small as the number of S's, then every P is an S. For if, in counting the P's, we begin with the S's (which are a part of them), and having counted all the S's arrive at the number n, there will remain over no P's not S's. For if there were any, the number of P's would count up to more than n. From this we deduce the validity of the following mode of inference:

Every Texan kills a Texan,
Nobody is killed by but one person,
Hence, every Texan is killed by a Texan,

supposing Texans to be a finite lot. For, by the first premise, every Texan killed by a Texan is a Texan killer of a Texan. By the second premise, the Texans killed by Texans are as many as the Texan killers of Texans. Whence we conclude that every Texan killer of a Texan is a Texan killed by a Texan, or, by the first premise, every Texan is killed by a Texan. This mode of reasoning is frequent in the theory of numbers.

D. ON THE ALGEBRA OF LOGIC: A CONTRIBUTION TO THE PHILOSOPHY OF NOTATION (*PEIRCE 1885*)

Peirce's next two papers, the brief note *1883* and the longer article, 'On the algebra of logic' (*1885*), presented his discovery of the quantifiers. Gottlob Frege, in his *Begriffschrift* (*1879*), had already made the same discovery (and had carried the analysis of number further than Peirce was to do); and Peirce's student O.H. Mitchell (*1883*) had, under Peirce's guidance, in effect developed a system of monadic quantification theory. But these discoveries had little impact at the time. It was Peirce's *1885* that successfully launched upon the world the theory of quantification *via* the three volumes of Schröder's *Vorlesungen über die Algebra der Logik* (*1890*, *1891*, *1895*). (These *Vorlesungen* were for some years the standard reference work in mathematical logic, and were largely based on Peirce's discoveries.)

'On the algebra of logic' is noteworthy for other reasons as well. It begins with an important passage (§2) on the propositional calculus, containing the first explicit use of two truth-values.[a] Peirce then describes a decision procedure for the truth of any formula of the sentential calculus: '[T]o find whether a formula is necessarily true substitute **f** and **v** for the letters and see whether

[a] Truth-values and truth-tables have their roots in the work of George Boole (*1854*, pp. 72–6). They are implicit in the work of Venn and Jevons (see the discussion in *Lewis 1918*, pp. 74 and 175 ff.). Truth-tables are also implicit in §5 of the *Begriffschrift*, although Frege did not introduce 'The True' and 'The False' until his *1891*. For further references on this topic, see *Post 1921*, reproduced in *van Heijenoort 1967*, pp. 264–83.

it can be supposed false by any such assignment of values.' He also gives a lucid defence of material implication, and shows how to define negation in terms of implication and a special symbol α for absurdity. Next (§3) Peirce treats first-order quantification theory. He coins the term 'quantifier' (probably derived from Sir William Hamilton's terminology of 'quantifying the predicate'); the propositional matrix of a quantified formula he calls its 'Boolian'. He uses the symbols Σ and Π to represent the existential and universal quantifiers. This felicitous notation—like his use of the Boolean sentential connectives—was a major advantage of his system, and enabled Peirce to discuss the rules for transforming a quantified formula into prenex normal form.[b] Peirce next (§4) proceeds to second-intentional logic. (Following the Schoolmen, he clearly distinguishes first-intentional logic from second-intentional.) He states the modern second-order definition of identity, avoiding Leibniz's confusion of use and mention.[c] The paper closes with his definition of a finite set as one which cannot be put into a one-to-one correspondence with any proper subset. (Dedekind's later independent definition of an infinite collection in his *1888* is equivalent to Peirce's.)

References to *Peirce 1885* should be to the section numbers, which appear in the original text.

I.—THREE KINDS OF SIGNS

Any character or proposition either concerns one subject, two subjects, or a plurality of subjects. For example, one particle has mass, two particles attract one another, a particle revolves about the line joining two others. A fact concerning two subjects is a dual character or relation; but a relation which is a mere combination of two independent facts concerning the two subjects may be called *degenerate*, just as two lines are called a degenerate conic. In like manner a plural character or conjoint relation is to be called degenerate if it is a mere compound of dual characters.

A sign is in a conjoint relation to the thing denoted and to the mind. If this triple relation is not of a degenerate species, the sign is related to its object only

[b] The Peircean notation was standard in the work of the Polish set-theoretic logicians of the 1920s and 1930s: see the papers of Kuratowski or Sierpinski, or any volume of *Fundamenta mathematicae* from that period. Löwenheim and Skolem continued to use the Peirce-Schröder notation well into the twentieth century; and as late as his '*Einkleidung der Mathematik in Schröderschen Relativkalkül*' (*Löwenheim 1940*), Löwenheim was urging the superiority of the Peirce-Schröder notation to that of Peano and Russell.

[c] Leibniz's definition of identity was as follows: 'Those things are the same of which one can be substituted for the other *salva veritate*'—'Eadem sunt quorum unum potest substitui alteri salva veritate' (*Leibniz 1875-90*, Vol. vii, pp. 228, 236). Quine observes that Aristotle and Aquinas had already given a similar definition (*Quine 1960*, p. 116).

in consequence of a mental association, and depends upon a habit. Such signs are always abstract and general, because habits are general rules to which the organism has become subjected. They are, for the most part, conventional or arbitrary. They include all general words, the main body of speech, and any mode of conveying a judgment. For the sake of brevity I will call them *tokens*.

But if the triple relation between the sign, its object, and the mind, is degenerate, then of the three pairs

sign	object
sign	mind
object	mind

two at least are in dual relations which constitute the triple relation. One of the connected pairs must consist of the sign and its object, for if the sign were not related to its object except by the mind thinking of them separately, it would not fulfil the function of a sign at all. Supposing, then, the relation of the sign to its object does not lie in a mental association, there must be a direct dual relation of the sign to its object independent of the mind using the sign. In the second of the three cases just spoken of, this dual relation is not degenerate, and the sign signifies its object solely by virtue of being really connected with it. Of this nature are all natural signs and physical symptoms. I call such a sign an *index*, a pointing finger being the type of the class.

The index asserts nothing; it only says "There!" It takes hold of our eyes, as it were, and forcibly directs them to a particular object, and there it stops. Demonstrative and relative pronouns are nearly pure indices, because they denote things without describing them; so are the letters on a geometrical diagram, and the subscript numbers which in algebra distinguish one value from another without saying what those values are.

The third case is where the dual relation between the sign and its object is degenerate and consists in a mere resemblance between them. I call a sign which stands for something merely because it resembles it, an *icon*. Icons are so completely substituted for their objects as hardly to be distinguished from them. Such are the diagrams of geometry. A diagram, indeed, so far as it has a general signification, is not a pure icon; but in the middle part of our reasonings we forget that abstractness in great measure, and the diagram is for us the very thing. So in contemplating a painting, there is a moment when we lose the consciousness that it is not the thing, the distinction of the real and the copy disappears, and it is for the moment a pure dream,—not any particular existence, and yet not general. At that moment we are contemplating an *icon*.

