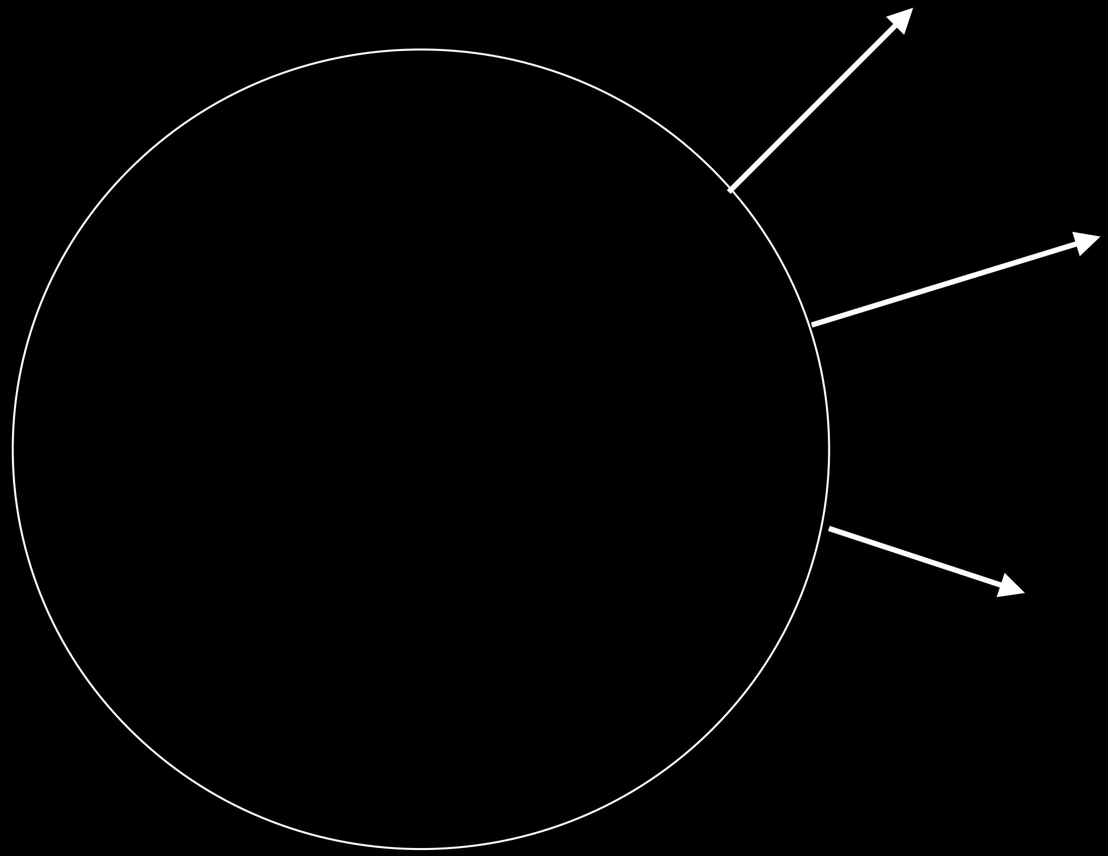
Physics and reality I: Origins of classical mechanics

Robert DiSalle
"The University of Western Ontario"

Measure of curvature: Extrinsic curvature is measured by the relation of the space to the ambient space. E.g. perpendiculars to the surface of a sphere or cylinder are not parallel.

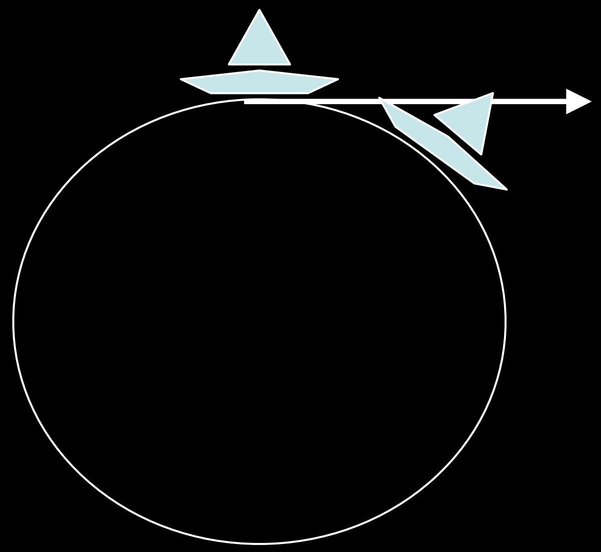


But if you roll a Euclidean surface into a cylinder, Euclidean figures remain Euclidean. The cylinder has no *intrinsic* curvature.

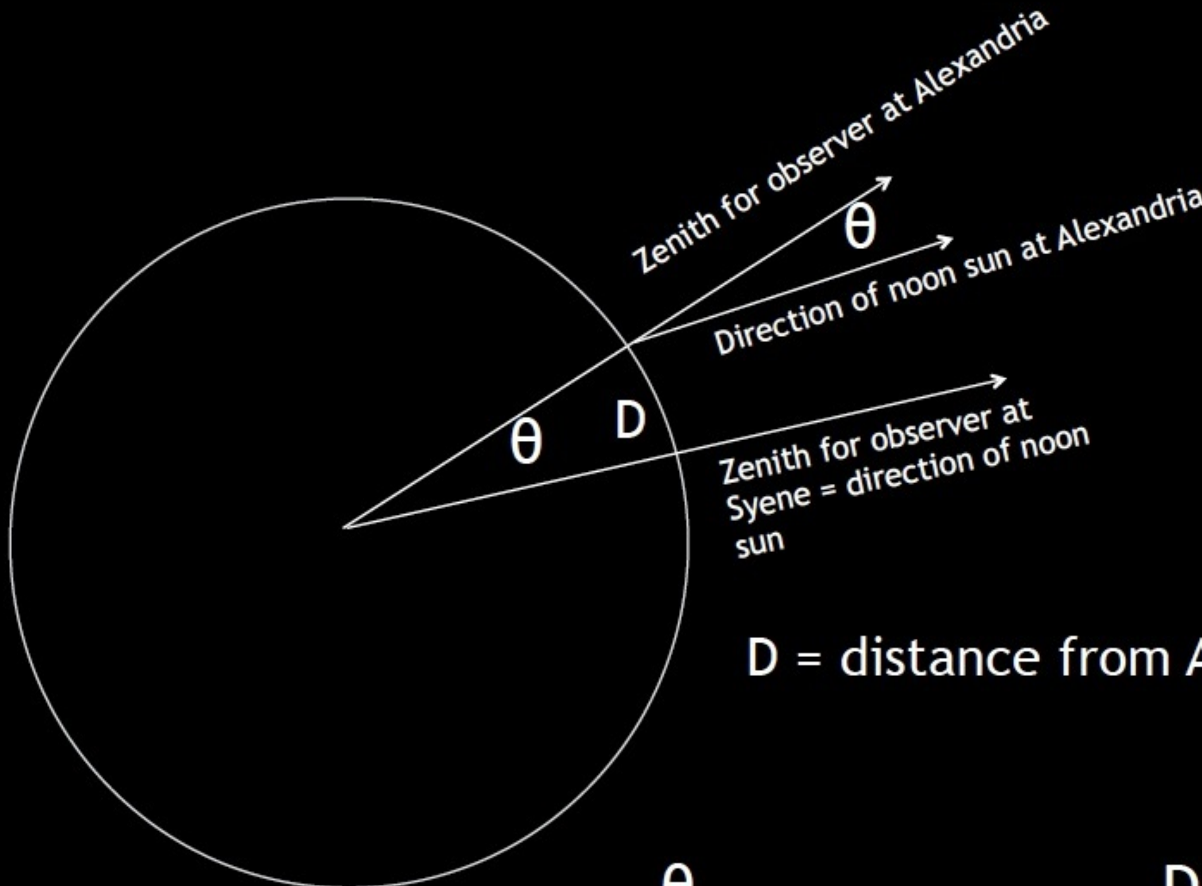
We understand and visualize the curvature of these 2-D surfaces, because we can picture them in relation to the straight lines in the ambient 3-D space.

The curvature of the earth was understood and measured more than 2000 years ago, by comparison of its surface to straight lines indicated by light rays— lines of sight— in the ambient space.

Example: A line of sight goes straight above the horizon, while the surface of the earth curves away. Hence a departing ship dips below the horizon as it gets further from the shore. Thus the ancient Greeks knew that the earth is not flat.



How Eratosthenes measured the earth (240 B.C.E.)



D = distance from Alexandria to Syene

$$\frac{\theta}{360} = \frac{D}{\text{circumference of the earth}}$$

(252,000 stadia, roughly 40,000 km)

Kepler’s model: the inner planets

The context for Newton:

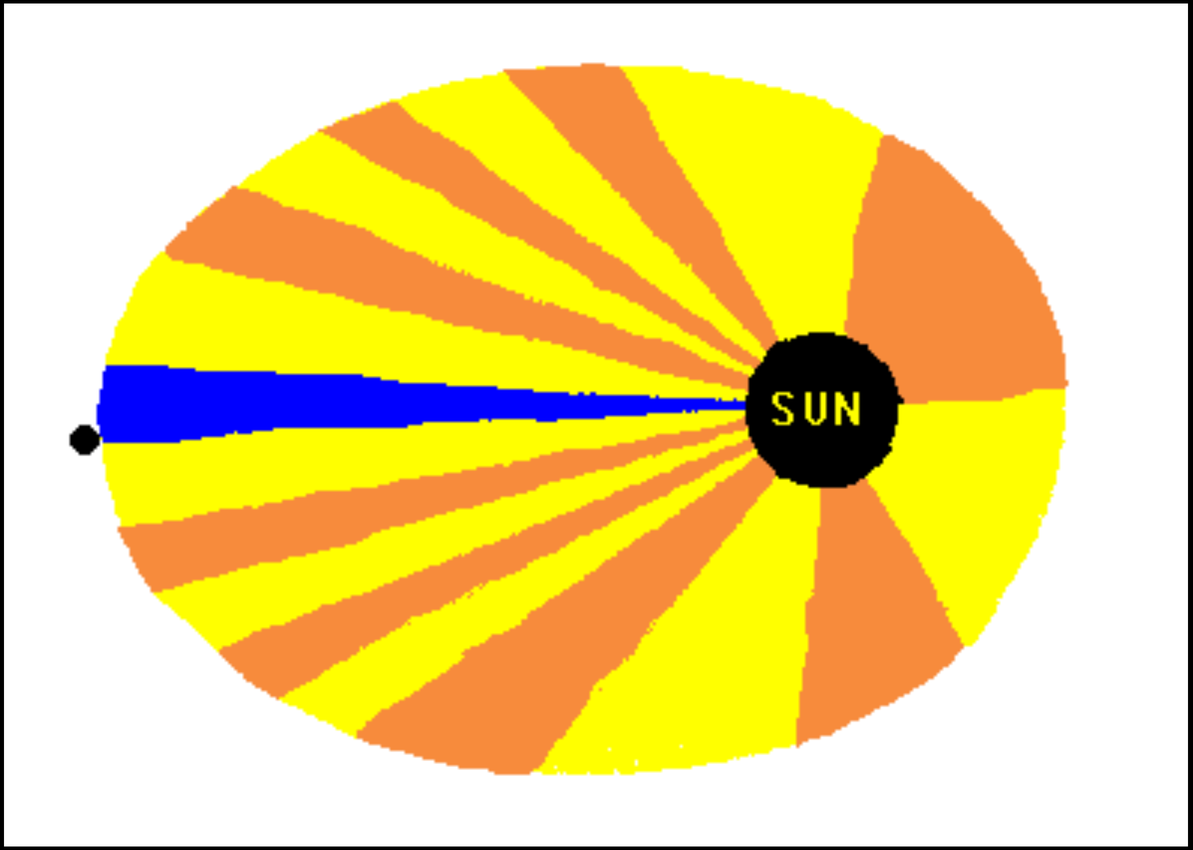
Galileo: An extension of the notion of evidence regarding the nature of physical reality
A mechanical-causal picture in the background
A mathematical method of solving dynamical problems

Kepler: An extension of the notion of accurate evidence regarding the nature of physical reality
A causal story of physical interactions

Descartes: A geometrical picture of the physical world
Laws of nature represented geometrically
A mechanical-causal story of physical interactions

Huygens: A mechanical-causal program for physical explanation
A mathematical method of solving dynamical problems

Kepler’s area law: The radius drawn from the sun to a planet sweeps out equal areas in equal times.



Kepler's "harmonic law": The periodic time ***t*** and the mean radius ***r*** of any planetary orbit are related as $t^2 \propto r^3$.

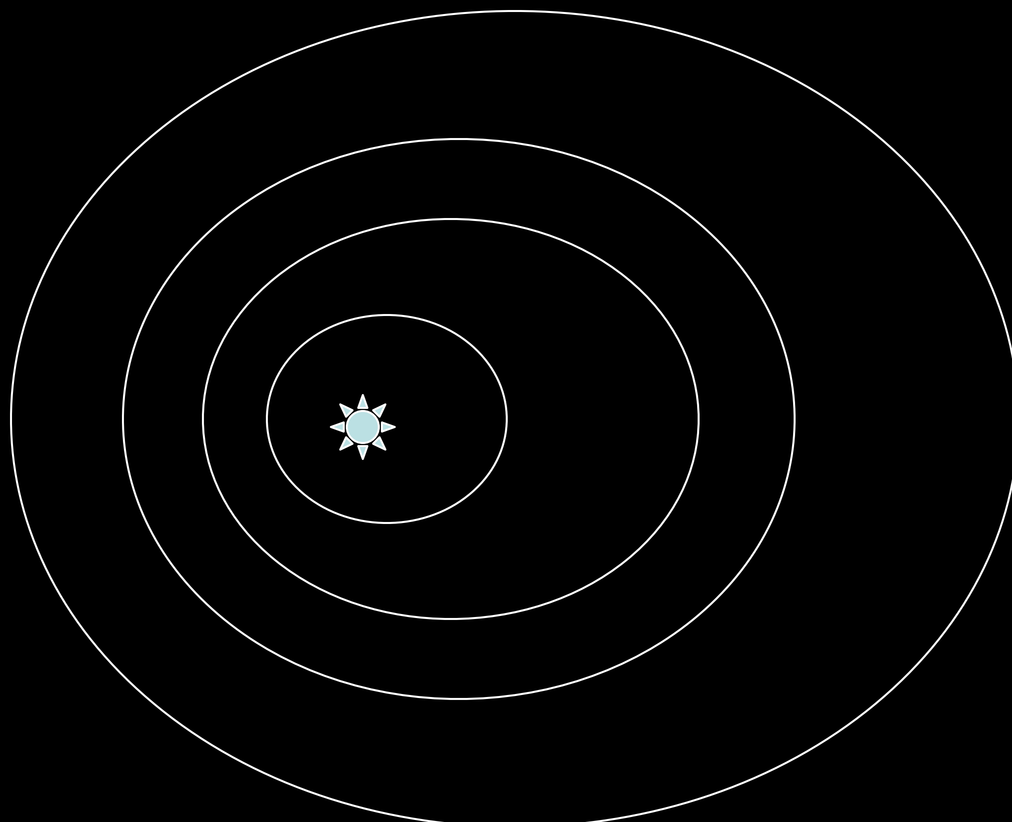
Or,

$$t \propto r^{3/2}$$

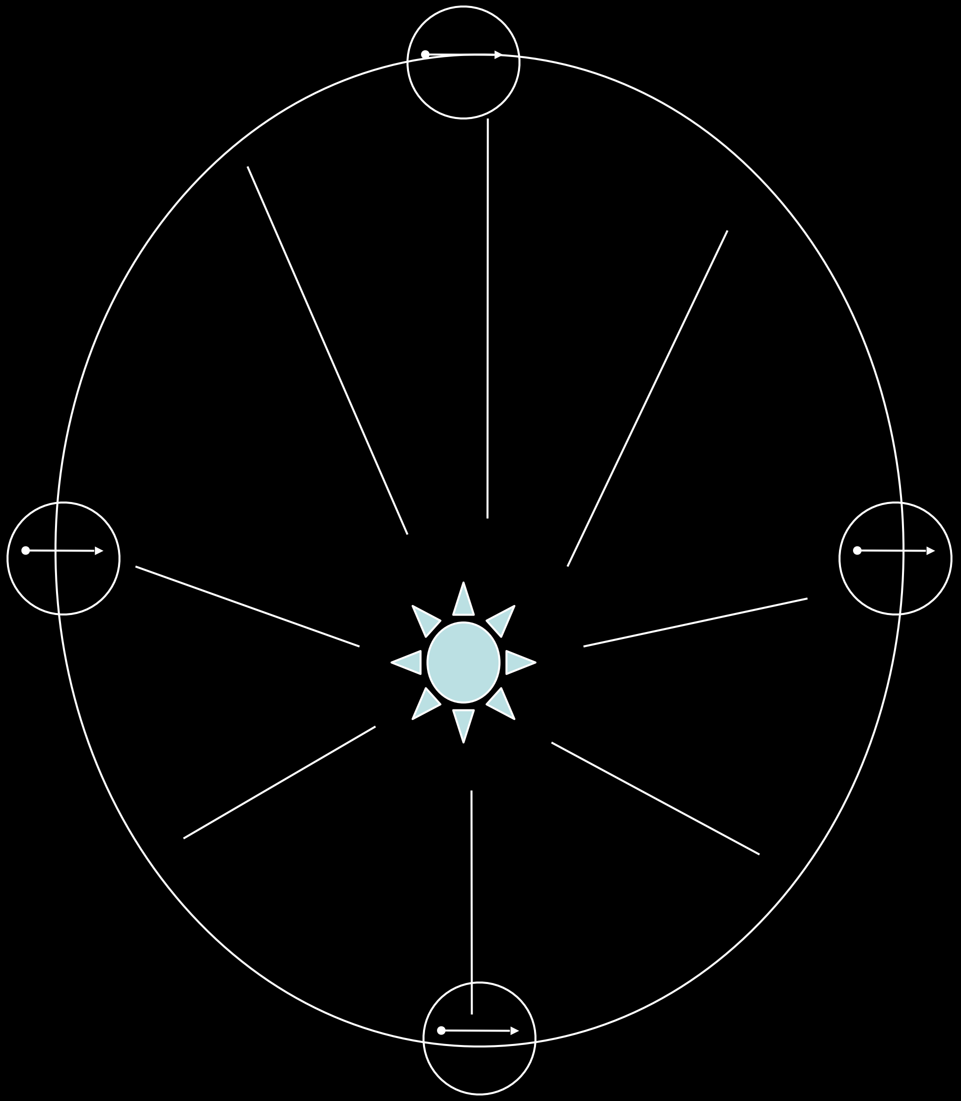
Or, for any two planets ***a*** and ***b***,

