



Calculus as method or calculus as rules? Boole and Frege on the aims of a logical calculus

David Waszek¹ · Dirk Schlimm¹

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Abstract

By way of a close reading of Boole and Frege's solutions to the same logical problem, we highlight an underappreciated aspect of Boole's work—and of its difference with Frege's better-known approach—which we believe sheds light on the concepts of 'calculus' and 'mechanization' and on their history. Boole has a clear notion of a logical *problem*; for him, the whole point of a logical calculus is to enable systematic and goal-directed solution methods for such problems. Frege's *Begriffsschrift*, on the other hand, is a visual tool to scrutinize concepts and inferences, and is a calculus only in the thin sense that every possible transition between sentences is fully and unambiguously specified in advance. While Frege's outlook has dominated much of philosophical thinking about logical symbolism, we believe there is value—particularly in light of recent interest in the role of notations in mathematics and logic—in reviving Boole's idea of an intrinsic link between, as he put it, a 'calculus' and a 'directive method' to solve problems.

Keywords Boole · Frege · Logic · Notations · Calculus · Mechanization

Introduction

The main contrast this paper aims at drawing is the following: Boole was looking for a systematic, goal-directed method to solve logical problems, and *saw this as essential to the very idea of a logical calculus*. For Frege, on the other hand, such a method was neither possible nor desirable; his idea of a calculus, though perhaps sharper in requiring fully explicit transformation rules, was also much thinner. As Mark Wilson

✉ David Waszek
david.waszek@posteo.net
Dirk Schlimm
dirk.schlimm@mcgill.ca

¹ Department of Philosophy, McGill University, Leacock Building, Room 414, 855 Sherbrooke St. W., Montreal, QC H3A 2T7, Canada

put it, ‘to Boole [. . .], the very phrase “logical method” implies a “heuristic”, not the less structured forms of non-algorithmic specification (like axiom schemes or natural deduction systems) that are usually called “logics” today.’¹ Genealogically, this makes Boole closer in outlook to twentieth-century computer science and automated theorem-proving than are the more canonical forefathers of modern logic. Philosophically, it draws attention to a phenomenon that the main orientations of mathematical logic, whose debt to Frege is significant, tend to relegate to the background—and that, despite some recent interest in the philosophy of mathematical practice, remains largely unstudied: in practice, what makes a symbolic language or notation valuable is not merely the availability of formal transformation rules, nor just the totality of what can be derived from these, but the practical problem-solving methods that the language lends itself to.

Our contrast between Boole and Frege is usually obscured by two tendencies in the historiography of nineteenth-century logic. The first is an inclination to view Boole’s work through the lens of our propositional calculus. Once stripped of its irrelevant algebraic dress—so the story goes—Boole’s calculus is but a limited part of Frege’s much more encompassing quantificational logic, which should thus be able to achieve everything the former can. In other words, Frege’s logic is taken to do strictly *more* than Boole’s. What we shall point out, however, is that Frege’s broader scope goes hand in hand with a retreat from Boole’s ambitions of a systematic method; moreover, shifting from Boole’s notation to Frege’s or contemporary ones—thereby eliminating the link with algebra—makes Boole’s methods, even in their limited domain, downright unintelligible. In this sense, the greater expressivity of Frege’s logic comes with a loss in computational power, albeit in a limited domain.

A second, more sophisticated obstacle is the opposition deployed by Frege himself, then reinterpreted and popularized by Van Heijenoort (1967), according to which Boole’s logic is merely a ‘*calculus ratiocinator*’ while Frege’s is *not just that, but also a ‘lingua characteri(sti)ca’*.² Whatever the merits of this contrast to point to differences between Frege and the Booleans, the problem for us is that it suggests that, on the ‘calculus’ side, Frege and Boole’s systems are similar: both would allow treating inferences as potentially mechanizable computations on signs—if anything, Frege is assumed to do it better than Boole, as he spells out his formal rules much more explicitly. This is misleading, as we shall show. The sense of ‘calculus’ (and hence the kind of mechanization) that is implicit in such discussions is different from the one that mattered most to Boole. Representing elementary inferences by symbol-pushing was not his central goal at all; rather, he sought a method that, starting from a given logical problem to be solved, indicated which elementary inferences to use and how they should be strung together.

¹ Wilson (2006, p. 523).

² These expressions should be approached with caution, as they have been used for various purposes and interpreted in numerous different ways; for further discussion, see Sect. 6 below, in particular note 50. Note that Frege generally writes *lingua characterica*, but that other authors typically use *lingua characteristica* instead. Nothing much hinges on this, but for reference, here is the source of the discrepancy: Leibniz, who coined the phrase, actually used the latter spelling; the former (mistaken) spelling comes from a spurious title for one of Leibniz’s manuscripts introduced by a much-circulated eighteenth-century edition (Raspe 1765), which was then used by Trendelenburg (1856), Frege’s main source on the topic. We thank an anonymous reviewer for this remark.

One possible way to approach our contrast between Boole and Frege is to look back on it, anachronistically, from the vantage point of later logic. As was proved in the 1930s, while modern logics comparable to Boole's system (propositional logic and monadic first-order logic) are *decidable*, systems as expressive as Frege's are not: there is no algorithmic procedure to determine whether a given sentence is logically valid. This can suggest the following (too) neat story. Earlier logicians, who had not grappled with the intricacies of relational, quantified logic, could still think that devising a logical *calculus* (in the sense of setting up symbols and manipulation rules on the model of algebra) and obtaining *algorithmic methods* to solve logical problems (on the model of the solution of simple algebraic equations) would go hand in hand; Frege, being the first to go far enough to see the full complexity of the problem, realized how inaccessible a general method to find the proof of a given proposition was. On this account, what is at stake in the gap between Boole and Frege is the emergence of a typical twentieth-century predicament: modern logic makes *checking* the validity of a fully formalized proof mechanical, but only highlights the difficulty of *finding* a proof given the sought-after conclusion.

Such an anachronistic reading obscures matters, however. Historically, it does not seem accurate to say that Frege retreated from Boole's grander ambitions just because, seeing further than Boole had, he realized they were out of reach in interesting cases. His goals were different from the start: rather than a systematic problem-solving method, what he sought were tools to improve mathematical concepts and clarify mathematical inferences. More importantly, recent work by Jamie Tappenden has unearthed that a salient motivation of Frege's work was to fight against a 'mechanical' view of arithmetic, and mathematics in general. From this point of view, Frege's vaunted 'formalization' of inference paradoxically comes to be seen as a way to *push back* against mechanization and confine it to a limited, subordinate role.

On the Boolean side, too, there is more to be learnt. Boole's conception of a calculus is not just a failed dream, historically doomed to fall apart when faced with the complexities of full predicate logic. It draws our attention to a theme that Frege's project obscured, but that is still highly relevant today: an important part of the practical value of a symbolic notation is its suitability to guide us effectively to the solution of problems.

To bring out our contrast between Boole and Frege, our strategy is to compare their solutions to the same logical problem; along the way, we reconstruct Boole's conception of what a logical problem is and highlight the goal-directedness of his solution method. Initially introduced by Boole in his *Laws of Thought*, the problem we focus on is one of the more sophisticated examples he considers in his book, and as such attracted the attention of a number of later authors eager to prove that their own system was at least as powerful as his.³ Among them, Frege—despite repeatedly putting the value of this kind of problems in doubt—spent several pages developing a solution from the perspective of his *Begriffsschrift*, thereby offering us a revealing point of comparison.

Sections 1–3 describe Boole's overall conception of what a logical problem is, introduce the sample problem we shall study, and sketch Boole's solution of it. Sec-

³ See below, especially footnote 14, p. 7.

tion 4 briefly goes over the reception of Boole's conception of logic in order to prepare our discussion, in Sect. 5, of Frege's solution of the same problem. Sections 6–8 then draw lessons from the comparison of Boole and Frege's solutions: we explore how they testify to different conceptions of what a calculus is, how they relate to different forms of mechanization, and what we can learn from reviving Boole's outlook. Finally, Sect. 9 tentatively sketches some further historical developments, mostly in Schröder and Peirce, so as to place our main contrast in a broader perspective.

1 Boole's conception of the problem of logic

To understand the goal-directed nature of Boole's logical method, we first need to explain what the problems are that his method is intended to solve. As it happens, Boole believed that there is a general form common to all logical problems. The easiest way to grasp this form is as a broad generalization of syllogistic, which was Boole's starting point. Accordingly, this section builds up to his conception of logical problems through successive generalizations from a simple syllogistic case.

Take two sentences in Aristotelian subject-predicate form, for example 'All horses are mammals' and 'All mammals are animals', that have a term in common (here 'mammals'). The standard problem of syllogistic inference (as Boole construes it)⁴ is to find what relation, if any, follows between 'horses' and 'animals', eliminating the term 'mammals' which is already present in both premises. (In this instance, the conclusion sought is 'All horses are animals.') Importantly, this does not amount to finding *all* possible consequences of the initial sentences, of which there are many others, such as 'Some mammals are horses': we are only interested in 'horses' and 'animals'. Nor would just any consequence do as long as it relates 'horses' and 'animals': neither 'Everything that is a horse but not an animal does not exist' nor 'Some horses are animals' would do, the first because it lacks the expected Aristotelian form, the second because it is not as strong as possible. Syllogistic inference, in short, comes with a constrained specification of the expected solution. Boole's problems do likewise, but in a broader setting.

First, Boole's propositions are more general than the Aristotelian subject-predicate forms, and are expressed as symbolic equations rather than in natural language. This aspect of Boole's work is well-known, so we shall be quick. Let us start with an example Boole discusses repeatedly in his *Investigation of the Laws of Thought*, namely the definition of *wealth* offered by the economist William Nassau Senior:

Wealth consists of things transferable, limited in supply, and either productive of pleasure or preventive of pain.⁵

To express this definition symbolically, Boole introduces letters that denote *classes* of things; in our case, he writes *w* for wealth, *t* for things transferable, *s* for things limited in supply, *p* for things productive of pleasure, and *r* for things preventive of pain. He then uses operations analogous to those of algebra to express combinations of

⁴ We are here glossing on Boole's own account (see in particular (Boole 1854, ch. XV, pp. 226–242), and not aiming for a historically accurate rendition of the goals of syllogistic.