I have taken pains to make my distinction* of icons, indices, and tokens clear, in order to enunciate this proposition: in a perfect system of logical notation signs of these several kinds must all be employed. Without tokens there would be no generality in the statements, for they are the only general signs; and generality

* See *Proceedings American Academy of Arts and Sciences*, Vol. VII, p. 294, May 14, 1867.

is essential to reasoning. Take, for example, the circles by which Euler represents the relations of terms. They well fulfil the function of icons, but their want of generality and their incompetence to express propositions must have been felt by everybody who has used them. Mr. Venn has, therefore, been led to add shading to them; and this shading is a conventional sign of the nature of a token. In algebra, the letters, both quantitative and functional, are of this nature. But tokens alone do not state what is the subject of discourse; and this can, in fact, not be described in general terms; it can only be indicated. The actual world cannot be distinguished from a world of imagination by any description. Hence the need of pronouns and indices, and the more complicated the subject the greater the need of them. The introduction of indices into the algebra of logic is the greatest merit of Mr. Mitchell's system.* He writes F_1 to mean that the proposition F is true of every object in the universe, and F_u to mean that the same is true of some object. This distinction can only be made in some such way as this. Indices are also required to show in what manner other signs are connected together. With these two kinds of signs alone any proposition can be expressed; but it cannot be reasoned upon, for reasoning consists in the observation that where certain relations subsist certain others are found, and it accordingly requires the exhibition of the relations reasoned with in an icon. It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. Various have been the attempts to solve the paradox by breaking down one or other of these assertions, but without success. The truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. For instance, take the syllogistic formula,

$$\begin{array}{rcl} \text{All } M & \text{is} & P \\ S & \text{is} & M \\ \therefore S & \text{is} & P. \end{array}$$

This is really a diagram of the relations of S, M, and P. The fact that the middle term occurs in the two premises is actually exhibited, and this must be done or the notation will be of no value. As for algebra, the very idea of the art is that it presents formulae which can be manipulated, and that by observing the effects of such manipulation we find properties not to be otherwise discerned. In such manipulation, we are guided by previous discoveries which are embodied in general formulae. These are patterns which we have the right to imitate in our procedure, and are the *icons par excellence* of algebra. The letters of applied

* *Studies in Logic*, by members of the Johns Hopkins University. Boston: Little & Brown, 1883.

algebra are usually tokens, but the x, y, z, etc. of a general formula, such as

$$(x + y)z = xz + yz,$$

are blanks to be filled up with tokens, they are indices of tokens. Such a formula might, it is true, be replaced by an abstractly stated rule (say that multiplication is distributive); but no application could be made of such an abstract statement without translating it into a sensible image.

In this paper, I purpose to develope an algebra adequate to the treatment of all problems of deductive logic, showing as I proceed what kinds of signs have necessarily to be employed at each stage of the development. I shall thus attain three objects. The first is the extension of the power of logical algebra over the whole of its proper realm. The second is the illustration of principles which underlie all algebraic notation. The third is the enuneration of the essentially different kinds of necessary inference; for when the notation which suffices for exhibiting one inference is found inadequate for explaining another, it is clear that the latter involves an inferential element not present to the former. Accordingly, the procedure contemplated should result in a list of categories of reasoning, the interest of which is not dependent upon the algebraic way of considering the subject. I shall not be able to perfect the algebra sufficiently to give facile methods of reaching logical conclusions: I can only give a method by which any legitimate conclusion may be reached and any fallacious one avoided. But I cannot doubt that others, if they will take up the subject, will succeed in giving the notation a form in which it will be highly useful in mathematical work. I even hope that what I have done may prove a first step toward the resolution of one of the main problems of logic, that of producing a method for the discovery of methods in mathematics.

II.—NON-RELATIVE LOGIC

According to ordinary logic, a proposition is either true or false, and no further distinction is recognized. This is the descriptive conception, as the geometers say; the metric conception would be that every proposition is more or less false, and that the question is one of amount. At present we adopt the former view.

Let propositions be represented by quantities. Let $\mathbf{v}$ and $\mathbf{f}$ be two constant values, and let the value of the quantity representing a proposition be $\mathbf{v}$ if the proposition is true and be $\mathbf{f}$ if the proposition is false. Thus, x being a proposition, the fact that x is either true or false is written

$$(x - \mathbf{f})(\mathbf{v} - x) = 0.$$

So

$$(x - \mathbf{f})(\mathbf{v} - y) = 0$$

will mean that either x is false or y is true. This may be said to be the same as 'if x is true, y is true'. A hypothetical proposition, generally, is not confined to stating what actually happens, but states what is invariably true throughout a universe of possibility. The present proposition is, however, limited to that one individual state of things, the Actual.

We are, thus, already in possession of a logical notation, capable of working syllogism. Thus, take the premises, 'if x is true, y is true', and 'if y is true, z is true'. These are written

$$(x - \mathbf{f})(\mathbf{v} - y) = 0$$

$$(y - \mathbf{f})(\mathbf{v} - z) = 0.$$

Multiply the first by $(\mathbf{v} - z)$ and the second by $(x - \mathbf{f})$ and add. We get

$$(x - \mathbf{f})(\mathbf{v} - \mathbf{f})(\mathbf{v} - z) = 0,$$

or dividing by $\mathbf{v} - \mathbf{f}$, which cannot be 0,

$$(x - \mathbf{f})(\mathbf{v} - z) = 0;$$

and this states the syllogistic conclusion, "if x is true, z is true".

But this notation shows a blemish in that it expresses propositions in two distinct ways, in the form of quantities, and in the form of equations; and the quantities are of two kinds, namely those which must be either equal to $\mathbf{f}$ or to $\mathbf{v}$, and those which are equated to *zero*. To remedy this, let us discard the use of equations, and perform no operations which can give rise to any values other than $\mathbf{f}$ and $\mathbf{v}$.

Of operations upon a simple variable, we shall need but one. For there are but two things that can be said about a single proposition, by itself; that it is true and that it is false,

$$x = \mathbf{v} \text{ and } x = \mathbf{f}.$$

The first equation is expressed by x itself, the second by any function, ϕ, of x, fulfilling the conditions

$$\phi\mathbf{v} = \mathbf{f} \quad \phi\mathbf{f} = \mathbf{v}.$$

The simplest solution of these equations is

$$\phi x = \mathbf{f} + \mathbf{v} - x.$$

A product of n factors of the two forms $(x - \mathbf{f})$ and $(\mathbf{v} - y)$, if not zero equals $(\mathbf{v} - \mathbf{f})^n$. Write P for the product. Then $\mathbf{v} - \dfrac{P}{(\mathbf{v} - \mathbf{f})^{n-1}}$ is the simplest function of the variables which becomes $\mathbf{v}$ when the product vanishes and $\mathbf{f}$ when it does not. By this means any proposition relating to a single individual can be expressed.

If we wish to use algebraical signs with their usual significations, the meanings of the operations will entirely depend upon those of $\mathbf{f}$ and $\mathbf{v}$. Boole chose $\mathbf{v} = 1$, $\mathbf{f} = 0$. This choice gives the following forms:

$$\mathbf{f} + \mathbf{v} - x = 1 - x$$

which is best written $\bar{x}$.

$$\mathbf{v} - \frac{(x - \mathbf{f})\,(\mathbf{v} - y)}{\mathbf{v} - \mathbf{f}} = 1 - x + xy = \overline{x\bar{y}}$$

$$\mathbf{v} - \frac{(\mathbf{v} - x)\,(\mathbf{v} - y)}{\mathbf{v} - \mathbf{f}} = x + y - xy$$

$$\mathbf{v} - \frac{(\mathbf{v} - x)\,(\mathbf{v} - y)\,(\mathbf{v} - z)}{(\mathbf{v} - \mathbf{f})^2} = x + y + z - xy - xz - yz + xyz$$

$$\mathbf{v} - \frac{(x - \mathbf{f})\,(y - \mathbf{f})}{\mathbf{v} - \mathbf{f}} = 1 - xy = \overline{xy}.$$

