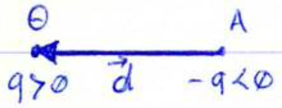
(ἀντί για §4.3)

Η Χαμιλτονιανή της αλληλεπίδρασης ΑΤΟΜΟΥ (διστάθμικού) — Η/Μ πεδίου

- $\Lambda Q \; \boxed{M} \; \omega \Rightarrow$ $$\boxed{E_x^m(z,t) = \left(\frac{2 M_m \omega_m^2}{\varepsilon_0 V}\right)^{1/2} \cdot \sin\left(\frac{m\pi z}{L}\right) q_m(t)}$$ ΛQ

$\Lambda \dot{Q} \; \boxed{M} \; \omega \Rightarrow$ $$\boxed{B_y^m(z,t) = \frac{1}{c}\left(\frac{2 M_m}{\varepsilon_0 V}\right)^{1/2} \cdot \cos\left(\frac{m\pi z}{L}\right) \dot{q}_m(t)}$$ $\Lambda \dot{Q}$

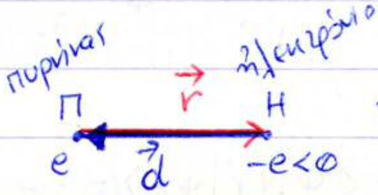
- electric dipole moment



$\vec{d} = \vec{A\Theta}$

$\vec{p} = q\vec{d}$

π.χ.



$\vec{p} = q\vec{d} = e(-\vec{r}) \Rightarrow$

$\vec{p} = -e\vec{r}$

potential energy

$U_\varepsilon = -\vec{p} \cdot \vec{\mathcal{E}}$

- transition (electric) dipole moment

$\vec{p} = e\vec{d} = e(-\vec{r}) \Rightarrow$

$\vec{p} = -e\vec{r}$

- transition (electric) dipole moment OPERATOR

The transition electric dipole moment operator in the basis of the energy eigenstates reads:

$$\boxed{\hat{\vec{d}} = \hat{\vec{p}} = \sum_{i=1}^{N} \sum_{j=1}^{N} \vec{d}_{ij} \, |\Phi_i\rangle\langle\Phi_j|}$$

ΕΝΑΛΛΑΚΤΙΚΟΙ ΣΥΜΒΟΛΙΣΜΟΙ

$$\boxed{\vec{d}_{ij} = -e\langle\Phi_i|\hat{\vec{r}}|\Phi_j\rangle \;✡\; = \vec{p}_{ij}}$$

ΕΝΑΛΛΑΚΤΙΚΟΙ ΣΥΜΒΟΛΙΣΜΟΙ

(electric) transition dipole moment matrix element between the levels $|\Phi_i\rangle$ and $|\Phi_j\rangle$

Υπενθύμιση: Dirac notation $|A\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\langle A| = (a_1^* \; a_2^* \; a_3^*)$

$\hat{\vec{r}}|\vec{r}\rangle = \vec{r}|\vec{r}\rangle$ $\quad \hat{\vec{r}}$ position operator

Ἀς δοῦμε τι κάνει ὁ $\hat{\vec{p}}$ σε διστάθμικό σύστημα

$$\hat{\vec{p}} = \vec{d}_{11}|\Phi_1\rangle\langle\Phi_1| + \vec{d}_{12}|\Phi_1\rangle\langle\Phi_2| + \vec{d}_{21}|\Phi_2\rangle\langle\Phi_1| + \vec{d}_{22}|\Phi_2\rangle\langle\Phi_2|$$

$$= \vec{d}_{11}\begin{pmatrix} 0 \\ 1 \end{pmatrix}(0 \; 1) + \vec{d}_{12}\begin{pmatrix} 0 \\ 1 \end{pmatrix}(1 \; 0) + \vec{d}_{21}\begin{pmatrix} 1 \\ 0 \end{pmatrix}(0 \; 1) + \vec{d}_{22}\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \; 0)$$

$$= \underbrace{\cancel{\vec{d}_{11}}}_{0} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \vec{d}_{12}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \vec{d}_{21}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \underbrace{\cancel{\vec{d}_{22}}}_{0}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

διαγώνιο στοιχείο ✡	τμήμα διαγώνιου πίνακα	μη διαγώνιο στοιχείο	ΤΜΗΜΑ ΑΝΤΙΔΙΑΓΩΝΙΟΥ ΠΙΝΑΚΑ	μη διαγώνιο στοιχείο	ΤΜΗΜΑ ΑΝΤΙΔΙΑΓΩΝΙΟΥ ΠΙΝΑΚΑ	διαγώνιο στοιχείο ✡	τμήμα διαγώνιου πίνακα

υπενθυμίζεται ότι

$$\langle \Phi_i|\vec{r}|\Phi_j\rangle = \sum_{\substack{|\vec{r}'\rangle,\\ |\vec{r}''\rangle}} \langle \Phi_i|\vec{r}'\rangle \underbrace{\langle \vec{r}'|\hat{\vec{r}}|\vec{r}''\rangle}_{=\vec{r}''\delta_{\vec{r}',\vec{r}''}} \langle \vec{r}''|\Phi_j\rangle$$

$$= \sum_{|\vec{r}'\rangle} \langle \Phi_i|\vec{r}'\rangle\, \vec{r}'\, \langle \vec{r}'|\Phi_j\rangle$$

$$= \int dV\, \Phi_i(\vec{r})^*\, \vec{r}\, \Phi_j(\vec{r})$$

Άρα

$$\vec{d}_{11} = -e\langle \Phi_1|\hat{\vec{r}}|\Phi_1\rangle = -e\int dV\, \underbrace{\Phi_1^*(\vec{r})\,\vec{r}\,\Phi_1(\vec{r})}_{\text{περιττή}} = 0$$

$$\left.\begin{aligned} \vec{d}_{12} &= -e\langle \Phi_1|\hat{\vec{r}}|\Phi_2\rangle = -e\int dV\, \Phi_1^*(\vec{r})\,\vec{r}\,\Phi_2(\vec{r}) \neq 0 \\ \vec{d}_{21} &= -e\langle \Phi_2|\vec{r}|\Phi_1\rangle = -e\int dV\, \Phi_2^*(\vec{r})\,\vec{r}\,\Phi_1(\vec{r}) \neq 0 \end{aligned}\right\} \vec{d}_{12} = \vec{d}_{21}$$

$$\vec{d}_{22} = -e\langle \Phi_2|\vec{r}|\Phi_2\rangle = -e\int dV\, \Phi_2^*(\vec{r})\,\vec{r}\,\Phi_2(\vec{r}) = 0$$

Άρα $\hat{\vec{\wp}} = \vec{d}_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. ΕΡΩΤΗΣΗ: Και τι κάνει ο τελεστής $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

ΑΠΑΝΤΗΣΗ

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Μεταφέρει από τη μία κατάσταση στην άλλη, όπως έπρεπε!

- potential energy operator

$$U_\varepsilon^m = -\vec{\wp}\cdot\vec{\mathcal{E}} \Rightarrow \hat{U}_\varepsilon^m = -\hat{\vec{\wp}}\cdot\hat{\vec{\mathcal{E}}} = -\sum_{i=1}^{N}\sum_{j=1}^{N} \vec{d}_{ij}\,|\Phi_i\rangle\langle\Phi_j| \cdot \hat{E}_x^m(z,t)\,\underset{\text{μοναδιαίο διάνυσμα άξονα x}}{\hat{i}}$$

$$\Rightarrow \hat{U}_\varepsilon^m = \vec{d}_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot \hat{E}_x^m(z,t)\,\hat{i}$$

$$\hat{E}_x^m(z,t) \overset{\text{ΓΕΝΙΚΑ}}{=\!=} \underset{\text{πλάτος}}{E_0^m}\,\hat{f}(t)$$

$$\vec{d}_{12}\cdot\hat{i} = -e\int dV\, \Phi_1^*(\vec{r})\,(\vec{r}\cdot\hat{i})\,\Phi_2(\vec{r}) = -e\,X_{12} = \wp_{x12} = \wp$$

$$\hat{U}_\varepsilon^m = \wp E_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{f}(t) \qquad \wp E_0^m = \hbar\,\Omega_R \quad \text{συχνότητα Rabi}$$

ΣΗΜΕΙΩΣΗ: Θεωρούμε ότι η αλληλεπίδραση ΑΤΟΜΟΥ – ΗΜ πεδίου έχει τη μορφή μηχανισμού ηλεκτρικού διπόλου. Αγνοούμε άλλες μορφές αλληλεπιδράσεις, όπως π.χ. ηλεκτρικού τετραπόλου ή μαγνητικού διπόλου εδώ.

$$\Omega_R = \frac{\wp E_0^m}{\hbar} \quad \text{συχνότητα Rabi}$$

Υπενθυμίζεται ακόμη ότι είχαμε αποδείξει τις σχέσεις (Λa†a) κ (Λ'a†a)

$$\boxed{\begin{aligned}\hat{E}_x^m(z,t) &= \left(\frac{\hbar\omega_m}{\varepsilon_0 V}\right)^{1/2}\cdot\sin\left(\frac{m\pi z}{L}\right)\left(\hat{a}_m^\dagger+\hat{a}_m\right) \quad (\Lambda a^\dagger a)\\ \hat{B}_y^m(z,t) &= \left(\frac{\hbar\omega_m}{\varepsilon_0 V}\right)^{1/2}\frac{1}{c}\cdot\cos\left(\frac{m\pi z}{L}\right) i\left(\hat{a}_m^\dagger-\hat{a}_m\right) \quad (\Lambda' a^\dagger a)\end{aligned}}$$

ΑΡΑ

$$\hat{U}_\varepsilon^m = -e x_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix}\left(\frac{\hbar\omega_m}{\varepsilon_0 V}\right)^{1/2}\cdot\sin\left(\frac{m\pi z}{L}\right)\cdot\left(\hat{a}_m^\dagger+\hat{a}_m\right) \quad (U1) \qquad \sim 4.38$$

ή από τη σχέση (Λa)

$$\hat{U}_\varepsilon^m = -e x_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix}\left(\frac{2M_m\omega_m^2}{\varepsilon_0 V}\right)^{1/2}\sin\left(\frac{m\pi z}{L}\right)\hat{q}_m(t) \quad (U2) \qquad \sim 4.37$$

Υπενθυμίζεται ακόμη ότι $\hat{S}_+ + \hat{S}_- = \begin{pmatrix}0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$

Άρα ενταξει

$$\hat{U}_\varepsilon^m = -e x_{12}\left(\frac{\hbar\omega_m}{\varepsilon_0 V}\right)^{1/2}\cdot\sin\left(\frac{m\pi z}{L}\right)\left(\hat{S}_+ + \hat{S}_-\right)\left(\hat{a}_m^\dagger+\hat{a}_m\right)$$

θέτω $\boxed{\hbar g^m = -e x_{12}\left(\frac{\hbar\omega_m}{\varepsilon_0 V}\right)^{1/2}\sin\left(\frac{m\pi z}{L}\right)}$ (g) $\Rightarrow$

$$\boxed{\hat{U}_\varepsilon^m = \hbar g^m\left(\hat{S}_+ + \hat{S}_-\right)\left(\hat{a}_m^\dagger+\hat{a}_m\right)}$$

$\hat{H}_{AF}$ χαμιλτονιανή αλληλεπιδράσεως ατόμου – ΗΜ πεδίου

εναλλακτικός συμβολισμός $\hat{H}_{AF}^m$

Άρα μέχρι τώρα είδαμε

$$\left.\begin{aligned}&\hat{H}_{HM}^m = \hbar\omega_m\left(\hat{a}_m^\dagger\hat{a}_m+\frac{1}{2}\right) \xrightarrow{\text{αγνοώντας } \frac{\hbar\omega_m}{2}} \hat{H}_{HM}^m = \hbar\omega_m\hat{a}_m^\dagger\hat{a}_m\\ &\hat{H}_{atom} = E_2\hat{S}_+\hat{S}_- + E_1\hat{S}_-\hat{S}_+ \xrightarrow[E_1=0]{\text{θέτοντας}} \hat{H}_{atom} = \hbar\Omega\hat{S}_+\hat{S}_-\\ &\hat{U}_\varepsilon^m = \hbar g^m\left(\hat{S}_+ + \hat{S}_-\right)\left(\hat{a}_m^\dagger+\hat{a}_m\right)\end{aligned}\right\}$$

$$\Rightarrow \boxed{\hat{H}^m = \hbar\omega_m\hat{a}_m^\dagger\hat{a}_m + \hbar\Omega\hat{S}_+\hat{S}_- + \hbar g^m\left(\hat{S}_+ + \hat{S}_-\right)\left(\hat{a}_m^\dagger+\hat{a}_m\right)} \qquad \sim 4.41$$

Rabi Hamiltonian describing the interaction of a two-level atom with a single-mode harmonic field

ΟΛΙΚΗ ΧΑΜΙΛΤΟΝΙΑΝΗ

$\boxed{|\uparrow, n_m\rangle \text{ και } |\downarrow, n_m\rangle}$ ΟΛΙΚΕΣ ΚΑΤΑΣΤΑΣΕΙΣ

Απ4

(M1)

<u>ΣΗΜΕΙΩΣΗ</u> $\hat{H}_{atom} = E_2 \hat{S}_+ \hat{S}_- + E_1 \hat{S}_- \hat{S}_+$

(M2) εναλλακτική γραφή π.χ. Rossi $\hat{H}_{atom} = E_2 \hat{a}_2^+ a_2 + E_1 \hat{a}_1^+ \hat{a}_1$

$\hat{a}_i^+, \hat{a}_i$ τελεστές δημιουργίας (αναβιβάζοντας) και καταστροφής (καταβιβάζοντας)

$\hat{S}_+ \hat{S}_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \hat{S}_- \hat{S}_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\hat{a}_2^+ \hat{a}_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{a}_2^+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \hat{a}_1^+ \hat{a}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{a}_1^+ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\hat{a}_2^+ \hat{a}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hat{a}_2^+ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \hat{a}_1^+ \hat{a}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hat{a}_1^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Steck εναλλακτική μορφή $\hat{H}_{atom} = E_2 |2\rangle\langle 2| + E_1 |1\rangle\langle 1|$ (M3)

ΑΝ $\hat{a}_2^+ = |2\rangle\langle 1|$ $\quad \hat{a}_2 = |1\rangle\langle 2|$ $\boxed{0}$

$\hat{a}_1^+ = |1\rangle\langle 2|$ $\quad \hat{a}_1 = |2\rangle\langle 1|$

$\Rightarrow \hat{a}_1^+ = \hat{a}_2$

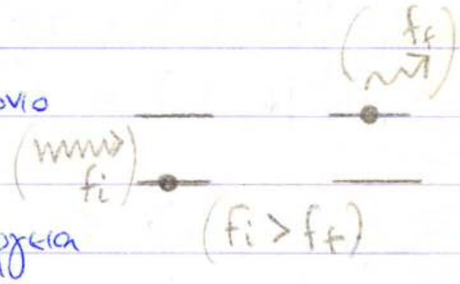
$\hat{a}_2^+ = \hat{a}_1$

(M2) $\hat{H}_{atom} = E_2 |2\rangle\langle 1|1\rangle\langle 2| + E_1 |1\rangle\langle 2|2\rangle\langle 1| \Rightarrow \hat{H}_{atom} = E_2 |2\rangle\langle 2| + E_1 |1\rangle\langle 1|$

$\boxed{0}$

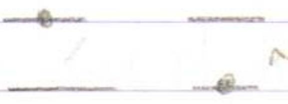
$$\hat{U}_\varepsilon^m = \hbar g^m (\hat{S}_+ + \hat{S}_-)(\hat{a}_m^\dagger + \hat{a}_m) =$$
$$= \hbar g^m \{ \hat{S}_+ \hat{a}_m^\dagger + \hat{S}_+ \hat{a}_m + \hat{S}_- \hat{a}_m^\dagger + \hat{S}_- \hat{a}_m \}$$