$$T_a^2 / T_b^2 = R_a^3 / R_b^3$$

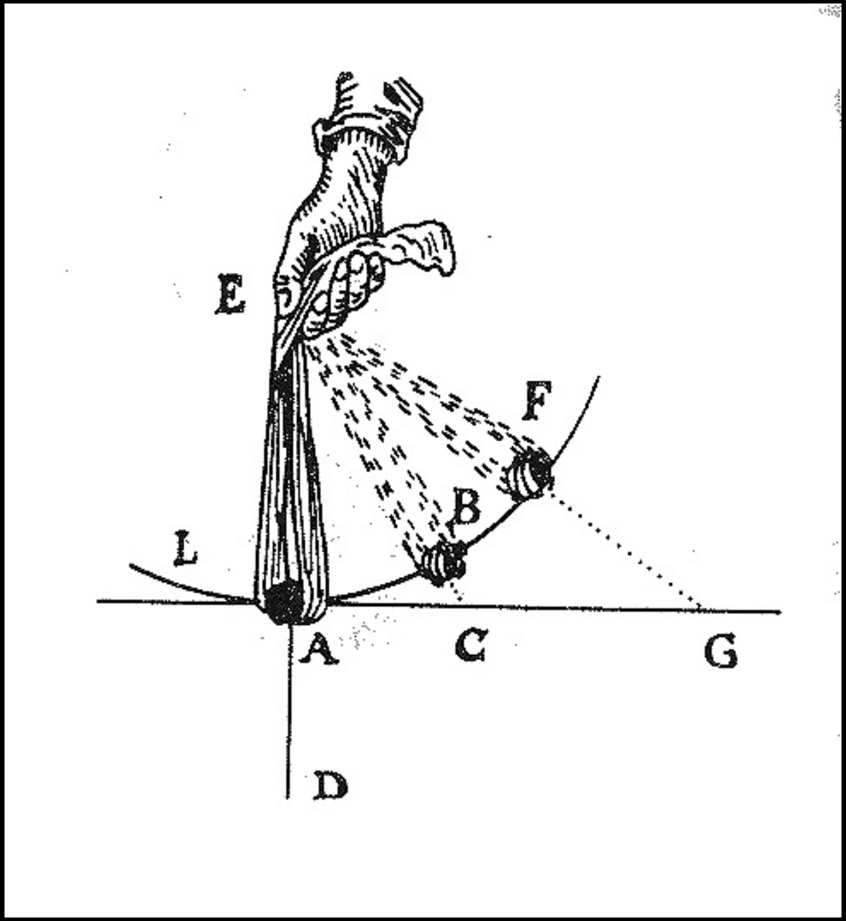
Kepler’s ellipse law: Planets orbit the sun in ellipses with the sun at their common focus.



Kepler’s physical astronomy

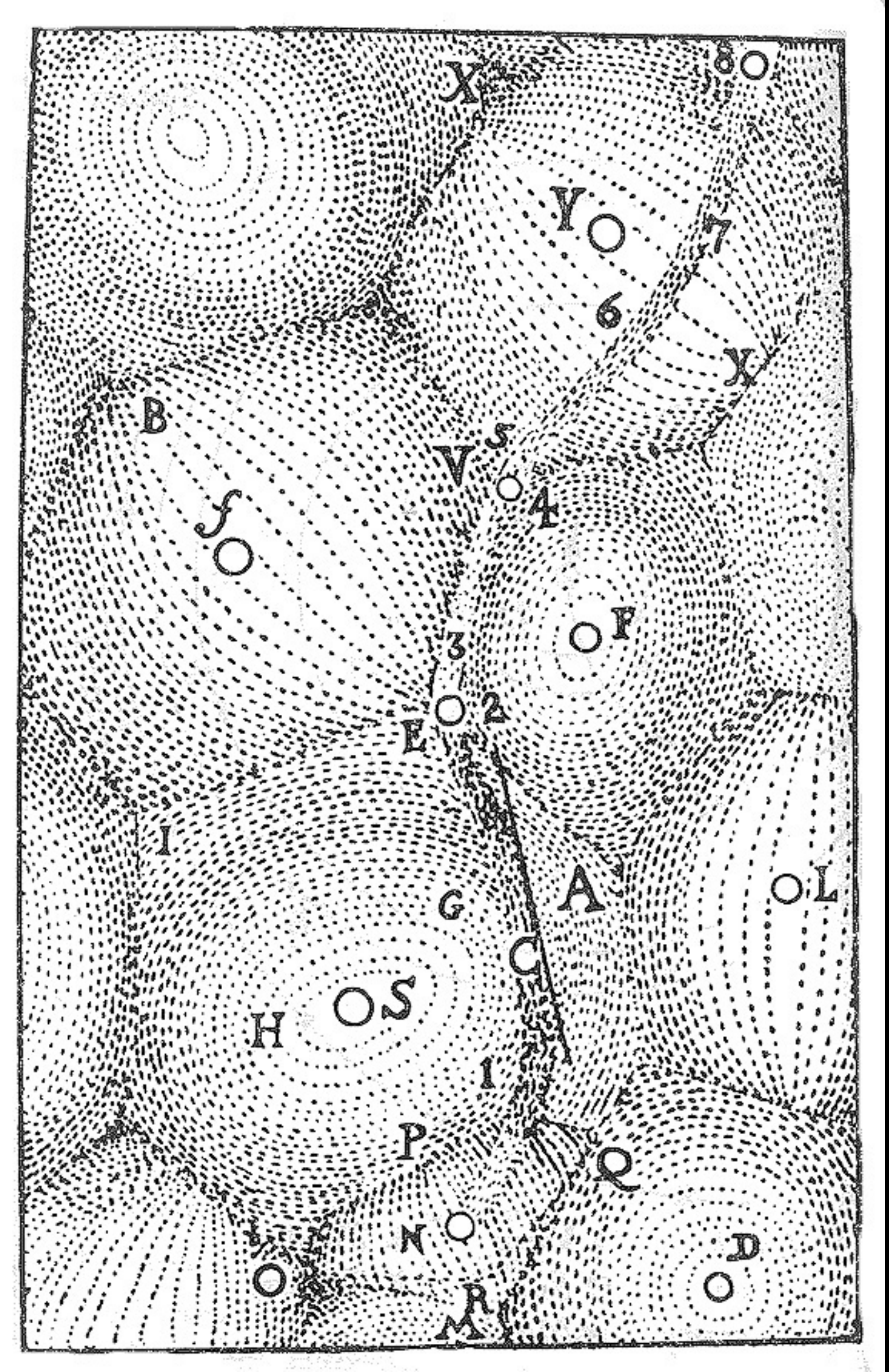


Descartes on the origin of centrifugal forces:



Descartes' vortex theory of planetary motion:

The universe is completely filled with vortices, each surrounding a rotating star



Descartes on the cause of gravity:

"But now I want you to consider what the weight of this Earth is, that is, what the force is that unites all its parts and makes them all tend toward the centre, each more or less according to the extent of its size and solidity. This force is nothing but, and consists in nothing but, the parts of the small heaven which surround it turning much faster than its own parts about its centre, and tending to move away with greater force from its centre, and as a result pushing the parts of the Earth back toward its centre." (*Le Monde*, Chapter II.)

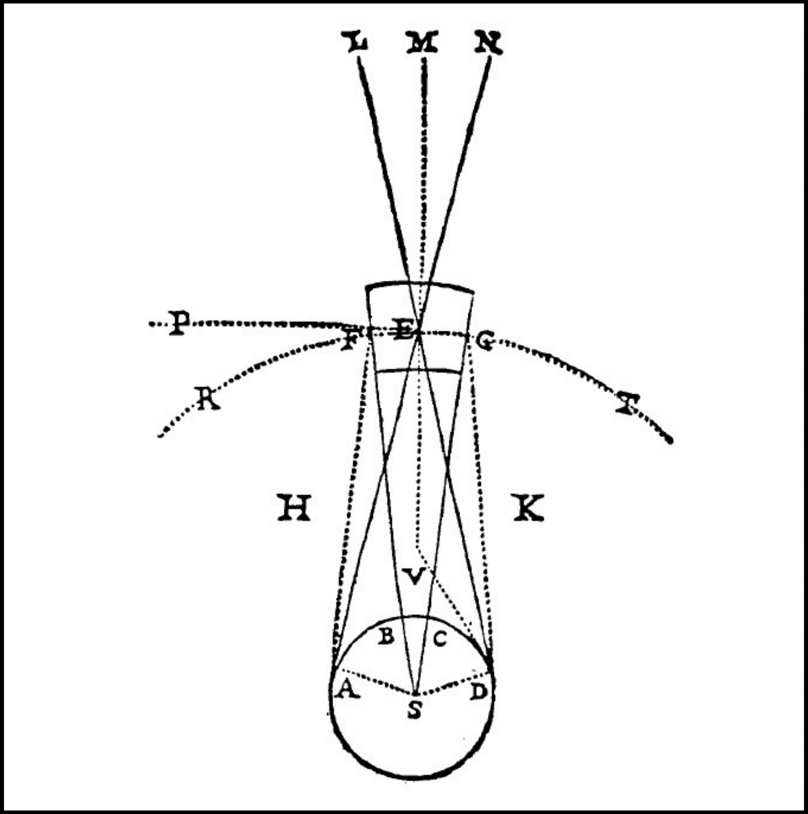
Huygens on explaining gravity:

In order to find an intelligible cause of gravity, we must see how gravity can come about while presupposing in nature only bodies that are made from a like matter, and considering in these neither any quality nor any tendency to draw near one another, but only their different magnitudes, figures, and motion. How might it still come about that several of these bodies tend directly toward a common centre, and are held together around it? This is the most extraordinary and most important phenomenon of what we call gravity.

(*Discourse on the cause of gravity*)

Descartes' theory of light as a pressure propagated through the celestial medium:

"And so, if it were the eye of a man that was at point E, it would really be pushed, both by the Sun and by all the celestial matter between the lines AF and DG. Now one must know that the men of this new world will be of such a nature that, when their eyes are pushed in this fashion, they have a sensation very similar to that which we have of light, as I shall explain more fully below." (Descartes, *Le Monde*, Chapter 13.)



Huygens on the methodology of the *Treatise on Light*:

There will be seen in it demonstrations of those kinds which do not produce as great a certitude as those of Geometry, and which even differ much therefrom, since whereas the Geometers prove their Propositions by fixed and incontestable Principles, here the Principles are verified by the conclusions to be drawn from them; the nature of these things not allowing of this being done otherwise.

It is inconceivable to doubt that light consists in the motion of some sort of matter. For whether one considers its production, one sees that here upon the Earth it is chiefly engendered by fire and flame which contain without doubt bodies that are in rapid motion, since they dissolve and melt many other bodies, even the most solid; or whether one considers its effects, one sees that when light is collected, as by concave mirrors, it has the property of burning as a fire does, that is to say it disunites the particles of bodies. This is assuredly the mark of motion, at least in the true Philosophy, in which one conceives the causes of all natural effects in terms of mechanical motions. This, in my opinion, we must necessarily do, or else renounce all hopes of ever comprehending anything in Physics. (*Treatise on Light*, chapter I.)

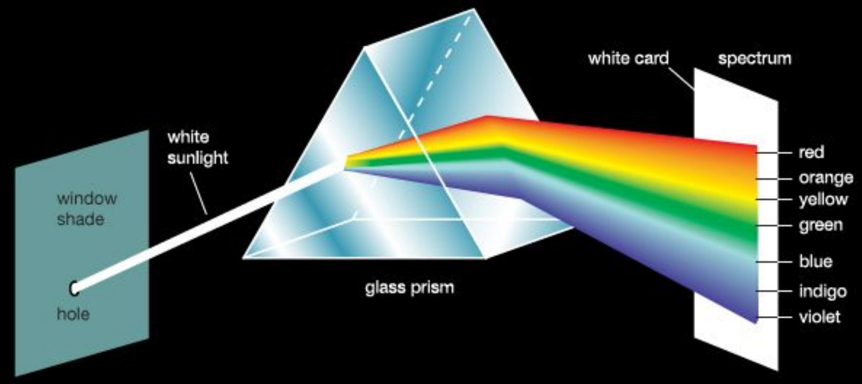
It is always possible to attain thereby to a degree of probability which very often is scarcely less than complete proof. To wit, when things which have been demonstrated by the Principles that have been assumed correspond perfectly to the phenomena which experiment has brought under observation; especially when there are a great number of them, and further, principally, when one can imagine and foresee new phenomena which ought to follow from the hypotheses which one employs, and when one finds that therein the fact corresponds to our prevision. (Huygens, from the Preface to *Treatise on Light)*

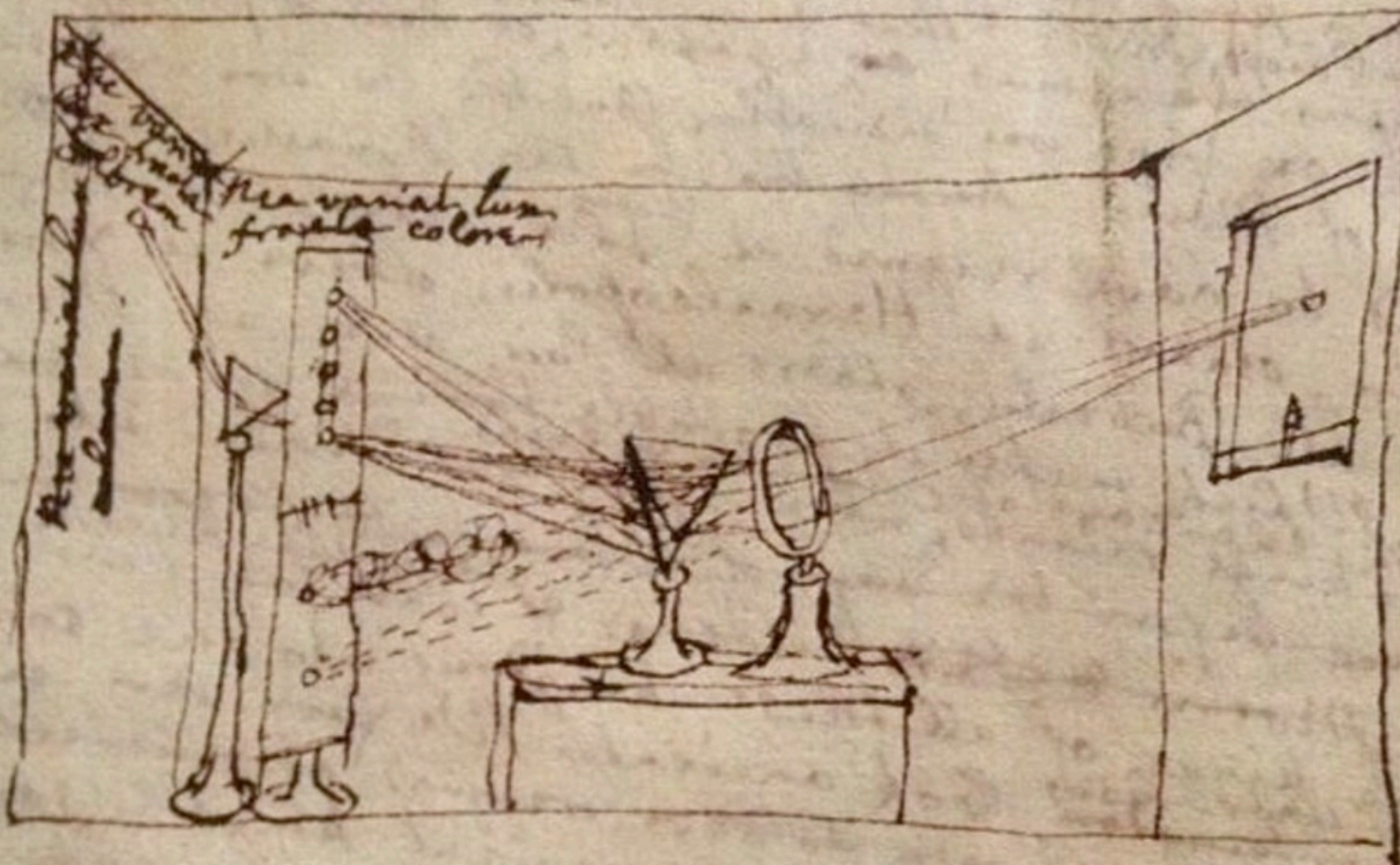
The Newtonian method (from *Opticks*):

As in Mathematicks, so in Natural Philosophy, the Investigation of difficult Things by the Method of Analysis, ought ever to precede the Method of Composition. This Analysis consists in making Experiments and Observations, and in drawing general Conclusions from them by Induction, and admitting of no Objections against the Conclusions, but such as are taken from Experiments, or other certain Truths. For Hypotheses are not to be regarded in experimental Philosophy….

