forall y\exists z(\psi \to \chi) \\
\text{c)} \quad \forall x\exists y\psi \to \exists z\chi & \equiv & \exists x(\exists y\psi \to \exists z\chi) \\
& \equiv & \exists x\exists z(\exists y\psi \to \chi) \\
& \equiv & \exists x\exists z\forall y(\psi \to \chi).
\end{array}
$$

3.6 Elements of model theory

Definition 3.6.1 Let $\mathfrak{A}$, $\mathfrak{B}$ be structures for $\mathcal{L}_1$ and $h : |\mathfrak{A}| \to |\mathfrak{B}|$. We say that h is

1) a *homomorphism of* $\mathfrak{A}$ *into* $\mathfrak{B}$ iff the following hold:
   a) For every n-ary predicate symbol P and $a_1, \ldots, a_n \in |\mathfrak{A}|$:
   $$\langle a_1, \ldots, a_n\rangle \in P^{\mathfrak{A}} \text{ iff } \langle h(a_1), \ldots, h(a_n)\rangle \in P^{\mathfrak{B}}$$
   b) For every n-ary function symbol f and $a_1, \ldots, a_n \in |\mathfrak{A}|$:
   $$h(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(h(a_1), \ldots, h(a_n))$$
   c) For every constant c: $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$.

   Instead of a) we say that h *preserves relations* and instead of b) we say that h *preserves functions*.

2) an *isomorphism of* $\mathfrak{A}$ *into* $\mathfrak{B}$ iff it is 1-1 and a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$

3) an *isomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{B}$ iff it is 1-1, onto $|\mathfrak{B}|$ and a homomorphsim of $\mathfrak{A}$ into $\mathfrak{B}$.

We say that $\mathfrak{A}$, $\mathfrak{B}$ are *isomorphic*, and we write $\mathfrak{A} \cong \mathfrak{B}$, iff there exists an isomorphism h of $\mathfrak{A}$ onto $\mathfrak{B}$ (denoted with $h : \mathfrak{A} \cong \mathfrak{B}$).

Examples. i) Let us consider the language $\mathcal{L}_1^{am}$ with only two binary function symbols $\oplus, \odot$ and the structures $\mathfrak{A}$, $\mathfrak{B}$ for $\mathcal{L}_1^{am}$ that are defined as follows:

$$
\begin{array}{ll}
|\mathfrak{A}| = \mathbb{N}, & \oplus^{\mathfrak{A}}, \odot^{\mathfrak{A}} \text{ the usual operations of addition and multiplication in } \mathbb{N} \\
|\mathfrak{B}| = \{e, o\}, & \oplus^{\mathfrak{B}} = \{\langle e,e,e\rangle, \langle e,o,o\rangle, \langle o,e,o\rangle, \langle o,o,e\rangle\} \\
& \odot^{\mathfrak{B}} = \{\langle e,e,e\rangle, \langle e,o,e\rangle, \langle o,e,e\rangle, \langle o,o,o\rangle\}.
\end{array}
$$

Then the function $h : |\mathfrak{A}| \to |\mathfrak{B}|$ with

$$h(n) = \begin{cases} e, & \text{if } n \text{ is even} \\ o, & \text{otherwise} \end{cases}$$

is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Indeed, by the definition of h we see that for every $m, n \in |\mathfrak{A}|$:

$$h(m \oplus^{\mathfrak{A}} n) = h(m) \oplus^{\mathfrak{B}} h(n) \quad \text{and} \quad h(m \odot^{\mathfrak{A}} n) = h(m) \odot^{\mathfrak{B}} h(n).$$

ii) Let $\mathfrak{A}$, $\mathfrak{B}$ be the structures for $\mathcal{L}_1^o$ that are defined as follows:

$|\mathfrak{A}| = \mathbb{N} - \{0\}$, $\quad \prec^{\mathfrak{A}}$ the usual relation "less than" in $\mathbb{N} - \{0\}$,

$|\mathcal{B}| = \mathbb{N}$, $\quad \prec^{\mathfrak{B}}$ the usual relation "less than" in $\mathbb{N}$.

Then the function $h : |\mathfrak{A}| \to |\mathfrak{B}|$ with $h(n) = n-1$ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, as we can easily verify.

The previous definition reminds us corresponding algebraic definitions. The next theorem links it with semantical notions.