It appears to me that if the strict Boolian system is used, the sign + ought to be altogether discarded. Boole and his adherent, Mr. Venn (whom I never disagree with without finding his remarks profitable), prefer to write $x + \bar{x}y$ in place of $\overline{\bar{x}\bar{y}}$. I confess I do not see the advantage of this, for the distributive principle holds equally well when written

$$\overline{\bar{x}\bar{y}}z = \overline{\overline{xz}\,\overline{yz}}$$

$$\overline{\overline{xy}\,\bar{z}} = \overline{\bar{x}\bar{z}}.\overline{\bar{y}\bar{z}}.$$

The choice of $\mathbf{v} = 1$, $\mathbf{f} = 0$, is agreeable to the received measurement of probabililies. But there is no need, and many times no advantage, in measuring probabilities in this way. I presume that Boole, in the formation of his algebra, at first considered the letters as denoting propositions or events. As he presents the subject, they are class-names; but it is not necessary so to regard them. Take, for example, the equation

$$t = n + hf,$$

which might mean that the body of taxpayers is composed of all the natives, together with householding foreigners. We might reach the signification by either of the following systems of notation, which indeed differ grammatically rather than logically.

Sign	Signification 1st System	Signification 2nd System
t	Taxpayer	He is a Taxpayer
n	Native	He is a Native
h	Householder	He is a Householder
f	Foreigner	He is a Foreigner

There is no *index* to show who the "He" of the second system is, but that makes no difference. To say that he is a taxpayer is equivalent to saying that he is a native or is a householder and a foreigner. In this point of view, the constants

1 and 0 are simply the probabilities, to one who knows, of what is true and what is false; and thus unity is conferred upon the whole system.

For my part, I prefer for the present not to assign determinate values to $\mathbf{f}$ and $\mathbf{v}$, nor to identify the logical operations with any special arithmetical ones, leaving myself free to do so hereafter in the manner which may be found most convenient. Besides, the whole system of importing arithmetic into the subject is artificial, and modern Boolians do not use it. The algebra of logic should be self-developed, and arithmetic should spring out of logic instead of reverting to it. Going back to the beginning, let the writing of a letter by itself mean that a certain proposition is true. This letter is a *token*. There is a general understanding that the actual state of things or some other is referred to. This understanding must have been established by means of an *index*, and to some extent dispenses with the need of other indices. The denial of a proposition will be made by writing a line over it.

I have elsewhere shown that the fundamental and primary mode of relation between two propositions is that which we have expressed by the form

$$\mathbf{v} - \frac{(x - \mathbf{f})(\mathbf{v} - y)}{\mathbf{v} - \mathbf{f}}.$$

We shall write this

$$x \prec y,$$

which is also equivalent to

$$(x - \mathbf{f})(\mathbf{v} - y) = 0.$$

It is stated above that this means "if x is true, y is true". But this meaning is greatly modified by the circumstance that only the actual state of things is referred to.

To make the matter clear, it will be well to begin by defining the meaning of a hypothetical proposition, in general. What the usages of language may be does not concern us; language has its meaning modified in technical logical formulae as in other special kinds of discourse. The question is what is the sense which is most usefully attached to the hypothetical proposition in logic? Now, the peculiarity of the hypothetical proposition is that it goes out beyond the actual state of things and declares what *would* happen were things other than they are or may be. The utility of this is that it puts us in possession of a rule, say that "if A is true, B is true", such that should we hereafter learn something of which we are now ignorant, namely that A is true, then, by virtue of this rule, we shall find that we know something else, namely, that B is true. There can be no doubt that the Possible, in its primary meaning, is that which may be true for aught we know, that whose falsity we do not know. The purpose is subserved, then, if, throughout the whole range of possibility, in every state of things in which A is true, B is true too. The hypothetical proposition may therefore be falsified by a single state of things, but only by one in which A is true while B is false. States of things in which A is false, as well as those in which B is true, cannot falsify it. If, then, B is a proposition true in every case throughout the whole range of possibility, the hypothetical proposition, taken in its logical sense, ought to be regarded as true, whatever may be the usage of ordinary speech. If, on the other hand, A is in no case true, throughout the range of possibility, it is a matter of indifference whether the hypothetical be understood to be true or not, since it is useless. But it will be more simple to class it among true propositions, because the cases in which the antecedent is false do not, in any other case, falsify a hypothetical. This, at any rate, is the meaning which I shall attach to the hypothetical proposition in general, in this paper.

The range of possibility is in one case taken wider, in another narrower; in the present case it is limited to the actual state of things. Here, therefore, the proposition

$$a \prec b$$

is true if a is false or if b is true, but is false if a is true while b is false. But though we limit ourselves to the actual state of things, yet when we find that a formula of this sort is true by logical necessity, it becomes applicable to any single state of things throughout the range of logical possibility. For example, we shall see that from $x \overline{\prec} y$ we can infer $z \prec x$. This does not mean that because in the actual state of things x is true and y false, therefore in every state of things either z is false or x true; but it does mean that in whatever state of things we find x true and y false, in that state of things either z is false or x is true. In that sense, it is not limited to the actual state of things, but extends to any single state of things.

The *first icon* of algebra is contained in the formula of identity

$$x \prec x.$$

This formula does not of itself justify any transformation, any inference. It only justifies our continuing to hold what we have held (though we may, for instance, forget how we were originally justified in holding it).

The *second icon* is contained in the rule that the several antecedents of a *consequentia* may be transposed; that is, that from

$$x \prec (y \prec z)$$

we can pass to

$$y \prec (x \prec z).$$

This is stated in the formula

$$\{x \prec (y \prec z)\} \prec \{y \prec (x \prec z)\}.$$

Because this is the case, the brackets may be omitted, and we may write

$$y \prec x \prec z.$$

By the formula of identity

$$(x \prec y) \prec (x \prec y);$$

and transposing the antecedents

$$x \prec \{(x \prec y) \prec y\}$$

or, omitting the unnecessary brackets

$$x \prec (x \prec y) \prec y.$$

This is the same as to say that if in any state of things x is true, and if the proposition "if x, then y" is true, then in that state of things y is true. This is the *modus ponens* of hypothetical inference, and is the most rudimentary form of reasoning.

To say that $(x \prec x)$ is generally true is to say that it is so in every state of things, say in that in which y is true; so that we may write

$$y \prec (x \prec x),$$

and then, by transposition of antecedents,

$$x \prec (y \prec x),$$

or from x we may infer $y \prec x$.

The *third icon* is involved in the principle of the transitiveness of the copula, which is stated in the formula

$$(x \prec y) \prec (y \prec z) \prec x \prec z.$$

According to this, if in any case y follows from x and z from y, then z follows from x. This is the principle of the syllogism in *Barbara*.

We have already seen that from x follows $y \prec x$. Hence, by the transitiveness of the copula, if from $y \prec x$ follows z, then from x follows z, or from

$$(y \prec x) \prec z$$

follows

$$x \prec z,$$

or

$$\{(y \prec x) \prec z\} \prec x \prec z.$$

The original notation $x \prec y$ served without modification to express the pure formula of identity. An enlargement of the conception of the notation so as to make the terms themselves complex was required to express the principle of the transposition of antecedents; and this new *icon* brought out new propositions. The third *icon* introduces the image of a chain of consequence. We must now again enlarge the notation so as to introduce negation. We have already seen that if a is true, we can write $x \prec a$, whatever x may be. Let b be such that we can write $b \prec x$ whatever x may be. Then b is false. We have here a *fourth icon*, which gives a new sense to several formulæ. Thus the principle of the interchange of antecedents is that from

$$x \prec (y \prec z)$$

we can infer

$$y \prec (x \prec z).$$

Since z is any proposition we please, this is as much as to say that if from the truth of x the falsity of y follows, then from the truth of y the falsity of x follows.