Newton’s new theory of light and colours:

I am purposing [to the Royal Society], to be considered of & examined, an account of a Philosophical discovery which induced me to the making of the said Telescope, & which I doubt not but will prove much more grateful than the communication of that instrument, being in my Judgment the oddest if not the most considerable detection which hath hitherto been made in the operations of Nature.” (Letter to Oldenburg, 1672)




Nec variat lux fracta colorem

Huygens on a fundamental disagreement with Newton's method and metaphysics:

Me thinks, that the most important Objection, which is made against him by way of Quære, is that, Whether there be more than two sorts of Colours. For my part, I believe, that an Hypothesis, that should explain mechanically and by the nature of motion the Colours Yellow and Blew, would be sufficient for all the rest, in regard that those others, being only more deeply charged (as appears by the Prismes of Mr. Hook,) do produce the dark or deep-Red and Blew; and that of these four all the other colors may be compounded. Neither do I see, why Mr. Newton doth not content himself with the two Colors, Yellow and Blew; for it will be much more easy to find an Hypothesis by Motion, that may explicate these two differences, than for so many diversities as there are of other Colors. And till he hath found this Hypothesis, he hath not taught us, what it is wherein consists the nature and difference of Colours, but only this accident (which certainly is very considerable,) of their different Refrangibility.

Newton’s reply to Huygens on mechanical explanations of the phenomena of colours:

Nor is it easier to frame an Hypothesis by assuming only two Original colors rather than an indefinit variety; unless it be easier to suppose, that there are but two figures, sizes and degrees of velocity or force of the Æthereal corpuscles or pulses, rather than indefinit variety; which certainly would be a harsh supposition. No man wonders at the indefinit variety of Waves of the Sea, or of sands on the shore; <6109> but, were they all but two sizes, it would be a very puzling phænomenon. And I should think it as unaccountable, if the several parts or corpuscles, of which a shining body consists, which must be suppos'd of various figures, sizes and motions, should impress but two sorts of motion on the adjacent Æthereal medium, or any other way beget but two sorts of Rays.

Newton on his “new theory of light and colours” (1672):

A naturalist would scearce expect to see ye science of [colours] become mathematicall, & yet I dare affirm that there is as much certainty in it as in any other part of Opticks.

Hooke’s reply:

I doe not therefore see any absolute necessity to believe his Theory demonstrated, since I can assure Mr. Newton, that I cannot only salve all the Phaenomena of Light and colours, by the Hypothesis, that I have formerly printed and now explicate yt by, but by two or three others….Nor would I be understood to have said all this against his theory as it is an hypothesis, for I doe most Readily agree with him in every part thereof, and esteem it very subtill and ingenious, and capable of salving all the phænomena of coulours; but I cannot think it to be the only hypothesis; not soe certain as mathematicall Demonstrations.

Newton's reply to Hooke's objection, that his theory depends on "the hypothesis that light be a body":

'Tis true, that from my Theory I argue the Corporeity of Light; but I do it without any absolute positiveness… and make it at most but a very plausible consequence of the Doctrine, and not a fundamental Supposition, nor so much as any part of it; which was wholly comprehended in the precedent Propositions….Had I intended any such Hypothesis, I should somewhere have explain'd it. But I knew, that the Properties, which I declar'd of Light, were in some measure capable of being explicated not only by that, but by many other Mechanical Hypotheses. And therefore I chose to decline them all, and to speak of Light in general terms, considering it abstractly, as something or other propagated every way in streight lines from luminous bodies, without determining, what that Thing is; whether a confused Mixture of difform qualities, or Modes of bodies, or of Bodies themselves, or of any Virtues, Powers, or Beings whatsoever.

Newton to Hooke, on a possible wave theory of light (1672):

I told you, that the Objectors Hypothesis, as to the fundamental part of it, is not against me. That fundamental Supposition is; That the parts of bodies, when briskly agitated, do excite Vibrations in the Æther, which are propagated every way from those bodies in streight lines.…Now, the most free and natural Application of this Hypothesis to the Solution of phænomena I take to be this: That the agitated parts of bodies, according to their several sizes, figures, and motions, do excite Vibrations in the æther of various depths or bignesses, which being promiscuously propagated through that Medium to our Eyes, effect in us a Sensation of Light of a White colour; but if by any means those of unequal bignesses be separated from one another, the largest beget a Sensation of a Red colour, the least or shortest, of a deep Violet, and the intermediat ones, of intermediat colors.…

Newton to Hooke on the mathematical character of his theory:

I said indeed that the Science of Colours was Mathematical & as certain as any other part of Optiques; but who does not know that Optiques and many other mathematical sciences depend as well on Physicall Principles as on Mathematicall Demonstrations: And the absolute certainty of a Science cannot exceed the absolute certainty of its Principles. Now the evidence by wch I asserted the propositions of Colours is in the next words expressed to be from Experiments & so but Physicall: Whence the Propositions themselves can be esteemed no more then Physicall Principles of a Science. And if those Principles be such that on them a Mathematician may determin all the Phænomena of colours that can be caused by refractions . . . I suppose the Science of Colours will be granted Mathematicall & as certain as any part of Optiques.

Both the genesis of the subject-matter of geometry, therefore, and the fabrication of its postulates pertain to mechanics. Any plane figures executed by God, nature or any technician you will are measured by geometry on the hypothesis that they are exactly constructed.... Geometry makes the unique demand that [its objects] be described exactly. It has now, however, come to be usual to regard as geometrical everything which is exact, and as mechanical all that proves not to be of the kind, as though nothing could possibly be mechanical and at the same time exact. But this common belief is a stupid one, and has its origin in nothing else than that geometry postulates an exact mechanical practice in the description of a straight line and circle, and moreover is exact in all its operations, while mechanics as it is commonly exercised is imperfect and without exact laws. It is from the ignorance and imperfection of mechanicians that the common opinion defines mechanics. On this reasoning a thing would be the more mechanical the more imperfect it was. Posit a mechanical thing to be perfect and you will correct the error. (Newton, *Geometry*)

Huygens on the programme of physical explanation:

In order to find an intelligible cause of gravity, we must see how gravity can come about while presupposing in nature only bodies that are made from a like matter, and considering in these neither any quality nor any tendency to draw near one another, but only their different magnitudes, figures, and motion. How might it still come about that several of these bodies tend directly toward a common centre, and are held together around it? This is the most extraordinary and most important phenomenon of what we call gravity.

(*Discourse on the cause of gravity*)

Newton on the programme of physical explanation:

I wish we could derive the rest of the phænomena of nature by the same kind of reasoning from mechanical principles; for I am induced by many reasons to suspect that they may all depend upon certain forces by which the particles of bodies, by some causes hitherto unknown, are either mutually impelled towards each other, and cohere in regular figures, or are repelled and recede from each other; which forces being unknown, philosophers have hitherto attempted the search of nature in vain; but I hope the principles here laid down will afford some light either to this or some truer method of philosophy.

Huygens on universal gravitation:

I do not agree with a Principle that he assumes in this calculation and elsewhere; namely, that all the little parts that one can imagine in two or more different bodies attract or tend to approach one another. This I cannot admit, because I believe I see clearly that the cause of such an attraction is in no way explicable by any principle of Mechanics or rules of motion; as I am moreover not persuaded of the necessity of the mutual attraction of entire bodies, having shown that, even if there were no earth at all, bodies would still, by what one calls their weight, tend towards a center. (*Discours de la cause de la pesanteur*)

Newton's reply to Huygens' objection to the argument for universal gravitation, and the speculative extension of Law III to gravitational attractions:

What that great man Huygens has remarked on my work is acute… But… since all the phenomena of the heavens and of the sea follow accurately, so far as I am aware, from gravity alone acting in accordance with the laws discovered by me, and nature is most simple; I myself have judged that all other causes are to be rejected and that the heavens are to be stripped as far as may be of all matter lest the motions of the planets and comets be impeded or rendered irregular. But if meanwhile someone explains gravity together with all its laws by the action of some subtle matter, and shows that the motions of the planets and comets will not be disturbed by this matter, I shall be far from objecting. (From a letter to Leibniz)

Precisely what did Newton mean by “the laws discovered by me,” and how is this connected with understanding gravity as a causal principle, or “power”?

To answer these questions we need to understand the interconnections among:

Newton’s understanding of physical causes

The method of Newton’s mathematical physics

Newton’s understanding of the problem of true motion.

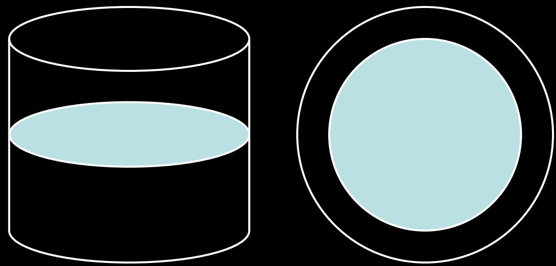
Descartes: What motion is, taking the term in its common use.

But motion…in the ordinary sense of the term, is nothing more than the action by which a body passes from one place to another. And just as we have remarked above that the same thing may be said to change and not to change place at the same time, so also we may say that the same thing is at the same time moved and not moved. Thus, for example, a person seated in a vessel which is setting sail, thinks he is in motion if he look to the shore that he has left, and consider it as fixed; but not if he regard the ship itself, among the parts of which he preserves always the same situation. Moreover, because we are accustomed to suppose that there is no motion without action, and that in rest there is the cessation of action, the person thus seated is more properly said to be at rest than in motion, seeing he is not conscious of being in action.

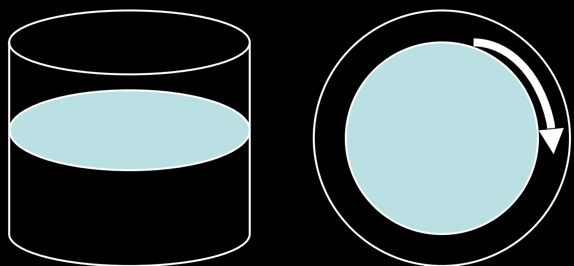
Descartes: What motion is properly so called (motion "in the philosophical sense")

But if, instead of occupying ourselves with that which has no foundation, unless in ordinary usage, we desire to know what ought to be understood by motion according to the truth of the thing, we may say, in order to give it a determinate nature, that it is THE TRANSPORTING OF ONE PART OF MATTER OR OF ONE BODY FROM THE VICINITY OF THOSE BODIES THAT ARE IN IMMEDIATE CONTACT WITH IT, OR WHICH WE REGARD AS AT REST, to the vicinity of other bodies. By a body as a part of matter, I understand all that which is transferred together, although it be perhaps composed of several parts, which in themselves have other motions….

Newton's bucket experiment: The Cartesian definition of motion vs. the dynamical measure of motion

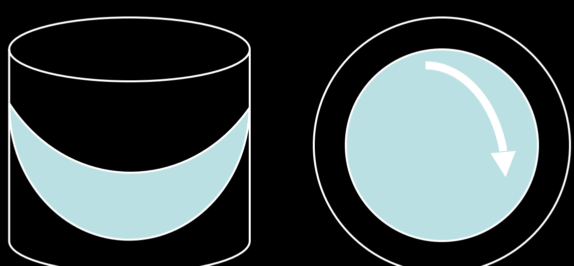


The bucket and water are at rest: No motion in Descartes' sense, and no dynamical effect



The bucket spins: Now the water moves in Descartes' sense, but no dynamical effect

The water spins along with the bucket: No motion in Descartes' sense, but an evident dynamical effect



The bucket stops and the water continues: The water moves in Descartes' sense, with the same dynamical effect

Newton's conclusion from the bucket experiment:

And therefore this endeavour does not depend upon any translation of the water in respect of the ambient bodies, nor can true circular motion be defined by such translation. There is only one real circular motion of any one revolving body, corresponding to only one power of endeavouring to recede from its axis of motion, as its proper and adequate effect ; but relative motions, in one and the same body, are innumerable, according to the various relations it bears to external bodies, and like other relations, arc altogether destitute of any real effect, any otherwise than they may perhaps partake of that one only true motion. And therefore in their system who suppose that our heavens, revolving below the sphere of the fixed stars, carry the planets along with them ; the several parts of those heavens, and the planets, which are indeed relatively at rest in their heavens, do yet really move. For they change their position one to another (which never happens to bodies truly at rest), and being carried together with their heavens, partake of their motions, and as parts of revolving wholes, endeavour to recede from the axis of their motions.

Newton:

It is indeed a matter of great difficulty to discover, and effectually to distinguish, the true motion of particular bodies from the apparent; because the parts of that immovable space, in which those motions are performed, do by no means come under the observation of our senses. Yet the thing is not altogether desperate; for we have some arguments to guide us, partly from the apparent motions, which are the differences of the true motions; partly from the forces, which are the causes and effects of the true motion. For instance, if two globes, kept at a given distance one from the other by means of a cord that connects them, were revolved about their common center of gravity, we might, from the tension of the cord, discover the endeavor of the globes to recede from the axis of their motion, and from thence we might compute the quantity of their circular motions.

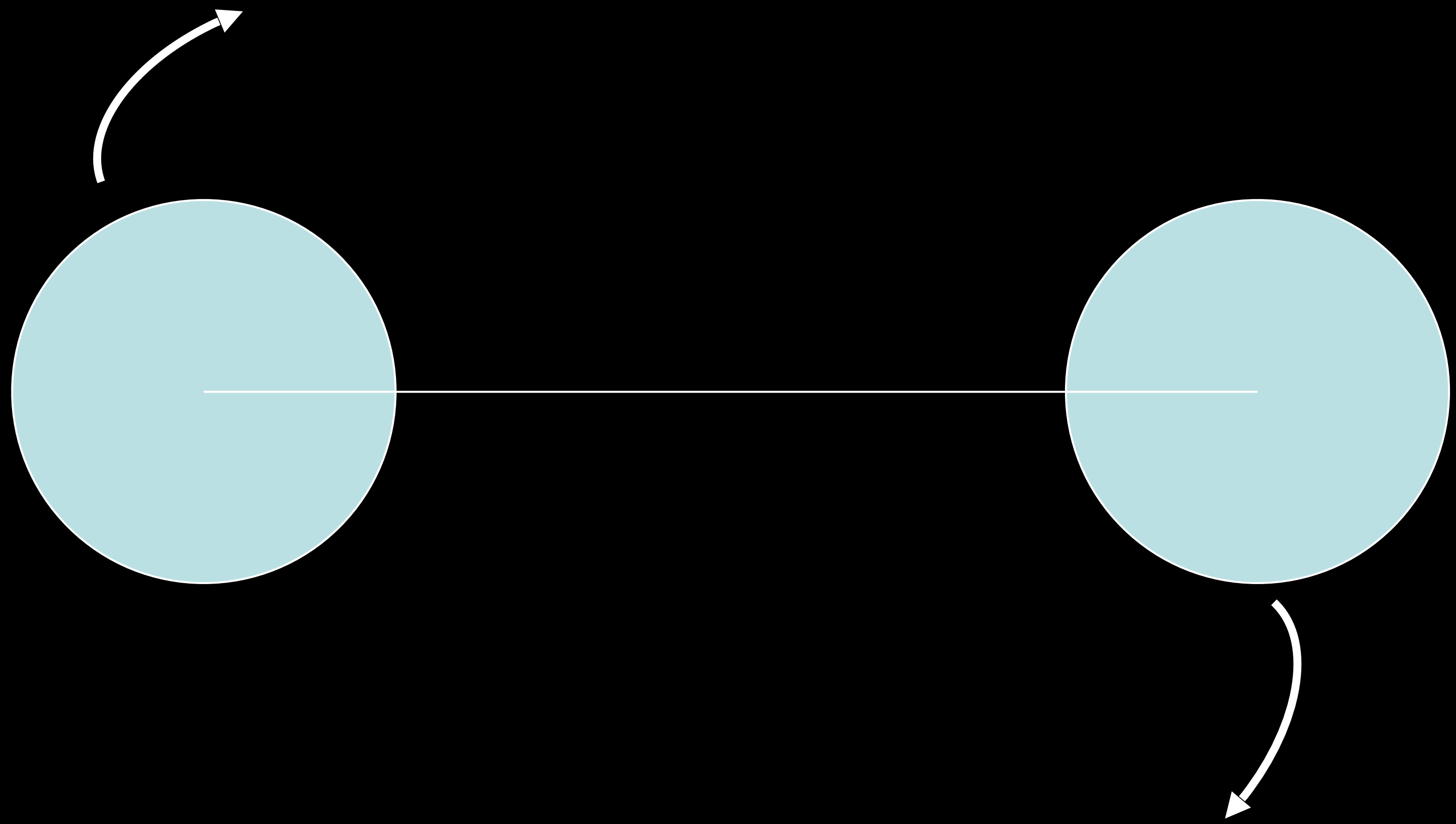
Newton:

“But how to collect the true motions from their causes, effects, and apparent differences, and conversely from the motions whether true or apparent from their causes and effects, shall be explained more fully in the sequel. For to this end I have composed the following treatise.”

Stein, 1967:

“The notion that the *Principia* was composed in order to explain how to determine absolute motion has been rejected by some with shocked and fervent rhetoric. But it seems to me that one does well, in a case like this, to read the book and see whether it does what the author says he intends it to do.”

Newton's thought-experiment on rotation: Even if there is nothing else in the universe-- therefore no relative motion-- the rotation of these spheres about their common centre of gravity can be known from the tension on the cord joining them.



Huygens's response to Newton (unpublished):

But the parts of a body can be moved with reference to one another (which is called whirling motion), preserving their distance on account of a bond or obstacle; a bond, in the case of a top or the composite of two bodies, connected by a cord; an obstacle, in the case of water swirled round in a vessel.

Now in the circulation of 2 bodies bound by the thread AB one knows that they have received impulsion which has produced their mutual relative motion or direction; but one cannot know, by considering them alone, whether they were pushed equally, or whether only one was pushed. For if A alone had been pushed, the circular motion and the tension of the thread would have followed all the same, although the circle would then have a progressive motion with respect to the other bodies at rest.

That I have therefore shown how in circular motion just as well as in free and straight motion there is nothing but what is relative – in such a way that that is all there is to know about motion, and also all that one has any need to know….

George Smith's conjecture:

Huygens discussed this objection with Newton in person, during their encounter.