Theorem 3.6.1 Let $\mathfrak{A}$, $\mathfrak{B}$ be structures for $\mathcal{L}_1$, h a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ and v a valuation in $\mathfrak{A}$. Then

a) For every term t, $h(\overline{v}(t)) = \overline{h \circ v}(t)$

b) For every open formula φ without the symbol $\approx$,

$$\mathfrak{A} \models \varphi[v] \text{ iff } \mathfrak{B} \models \varphi[h \circ v]$$

c) If h is 1-1, then for every open formula φ,

$$\mathfrak{A} \models \varphi[v] \text{ iff } \mathfrak{B} \models \varphi[h \circ v]$$

d) If h is onto, then for every formula φ without the symbol $\approx$ we have

$$\mathfrak{A} \models \varphi[v] \text{ iff } \mathfrak{B} \models \varphi[h \circ v].$$

Proof. a) Exercise 71.

b) We use the principle of induction for the set of formulas constructed from atomic ones without the symbol $\approx$ by means of the symbols $\neg, \to$.

1) Let us consider a formula of the form $P(t_1, \ldots, t_n)$. Then

$$\begin{aligned} \mathfrak{A} \models P(t_1, \ldots, t_n)[v] \quad &\text{iff} \quad \langle \overline{v}(t_1), \ldots, \overline{v}(t_n) \rangle \in P^{\mathfrak{A}} \text{ (Tarski's def.)} \\ &\text{iff} \quad \langle h(\overline{v}(t_1)), \ldots, h(\overline{v}(t_n)) \rangle \in P^{\mathfrak{B}} \text{ (h homom.)} \\ &\text{iff} \quad \langle \overline{h \circ v}(t_1), \ldots, \overline{h \circ v}(t_n) \rangle \in P^{\mathfrak{B}} \text{ (by a))} \\ &\text{iff} \quad \mathfrak{B} \models P(t_1, \ldots, t_n)[h \circ v] \text{ (Tarski's def.)} \end{aligned}$$

2) Suppose that φ is of the form $\neg\psi$, where $\mathfrak{A} \models \psi[v]$ iff $\mathfrak{B} \models \psi[h \circ v]$. Then, by Tarski's definition,

$$\mathfrak{A} \models \varphi[v] \text{ iff } \mathfrak{A} \not\models \psi[v] \text{ iff } \mathfrak{B} \not\models \psi[h \circ v] \text{ iff } \mathfrak{B} \models \varphi[h \circ v].$$

3) If φ is of the form $\psi \to \chi$, then we work as in 2).

c) We work as in b), without excluding the case in which the formula is of the form $t_1 \approx t_2$, where t_1, t_2 are terms. But

$$\begin{array}{lll} \mathfrak{A} \models t_1 \approx t_2[v] & \text{iff} \quad \overline{v}(t_1) = \overline{v}(t_2) & \text{(Tarski's def.)} \\ & \text{iff} \quad h(\overline{v}(t_1)) = h(\overline{v}(t_2)) & \text{(since } h \text{ is 1-1)} \\ & \text{iff} \quad \overline{h \circ v}(t_1) = \overline{h \circ v}(t_2) & \text{(by a))} \\ & \text{iff} \quad \mathfrak{B} \models t_1 \approx t_2[h \circ v] & \text{(Tarski's def.),} \end{array}$$

and so the claim holds.
d) We work as in b). The case that remains to be considered is when φ is of the form $\forall x\psi$ and $\mathfrak{A} \models \psi[v]$ iff $\mathfrak{B} \models \psi[h \circ v]$, for all valuations v in $\mathfrak{A}$.

1) Suppose that $\mathfrak{B} \models \forall x\psi[h \circ v]$. Then $\mathfrak{B} \models \psi[(h \circ v)(x|b)]$ for every $b \in |\mathfrak{B}|$. Since h is onto, it follows that $\mathfrak{B} \models \psi[(h \circ v)(x|h(a))]$ for every $a \in |\mathfrak{A}|$. But the functions $(h \circ v)(x|h(a))$, $h \circ (v(x|a))$ are equal, hence $\mathfrak{B} \models \psi[h \circ (v(x|a))]$ for every $a \in |\mathfrak{A}|$, and so, by the inductive hypothesis, $\mathfrak{A} \models \psi[v(x|a)]$ for every $a \in |\mathfrak{A}|$. But then we deduce that $\mathfrak{A} \models \forall x\psi[v]$ by Tarski's definition, i.e., the claim holds.