⁵ Boole (1854, p. 59).

these classes. The juxtaposition of two class symbols, akin to algebraic multiplication, denotes the class of things common to both (in contemporary terms, their *intersection*), so that st denotes things both limited in supply and transferable. The addition of two class symbols expresses the class formed by taking the elements of both together (today called their *union*), except that Boole only allows this operation on classes that are disjoint, that is, have no elements in common. Thus, if nothing is both productive of pleasure and preventive of pain, $p + r$ would correspond to things that are either one or the other. The subtraction of two class symbols $x - y$ expresses the class of elements of x that are not elements of y , where it is assumed that y is included in x . Finally, the symbol 1 denotes the universe of discourse, so that for instance $1 - p$ denotes things *not* productive of pleasure. (Boole sometimes abbreviates $1 - p$ as $\bar{p}$.) This allows Boole to express Senior's definition as:

$$w = st\{p + r(1 - p)\}$$

or in words: wealth is things that are at the same time limited in supply, transferable, and either productive of pleasure, or preventive of pain and not productive of pleasure (the complexity of this last clause being required by Boole's restriction that addition can only be performed on disjoint classes). One last device that Boole introduces is 'indefinite' class symbols (which he often writes v , but sometimes with other letters as well, such as q), that is symbols denoting an unspecified class. These allow him to represent inclusions like 'All horses are mammals' in the form of equalities, such as $h = vm$ (where h stands for the class of horses, m for that of mammals and v for an indefinite class); such equations can also be understood as 'conditionals' rather than inclusions, e.g., as 'If something is a horse, then it is a mammal.'⁶

Second, Boole admits not just two premises involving three terms, as in syllogistic inference, but any number of premises involving any number of terms. The problem analogous to syllogistic inference then becomes that of finding a relation—or more precisely, the strongest possible relation—among any number of terms selected among those appearing in the premises. One example Boole discusses is a piece of reasoning from Aristotle's *Nicomachean Ethics*.⁷ Aristotle asks whether virtue is a passion, a faculty, or a habit. The six premises, as reconstructed by Boole, express various properties of virtue as well as of passions, faculties, and habits, involving several auxiliary properties, for instance 'things according to which we are praised or blamed' (according to Aristotle, we can be praised for our virtue, but not for our inborn faculties). The goal here, in Boole's terms, is to find the relation between virtue, passions, faculties, and habits, eliminating all other terms—the conclusion, as it turns out, being that virtue is a habit, but not a faculty nor a passion.

⁶ In fact, Boole uses indefinite class symbols ambiguously, a difficulty that we shall point out here but ignore in the sequel, as it does not bear on our main points. In most settings, he takes such symbols to be absolutely indefinite, that is, to denote a class that can be empty, equal to the full universe of discourse, or anything in between. But Boole also translates the Aristotelian form 'Some A is B ' as $va = vb$, in which—if the traditional interpretation of such forms is to be preserved— v has to be interpreted as an indefinite *non-empty* class. For a careful discussion written from a Boolean perspective, see Venn (1881, ch. VI–VII).

⁷ Boole (1854, pp. 134–137). We follow Boole's rendition of Aristotle; the passage in question is *Nicomachean Ethics* II.5 (1105b20–1106a15).

Third, the broader range of possible propositional forms in Boole's system allows him to put further constraints on the conclusion that is sought. In the syllogistic case, it is simply expected that the conclusion will be in one of the Aristotelian forms ('All A is B ', 'Some A is B ', or their negations).⁸ Boole is able to be more specific. In the example of virtue, what is expected—and can be provided by Boole's method—is not just any equation linking virtue (denoted by v) with passions (p), faculties (f), and habits (h), but rather an equation of the form $v = \dots$ (in which the right-hand side only contains p , f and h). This is the simplest and most common case, but other forms can also be requested (and obtained). Returning to Senior's definition of wealth, one could for instance ask about what can be concluded about 'wealth that is preventive of pain' in terms of 'things transferrable' and 'things limited in supply'—in other words, ask for a conclusion of the form $wr = \dots$, where the right-hand side only contains s and t .⁹ In Boole's words, the relation sought is 'that full relation which, in virtue of the premises, connects any elements selected out of the premises at will, and which, moreover, expresses that relation in any desired form and order.'¹⁰

We have focused so far on what Boole calls 'primary propositions', in which symbols denote classes. His system can also treat 'secondary propositions', in which the symbols already denote propositions; it is this 'secondary' part of his system that is closest to our contemporary propositional calculus. Boole derives his secondary propositions from his primary class-based ones by introducing, for a given proposition X , a class symbol x denoting 'that portion of time for which the proposition X is true.'¹¹ This *extensional* account of propositions allows treating relations between propositions just like relations between classes. The only differences lie in the interpretation of equations: in this new context, $x = 0$ and $x = 1$ mean that the proposition X is (always, i.e., 'throughout the whole duration of the time to which our discourse refers'¹²) true and (always) false, respectively; equations containing an indefinite class symbol, such as $x = vy$, are interpreted as implications ('If X , then Y '). As far as the general formulation of logical problems is concerned, however, the move to 'secondary' propositions changes very little: premises are still expressed by equations,

⁸ In traditional tables of the canonical syllogistic forms, there are other restrictions, which Boole neglects (it is expected that the major term will come first, for instance). The discrepancy arises because Boole does not fully do justice to traditional logic: he takes syllogistic's *classification* of inference forms as a full-fledged theory of reasoning, whereas traditional textbooks would also contain a broader theory exploring how non-canonical pieces of reasoning are to be brought into one of the standard forms. This need not detain us further, as we are only concerned with Boole's own portrayal of syllogistic.

⁹ There is yet another form that can be requested of the conclusion: one may want the list of those combinations of the selected terms that are *excluded* by the premises, i.e., that correspond to classes that the premises force to be empty (for more on this, see Sect. 4 below). In Boole's system, this amounts to seeking an equation of the form $V = 0$, in which V is a sum of combinations of class symbols or their negations; such an equation is equivalent to having each of the members of the sum be separately equal to 0. This form is the most exhaustive, being equivalent to the premises (or to the premises once elimination has been performed, if some terms have been eliminated), whereas equations of the form $x = \dots$ will usually be weaker.

¹⁰ Boole (1854, p. 10). Note that Boole uses the word 'premise' to refer to the data, or given relations, of the kind of logical problems he considers; the word thus takes on a broader meaning than may be usual (i.e., it does not refer merely to the starting points of some individual inference).

¹¹ Boole (1854, p. 165).

¹² Boole (1854, pp. 168–169)

and the goal is again to obtain an equation of a specified form relating a subset of the symbols appearing in the premises.¹³

2 A sample problem

Let us now turn to the sample problem that our comparison of Boole and Frege will focus on below. Among the examples discussed by Boole in his *Investigation of the Laws of Thought*, this problem is of particular interest, not just because it is one of the most intricate, but also because—for this very reason—it was repeatedly addressed by later authors, including Schröder, McColl, and Frege,¹⁴ to show that their system was able to do as much as Boole's. Its formulation is quite abstract. It is about a class of 'natural productions' (which, in this particular case, will serve as the universe of discourse) whose members can display five properties A , B , C , D and E , with three relations between them that will serve as premises.

Before turning to the premises, a caveat is in order. Symbolically, Boole writes x for the property A , y for B , and so on. Strictly speaking, as we shall see below, following his method largely does not require referring back to the meaning of the symbols (except when initially translating the premises into symbols and when interpreting the final equation), so the discrepancy between the names of the properties and the corresponding symbols is tolerable. Moreover, this discrepancy is justified by Boole's algebraic model, in which it is customary, since Descartes, to write the unknowns using letters from the end of the alphabet and the knowns (such as coefficients) using letters from the beginning. Nevertheless, since later authors—in particular Schröder and through him Frege, which we shall discuss at length—revert to the more straightforward convention of writing a for property A , etc., keeping Boole's notation would make the discussion below exceedingly confusing. In breach of the spirit of our paper, which strives to be faithful to the algebraic spirit of Boole's method, we therefore decided to alter his choice of symbols here. We also chose to effect a minor change in Schröder's notation: in this problem, Boole writes the negation of a symbol a as $\bar{a}$, while Schröder writes it a_1 —we chose to write $\bar{a}$ throughout. All other notations are unchanged.

The three premises of Boole's problem, then, are the following:¹⁵

- i. 'That in whichsoever of these productions the properties A and C are missing, the property E is found, together with one of the properties B and D , but not with both.' In symbols,

$$\bar{a}\bar{c} = qe(b\bar{d} + d\bar{b})$$

¹³ Boole (1854, pp. 178–179).

¹⁴ Schröder (1877, pp. 25–28), Lotze (1884, pp. 219–221) = Lotze (1912, pp. 265–267), Wundt (1880, vol. 1, pp. 356–357), Frege (1979, pp. 39–45) = Frege (1969, pp. 45–51), Venn (1881, pp. 280–281), and McColl (1878, pp. 23–25). That this single problem has been solved by logicians of various outlooks whose solutions would repay further comparison has already been noted by Gabriel (1989, p XXIII).

¹⁵ See Boole (1854), pp. 146–147.

where q is an indefinite class symbol, which can be read equivalently as an inclusion ('the class of productions without properties A and C is *a certain part of* the class of productions with property E etc.') or as a conditional ('if a production lacks properties A and C , then it has property E etc.').