Huygens to Leibniz (1694):

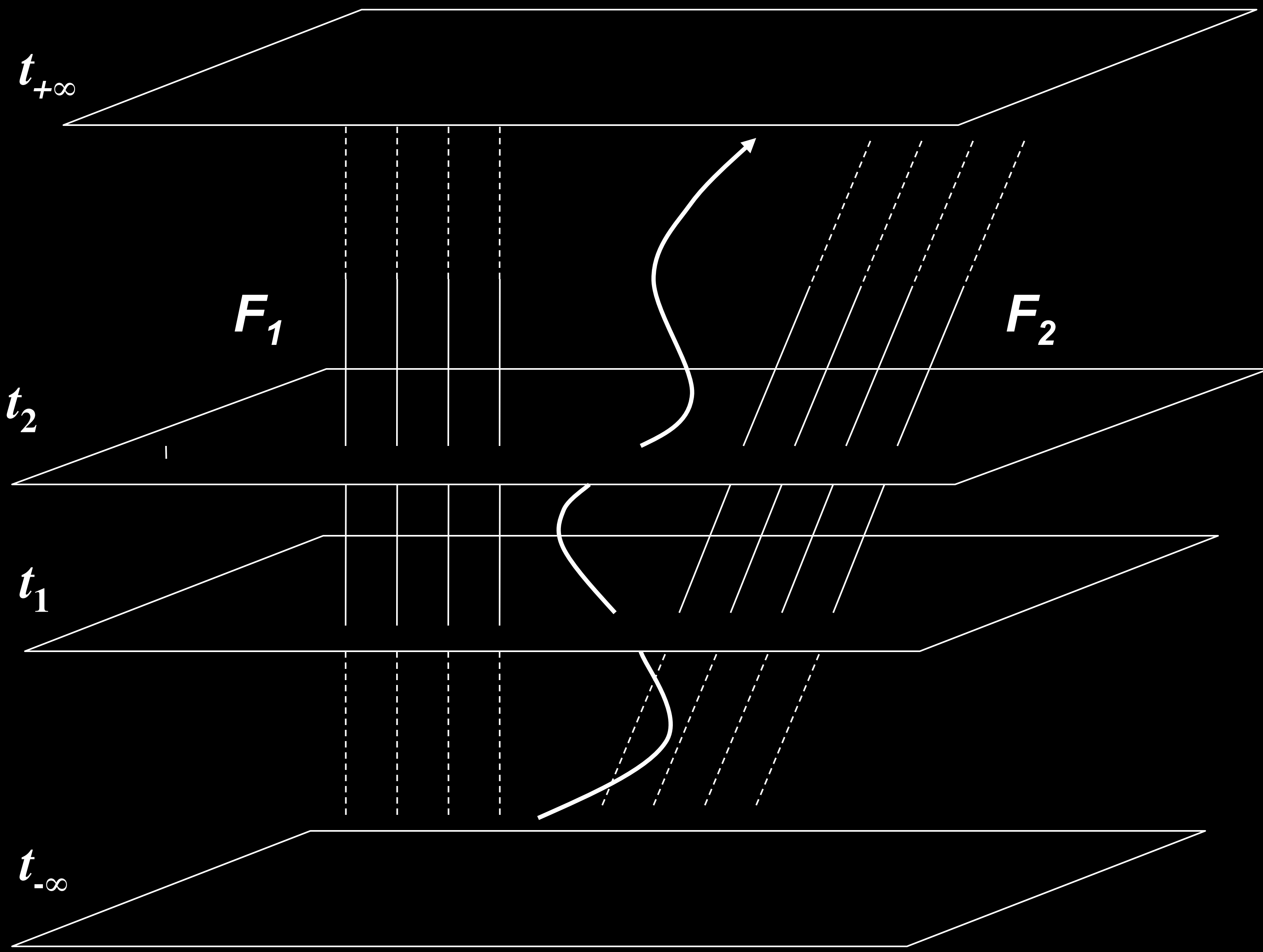
I have noticed in your notes on des Cartes that you believe it to be discordant that no real motion is given, but only relative. Yet I hold this to be very sure, and am not checked by the argument and experiments of Mr. Newton in his Principles of Philosophy, which I know to be in error; and I am eager to see whether he will not make a retraction in the new edition of this book, which David Gregorius is to procure. Des Cartes did not sufficiently understand this matter.

Why did Newton need (or think he needed) absolute space for his theory of motion?

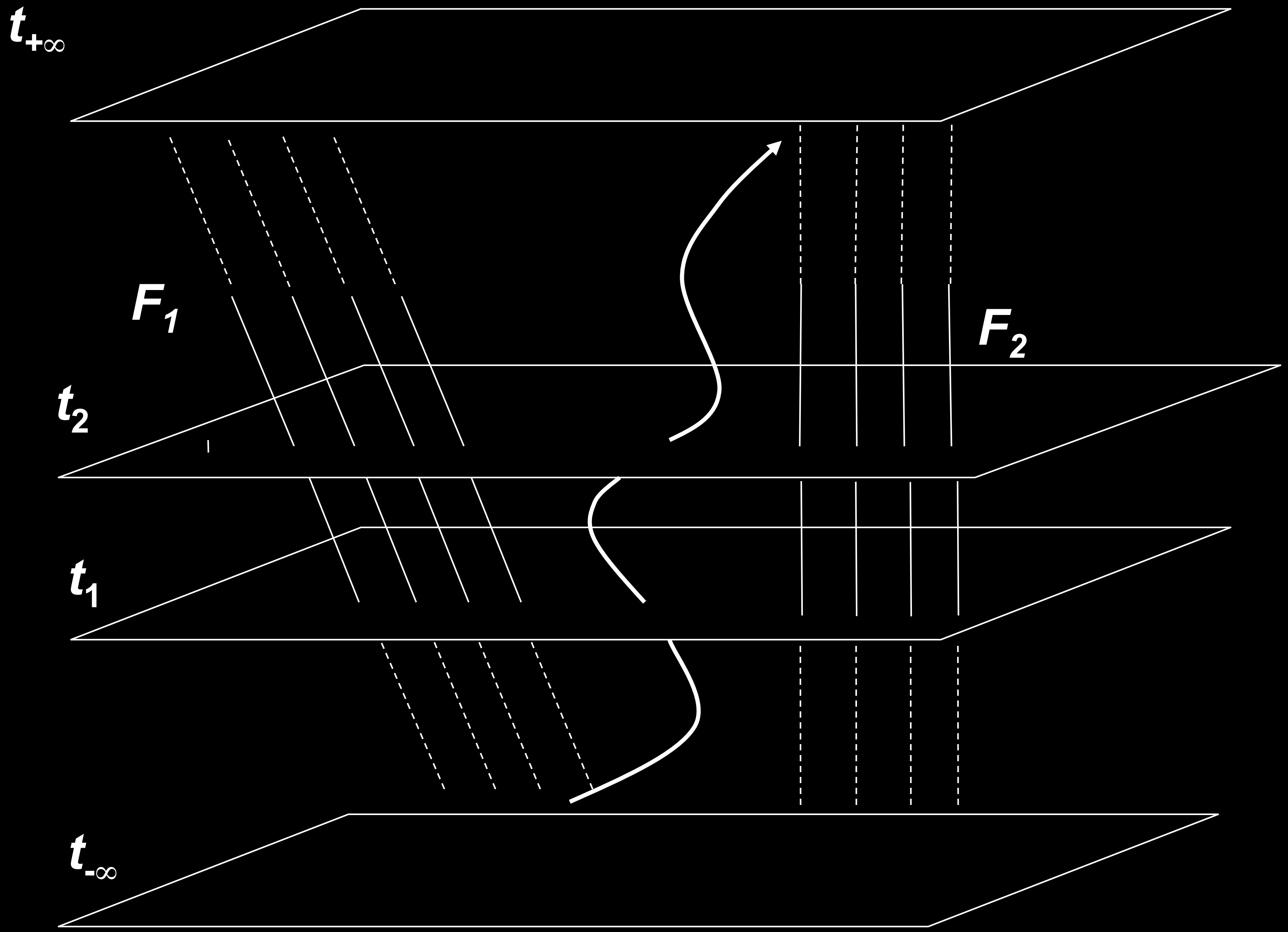
...[A]ll motions, from places in motion, are no other than parts of entire and absolute motions; and every entire motion is composed of the motion of the body out of its first place, and the motion of this place out of its place; and so on, until we come to some immovable place....Wherefore, entire and absolute motions can be no otherwise determined than by immovable places; and for that reason I did before refer those absolute motions to immovable places, but relative ones to movable places. Now no other places are immovable but those that, from infinity to infinity, do all retain the same given position to one another; and upon this account must ever remain unmoved; and do thereby constitute immovable space.

(*Principia*, Scholium to the Definitions)

Newtonian relativity: F_1 and F_2 are physically equivalent



The same physical situation



A philosophical commitment to relativity:

A conviction, on grounds of epistemology or metaphysics or both, that motion is can be nothing but the observable changes of relations among bodies.

Privileged states of motion, and motion with respect to space, are scientifically and philosophically illegitimate notions.

Space and time are philosophically suspect theoretical entities, based on an illegitimate “inference to the best explanation”

Noteworthy features of the philosophical idea of relativity:

1. It asserts, for a system of interacting bodies, the complete arbitrariness of its configuration. Motion and rest may be freely assigned to any body in the system.

2. It separates the problem of the relativity of motion, which concerns phenomena, from the question of causes, which pertains to a deeper metaphysical level.

3. It suggests (in spite of 1 and 2) that the study of mechanical causes makes (e.g.) the Copernican hypothesis more useful.

We speak as the situation requires,in accordance with the more appropriate and simpler explanation of the phenomena. It is just in this sense that we use the motion of the *primum mobile* in spherical astronomy, while in the theoretical study of the planets we ought to use the Copernican hypothesis. As an immediate consequence of this view, those disputes conducted with such enthusiasm...completely disappear. For even though force is something real and absolute, motion belongs among phenomena and relations, and we must seek truth not so much in the phenomena as in their causes.

(Leibniz, *Specimen dynamicum*)

A *theory* of relativity:

1. An account, from the laws of physics, of fundamental theoretical magnitudes-- those that matter in physical interactions and their empirical measures;

2. A principled distinction (derived from 1) between the invariant features of a physical system, and those that depend solely on the mode of description;

3. A critical analysis of familiar concepts--either from common sense or from scientific discourse--revealing the extent to which they represent partial or relative perspectives on the invariant quantities.

***Newton's theory of relativity:* unites the relativity of motion with the pursuit of causal explanations, in a mathematical account of interaction:**

1. An account of the fundamental concepts of inertia and force that distinguishes their invariant features from those that depend on a particular relative perspective

2. A critical analysis of the familiar uses of these concepts--revealing the relativistic aspect of "inertial mass," and extent to which the usual notions represent partial or relative perspectives on the invariant quantity

3. An account of how the mathematical treatment of forces permits the separation of causal analysis within a system of bodies from the motion of the system as a whole

Absolute space is introduced as the background for explicating this theory of relativity.

De Gravitatione: Descartes' definition of motion "in the philosophical sense" makes nonsense of the basic law of motion:

[N]o one can assign the place…at which the body was in the beginning of the accomplished motion, or rather he has not said whence it is possible a body will be moved. And the reason is that according to Descartes it is not possible to define and assign the place except from the position of the surrounding bodies, and that after any motion having been accomplished the position of the surrounding bodies remains no more the same as it was before. For example, if the place of the planet Jupiter were where (it was) the year before, then having been accomplished it would be at rest; by what reasoning, I ask, will the philosopher, Descartes, describe it?

.…..It follows that Cartesian motion is not motion, for it has no velocity, no definition, and there is no space or distance traversed by it. So it is necessary that the definition of places, and hence of local motion, be referred to some motionless thing such as extension alone or space insofar as it is seen to be truly distinct from bodies.

This argument does not appear in the Scholium. Here Newton shows that Descartes' definition of motion is compatible with the principle of inertia, but on *causal* grounds:

"True motion is neither generated nor altered, but by some force impressed upon the body moved; but relative motion may be generated or altered without any force impressed upon the body."

The missing argument from *De Gravitatione* is arguably a metaphysical one: the principle of inertia requires the existence of a definite path through space.

De motu Sphaericorum Corporum in fluidis, 1684:

Lex 3. The motions of bodies within a given space are the same among themselves whether that space rests or moves perpetually and uniformly in a right line without any circular motion.

Lex 4. By the mutual actions between bodies their common centre of gravity does not change its state of motion or rest.

Scholium: Moreover the whole space of the planetary heavens either rests (as is commonly believed) or moves uniformly in a straight line, and hence the communal centre of gravity of the planets (by Law 4) either rests or moves along with it. In both cases (by Law 3) the relative motions of the planets are the same, and their common centre of gravity rests in relation to the whole space, and so can certainly be taken for the still centre of the whole planetary system. Hence truly the Copernican system is proved *a priori*. For if the common centre of gravity is calculated for any position of the planets it either falls in the body of the Sun or will always be very close to it.

De Motu Corporum In Medijs Regulariter Cedentibus (1684-85?)

(On the Motion of Bodies in regularly yielding media)

Def. 1. Absolute time is that which by its own nature without relation to anything else flows uniformly. Such it is whose equation Astronomers investigate, and by another name is called Duration.

Def. 2. Time looked at relatively is that which from something some other sensible passage or another flow or passage is measured in respect to the flow or passage of any sensible thing is considered as uniform. Such is the time of days, months, and other heavenly periods, which on that account are believed to have begun with this world under the hypothesis that these periods are equal which common people consider as equal∧among common people.

Def. 3. Absolute space so-called is that which by its own nature and unrelated to any other thing whatsoever always remains immobile. As the order of the parts of time is immutable, so also is that of the parts of space. Were these to be moved from their places they would be moved out of themselves. For times and spaces are just the places of themselves and all things. All things are located in time as long as the order of succession and in space as far as the order of position. The essence of those is that they are to be places, and for primary places to be moved is absurd. Moreover, were one part of space to be moved by a certain force, the whole of space will be moved by the same such a force applied to all parts to infinity which again is absurd.

Def. 4. Relative space is that which is considered immobile with respect to another ∧any sensible∧ thing: such as the space of our air with respect to the earth. These spaces, however, are in fact distinguished from one another by the descent of heavy bodies [gravium] which in absolute space seek the center directly but in relative [space] rotating absolutely are deflected to the sides [ad latus].

The aim of explicating [explicare] all these at length is that the Reader ∧freed from certain common [vulgaribus] prejudices∧ and imbued with clear and distinct conceptions of Mechanical principles may agree to what follows. Moreover to sharply diligently distinguish absolute and relative quantities from one another I am compelled [coactus sum] ∧has been necessary∧ since all phenomena depend on the absolute, yet ordinary people who do not know to abstract thought from sensible appearances always speak of the relative, so much so that it would be absurd for either wise men or ∧even∧ Prophets to speak to them otherwise.

The definition of inertia: An example of the (gradual) removal of prejudices:

De Gravitatione:

Definition 6. Conatus (endeavor) is resisted force, or force in so far as it is resisted.

Definition 7. Impetus is force in so far as it is impressed on something.

Definition 8: Inertia is the internal force of a body, so that its state should not be easily changed by an external force.

De Motu Corporum In Medijs Regulariter Cedentibus

Definition 12: The internal, innate, and essential force of a body is the power by which it perseveres in its state of rest or uniform motion in a straight line. It is proportional to the quantity of the body, and is truly exercised in proportion to the change of state, and insofar as it is exercised may be said to be the exercised force of the body...

Definition 13 (struck out): The force of a body arising from its motion is that by which the body endeavors to preserve the total quantity of its motion. It is commonly called impetus, and is proportional to the motion, and according to its kind is called absolute or relative. The centrifugal force of rotating bodies is to be referred to the absolute kind.

De Motu Corporum (1685), **Newton's fully relativistic definition of inertia (Definition 3):**

The innate force of matter is the power of resisting whereby each individual body, insofar as in it lies, perseveres in its state of resting or of moving uniformly straight forward: it is, furthermore, proportional to the body, nor does it differ at all from the inertia of its mass other than in the manner of our conceiving it. A body in fact exerts this force only during a change of its state effected by the impress of another force upon it.

The exercise of this force is either Resistance or Impetus, which are distinct only in relation to one another: it is resistance insofar as the body opposes an impressed force, impetus insofar as the body, by yielding with difficulty attempts to change the state of another body. Resistance is commonly attributed to bodies at rest and impetus to those in motion: but motion and rest are distinct only in relation to each other; nor do those things truly rest which are commonly regarded as being at rest.

Principia, Definition III:

The inherent force of matter is a power of resisting, by which every body, as far as it is able, perseveres in its present state, whether it be of rest, or of moving uniformly straight forward.

This force is alway proportional to the body and does not differ from the inertia of the mass except in the manner in which it is conceived. Because of the inertia of matter, a body is only with difficulty put out of its state of resting or of moving. Consequently, the inherent force may also be called by the very significant name of *vis inerti*ae, or force of inactivity.

[A] body exerts this force only during a change of its state, caused by another force impressed upon it, and the exercise of this force is, depending on viewpoint, both resistance and impetus: resistance in so far as the body, in order to maintain its state, strives against the impressed force, and impetus in so far as the same body, yielding only with difficulty to the force of a resisting obstacle, endeavors to change the state of that obstacle. Resistance is commonly attributed to resting bodies and impetus to moving bodies; but motion and rest, in the popular sense of the term, are distinguished from each other only by point of view, and bodies commonly regarded as being at rest are not always truly at rest.