2) As in 1), we show that if $\mathfrak{B} \not\models \forall x\psi[h \circ v]$, then $\mathfrak{A} \not\models \forall x\psi[v]$, nd hence the required equivalence holds.

Remark. A consequence of the previous theorem is that if $h : \mathfrak{A} \cong \mathfrak{B}$, then for every formula φ and valuation v in $\mathfrak{A}$ the following holds:

$$\mathfrak{A} \models \varphi[v] \quad \text{iff} \quad \mathfrak{B} \models \varphi[h \circ v].$$

In particular, for every sentence φ: $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$.

Definition 3.6.2 Let $\mathfrak{A}$, $\mathfrak{B}$ be structures for $\mathcal{L}_1$. We say that

1) η $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$, denoted by $\mathfrak{A} \subseteq \mathfrak{B}$, iff $|\mathfrak{A}| \subseteq |\mathfrak{B}|$ and

   α) for every n-ary predicate symbol P: $P^{\mathfrak{A}} = P^{\mathfrak{B}} \cap |\mathfrak{A}|^n$

   β) for every n-ary function symbol f: $f^{\mathfrak{A}} = f^{\mathfrak{B}} \lceil |\mathfrak{A}|^n$

   γ) for any constant symbol c: $c^{\mathfrak{A}} = c^{\mathfrak{B}}$.

2) $\mathfrak{A}$, $\mathfrak{B}$ are *elementarily equivalent*, denoted by $\mathfrak{A} \equiv \mathfrak{B}$, iff $\mathfrak{A}$, $\mathfrak{B}$ they satisfy the same sentences, i.e., for every sentence φ:

$$\mathfrak{A} \models \varphi \text{ iff } \mathfrak{B} \models \varphi.$$

An example of structures $\mathfrak{A}$, $\mathfrak{B}$ for which $\mathfrak{A} \subseteq \mathfrak{B}$ is given in Example ii), page 65. In this case we also have $\mathfrak{A} \cong \mathfrak{B}$, but it is not true in general that if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \cong \mathfrak{B}$. Indeed, for the structures $\mathfrak{A}$, $\mathfrak{B}$ of Exercise 39 it is obviously true that $\mathfrak{A} \subseteq \mathfrak{B}$, but it is not true even that $\mathfrak{A} \equiv \mathfrak{B}$, since there exists a sentence φ such that $\mathfrak{A} \models \varphi$ but $\mathfrak{B} \models \neg\varphi$.

From the previous Remark it follows that if $\mathfrak{A} \cong \mathfrak{B}$ then $\mathfrak{A} \equiv \mathfrak{B}$. The converse is not true. Indeed, let $\mathfrak{M}$ be a non-standard model of arithmetic (Exercise 66). Obviously it holds that $\mathfrak{M} \equiv \mathfrak{N}$, but we will show that $\mathfrak{M} \ncong \mathfrak{N}$. So suppose that $h : \mathfrak{M} \cong \mathfrak{N}$. By the hypothesis, there exists a valuation v in $\mathfrak{M}$ such that

$$\mathfrak{M} \models x_0 \neq \underline{n}[v] \text{ for every } n \in \mathbb{N}.$$

Hence, by the Remark,

$$\mathfrak{N} \models x_0 \neq \underline{n}[h \circ v] \text{ for every } n \in \mathbb{N},$$
$$\text{that is,} \quad \overline{h \circ v}(x_0) \neq \overline{h \circ v}(\underline{n}) \text{ for every } n \in \mathbb{N}.$$

Using (ordinary) induction we show that $\overline{h \circ v}(\underline{n}) = n$ for every $n \in \mathbb{N}$. Hence $\overline{h \circ v}(x_0) \neq n$ for every $n \in \mathbb{N}$, which is impossible, since $\overline{h \circ v}(x_0) \in |\mathfrak{N}| = \mathbb{N}$. Therefore, there is no function $h : \mathfrak{M} \cong \mathfrak{N}$.