- ii. 'That wherever the properties A and D are found while E is missing, the properties B and C will either both be found, or both be missing.' In symbols,

$$ad\bar{e} = q(bc + \bar{b}\bar{c}).$$

- iii. 'That wherever the property A is found in conjunction with either B or E , or both of them, there either the property C or the property D will be found, but not both of them. And conversely, wherever the property C or D is found singly, there the property A will be found in conjunction with either B or E , or both of them.' This premise is formulated as an equivalence (double implication); Boole thus translates it as an equality without indefinite class symbols:

$$ab + a\bar{b}e = \bar{c}d + c\bar{d}.$$

It may seem more straightforward to write this as Schröder later does:

$$a(b + e) = \bar{c}d + c\bar{d}$$

The reason for the difference is that Boole's '+' only allows for the addition of disjoint classes: writing $a\bar{b}e$ guarantees that it is disjoint from ab . Schröder, who adopts an inclusive interpretation of '+', can accordingly dispense with this extra factor.

Now, Boole asks for the relation between the properties A , B , C , and D ,—thus eliminating E ¹⁶—and this in two different forms. First, with interest for 'what may be concluded from the ascertained presence of the property A , with reference to the properties B , C , and D ', the relation is sought in the form $a = \dots$; second, looking for 'what may be concluded in like manner respecting the property B , and the properties A , C , and D ', the relation is sought in the form $b = \dots$.

Additionally, Boole asks for 'whether any relations exist independently among the properties B , C , and D ' (which are those that A is to be expressed in terms of in the first half of the problem) and likewise among the properties A , C , and D . While these two questions can be seen as instances of the general problem of logic (the first amounts to eliminating both A and E from the premises, the second both B and E , but with no particular form prescribed for the relation sought between the remaining terms), their presence here is somewhat peculiar. One way of reading them is as asking for what information has had to be discarded about the relation between—taking the first case as an example— A , B , C and D in order to express it under the particular form

¹⁶ As Boole puts it, 'It will be observed, that in each of the three data, the information conveyed respecting the properties A , B , C , and D , is complicated with another element, E , about which we desire to say nothing in our conclusion. It will hence be requisite to eliminate the symbol representing the property E [...]' (Boole 1854, p. 146).

$a = \dots$. The main reason why they appear here, however, may just be that Boole's method for obtaining the solution $a = \dots$ gives this further relation for free, as we shall see.

3 Boole's solution

In order to bring to the fore the goal-directedness of Boole's method, we now briefly describe how he applied it to the foregoing problem. (We briefly discuss Schröder's modified treatment, too, because Frege referred to it as well.) A synopsis will have to suffice for our purposes, but readers should be warned that we are skimming over the entire substance of Boole's system, going so far as to omit every proof and justification.

Here, then, is how Boole and Schröder proceed:

- i. Transform and bring together the premises, so as to obtain a single equation of the form $\Delta = 0$.¹⁷
- ii. Eliminate the terms that should not appear in the solution (in our case e). This leads to a new equation $\Delta' = 0$, where Δ' does not contain e .
- iii. If the desired solution is of the form $a = \dots$ (say), first factor the preceding equation by a and $\bar{a}$, yielding in our case

$$(cd + b\bar{c}\bar{d})a + (\bar{c}d + c\bar{d} + \bar{b}\bar{c}\bar{d})\bar{a} = 0, \quad (1)$$

then expand $\bar{a}$ as $1 - a$ and proceed as if algebraically solving for a :

$$a = \frac{\bar{c}d + c\bar{d} + \bar{b}\bar{c}\bar{d}}{\bar{c}d + c\bar{d} + \bar{b}\bar{c}\bar{d} - cd - b\bar{c}\bar{d}}.$$

As is the case here, this typically results in division signs of rather unclear meaning on the right-hand side. Schröder, whose solution follows Boole's up to this point (in broad outline at least), avoids this murky division and stops at equation (1) above.

- iv. At this stage, Boole and Schröder split ways. Boole performs a process he calls 'development' to get rid of the fraction he has just introduced; generically, this gets the right-hand side into the strange-looking form

$$U + \frac{0}{0}V + \frac{1}{0}W,$$

where U , V and W are sums of terms: for our problem,

$$a = c\bar{d} + \bar{c}d + \bar{b}\bar{c}\bar{d} \quad (\text{i.e., } V = W = 0), \quad (2)$$

$$b = \bar{a}\bar{c}\bar{d} + \frac{0}{0}(\bar{a}cd + a\bar{c}d + a\bar{c}\bar{d}) + \frac{1}{0}(acd + \bar{a}\bar{c}d + \bar{a}c\bar{d}). \quad (3)$$

¹⁷ Note that Δ is our own notation to denote some arbitrary algebraic expression.

Equation (2) has no unusual symbols, hence straightforwardly answers the question (in words, property A is to be found exactly when one but not both of properties C and D are found, or when none of B , C and D are found). In cases where, as in equation (3), V and W do not vanish, Boole does two things: he interprets $\frac{0}{0}$ as an indeterminate class symbol and splits off the term $\frac{1}{0}W$ into a separate equation $W = 0$. His interpretation of (3), then, is that B has the same extension as the class expressed by U plus *part of* the class expressed by V (remember that indeterminate class symbols are used for inclusions), and that the equation $W = 0$ expresses the *independent relations* between A , C and D (asked for in the statement of the problem, above).

Schröder, for his part, avoids Boole's perplexing symbolic manipulations by way of a general theorem, which allows jumping straight from equations like (1) to solutions equivalent to Boole's. In essence (taking for instance the first question, aiming at $a = \dots$), from an equation of the form

$$Sa + T\bar{a} = 0,$$

Schröder directly expresses our U , V and W above in terms of S and T :

$$U = T, \quad V = \bar{S}, \quad W = ST$$

so that one gets the full solution $a = T + u\bar{S}$ (with u an indeterminate class symbol) and the independent relation $ST = 0$.¹⁸

For simplicity, we described Boole's method in the context of a particular problem, but its outline is general, with minor variants to cover special cases.¹⁹ On this basis, the two features of Boole's approach we want to highlight are that it is *systematic* and *goal-directed*. As shown in the previous section, Boole delineates a well-defined class of problems, and the method just sketched allows for the *systematic* solution of any of them, *guided by* the particular problem to be solved. In the synopsis above, steps 2 and 3 are where this goal-dependence appears: the elimination of unwanted terms depends on the particular relation sought, as do the algebraic manipulations of step 3, where preexisting experience with first-degree algebraic equations points us to the transformations needed in order to solve for a particular variable.

¹⁸ In our case, we indeed get $U = T = \bar{c}d + c\bar{d} + \bar{b}c\bar{d}$ and $W = ST = 0$ (it is easy to see that the product ST vanishes, because each combination of a summand from S and a summand from T multiplies out to a term containing both a symbol and its negation, e.g. c and $\bar{c}$, and thus to zero), but on the face of it, we obtain a nonzero term for $V = \bar{S}$. However, one can also see that $\bar{S}\bar{T} = 0$, which means that $V\bar{U} = 0$: in terms of classes, V is included in U , so that in the solution $a = U + uV$, where u is an indefinite class term, V is redundant and can be discarded—thereby correctly falling back on Boole's solution (Schröder 1877, p. 27).

¹⁹ For instance, as discussed in the previous section, one might ask for the expression of ab in terms of c and d , that is, seek a relation of the form $ab = \dots$ instead of the simpler $a = \dots$ or $b = \dots$. Boole would then introduce an auxiliary term t with an additional premise $t = ab$, then eliminate a and b as well as other unwanted terms and proceed as above, seeking a solution of the form $t = \dots$, where t can ultimately be replaced by ab again (Boole 1854, pp. 140–142).

4 A combinatorial problem?

Before turning to Frege's solution, and in preparation for tackling it, it is useful to dwell a little on a remark made repeatedly by followers of Boole as well as by critics. The point, as we would phrase it nowadays, is that problems like the above are at bottom *combinatorial*. Some, like Jevons and Venn, emphasized this in order to offer other methods to complement Boole's algebraic machinery or even replace it altogether. Others, including Frege, concluded that Boole's logical problems were of little interest.

The discussion focuses on logical problems that, like the above, only have *universal* propositions as premises. (In practice, Boole had some trouble dealing with existential propositions—like the Aristotelian 'Some As and Bs'—and most of his immediate successors preferred to leave those aside.²⁰) As Boole already noted, any universal premise amounts to the assertion that certain combinations of classes are empty: for instance, 'All As are Bs' is equivalent to $a\bar{b} = 0$, i.e., the class of things that are both A and not-B is empty. More generally, take a problem involving n classes. Each class divides the universe into two parts (those elements in it, and those not in it), so that the problem determines 2^n subdivisions overall—e.g., for two classes denoted by a and b , we get ab , $a\bar{b}$, $\bar{a}b$ and $\bar{a}\bar{b}$. Each (universal) premise is then equivalent to what Boole calls 'a system of denials':²¹ the assertion that a certain number of these combinations are empty.

Accordingly, one can solve a problem like the one we worked through above by (i) going through each of the possible combinations of classes and checking it against every premise to determine whether it has to be empty; then (ii) inspecting the resulting list of denials to answer the question. This is basically the method offered by Jevons (1864), who famously went on to design a machine automating step (i).²² Step (ii) is more delicate, as the simplest way to summarize a given list of denials may not be easy to see. Jevons tried drawing up exhaustive tables in which solutions could be looked up; Venn suggested that the best approach was to use 'tact and judgment, aided by graphical methods', that is, his well-known diagrams, which were supposed to make the denials more surveyable and potential patterns among them clearer.²³ In either case, a good amount of the goal-directedness of Boole's method gets lost: instead of a systematic procedure building up to a conclusion of a specified form, say $a = \dots$, we get techniques for enumerating and surveying all elementary denials that follow from given premises.

Despite emphasizing (what we might call) 'combinatorial' aspects of it, Boole's English followers—like Jevons or Venn—still largely celebrated his general conception of the problem of logic: according to Jevons, Boole was the 'first to set forth in its full extent the problem of logic'²⁴ and 'it will probably be allowed that [he]

²⁰ See note 6 above.