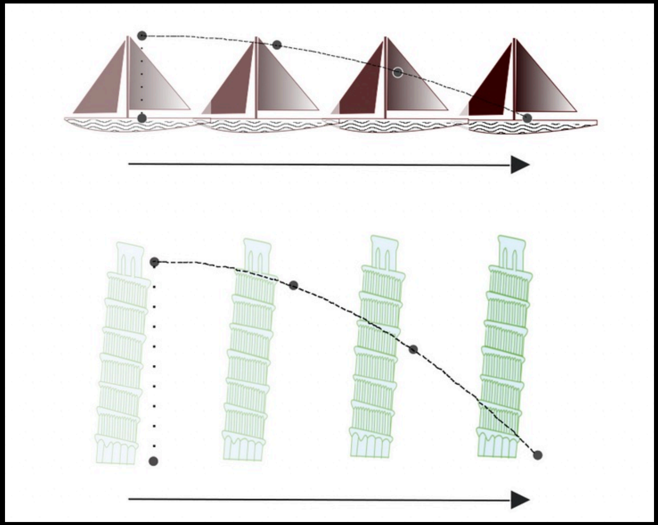
The relativity theory of Newton's Principia:

Corollary V: "When bodies are enclosed in a given space, their motions among themselves are the same whether the space is at rest or whether it is moving uniformly straight forward without circular motion."

Corollary VI: "If bodies are moving in any way whatsoever among themselves and are urged by equal accelerative forces along parallel lines, they will all continue to move with respect to one another in the same way as they would if they were not acted on by those forces."

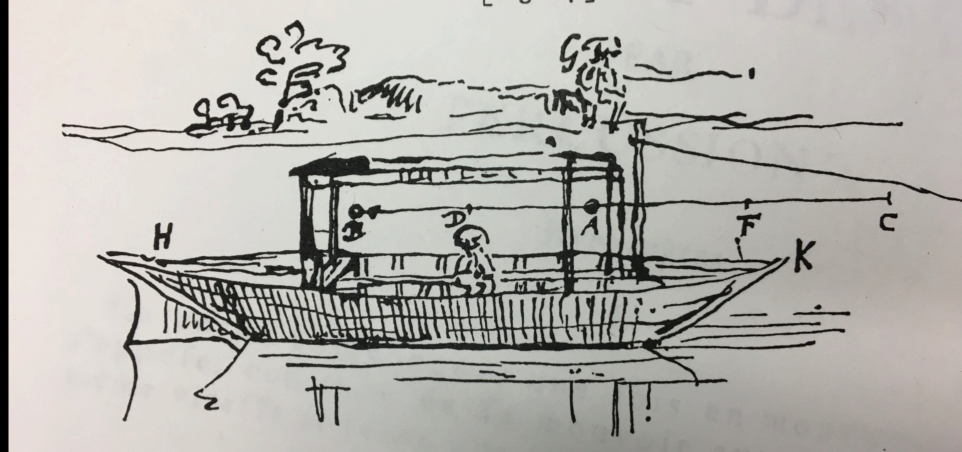
Galilean relativity: If the earth is moving rapidly through space, and turning on its axis, why don't we notice any effects of motion?

Because the motions and interactions among bodies within a frame of reference are combined with the earth's motions. Those interactions are invariant, while the motion of the whole is relative.



On a smoothly moving ship, a stone dropped from the mast will appear to fall along the mast because the stone and the ship share the same smooth motion. A stone falls to the base of a tower on the earth, because the stone, the tower, and the earth share the same smooth motion.

Christiaan Huygens’s relativity principle (1654):
The motions of bodies, and their speeds equal or unequal, are to be understood respectively, in relation to other bodies which are considered as at rest, even though perhaps both the former and the latter are subject to another motion that is common to them. Consequently, when two bodies collide with one another, even if both together undergo another equable motion, they will move each other no differently, with respect to a body that is carried by the same common motion, than if this extraneous motion were absent from all of them.



[Fig. 14.]

Why Corollary VI is *not* a relativity principle in the sense of Corollary V:

Corollary V: An account, from the laws of physics, of fundamental theoretical magnitudes-- those that matter in physical interactions and their empirical measures— and a principled distinction between the invariant features of a physical system, and those that depend solely on the mode of description

Corollary VI: The common acceleration of a system of bodies, though locally *nearly* indistinguishable from a common uniform motion, is the effect of a ***physical interaction*** that must be treated as any other physical interaction: its source must be found, and an upper bound placed on its differential effects.

Corollary V: An account, from the laws of physics, of fundamental theoretical magnitudes-- those that matter in physical interactions and their empirical measures— and a principled distinction between the invariant features of a physical system, and those that depend solely on the mode of description

Corollary VI: The common acceleration of a system of bodies, though locally *nearly* indistinguishable from a common uniform motion, is the effect of a ***physical interaction*** that must be treated as any other physical interaction: its source must be found, and an upper bound placed on its differential effects.

How Corollary VI nonetheless functions as something like a relativity principle:

In the most obvious sense, it allows Newton to treat states of motion as equivalent, even though they are dynamically quite distinct, because they are approximately indistinguishable.

More important: it allows Newton to give a robust account of the forces acting within a system of interacting bodies, independent of actual or possible interactions in which the system as a whole may be involved, with other known or unknown bodies or systems.

Principia, Proposition III, Theorem III

Every body, that, by a radius drawn to the centre of another body, howsoever moved, describes areas about that centre proportional to the times, is urged by a force compounded out of the centripetal force tending to that other body, and of all the accelerative force by which that other body is impelled.

[If T is moving anyhow, and L orbits T describing equal areas in equal times, then the force acting on L is the force toward T combined with whatever forces are acting on T.]

[Or: If L orbits T, then there is a centripetal force toward T; if L's orbit around T obeys Kepler's 2nd law , then L must be acted upon by the same forces that act upon T, whatever they are.]

Principia, Proposition III, Theorem III

Let L represent the one, and T the other body; and (by Cor. 6 of the Laws) if both bodies are urged in the direction of parallel lines, by a new force equal and contrary to that by which the second body T is urged, the first body L will go on to describe about the other body T the same areas as before: but the force by which that other body T was urged will be now destroyed by an equal and contrary force; and therefore (by Law I.) that other body T, now left to itself, will either rest, or move uniformly forward in a right line: and the first body L impelled by the difference of the forces, that is, by the force remaining, will go on to describe about the other body T areas proportional to the times...

Cor. 2. And, if these areas are proportional to the times nearly, the remaining force will tend to the other body T nearly.

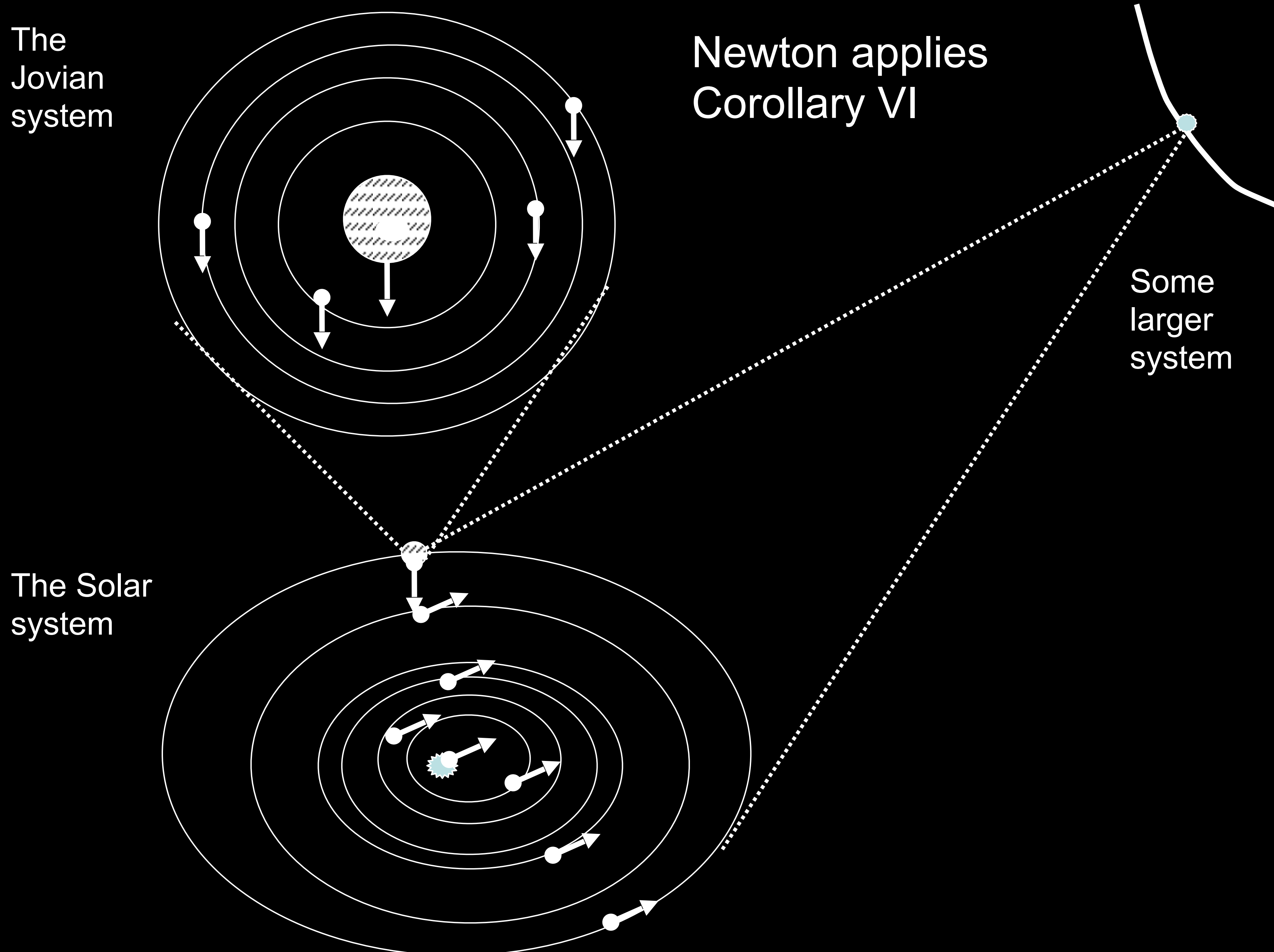
Cor. 3. And vice versa, if the remaining force, tends nearly to the other body T, those areas will be nearly proportional to the times.

Cor. 4. If the body L, by a radius drawn to the other body T, describes areas, which, compared with the times, are very unequal ; and that other body T be either at rest, or moves uniformly forward in a right line: the action of the centripetal force tending to that other body T is either none at all, or it is mixed and compounded with very powerful actions of other forces: and the whole force compounded of them all, if they are many, is directed to another (immovable or moveable) centre. The same thing obtains, when the other body is moved by any motion whatsoever; provided that centripetal force is taken, which remains after subducting that whole force acting upon that other body T.

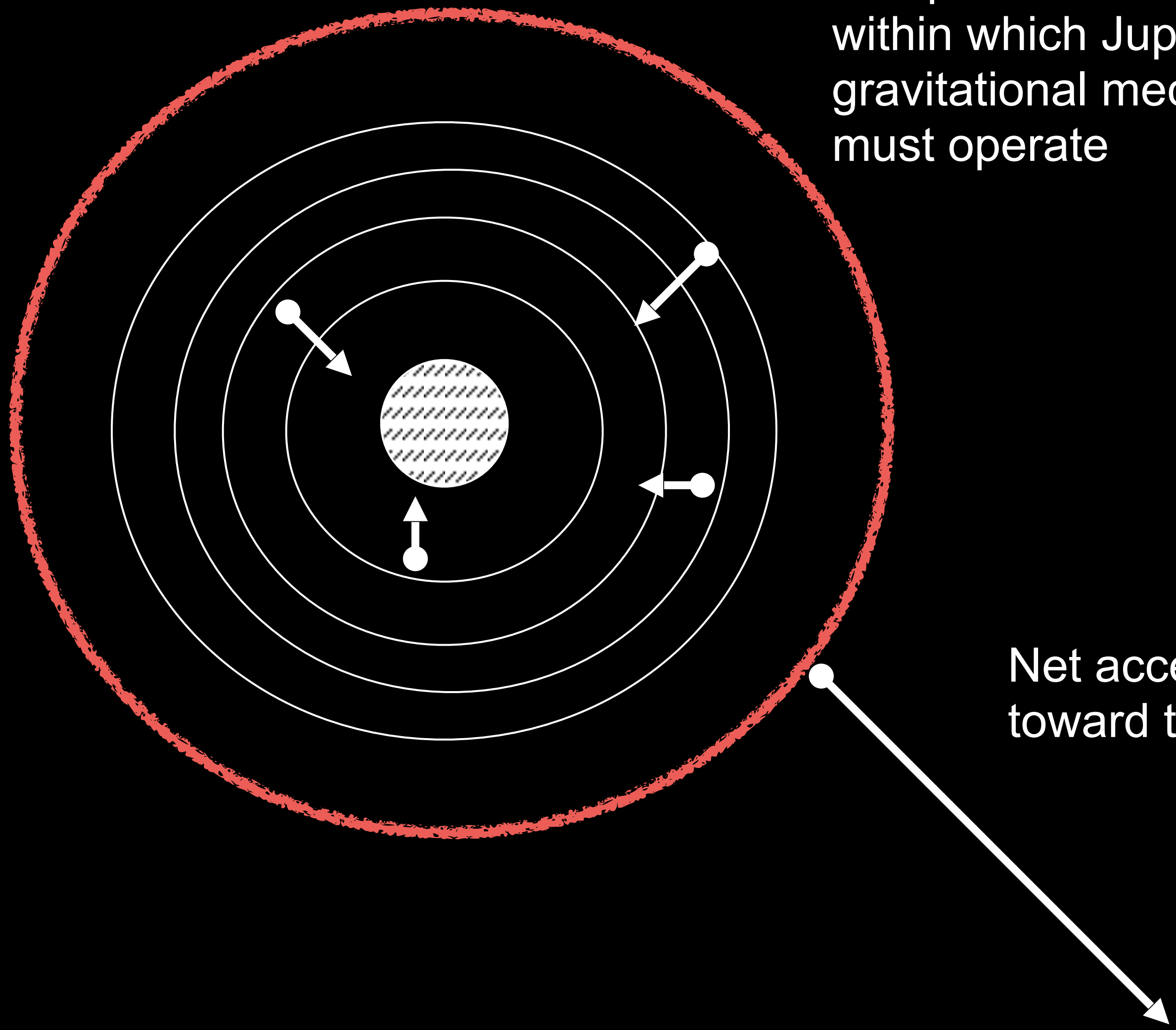
SCHOLIUM.

Because the equable description of areas indicates that a centre is respected by that force with which the body is most affected, and by which it is drawn back from its rectilinear motion, and retained in its orbit; why may we not be allowed, in the following discourse, to use the equable description of areas as an indication of a centre, about which all circular motion is performed in free spaces?

Newton applies Corollary VI



The Jovian system according to Huygens



The System of the Universe, proposition 8: The force which governs the superior planets is not directed toward the earth. It is directed toward the sun.

It is necessary to seek (by prop. 2 and 3 and the corollaries to the latter) another center of these forces about which the description of areas (by radii drawn from that center to the planets) is uniform. And that this is the sun has already been more or less proved for Mars and Saturn, and has been proved more than exactly enough for Jupiter. It is possible to imagine that the sun and planets are urged equally and along parallel lines by any other forces. But (by corol. 6 of the laws) the situation of the planets with regard to one another will not be changed by such a force, and no sensible effect will be produced. But we are dealing with the causes of sensible effects. Therefore let every force of this kind be rejected as precarious and having nothing to do with the phenomena of the heavens; then all the remaining force by which the planet [lit. star] Jupiter is urged will tend (by prop. 3, corol. 1) toward the center of the sun.

Newton’s Proposition 43 (intended for the second edition of *Principia*, but never published):

“In order for the Earth to be at rest in the center of the system of the Sun, Planets, and Comets, there is required both universal gravity and another force in addition that acts on all bodies equally according to the quantity of matter in each of them and is equal and opposite to the accelerative gravity with which the Earth tends to the Sun, tending along parallel lines on the same flat surface with the line drawn from the center of the Sun to the center of the Earth.

For, such a force, acting on all bodies equally and along parallel lines, does not change their position among themselves, but permits bodies to move among themselves by the force of universal gravity in the same way as if it were not acting on them. Since this force is equal and opposite to its gravity toward the Sun, the Earth can in truth remain in equilibrium between these two forces and be at rest. And thus celestial bodies can move around the Earth at rest, as in the Tychonic System.”

Why was Huygens not struck by Corollary VI?

What Huygens claimed to know about the action of gravity:

So I have explained, with one hypothesis that contains nothing impossible, why terrestrial bodies tend to the centre; why the action of gravity cannot be prevented by any known body; why the parts within each body all contribute to its gravity; and finally why falling bodies constantly increase their velocity in proportion to the times. Such are the properties of gravity as we have distinguished them so far.
There still remains one property…namely that bodies weigh as much in one place on earth as they do in another.
(Discourse)

What Newton claimed to know about the action of gravity:

that it must proceed from a cause that penetrates to the very centres of the sun and planets, without suffering the least diminution of its force;

that it operates not according to the quantity of the surfaces of the particles upon which it acts (as mechanical causes use to do), but according to the quantity of the solid matter which they contain;
that it propagates its virtue on all sides to immense distances, decreasing always in the duplicate proportion of the distances…

….in receding from the sun [it] decreases accurately in the duplicate proportion of the distances as far as the orb of Saturn, as evidently appears from the quiescence of the aphelions of the planets; nay, and even to the remotest aphelions of the comets; if those aphelions are also quiescent.

I likewise call Attractions and Impulses, in the same sense, Accelerative, and Motive; and use the words Attraction, Impulse, or Propensity of any sort towards a centre, promiscuously, and indifferently, one for another; considering those forces not Physically, but Mathematically: wherefore the reader is not to imagine, that by those words I anywhere take upon me to define the kind, or the manner of any Action, the causes or the physical reason thereof, or that I attribute Forces, in a true and Physical sense, to certain centres (which are only Mathematical points); when at any time I happen to speak of centres as attracting, or as endued with attractive powers.

Newton, Proposition VI. Theorem VI.

That all bodies gravitate towards every planet; and that the weights of bodies towards any the same planet, at equal distances from the centre of the planet, are proportional to the quantities of matter which they severally contain.