1ος 2ος 3ος 4ος

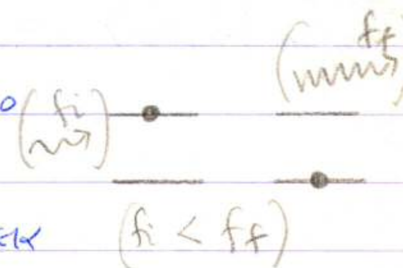


1ος ΟΡΟΣ Το ηλεκτρόνιο ανεβαίνει και δημιουργείται (εκπέμπεται) φωτόνιο
Δ(Ενέργειας) > 0
αυτός ο όρος μόνος του δεν διατηρεί την ενέργεια

2ος ΟΡΟΣ Το ηλεκτρόνιο ανεβαίνει και απορροφάται (καταστρέφεται) φωτόνιο



3ος ΟΡΟΣ Το ηλεκτρόνιο κατεβαίνει και εκπέμπεται (δημιουργείται) φωτόνιο



4ος ΟΡΟΣ Το ηλεκτρόνιο κατεβαίνει και καταστρέφεται (απορροφάται) φωτόνιο
Δ(Ενέργειας) < 0
αυτός ο όρος μόνος του δεν διατηρεί την ενέργεια

Αν αγνοήσουμε τον 1ο κ 4ο όρο που ο καθένας μόνος του δεν διατηρεί την ενέργεια

$$U_\varepsilon^m \simeq \hbar g^m \{ \hat{S}_+ \hat{a}_m + \hat{S}_- \hat{a}_m^\dagger \}$$

Η προσέγγιση είναι κατά κάποιο τρόπο ανάλογη της RWA στις εξισώσεις Rabi όπου κρατήσαμε μόνο τους αργούς όρους όπου ΗΜ πεδίο και άτομο βρίσκονται περίπου σε συντονισμό ($\Omega \simeq \omega$)

Τότε κρατήσαμε $e^{\pm i(\Omega - \omega)t}$ αργοί όροι

και αγνοήσαμε $e^{\pm i(\Omega + \omega)t}$ γρήγοροι όροι

Rabi Hamiltonian

$$\hat{H}_R^m = \hbar\omega_m \hat{a}_m^\dagger \hat{a}_m + \hbar\Omega \hat{S}_+ \hat{S}_- + \hbar g^m (\hat{S}_+ \hat{a}_m^\dagger + \hat{S}_+ \hat{a}_m + \hat{S}_- \hat{a}_m^\dagger + \hat{S}_- \hat{a}_m)$$

"the so-called counter-rotating term" $\hbar g^m (\hat{S}_+ \hat{a}_m^\dagger + \hat{S}_- \hat{a}_m)$ αγνοείται σε πρώτη προσέγγιση

$$\hat{H}_{JC}^m = \hbar\omega_m \hat{a}_m^\dagger \hat{a}_m + \hbar\Omega \hat{S}_+ \hat{S}_- + \hbar g^m (\hat{S}_+ \hat{a}_m + \hat{S}_- \hat{a}_m^\dagger)$$

Jaynes-Cummings Hamiltonian

ΑΣΚΗΣΗ Βρείτε τι κάνουν οι όροι $\hat{S}_+\hat{a}_m^+$, $\hat{S}_+\hat{a}_m$, $\hat{S}_-\hat{a}_m^+$, $\hat{S}_-\hat{a}_m$ στις καταστάσεις $|\uparrow, n_m\rangle$ και $|\downarrow, n_m\rangle$

$$\hat{S}_+\hat{a}_m^+|\uparrow, n_m\rangle = \sqrt{n_m+1}\,|0, n_m+1\rangle$$
$$\hat{S}_+\hat{a}_m^+|\downarrow, n_m\rangle = \sqrt{n_m+1}\,|\uparrow, n_m+1\rangle \qquad \Delta E > 0$$

$$\hat{S}_+\hat{a}_m|\uparrow, n_m\rangle = \sqrt{n_m}\,|0, n_m-1\rangle$$
$$\hat{S}_+\hat{a}_m|\downarrow, n_m\rangle = \sqrt{n_m}\,|\uparrow, n_m-1\rangle$$

$$\hat{S}_-\hat{a}_m^+|\uparrow, n_m\rangle = \sqrt{n_m+1}\,|\downarrow, n_m+1\rangle$$
$$\hat{S}_-\hat{a}_m^+|\downarrow, n_m\rangle = \sqrt{n_m+1}\,|0, n_m+1\rangle$$

$$\hat{S}_-\hat{a}_m|\uparrow, n_m\rangle = \sqrt{n_m}\,|\downarrow, n_m-1\rangle \qquad \Delta E < 0$$
$$\hat{S}_-\hat{a}_m|\downarrow, n_m\rangle = \sqrt{n_m}\,|0, n_m-1\rangle$$

$\Downarrow 2\hat{a}_r^+\hat{a}_r^+ = 0 \Rightarrow \hat{a}_r^+\hat{a}_r^+ = 0$ Pauli's exclusion principle

α

Σχέσεις μεταθέσεως και αντιμεταθέσεως

$$[A,B]_\pm := AB \pm BA$$

FERMIONS + anticommutator αντιμεταθέτης $[A,B]_+ \equiv \{A,B\}$

BOSONS − commutator μεταθέτης $[A,B]_- \equiv [A,B]$

FERMIONS π.χ. ηλεκτρόνια $\{\hat{a}_i^+, \hat{a}_j\} = \delta_{ij}$ $\quad \{\hat{a}_i, \hat{a}_j\} = 0 \Rightarrow \hat{a}_i\hat{a}_j = -\hat{a}_j\hat{a}_i$ $\quad \{\hat{a}_i^+, \hat{a}_j^+\} = 0$

ἐξ οὗ το όνομα "αντιμεταθέτης"

σχέσεις μεταθέσεως για φερμιόνια

BOSONS π.χ. φωτόνια $[\hat{a}_m, \hat{a}_n^+] = \delta_{mn}$ $\quad [\hat{a}_m, \hat{a}_n] = 0 \Rightarrow \hat{a}_m\hat{a}_n = \hat{a}_n\hat{a}_m$ $\quad [\hat{a}_m^+, \hat{a}_n^+] = 0$

ἐξ οὗ το όνομα "μεταθέτης"

σχέσεις μεταθέσεως για μποζόνια

Χαμιλτονιανές

1 $\hat{H}_{HM}^m = \hbar\omega_m\left(\hat{a}_m^+\hat{a}_m + \frac{1}{2}\right) \rightsquigarrow \hbar\omega_m\hat{a}_m^+\hat{a}_m$ αγνοώντας το $\frac{\hbar\omega_m}{2}$ μποζόνια

m τρόπος ΗΜ πεδίου

2 $\hat{H}_{atom} = E_2\hat{S}_+\hat{S}_- + E_1\hat{S}_-\hat{S}_+ \rightsquigarrow \hbar\Omega\hat{S}_+\hat{S}_-$ θέτοντας $E_1 = 0$ φερμιόνια

ηλεκτρόνιο στο άτομο

$$\overset{\text{ή}}{=} E_2\hat{a}_2^+\hat{a}_2 + E_1\hat{a}_1^+\hat{a}_1$$

$$\overset{\text{ή}}{=} E_2|2\rangle\langle 2| + E_1|1\rangle\langle 1| = E_2|2\rangle\langle 1|1\rangle\langle 2| + E_1|1\rangle\langle 2|2\rangle\langle 1|$$

δηλαδή $\hat{a}_2^+ = \hat{a}_1 = \hat{S}_+$ $\quad \hat{a}_2^+ = |2\rangle\langle 1|$ $\quad \hat{S}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\hat{a}_1^+ = \hat{a}_2 = \hat{S}_-$ $\quad \hat{a}_1^+ = |1\rangle\langle 2|$ $\quad \hat{S}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

γενικότερα $\hat{H}_{atom} = \sum_{i=1}^{N} E_i\hat{a}_i^+\hat{a}_i$

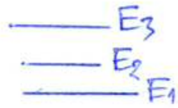
3 $\hat{H}_{AF}^m = \hbar g^m(\hat{S}_+ + \hat{S}_-)(\hat{a}_m^+ + \hat{a}_m) \rightsquigarrow \hbar g^m(\hat{S}_+\hat{a}_m + \hat{S}_-\hat{a}_m^+)$

$\hbar g^m = -e x_{12}\left(\frac{\hbar\omega_m}{\varepsilon_0 V}\right)^{1/2} \cdot \sin\left(\frac{m\pi z}{L}\right)$ $\qquad \hat{H}_{AF}^m \equiv \hat{U}_E^m$

αλληλεπίδραση ατόμου − ΗΜ πεδίου (m τρόπος)

2

• Ας δούμε παράδειγμα με 3 στάθμες



$$\hat{H}_{atom} = E_3 \hat{a}_3^{+} \hat{a}_3 + E_2 \hat{a}_2^{+} \hat{a}_2 + E_1 \hat{a}_1^{+} \hat{a}_1$$

Ισχυρίζομαι ότι $\hat{a}_2^{+} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \hat{a}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ μορφή πίνακα

έλεγχος

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{a}_2^{+}|3\rangle = |2\rangle \qquad \hat{a}_2^{+}|2\rangle = |0\rangle \qquad \hat{a}_2^{+}|1\rangle = |2\rangle$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{a}_2|3\rangle = |0\rangle \qquad \hat{a}_2|2\rangle = |1\rangle + |3\rangle \qquad \hat{a}_2|1\rangle = |0\rangle$$

Ανάλογως $\hat{a}_1^{+} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow \hat{a}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

έλεγχος

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\hat{a}_1^{+}|1\rangle = |0\rangle \qquad \hat{a}_1^{+}|2\rangle = |1\rangle \qquad \hat{a}_1^{+}|3\rangle = |1\rangle$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{a}_1|1\rangle = |2\rangle + |3\rangle \qquad \hat{a}_1|2\rangle = |0\rangle \qquad \hat{a}_1|3\rangle = |0\rangle$$

και επίσης $\hat{a}_3^{+} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \hat{a}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

έλεγχος

μορφή ket bra

$$\hat{a}_2^{+} = |2\rangle\langle 1| + |2\rangle\langle 3|$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{a}_2 = |1\rangle\langle 2| + |3\rangle\langle 2|$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{a}_2^+ = \underline{|2\rangle\langle 1|} + \underline{|2\rangle\langle 3|}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix}(0\ 0\ 1) + \begin{pmatrix}0\\1\\0\end{pmatrix}(1\ 0\ 0) = \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}0&0&0\\1&0&1\\0&0&0\end{pmatrix}$$

Τι σχέση έχουν τα $\hat{a}_2^+$ $\hat{a}_1, \hat{a}_3$

$$\hat{a}_2 = |1\rangle\langle 2| + |3\rangle\langle 2|$$

$$\hat{a}_1 = \underline{|2\rangle\langle 1|} + |3\rangle\langle 1|$$

$$T_{13} = |3\rangle\langle 1| + |1\rangle\langle 3|$$

$$\hat{a}_3 = |1\rangle\langle 3| + \underline{|2\rangle\langle 3|}$$

ΥΠΗΡΞΕ ΕΡΩΤΗΣΗ

$$\hat{a}_1 + \hat{a}_3 = \hat{a}_2^+ + \hat{T}_{13}$$

$$\hat{H}_R^m = \hbar\omega_m \hat{a}_m^+ \hat{a}_m + \hbar\Omega \hat{S}_+\hat{S}_- + \hbar g^m (\hat{S}_+ + \hat{S}_-)(\hat{a}_m^+ + \hat{a}_m)$$

Rabi Hamiltonian describing the interaction of a two level atom with a single-mode harmonic field

$\hbar g^m(\hat{S}_+ \hat{a}_m^+ + \hat{S}_- \hat{a}_m)$ the so-called counter-rotating terms (αγνοούμε)

$$\hat{H}_{JC}^m = \hbar\omega_m \hat{a}_m^+ \hat{a}_m + \hbar\Omega \hat{S}_+\hat{S}_- + \hbar g^m (\hat{S}_+ \hat{a}_m + \hat{S}_- \hat{a}_m^+)$$

Jaynes - Cummings Hamiltonian

Να υπολογιστούν τα $\langle \hat{a}_m^+ \hat{a}_m \rangle$, $\langle \hat{S}_+\hat{S}_- \rangle$, $\langle \hat{S}_+ \hat{a}_m \rangle$, $\langle \hat{S}_- \hat{a}_m \rangle$ για τις καταστάσεις

Ⓐ $|\psi_A(t)\rangle = c_1(t)|\downarrow, n\rangle + c_2(t)|\uparrow, n-1\rangle$

Ⓔ $|\psi_E(t)\rangle = c_1(t)|\downarrow, n+1\rangle + c_2(t)|\uparrow, n\rangle$

Ⓐ

$$\langle \hat{a}_m^+ \hat{a}_m \rangle_{A} = \langle \psi_A(t)| \hat{a}_m^+ \hat{a}_m |\psi_A(t)\rangle = \{c_1^*\langle\downarrow,n| + c_2^*\langle\uparrow,n-1|\} \hat{a}_m^+ \hat{a}_m \{c_1|\downarrow,n\rangle + c_2|\uparrow,n-1\rangle\}$$

$$= |c_1|^2 \langle\downarrow,n|\hat{a}_m^+ \hat{a}_m|\downarrow,n\rangle + c_1^* c_2 \langle\downarrow,n|\hat{a}_m^+ \hat{a}_m|\uparrow,n-1\rangle$$

$$+ c_2^* c_1 \langle\uparrow,n-1|\hat{a}_m^+ \hat{a}_m|\downarrow,n\rangle + |c_2|^2 \langle\uparrow,n-1|\hat{a}_m^+ \hat{a}_m|\uparrow,n-1\rangle$$

$$= |c_1|^2 \sqrt{n}\sqrt{n}\langle\downarrow,n|\downarrow,n\rangle + c_1^* c_2 \sqrt{n-1}\sqrt{n-1}\langle\downarrow,n|\uparrow,n-1\rangle$$

$$+ c_2^* c_1 \sqrt{n}\sqrt{n}\langle\uparrow,n-1|\downarrow,n\rangle + |c_2|^2\sqrt{n-1}\sqrt{n-1}\langle\uparrow,n-1|\uparrow,n-1\rangle =$$

$$= n\cdot|c_1|^2\cdot 1 + c_1^* c_2 (n-1)\cdot 0 + c_2^* c_1 \cdot n \cdot 0 + (n-1)|c_2|^2\cdot 1 =$$

$$= n|c_1|^2 + n|c_2|^2 - |c_2|^2 = n(|c_1|^2 + |c_2|^2) - |c_2|^2 = n - |c_2|^2 \Rightarrow$$

$$\boxed{\langle \hat{a}_m^+ \hat{a}_m \rangle_{A} = n - |c_2(t)|^2} \checkmark \qquad (4.49)$$

$$\langle \hat{S}_+\hat{S}_- \rangle_{A} = \langle \psi_A(t)| \hat{S}_+\hat{S}_- |\psi_A(t)\rangle = \{c_1^*\langle\downarrow,n| + c_2^*\langle\uparrow,n-1|\} \hat{S}_+\hat{S}_- \{c_1|\downarrow,n\rangle + c_2|\uparrow,n-1\rangle\}$$

$$= |c_1|^2\langle\downarrow,n|\hat{S}_+\hat{S}_-|\downarrow,n\rangle + c_1^* c_2 \langle\downarrow,n|\hat{S}_+\hat{S}_-|\uparrow,n-1\rangle$$

$$+ c_2^* c_1 \langle\uparrow,n-1|\hat{S}_+\hat{S}_-|\downarrow,n\rangle + |c_2|^2\langle\uparrow,n-1|\hat{S}_+\hat{S}_-|\uparrow,n-1\rangle =$$

$$= |c_1|^2\cdot 0 + c_1^* c_2 \underbrace{\langle\downarrow,n|\uparrow,n-1\rangle}_{0} + c_2^* c_1 \cdot 0 + |c_2|^2 \underbrace{\langle\uparrow,n-1|\uparrow,n-1\rangle}_{1} \Rightarrow$$

$$\boxed{\langle \hat{S}_+\hat{S}_- \rangle_{A} = |c_2(t)|^2} \checkmark \qquad (4.49)$$