²¹ See Boole (1854, p. 84): 'Every primary proposition can thus be resolved into a series of denials of the existence of certain defined classes of things, and may, from that system of denials, be itself reconstructed.'

²² See Jevons (1870). For an introduction and further references, see Gardner (1958, ch. 5).

²³ See Jevons (1874, in particular p. 135 and ch. VII); Venn (1881, pp. 255–260) discusses Jevons' table, then compares it with his diagrams.

²⁴ Jevons (1874, p. 130); Jevons (1887, p. 113).

discovered the true and general form of logic.’²⁵ Venn, for his part, wrote that ‘[t]he general solution of this problem was probably first conceived, and almost certainly first effected, by Boole [. . .]. It cannot often be the lot of any one to conceive and so completely to carry out such a generalization in an old and well-studied subject.’²⁶

Boole’s German reception was markedly different. Hermann Lotze, discussing the very problem presented above, noted that tailor-made algebraic methods seemed quite unnecessary and proceeded to offer a combinatorial solution, somewhat reminiscent of Jevons’s (who for his part kept to simpler cases and never tackled this one).²⁷ But, while Lotze basically started from the same observation as Jevons—that Boolean problems are merely about recombining fragments delineated by already-defined classes—he came to a starkly different conclusion: for him, such problems were ultimately of little interest. Such, too, was Frege’s opinion:²⁸ Boole’s problems, he wrote, ‘mostly seem to have been invented only for the purpose of being solved by his formulas’²⁹ and ‘will seldom, if ever, occur in science.’³⁰

Both Lotze and Frege’s criticisms of Boolean problems rely on a broader discussion of concept formation (see Heis 2013). Let us concentrate on Frege, whose views are the most important for us here. His argument is that interesting new concepts are not formed merely by assuming as given ‘a system of concepts, or speaking metaphorically, a network of lines’ and then by ‘[using] the lines that are already there in a new way to fully describe the boundary of pieces of area.’³¹ Focusing on such limited cases, Frege forcefully argues, can only be sterile:

It is the fact that attention is principally given to this way of forming new concepts from old ones, while other more fruitful ones are neglected, which surely is responsible for the impression one easily gets in logic that for all our to-ing and fro-ing, we never really leave the same spot.³²

The apparatus of function-argument analysis and of quantification is, of course, what allowed Frege to offer a new and fruitful view of what concept formation can be. Nevertheless, he sought to prove that his *Begriffsschrift* could tackle Boole’s problems anyway. Let us now turn to his solution.

²⁵ Jevons (1887, p. 113); this sentence is missing from the 1874 edition.

²⁶ Venn (1881, p. 301).

²⁷ See Lotze (1884, pp. 219–221) = Lotze (1912, pp. 265–267). This solution is included in a ‘Note on the Logical Calculus’ that Lotze only added to the second, 1880 edition of his *Logic*.

²⁸ For a fuller discussion of Lotze’s outlook on this point as well as of parallels with Frege, see Heis (2013) as well as Gabriel (1989). Note that Frege, although he knew Lotze’s book, may not have been aware that the second edition contained an addendum treating the very problem he chose to discuss in his manuscript on Boole; see Gabriel, *op. cit.*, p. XX and XXIII.

²⁹ Frege (1993b, p. 97).

³⁰ Frege (1979, p. 46).

³¹ Frege (1979, p. 34, translation altered) = Frege (1969, p. 38). The ‘metaphor’ here refers to something like Euler diagrams, described by Frege earlier, that represent concepts by the area inside a circle or closed curve.

³² Ibid. Frege makes a similar point, but framed as a criticism of Kant, in the *Grundlagen*, §88 (Frege 1884, pp. 99–101) = (Frege 1953, pp. 99–101).

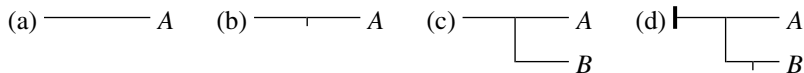
5 Frege's solution

At the end of a manuscript comparing his *Begriffsschrift* with Boole's system, Frege tackles the very problem discussed above in order to show that 'if in fact science were to require the solution of such problems, the concept-script would be able to cope with them without any difficulty.'³³ Yet as we shall see, his solution is rather haphazard, and is much more akin to an exhaustive (though intelligent) search through the space of all possible proofs from the premises than to Boole's algebraic method.

As a preliminary, we need to dispose of two slight complications. First, Frege changes notations a little: he uses the Greek capital letters A, B, Γ, Δ, E to refer to the presence of the properties A, B, C, D, E respectively. To avoid needlessly complicating the comparison, we decided to revert to Roman capitals. Second, as pointed out by the editors of Frege's manuscript, some mistakes in his premises—partly due to his following Schröder, whose initial phrasing of the problem contains a minor misprint—appear to put his solution in jeopardy, even though he gets to the right conclusions (unsurprisingly, given that he had the correct answer at hand). This, however, is inessential: Frege's solution can be corrected and carried through with only minor changes, and we shall proceed assuming such amendments (spelled out in footnotes below).

Frege starts his solution by decomposing Boole's three premises into as many as thirteen;³⁴ as he puts it, 'whereas the dominant procedure in Boole is the unification of different judgements into a single expression, I analyse the data into simple judgements.'³⁵ To understand what Frege's 'simple' judgements are, we need to briefly review his notation, which we shall do through a few examples.³⁶

In Frege's *Begriffsschrift* notation, the formulas below stand for, respectively, (a) the proposition A ; (b) the negation of A ; (c) the material conditional we would write as $B \rightarrow A$, which is Frege's only device for combining different propositions into more complex ones (conjunction and disjunction being obtained using the conditional together with negation).³⁷



Frege explicates the conditional (c) as the proposition that one cannot have that *B* is asserted and *A* denied; this parallels the truth-functional analysis of the conditional as excluding a single one of the four possible combinations of truth-values for *A* and

³³ Frege (1979, p. 45) = Frege (1969, p. 51).

³⁴ This is assuming his solution is corrected as per footnote 42 below; his own version has only twelve premises.

³⁵ Frege (1979, p. 45) = Frege (1969, p. 51).

³⁶ For a quick introduction to Frege's *Begriffsschrift* notation, see for instance von Plato (2017, ch. 4) or Schlimm (2018, pp. 54–65).

³⁷ Frege (1993a, pp. 11–12) = Frege (1972a, pp. 121–122); see also Schlimm (2018).

B (but phrased in terms of ‘assertion’ and ‘denial’ rather than of truth and falsity).³⁸ In fact, in his manuscript on Boole, Frege argues that it is precisely because the conditional only excludes one out of four such combinations that it is ‘simpler’ than Boole’s equality sign (indeed, a Boolean equality $a = b$, which in today’s notation corresponds to two conditionals $A \rightarrow B$ and $B \rightarrow A$, excludes not just one but two possible combinations).³⁹ Finally, a thick vertical stroke to the left of a proposition, as in (d), turns it into a *judgement*, that is, means that the proposition is *asserted*. Putting everything together, (d) thus stands for the judgement that ‘ A and B cannot both be denied’.⁴⁰

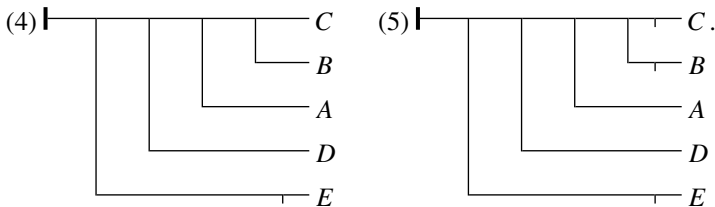
What Frege calls ‘simple’ judgements in the context of our problem are formed from conditionals like in (c) or (d), but nested. Take for example Boole’s premise (ii), that ‘wherever the properties A and D are found while E is missing, the properties B and C will either both be found, or both be missing’, which he wrote $\bar{e}ad = q(bc + \bar{b}\bar{c})$. The indefinite class symbol q on the right-hand side allows translating this as a conditional, namely, in contemporary notation,

$$(\neg E \wedge A \wedge D) \rightarrow ((B \wedge C) \vee (\neg B \wedge \neg C)).$$

To understand Frege’s translation, notice, first, that the consequent states that B and C always go together, and so is equivalent to the conjunction of $B \rightarrow C$ and $\neg B \rightarrow \neg C$. Splitting this consequent, this leads to the two formulas

$$(\neg E \wedge A \wedge D) \rightarrow (B \rightarrow C) \quad \text{and} \quad (\neg E \wedge A \wedge D) \rightarrow (\neg B \rightarrow \neg C).$$

Finally, a conjunctive antecedent like $\neg E \wedge A \wedge D$ can be replaced by nested conditionals (the order of nesting being indifferent); hence Frege’s translations (the numbering is his):



Let us consider a more intricate example, namely Boole’s premise (iii), which he wrote $ab + ae\bar{b} = d\bar{c} + c\bar{d}$. Taking $+$ in a non-exclusive sense, as Schröder and later authors do, it can be rewritten $ab + ae = d\bar{c} + c\bar{d}$. This has the form of an equation without indefinite class symbols, so that in modern terms, it is an *equivalence* and first needs to be split into two implications. The first half can then be translated as

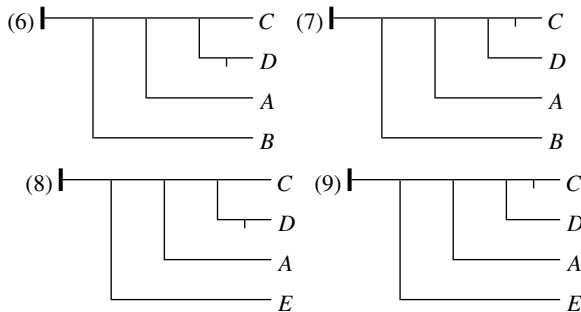
³⁸ Frege (1993a, p. 5) = Frege (1972a, pp. 114–115). Note that we are discussing the *Begriffsschrift* here; in the later system of the *Grundgesetze*, Frege is able to define his conditional directly in terms of truth and falsity (Frege 1893, §12).