Again the formula

$$x \prec \{(x \prec y) \prec y\}$$

is seen to mean that from x we can infer that anything we please follows from that things ⟦*sic*⟧ following from x, and *a fortiori* from everything following from x. This is, therefore, to say that from x follows the falsity of the denial of x; which is the principle of contradiction.

Again the formula of the transitiveness of the copula, or

$$\{x \prec y\} \prec \{(y \prec z) \prec (x \prec z)\}$$

is seen to justify the inference

$$x \prec y$$
$$\therefore\ \bar{y} \prec \bar{x}.$$

The same formula justifies the *modus tollens*,

$$x \prec y$$
$$\bar{y}$$
$$\therefore\ \bar{x}.$$

So the formula

$$\{(y \prec x) \prec z\} \prec (x \prec z)$$

shows that from the falsity of $y \prec x$ the falsity of x may be inferred.

All the traditional moods of syllogism can easily be reduced to *Barbara* by this method.

A *fifth icon* is required for the principle of excluded middle and other propositions connected with it. One of the simplest formulæ of this kind is

$$\{(x \prec y) \prec x\} \prec x.$$

This is hardly axiomatical. That it is true appears as follows. It can only be false by the final consequent x being false while its antecedent $(x \prec y) \prec x$ is true. If this is true, either its consequent, x, is true, when the whole formula would be true, or its antecedent $x \prec y$ is false. But in the last case the antecedent of $x \prec y$, that is x, must be true.*

From the formula just given, we at once get

$$\{(x \prec y) \prec \alpha\} \prec x,$$

where the α is used in such a sense that $(x \prec y) \prec \alpha$ means that from $(x \prec y)$ every proposition follows. With that understanding, the formula states the principle of excluded middle, that from the falsity of the denial of x follows the truth of x.

The logical algebra thus far developed contains signs of the following kinds:

1st, Tokens; signs of simple propositions, as t for 'He is a taxpayer', etc.

2d, The single operative sign $\prec$; also of the nature of a token.

3d, The juxtaposition of the letters to the right and left of the operative sign. This juxtaposition fulfils the function of an index, in indicating the connections of the tokens.

4th, The parentheses, subserving the same purpose.

5th, The letters α, β, etc. which are indices of no matter what tokens, used for expressing negation.

6th, The indices of tokens, x, y, z, etc. used in the general formulæ.

7th, The general formulæ themselves, which are *icons*, or exemplars of algebraic proceedings.

8th, The fourth *icon* which affords a second interpretation of the general formulæ.

We might dispense with the fifth and eighth species of signs—the devices by

* It is interesting to observe that this reasoning is dilemmatic. In fact, the dilemma involves the fifth icon. The dilemma was only introduced into logic from rhetoric by the humanists of the *renaissance*; and at that time logic was studied with so little accuracy that the peculiar nature of this mode of reasoning escaped notice. I was thus led to suppose that the whole non-relative logic was derivable from the principles of the ancient syllogistic, and this error is involved in Chapter II of my paper in the third volume of this Journal. My friend, Professor Schröder, detected the mistake and showed that the distributive formulæ

$$(x + y)z \prec xz + yz$$
$$(x + z)(y + z) \prec xy + z$$

could not be deduced from syllogistic principles. I had myself independently discovered and virtually stated the same thing. (*Studies in Logic*, p. 189). There is some disagreement as to the definition of the dilemma (see Keynes's excellent *Formal Logic*, p. 241); but the most useful definition would be a syllogism depending on the above distribution formulæ. The distribution formulæ

$$xz + yz \prec (x + y)z$$
$$xy + z \prec (x + z)(y + z)$$

are strictly syllogistic. DeMorgan's added moods are virtually dilemmatic, depending on the principle of excluded middle.

which we express negation—by adopting a second operational sign $\overline{\prec}$, such that

$$x \overline{\prec} y$$

should mean that $x = \mathbf{v}$, $y = \mathbf{f}$. With this we should require new indices of connections, and new general formulae. Possibly this might be the preferable notation. We should thus have two operational signs but no sign of negation. The forms of Boolian algebra hitherto used, have either two operational signs and a special sign of negation, or three operational signs. One of the operational signs is in that case superfluous. Thus, in the usual notation we have

$$\overline{x + y} = \bar{x}\bar{y}$$

$$\bar{x} + \bar{y} = \overline{xy}$$

showing two modes of writing the same fact. The apparent balance between the two sets of theorems exhibited so strikingly by Schröder, arises entirely from this double way of writing everything. But while the ordinary system is not so analytically fitted to its purpose as that here set forth, the character of superfluity here, as in many other cases in algebra, brings with it great facility in working.

The general formulæ given above are not convenient in practice. We may dispense with them altogether, as well as with one of the indices of tokens used in them, by the use of the following rules. A proposition of the form

$$x \prec y$$

is true if $x = \mathbf{f}$ or $y = \mathbf{v}$. It is only false if $y = \mathbf{f}$ and $x = \mathbf{v}$. A proposition written in the form

$$x \overline{\prec} y$$

is true if $x = \mathbf{v}$ and $y = \mathbf{f}$, and is false if either $x = \mathbf{f}$ or $y = \mathbf{v}$. Accordingly, to find whether a formula is necessarily true substitute $\mathbf{f}$ and $\mathbf{v}$ for the letters and see whether it can be supposed false by any such assignment of values. Take, for example, the formula

$$(x \prec y) \prec \{(y \prec z) \prec (x \prec z)\}.$$

To make this false we must take

$$(x \prec y) = \mathbf{v}$$

$$\{(y \prec z) \prec (x \prec z)\} = \mathbf{f}.$$

The last gives $(y \prec z) = \mathbf{v}$, $(x \prec z) = \mathbf{f}$, $x = \mathbf{v}$, $z = \mathbf{f}$. Substituting these values in

$$(x \prec y) = \mathbf{v} \quad (y \prec z) = \mathbf{v}$$

we have $$(\mathbf{v} \prec y) = \mathbf{v} \quad (y \prec \mathbf{f}) = \mathbf{v},$$

which cannot be satisfied together.

As another example, required the conclusion from the following premises. Any one I might marry would be either beautiful or plain; any one whom I might marry would be a woman; any beautiful woman would be an ineligible wife; any plain woman would be an ineligible wife. Let

m be any one whom I might marry,
b, beautiful,
p, plain,
w, woman,
i, ineligible.

Then the premises are

$$
\begin{aligned}
&m \prec (b \prec \mathbf{f}) \prec p,\\
&m \prec w,\\
&w \prec b \prec i,\\
&w \prec p \prec i.
\end{aligned}
$$

Let x be the conclusion. Then

$$
[m \prec (b \prec \mathbf{f}) \prec p] \prec (m \prec w) \prec (w \prec b \prec i) \prec (w \prec p \prec i) \prec x
$$

is necessarily true. Now if we suppose $m = \mathbf{v}$, the proposition can only be made false by putting $w = \mathbf{v}$ and either b or $p = \mathbf{v}$. In this case the proposition can only be made false by putting $i = \mathbf{v}$. If therefore, x can only be made $\mathbf{f}$ by putting $m = \mathbf{v}$, $i = \mathbf{f}$, that is if $x = (m \prec i)$ the proposition is necessarily true.