ΑΡΑ $\boxed{\langle \hat{a}_m^+ \hat{a}_m \rangle_{A} + \langle \hat{S}_+\hat{S}_- \rangle_{A} = n} \checkmark \qquad (4.50)$

$$\langle \hat{S}_+ \hat{a}_m \rangle_{A} = \langle \psi_A(t) | \hat{S}_+ \hat{a}_m | \psi_A(t) \rangle = \{ c_1^* \langle \downarrow, n | + c_2^* \langle \uparrow, n-1 | \} \hat{S}_+ \hat{a}_m \{ c_1 | \downarrow, n \rangle + c_2 | \uparrow, n-1 \rangle \}$$
$$= |c_1|^2 \langle \downarrow, n | \hat{S}_+ \hat{a}_m | \downarrow, n \rangle + c_1^* c_2 \langle \downarrow, n | \hat{S}_+ \hat{a}_m | \uparrow, n-1 \rangle +$$
$$+ c_2^* c_1 \langle \uparrow, n-1 | \hat{S}_+ \hat{a}_m | \downarrow, n \rangle + |c_2|^2 \langle \uparrow, n-1 | \hat{S}_+ \hat{a}_m | \uparrow, n-1 \rangle =$$
$$= |c_1|^2 \sqrt{n} \langle \downarrow, n | \uparrow, n-1 \rangle + c_1^* c_2 \cdot \sqrt{n-1} \cdot \langle \downarrow, n | \hat{S}_+ | \uparrow, n-2 \rangle$$
$$+ c_2^* c_1 \sqrt{n} \langle \uparrow, n-1 | \uparrow, n-1 \rangle + |c_2|^2 \sqrt{n-1} \langle \uparrow, n-1 | \underbrace{\hat{S}_+ | \uparrow, n-2 \rangle}_{0} \Rightarrow$$

$$\boxed{\langle \hat{S}_+ \hat{a}_m \rangle_A = c_2^*(t) c_1(t) \cdot \sqrt{n}}$$

$$\langle \hat{S}_- \hat{a}_m^+ \rangle_{A} = \langle \psi_A(t) | \hat{S}_- \hat{a}_m^+ | \psi_A(t) \rangle = \{ c_1^* \langle \downarrow, n | + c_2^* | \uparrow, n-1 \rangle \} \hat{S}_- \hat{a}_m^+ \{ c_1 | \downarrow, n \rangle + c_2 | \uparrow, n-1 \rangle \}$$
$$= |c_1|^2 \langle \downarrow, n | \hat{S}_- \hat{a}_m^+ | \downarrow, n \rangle + c_1^* c_2 \langle \downarrow, n | \hat{S}_- \hat{a}_m^+ | \uparrow, n-1 \rangle +$$
$$c_2^* c_1 \langle \uparrow, n-1 | \hat{S}_- \hat{a}_m^+ | \downarrow, n \rangle + |c_2|^2 \langle \uparrow, n-1 | \hat{S}_- \hat{a}_m^+ | \uparrow, n-1 \rangle =$$
$$= |c_1|^2 \sqrt{n+1} \langle \downarrow, n | \underbrace{\hat{S}_- | \downarrow, n+1 \rangle}_{0} + c_1^* c_2 \sqrt{n} \langle \downarrow, n | \downarrow, n \rangle$$
$$+ c_2^* c_1 \langle \uparrow, n-1 | \underbrace{\hat{S}_- | \downarrow, n+1 \rangle}_{0} \sqrt{n+1} + |c_2|^2 \langle \uparrow, n-1 | \downarrow, n \rangle \sqrt{n} \Rightarrow$$

$$\boxed{\langle \hat{S}_- \hat{a}_m^+ \rangle_{A} = c_1^*(t) c_2(t) \sqrt{n}}$$

E

$$\langle \hat{a}_m^+ \hat{a}_m \rangle_E = \langle \psi_E(t) | \hat{a}_m^+ \hat{a}_m | \psi_E(t) \rangle = \{ c_1^* \langle \downarrow, n+1 | + c_2^* \langle \uparrow, n | \} | \hat{a}_m^+ \hat{a}_m | \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \}$$
$$= |c_1|^2 \langle \downarrow, n+1 | \hat{a}_m^+ \hat{a}_m | \downarrow, n+1 \rangle + c_1^* c_2 \langle \downarrow, n+1 | \hat{a}_m^+ \hat{a}_m | \uparrow, n \rangle$$
$$+ c_2^* c_1 \langle \uparrow, n | \hat{a}_m^+ \hat{a}_m | \downarrow, n+1 \rangle + |c_2|^2 \langle \uparrow, n | \hat{a}_m^+ \hat{a}_m | \uparrow, n \rangle$$
$$= |c_1|^2 \sqrt{n+1}\sqrt{n+1} \langle \downarrow, n+1 | \downarrow, n+1 \rangle + c_1^* c_2\, n \langle \downarrow, n+1 | \uparrow, n \rangle$$
$$+ c_2^* c_1 (n+1) \langle \uparrow, n | \downarrow, n+1 \rangle + |c_2|^2 n \langle \uparrow, n | \uparrow, n \rangle =$$
$$= |c_1|^2 (n+1) + n |c_2|^2 = n(|c_1|^2 + |c_2|^2) + |c_1|^2 \Rightarrow$$

$$\boxed{\langle \hat{a}_m^+ \hat{a}_m \rangle_E = n + |c_1(t)|^2} \checkmark \quad (4.60)$$

$$\langle \hat{S}_+ \hat{S}_- \rangle_E = \langle \psi_E(t) | \hat{S}_+ \hat{S}_- | \psi_E(t) \rangle = \{ c_1^* \langle \downarrow, n+1 | + c_2^* \langle \uparrow, n | \} \hat{S}_+ \hat{S}_- \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \}$$
$$= |c_1|^2 \cdot 0 + c_1^* c_2 \langle \downarrow, n+1 | \uparrow, n \rangle + c_2^* c_1 \cdot 0 + |c_2|^2 \langle \uparrow, n | \uparrow, n \rangle \Rightarrow$$

$$\boxed{\langle \hat{S}_+ \hat{S}_- \rangle_E = |c_2(t)|^2} \checkmark \quad (4.60)$$

$$\langle \hat{S}_+ \hat{a}_m \rangle_E = \langle \psi_E(t) | \hat{S}_+ \hat{a}_m | \psi_E(t) \rangle = \{ c_1^* \langle \downarrow, n+1 | + c_2^* \langle \uparrow, n | \} \hat{S}_+ \hat{a}_m \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \}$$
$$= |c_1|^2 \langle \downarrow, n+1 | \hat{S}_+ \hat{a}_m | \downarrow, n+1 \rangle + c_1^* c_2 \langle \downarrow, n+1 | \hat{S}_+ \hat{a}_m | \uparrow, n \rangle +$$
$$c_2^* c_1 \langle \uparrow, n | \hat{S}_+ \hat{a}_m | \downarrow, n+1 \rangle + |c_2|^2 \langle \uparrow, n | \hat{S}_+ \hat{a}_m | \uparrow, n \rangle$$
$$= |c_1|^2 \langle \downarrow, n+1 | \uparrow, n \rangle \sqrt{n+1} + c_1^* c_2 \cdot 0 + c_2^* c_1 \langle \uparrow, n | \uparrow, n \rangle \sqrt{n+1} + |c_2|^2 0$$

$$\boxed{\langle \hat{S}_+ \hat{a}_m \rangle_E = c_2^*(t)\, c_1(t) \cdot \sqrt{n+1}}$$

$$\langle \hat{S}_- \hat{a}_m^+ \rangle_E = \langle \psi_E(t) | \hat{S}_- \hat{a}_m^+ | \psi_E(t) \rangle = \{ c_1^* \langle \downarrow, n+1 | + c_2^* \langle \uparrow, n | \} \hat{S}_- \hat{a}_m^+ \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \}$$
$$= |c_1|^2 \langle \downarrow, n+1 | \hat{S}_- \hat{a}_m^+ | \downarrow, n+1 \rangle + c_1^* c_2 \langle \downarrow, n+1 | \hat{S}_- \hat{a}_m^+ | \uparrow, n \rangle$$
$$+ c_2^* c_1 \langle \uparrow, n | \hat{S}_- \hat{a}_m^+ | \downarrow, n+1 \rangle + |c_2|^2 \langle \uparrow, n | \hat{S}_- \hat{a}_m^+ | \uparrow, n \rangle =$$
$$= |c_1|^2 \langle \downarrow, n+1 | \uparrow, n+2 \rangle \sqrt{n+2} + c_1^* c_2 \langle \downarrow, n+1 | \downarrow, n+1 \rangle \sqrt{n+1}$$
$$+ c_2^* c_1 \cdot 0 + |c_2|^2 \langle \uparrow, n | \downarrow, n+1 \rangle \sqrt{n+1} \Rightarrow$$

$$\boxed{\langle \hat{S}_- \hat{a}_m^+ \rangle_E = c_1^*(t) \cdot c_2(t) \cdot \sqrt{n+1}}$$

APA $\boxed{\langle \hat{a}_m^+ \hat{a}_m \rangle_E + \langle \hat{S}_+ \hat{S}_- \rangle_E = n+1}$ ✓ (4.61)

$$|\psi_A(t)\rangle = c_1(t)|\downarrow,n\rangle + c_2(t)|\uparrow,n-1\rangle$$

ΠΡΟΒΛΗΜΑ ΑΠΟΡΡΟΦΗΣΗΣ ΦΩΤΟΝΙΟΥ

$$i\hbar\frac{\partial}{\partial t}|\psi_A(t)\rangle = \hat{H}|\psi_A(t)\rangle$$

$$\hat{H} = \hat{H}^m_{JC} = \hbar\omega_m \hat{a}_m^+\hat{a}_m + \hbar\Omega\hat{S}_+\hat{S}_- + \hbar g^m(\hat{S}_+\hat{a}_m + \hat{S}_-\hat{a}_m^+)$$

Α.Σ. $c_1(0)=1 \quad c_2(0)=0 \qquad (4.47)$

$$A' = i\hbar\frac{\partial}{\partial t}|\psi_A(t)\rangle = i\hbar\dot{c}_1|\downarrow,n\rangle + i\hbar\dot{c}_2|\uparrow,n-1\rangle$$

$$\begin{aligned}\Delta' = \hat{H}|\psi_A(t)\rangle &= \left(\hbar\omega_m \hat{a}_m^+\hat{a}_m + \hbar\Omega\hat{S}_+\hat{S}_- + \hbar g^m\hat{S}_+\hat{a}_m + \hbar g^m\hat{S}_-\hat{a}_m^+\right)\left(c_1|\downarrow,n\rangle + c_2|\uparrow,n-1\rangle\right)\\
&= c_1\hbar\omega_m n|\downarrow,n\rangle + c_1\hbar\Omega\cdot 0 + c_1\hbar g^m|\uparrow,n-1\rangle\sqrt{n} + c_1\hbar g^m 0\\
&\quad c_2\hbar\omega_m(n-1)|\uparrow,n-1\rangle + c_2\hbar\Omega|\uparrow,n-1\rangle + \hbar g^m\cdot 0 + \hbar g^m|\downarrow,n\rangle\sqrt{n}\,c_2\\
&= c_1\hbar\omega_m n|\downarrow,n\rangle + c_1\hbar g^m|\uparrow,n-1\rangle\sqrt{n}\\
&\quad c_2\hbar\omega_n(n-1)|\uparrow,n-1\rangle + c_2\hbar\Omega|\uparrow,n-1\rangle + \hbar g^m|\downarrow,n\rangle\sqrt{n}\,c_2\end{aligned}$$

επί $\langle\downarrow,n|$

$$\left.\begin{aligned}A' &= i\hbar\dot{c}_1\\ \Delta' &= c_1\hbar\omega_m n + \hbar g^m\sqrt{n}\,c_2\end{aligned}\right\} \Rightarrow i\dot{c}_1 = n\omega_m c_1 + g^m\sqrt{n}\,c_2$$

επί $\langle\uparrow,n-1|$

$$\left.\begin{aligned}A' &= i\hbar\dot{c}_2\\ \Delta' &= c_1\hbar g^m\sqrt{n} + c_2\hbar\omega_m(n-1) + c_2\hbar\Omega\end{aligned}\right\} \Rightarrow i\dot{c}_2 = g^m\sqrt{n}\,c_1 + (\Omega+(n-1)\omega_m)c_2$$

Δηλαδή

$$\boxed{i\begin{pmatrix}\dot{c}_1\\ \dot{c}_2\end{pmatrix} = \begin{pmatrix}n\omega_m & g^m\sqrt{n}\\ g^m\sqrt{n} & \Omega+(n-1)\omega_m\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}}$$

σύστημα διαφορικών εξισώσεων (4.51)

ορίζουμε

$$\boxed{\Omega_n = \left[\left(\frac{\Omega-\omega}{2}\right)^2 + g^2 n\right]^{1/2}}$$

συχνότητα Rabi (4.52)

ΕΠΙΛΥΟΝΤΑΣ ⇒ ... ⇒ ΑΛΛΟΥ

$$c_1(t) = \exp\left[-i\left(n\omega + \frac{\Omega-\omega}{2}\right)t\right]\cdot\left\{\cos(\Omega_n t) + i\frac{\Omega-\omega}{2\Omega_n}\sin(\Omega_n t)\right\}$$

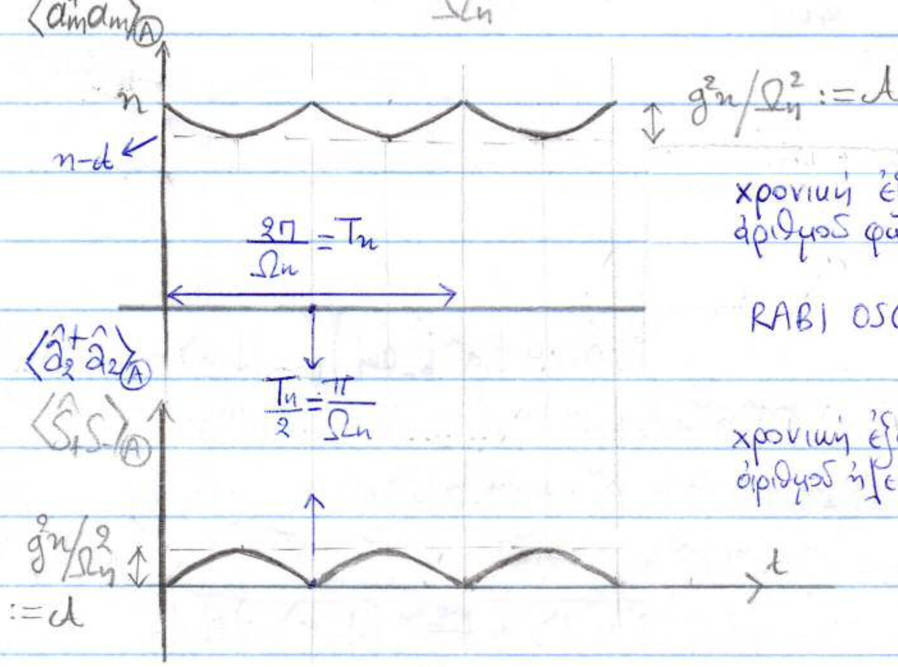
(4.53)

$$c_2(t) = \exp\left[-i\left(n\omega + \frac{\Omega-\omega}{2}\right)t\right]\cdot\left\{-i\frac{g\sqrt{n}}{\Omega_n}\sin(\Omega_n t)\right\}$$

$$\Rightarrow \boxed{|c_2(t)|^2 = \frac{n g^2}{\Omega_n^2}\sin^2(\Omega_n t)} \qquad |c_1(t)|^2 = 1 - |c_2(t)|^2 = \dots \qquad (4.54)$$