Definition 3.6.3 Let $\mathfrak{A}$ be a structure for $\mathcal{L}_1$, $a_1, \ldots, a_n \in |\mathfrak{A}|$, $R \subseteq |\mathfrak{A}|^n$ and $\varphi(x_1, \ldots, x_n)$ a formula – when writing $\varphi(x_1, \ldots, x_n)$ we mean that all the variables occurring free in φ are among $x_1, \ldots, x_n$. We say that

1) φ *is satisfied by* $a_1, \ldots, a_n$ *in* $\mathfrak{A}$ and we write $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$, if there exists a valuation v in $\mathfrak{A}$ such that $\mathfrak{A} \models \varphi[v]$ and $v(x_i) = a_i$ for $i = 1, \ldots, n$ (or, equivalently, for every valuation v in $\mathfrak{A}$, if $v(x_i) = a_i$ for $i = 1, \ldots, n$, then $\mathfrak{A} \models \varphi[v]$)

2) φ *defines* R *in* $\mathfrak{A}$ iff it holds that

$$R = \{\langle a_1, \ldots, a_n \rangle \in |\mathfrak{A}|^n : \mathfrak{A} \models \varphi[a_1, \ldots, a_n]\}$$

3) R is *definable in* $\mathfrak{A}$ iff there exists a formula $\varphi(x_1, \ldots, x_n)$ which defines R in $\mathfrak{A}$.

Examples. Let us consider the structure $\mathfrak{N}$ for $\mathcal{L}_1^{nt}$.

a) The relation $\{\langle m, n \rangle \in \mathbb{N}^2 : m <^{\mathbb{N}} n\}$ is defined in $\mathfrak{N}$ by the formula $\exists x_3\, x_1 \oplus x_3' \approx x_2$.

b) For an arbitrary $n \in \mathbb{N}$, the set $\{n\}$ is definable in $\mathfrak{N}$ by the formula $x_1 \approx \underline{n}$.

c) The set of prime numbers is definable in $\mathfrak{N}$ by the formula

$$\exists x_3\, \mathbf{0}'' \oplus x_3 \approx x_1 \wedge \forall x_2 \forall x_3 (x_1 \approx x_2 \odot x_3 \rightarrow x_2 \approx \mathbf{0}' \vee x_3 \approx \mathbf{0}').$$

To show that a relation is not definable in a structure we use the following

Corollary 3.6.1 Let $\mathfrak{A}$ be a structure for $\mathcal{L}_1$, h an automorphism of $\mathfrak{A}$, i.e., $h : \mathfrak{A} \cong \mathfrak{A}$, and $R \subseteq |\mathfrak{A}|^n$ definable in $\mathfrak{A}$. Then for every $a_1, \ldots, a_n \in |\mathfrak{A}|$:

$$\langle a_1, \ldots, a_n \rangle \in R \text{ iff } \langle h(a_1), \ldots, h(a_n) \rangle \in R.$$

Proof. Let φ be the formula which defines R in $\mathfrak{A}$ and v an arbitrary valuation in $\mathfrak{A}$ with $v(x_i) = a_i$, $i = 1, \ldots, n$. Then

$\langle a_1, \ldots, a_n \rangle \in R$	iff $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$	(since φ defines R)
	iff $\mathfrak{A} \models \varphi[v]$	(because of notation)
	iff $\mathfrak{A} \models \varphi[h \circ v]$	(by Theorem 3.6.1)
	iff $\mathfrak{A} \models \varphi[h(a_1), \ldots, h(a_n)]$	(notation)
	iff $\langle h(a_1), \ldots, h(a_n) \rangle \in R$,	

and so the claim holds. □

Example. Let $\mathfrak{R}_o$ be the structure for $\mathcal{L}_1^o$ which is defined as follows:

$$|\mathfrak{R}_o| = \mathbb{R} \text{ and } \prec^{\mathfrak{R}_o} = \text{ the usual relation "smaller than" in } \mathbb{R}.$$

Using the previous corollary, we will show that the set $\mathbb{N}$ is not definable in $\mathfrak{R}_o$ (– $\mathbb{N}$ is considered as a unary relation $R \subseteq |\mathfrak{R}_o| = \mathbb{R}$). So suppose that $\mathbb{N}$ is definable. Then, because of Corollary 3.6.1, for every $a \in \mathbb{R}$ it holds that

$$a \in \mathbb{N} \text{ iff } h(a) \in \mathbb{N},$$

where h is an arbitrary automorphism of $\mathfrak{R}_o$. But it is easy to see that the function $h : \mathbb{R} \rightarrow \mathbb{R}$ with $h(x) = x^3$ is an automorphism of $\mathfrak{R}_o$. Hence for every $a \in \mathbb{R}$:

$$a \in \mathbb{N} \text{ iff } a^3 \in \mathbb{N}.$$

But this is not true, since, e.g., $2 \in \mathbb{N}$, while $\sqrt[3]{2} \notin \mathbb{N}$. Hence $\mathbb{N}$ is not definable in $\mathfrak{R}_o$.