It has been, now of a long time, observed by others, that all sorts of heavy bodies (allowance being made for…resistance in the air) descend to the earth from equal heights in equal times; and that equality of times we may distinguish to a great accuracy, by the help of pendulums. I tried the thing in gold, silver, lead, glass, sand, common salt, wood, water, and wheat. I provided two wooden boxes, round and equal: I filled the one with wood, and suspended an equal weight of gold (as exactly as I could) in the centre of oscillation of the other. The boxes hanging by equal threads of 11 feet made a couple of pendulums perfectly equal in weight and figure, and equally receiving the resistance of the air.

….And, placing the one by the other, I observed them to play together forward and backward, for a long time, with equal vibrations. And therefore the quantity of matter in the gold (by Cor. 1 and 6, Prop. XXIV, Book II) was to the quantity of matter in the wood as the action of the motive force (or vis motrix) upon all the gold to the action of the same upon all the wood: that is, as the weight of the one to the weight of the other: and the like happened in the other bodies. By these experiments, in bodies of the same weight, I could manifestly have discovered a difference of matter less than the thousandth part of the whole, had any such been. But, without all doubt, the nature of gravity towards the planets is the same as towards the earth. (Newton, 1687)

Book I, Prop. LXV, Case 2. Let us imagine a system of lesser bodies revolving about a very great one in the manner just described…and in the mean time to be impelled sideways by the force of another vastly greater body situate at a great distance. And because the equal accelerative forces with which the bodies are impelled in parallel directions do not change the situation of the bodies with respect to each other, but only oblige the whole system to change its place while the parts still retain their motions among themselves, it is manifest that no change in those motions of the attracted bodies can arise from their attractions towards the greater, unless by the inequality of the accelerative attractions, or by the inclinations of the lines towards each other, in whose directions the attractions are made.

Cor. 3. Hence if the parts of this system move in ellipses or circles without any remarkable perturbation, it is manifest that, if they are at all impelled by accelerative forces tending to any other bodies, the impulse is very weak, or else is impressed very near equally and in parallel directions upon all of them.

Book III, Prop. VI:

Further, that the weights of Jupiter and of his satellites towards the sun are proportional to the several quantities of their matter, appears from the exceedingly regular motions of the satellites (by Cor. 3, Prop. LXV, Book 1)…..Therefore if, at equal distances from the sun, the accelerative gravity of any satellite towards the sun were greater or less than the accelerative gravity of Jupiter towards the sun but by one 1⁄1000 part of the whole gravity, the distance of the centre of the satellite's orbit from the sun would be greater or less than the distance of Jupiter from the sun by one 1⁄2000 part of the whole distance…an eccentricity of the orbit which would be very sensible. But the orbits of the satellites are concentric to Jupiter, and therefore the accelerative gravities of Jupiter, and of all its satellites towards the sun, are equal among themselves….

What Newton claimed to know about the action of gravity: The mathematical theory vs. the physical facts

that it must proceed from a cause that penetrates to the very centres of the sun and planets, without suffering the least diminution of its force;

that it operates not according to the quantity of the surfaces of the particles upon which it acts (as mechanical causes use to do), but according to the quantity of the solid matter which they contain; that it propagates its virtue on all sides to immense distances, decreasing always in the duplicate proportion of the distances…

….in receding from the sun [it] decreases accurately in the duplicate proportion of the distances as far as the orb of Saturn, as evidently appears from the quiescence of the aphelions of the planets; nay, and even to the remotest aphelions of the comets; ***if those aphelions are also quiescent.*** [Emphasis added]

Newton's ontological lesson regarding space:

Leibniz: Since space has the properties of a mathematical continuum, and not of an aggregate composed of its points, it does not satisfy any reasonable metaphysical notion of substance. It must be something ideal and not real.

Newton: Since space has the properties of a mathematical continuum, and lacks the properties of a substance, and yet is evidently something real, the traditional categories of substance and accident are clearly inadequate.

An analogous lesson from universal gravitation:

Newton claimed to provide a theoretical description of a notable feature of nature, a description that can be empirically applied and evaluated *even in the absence of any satisfactory view of its ultimate metaphysical basis*-- regardless of any scruples regarding action at a distance.

Leibniz: The mode of action of gravity, as proposed by Newton, has features that cannot belong to any genuine physical interaction

Newton: These features suggest that gravity may be a natural “power” of a hitherto-unexpected kind, so that our conception of what is physically intelligible may have to be revisited.

The ancients and the moderns, who own that gravity is an occult quality, are in the right, if they mean by it that there is a certain mechanism unknown to them whereby all motions tend towards the center of the earth. But if they mean that the thing is performed without any mechanism by a simple primitive quality or by a law of God who produces that effect without using any intelligible means, it is an unreasonable an occult quality, and so very occult that it is impossible that it should ever be done though an angel or God himself should undertake to explain it.

(Leibniz to Hartsoeker, 1712)

The same ought to be said of hardness. So then gravity and hardness must go for unreasonable occult qualities unless they can be explained mechanically. And why may not the same be said of the *vis inertiae* and the extension, the duration, and mobility of bodies, and yet no man ever attempted to explain these qualities mechanically, or took them for miracles or supernatural things or fictions or occult qualities. They are the natural, real, reasonable, manifest qualities of all bodies seated in them by the will of God from the beginning of creation and perfectly incapable of being explained mechanically, and so may be the hardness of primitive particles of bodies.

(Newton to the editor of *Memoirs of Literature*, 1712)

But [Leibniz] goes on and tells us that God *could not create planets that* should move round of themselves without any cause that should prevent their removing through the tangent. For a miracle at least must keep *the planet in*. But certainly God could create planets that should move round of themselves without any other cause than gravity that should prevent their removing through the tangent. For gravity without a miracle may keep the planets in. And to understand this without knowing the cause of gravity, is as good a progress in philosophy as to understand the frame of a clock and the dependence of the wheels upon one another without knowing the cause of the gravity of the weight which moves the machine is in the philosophy of clockwork…

Newton on the role of mathematical "formalism" in representing physical interactions:

The mathematical representation of motions ***and forces*** is crucial to understanding the causal role that forces play.

The causal role of forces can only be understood through the mathematical analysis that isolates particular interactions among bodies from the motion of systems of bodies

The insight that we thus obtain into real causal relations is robust even if our mathematical theory is not correct.

The i***nstrumental value*** of the "formalism" consists in what it reveals about ***real*** causal relations

What is non-Euclidean geometry?
Starting from Euclidean geometry, we have Euclid’s postulates:

Postulate 1: To draw a line from any point to any point.
Postulate 2: To extend any line indefinitely in either direction.
Postulate 3: To draw a circle of any radius about any point.
Postulate 4: All right angles are congruent.
Postulate 5: The parallel postulate.

Kant (1781): The postulates are synthetic a priori truths, whose truth we understand through the form of spatial intuition.

Bolyai (1823): The first four postulates constitute “the absolute science of space.” They are the (something like) the conditions of the possibility of geometrical reasoning. Geometries compatible with these are all possibiy true. Only experience can distinguish among them.

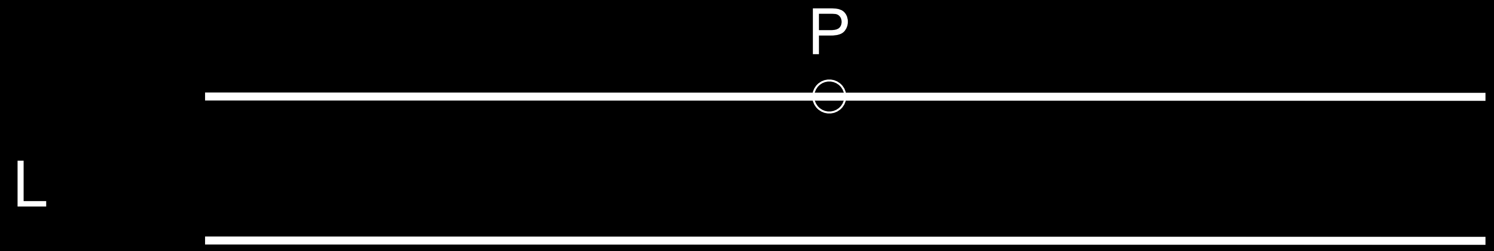
K.F. Gauss on Bolyai:

“I regard this young geometer Bolyai as a genius of the first order.”

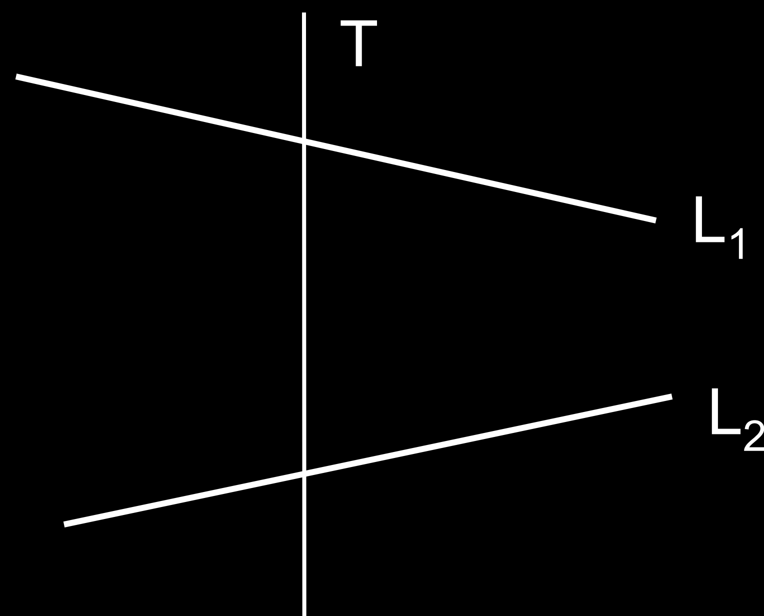
Gauss to Bolyai’s father:

“To praise it would amount to praising myself. For the entire content of the work...coincides almost exactly with my own meditations which have occupied my mind for the past thirty or thirty-five years.”

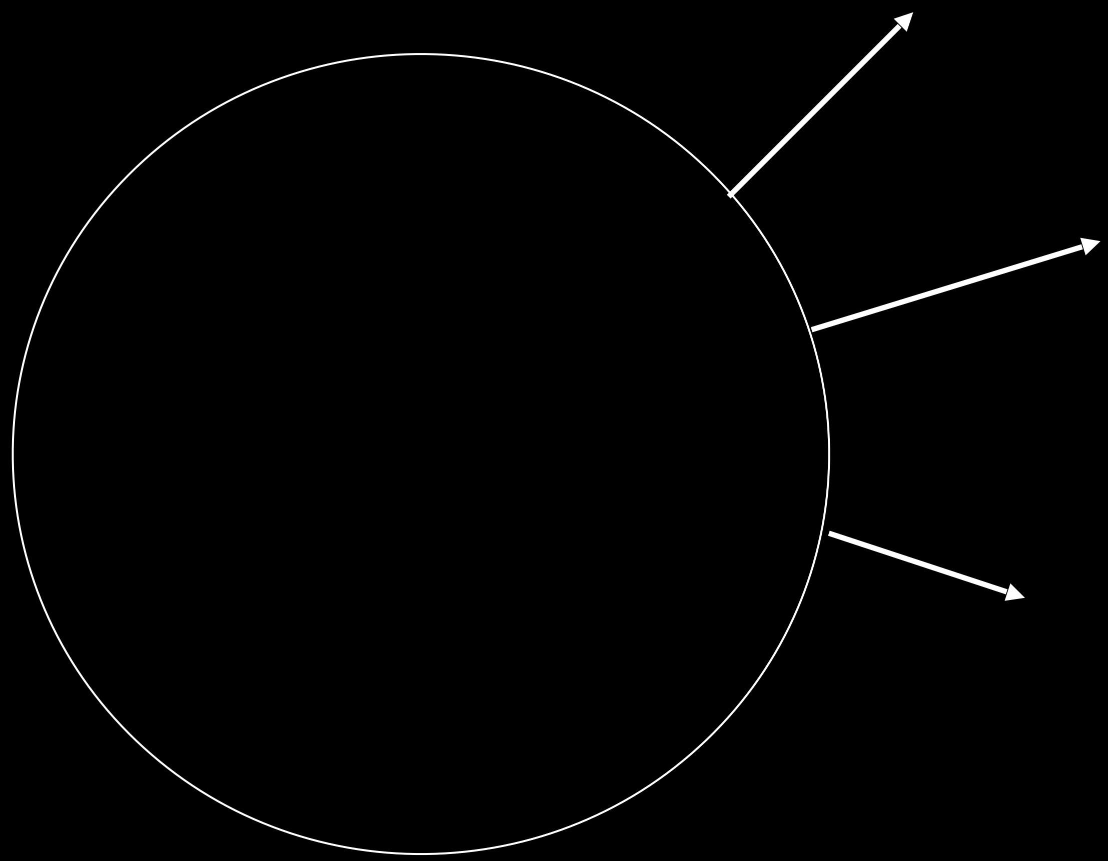
Parallel postulate: Given a line L and a point P not on L, there is exactly one line through P that does not intersect L. (From Proclus, 500-something C.E.)



Equivalently, if lines L_1 and L_2 cross a line T, L_1 and L_2 will meet on that side of T where their internal angles with T are less than two right angles. (This is Euclid's version.)



Measure of curvature: Extrinsic curvature is measured by the relation of the space to the ambient space. E.g. perpendiculars to the surface of a sphere or cylinder are not parallel.



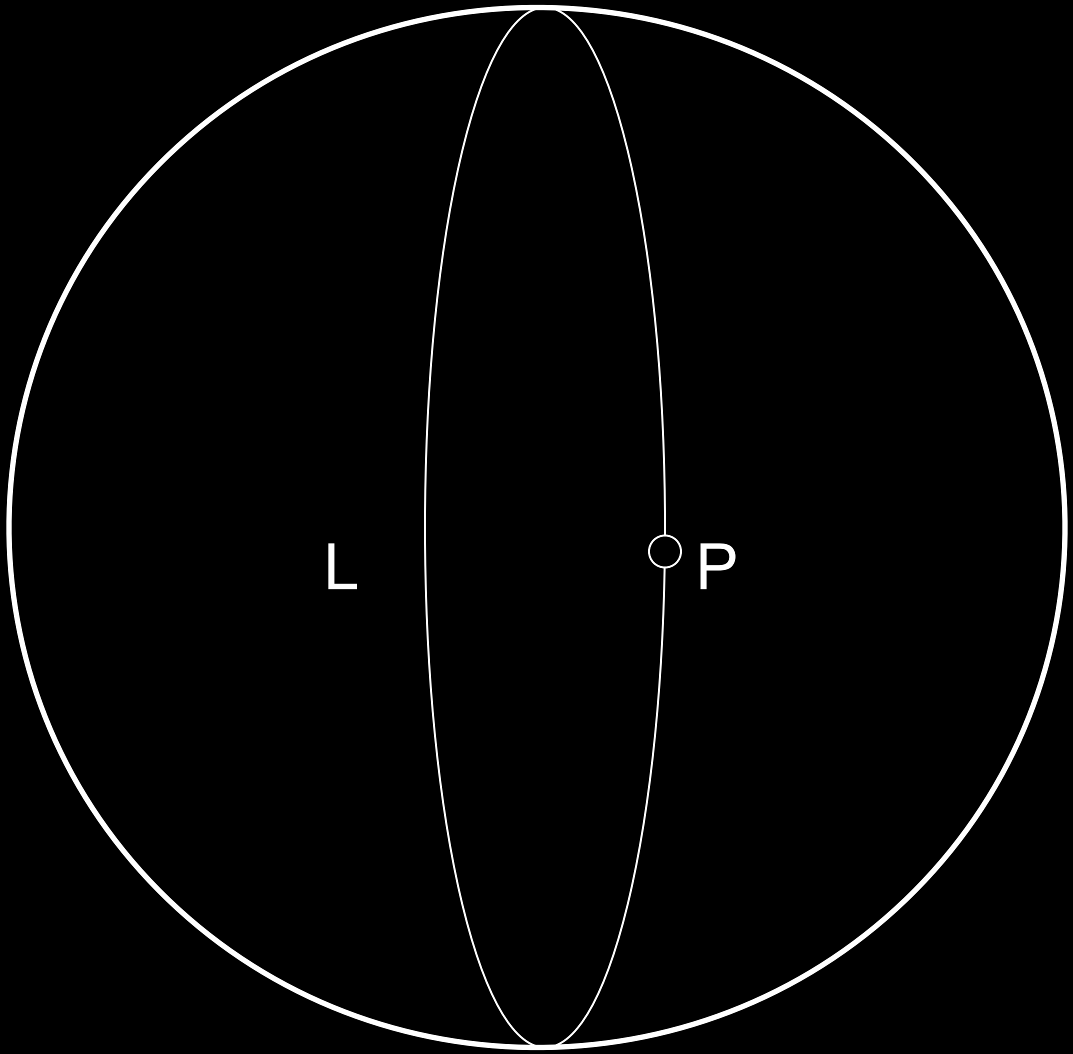
But if you roll a Euclidean surface into a cylinder, Euclidean figures remain Euclidean. The cylinder has no *intrinsic* curvature.

Intrinsic curvature is measured by features of the surface, or space, itself without regard to the ambient space. Such features are introduced by the failure of Euclid’s parallel postulate.