ΑΡΑ $\langle \hat{a}_m^{+}\hat{a}_m \rangle_{Ⓐ} = n - \dfrac{g^2 n \cdot \sin^2(\Omega_n t)}{\Omega_n^2}$

$$\langle \hat{S}_+ \hat{S}_- \rangle_{Ⓐ} = \frac{g^2 n \cdot \sin^2(\Omega_n t)}{\Omega_n^2}$$



χρονική εξέλιξη [μέσης] τιμής (αναμενόμενης) αριθμός φωτονίων στην κοιλότητα

RABI OSCILLATIONS

χρονική εξέλιξη [μέσης] τιμής (αναμενόμενης) αριθμού ηλεκτρονίων στην E_2 στάθμη δηλ. στην άνω στάθμη

ΣΥΣΤΗΜΑ ΔΙΑΦΟΡΙΚΩΝ ΕΞΙΣΩΣΕΩΝ

$$\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = (-i)\begin{pmatrix} n\omega & g\sqrt{n} \\ g\sqrt{n} & \Omega + (n-1)\omega \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

ΜΟΡΦΗΣ $\dot{\vec{X}}(t) = \overleftrightarrow{A}\,\vec{X}(t)$

$$\dot{\vec{X}}(t) = \begin{pmatrix} \dot{c}_1(t) \\ \dot{c}_2(t) \end{pmatrix} \quad \vec{X}(t) = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} \quad \overleftrightarrow{A} = (-i)\begin{pmatrix} n\omega & g\sqrt{n} \\ g\sqrt{n} & \Omega + (n-1)\omega \end{pmatrix}$$

$$d = \frac{g^2 n}{\Omega_n^2} = \frac{g^2 n}{\left(\frac{\Omega - \omega}{2}\right)^2 + g^2 n}$$

ΑΡΑ για $\Omega = \omega$ (συντονισμός) $\Rightarrow d = 1$

για $\Omega \neq \omega$ (μη συντονισμός) $\Rightarrow d < 1$

Η συχνότητα Rabi καθορίζει (I) το πλάτος $d = g^2 n/\Omega_n^2$

(II) το χρόνο μεταξύ διαδοχικών μεγίστων ή ελαχίστων $\frac{T_n}{2} = \frac{\pi}{\Omega_n}$

$|\psi_E(t)\rangle = c_1(t)\,|\downarrow, n+1\rangle + c_2(t)\,|\uparrow, n\rangle$

ΠΡΟΒΛΗΜΑ ΕΚΠΟΜΠΗΣ ΦΩΤΟΝΙΟΥ

$i\hbar \frac{\partial}{\partial t}|\psi_E(t)\rangle = \hat{H}\,|\psi_E(t)\rangle$

$\hat{H} = \hat{H}^m_{JC} = \hbar\omega_m \hat{a}^+_m \hat{a}_m + \hbar\Omega \hat{S}_+\hat{S}_- + \hbar g^m(\hat{S}_+\hat{a}_m + \hat{S}_-\hat{a}^+_m)$

Α.Σ. $c_1(0)=0\ ,\ c_2(0)=1 \quad (4.59)$

$A' = i\hbar\frac{\partial|\psi_E(t)\rangle}{\partial t} = i\hbar\dot{c}_1|\downarrow, n+1\rangle + i\hbar\dot{c}_2|\uparrow, n\rangle$

$$\begin{aligned}\Delta' &= \left(\hbar\omega_m \hat{a}^+_m\hat{a}_m + \hbar\Omega\hat{S}_+\hat{S}_- + \hbar g^m(\hat{S}_+\hat{a}_m + \hat{S}_-\hat{a}^+_m)\right)\left(c_1|\downarrow, n+1\rangle + c_2|\uparrow, n\rangle\right) = \\ &= \hbar\omega_m c_1(n+1)|\downarrow, n+1\rangle + \hbar\Omega c_1\cdot 0 + \hbar g^m c_1|\uparrow, n\rangle\sqrt{n+1} + \hbar g^m c_1\cdot 0 \\ &+ \hbar\omega_m c_2 n|\uparrow, n\rangle + \hbar\Omega c_2|\uparrow, n\rangle + \hbar g^m c_2\cdot 0 + \hbar g^m c_2|\downarrow, n+1\rangle\sqrt{n+1} = \\ &= \hbar\omega_m c_1(n+1)|\downarrow, n+1\rangle + \hbar g^m c_1\sqrt{n+1}\,|\uparrow, n\rangle \\ &+ \hbar\omega_m c_2 n|\uparrow, n\rangle + \hbar\Omega c_2|\uparrow, n\rangle + \hbar g^m c_2\sqrt{n+1}\,|\downarrow, n+1\rangle\end{aligned}$$

επί $\langle\downarrow, n+1|$ $\quad A' = i\hbar\dot{c}_1$, $\quad \Delta' = \hbar\omega_m c_1(n+1) + \hbar g^m c_2\sqrt{n+1}$ $\Rightarrow i\dot{c}_1 = \omega_m(n+1)c_1 + g^m\sqrt{n+1}\,c_2$

επί $\langle\uparrow, n|$ $\quad A' = i\hbar\dot{c}_2$, $\quad \Delta' = \hbar g^m c_1\sqrt{n+1} + \hbar\omega_m c_2 n + \hbar\Omega c_2$ $\Rightarrow i\dot{c}_2 = g^m\sqrt{n+1}\,c_1 + (n\omega_m + \Omega)c_2$

Δηλαδή
$$\boxed{i\begin{pmatrix}\dot{c}_1\\ \dot{c}_2\end{pmatrix} = \begin{pmatrix}(n+1)\omega_m & g^m\sqrt{n+1}\\ g^m\sqrt{n+1} & \Omega + n\omega_m\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}}$$
σύστημα διαφορικών εξισώσεων

$$\boxed{\Omega_{n+1} = \left[\left(\frac{\Omega-\omega}{2}\right)^2 + g^2(n+1)\right]^{1/2}}$$
συχνότητα Rabi (4.64)

ΕΠΙΛΥΟΝΤΑΣ $\Rightarrow \ldots \Rightarrow$ ΑΠΛΟΥ

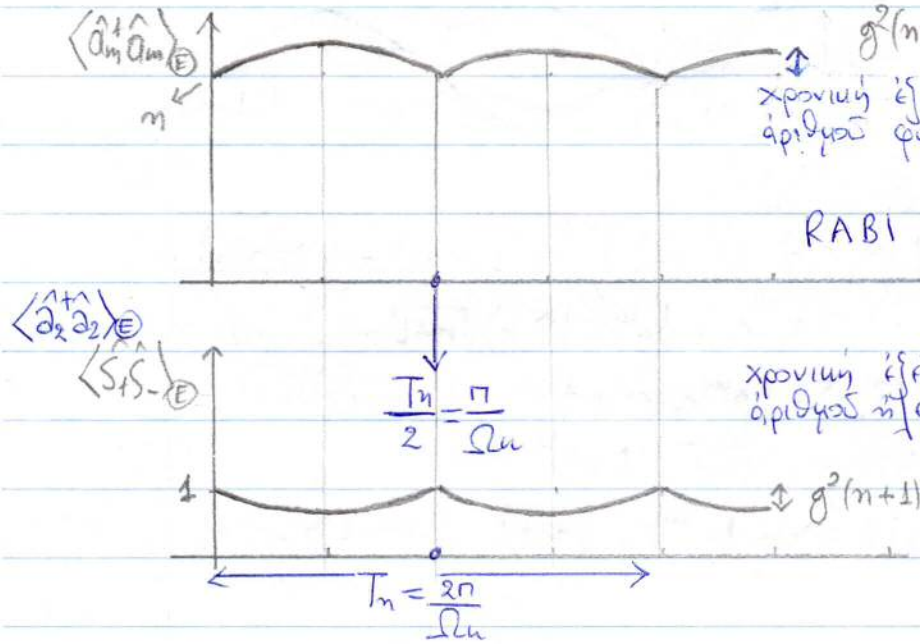
$$c_1(t) = \exp\left[-i\left((n+1)\omega + \frac{\Omega-\omega}{2}\right)t\right]\left[-i\,\frac{g\sqrt{n+1}}{\Omega_{n+1}}\sin(\Omega_{n+1}t)\right] \quad (4.62)$$

$$c_2(t) = \exp\left[-i\left((n+1)\omega + \frac{\Omega-\omega}{2}\right)t\right]\left[\cos(\Omega_{n+1}t) - i\,\frac{\Omega-\omega}{2\Omega_{n+1}}\cdot\sin(\Omega_{n+1}t)\right]$$

$$\Rightarrow \boxed{|c_1(t)|^2 = \frac{g^2(n+1)}{\Omega^2_{n+1}}\sin^2(\Omega_{n+1}t)} \quad \underline{\text{και}} \quad |c_2(t)|^2 = 1 - |c_1(t)|^2 = \ldots \quad (4.63)$$

ΑΡΑ $\langle \hat{a}_m^{+} \hat{a}_m \rangle_{(E)} = n + \dfrac{g^2(n+1)}{\Omega_{n+1}^2} \sin^2(\Omega_{n+1} t)$

$$\langle \hat{S}_+ \hat{S}_- \rangle_{(E)} = 1 - \frac{g^2(n+1)}{\Omega_{n+1}^2} \cdot \sin^2(\Omega_{n+1} t)$$



$g^2(n+1)/\Omega_{n+1}^2 = d$

χρονική εξέλιξη αναμενόμενης τιμής αριθμού φωτονίων στην κοιλότητα

RABI OSCILLATIONS

χρονική εξέλιξη αναμενόμενης τιμής αριθμού ηλεκτρονίων στην E_2 στάθμη δηλ στην άνω στάθμη

$g^2(n+1)/\Omega_{n+1}^2 = d$

ΣΥΣΤΗΜΑ ΔΙΑΦΟΡΙΚΩΝ ΕΞΙΣΩΣΕΩΝ

$$\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = (-i) \begin{pmatrix} (n+1)\omega & g\sqrt{n+1} \\ g\sqrt{n+1} & \Omega + n\omega \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$ ΜΟΡΦΗΣ $\dot{\vec{X}}(t) = \overleftrightarrow{A} \vec{X}(t)$

$$\dot{\vec{X}}(t) = \begin{pmatrix} \dot{c}_1(t) \\ \dot{c}_2(t) \end{pmatrix} \quad \vec{X}(t) = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} \quad \overleftrightarrow{A} = (-i) \begin{pmatrix} (n+1)\omega & g\sqrt{n+1} \\ g\sqrt{n+1} & \Omega + n\omega \end{pmatrix}$$

Όμοιως ...

ΣΧΟΛΙΑΣΜΟΣ @ ΠΡΟΒΛΗΜΑ ΑΠΟΡΡΟΦΗΣΗΣ ΦΩΤΟΝΙΟΥ

$$|\psi_A(t)\rangle = c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow n-1\rangle$$

Α.Σ. $c_1(t)=1$, $c_2(0)=0$

$$i\hbar\frac{\partial|\psi_A(t)\rangle}{\partial t} = \hat{H}|\psi_A(t)\rangle$$

$$\hat{H} = \hat{H}^m_{JC} = \hbar\omega_m \hat{a}^+_m \hat{a}_m + \hbar\Omega\hat{S}_+\hat{S}_- + \hbar g^m(\hat{S}_+\hat{a}_m + \hat{S}_-\hat{a}^+_m)$$

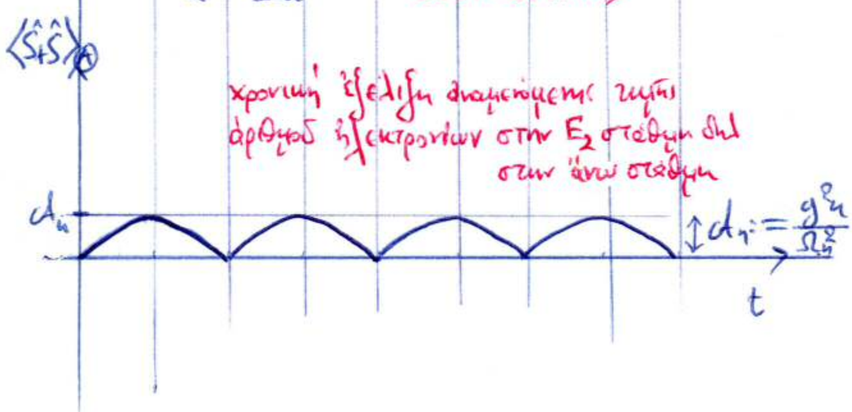
Jaynes-Cummings Hamiltonian

ΕΝΝΟΕΙΤΑΙ...