EXERCISES

28. Construct the tree diagrams of the formulas

$$\exists x_1(Q(x_1) \to \forall x_2 Q(x_2)),\ [(\forall x(\varphi \to \psi) \wedge \exists x\varphi) \to \exists x\psi],$$
$$(\exists x_1 \forall x_2 P(x_1, x_2) \to \forall x_2 \exists x_1 P(x_1, x_2)).$$

29. Which variables occur free in each of the following formulas? Which of these formulas are sentences?

$$(\forall x_1 Q(x_1) \leftrightarrow \exists x_2 Q(x_2)),\ (\exists x_1 Q(x_1) \wedge Q(x_1)),\ (x_1 \approx x_2 \vee x_1 \not\approx x_2)$$
$$\exists x_3(Q(x_3) \to \forall x_4 Q(x_4)),\ (Q(x_2) \to \forall x_1 P(x_1, x_2)).$$

30. Prove that every formula has the same number of right and left parentheses.

31. Which are the formulas that, after omitting parentheses, are written as

$$\forall x_1 Q(x_1) \wedge P(x_1, x_2) \to \exists x_2 \neg P(x_1, x_2),$$
$$\neg \exists x_1 \forall x_2 P(x_1, x_2) \vee \forall x_1 Q(x_1) \leftrightarrow Q(x_2);$$

32. a) Find formulas of $\mathcal{L}_1^{st}$ corresponding to the propositions: "There exists a set which has no elements", "Every set is an element of some set".

    b) Find formulas of $\mathcal{L}_1^{nt}$ that correspond to the propositions: "There exists a natural number that is not the successor of any natural number", "The product of any two arbitrary natural numbers is greater than their sum".

33. We consider the following formulas of $\mathcal{L}_1^{st}$:

$$x_1 \in x_2 \vee x_1 \in x_3,\ \exists x_4(x_4 \notin x_1 \wedge x_4 \notin x_3)$$
$$x_1 \in x_2 \wedge x_2 \in x_3 \to x_1 \in x_3,\ \forall x_2\, x_1 \in x_2.$$

    - Let v be a valuation in the structure $\mathfrak{N}_s$ such that $v(x_1) = 2$, $v(x_2) = 4$ and $v(x_3) = 7$. Which of these formulas are true for v in $\mathfrak{N}_s$?
    - Let v^* be a valuation in the structure $\mathfrak{N}_s^*$ such that $v^*(x_1) = 5$, $v^*(x_2) = 1$ and $v^*(x_3) = 3$. Which of these formulas are true for v^* in $\mathfrak{N}_s^*$?

34. Let v be a valuation in the structure $\mathfrak{N}^*$ for $\mathcal{L}_1^{nt}$ such that $v(x_1) = -3$, $v(x_2) = -4$ and $v(x_3) = 0$.

- Determine the values of the following terms for v in $\mathfrak{N}^*$:

$$(x_1' \oplus x_2'')' \oplus (x_1 \odot x_3),\ x_2 \odot (x_1 \oplus (x_3'' \odot \mathbf{0})).$$

- Which of the following formulas are true for v in $\mathfrak{N}^*$;

$$\exists x_1\ \mathbf{0}'' \odot x_1 \approx x_2,\ \forall x_1\, x_1 \not\approx \mathbf{0} \to x_3 \approx x_2.$$

35. Which of the following sentences of $\mathcal{L}_1^{nt}$ are true in $\mathfrak{N}$ and why?

$$\exists x_1\ x_1 \oplus x_1 \approx \mathbf{0},\ \forall x_1\, x_1 \approx x_1 \vee \exists x_0 x_0 \not\approx x_0,$$
$$\forall x_1\ x_1 \not\approx x_1 \to \forall x_1 \forall x_2\ x_1 \oplus x_2 \approx x_2 \oplus x_1.$$

36. Let Σ be a set of formulas of $\mathcal{L}_1$ and $\mathfrak{A}$ a structure for $\mathcal{L}_1$. Is it true that if every element of Σ is satisfiable in $\mathfrak{A}$ then Σ is also satisfiable in $\mathfrak{A}$?

37. Let φ be the sentence of $\mathcal{L}_1^{st}$ which expresses "The powerset of every set exists", i.e., the sentence

$$\forall x_1 \exists x_2 \forall x_3 (x_3 \in x_2 \leftrightarrow \forall x_4 (x_4 \in x_3 \to x_4 \in x_1)).$$

Is it true that $\mathfrak{N}_s \models \varphi$ or not?