³⁹ Frege (1979, p. 36) = Frege (1969, p. 40).

⁴⁰ Frege (1993a, pp. 10–11) = Frege (1972a, p. 121).

$$((A \wedge B) \vee (A \wedge E)) \rightarrow ((D \wedge \neg C) \vee (C \wedge \neg D)).$$

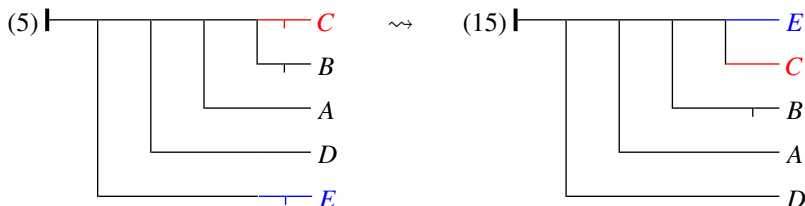
As above, the consequent may be broken down into $D \rightarrow \neg C$ and $\neg D \rightarrow C$. But in this case, the disjunction in the antecedent also requires splitting up, namely into the judgements that the consequent holds given A and B , and that it holds given A and E . All in all, the first implication of Boole's (iii) yields four different judgements:



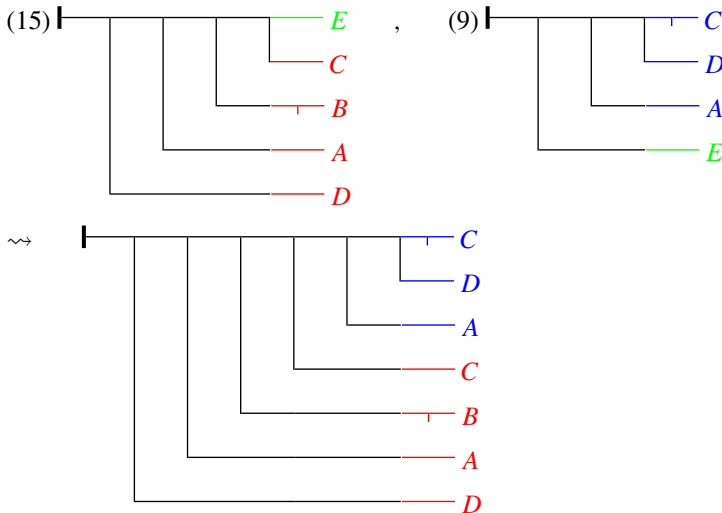
The second implication of (iii)—Frege's mistakes aside—also yields four judgements, which makes eight in total for a single Boolean equation.

In what sense does Frege see his version of the premises as 'simple'? Remember that he dubbed the conditional 'simple' because it excluded a single combination of assertions and denials of the terms involved. The same can be said here, only with more propositions. For instance, (7) excludes that B , A , D be asserted and the negation of C denied, that is, excludes that B , A , D , and C all be asserted together. In this sense, Frege's premises are similar to the 'atomic denials' into which the 'combinatorial' solutions discussed above broke down the data of the problem. The main difference is that Frege's simple judgements do not always contain all five of the terms involved, as shown by formulas (6)–(9); still, they can easily be used to generate the full list of combinations excluded by the premises. Frege, however, uses the data in a strikingly different way: as will appear presently, his approach could be described as an *inferential* recasting of the combinatorial solutions.

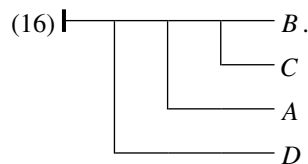
To solve the problem from his list of 'simple judgements', Frege essentially uses two kinds of transformations, which we may call *contraposition* and *cut*. First, contraposition: in a nested conditional, the consequent (written on the top line) may be switched with any antecedent (written on any of the other lines) while negating both. For instance, Frege transforms formula (5) above into (15) (the numbering is still his):



(As mentioned already, the order of antecedents does not matter in such formulas: lines other than the top one can be reordered freely.⁴¹ Strictly speaking, this could be considered as a third transformation Frege uses.) Second, a rule akin to *modus ponens*, or more precisely, *cut*: two conditionals can be combined when the consequent of one is among the antecedents of the other, as *E* (shown in green) is in (15) and (9) above:



Here, the antecedents of *E* in (15), shown in red, have been plugged into (9) at the place of *E*. The result can then be simplified, using contraposition to switch the consequent $\vdash C$ with $\vdash B$ and eliminating redundant antecedents; hence, still using Frege's own numbering,



With these tools in hand, we can tackle Boole's first question, namely to find what follows from *A* regarding *B*, *C* and *D*. Frege's strategy is to search for every possible judgement inferrable from the premises that has *A* as antecedent and does not contain *E*. Two such judgements (to wit, (6) and (7) above) are already found among the premises. For the rest, since Frege relies (in this context) on a single inference rule that allows *combining* different judgements, the problem boils down to surveying every possible application of it. This is exactly what he does: he looks for every possible application of his cut-like rule that would eliminate *E* from some premises. In order to do this efficiently and systematically, Frege first uses his contraposition-like rule

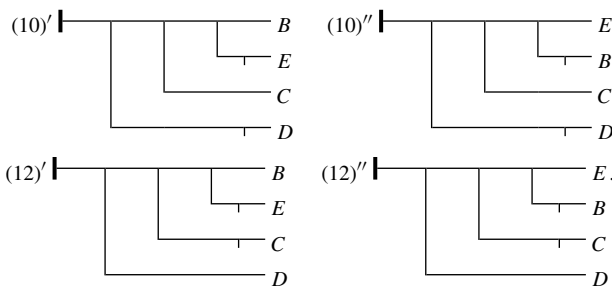
⁴¹ Note that this is only true because of the specific form of the nested conditionals Frege uses here; if there were, say, negation signs anywhere else on the top line than just before the letter, this would cease to hold.

to rewrite the premises so that E never appears negated—he thus transforms (5) into (15), as shown above, and proceeds similarly for (4).

At this stage, the premises can be sorted into three groups, according to the occurrence of E : those that have E as an antecedent, those that have it as consequent, and those in which it does not occur at all. Members of the third group are either already part of the solution, if they contain A , like (6) and (7), or are of no use if they do not. As for the rest, cut allows combining every premise of the first group with every premise of the second group. Most of these combinations yield judgements which, in Frege's words, hold 'independently of the contents'—that is, tautologies (this can happen either because the consequent is already among the antecedents, or because two of the antecedents contradict each other, like $\vdash B$ and $\vdash \neg B$). As it happens, in our case, ten combinations have to be surveyed,⁴² and the only fruitful one is the combination of (9) and (15) shown above, which yields (16). In the end, the full solution is given by (6), (7), and (16).⁴³ (In passing, note that Frege's solution is, in fact, weaker than Boole's: in contemporary terms, the latter—being an equation with no indefinite class terms—corresponds to a biconditional of which the former is only the first half.)

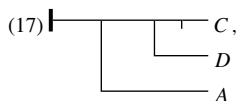
The nature of Frege's solution should be clear by now: it is, essentially, a systematic search through a space of possible proofs. Despite the superficial similarities, noted above, of his 'simple judgements' with combinations of terms excluded by Boole's premises, the spirit of his solution is inferential rather than combinatorial: his goal is, essentially, to show that Boole's problem can be solved through simple logical

⁴² This is assuming that Frege's mistakes are corrected. For the record, here is how this should be done. The editors (see Frege 1979, note 1 p. 41) suggest *adding* a premise (10)' and *replacing* Frege's premise (12) by their (12)' (shown below). Additionally, the following amendments are required. First, one should introduce equivalent variants of (10)' and (12)', namely (10)'' and (12)'':



Second, in Frege's solution of the first question, one should check that combining either of (10)'' or (12)'' with either of (8) or (9) only produces tautologies. Third, in the list at the top of p. 44, all occurrences of '(17)' should be replaced by '(19)'.

⁴³ In fact, Frege notices that a further simplification is possible: B can be eliminated by cut from (7) and (16), yielding



so that (6) and (17) together contain the full solution to Boole's first question.

inferences, with no tailor-made method and, as he puts it, ‘practically no theoretical preparation at all’.⁴⁴

Admittedly, approaching the *Begriffsschrift* through the lens of Boole’s problem may seem unfair. After all, before offering his own solution, Frege writes that ‘it would not be surprising, and I would have no reluctance to concede the point, if Boolean logic were better suited than my concept-script to solve the kind of problems it was specifically designed for, or that were specifically invented for it’ (though, he adds, ‘perhaps not even this is the case’).⁴⁵ Among the points he makes in his comparison with Boole, he even describes the fact that his system can handle such problems just as well as Boole’s as ‘the point to which I attach least importance’ (since in his eyes they are of little use anyway).⁴⁶

Yet approaching Frege’s system on its own terms only confirms that it contains nothing comparable to Boole’s conception of what a logical problem is. In fact, as the rest of Frege’s paper on Boole makes abundantly clear, the *Begriffsschrift* is not meant to solve problems at all: rather, it is a visual tool to scrutinize concepts and inferences. In the famous formulation of the *Grundlagen*, the *Begriffsschrift* ‘is intended to achieve greater economy and surveyability of expression and to be used in a few fixed forms in the manner of a calculus, so that no transition is permitted that is not in accord with the rules that are laid down once and for all’.⁴⁷

6 Two conceptions of ‘calculus’

We now come to our main claim, which is that the striking difference we described between Boole and Frege’s solution methods—the former offering a goal-directed algorithm, the latter an exhaustive search through possible proofs—reflects broader philosophical conceptions: the two logicians hold very different views of what a logical ‘calculus’ should be.