(m) $\Rightarrow \langle\hat{a}^+\hat{a}\rangle_Ⓐ = n - \frac{g^2 n \sin^2(\Omega_n t)}{\Omega_n^2}$

$$\langle\hat{S}_+\hat{S}_-\rangle_Ⓐ = \frac{g^2 n \sin^2(\Omega_n t)}{\Omega_n^2}$$

RABI OSCILLATIONS



$A_n := \frac{g^2 n}{\Omega_n^2}$ πλάτος της ταλαντώσεως

$$\Omega_n = \left[\left(\frac{\Omega-\omega}{2}\right)^2 + g^2 n\right]^{1/2}$$ γενικευμένη generalized Rabi frequency

$\Delta = \Omega - \omega$ detuning

$$\Rightarrow A_n = \frac{g^2 n}{\left(\frac{\Omega-\omega}{2}\right)^2 + g^2 n}$$

ΑΡΑ για $\Omega = \omega$ (συντονισμός) $\Rightarrow A_n = 1$

για $\Omega \neq \omega$ (μη συντονισμός) $\Rightarrow A_n < 1$

ΠΑΡΑΤΗΡΟΥΜΕ ότι η γενικευμένη συχνότητα Rabi, Ω_n, καθορίζει:

(Ι) το πλάτος της ταλαντώσεως $A_n := \frac{g^2 n}{\Omega_n^2}$

(ΙΙ) το χρόνο μεταξύ διαδοχικών μεγίστων ή ελαχίστων, $\frac{T_n}{2} = \frac{\pi}{\Omega_n}$

ΔΙΕΥΚΡΙΝΗΣΕΙΣ @ ΟΡΙΣΜΟΥΣ

Ε!! με ορίσει $\boxed{\Omega_R = \frac{\wp E_0^m}{\hbar}}$ συχνότητα Rabi frequency

όπου $\wp = -e x_{12}$

$$\hat{E}^m_x(z,t) \xrightarrow{\text{ΓΕΝΙΚΑ}} \underbrace{E_0^m}_{\text{"ΠΛΑΤΟΣ"}} \hat{f}(t) = \left(\frac{\hbar\omega_m}{\varepsilon_0 V}\right)^{1/2} \cdot \sin\left(\frac{m\pi z}{L}\right) \cdot (\hat{a}^+_m + \hat{a}_m)$$ Λάθε

$$\hbar g^m = -e x_{12}\left(\frac{\hbar\omega_m}{\varepsilon_0 V}\right)^{1/2} \cdot \sin\left(\frac{m\pi z}{L}\right) \quad (g)$$

Άρα $\hbar g^m = \wp E_0^m \Rightarrow \boxed{g^m = \frac{\wp E_0^m}{\hbar} = \Omega_R}$ Δηλαδή το g^m δεν είναι παρά η Ω_R

ΑΡΑ $$\Omega_n = \left[\frac{\Delta^2}{4} + \Omega_R^2 n\right]^{1/2}$$

ΣΗΜΕΙΩΣΗ: Μερικές φορές ορίζεται ως γενικευμένη συχνότητα Rabi η Ω_n':

$$\Omega_n = \left[\frac{\Delta^2}{4} + \frac{4\Omega_R^2 n}{4}\right]^{1/2} = \frac{1}{2}\left[\Delta^2 + 4n\Omega_R^2\right]^{1/2} = \frac{\Omega_n'}{2}$$

$$\left.\begin{array}{l}\dot{\vec{x}}(t) = \overleftrightarrow{A}\vec{x}(t) \\ \vec{x}(t) = \vec{u}\, e^{\lambda t} \\ \dot{\vec{x}}(t) = \vec{u}\lambda e^{\lambda t}\end{array}\right\} \vec{u}\lambda e^{\lambda t} = \overleftrightarrow{A}\vec{u} e^{\lambda t} \Leftrightarrow \overleftrightarrow{A}\vec{u} = \lambda\vec{u} \Leftrightarrow \overleftrightarrow{A}\vec{u} = \lambda\overleftrightarrow{I}$$

(c) ΠΡΟΒΛΗΜΑ ΑΠΟΡΡΟΦΗΣΗΣ ΦΩΤΟΝΙΟΥ

$$\overleftrightarrow{A} = (-i)\begin{pmatrix} n\omega & g\sqrt{n} \\ g\sqrt{n} & \Omega + (n-1)\omega \end{pmatrix} \quad \text{και για } n=1 \quad \overleftrightarrow{A} = (-i)\begin{pmatrix} \omega & g \\ g & \Omega \end{pmatrix}$$

$$p(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0 \qquad a_{11} = (-i)\omega \quad a_{12} = (-i)g$$

$$a_{11} + a_{22} = (-i)(\omega + \Omega) \qquad a_{21} = (-i)g \quad a_{22} = (-i)\Omega$$

$$a_{11}a_{22} = -\omega\Omega \qquad a_{12}a_{21} = -g^2$$

$$\Rightarrow p(\lambda) = \lambda^2 + i(\omega+\Omega)\lambda + (-\omega\Omega + g^2) = 0$$

$$\Delta = -(\omega+\Omega)^2 - 4(-\omega\Omega + g^2) = -\omega^2 - \Omega^2 - 2\omega\Omega + 4\omega\Omega - 4g^2 \Rightarrow$$

$$\Delta = -(\omega - \Omega)^2 - 4g^2 < 0$$

$$\lambda_{1,2} = \frac{-\beta \pm i\sqrt{-\Delta}}{2a} = \frac{-i(\omega+\Omega) \pm i\sqrt{(\omega-\Omega)^2 + 4g^2}}{2} \Rightarrow$$

$$\lambda_{1,2} = -i\,\frac{\omega+\Omega}{2} \pm i\underbrace{\sqrt{\left(\frac{\omega-\Omega}{2}\right)^2 + g^2}}_{}$$

$$\Omega_1 = \left[\left(\frac{\Omega-\omega}{2}\right)^2 + g^2 \cdot 1\right]^{1/2}$$

ΠΡΟΒΛΗΜΑ ΑΠΟΡΡΟΦΗΣΕΩΣ ΦΩΤΟΝΙΟΥ

$$A=(-i)\begin{pmatrix} n\omega & g\sqrt{n} \\ g\sqrt{n} & \Omega+(n-1)\omega \end{pmatrix} \qquad a_{11}=(-i)n\omega \quad a_{12}=(-i)g\sqrt{n}=a_{21}$$

$$a_{22}=(-i)[\Omega+(n-1)\omega]$$

$$p(\lambda)=\lambda^2-(a_{11}+a_{22})\lambda+(a_{11}a_{22}-a_{12}a_{21})=0$$

$$a_{11}+a_{22}=(-i)[n\omega+\Omega+(n-1)\omega]$$

$$a_{11}a_{22}=-n\omega[\Omega+(n-1)\omega] \qquad a_{12}a_{21}=-g^2n$$

$$p(\lambda)=\lambda^2+i(n\omega+\Omega+(n-1)\omega)\lambda+(-n\omega[\Omega+(n-1)\omega]+g^2n)=0$$

$$\Delta=-(n\omega+\Omega+(n-1)\omega)^2+4n\omega[\Omega+(n-1)\omega]-4g^2n$$

$$-[\Omega+(n-1)\omega]^2-(n\omega)^2-2[\Omega+(n-1)\omega](n\omega)$$
$$+4[\Omega+(n-1)\omega]\cdot(n\omega)-4g^2n$$

$$\Delta=-[\Omega+(n-1)\omega-n\omega]^2-4g^2n<0$$

για $n=1$ $\Delta=-[\Omega-\omega]^2-4g^2$

$$\Delta=-[\Omega-\omega]^2-4g^2n<0 \qquad \lambda_{1,2}=\frac{-\beta\pm i\sqrt{-\Delta}}{2\alpha}$$

$$\lambda_{1,2}=\frac{-i(n\omega+\Omega+(n-1)\omega)\pm i\sqrt{[\Omega-\omega]^2+4g^2n}}{2}$$

$$\lambda_{1,2}=\frac{-i(n\omega+\Omega+(n-1)\omega)}{2}\pm i\underbrace{\sqrt{\left(\frac{\Omega-\omega}{2}\right)^2+g^2n}}_{\Omega_n}$$

$$\lambda_{1,2}=-\frac{i}{2}(n\omega+\Omega+(n-1)\omega)\pm i\Omega_n=-i\Pi_n\pm i\Omega_n=(-i)[\Pi_n\mp\Omega_n]$$

$$\Pi_n=\frac{n\omega+\Omega+(n-1)\omega}{2}$$

ΑΡΑ

$$\lambda_{1,2}=(-i)[\Pi_n\mp\Omega_n]$$

ΛΥΣΗ $\vec{X}_h(t)=\sigma_1\vec{u}_1e^{\lambda_1t}+\sigma_2\vec{u}_2e^{\lambda_2t}$

$$\lambda_1=-i\Pi_n+i\Omega_n$$

$$\cancel{(-i)}\begin{pmatrix} n\omega & g\sqrt{n} \\ g\sqrt{n} & \Omega+(n-1)\omega \end{pmatrix}\begin{pmatrix} u_{11} \\ u_{21}\end{pmatrix}=\lambda_1\begin{pmatrix} u_{11} \\ u_{21}\end{pmatrix}=(\Pi_n-\Omega_n)\cancel{(-i)}\begin{pmatrix} u_{11} \\ u_{21}\end{pmatrix}$$

$$n\omega u_{11}+g\sqrt{n}u_{21}=(\Pi_n-\Omega_n)u_{11}$$
$$g\sqrt{n}u_{11}+[\Omega+(n-1)\omega]u_{21}=(\Pi_n-\Omega_n)u_{21}$$

$$g\sqrt{n}\,u_{21}=(\Pi_n-\Omega_n-n\omega)u_{11}$$
$$g\sqrt{n}\,u_{11}=(\Pi_n-\Omega_n-[\Omega+(n-1)\omega])u_{21}$$

$$g\sqrt{\Gamma_4}\, U_{21} = (J\Gamma_4 - \Omega_4 - n\omega)\frac{(J\Gamma_4 - \Omega_4 - [\Omega + (n-1)\omega])\, U_{21}}{g\sqrt{\Gamma_4}}$$

$$g^2 n\, U_{21} = (J\Gamma_4 - \Omega_4 - n\omega)(J\Gamma_4 - \Omega_4 - [\Omega + (n-1)\omega])\, U_{21}$$

ή $U_{21} = 0$ ή $g^2 n = (J\Gamma_4 - \Omega_4 - n\omega)(J\Gamma_4 - \Omega_4 - [\Omega + (n-1)\omega])$

$\Downarrow$

$U_{11} = 0$

$\Downarrow$

$\vec{U}_1 = \vec{0}$ ΑΤΟΠΟ

$$2J\Gamma_4 = \Omega + (n-1)\omega + n\omega \Rightarrow \Omega + (n-1)\omega = 2J\Gamma_4 - n\omega$$

ΑΡΑ $g^2 n = (J\Gamma_4 - \Omega_4 - n\omega)(J\Gamma_4 - \Omega_4 - 2J\Gamma_4 + n\omega)$

$$g^2 n = (J\Gamma_4 - \Omega_4 - n\omega)(-\Omega_4 - J\Gamma_4 + n\omega)$$

$$= (J\Gamma_4 - \Omega_4 - n\omega)(-1)(\Omega_4 + J\Gamma_4 - n\omega) \qquad (\alpha-\beta)(\alpha+\beta) = \alpha^2 - \beta^2$$

$$g^2 n = (-1)\left[(J\Gamma_4 - n\omega)^2 - \Omega_4^2\right]$$

$$J\Gamma_4 - n\omega = \frac{n\omega + \Omega + (n-1)\omega - 2n\omega}{2} = \frac{\Omega - \omega}{2}$$

U_{21} = ΟΤΙΔΗΠΟΤΕ

π.χ. $\boxed{U_{21} = 1}$

$$U_{11} = \frac{J\Gamma_4 - \Omega_4 - [\Omega + (n-1)\omega]}{g\sqrt{\Gamma_4}}$$

$$J\Gamma_4 - [\Omega + (n-1)\omega] = \frac{n\omega + \Omega + n\omega - \omega - 2\Omega - 2n\omega + 2\omega}{2} = \frac{\omega - \Omega}{2}$$

$$-g^2 n = \left(\frac{\Omega - \omega}{2}\right)^2 - \Omega_4^2$$

$$-g^2 n = \left(\frac{\Omega - \omega}{2}\right)^2 - \left(\frac{\Omega - \omega}{2}\right)^2 - g^2 n \Rightarrow 0 = 0 \Rightarrow \boxed{U_{21} \text{ ΟΤΙΔΗΠΟΤΕ}}$$

έστω $\boxed{U_{21} = 1} \Rightarrow U_{11} = \dfrac{(J\Gamma_4 - \Omega_4 - [\Omega + (n-1)\omega])}{g\sqrt{\Gamma_4}}$ $\boxed{\vec{U}_1}$

$$\frac{n\omega + \Omega + (n-1)\omega - 2[\Omega + (n-1)\omega]}{2}$$

$$\frac{\cancel{n\omega} + \Omega + \cancel{n\omega} - \omega - 2\Omega - \cancel{2n\omega} + 2\omega}{2} = \frac{\omega - \Omega}{2}$$

$$U_{11} = \frac{\frac{\omega - \Omega}{2} - \Omega_4}{g\sqrt{\Gamma_4}}$$

$$\vec{U}_1 = \begin{pmatrix} \dfrac{\frac{\omega - \Omega}{2} - \Omega_4}{g\sqrt{\Gamma_4}} \\ 1 \end{pmatrix}$$

$$U_{11} = \frac{\frac{\omega - \Omega}{2} - \Omega_4}{g\sqrt{\Gamma_4}}$$

$$\lambda_2 = -i\Pi_n - i\Omega_n = (-i)\left[\Pi_n + \Omega_n\right]$$

$$\cancel{(-i)}\begin{pmatrix} n\omega & g\sqrt{n} \\ g\sqrt{n} & \Omega+(n-1)\omega \end{pmatrix}\begin{pmatrix} v_{12} \\ v_{22}\end{pmatrix} = \cancel{(-i)}\left[\Pi_n+\Omega_n\right]\begin{pmatrix} v_{12} \\ v_{22}\end{pmatrix}$$

$$n\omega\, v_{12} + g\sqrt{n}\, v_{22} = \left[\Pi_n+\Omega_n\right] v_{12}$$

$$g\sqrt{n}\, v_{12} + \left[\Omega+(n-1)\omega\right] v_{22} = \left[\Pi_n+\Omega_n\right] v_{22}$$

$$v_{12} = \frac{\frac{\omega-\Omega}{2}+\Omega_n}{g\sqrt{n}}$$

$$g\sqrt{n}\, v_{22} = \left[\Pi_n+\Omega_n-n\omega\right] v_{12}$$

$$g\sqrt{n}\, v_{12} = \left[\Pi_n+\Omega_n-\left[\Omega+(n-1)\omega\right]\right] v_{22}$$

$$g\sqrt{n}\, v_{22} = \frac{\left[\Pi_n+\Omega_n-n\omega\right]\left[\Pi_n+\Omega_n-\left[\Omega+(n-1)\omega\right]\right] v_{22}}{g\sqrt{n}}$$

$$\vec{v}_2 = \begin{pmatrix} \frac{\frac{\omega-\Omega}{2}+\Omega_n}{g\sqrt{n}} \\ 1 \end{pmatrix}$$

$v_{22} = 0$;
$\Downarrow$
$v_{12} = 0$
$\Downarrow$
$\vec{v}_2 = 0$

$$g^2 n = \left[\Pi_n+\Omega_n-n\omega\right]\left[\Pi_n+\Omega_n-\left[\Omega+(n-1)\omega\right]\right]$$

$$\Pi_n - \left[\Omega+(n-1)\omega\right] = \frac{n\omega+\Omega+(n-1)\omega-2\Omega\ \overbrace{-2(n-1)\omega}}{2}$$

$$= \frac{-\Omega+n\omega-(n-1)\omega}{2} = \frac{-\Omega+\cancel{n\omega}-\cancel{n\omega}+\omega}{2} = \frac{\omega-\Omega}{2}$$

$$\frac{n\omega+\Omega+(n-1)\omega-2n\omega}{2}$$

$$\Pi_n - n\omega = \frac{\Omega-\omega}{2}$$

$$\text{APA}\quad g^2 n = \left(\frac{\Omega-\omega}{2}+\Omega_n\right)\left(\frac{\omega-\Omega}{2}+\Omega_n\right)$$

$$\Rightarrow\ g^2 n = \left(\Omega_n+\frac{\omega-\Omega}{2}\right)\left(\Omega_n-\frac{\omega-\Omega}{2}\right) = \Omega_n^2-\left(\frac{\omega-\Omega}{2}\right)^2$$

$$g^2 n = \left(\frac{\Omega-\omega}{2}\right)^2 + g^2 n - \left(\frac{\omega-\Omega}{2}\right)^2 \Rightarrow 0 = 0$$

v_{22} = LIBOVOLNE

napr. $v_{22} = 1$

$$v_{12} = \frac{\left[\Pi_n+\Omega_n-\left[\Omega+(n-1)\omega\right]\right]\cdot 1}{g\sqrt{n}}$$

$$\frac{\omega-\Omega}{2}$$

$$\Pi_n - \left[\Omega+(n-1)\omega\right] = \frac{n\omega+\Omega+(n-1)\omega-2\Omega-2(n-1)\omega}{2} =$$

$$\vec{X}_h(t) = \sigma_1 \begin{pmatrix} \dfrac{\frac{\omega-\Omega}{2} - \Omega_n}{g\sqrt{n}} \\ 1 \end{pmatrix} e^{\lambda_1 t} + \sigma_2 \begin{pmatrix} \dfrac{\frac{\omega-\Omega}{2} + \Omega_n}{g\sqrt{n}} \\ 1 \end{pmatrix} e^{\lambda_2 (t)} = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