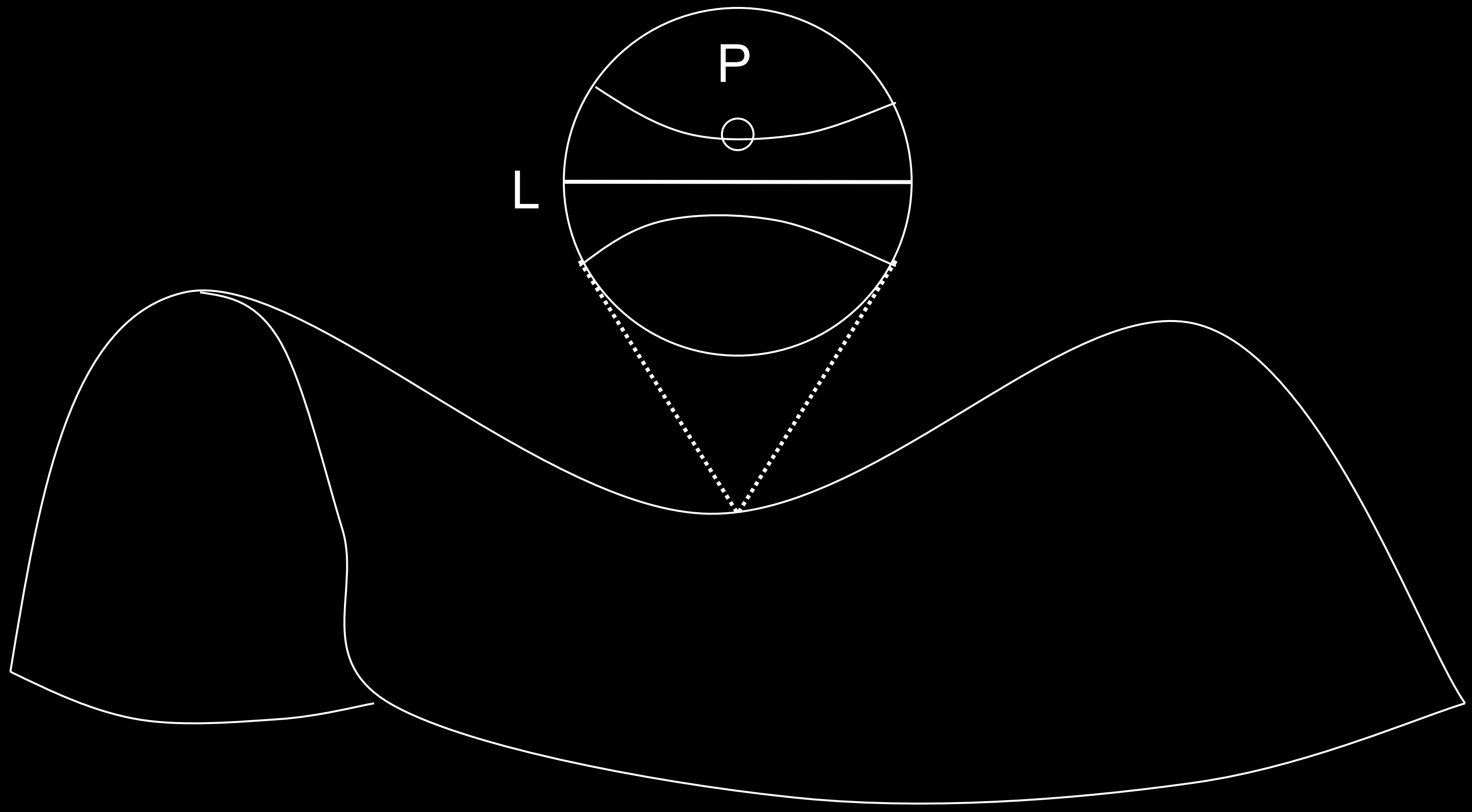
Positively curved space (spherical, elliptical, or “Riemannian” space”): Given a line L and a point P not on L, there is no line through P that fails to meet L.

Negatively curved space (Bolyai-Lobatchevsky geometry, or “pseudo-spherical” space: Given a line L and a point P not on L, there is more than one line through P that fails to meet L.

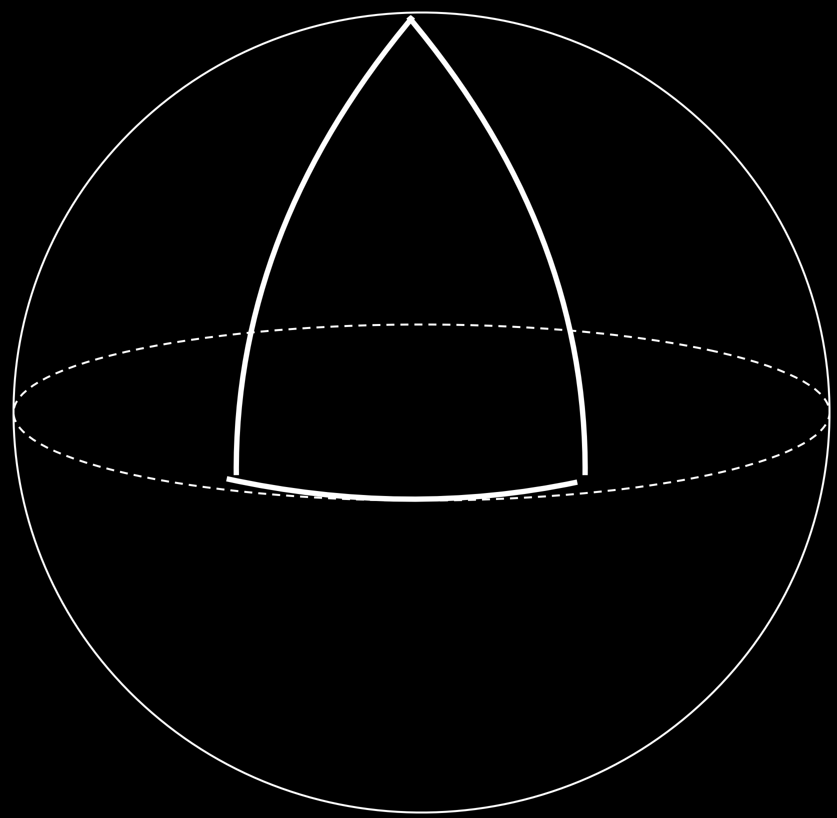
On a spherical surface, every line (“great circle”) through P will intersect L.



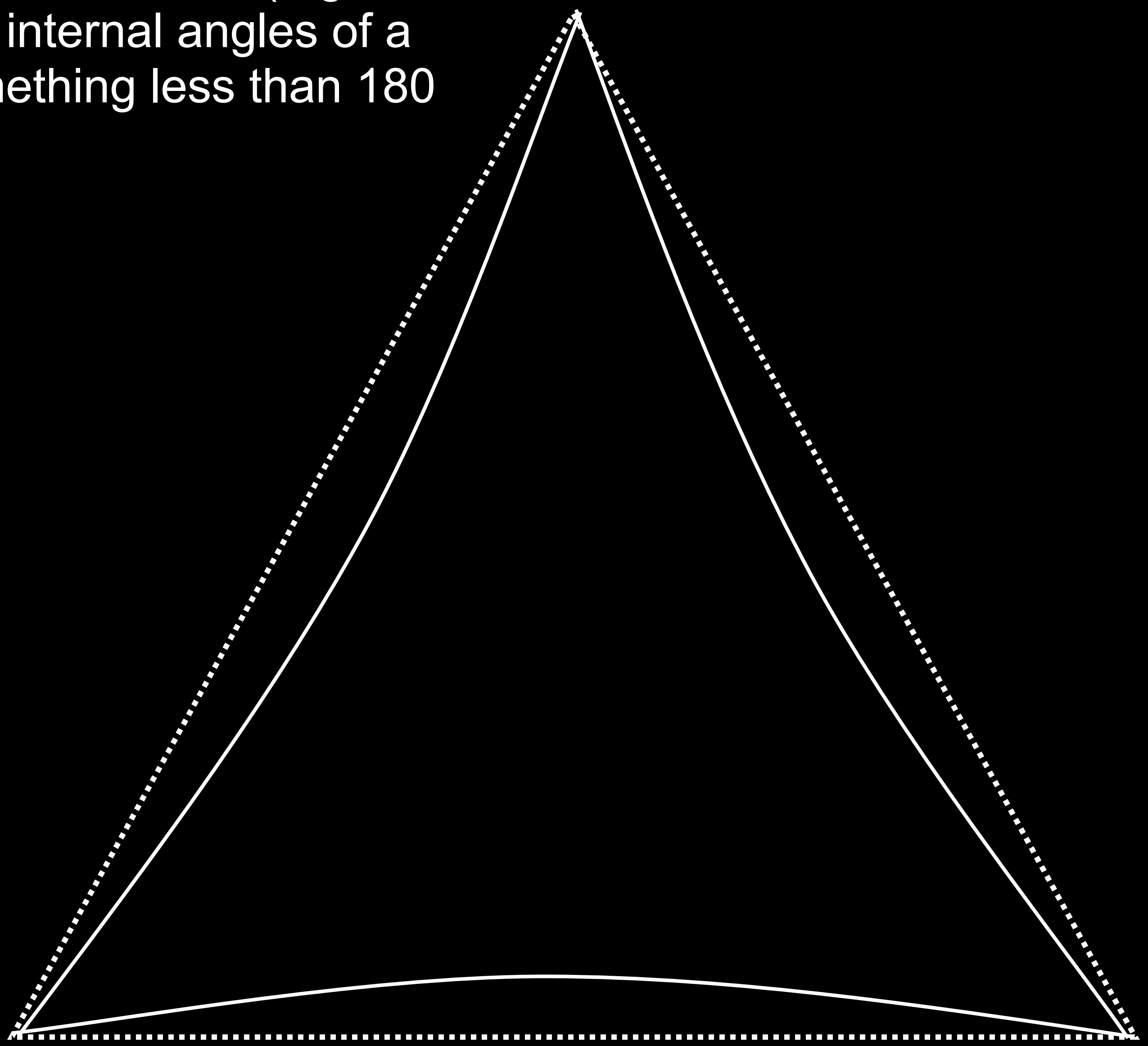
On a saddle surface, there may be infinitely many lines through P that do not intersect L.



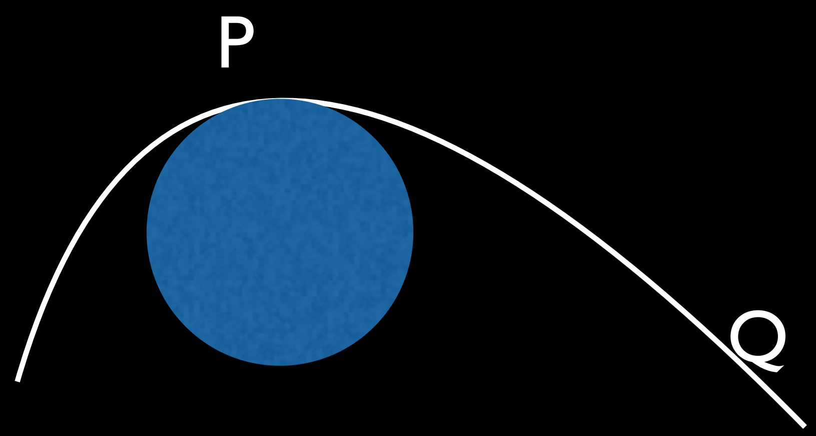
Comparison: On the surface of the sphere, the internal angles of a triangle sum to more than 180 degrees, and the excess depends on the size of the triangle. Each line of longitude forms a right angle with the equator.



On a negatively curved surface (e.g. a saddle surface, the internal angles of a triangle sum to something less than 180 degrees.

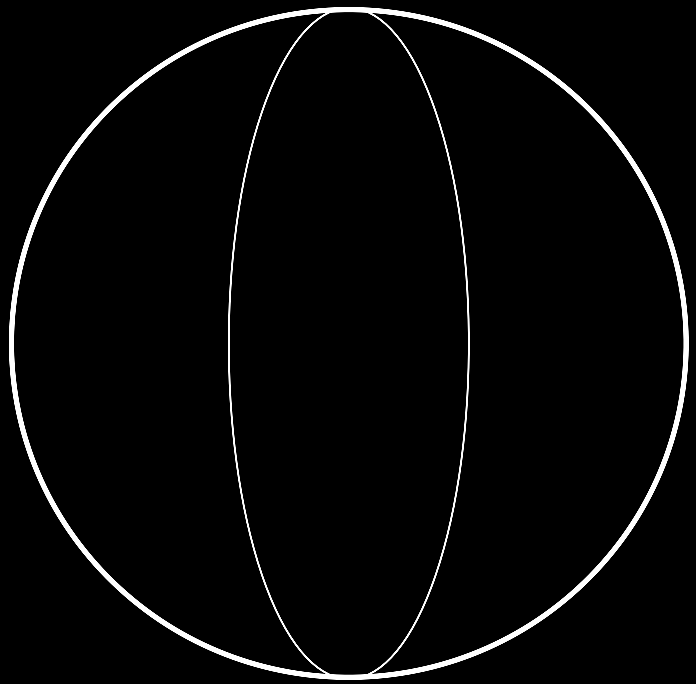


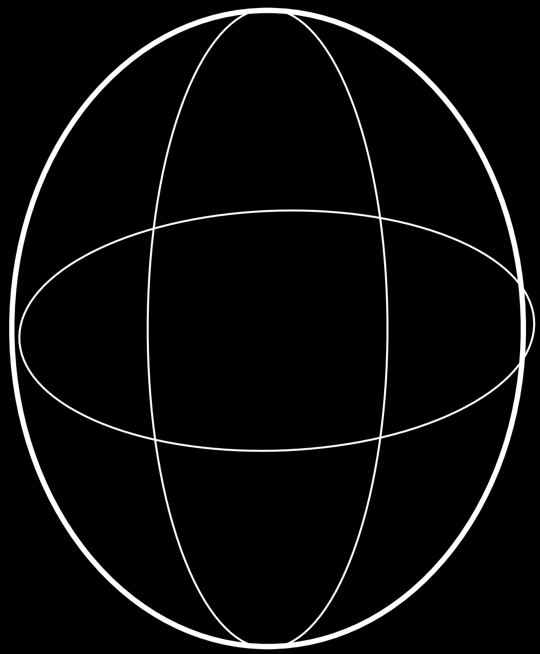
Intrinsic curvature is determined by the product of the ***greatest and least curvatures*** at a point. Each of these is determined by the radius of curvature, or the radius of the circle that best approximates the surface at a point (the “osculating circle”).



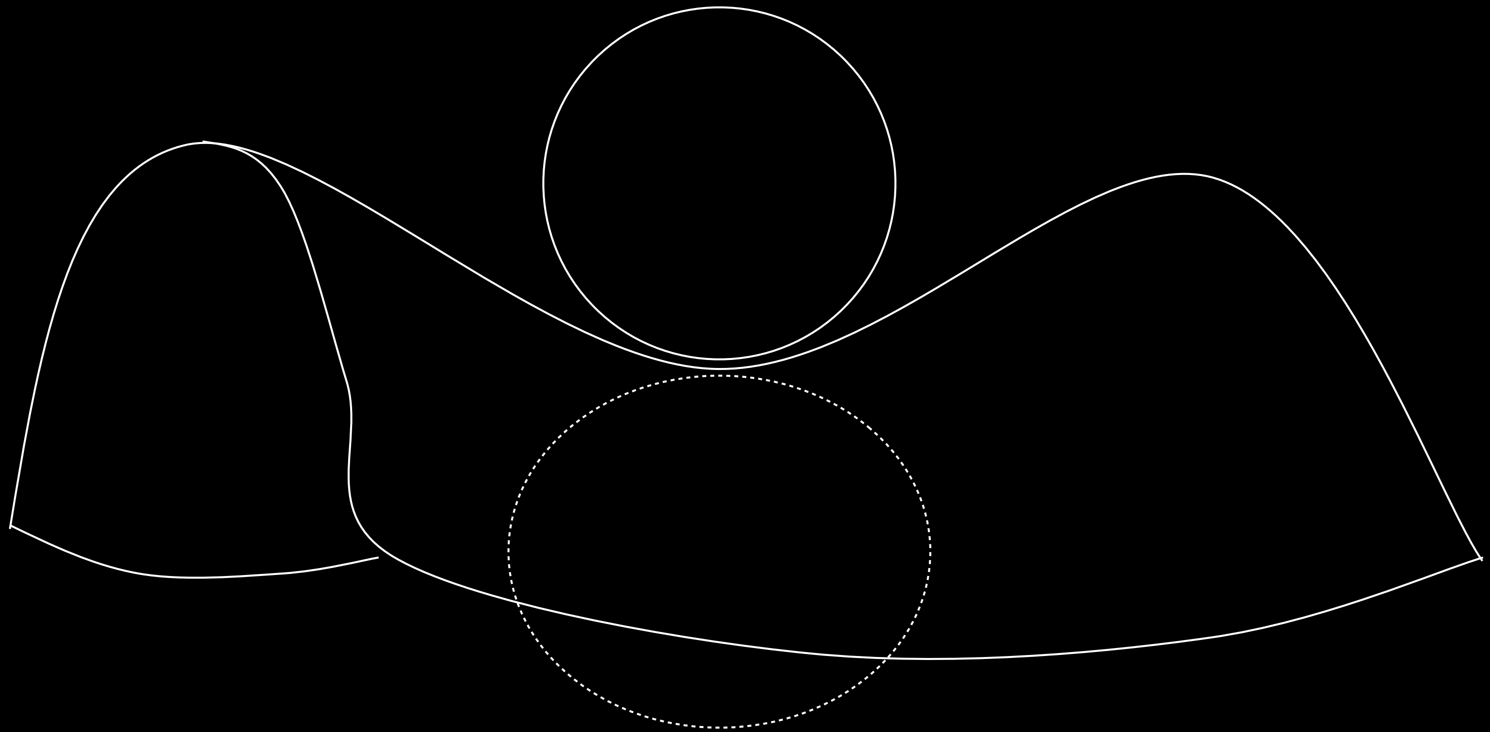
Evidently the osculating circle at Q must be greater than that at P. But the product of their radii will be a positive number.

On a spherical surface, the radius of curvature is the radius of the sphere itself, and it is the same everywhere on a perfect sphere, varying on (e.g.) an ellipsoidal surface.





On a saddle surface, at any point there are curvatures in opposite directions. Hence their product is negative.



Homogeneous geometry: The geometry of spaces of constant curvature.

Helmholtz-Lie theorem (***"free mobility"***): If a figure may be moved freely through space without changing its dimensions, then there is a quadratic function of the coordinates that is unchanging over space, and the curvature of the space is constant. (In the special case of Pythagoras's theorem, the quadratic function takes its simplest form, as the square root of the product of the squares of the coordinate-differences.)