Start with Boole: his main goal, as we have explained, was to find a general method to solve logical problems, and he says so explicitly. The point comes out particularly clearly in a passage of the introduction to the *Laws of Thought* in which he discusses the shortcomings of Aristotelian logic. After disputing that the traditional rules of syllogism and conversion⁴⁸ are the ‘ultimate processes’ of logic, he argues that *even if* all inference were in fact reducible to Aristotelian forms, a general method would still be needed:

If all inference were reducible to [syllogism and conversion], there would still exist the same necessity for a general method. For it would still be requisite to determine in what order the processes should succeed each other, as well as their particular nature, in order that the desired relation should be obtained. [. . .] If we

⁴⁴ Frege (1979, p. 44) = Frege (1969, p. 49).

⁴⁵ Frege (1979, p. 39, translation altered) = Frege (1969, p. 44).

⁴⁶ Frege (1979, p. 46) = Frege (1969, p. 52).

⁴⁷ See §91; Frege (1884, p. 103) = Frege (1953, p. 103), translation from Beaney (1997, p. 124).

⁴⁸ In Aristotelian logic, ‘converting’ means inverting subject and predicate, e.g., going from ‘Some A is B’ to ‘Some B is A’ or from ‘All A are B’ to ‘Some B are A’.

may judge from the mathematical sciences, which are the most perfect examples of method known, this *directive* function of Method constitutes its chief office and distinction. The fundamental processes of arithmetic, for instance, are in themselves but the elements of a possible science. To assign their nature is the first business of its method, but to arrange their succession is its subsequent and higher function. In the more complex examples of logical deduction [. . .] the aid of a directive method, such as a Calculus alone can supply, is indispensable.⁴⁹

To take a geographical metaphor, suppose logical problem-solving is a journey from a city *A* (the premises) to a city *B* (the ‘relation’ sought between certain terms appearing in the premises, under a given form—see Sect. 1). Merely knowing the ‘fundamental processes’ of logic is like seeing a number of roads in front of us that we could try, but with no signposts telling us which one leads to our destination. A logical method, for Boole, tells us which road to take. In the same way, just knowing that addition and multiplication are associative and commutative, or that one can add a number to both sides of an equality, is going to be of little help to solve even the simplest of algebraic equations: one needs some idea of how to string together such simple rules in order to reach the solution, which is precisely what algebra is all about (e.g., to solve for an unknown x , we learn to add or subtract the same quantities on each side of the equation until all x ’s are on the same side, to factor by x , and then to divide each side by the coefficient of x). What Boole calls a ‘directive’ method does just that. Moreover, as the last sentence in the quote above indicates, he sees such a method as an integral part of the very idea of a ‘calculus’—an idea that deserves deeper discussion, which we shall postpone until Sect. 8.

In Frege, the concept of calculus is still at the forefront, but takes on a decidedly thinner meaning. To get clearer on it, we need to briefly review his all too well-known opposition between *calculus ratiocinator* and *lingua characterica*. This contrast has been much written about, mostly because it has been used as a way to oppose, not just Frege and Boole (or the ‘Booleans’), but two broader lineages in the history of logic.⁵⁰ We shall sidestep these debates here: our only concern is the way Frege himself used the phrases. As it happens, he only deployed them when positioning his work with respect to competing programs, and this in three polemical pieces, two on Boole and one on Peano.⁵¹

When discussing Boole, Frege is very clear that he meant his system to be both *calculus* and *lingua*:

⁴⁹ Boole (1854, pp. 10–11). (Emphasis in the original.)

⁵⁰ The idea to use these phrases to describe distinct traditions in logic goes back, as far as we know, to Philip Jourdain in his preface to Couturat (1914); it was sharpened and popularized by Van Heijenoort (1967), then taken up by various authors to shed light on other aspects of the history of logic (for further references, see for instance Kusch 1989, notes 5 and 6 p. 289). A further complication is that already in Frege’s time, the phrases (which ultimately come from Leibniz through the reconstruction by Trendelenburg 1856) were used in senses quite different from his, most notably by Schröder in his review of the *Begriffsschrift*. For discussion, see Sluga (1987) and Peckhaus (2004).

⁵¹ The two papers on Boole (the long, only posthumously-published manuscript containing the solution discussed in Sect. 5, and a much briefer version that Frege managed to get published) are Frege (1979) = Frege (1969) and Frege (1972b) = Frege (1993b); the one on Peano is Frege (1984) = Frege (1897).

In fact, I wished to produce, not a mere *calculus ratiocinator*, but a *lingua characterica* in the Leibnizian sense. In doing so, however, I recognize that a deductive calculus of the former sort is a necessary part of a conceptual notation. If this was misunderstood, perhaps it is because I let the abstract logical aspect stand too much in the foreground.⁵²

So what exactly is this ‘calculus’ feature that Frege thought his system had in common with Boole’s? Helpful here are his remarks on Peano’s notation, which he thought lacking because of ‘the absence of rules of inference.’⁵³ In this respect, writes Frege, ‘we can recognize a closer affinity between Boolean logic and my conceptual notation, in as much as the main emphasis is on inference, which is not stressed so much in the Peano logical calculus’, so that ‘Boole’s logic is a *calculus ratiocinator* but not a *lingua characterica*; Peano’s mathematical logic is mainly a *lingua characterica*, and secondarily also a *calculus ratiocinator*; whereas my conceptual notation is both with equal emphasis.’⁵⁴ In the same article, he also explains unambiguously, by contrast with Peano’s, the sense in which his own system is a calculus:

In my conceptual notation inference is conducted like a calculation. I do not mean this in the narrow sense, as if it were subject to an algorithm the same as or similar to that of ordinary addition and multiplication, but only in the sense that there exists an algorithm at all, i.e. a totality of rules which govern the transition from one sentence or from two sentences to a new one in such a way that nothing happens except in conformity with these rules.⁵⁵

This reflects the fact that for Frege, the ‘chief purpose’ of the *Begriffsschrift* is ‘to test in the most reliable manner the validity of a chain of reasoning’:⁵⁶ it is a calculus merely in the sense that the transformation rules are fully specified, in a way that makes the validity of any given inference easily surveyable (see also the end of Section 5 above). Actual *algorithms* (in today’s customary sense of the word) for solving certain kinds of logical problems or finding the proofs of sentences are altogether absent from the picture.

The contrast should be clear by now. For Boole, the value of a logical calculus lies in the solution methods it affords access to, whereas for Frege, what matters is merely that inference rules be fully and formally specified, so that inferences can be scrutinized rigorously.

7 Enabling mechanization and fighting it

From a contemporary point of view, it is tempting to read both Boole and Frege as making progress toward some kind of mechanization of logic. Of course, retrospective

⁵² Frege (1972b, p. 91, translation altered) = Frege (1993b, p. 98).

⁵³ Frege (1984, p. 238) = Frege (1897, p. 366). As noted by von Plato (2017, pp. 55–56), one should not overstate this criticism: appropriate rules can easily be supplied. See also Schlimm (2021).

⁵⁴ Frege (1984, p. 242, translation altered) = Frege (1897, pp. 370–371).

⁵⁵ Frege (1984, p. 237, translation altered) = Frege (1897, pp. 364–365).

⁵⁶ Frege (1972a, p. 104) = Frege (1993a, p. IV).

judgements of this kind are anachronistic in nature, and indeed, we shall see that in their historical context, both authors explicitly wrote *against* mechanization. In and of itself, though, this is no objection: we can describe the mechanical *potential* of their work whatever their own views. Yet closer examination makes such a retrospective reading difficult to defend. On the one hand, even insofar as both authors can be read as advancing *some* kind of mechanization, it would not be the same kind in both cases, nor one straightforwardly reducible to today's use of the word. On the other hand, such a retrospective reading underestimates how much Frege's entire project amounted, in a sense, to using formalization *against* mechanization. Charting the complex relation of Boole and Frege to mechanization in light of their conceptions of 'calculus', discussed above, is the goal of this section.

Our discussion until now already makes clear that Boole and Frege's contrasting conceptions of what a logical calculus should do—afford algorithmic methods for solving problems or allow for the rigorous inspection of individual inferential steps—would lead to very different things being mechanized: in Boole's case, entire problem-solving processes; in Frege's, the verification of (already given) proofs. It is hard to escape the feeling that the latter goal is distinctly less ambitious. As it turns out, this is no coincidence. First, as has been shown by Jamie Tappenden's recent recontextualization efforts, much of Frege's enterprise was positioned against computational or mechanical views of mathematics. Second, on the basis of our comparison with Boole, we want to argue further that Frege did not merely have anti-mechanical *motivations*, but actually managed to shift the focus away from mechanization and push it into a secondary role.

Before turning to Frege, we need to say a few words about Boole, who, regarding mechanization, had an ambiguous attitude of his own. Historian Ivor Grattan-Guinness, comparing him to Babbage, went so far as to say that Boole 'would not have welcomed the association of his logic with the *repetitive* actions of computing; whether Babbagean mechanical or modern electrical'⁵⁷ and even that 'he would have found repellent the mechanical aspects of Babbage's engines.'⁵⁸ The case, we believe, should be approached with caution. The main piece of evidence for it comes from Boole's first foray into logic, *The Mathematical Analysis of Logic* (Boole 1847), more precisely from the preface and introduction (quoted below). However, such preliminary texts deserve careful contextualization: we should keep in mind that Boole's main concern there would have been to forestall the immediate hostility his 'mathematical' approach to logic was likely to provoke among more traditional practitioners of the field. Of particular relevance are the complex debates going on in Britain at the time about the respective educational value of mathematics and logic, a large topic that we can only hint at here (see Durand-Richard 2000). In a nutshell, that Boole's use of algebra in logic could be understood as a form of 'mechanization' went without saying for his contemporaries,⁵⁹ but he saw this as a threat rather than a selling point: he was at pains to counter the charge that he was reducing logic (then praised by some as a

⁵⁷ Grattan-Guinness (1997, p. xlv), his emphasis.