In this method, we introduce the two special tokens of second intention $\mathbf{f}$ and $\mathbf{v}$, we retain two indices of tokens x and y, and we have a somewhat complex *icon*, with a special prescription for its use.

A better method may be found as follows. We have seen that

$$x \prec (y \prec z)$$

may be conveniently written

$$x \prec y \prec z;$$

while

$$(x \prec y) \prec z$$

ought to retain the parenthesis. Let us extend this rule, so as to be more general, and hold it necessary *always* to include the antecedent in parenthesis. Thus, let us write

$$(x) \prec y$$

instead of $x \prec y$. If now, we merely change the external appearance of two signs; namely, if we use the vinculum instead of the parenthesis, and the sign $+$ in place of $\prec$, we shall have

$$x \prec y \quad \text{written} \quad \bar{x} + y$$
$$x \prec y \prec z \quad '' \quad \bar{x} + \bar{y} + z$$
$$(x \prec y) \prec z \quad '' \quad \overline{\bar{x} + y} + z, \text{ etc.}$$

We may further write for $x \overline{\prec} y$, $\overline{\bar{x} + y}$ implying that $\bar{x}+y$ is an antecedent for whatever consequent may be taken, and the vinculum becomes identified with the sign of negation. We may also use the sign of multiplication as an abbreviation, putting

$$xy = \overline{\bar{x} + \bar{y}} = \overline{x \prec \bar{y}}.$$

This subjects addition and multiplication to all the rules of ordinary algebra, and also to the following:

$$y + x\bar{x} = y \quad y(x + \bar{x}) = y$$
$$x + \bar{x} = \mathbf{v} \quad \bar{x}x = \mathbf{f}$$
$$xy + z = (x + z)(y + z).$$

To any proposition we have a right to add any expression at pleasure; also to strike out any factor of any term. The expressions for different propositions separately known may be multiplied together. These are substantially Mr Mitchell's rules of procedure. Thus the premises of *Barbara* are

$$\bar{x} + y \text{ and } \bar{y} + z.$$

Multiplying these, we get $(\bar{x} + y)(\bar{y} + z) = \bar{x}\bar{y} + yz$.
Dropping $\bar{y}$ and y we reach the conclusion $\bar{x} + z$.

III.—First-intentional logic of relations

The algebra of Boole affords a language by which anything may be expressed which can be said without speaking of more than one individual at a time. It is true that it can assert that certain characters belong to a whole class, but only such characters as belong to each individual separately. The logic of relatives considers statements involving two and more individuals at once. Indices are here required. Taking, first, a degenerate form of relation, we may write $x_i y_j$ to signify that x is true of the individual i while y is true of the individual j. If z be a relative character z_{ij} will signify that i is in that relation to j. In this way we can express relations of considerable complexity. Thus, if

1, 2, 3,
4, 5, 6,
7, 8, 9,

are points in a plane, and l_{123} signifies that 1, 2, and 3 lie on one line, a well-known proposition of geometry may be written

$$l_{159} \prec l_{267} \prec l_{348} \prec l_{147} \prec l_{258} \prec l_{369} \prec l_{123} \prec l_{456} \prec l_{789}.$$

In this notation is involved a *sixth icon.*

We now come to the distinction of *some* and *all*, a distinction which is precisely on a par with that between truth and falsehood; that is, it is descriptive, not metrical.

All attempts to introduce this distinction into the Boolian algebra were more or less complete failures until Mr. Mitchell showed how it was to be effected. His method really consists in making the whole expression of the proposition consist of two parts, a pure Boolian expression referring to an individual and a Quantifying part saying what individual this is. Thus, if k means 'he is a king', and h, 'he is happy', the Boolian

$$(\bar{k} + h)$$

means that the individual spoken of is either not a king or is happy. Now, applying the quantification, we may write

$$\text{Any } (\bar{k} + h)$$

to mean that this is true of any individual in the (limited) universe, or

$$\text{Some } (\bar{k} + h)$$

to mean that an individual exists who is either not a king or is happy. So

$$\text{Some } (kh)$$

means some king is happy, and

$$\text{Any } (kh)$$

means every individual is both a king and happy. The rules for the use of this notation are obvious. The two propositions

$$\text{Any } (x) \quad \text{Any } (y)$$

are equivalent to

$$\text{Any } (xy).$$

From the two propositions

$$\text{Any } (x) \quad \text{Some } (y)$$

we may infer

$$\text{Some } (xy).^*$$

* I will just remark, quite out of order, that the quantification may be made numerical; thus producing the numerically definite inferences of DeMorgan and Boole. Suppose at least $\frac{2}{3}$ of the company have white neckties and at least $\frac{3}{4}$ have dress coats. Let w mean 'he has a white necktie', and d 'he has a dress coat'. Then, the two propositions are

$$\tfrac{2}{3}\,(w) \text{ and } \tfrac{3}{4}\,(d).$$

Mr. Mitchell has also a very interesting and instructive extension of his notation for *some* and *all*, to a two-dimensional universe, that is, to the logic of relatives. Here, in order to render the notation as iconical as possible we may use Σ for *some*, suggesting a sum, and Π for *all*, suggesting a product. Thus $\Sigma_i x_i$ means that x is true of some one of the individuals denoted by i or

$$\Sigma_i x_i = x_i + x_j + x_k + \text{etc.}$$

In the same way, $\Pi_i x_i$ means that x is true of all these individuals, or

$$\Pi_i x_i = x_i x_j x_k, \text{etc.}$$

If x is a simple relation, $\Pi_i \Pi_j x_{ij}$ means that every i is in this relation to every j, $\Sigma_i \Pi_j x_{ij}$ that some one i is in this relation to every j, $\Pi_j \Sigma_i x_{ij}$ that to every j some i or other is in this relation, $\Sigma_i \Sigma_j x_{ij}$ that some i is in this relation to some j. It is to be remarked that $\Sigma_i x_i$ and $\Pi_i x_i$ are only *similar* to a sum and a product; they are not strictly of that nature, because the individuals of the universe may be innumerable.

At this point, the reader would perhaps not otherwise easily get so good a conception of the notation as by a little practice in translating from ordinary language into this system and back again. Let l_{ij} mean that i is a lover of j, and b_{ij} that i is a benefactor of j. Then

$$\Pi_i \Sigma_j l_{ij} b_{ij}$$

means that everything is at once a lover and a benefactor of something; and

$$\Pi_i \Sigma_j l_{ij} b_{ji}$$

that everything is a lover of a benefactor of itself.

$$\Sigma_i \Sigma_k \Pi_j (l_{ij} + b_{jk})$$

means that there are two persons, one of whom loves everything except benefac-

---

These are to be multiplied together. But we must remember that xy is a mere abbreviation for $\overline{\bar{x} + \bar{y}}$, and must therefore write

$$\overline{\overline{\tfrac{2}{3}\, w} + \overline{\tfrac{3}{4}\, d}}.$$

Now $\overline{\frac{2}{3}\, w}$ is the denial of $\frac{2}{3}\, w$, and this denial may be written $(>\frac{1}{3})\bar{w}$, or more than $\frac{1}{3}$ of the universe (the company) have not white neckties. So $\overline{\frac{3}{4}d} = (>\frac{1}{4})\bar{d}$. The combined premises thus become

$$\overline{(>\tfrac{1}{3})\bar{w} + (>\tfrac{1}{4})\bar{d}}.$$

Now $(>\frac{1}{3})\bar{w} + (>\frac{1}{4})\bar{d}$ gives May be $(\frac{1}{3} + \frac{1}{4})\,(\bar{w} + \bar{d})$.