38. Let $\mathfrak{N}_d$ be the structure for $\mathcal{L}_1^{st}$ defined as follows:

a) $|\mathfrak{N}_d| = \mathbb{N}$
b) $\in^{\mathfrak{N}_d} = \{\langle m, n\rangle \in \mathbb{N}^2 : m \text{ divides } n\}$.

Let now ψ be the sentence expressing that "For any sets x, y the pair $\{x, y\}$ exists", i.e.,

$$\forall x_1 \forall x_2 \exists x_3 \forall x_4 (x_4 \in x_3 \leftrightarrow x_4 \approx x_1 \vee x_4 \approx x_2).$$

Is it true that $\mathfrak{N}_d \models \psi$ or not?

39. Let $\mathcal{L}_1^o$ be the first-order language of the theory of order, that is the first-order language having as its only non-logical symbol the binary predicate symbol $\prec$. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be the structures for $\mathcal{L}_1^o$ that are defined as follows: $|\mathfrak{A}| = \mathbb{N}$, $|\mathfrak{B}| = \mathbb{Z}$, $|\mathfrak{C}| = \mathbb{Q}$ and $\prec^{\mathfrak{A}} = <^{\mathbb{N}}$, $\prec^{\mathfrak{B}} = <^{\mathbb{Z}}$, $\prec^{\mathfrak{C}} = <^{\mathbb{Q}}$.
Find sentences φ, ψ of $\mathcal{L}_1^o$ such that

$$\alpha)\ \mathfrak{B} \models \varphi \ \text{ and } \ \mathfrak{A} \models \neg\varphi$$
$$\beta)\ \mathfrak{C} \models \psi \ \text{ and } \ \mathfrak{B} \models \neg\psi.$$

40. Let Σ be a set of formulas. Prove that for every φ, ψ:

- $\Sigma \cup \{\varphi\} \models \psi$ iff $\Sigma \models \varphi \to \psi$
- $\varphi \equiv \psi$ iff $\models \varphi \leftrightarrow \psi$.

41. Prove that for any binary predicate symbol P:

$$\exists x_1 \forall x_2 P(x_1, x_2) \models \forall x_2 \exists x_1 P(x_1, x_2)$$
$$\forall x_2 \exists x_1 P(x_1, x_2) \not\models \exists x_1 \forall x_2 P(x_1, x_2).$$

42. We consider the following propositions:
    1) For every set x there exists a set y whose cardinal number is greater than that of x.
    2) If x is a subset of y, then the cardinal number of x is smaller than or equal to that of y.
    3) Every set is a subset of V.
    4) V is not a set.

    Using corresponding sentences of a first-order language, show that 1) – 3) logically imply 4).

43. We consider the following propositions:
    1) Every natural number is an integral number.
    2) n is an integral number.
    3) n is a natural number.

    Using corresponding sentences of a first-order language, show that 1), 2) do not logically imply 3).

44. Prove that for all formulas φ, ψ: $\{\forall x(\varphi \to \psi), \forall x\varphi\} \models \forall x\psi$.

45. Prove that if x does not occur free in φ, then $\varphi \models \forall x\varphi$.

46. Prove that for every formula φ: φ is valid iff $\forall x\varphi$ is valid.

47. Let S be a set of sentences such that for every sentence φ: $S \models \varphi$ or $S \models \neg\varphi$. Let also $\mathfrak{A}$ be a model of S. Prove that for every sentence φ: $\mathfrak{A} \models \varphi$ iff $S \models \varphi$.

48. Which of the following formulas are logical axioms? (P is a unary and Q a binary predicate symbol.)
    $[(\forall x P(x) \to \forall y P(y)) \to P(z)] \to [\forall x P(x) \to (\forall y P(y) \to P(z))]$,
    $\forall y[\forall x(P(x) \to P(x)) \to (P(c) \to P(c))]$, $\forall x \exists y Q(x, y) \to \exists y Q(y, y)$.

49. Prove that for all formulas φ, ψ:
    - $\vdash_{\mathcal{A}_1} \forall x\varphi \to \exists x\varphi$ and
    - if $\vdash_{\mathcal{A}_1} \varphi \to \psi$, then $\vdash_{\mathcal{A}_1} \forall x\varphi \to \forall x\psi$.