Spaces characterized by Euclid's postulates, excluding the parallel postulate, are spaces of constant curvature. Classical proofs with compass and straight-edge are possible.

Free mobility implies constant curvature, and vice-versa.

What about ***inhomogeneous geometry*****?**

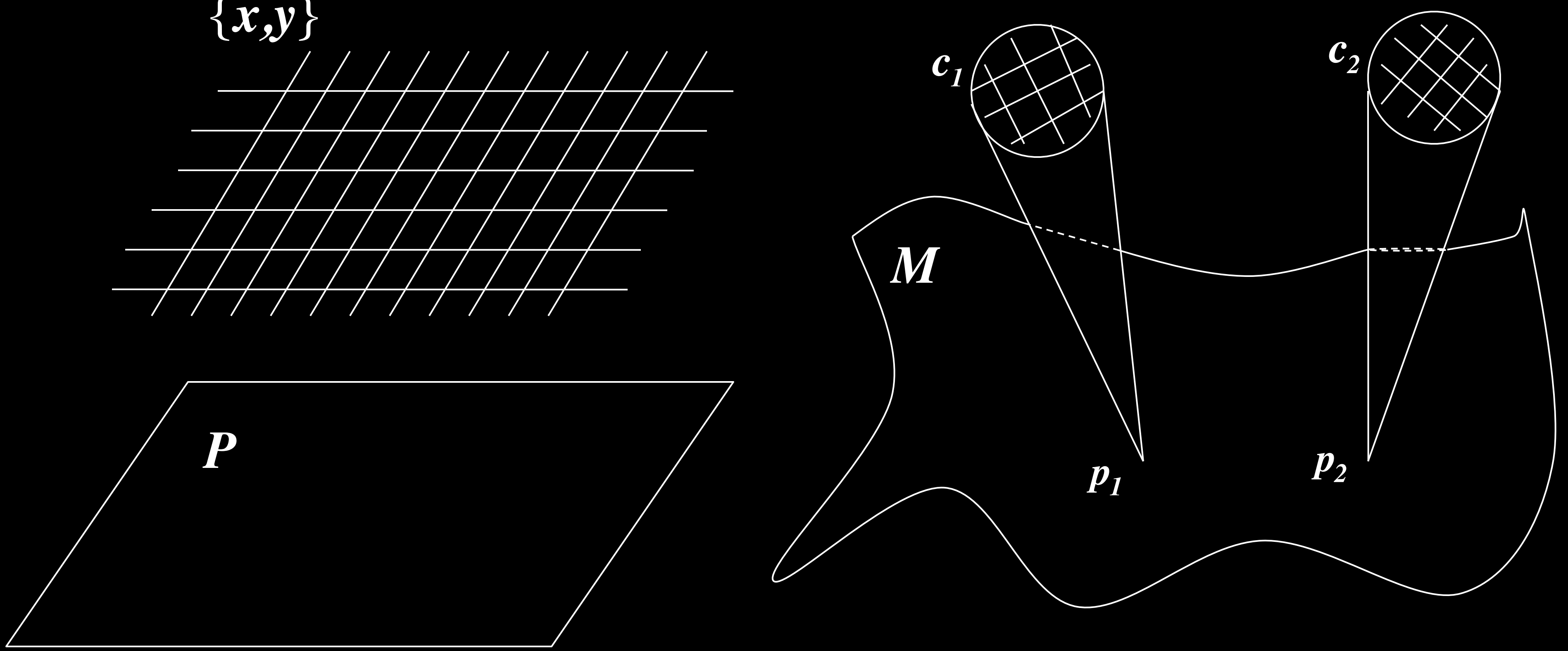
How do we describe spaces in which the curvature varies from point to point?

Bernhard Riemann (1826-66) recognized that geometry of constant curvature, in which free mobility is possible, is just a special case of a more general kind of geometry.

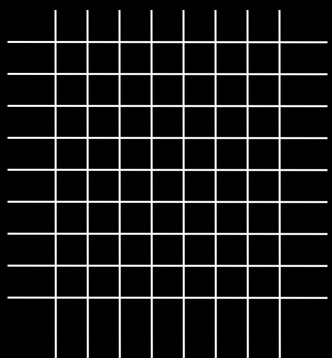
“Riemannian geometry” is the study of spaces which, at any point, have (“infinitesimally”) the structure of the Euclidean plane, but at different points, have variable curvature.

Over a vanishingly small region, a Cartesian coordinate system may be constructed. But the Cartesian coordinates at one point can’t be assumed to be extendible to any finite distance. (An irregular surface such as an apple can only be covered by a large number of very small stickers.)

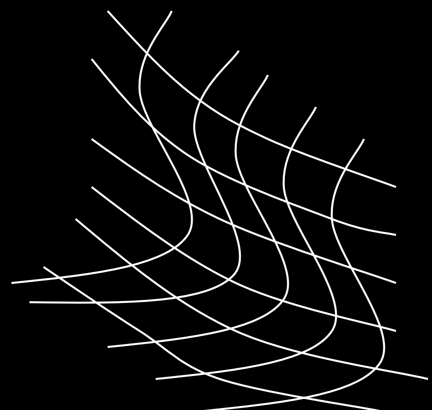
Spaces that are inhomogeneous:



The Cartesian rigid reference-frame

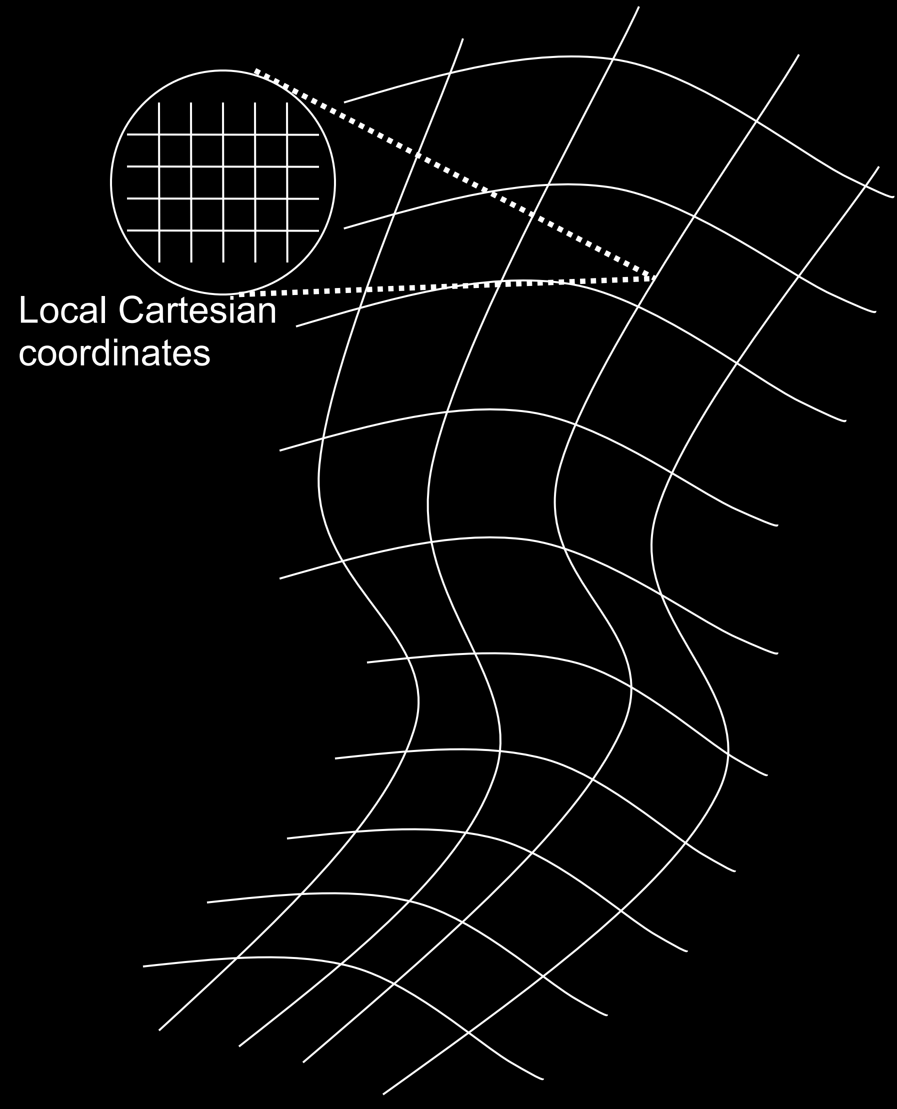


The Einsteinian “reference-mollusc”*



*Actual mollusc may differ

Large-scale and local structure of a space-time manifold



Two things to note about non-Euclidean geometry and Einstein's General Theory of Relativity:

The theory, as we will see, crucially relies on ***differential geometry***— the theory of very small variations in curvature from space-time point to space-time point, as developed by Riemann— because the variation of curvature is essential to the connection between curvature and gravitation.

The theory is not fundamentally about the curvature of ***space***, though spatial curvature plays an important role. The more fundamental notion is the curvature of ***space-time.*** The features of ***space-time geodesics*** provide the chief motivations to connect gravity with space-time curvature.

W**hat is the most general concept of space?**

Helmholtz-Poincaré: The most general concept is what is common to all spaces in which classical geometry is possible, i.e. all spaces in which it is possible to carry out classical Euclidean constructions, using a compass and straight-edge. (These assumptions characterize what Bolyai called "the absolute science of space".)

Geometrically: These are the spaces of constant curvature (homogeneous spaces) in which there is an invariant measure of length.

Physically: these are the spaces in which a measuring-stick may be displaced in any way without changing its dimensions. Metric invariance in spaces of constant curvature (like Euclid's0) corresponds to the free mobility of rigid bodies.

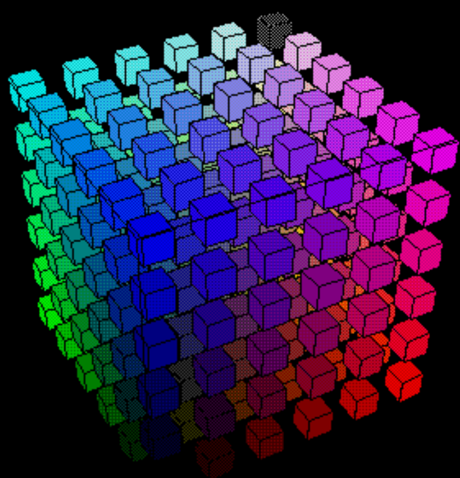
Riemann: Space as understood by Helmholtz-Poincaré is a very special case of a much more general concept.

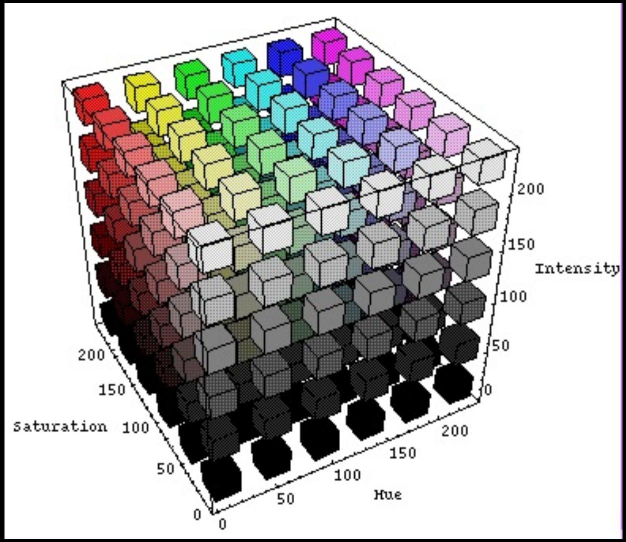
Riemann's general conception of space ("manifold"):

An n-dimensional space is an "n-fold extended aggregate"

—i.e. any aggregate in which ***n*** values are required to specify an individual. 3-D space is a three-fold extended aggregate in which three values are required to specify an individual.

"Colour space": Every colour lies in a 3-D space whose dimensions are (e.g.) RGB, or HSI ("hue, saturation, intensity")

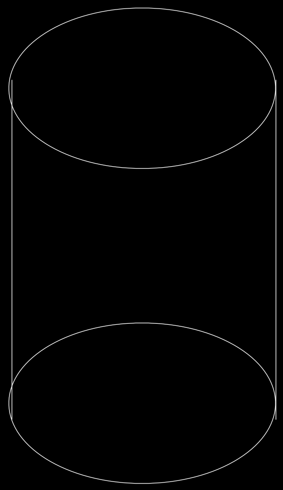


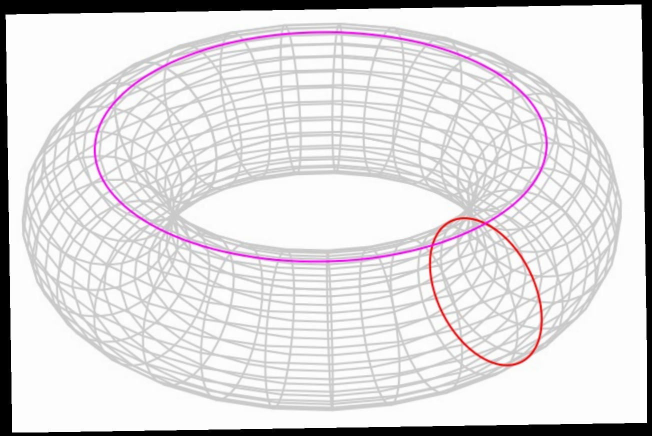


Euclidean space is a manifold whose elements can be thought of as ordered triples of real numbers, i.e., as the space $\mathbb{R}^3$ or the Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

A **cylinder** is a manifold $\mathbb{R} \times \mathbb{S}$, where $\mathbb{S}$ is the circle.

A **torus** is a manifold $\mathbb{S} \times \mathbb{S}$, so that every point lies somewhere on one circle and somewhere on another circle.





To consider 3-D space as “geometrical” space, we need to make the extra assumption that **lengths can be compared**.

To consider a 3-D homogeneous geometrical space, we need to add the further assumption of **free mobility of rigid bodies**.

In a **non-homogeneous space**, i.e. a space of variable curvature, we assume only that “infinitesimal” lengths can be compared, and that the curvature varies from point to point.

A **“differentiable manifold”** is a manifold on which all derivatives are defined, i.e. on which calculus is possible.

A **“Riemannian manifold”** is any continuous manifold that is “locally flat,” or locally Euclidean, but whose geometry varies continuously from point to point.

A **“Lorentzian manifold”** is one that is 4-D and **locally Minkowskian**, rather than Euclidean.

Scalar quantity: one that can be specified by a simple magnitude. An example of a **scalar field** is the distribution of heat on the surface of a frying pan. At every point on the disc, there is a value for the temperature at that point.

Vector quantity: one that must be specified by both by a magnitude and by a direction. An example of a **vector field** is the flow of heat in a convection oven. At every point inside the oven, there is a value for both the temperature at that point and the direction in which heat is flowing.

Tensor quantity: one that is a function of some number of vectors and yields a real number. An example of a tensor is the inner product, which takes two vectors and yields a real number. An example of a **tensor field** is the stress on a body that is subject to multiple forces (e.g. sagging shelf in a gravitational field). At every point there are stresses pulling in three independent directions.

$$ds^2 = \sum_{a,b} g_{ab}\,dx_a dx_b = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{12} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} \begin{pmatrix} dx_1dx_1 & dx_1dx_2 & dx_1dx_3 & dx_1dx_4 \\ dx_2dx_1 & dx_2dx_2 & dx_2dx_3 & dx_2dx_4 \\ dx_3dx_1 & dx_3dx_2 & dx_3dx_3 & dx_3dx_4 \\ dx_4dx_1 & dx_4dx_2 & dx_4dx_3 & dx_4dx_4 \end{pmatrix}$$

$$\begin{aligned} &= g_{11}dx_1dx_1 + g_{12}dx_1dx_2 + g_{13}dx_1dx_3 + g_{14}dx_1dx_4 \\ &+ g_{21}dx_2dx_1 + g_{22}dx_2dx_2 + g_{23}dx_2dx_3 + g_{24}dx_2dx_4 \\ &+ g_{31}dx_3dx_1 + g_{32}dx_3dx_2 + g_{33}dx_3dx_3 + g_{34}dx_3dx_4 \\ &+ g_{41}dx_4dx_1 + g_{42}dx_4dx_2 + g_{43}dx_4dx_3 + g_{44}dx_4dx_4 \end{aligned}$$

$$ds^2 = \sum_{a,b} g_{ab}\,dx_a dx_b = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{12} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} \begin{pmatrix} dx_1dx_1 & dx_1dx_2 & dx_1dx_3 & dx_1dx_4 \\ dx_2dx_1 & dx_2dx_2 & dx_2dx_3 & dx_2dx_4 \\ dx_3dx_1 & dx_3dx_2 & dx_3dx_3 & dx_3dx_4 \\ dx_4dx_1 & dx_4dx_2 & dx_4dx_3 & dx_4dx_4 \end{pmatrix}$$

$$\begin{aligned} &= g_{11}dx_1dx_1 + g_{12}dx_1dx_2 + g_{13}dx_1dx_3 + g_{14}dx_1dx_4 \\ &+ g_{21}dx_2dx_1 + g_{22}dx_2dx_2 + g_{23}dx_2dx_3 + g_{24}dx_2dx_4 \\ &+ g_{31}dx_3dx_1 + g_{32}dx_3dx_2 + g_{33}dx_3dx_3 + g_{34}dx_3dx_4 \\ &+ g_{41}dx_4dx_1 + g_{42}dx_4dx_2 + g_{43}dx_4dx_3 + g_{44}dx_4dx_4 \end{aligned}$$

We can arrive at the Minkowski metric as the special case where the matrix is:

$$g_{ac} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

so that all but the diagonal terms cancel, and we have the familiar Minkowski formula:

$$ds^2 = dx_1^2 - dx_2^2 - dx_3^2 - dx_4^2$$