Thus we have

$$\overline{\text{May be } \left(\tfrac{7}{12}\right)\left(\bar{w} + \bar{d}\right)},$$

and this is

$$\left(\text{At least } \tfrac{5}{12}\right)\left(\overline{\bar{w} + \bar{d}}\right),$$

which is the conclusion.

tors of the other (whether he loves any of these or not is not stated). Let g_i mean that i is a griffin, and c_i that i is a chimera, then

$$\Sigma_i \Pi_j (g_i l_{ij} + \bar{c}_j)$$

means that if there be any chimeras there is some griffin that loves them all; while

$$\Sigma_i \Pi_j g_i (l_{ij} + \bar{c}_j)$$

means that there is a griffin and he loves every chimera that exists (if any exist). On the other hand,

$$\Pi_j \Sigma_i g_i (l_{ij} + \bar{c}_j)$$

means that griffins exist (one, at least), and that one or other of them loves each chimera that may exist; and

$$\Pi_j \Sigma_i (g_i l_{ij} + \bar{c}_j)$$

means that each chimera (if there is any) is loved by some griffin or other.

Let us express: every part of the world is either sometimes visited with cholera, and at others with small-pox (without cholera), or never with yellow fever and the plague together. Let

c_{ij}	mean the place i has cholera at the time j.				
s_{ij}	"	"	"	small-pox	" "
y_{ij}	"	"	"	yellow fever	" "
p_{ij}	"	"	"	plague	" "

Then we write

$$\Pi_i \Sigma_j \Sigma_k \Pi_l (c_{ij} \bar{c}_{ik} s_{ik} + \bar{y}_{il} + \bar{p}_{il}).$$

Let us express this: one or other of two theories must be admitted, 1st, that no man is at any time unselfish or free, and some men are always hypocritical, and at every time some men are friendly to men to whom they are at other times inimical, or 2d, at each moment all men are alike either angels or fiends. Let

u_{ij}	mean the man i is unselfish at the time j,				
f_{ij}	"	"	"	free	" "
h_{ij}	"	"	"	hypocritical	" "
a_{ij}	"	"	"	an angel	" "
d_{ij}	"	"	"	a fiend	" "
p_{ijk}	"	"	"	friendly	" " to the man k,
e_{ijk}	the man i is an enemy at the time j to the man k;				
1_{jm}	the two objects j and m are identical.				

Then the proposition is

$$\Pi_i \Sigma_h \Pi_j \Sigma_k \Sigma_l \Sigma_m \Pi_n \Pi_p \Pi_q (\bar{u}_{ij} \bar{f}_{ij} h_{hj} p_{kjl} e_{kml} \bar{1}_{jm} + a_{pn} + d_{qn}).$$

We have now to consider the procedure in working with this calculus. It is far from being true that the only problem of deduction is to draw a conclusion from given premises. On the contrary, it is fully as important to have a method for ascertaining what premises will yield a given conclusion. There are besides other problems of transformation, where a certain system of facts is given, and it is required to describe this in other terms of a definite kind. Such, for example, is the problem of the 15 young ladies, and others relating to synthemes. I shall, however, content myself here with showing how, when a set of premises are given, they can be united and certain letters eliminated. Of the various methods which might be pursued, I shall here give the one which seems to me the most useful on the whole.

1st. The different premises having been written with distinct indices (the same index not used in two propositions) are written together, and all the Π's and Σ's are to be brought to the left. This can evidently be done, for

$$\Pi_i x_i . \Pi_j x_j = \Pi_i \Pi_j x_i x_j$$
$$\Sigma_i x_i . \Pi_j x_j = \Sigma_i \Pi_j x_i x_j$$
$$\Sigma_i x_i . \Sigma_j x_j = \Sigma_i \Sigma_j x_i x_j .$$

2d. Without deranging the order of the indices of any one premise, the Π's and Σ's belonging to different premises may be moved relatively to one another, and as far as possible the Σ's should be carried to the left of the Π's. We have

$$\Pi_i \Pi_j x_{ij} = \Pi_j \Pi_i x_{ij}$$
$$\Sigma_i \Sigma_j x_{ij} = \Sigma_j \Sigma_i x_{ij}$$

and also

$$\Sigma_i \Pi_j x_i y_j = \Pi_j \Sigma_i x_i y_j .$$

But this formula does not hold when the i and j are not separated. We do have, however,

$$\Sigma_i \Pi_j x_{ij} \prec \Pi_j \Sigma_i x_{ij} .$$

It will, therefore, be well to begin by putting the Σ's to the left, as far as possible, because at a later stage of the work they can be carried to the right but not to the left. For example, if the operators of the two premises are $\Pi_i \Sigma_j \Pi_k$ and $\Sigma_x \Pi_y \Sigma_z$, we can unite them in either of the two orders

$$\Sigma_x \Pi_y \Sigma_z \Pi_i \Sigma_j \Pi_k$$
$$\Sigma_x \Pi_i \Sigma_j \Pi_y \Sigma_z \Pi_k ,$$

and shall usually obtain different conclusions accordingly. There will often be room for skill in choosing the most suitable arrangement.

3d. It is next sometimes desirable to manipulate the Boolian part of the expression, and the letters to be eliminated can, if desired, be eliminated now. For this purpose they are replaced by relations of second intention, such as "other than", etc. If, for example, we find anywhere in the expression

$$a_{ijk}\bar{a}_{xyz},$$

this may evidently be replaceable by

$$(n_{ix} + n_{jy} + n_{kz})$$

where, as usual, n means not or other than. This third step of the process is frequently quite indispensable, and embraces a variety of processes; but in ordinary cases it may be altogether dispensed with.

4th. The next step, which will also not commonly be needed, consists in making the indices refer to the same collections of objects, so far as this is useful. If the quantifying part, or Quantifier, contains Σ_x, and we wish to replace the x by a new index i, not already in the Quantifier, and such that every x is an i, we can do so at once by simply multiplying every letter of the Boolian having x as an index by x_i. Thus, if we have "some woman is an angel" written in the form $\Sigma_w a_w$ we may replace this by $\Sigma_i(a_i w_i)$. It will be more often useful to replace the index of a Π by a wider one; and this will be done by adding $\bar{x}_i$ to every letter having x as an index. Thus, if we have "all dogs are animals, and all animals are vertebrates" written thus

$$\Pi_d \alpha_d \Pi_a v_a,$$

where a and α alike mean animal, it will be found convenient to replace the last index by i, standing for any object, and to write the proposition

$$\Pi_i(\bar{\alpha}_i + v_i).$$

5th. The next step consists in multiplying the whole Boolian part, by the modification of itself produced by substituting for the index of any Π any other index standing to the left of it in the Quantifier. Thus, for

$$\Sigma_i \Pi_j l_{ij},$$

we can write $$\Sigma_i \Pi_j l_{ij} l_{ii}.$$

6th. The next step consists in the re-manipulation of the Boolian part, consisting, 1st, in adding to any part any term we like; 2d, in dropping from any part any factor we like, and 3d, in observing that

$$x\bar{x} = \mathbf{f}, \qquad x + \bar{x} = \mathbf{v},$$

so that $$x\bar{x}y + z = z, \ (x + \bar{x} + y)z = z.$$

7th. Π's and Σ's in the Quantifier whose indices no longer appear in the Boolian are dropped.

The fifth step will, in practice, be combined with part of the sixth and seventh. Thus, from $\Sigma_i \Pi_j l_{ij}$ we shall at once proceed to $\Sigma_i l_{ii}$ if we like.