⁵⁸ Grattan-Guinness (1992, p. 44).

⁵⁹ A good reference point here is John Stuart Mill, whom Boole quotes at the beginning of his first logical tract (see below). Mill argues that the language of algebra is prototypically mechanical: 'The complete or extreme case of the mechanical use of language, is when it is used without any consciousness of a meaning

tool to train the highest faculties) to a mere matter of mechanical rote, liable to deaden the mind rather than improve it. Since he could hardly deny that he was offering a symbol-pushing method for solving logical problems, his overall strategy was to shift the question: the real intellectual exercise, he argued, lay not in solving individual problems, but in coming to understand the workings and general applicability of his symbolic method itself—and this, he claimed, offered an intellectual discipline much higher than any in the traditional logic.

In this light, let us have a closer look at Boole's remarks on calculation and mechanization. In the aforementioned preface to *The Mathematical Analysis of Logic*, he quotes the following passage by Mill, implying that logical problems can indeed be solved mechanically and that he is offering a (symbolic) language to do just that:

Whenever the nature of the subject permits the reasoning process to be without danger carried on mechanically, the language should be constructed on as mechanical principles as possible; while in the contrary case it should be so constructed, that there shall be the greatest possible obstacle to a mere mechanical use of it.⁶⁰

But Boole immediately qualifies, adding that this is not his main purpose:

In one respect, the science of Logic differs from all others; the perfection of its method is chiefly valuable as an evidence of the speculative truth of its principles. To supersede the employment of common reason, or to subject it to the rigour of technical forms, would be the last desire of one who knows the value [of] intellectual toil [. . .].⁶¹

In other words, the main point of his method is not to actually replace ordinary reasoning by mindless calculations: what is most valuable is not its actual use to solve problems, but the understanding of how and why it works in general. A few pages later, in the introduction, Boole makes this more explicit, and links it with the educational debates we hinted at earlier:

In discussing the [. . .] question of the influence of the use of symbols upon the discipline of the intellect [. . .] it is of most material consequence, whether those symbols are used with a full understanding of their meaning [. . .] or whether they are mere unsuggestive characters, the use of which is suffered to rest upon authority. [. . .] In the former case an intellectual discipline of a high order is provided, an exercise not only of reason, but of the faculty of generalization. In the latter case there is no mental discipline whatever. (Boole 1847, p. 10)

[. . .]. This extreme case is nowhere realized except in the figures of arithmetic, and still more, the symbols of algebra [. . .]. Its perfection consists in the completeness of its adaptation to a purely mechanical use. The symbols are mere counters [. . .]. There is nothing, therefore, to distract the mind from the set of mechanical operations which are to be performed upon the symbols [. . .].’ (Mill 1974, pp. 707–708.) But he goes on to argue that such a mechanical language is in fact radically inappropriate outside of arithmetic: ‘On all other subjects, instead of contrivances to prevent our attention from being distracted by thinking of the meaning of our signs, we ought to wish for contrivances to make it impossible that we should ever lose sight of that meaning even for an instant.’ (Ibid, p. 710.)

⁶⁰ Mill (1974, p. 707), quoted in Boole (1847, p. 2).

⁶¹ Boole (1847, p. 2).

Nevertheless, these quotations—and accordingly, Boole’s opposition to mechanization, in so far as he could conceive of it—should not be overinterpreted. There is no reason to doubt that Boole, as mathematician, was more interested in the general method itself than in the individual calculations it permitted; nor that, from an educational point of view, he would have wanted to impart to students a full understanding of the method rather than the mere mastery of computational tricks. But his insistence on these themes at this particular juncture—the beginning of his very first publication on logic—should be approached with caution, particularly as hardly anything of the kind is to be found in his more expansive *Investigation into the Laws of Thought* a few years later. While he likely did not believe that the entirety of human thinking was captured by his system or could be made mechanical⁶² (and may well have found ‘repellent’ indeed the prospect of such a complete mechanization), he certainly valued the algorithmic aspect of his logical method; moreover, remarks by his friend Joseph Hill—recently published by MacHale and Cohen in their biographical volume—make it quite probable that Boole was aware of the potential in his work for the eventual construction of logical machines, and was not particularly disturbed by the idea.⁶³

Boole, then, may have written against mechanization in specific contexts, but his work nevertheless fosters it, in the particular sense of providing algorithmic solution methods for a class of problems—something he likely would not have disputed. Frege’s opposition to mechanization, on the other hand, seems quite deep-seated.

For a start, Jamie Tappenden has recently emphasized that a motivating—though oblique—theme of Frege’s *Grundlagen* is, roughly speaking, a polemic against views of arithmetic as mechanical or computational.⁶⁴ The context here, in a way reminiscent of Boole, are contemporaneous discussions about the *value* of mathematics, in particular for education: when Frege, in his introduction, allusively criticizes the (now forgotten) view of Kuno Fischer that computation is nothing but ‘aggregative, mechanical thinking’,⁶⁵ he is in fact—as Tappenden shows—defending mathematics against the idea that it is of little educational worth because it is a matter of mere mechanical rote. Frege’s argumentative strategy is to show, by way of his conceptual notation, that ‘one has to recognize to the concept of number a finer structure than most of the concepts of the other sciences, even though it is still one of the simplest in arithmetic.’⁶⁶ In other words, he aims at using what we might call ‘formalization’ to reveal the exquisite intricacies of thought that make up the simplest arithmetical concepts—and so to display the worth of mathematical thinking.

This general picture of (some of) Frege’s motivations in the *Grundlagen* ties in with his remarks on concept formation in his posthumously-published piece on Boole, which were already briefly discussed above (see Sect. 4). After arguing that there is little to be gained by mere extensional recombination of already given concepts—as opposed to the real work of forming and scrutinizing concepts using the apparatus

⁶² As Grattan-Guinness points out, Boole credits the mind with a capacity to grasp ‘general laws from particular cases’, for instance, which appears to be outside the scope of his system (Grattan-Guinness 1997, p. xliv).

⁶³ See Machale (2018, pp. 182–184).

⁶⁴ See Tappenden (2020, 2021).

⁶⁵ Frege (1884, p. iii) = Frege (1953, p. xv, translation altered).

⁶⁶ Frege (1884, p. iv) = Frege (1953, p. xvi, translation altered).

of the *Begriffsschrift*—Frege goes on to claim that it is precisely because Boole only plays around with already given concepts without looking into their formation that he is able to (spuriously) mechanize inference:

I believe almost all errors made in inference to have their roots in the imperfection of the concepts. Boole presupposes logically perfect concepts as ready to hand, and hence the most difficult part of the task as having been already discharged; he can then draw his inferences from the given assumptions by a mechanical process of computation. Stanley Jevons has in fact invented a machine to do this. But if we have perfect concepts whose content we do not need to refer back to, we can easily guard ourselves from error, even without computation. [...] Boolean formula-language only represents a part of our thinking; our thinking as a whole can never be coped with by a machine or replaced by purely mechanical activity.⁶⁷

In other words, the actual work of thinking is about clarifying the underlying structure of concepts—something Frege’s *Begriffsschrift* allows making explicit and visible. But *this* work cannot be made mechanical. We thus see how Frege uses his more sophisticated symbolic language *against* the idea that mathematics is mechanical.⁶⁸

This is not to say that, for Frege, there is no mechanical potential at all in the rule-bound use of symbols, or that such a ‘mechanism’ is entirely bad. Although Frege insists that symbols should never be used in a fully thoughtless way, he grants that symbol-pushing can offer important advantages; as he writes in a (later) letter to Hilbert:

A mere mechanical operation with formulas is dangerous (1) for the truth of the results and (2) for the fruitfulness of the science. The first danger can probably be avoided almost entirely by making the system of signs logically perfect. As far as the second danger is concerned, science would come to a standstill if the mechanism of formulas were to become so rampant as to stifle all thought. Yet I would not want to regard such a mechanism as completely useless or harmful. On the contrary, I believe that it is necessary. The natural course of events seems to be as follows: what was originally saturated with thought hardens in time into a mechanism which partly relieves the scientist from having to think. [...] I should like to compare this to the process of lignification. Where a tree lives and grows it must be soft and succulent. But if what was succulent did not in time turn into wood, the tree could not reach a significant height. On the other hand, when all that was green has turned into wood, the tree ceases to grow.⁶⁹

Importantly, though, not only is there a part of mathematical thinking that is not and cannot be made mechanical, but also, as discussed above, such non-mechanical aspects are made more visible precisely by the very specific kind of symbolic language that the *Begriffsschrift* offers, with its ability to unveil the richness of mathematical concept-formation.

⁶⁷ Frege (1979, pp. 34–35) = Frege (1969, p. 39).

⁶⁸ For further discussion on this, see Rohr (2020, section 1.5).

⁶⁹ Letter to Hilbert, dated 1 October 1895 (Frege 1980, p. 33 = Frege 1976, pp. 58–59).

In any case, insofar as Frege's work could be taken (from an anachronistic perspective) to be conducive to mechanization, it would only be to that of the *checking* of proofs—or possibly, though this is more of a stretch, that of the systematic search through the space of possible proofs defined by given inference rules. The distance with the kind of problem-solving mechanization sought by the Booleans is obvious, and indeed, the literature on 'logical machines' tends to ignore Frege completely (see, e.g., Gardner 1958).

8 Calculus, notations, and 'directive method'

The previous sections compared and contrasted Boole's and Frege's solutions to the same problem, as well as their broader outlooks on logic, computation, and mechanization. We now want to dwell on a particular feature of Boole's view: the intrinsic link that he establishes (and Frege breaks) between the concepts of 'calculus' and of 'directive method':

In the more complex examples of logical deduction [...] the aid of a directive method, such as a Calculus alone can supply, is indispensable.⁷⁰

At first sight, Boole's remark may seem commonplace, even trite: of course—one might think—calculi (or in other words, symbolic notations) like that of algebra are very useful in problem-solving. Yet further thought makes it puzzling. Why, indeed, is symbolic language so useful? A simple answer would be that it makes simple operations easy and fast by allowing one to perform them without thinking, by mere symbol-pushing. Notice, though, that this is *not* what Boole says. What he points to here is not that a calculus makes individual steps easier, but that it helps to methodically identify which steps to take next.