Αρχικές συνθήκες $c_1(0) = 1 \quad c_2(0) = 0$

$$\Rightarrow \sigma_1 \frac{\frac{\omega-\Omega}{2} - \Omega_n}{g\sqrt{n}} + \sigma_2 \frac{\frac{\omega-\Omega}{2} + \Omega_n}{g\sqrt{n}} = 1$$

$$\sigma_1 + \sigma_2 = 0 \Rightarrow \sigma_2 = -\sigma_1$$

$$\sigma_1 \frac{\frac{\omega-\Omega}{2} - \Omega_n}{g\sqrt{n}} - \sigma_1 \frac{\frac{\omega-\Omega}{2} + \Omega_n}{g\sqrt{n}} = 1$$

$$\frac{\frac{\omega-\Omega}{2} - \Omega_n - \frac{\omega-\Omega}{2} - \Omega_n}{g\sqrt{n}} = 1 \Rightarrow \frac{-2\Omega_n}{g\sqrt{n}} = \frac{1}{\sigma_1}$$

$$\boxed{\sigma_1 = \frac{g\sqrt{n}}{-2\Omega_n}} \qquad \boxed{\sigma_2 = \frac{g\sqrt{n}}{2\Omega_n}}$$

$$c_1(t) = \frac{\cancel{g\sqrt{n}}}{-2\Omega_n} \left(\frac{\frac{\omega-\Omega}{2} - \Omega_n}{\cancel{g\sqrt{n}}} \right) \cdot e^{\lambda_1 t} + \frac{\cancel{g\sqrt{n}}}{2\Omega_n} \frac{\frac{\omega-\Omega}{2} + \Omega_n}{\cancel{g\sqrt{n}}} e^{\lambda_2 t}$$

$$c_1(t) = \frac{-\frac{\omega-\Omega}{2} + \Omega_n}{+2\Omega_n} e^{\lambda_1 t} + \frac{\frac{\omega-\Omega}{2} + \Omega_n}{2\Omega_n} e^{\lambda_2 t}$$

$$c_1(t) = \left(\frac{\Omega-\omega}{2 \cdot 2\Omega_n} + \frac{\Omega_n}{2\Omega_n} \right) e^{\lambda_1 t} + \left(\frac{\omega-\Omega}{2 \cdot 2\Omega_n} + \frac{\Omega_n}{2\Omega_n} \right) e^{\lambda_2 t}$$

$$c_1(t) = \left(\frac{\Omega-\omega}{4\Omega_n} + \frac{1}{2} \right) e^{\lambda_1 t} + \left(\frac{\omega-\Omega}{4\Omega_n} + \frac{1}{2} \right) e^{\lambda_2 t}$$

$$c_1(t) = \left(\frac{\Omega - \omega}{4\Omega_n} + \frac{1}{2}\right) e^{(-i)(\mathcal{T}_n - \Omega_n)t} + \left(\frac{\omega - \Omega}{4\Omega_n} + \frac{1}{2}\right) e^{(-i)(\mathcal{T}_n + \Omega_n)t}$$ (5)

$$\mathcal{T}_n - \Omega_n = \frac{n\omega + \Omega + (n-1)\omega}{2} - \Omega_n \qquad e^{-i\mathcal{T}_n t}\, e^{+i\Omega_n t} \quad e^{-i\mathcal{T}_n t}\, e^{-i\Omega_n t}$$

$$e^{-i\mathcal{T}_n t}$$

$$c_2(t) = \frac{g\sqrt{n}}{-2\Omega_n} \cdot 1\, e^{\lambda_1 t} + \frac{g\sqrt{n}}{2\Omega_n} \cdot 1 \cdot e^{\lambda_2 t} = \frac{g\sqrt{n}}{2\Omega_n}\left(e^{\lambda_2 t} - e^{\lambda_1 t}\right)$$

$$= \frac{g\sqrt{n}}{2\Omega_n}\left(e^{(-i)(\mathcal{T}_n + \Omega_n)t} - e^{(-i)(\mathcal{T}_n - \Omega_n)t}\right)$$

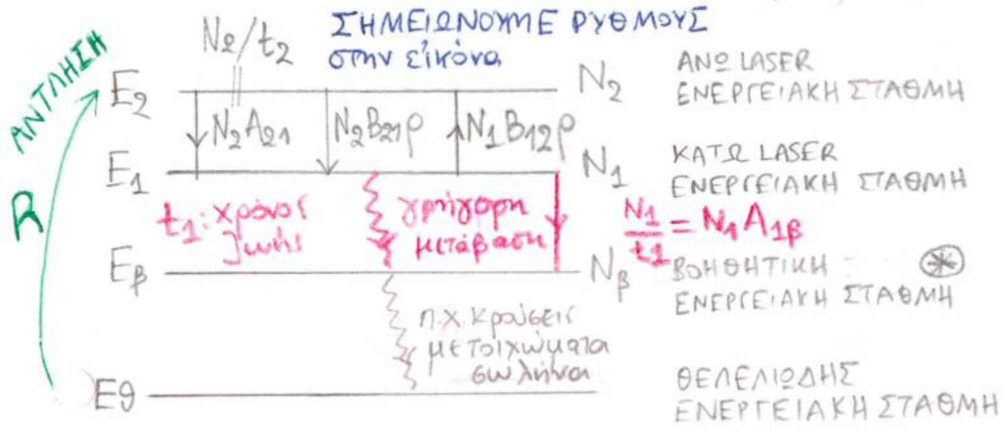
$$= \frac{g\sqrt{n}}{2\Omega_n}\left(e^{-i\mathcal{T}_n t} e^{-i\Omega_n t} - e^{-i\mathcal{T}_n t} e^{+i\Omega_n t}\right) = e^{-i\mathcal{T}_n t} \frac{g\sqrt{n}}{2\Omega_n}\left(e^{-i\Omega_n t} - e^{i\Omega_n t}\right)$$

$$\cancel{\cos \Omega_n t} - i\sin(\Omega_n t)$$
$$-\cancel{\cos \Omega_n t} - i\sin(\Omega_n t)$$

$$\frac{g\sqrt{n}}{2\Omega_n}(-2i)\sin(\Omega_n t)\, e^{-i\mathcal{T}_n t}$$

$$c_2(t) = \left\{(-i)\frac{g\sqrt{n}}{\Omega_n} \cdot \sin(\Omega_n t)\right\} e^{-i\mathcal{T}_n t}$$

ΕΞΙΣΩΣΕΙΣ ΡΥΘΜΩΝ σε Laser με "δισταθμικά άτομα"



∗ για να μη μένουν τα ηλεκτρόνια στην ① ώστε να διευκολύνεται η ΑΝΑΣΤΡΟΦΗ ΠΛΗΘΥΣΜΟΥ μεταξύ ① κ ②

ΑΝΩ LASER ΕΝΕΡΓΕΙΑΚΗ ΣΤΑΘΜΗ

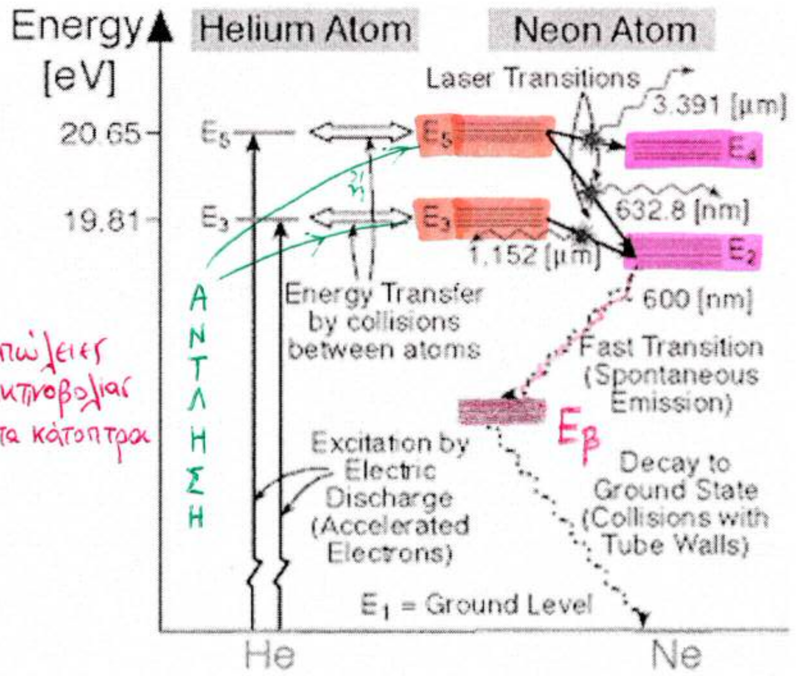
ΚΑΤΩ LASER ΕΝΕΡΓΕΙΑΚΗ ΣΤΑΘΜΗ

$[\rho(\nu)] = \frac{J}{m^3 Hz} = \frac{Js}{m^3}$

$[A] = \frac{1}{s}$

$[B] = \frac{m^3}{Js^2}$

$-\frac{\rho(\nu)}{t_0}$: απώλειες ακτινοβολίας στα κάτοπτρα



①

- $dW^{αυθ}_{εκπ\, 1\to\beta} = A_{1\beta}dt$ πιθανότητα αυθόρμητης εκπομπής $1\to\beta$ σε χρόνο dt (για ένα άτομο)

$1 := A_{1\beta}\cdot t_1 \Rightarrow \boxed{t_1 = \frac{1}{A_{1\beta}}}$

t_1: χρόνος ζωής της ①

$R_2 = R$: ρυθμός άντλησης στη σταθμη 2 $[R] = \frac{\#}{s}$

$[ρυθμός] = \frac{\#}{s}$, # = ΑΡΙΘΜΟΣ

$dW^{αυθ}_{εκπ\, 1\to\beta} = \frac{dt}{t_1}$

$dN_{1\to\beta} = N_1\frac{dt}{t_1}$ αριθμός ατόμων που μεταβαίνουν $1\to\beta$ σε χρόνο dt με αυθ.εκπ.

$\frac{dN_{1\to\beta}}{dt} = \frac{N_1}{t_1}$ ρυθμός μεταβάσεως των ατόμων $1\to\beta$ με αυθ.εκπ

Όμοιως...

- $dW^{αυθ}_{εκπ\, 2\to1} = A_{21}dt$ πιθανότητα αυθόρμητης εκπομπής $2\to1$ σε χρόνο dt (για ένα άτομο)

$1 := A_{21}t_2 \Rightarrow \boxed{t_2 = \frac{1}{A_{21}}}$ t_2: χρόνος ζωής της ②

$dN^{αυθ}_{εκπ\, 2\to1} = N_2A_{21}dt$ αριθμός ατόμων που μεταβαίνουν $2\to1$ με αυθ.εκπ. σε χρόνο dt

$\frac{dN^{αυθ}_{εκπ\, 2\to1}}{dt} = \boxed{N_2A_{21}}$ ρυθμός μεταβάσεως των ατόμων $2\to1$ με αυθ.εκπ.

- $dW^{εξ}_{εκπ\, 2\to1} = B_{21}\rho(\nu)dt$ πιθανότητα εξαναγκασμένης εκπομπής $2\to1$ σε χρόνο dt (για ένα άτομο)

$dN^{εξ.εκπ.}_{2\to1} = N_2B_{21}\rho(\nu)dt$ αριθμός ατόμων που μεταβαίνουν με εξ.εκπ $2\to1$ σε χρόνο dt

$\frac{dN^{εξ.εκπ}_{2\to1}}{dt} = \boxed{N_2B_{21}\rho(\nu)}$ ρυθμός μεταβάσεως $2\to1$ με εξ.εκπ.

- $dW^{(εξ)απορ.}_{1\to2} = B_{12}\rho(\nu)dt$ πιθανότητα (εξ.) απορροφήσεως $1\to2$ σε χρόνο dt (για ένα άτομο)

$dN^{(εξ.)απορ.}_{1\to2} = N_1B_{12}\rho(\nu)dt$ αριθμός ατόμων που μεταβαίνουν $1\to2$ σε χρόνο dt

$\frac{dN^{εξ.απορ.}_{1\to2}}{dt} = \boxed{N_1B_{12}\rho(\nu)}$ ρυθμός μεταβάσεως $1\to2$ με (εξ.) απορ.

● Αν είχα Θερμοδυναμική Ισορροπία ΧΩΡΙΣ ΑΠΩΛΕΙΕΣ κ ΑΝΤΛΗΣΗ ⇒

$$dN_{1\to 2} = dN_{2\to 1} \Leftrightarrow N_1\, dW^{εξ}_{απορ} = N_2\left(dW^{εξ}_{εκπ} + dW^{αυθ}_{εκπ}\right) \Leftrightarrow$$

$$\left.\begin{array}{r} N_1 B_{12}\rho\, dt = N_2\left(B_{21}\rho\, dt + A_{21}\, dt\right) \\ A_{21} = A \\ B_{21} = B_{12} = B \end{array}\right\} \Rightarrow \cdots \frac{A}{B} = \frac{8\pi h}{c^3}\nu^3$$

με σύγκριση με ν. Planck

● ΤΩΡΑ ΕΧΟΥΜΕ <u>ΑΠΩΛΕΙΕΣ</u> και <u>ΑΝΤΛΗΣΗ</u>

● (Ε1)(Ε2)(Ε3) ή (Ε1′)(Ε2′)(Ε3′) ή (ε1)(ε2)(ε3) } διαφορικές εξισώσεις ρυθμών laser

Θα δούμε ότι $N_1 = f_1(R)$

$N_2 = f_2(R)$

$\rho = f(R)$

θέτοντας $A_{21} = A \quad B_{21} = B_{12} = B$

"εξισώσεις της ύλης"

$$\frac{dN_1}{dt} = -\frac{N_1}{t_1} - N_1B_{12}\rho + N_2B_{21}\rho + N_2A_{21} \Rightarrow \boxed{\frac{dN_1}{dt} = AN_2 + B\rho(N_2 - N_1) - \frac{N_1}{t_1}} \quad (E1) \quad 5.22$$

$$\frac{dN_2}{dt} = R + N_1B_{12}\rho - N_2A_{21} - N_2B_{21}\rho \Rightarrow \boxed{\frac{dN_2}{dt} = R + B\rho(N_1 - N_2) - AN_2} \quad (E2) \quad 5.23$$

"εξίσωση πεδίου"

$$\frac{d\rho}{dt} = -\frac{\rho}{t_0} + \left\{-N_1B_{12}\rho + N_2B_{21}\rho + N_2A'_{21}\right\}\frac{\hbar\omega}{V}F(\omega)$$

απώλεια ακτινοβολίας στα κάτοπτρα

συνάρτηση φασματική γραμμή

η γραμμή έχει εύρος

FWHM Full Width at Half Maximum

Το V (όγκος κοιλότητας) χρειάζεται εφ' όσον τα Ni είναι αριθμοί και όχι πυκνότητες και το $[F] = 1/\mathrm{Hz}$

$$\Rightarrow \boxed{\frac{d\rho}{dt} = -\frac{\rho}{t_0} + \left\{A'N_2 + B\rho(N_2 - N_1)\right\}\frac{\hbar\omega}{V}F(\omega)} \quad (E3) \quad 5.25$$

$h\nu F(\nu)$

η συνάρτηση φασματική γραμμή μπορεί π.χ να είναι Lorentzian

$$F(\omega) = \frac{T_2}{\pi}\,\frac{1}{1 + (\omega - \omega_0)^2 T_2^2}$$

"όπου T_2 = χρόνος μεταξύ δύο συγκρούσεων" ή Gaussian

$$F(\omega) = \frac{c}{\omega_0}\left(\frac{M}{2\pi k_B T}\right)^{1/2}\cdot \exp\left[-\frac{Mc^2}{2k_BT}(\omega-\omega_0)^2/\omega_0^2\right]$$

"όπου M = μάζα ατόμου αερίου"

$A' \simeq 10^{-9}A$ διότι κάτοπτρα υπάρχουν μόνο στον άξονα z, το μεγαλύτερο μέρος της κοιλότητας δεν περικλείεται από κάτοπτρα, άρα τα περισσότερα φωτόνια αυθόρμητης εκπομπής χάνονται μια και εκπέμπονται σε άσχετες με τον άξονα z διευθύνσεις

$\frac{A'}{A} = 0.15\times 10^{-9}$ πχ Tiley

"Η παράμετρος t_0 χαρακτηρίζει το χρόνο που χρειάζεται για να αδειάσει λόγω απωλειών στα κάτοπτρα η κοιλότητα από την υπάρχουσα αποθηκευμένη ακτινοβολία, ρ, αν <u>δεν</u> υπάρχει ενεργό υλικό laser." "Όσο <u>μικρότερη</u> είναι η ανακλαστικότητα των κατόπτρων τόσο <u>μικρότερος</u> είναι ο χρόνος t_0"

ΕΝΑΛΛΑΚΤΙΚΑ

$$\frac{dn_1}{dt} = An_2 + B\rho(n_2 - n_1) - \frac{n_1}{t_1} \quad (E1')$$

$$\frac{dn_2}{dt} = r + B\rho(n_1 - n_2) - An_2 \quad (E2')$$

$$\frac{d\rho}{dt} = -\frac{\rho}{t_0} + \left\{A'n_2 + B\rho(n_2 - n_1)\right\}h\nu F(\nu) \quad (E3')$$

(E4′) $n_i = \frac{N_i}{V}$ $\qquad r = \frac{R}{V}$ (E5′)

- $\boxed{\omega_m = \frac{m\pi c}{L}, m=1,2,3,\ldots \text{ και } k_m = \frac{m\pi}{L}}$ διαμήκεις τρόποι longitudinal modes

Άρα οι τρόποι (modes) που θα υποστηριχθούν στην κοιλότητα εξαρτώνται από την απόσταση των κατόπτρων L. Έστω ότι διαλέγουμε να υποστηρίξουμε τα ερυθρά φωτόνια με $\lambda \simeq 632.8$ nm και όχι κάποια από τα υπέρυθρα 3.391 μm και 1.152 μm.