The following examples will be sufficient.

From the premises $\Sigma_i a_i b_i$ and $\Pi_j(\bar{b}_j + c_j)$, eliminate b. We first write

$$\Sigma_i \Pi_j a_i b_i(\bar{b}_j + c_j).$$

The distributive process gives

$$\Sigma_i \Pi_j a_i (b_i \bar{b}_j + b_i c_j).$$

But *we always have a right to drop a factor or insert an additive term.* We thus get

$$\Sigma_i \Pi_j a_i (b_i \bar{b}_j + c_j).$$

By the third process, we can, if we like, insert n_{ij} for $b_i \bar{b}_j$. In either case, we identify j with i and get the conclusion

$$\Sigma_i a_i c_i.$$

Given the premises

$$\Sigma_h \Pi_i \Sigma_j \Pi_k (\alpha_{hik} + s_{jk} l_{ji})$$

$$\Sigma_u \Sigma_v \Pi_x \Pi_y (\varepsilon_{uyx} + \bar{s}_{yv} b_{vx}).$$

Required to eliminate s. The combined premise is

$$\Sigma_u \Sigma_v \Sigma_h \Pi_i \Sigma_j \Pi_x \Pi_k \Pi_y (\alpha_{hik} + s_{jk} l_{ji}) (\varepsilon_{uyx} + \bar{s}_{yv} b_{vx}).$$

Identify k with v and y with j, and we get

$$\Sigma_u \Sigma_v \Sigma_h \Pi_i \Sigma_j \Pi_x (\alpha_{hiv} + s_{jv} l_{ji}) (\varepsilon_{ujx} + \bar{s}_{jv} b_{vx}).$$

The Boolian part then reduces, so that the conclusion is

$$\Sigma_u \Sigma_v \Sigma_h \Pi_i \Sigma_j \Pi_x (\alpha_{hiv} \varepsilon_{ujx} + \alpha_{hiv} b_{vx} + \varepsilon_{ujx} l_{ji}).$$

IV.—Second-Intentional Logic

Let us now consider the logic of terms taken in collective senses. Our notation, so far as we have developed it, does not show us even how to express that two indices, i and j, denote one and the same thing. We may adopt a special token of second intention, say 1, to express identity, and may write 1_{ij}. But this relation of identity has peculiar properties. The first is that if i and j are identical, whatever is true of i is true of j. This may be written

$$\Pi_i \Pi_j \{ \bar{1}_{ij} + \bar{x}_i + x_j \}.$$

The use of the general index of a token, x, here, shows that the formula is iconical. The other property is that if everything which is true of i is true of j, then i and j are identical. This is most naturally written as follows: Let the token, q, signify the relation of a quality, character, fact, or predicate to its subject. Then the property we desire to express is

$$\Pi_i \Pi_j \Sigma_k (1_{ij} + \bar{q}_{ki} q_{kj}).$$

And identity is defined thus

$$1_{ij} = \Pi_k (q_{ki} q_{kj} + \bar{q}_{ki} \bar{q}_{kj}).$$

That is, to say that things are identical is to say that every predicate is true of both or false of both. It may seem circuitous to introduce the idea of a quality to express identity; but that impression will be modified by reflecting that $q_{ki}q_{kj}$ merely means that i and j are both within the class or collection k. If we please, we can dispense with the token q, by using the index of a token and by referring to this in the Quantifier just as subjacent indices are referred to. That is to say, we may write

$$1_{ij} = \Pi_x(x_i x_j + \bar{x}_i \bar{x}_j).$$

The properties of the token q must now be examined. These may all be summed up in this, that taking any individuals i_1, i_2, i_3, etc., and any individuals, j_i, j_2, j_3, etc., there is a collection, class, or predicate embracing all the i's and excluding all the j's except such as are identical with some one of the i's. This might be written

$$(\Pi_\alpha \Pi_{i_\alpha})\,(\Pi_\beta \Pi_{j_\beta}) \Sigma_k (\Pi_\alpha \Sigma_{i'_\alpha}) \Pi_l\, q_{ki_\alpha}(\bar{q}_{kj_\beta} + q_{li'_\alpha} q_{lj_\beta} + \bar{q}_{li'_\alpha} \bar{q}_{lj_\beta}),$$

where the i's and the i''s are the same lot of objects. This notation presents indices of indices. The $\Pi_\alpha \Pi_{i_\alpha}$ shows that we are to take any collection whatever of i's, and then any i of that collection. We are then to do the same with the j's. We can then find a quality k such that the i taken has it, and also such that the j taken wants it unless we can find an i that is identical with the j taken. The necessity of some kind of notation of this description in treating of classes collectively appears from this consideration: that in such discourse we are neither speaking of a single individual (as in the non-relative logic) nor of a small number of individuals considered each for itself, but of a whole class, perhaps an infinity of individuals. This suggests a relative term with an indefinite series of indices as $x_{ijkl\ldots}$. Such a relative will, however, in most, if not in all cases, be of a degenerate kind and is consequently expressible as above. But it seems preferable to attempt a partial decomposition of this definition. In the first place, any individual may be considered as a class. This is written,

$$\Pi_i \Sigma_k \Pi_j\, q_{ki}(\bar{q}_{kj} + 1_{ij}).$$

This is the *ninth icon*. Next, given any class, there is another which includes all the former excludes and excludes all the former includes. That is,

$$\Pi_l \Sigma_k \Pi_i (q_{li}\bar{q}_{ki} + \bar{q}_{li} q_{ki}).$$

This is the *tenth icon*.Next, given any two classes, there is a third which includes all that either includes and excludes all that both exclude. That is

$$\Pi_l \Pi_m \Sigma_k \Pi_i (q_{li} q_{ki} + q_{mi} q_{ki} + \bar{q}_{li} \bar{q}_{mi} \bar{q}_{ki}).$$

This is the *eleventh icon*. Next, given any two classes, there is a class which includes the whole of the first and any one individual of the second which there may be not included in the first and nothing else. That is,

$$\Pi_l \Pi_m \Pi_i \Sigma_k \Pi_j \{ q_{li} + \bar{q}_{mi} + q_{ki}(q_{kj} + \bar{q}_{lj}) \}.$$

This is the *twelfth icon.*

To show the manner in which these formulæ are applied let us suppose we have given that everything is either true of i or false of j. We write

$$\Pi_k(q_{ki} + \bar{q}_{kj}).$$

The tenth icon gives

$$\Pi_l\Sigma_k(q_{li}\bar{q}_{ki} + \bar{q}_{li}q_{ki})\,(q_{lj}\bar{q}_{kj} + \bar{q}_{lj}q_{kj}).$$

Multiplication of these two formulæ gives

$$\Pi_l\Sigma_k(q_{ki}\bar{q}_{li} + q_{lj}\bar{q}_{kj}),$$

or, dropping the terms in k

$$\Pi_l(\bar{q}_{li} + q_{lj}).$$

Mutliplying this with the original datum and identifying l with k, we have

$$\Pi_k\,(q_{ki}q_{kj} + \bar{q}_{ki}\bar{q}_{kj}).$$

No doubt, a much more direct method of procedure could be found.