50. Prove that for arbitrary predicate symbols P, Q:

$$\{Q(x), \forall y(Q(y) \to \forall z P(z))\} \vdash_{\mathcal{A}_1} \forall x P(x).$$

51. Prove that for every binary predicate symbol P:
    - $\vdash_{\mathcal{A}_1} \forall x \forall y \forall z (x \approx y \to (y \approx z \to x \approx z))$ and
    - $\vdash_{\mathcal{A}_1} \forall y_1 \forall y_2 \forall z_1 \forall z_2 [y_1 \approx z_1 \to (y_2 \approx z_2 \to (P(y_1, y_2) \to P(z_1, z_2)))]$.

52. Prove that for any terms t, r, s:
    - $\vdash_{\mathcal{A}_1} t \approx t$
    - $\vdash_{\mathcal{A}_1} t \approx s \to s \approx t$
    - $\vdash_{\mathcal{A}_1} t \approx s \to (s \approx r \to t \approx r)$
    - $\vdash_{\mathcal{A}_1} s \approx t \to (r \approx t \to s \approx r)$.

53. Prove that for every formula φ, if x is substitutable by y in φ, then

$$\vdash_{\mathcal{A}_1} x \approx y \to (\varphi \to \varphi^*),$$

    where φ^* is the formula obtained from φ by replacing x, in some or all its free occurrences, by y.

54. Prove that for any binary predicate symbol P:

$$\forall x \forall y P(x, y) \vdash_{\mathcal{A}_1} \forall y \forall x P(y, x).$$

55. Prove that for any unary predicate symbol Q:

$$\vdash_{\mathcal{A}_1} Q(y) \leftrightarrow \forall x (x \approx y \to Q(x)).$$

56. Prove Generalization on Constants.
    Hint. Suppose that $\varphi_1, ..., \varphi_n = \varphi$ is a formal proof from T and y a variable that does not occur in any of the formulas $\varphi_1, \ldots, \varphi_n$. Prove that the sequence $(\varphi_1)^c_y, \ldots, (\varphi_n)^c_y$ is a formal proof of φ^c_y from T.
57. Using examples, show that the conditions on c that are mentioned in Theorems 3.3.5, 3.3.6 are necessary.
58. Let φ be a formula and x, y variables. Prove that
    - It is not always true that $(\varphi^x_y)^y_x$ is identical to φ.
    - If y does not occur at all in φ, then y is substitutable by x in φ^x_y and $(\varphi^x_y)^y_x = \varphi$.

59. Prove the following corollary of Generalization on Constants: If $T \vdash_{\mathcal{A}_1} \varphi^y_c$ and c does not occur in any element of $T \cup \{\varphi\}$, then $T \vdash_{\mathcal{A}_1} \forall y \varphi$.

60. Prove that the following formulas are formal theorems of $\mathcal{A}_1$, for any variable x and any formulas φ, ψ, χ:
    1) $\exists x(\varphi \vee \psi) \leftrightarrow \exists x\varphi \vee \exists x\psi$
    2) $\forall x\varphi \vee \forall x\psi \rightarrow \forall x(\varphi \vee \psi)$
    3) $\exists x(\varphi \wedge \psi) \rightarrow \exists x\varphi \wedge \exists x\psi$
    4) $\forall x(\varphi \wedge \psi) \leftrightarrow \forall x\varphi \wedge \forall x\psi$
    5) $\forall x(\varphi \rightarrow \psi) \rightarrow (\exists x\varphi \rightarrow \exists x\psi)$.

61. Prove that the logical axioms are valid formulas and that Modus Ponens preserves logical implication (that is, if $T \models \psi$ and $T \models \psi \rightarrow \chi$, then $T \models \chi$).

62. Prove that the second part of the Soundness Theorem implies the first part.

63. Prove that the first part of the Completeness Theorem is equivalent to the second.

64. Let $\mathcal{L}_1, \mathcal{L}_1'$ be first-order languages such that $\mathcal{L}_1'$ is an extension of $\mathcal{L}_1$, that is, it includes all the non-logical symbols of $\mathcal{L}_1$. If $\mathfrak{A}'$ is a structure for $\mathcal{L}_1'$ and $\mathfrak{A}' \lceil \mathcal{L}_1$ is the restriction of $\mathfrak{A}'$ to the language $\mathcal{L}_1$ (that is, $\mathfrak{A}', \mathfrak{A}$ have the same universe and the same interpretations for the non-logical symbols of $\mathcal{L}_1$) and v' a valuation in $\mathfrak{A}'$, then for every formula φ of $\mathcal{L}_1$:

$$\mathfrak{A}' \models \varphi[v'] \quad \text{iff} \quad \mathfrak{A}' \lceil \mathcal{L}_1 \models \varphi[v'].$$