To make the question sharper and better understand the value of Boole's remarks, let us briefly survey the recent literature on why symbolic languages can be so useful in mathematics and logic. Much of it has taken a psychological route, focusing on externalization and extended cognition. Two ideas stand out. One is that symbolic languages allow recruiting powerful 'perceptual and sensorimotor' abilities in the service of problem-solving:⁷¹ symbols are not mere clothing for reasoning processes that somehow happen in the mind independently of the symbols, but the reasoning is done through the sensorimotor manipulation of physical or imagined inscriptions and nowhere else. In essence, this line of research gives experimental psychology substance to the old idea that symbolic languages are about replacing thinking by symbol-pushing, except that contemporary proponents tend to suspect that there is no pure thinking beforehand to be replaced, or that if there is, externalization transforms it into something substantially different.⁷² The other idea goes further: by offloading problem-solving to sensorimotor abilities, we are not merely making it more efficient (or even possible in practice), but we are actually avoiding biases built into our more

⁷⁰ Boole (1854, p. 11).

⁷¹ See for instance Landy et al. (2014), which also surveys some of the relevant experimental literature.

⁷² See for instance Vold and Schlimm (2020).

spontaneous internal reasoning processes. In the case of logic, such a point has recently been made by Dutilh Novaes (2012), who argued that formalization—by shifting our attention to symbols whose meaning we are able to disregard—allows us to bypass our widespread ‘belief bias’ (that is, the fact that we automatically bring previous knowledge to bear when reasoning). This is basically a recasting in the language of psychology of Frege’s idea that formal languages help us avoid smuggling assumptions or intuitions unnoticed.

Can these lines of research help make sense of Boole’s hints? What needs explaining is the way symbolic languages offer systematic help in identifying which steps to take next when engaged in problem-solving. There is indeed a way in which the aforementioned research could account for this: if, say, sensorimotor skills make us proficient at imagining algebraic symbols moving around, we may be able to quickly see which steps are likely to be helpful and which will bring no benefit (presumably because, consciously or not, we are performing them mentally without having to try them out on paper), in a way we could never match without recourse to symbolic tools. On this account, the use of symbols affords us a mere gain in efficiency, but this is enough to make it *seem* as though we are better at finding our way forward—even though, in fact, we are no better at orientation, but just quicker at systematically trying out various paths.

But Boole’s remarks surely point to something more fundamental than psychology. First, while there can be no doubt that our particular sensorimotor faculties are what makes the particular symbolic languages we use suitable and appropriate, this psychological advantage presupposes a non-psychological fact, namely that calculi codify possible steps toward a solution in the form of a limited number of symbol-pushing moves—thus making it possible to survey possible problem-solving strategies. This, however, is true of Frege’s system just as much as of Boole’s. The comparison between their solutions to the same problem suggests a second and more important point: what makes a calculus valuable is not merely the fact that it fully codifies possible moves and that the solution of any problem can, *in principle*, be derived from it. Rather, what matters is that the calculus gives us easy access, in some sense to be clarified, to the moves appropriate for solving the problems we care about. Indeed, as we have seen, while Frege’s system can solve Boole’s problems, it only does so in a much clumsier fashion—and, in fact, it appears that Boole’s ‘directive method’ would stop making much sense if translated into Frege’s *Begriffsschrift*, as it is closely bound up with Boole’s notation.

While we are not proposing to embark on a philosophical study of such notational differences here, we believe that Boole’s insight usefully points to the task that such a discussion should accomplish: explore how particular symbolic languages or notations can organize the space of possible expressions, so to speak, in such a way as to make solution strategies for certain kinds of problems accessible in practice.

9 Broader developments

Although we can offer no more than tentative hints here, we believe that the contrast we have drawn between Boole and Frege is useful to think about the broader history

of logic in the late 19th and early 20th centuries. First, the problem-solving approach pioneered by Boole shaped large parts of what is often called the ‘algebra of logic’ tradition, including in settings we would now call ‘first-order’, like Schröder’s work on relatives and Skolem’s results on quantifier elimination. Second, a shift intriguingly parallel to Frege’s—away from the algebraic, problem-solving approach to logic and toward a focus on analysis and inference—seems to have happened inside the ‘algebra of logic’ tradition too, in the work of Peirce.

As already hinted in the introduction, one false impression one might take away from the foregoing is that our contrast between Boole and Frege tracks that between decidable propositional logic and undecidable first-order logic. Later work in the Boolean tradition disproves this misapprehension. As a preliminary, it should be kept in mind that quantification *was* also introduced by authors working in a broadly Boolean framework, namely by Peirce and his students Christine Ladd-Franklin and O. H. Mitchell; that their ideas were brought together in Schröder’s calculus of relatives, which makes up the third volume of his expansive *Lectures on the Algebra of Logic*; ⁷³ and that this line of inquiry—and *not* Frege’s work, nor Russell’s—was the actual point of departure of crucial work on quantification by Löwenheim and Skolem. ⁷⁴ Now, what matters for our purposes is that Schröder built his calculus of relatives around what he called ‘solution-problems’ (‘*Auflösungsprobleme*’), essentially a relational analog to Boole’s logical problems. ⁷⁵

Interestingly, something akin to Frege’s criticism of Boole’s approach to logic is also found in Peirce: although Peirce’s starting-point was Boolean, he moved toward an understanding of logic as having to do with analysis rather than with the calculational solution of logical problems. ⁷⁶ Against Schröder’s aforementioned ‘solution-problems’, Peirce objected: ⁷⁷

Professor Schröder chiefly occupies himself with what he calls “solution-problems” [. . .]. While I am not at all disposed to deny that [such problems], consisting in the ascertainment of the general forms of relatives which satisfy given conditions, are often of considerable importance, I cannot admit that the interest of logical study centres in them. I hold that it is usually much more to the purpose to express in the simplest way what a given premise discloses in regard to the constitution of a relative [. . .]. ⁷⁸

In other words, what matters most is not to methodically solve well-defined problems, but to provide a logical analysis of premises and display this analysis in the most conspicuous way possible.

More broadly, Peirce later wrote in an encyclopedia entry on ‘Logic (exact)’:

⁷³ See Schröder (1895).

⁷⁴ See Brady (2000), in particular ch. 8 and 9.

⁷⁵ See Schröder (1895), in particular the fifth lecture; for a synopsis, see Brady (2000, pp. 152–153).

⁷⁶ On Peirce’s view of analysis as the main task of logic, see for instance Pietarinen (2016).

⁷⁷ Incidentally, Peirce also developed technical criticisms of Schröder’s approach of the topic, on which see Brady (2000, pp. 153–155).

⁷⁸ Peirce (1897, pp. 193–194).

There are those, not merely outside the ranks of exact logic, but even within it, who seem to suppose that the aim is to produce a calculus, or semi-mechanical method, for performing all reasoning, or all deductive inquiry; but there is no reason to suppose that such a project [...] can ever be realized. The real aim is to find an indisputable theory of reasoning by the aid of mathematics.⁷⁹

Right after this passage, he warns that the use of algebra in logic ‘is open to the danger of degenerating into idle trifling’—which may well refer to complex and artificial problems of the kind we discussed above, since he then defends his own logical graphs by saying that they ‘cannot, it is true, be applied to cases of great complexity; but for that very reason they are less liable to serve the purposes of the logical trifler.’ In Peirce’s views, if algebra is good for something in logic, it is not for its problem-solving advantages, but because it forces us ‘to reason explicitly and definitely, if at all’ so that ‘it may afford very considerable aid to analysis.’⁸⁰ A shift in emphasis comparable to Frege’s, albeit without the logicist ambitions and the focus on arithmetic, clearly seems at work here.

Conclusion

Histories of logic often present Boole and Frege as two steps—Frege usually the bigger one—on the road to formalization, and hence to mechanization. Such a retrospective view distorts the historical record considerably. While both authors do, in some sense, offer a ‘calculus’ for logic, the word takes on very different meanings in each: for Boole, the value of a calculus lies in its offering the proper kind of *guidance for problem-solving*; for Frege, designing a calculus merely means codifying elementary inferences symbolically, so that the *correctness* of proofs can be checked visually. This is why isolating elementary inferences and codifying them as rules was not central for Boole—what mattered to him was the overall architecture of his solution methods—while Frege, conversely, had no notion of logical problem-solving at all; insofar as his system lends itself (anachronistically speaking) to any kind of mechanization, it can only be that of the *checking* of proofs, and no longer of the methodical solution of problems, as Boole’s did (a shift in emphasis which, incidentally, dovetails with Frege’s explicit positioning of his work in opposition to ‘mechanical’ views of arithmetic, and mathematics more broadly). This contrast comes out clearly, as we have shown, in Frege’s attempt to solve one of Boole’s logical problems using the *Begriffsschrift*: in place of Boole’s sophisticated and goal-directed symbolic methods, what Frege offers is essentially a systematic proof-search—a method only in the thinnest of senses.

A view of the role of symbolic notations closer to Frege had an outsize influence on logic and analytic philosophy. In effect, his focus on the strict codification of elementary inferences (as well as, on the *lingua* side, on the analysis of concepts) drew attention away from the kind of reflections developed by Boole earlier on the value of symbolic languages for methodical problem-solving. We believe, however, that Boole’s outlook—which establishes an intrinsic link between a ‘calculus’ and the

⁷⁹ Peirce (1902, p. 24).

⁸⁰ Ibid.

systematic problem-solving methods it lends itself to—deserves to be revived, and can pave the way for a better understanding of the role of notations, and of symbolic languages in general, in mathematics and logic.

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