Just as q signifies the relation of predicate to subject, so we need another token, which may be written r, to signify the conjoint relation of a simple relation, its relate and its correlate. That is, $r_{j_\alpha i}$ is to mean that i is in the relation α to j. Of course, there will be a series of properties of r similar to those of q. But it is singular that the uses of the two tokens are quite different. Namely, the chief use of r is to enable us to express that the number of one class is at least as great as that of another. This may be done in a variety of different ways. Thus, we may write that for every a there is a b, in the first place, thus:

$$\Sigma_\alpha\Pi_i\Sigma_j\Pi_h\{\bar{a}_i + b_jr_{j_\alpha i}(\bar{r}_{j_\alpha h} + \bar{a}_h + 1_{ih})\}.$$

But, by an icon analogous to the eleventh, we have

$$\Pi_\alpha\Pi_\beta\Sigma_\gamma\Pi_u\Pi_v(r_{u\alpha v}r_{u\gamma v} + r_{u\beta v}r_{u\gamma v} + \bar{r}_{u\alpha v}\bar{r}_{u\beta v}\bar{r}_{u\gamma v}).$$

From this, by means of an icon analogous to the *tenth*, we get the general formula

$$\Pi_\alpha\Pi_\beta\Sigma_\gamma\Pi_u\Pi_v\{r_{u\alpha v}r_{u\beta v}r_{u\gamma v} + \bar{r}_{u\gamma v}(\bar{r}_{u\alpha v} + \bar{r}_{u\beta v})\}.$$

For $r_{u\beta v}$ substitute a_u and multiply by the formula the last but two. Then, identifying u with h and v with j, we have

$$\Sigma_a\Pi_i\Sigma_h\Pi_h\{\bar{a}_i + b_jr_{jai}(\bar{r}_{jah} + 1_{ih})\}$$

a somewhat simpler expression. However, the best way to express such a proposition is to make use of the letter c as a token of a one-to-one correspondence. That is to say, c will be defined by the three formulæ,

$$\Pi_a\Pi_u\Pi_v\Pi_w(\bar{c}_a + \bar{r}_{uav} + \bar{r}_{uaw} + 1_{vw})$$

$$\Pi_a\Pi_u\Pi_v\Pi_w(\bar{c}_a + \bar{r}_{uaw} + r_{vaw} + 1_{uv})$$

$$\Pi_a\Sigma_u\Sigma_v\Sigma_w(c_a + r_{uav}r_{uaw}\,\bar{1}_{vw} + r_{uaw}r_{vaw}\,\bar{1}_{uv}).$$

Making use of this token, we may write the proposition we have been considering in the form

$$\Sigma_a \Pi_i \Sigma_j c_a(\bar{a}_i + b_j r_{j\alpha i}).$$

In an appendix to his memoir on the logic of relatives, DeMorgan enriched the science of logic with a new kind of inference, the syllogism of transposed quantity. DeMorgan was one of the best logicians that ever lived and unquestionably the father of the logic of relatives. Owing, however, to the imperfection of his theory of relatives, the new form, as he enunciated it, was a down-right paralogism, one of the premises being omitted. But this being supplied, the form furnishes a good test of the efficacy of a logical notation. The following is one of DeMorgan's examples:

> Some X is Y,
> For every X there is something neither Y nor Z;
> Hence, something is neither X nor Z.

The first premise is simply

$$\Sigma_a x_a y_a.$$

The second may be written

$$\Sigma_a \Pi_i \Sigma_j c_a(\bar{x}_i + r_{jai}\bar{y}_j\bar{z}_j).$$

From these two premises, little can be inferred. To get the above conclusion it is necessary to add that the class of X's is a finite collection; were this not necessary the following reasoning would hold good (the limited universe consisting of numbers); for it precisely conforms to DeMorgan's scheme.

> Some odd number is prime;
> Every odd number has its square, which is neither prime nor even;
> Hence, some number is neither odd nor even.*

Now, to say that a lot of objects is finite, is the same as to say that if we pass through the class from one to another we shall necessarily come round to one of those individuals already passed; that is, if every one of the lot is in any one-to-one relation to one of the lot, then to every one of the lot some one is in this same relation. This is written thus:

$$\Pi_\beta \Pi_u \Sigma_v \Sigma_s \Pi_t \{\bar{c}_\beta + \bar{x}_u + x_v r_{u\beta v} + x_s(\bar{x}_t + \bar{r}_{t\beta s})\}$$

Uniting this with the two premises and the second clause of the definition of c, we have

* Another of DeMorgan's examples is this: "Suppose a person, on reviewing his purchases for the day, finds, by his counterchecks, that he has certainly drawn as many checks on his banker (and maybe more) as he has made purchases. But he knows that he paid some of his purchases in money, or otherwise than by checks. He infers then that he has drawn checks for something else except that day's purchases. He infers rightly enough". Suppose, however, that what happened was this: He bought something and drew a check for it; but instead of paying with the check, he paid cash. He then made another purchase for the same amount, and drew another check. Instead, however, of paying with that check, he paid with the one previously drawn. And thus he continued without cessation, or *ad infinitum*. Plainly the premises remain true, yet the conclusion is false.

$$\Sigma_a\Sigma_\alpha\Pi_\beta\Pi_u\Sigma_v\Sigma_s\Pi_i\Sigma_j\Pi_t\Pi_\gamma\Pi_e\Pi_f\Pi_g x_a y_a c_\alpha(\bar{x}_i + r_{j\alpha i}\bar{y}_j\bar{z}_j)$$
$$\{\bar{c}_\beta + \bar{x}_u + x_v r_{u\beta v} + x_s(\bar{x}_t + \bar{r}_{t\beta s})\}(\bar{c}_\gamma + \bar{r}_{e\gamma g} + \bar{r}_{f\gamma w} + 1_{ef}).$$

We now substitute α for β and for γ, a for u and for e, j for t and for f, v for g. The factor in i is to be repeated, putting first s and then v for i. The Boolian part thus reduces to

$$(\bar{x}_s + r_{j\alpha s}\bar{y}_j\bar{z}_j)c_\alpha x_a y_a r_{a\alpha v} x_v r_{j\alpha v}\bar{y}_j\bar{z}_j 1_{aj} + r_{j\alpha s}\bar{y}_j\bar{z}_j x_s \bar{x}_j(\bar{x}_v + r_{j\alpha v}\bar{y}_j\bar{z}_j)$$
$$(\bar{r}_{a\alpha v} + \bar{r}_{j\alpha v} + 1_{aj}),$$

which, by the omission of factors, becomes

$$y_a\bar{y}_j 1_{aj} + \bar{x}_j\bar{z}_j.$$

Thus we have the conclusion

$$\Sigma_j\bar{x}_j\bar{z}_j.$$

It is plain that by a more iconical and less logically analytical notation this procedure might be much abridged. How minutely analytical the present system is, appears when we reflect that every substitution of indices of which nine were used in obtaining the last conclusion is a distinct act of inference. The annulling of $(y_a\bar{y}_j 1_{aj})$ makes ten inferential steps between the premises and conclusion of the syllogism of transposed quantity.

E. THE LOGIC OF MATHEMATICS IN RELATION TO EDUCATION (*PEIRCE 1898*)

The following article appeared in the *Educational review*; despite the concluding declaration that the series was to be continued, no more articles appeared.

References to *Peirce 1898* should be to the paragraph numbers, which have been added in this edition.

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§1 Of mathematics in general

[1] In order to understand what number is, it is necessary first to acquaint ourselves with the nature of the business of mathematics in which number is employed.

[2] I wish I knew with certainty the precise origin of the definition of mathematics as the science of quantity. It certainly cannot be Greek, because the Greeks were advanced in projective geometry, whose problems are such as these: whether or not four points obtained in a given way lie in one plane; whether or not four planes have a point in common; whether or not two rays