- Το εύρος της ερυθρής γραμμής είναι γύρω στα 1.7 GHz και ας υποθέσουμε ότι μιλάμε για το FWHM.

Αν $\lambda_0 \simeq 632.8$ nm $\quad \omega_0 = 2\pi\nu_0 = 2\pi \frac{c}{\lambda_0} = \frac{2\pi\, 3\cdot 10^8\, m\cdot s^{-1}}{632.8\cdot 10^{-9} m} = \frac{6\pi}{6328}\cdot 10^{17} s^{-1} \simeq 0.02979\times 10^{+17} s^{-1} \Rightarrow$

$c = \lambda_0\nu_0$

κυκλική κεντρική συχνότητα $\quad \omega_0 \simeq 2.979\times 10^{15} s^{-1}$

ή κεντρική συχνότητα $\quad \nu_0 \simeq 0.474\times 10^{15}$ Hz

έστω FWHM (εύρος) $\quad \Delta\nu^{FWHM} \simeq 1.7$ GHz

$$\Rightarrow \boxed{\frac{\Delta\nu^{FWHM}}{\nu_0} \simeq \frac{1.7\times 10^9}{0.474\times 10^{15}} = 3.59\times 10^{-6}}$$

- Το εύρος κάθε $\omega_m = 2\pi\nu_m$ περιγράφεται ποιοτικά από $\Delta\nu_m^{FWHM} \simeq \frac{1}{2\pi t_0}$ και είναι της τάξεως των 1–10 MHz δηλαδή πολύ μικρότερο από το εύρος της ερυθρής γραμμής ($\Delta\nu^{FWHM} \sim 1.7$ GHz)

$\omega_m = \frac{m\pi c}{L} \Rightarrow 2\pi\nu_m = \frac{m\pi c}{L} \Rightarrow \nu_m = \frac{mc}{2L} \Rightarrow$ οι διαμήκεις τρόποι απέχουν μεταξύ τους κατά $\Delta\nu_m = \frac{c}{2L}$

διαμήκεις τρόποι

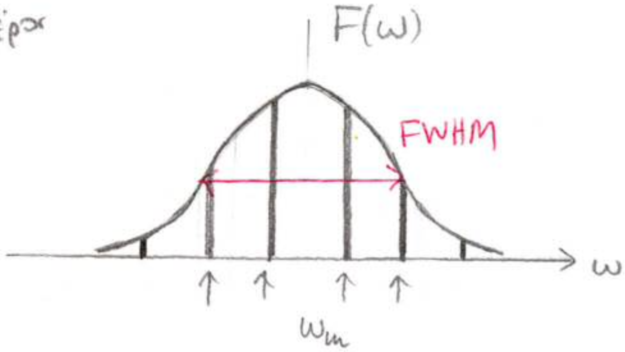
Άρα $\Delta\nu_m = \frac{3\cdot 10^8\, ms^{-1}}{2\; 0.4m} = \frac{15}{4} 10^8 s^{-1} \Rightarrow \Delta\nu_m = 375$ MHz

για $L = 0.4$m

δηλ μέσα στο $\Delta\nu^{FWHM} = 1.7$ GHz χωράνε

$\left[\frac{1.7\text{ GHz}}{0.375\text{ GHz}}\right] \sim 4$ διαμήκεις τρόποι

ακέραιο μέρος



Στη στάσιμη κατάσταση (steady state) έχουμε $\frac{dN_1}{dt}=\frac{dN_2}{dt}=0=\frac{d\rho}{dt}$ $\qquad n_i=\frac{N_i}{V} \qquad r=\frac{R}{V}$ (4)

$$\boxed{AN_2 + B\rho(N_2-N_1) - \frac{N_1}{t_1}=0} \quad 5.28 \quad (A) \qquad An_2 + B\rho(n_2-n_1)-\frac{n_1}{t_1}=0 \quad (A')$$

$$\boxed{R + B\rho(N_1-N_2) - AN_2 = 0} \quad 5.29 \quad (B) \qquad r+B\rho(n_1-n_2)-An_2=0 \quad (B')$$

$$-\frac{\rho}{t_0}+B\rho(N_2-N_1)\frac{h\nu}{V}F(\nu)=0 \quad 5.30$$ αν αγνοήσουμε το $A'<<A$

$$\Leftrightarrow \boxed{B\rho(N_2-N_1)=\frac{\rho}{t_0\left(\frac{h\nu}{V}\right)F(\nu)}} \quad (\Gamma) \qquad B\rho(n_2-n_1)=\frac{\rho}{t_0 h\nu F(\nu)} \quad (\Gamma')$$

(A)(B) πρόσθεση κατά μέλη $R=\frac{N_1}{t_1}\Leftrightarrow \boxed{N_1=t_1\cdot R}$ (Λ1)

(B)(Γ) πρόσθεση κατά μέλη $R-AN_2=\frac{\rho}{t_0(h\nu/V)F(\nu)}$ (Δ) $\Leftrightarrow N_2=\frac{R}{A}-\frac{\rho}{At_0(h\nu/V)F(\nu)}$ (Δ')

Π1 $\rho>0$

$$(\Delta)\Leftrightarrow R>AN_2 \overset{(\Lambda 1)}{\Leftrightarrow} \frac{N_1}{t_1}>\frac{N_2}{t_2}\Leftrightarrow \boxed{\frac{t_2}{N_2}>\frac{t_1}{N_1}} \quad (E)$$

$$(\Gamma)(\Delta')(\Lambda 1) \Rightarrow B\left(\frac{R}{A}-\frac{\rho}{At_0(h\nu/V)F(\nu)}\right)-Bt_1R=\frac{1}{t_0(h\nu/V)F(\nu)} \Leftrightarrow$$

$$BRt_2 - BRt_1 - \frac{B\rho}{At_0(h\nu/V)F(\nu)} = \frac{1}{t_0(h\nu/V)F(\nu)} \Leftrightarrow \frac{B\rho}{At_0(h\nu/V)F(\nu)} = B(t_2-t_1)R - \frac{1}{t_0\left(\frac{h\nu}{V}\right)F(\nu)}$$

$$\Leftrightarrow \rho = At_0\left(\frac{h\nu}{V}\right)F(\nu)\cdot(t_2-t_1)R-\frac{A}{B} \quad (ϛ)$$

Αλλά $\boxed{\rho>0}\Leftrightarrow R>\frac{A}{BAt_0(h\nu/V)F(\nu)(t_2-t_1)}\Leftrightarrow \boxed{R>\frac{1}{Bt_0(t_2-t_1)(h\nu/V)F(\nu)}:=R_c}$

ΠΡΟΣΟΧΗ: για $R_c>0 \Leftrightarrow t_2>t_1$ (λ)

οπότε το ρ μπορεί να γραφτεί

$$\rho = A\frac{1}{BR_c}\cdot R-\frac{A}{B}\Rightarrow \boxed{\rho=\frac{AR}{BR_c}-\frac{A}{B}} \quad (Z)$$

$$(Z)(\Delta') \Rightarrow N_2=\frac{R}{A}-\frac{R_cB(t_2-t_1)}{A}\left(\frac{AR}{BR_c}-\frac{A}{B}\right)\Rightarrow N_2=\frac{R}{A}-\frac{\cancel{R_c}\cancel{B}(t_2-t_1)\cancel{A}R}{\cancel{A}\,\cancel{B}\cancel{R_c}}+\frac{R_cB(t_2-t_1)\cancel{A}}{\cancel{A}\,\cancel{B}}$$

$$\Rightarrow N_2=t_2R-(t_2-t_1)R+(t_2-t_1)R_c \Rightarrow N_2=\cancel{t_2R}-\cancel{t_2R}+t_1R+t_2R_c-t_1R_c$$

$$\Rightarrow \boxed{N_2=t_1\cdot R+(t_2-t_1)R_c} \quad (H)$$

Π2 $\rho=0$

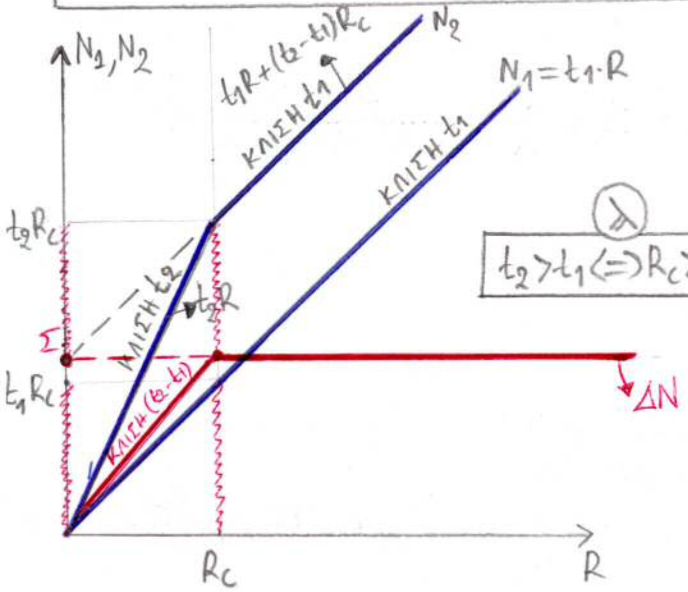
$$(\Delta) \Rightarrow R=AN_2 \Rightarrow \boxed{N_2=t_2R} \quad (\Theta)$$

Συνοψίζοντας, οἱ λύσεις τῶν ἐξισώσεων (Α)(Β)(Γ) εἶναι:

$$\boxed{N_1 = t_1 \cdot R, \quad \forall R} \quad (\Lambda 1)$$

$$\boxed{N_2 = \begin{cases} t_2 R, & \forall R < R_c \\ t_1 R + (t_2 - t_1) R_c, & \forall R > R_c \end{cases}} \quad (\Lambda 2)$$

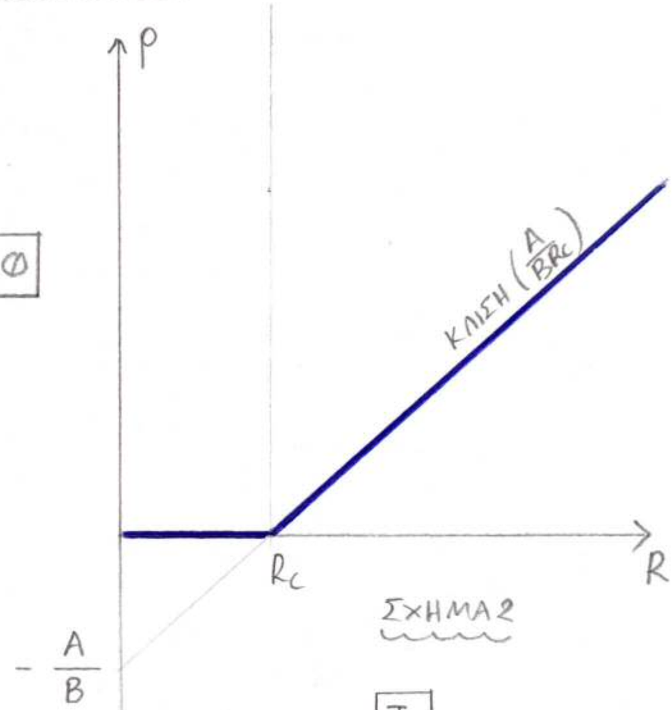
$$\boxed{\rho = \begin{cases} \varnothing, & \forall R < R_c \\ \dfrac{AR}{BR_c} - \dfrac{A}{B} = \dfrac{1}{Bt_2R_c} \cdot R - \dfrac{1}{Bt_2}, & \forall R > R_c \end{cases}} \quad (\Lambda 3)$$



(λ) $\boxed{t_2 > t_1 \Leftrightarrow R_c > \varnothing}$

ΣΧΗΜΑ 1

Σ • $(t_2 - t_1) R_c$



ΣΧΗΜΑ 2

Π1 • Ἡ ἀναστροφή πληθυσμοῦ $\Delta N := N_2 - N_1$

Ἄν $R < R_c \Rightarrow \Delta = t_2 R - t_1 R \Rightarrow \Delta = (t_2 - t_1) R$

Ἄν $R > R_c \Rightarrow \Delta = t_1 R + (t_2 - t_1) R_c - t_1 R \Rightarrow \Delta = (t_2 - t_1) R_c$

(I) $$\boxed{\Delta N = \begin{cases} (t_2 - t_1) R, & \forall R < R_c \\ (t_2 - t_1) R_c, & \forall R > R_c \end{cases}}$$

Ὁπότε ἄν θέλουμε $\boxed{\Delta N > \varnothing \Leftrightarrow t_2 > t_1}$

Π2 • $$\boxed{R_c = \frac{1}{Bt_0 (t_2 - t_1) \left(\frac{h\nu}{V}\right) F(\nu)}} \quad (5)$$

* ἄν $t_0 \uparrow \Leftrightarrow \dfrac{\rho}{t_0}$ ἀπώλειες στα κάτοπτρα $\downarrow \Leftrightarrow R_c \downarrow$

* $R_c > \varnothing \Leftrightarrow t_2 > t_1$ (λ)

* ἄν $t_2 >> t_1 \Rightarrow R_c$ μικρό

Π4 • Ὅλα αὐτά ἔχουν νόημα ἐφ' ὅσον ἐπιτρέπεται ἡ μετάπτωση ἀπό την ΑΝΩ ΣΤΑΘΜΗ ② στην κάτω στάθμη ① με ἐκπομπή φωτονίου. Θα πρέπει

transition (electric) dipole matrix element

$$\vec{P}_{12} = \int dV\, \Phi_1^*(\vec{r}) (-e) \vec{r}\, \Phi_2(\vec{r}) \neq \varnothing \Leftrightarrow$$

$$\vec{r}_{12} = \int dV\, \Phi_1^*(\vec{r})\, \vec{r}\, \Phi_2(\vec{r}) \neq \varnothing$$