65. Suppose that T, S are sets of sentences of $\mathcal{L}_1$. Prove that if T and S have no common model (that is, there is no structure $\mathfrak{A}$ such that $\mathfrak{A} \models T$ and $\mathfrak{A} \models S$), then there exists a sentence φ such that $T \vdash_{\mathcal{A}_1} \varphi$ and $S \vdash_{\mathcal{A}_1} \neg\varphi$.

66. We consider the set $T = Th\mathfrak{N} \cup \{x_0 \not\approx \underline{n} : n \in \mathbb{N}\}$, where $Th\mathfrak{N}$ denotes the set of sentences of $\mathfrak{L}_1^{nt}$ which are true in $\mathfrak{N}$ and $\underline{n}$ is the term $\mathbf{0}'^{\cdots'}$ (n times). Prove that T is satisfiable.
    Note. The structures in which T is satisfied are called *non-standard models of arithmetic*.

67. **Definition.** Let T be a set of sentences of $\mathcal{L}_1$ and K a class of structures for $\mathcal{L}_1$.
    a) $ModT$ denotes the class of models of T.
    b) ThK denotes the set
    $$\{\varphi \in S(\mathcal{L}_1) : \mathfrak{A} \models \varphi \text{ for every structure } \mathfrak{A} \text{ in } K\}.$$

Prove that for any sets T, S of sentences of $\mathcal{L}_1$ and arbitrary classes of structures K, L the following hold:

1) If $T \subseteq S$, then $ModS \subseteq ModT$
2) If $K \subseteq L$, then $ThL \subseteq ThK$
3) $T \subseteq ThModT$ and $K \subseteq ModThK$
4) $ModT = ModThModT$ and $ThK = ThModThK$.

68. Let T, S be sets of sentences such that $T \subseteq S$, T is complete and S is satisfiable. Show that $T = S$.
69. Prove that a), b) in the first case of the proof of Theorem 3.5.1 hold for the formula $\overline{\varphi}$.
70. Find formulas in prenex normal form that are logically equivalent to the formulas

$$\forall x(P(x) \to x \approx y) \to (\exists z P(z) \to \exists u z \approx u),$$
$$\exists x x \approx y \to (P(x) \to \neg \exists z x \approx z).$$

71. Prove part a) of Theorem 3.6.1.
72. Let $\mathfrak{A}$ be a structure for $\mathcal{L}_1$ and g a 1-1 function with domain $|\mathfrak{A}|$. Prove that there exists a unique structure $\mathfrak{B}$ such that $g : \mathfrak{A} \cong \mathfrak{B}$.
73. Let $\mathfrak{Z}$ be the structure for $\mathcal{L}_1^{nt}$ that is defined as follows:

$\mathfrak{Z} = \mathbb{Z}$,
$'^{\mathfrak{Z}}$ is the function with $'^{\mathfrak{Z}}(n) = n +^{\mathbb{Z}} 1$ and $\mathbf{0}^{\mathfrak{Z}} = 0$,
$\oplus^{\mathfrak{Z}}$ is the usual operation of addition on $\mathbb{Z}$,
$\odot^{\mathfrak{Z}}$ is the usual operation of multiplication on $\mathbb{Z}$.

a) Is it true that $\mathfrak{N} \subseteq \mathfrak{Z}$;
b) Is it true that $\mathfrak{N} \equiv \mathfrak{Z}$;
c) Is it true that $\mathfrak{N} \cong \mathfrak{Z}$; If not, is there an isomorphism of $\mathfrak{N}$ into $\mathfrak{Z}$;

74. Let $\mathfrak{N}_{am}$ be the structure for $\mathcal{L}_1^{am}$ that is defined as follows:

$$|\mathfrak{N}_\pi| = \mathbb{N} \text{ and } \oplus^{\mathfrak{N}}, \odot^{\mathfrak{N}} \text{ are the usual operations of addition and multiplication on } \mathbb{N}.$$

Prove that the following relations are definable in $\mathfrak{N}_{am}$:

$$\{1\}, \{\langle m, n\rangle : n = m'\}, \{\langle m, n\rangle : m < n\}.$$

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