Π3 • $$\frac{A}{B} = \frac{8\pi h \nu^3}{c^3} \Rightarrow \frac{1}{B} = \frac{8\pi h t_2 \nu^3}{c^3} \overset{(5)}{\Rightarrow} R_c = \frac{8\pi h t_2 \nu^3}{c^3 t_0 (t_2 - t_1) \frac{h\nu}{V} F(\nu)} \propto \nu^2$$

δηλ αὐτός εἶναι ἕνας λόγος ὥστε

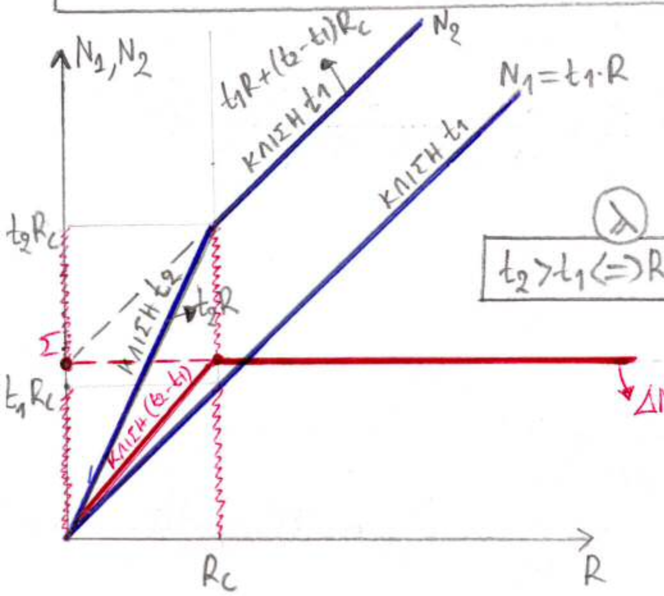
R_c (μικροκύματα) $< R_c$ (ὁρατό)

Συνοψίζοντας, οι λύσεις των εξισώσεων ⟨A⟩⟨B⟩⟨Γ⟩ είναι:

$$\boxed{N_1 = t_1 \cdot R, \quad \forall R} \qquad \langle \Lambda 1 \rangle$$

$$\boxed{N_2 = \begin{cases} t_2 R, & \forall R < R_c \\ t_1 R + (t_2 - t_1) R_c, & \forall R > R_c \end{cases}} \qquad \langle \Lambda 2 \rangle$$

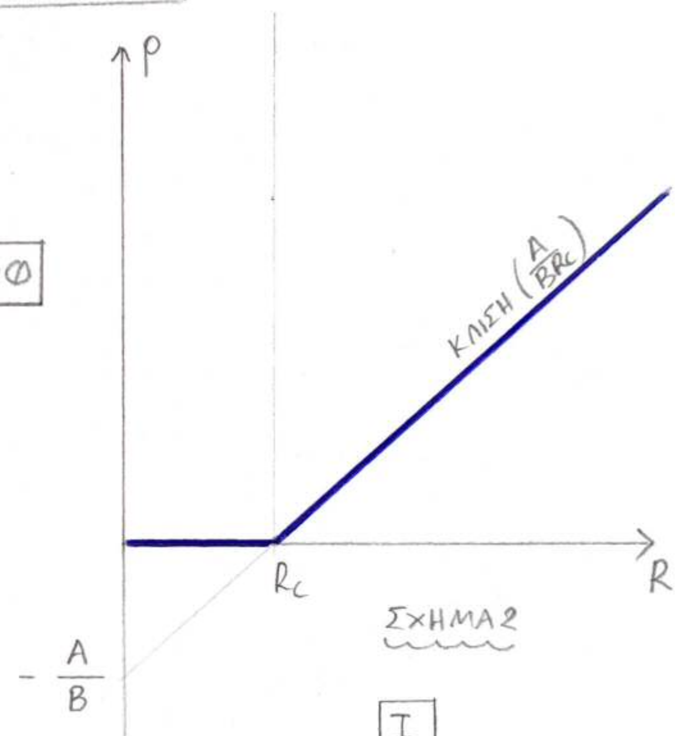
$$\boxed{\rho = \begin{cases} 0, & \forall R < R_c \\ \dfrac{AR}{BR_c} - \dfrac{A}{B} = \dfrac{1}{Bt_2R_c} \cdot R - \dfrac{1}{Bt_2}, & \forall R > R_c \end{cases}} \qquad \langle \Lambda 3 \rangle$$



ΣΧΗΜΑ 1

Σ • $(t_2-t_1)R_c$

(λ) $\boxed{t_2 > t_1 \Leftrightarrow R_c > 0}$



ΣΧΗΜΑ 2

Π1 • Η αναστροφή πληθυσμού $\Delta N := N_2 - N_1$

Αν $R < R_c \Rightarrow \Delta = t_2 R - t_1 R \Rightarrow \Delta = (t_2 - t_1) R$

Αν $R > R_c \Rightarrow \Delta = t_1 R + (t_2 - t_1) R_c - t_1 R \Rightarrow \Delta = (t_2 - t_1) R_c$

I
$$\boxed{\Delta N = \begin{cases} (t_2 - t_1) R, & \forall R < R_c \\ (t_2 - t_1) R_c, & \forall R > R_c \end{cases}}$$

Οπότε αν θέλουμε $\boxed{\Delta N > 0 \Leftrightarrow t_2 > t_1}$

Π2 • $$\boxed{R_c = \frac{1}{Bt_0 (t_2 - t_1) \left(\frac{h\nu}{V}\right) F(\nu)}} \qquad \langle 5 \rangle$$

* αν $t_0 \uparrow \Leftrightarrow \frac{\rho}{t_0}$ απώλειες στα κάτοπτρα $\downarrow \Leftrightarrow R_c \downarrow$

* $R_c > 0 \Leftrightarrow t_2 > t_1$ (λ)

* αν $t_2 >> t_1 \Rightarrow R_c$ μικρό

Π3 • $$\frac{A}{B} = \frac{8\pi h \nu^3}{c^3} \Rightarrow \frac{1}{B} = \frac{8\pi h t_2 \nu^3}{c^3} \overset{\langle 5 \rangle}{\Rightarrow} R_c = \frac{8\pi h t_2 \nu^3}{c^3 t_0 (t_2 - t_1) \frac{h\nu}{V} F(\nu)} \propto \nu^2$$

δηλ αυτός είναι ένας λόγος ώστε

R_c (μικροκύματα) $< R_c$ (ορατό)

Π4 • Όλα αυτά έχουν νόημα εφ' όσον επιτρέπεται η μετάπτωση από την ΑΝΩ ΣΤΑΘΜΗ ② στην κάτω στάθμη ① με εκπομπή φωτονίου. Θα πρέπει

transition (electric) dipole matrix element

$$\vec{P}_{12} = \int dV \, \Phi_1^*(\vec{r}) (-e) \vec{r} \, \Phi_2(\vec{r}) \neq 0 \Leftrightarrow$$

$$\vec{r}_{12} = \int dV \, \Phi_1^*(\vec{r}) \vec{r} \, \Phi_2(\vec{r}) \neq 0$$

Ας εφαρμόσουμε μια αλλαγή μεταβλητών ώστε να κάνουμε τις εξισώσεις Λ1 Λ2 Λ3 αδιάστατες.

• Πρώτα βρίσκουμε τα μεγέθη

$$n_i := \frac{N_i}{V}, \quad r := \frac{R}{V}, \quad r_c := \frac{R_c}{V}$$

οπότε οι Λ1 Λ2 Λ3 γίνονται:

$$\boxed{n_1 = t_1 r, \quad \forall r} \quad (\Lambda 1')$$

$$\boxed{n_2 = \begin{cases} t_2 r, & \forall r < r_c \\ t_1 r + (t_2 - t_1)\cdot r_c, & \forall r > r_c \end{cases}} \quad (\Lambda 2')$$

$$\boxed{\rho = \begin{cases} 0, & \forall r < r_c \\ \frac{1}{B t_2 r_c}\cdot r - \frac{1}{B t_2}, & \forall r > r_c \end{cases}} \quad (\Lambda 3')$$

όμοιως $\boxed{I} \Rightarrow$

$$\boxed{\Delta_n = \begin{cases} (t_2 - t_1)\cdot r, & \forall r < r_c \\ (t_2 - t_1)\cdot r_c, & \forall r > r_c \end{cases}} \quad (I')$$

$$\tau := \frac{t}{t_2}$$

$$\frac{\Delta n}{n_0} = \Delta\nu = \begin{cases} \frac{(t_2 - t_1) r}{t_2 r_c}, & \forall r < r_c \\ \frac{(t_2 - t_1) r_c}{t_2 r_c}, & \forall r > r_c \end{cases} \Rightarrow$$

$$\boxed{\Delta\nu = \begin{cases} (1 - \tau_1)\cdot r_N, & \forall r_N < 1 \\ (1 - \tau_1), & \forall r_N > 1 \end{cases}} \quad (II'')$$

• Κατόπιν τις κάνουμε αδιάστατες με τη βοήθεια των μεγεθών

$$[\varrho] = \frac{m^3}{J\cdot s^2}\cdot s \cdot \frac{J}{m^3 \cdot s} = 1$$

$$n_0 := t_2 r_c \qquad [n_0] = s\cdot\frac{\#}{s\cdot m^3} = \frac{1}{m^3} \qquad \varrho = B t_2 \rho$$

$$\nu_1 := \frac{n_1}{n_0} \qquad \nu_2 := \frac{n_2}{n_0} \qquad \frac{t_1}{t_2} := \tau_1 \qquad \frac{r}{r_c} := r_N$$

$$(\Lambda 1') \Rightarrow \frac{n_1}{n_0} = \frac{t_1 r}{t_2 r_c} \Rightarrow \boxed{\nu_1 = \tau_1 \cdot r_N, \quad \forall r_N} \quad (\Lambda 1'')$$

$$(\Lambda 2') \Rightarrow \frac{n_2}{n_0} = \begin{cases} \frac{t_2 r}{t_2 r_c}, & \forall r_N < 1 \\ \frac{t_1 r}{t_2 r_c} + \frac{(t_2 - t_1) r_c}{t_2 r_c}, & \forall r_N > 1 \end{cases} \Rightarrow$$

$$\Rightarrow \boxed{\nu_2 = \begin{cases} r_N, & \forall r_N < 1 \\ \tau_1 r_N + (1 - \tau_1), & \forall r_N > 1 \end{cases}} \quad (\Lambda 2'')$$

$$(\Lambda 3') \Rightarrow \varrho = B t_2 \rho = \begin{cases} 0 & \forall r_N < 1 \\ r_N - 1, & \forall r_N > 1 \end{cases}$$

$$\boxed{\varrho = \begin{cases} 0, & \forall r_N < 1 \\ r_N - 1, & \forall r_N > 1 \end{cases}} \quad (\Lambda 3'')$$

Στις εξισώσεις (Λ1'') (Λ2'') (Λ3'') όλα τα μεγέθη είναι αδιάστατα.

Τα $\nu_1, \nu_2, \Delta\nu$ εξαρτώνται μόνο από τα τ_1, r_N.
Το ϱ εξαρτάται μόνο από το r_N

π.χ. για $\tau_1 = 0.5$ και $r_N = 1.5 \Rightarrow$

$$\nu_1 = 0.5 \cdot 1.5 = 0.75$$

$$\nu_2 = 0.5\cdot 1.5 + (1 - 0.5) = 1.25 \qquad \Delta\nu = 0.5$$

$$\varrho = 1.5 - 1 = 0.5$$

ενώ για $\tau_1 = 0.5$ και $r_N = 0.5 \Rightarrow$

$$\nu_1 = 0.5 \cdot 0.5 = 0.25$$

$$\nu_2 = 0.5 \qquad \Delta\nu = 0.25$$

$$\varrho = 0$$

Ας ξαναθυμηθούμε τις σχετικές διαφορικές εξισώσεις (E1)(E2)(E3) μετατρέποντάς τες με τη βοήθεια των μεγεθών $\boxed{n_i := \frac{N_i}{V}}$ (n) και $\boxed{r := \frac{R}{V}}$ (r) ρυθμών laser

$$\frac{dn_1}{dt} = An_2 + B\rho(n_2 - n_1) - \frac{n_1}{t_1} \quad (E1')$$

ορίζουμε και $r_c := \frac{R_c}{V} = \frac{1}{Bt_0(t_2-t_1)h\nu F(\nu)}$

$$\frac{dn_2}{dt} = r + B\rho(n_1 - n_2) - An_2 \quad (E2')$$

$$\frac{d\rho}{dt} = -\frac{\rho}{t_0} + \left\{A'n_2 + B\rho(n_2 - n_1)\right\} h\nu F(\nu) \quad (E3')$$

Μπορούμε να λύσουμε το σύστημα των διαφορικών εξισώσεων (E1')(E2')(E3') είτε ως έχει είτε κάνοντάς τες πρώτα αδιάστατες. Για το λόγο αυτό ορίζουμε μερικά βοηθητικά αδιάστατα μεγέθη

$n_0 := t_2 r_c \quad [n_0] = s \cdot \frac{1}{sm^3} = \frac{1}{m^3} \qquad \nu_i := \frac{n_i}{n_0} \Leftrightarrow n_i = \nu_i n_0 \qquad [\nu_i] = \frac{\frac{1}{m^3}}{\frac{1}{m^3}} = 1$

$\wp = B\rho t_2 \quad [\wp] = \frac{m^3}{Js^2}\frac{J\cdot s}{m^3}\cdot s = 1 \qquad \tau := \frac{t}{t_2} \Leftrightarrow t = t_2\tau$

$\varpi := Vt_2 r \quad \varpi_c := Vt_2 r_c \quad \frac{\varpi}{\varpi_c} = \frac{r}{r_c} := r_N \quad t_1 = \boxed{\frac{t_1}{t_2}}t_2 = \tau_1 t_2 \quad t_0 = \boxed{\frac{t_0}{t_2}}t_2 = \tau_0 t_2$

(E1') $\frac{d(\nu_1 n_0)}{d(t_2\tau)} = \frac{\nu_2 n_0}{t_2} + \frac{\wp}{t_2}(\nu_2 - \nu_1)n_0 - \frac{\nu_1 n_0}{t_1 t_2}t_2 \Rightarrow \boxed{\frac{d\nu_1}{d\tau} = \nu_2 + \wp(\nu_2 - \nu_1) - \frac{\nu_1}{\tau_1}}$ (ε1)

(E2') $\frac{d(\nu_2 n_0)}{d(t_2\tau)} = \frac{\varpi n_0}{Vt_2 n_0} + \frac{\wp}{t_2}(\nu_1 n_0 - \nu_2 n_0) - \frac{\nu_2 n_0}{t_2} \Rightarrow \boxed{\frac{d\nu_2}{d\tau} = r_N + \wp(\nu_1 - \nu_2) - \nu_2}$ (ε2)

(E3') $\frac{d\wp}{d\tau}\frac{1}{Bt_2^2} = -\frac{\wp}{Bt_2^2\frac{t_0}{t_2}} + \left\{A'\nu_2 n_0 + \frac{\wp}{t_2}(\nu_2 n_0 - \nu_1 n_0)\right\}\frac{h\nu F(\nu)\,Bt_2^2}{Bt_2^2}$

$$\frac{d\wp}{d\tau} = -\frac{\wp}{\tau_0} + \left\{\frac{A'\nu_2 n_0 t_2}{t_2} + \frac{\wp}{t_2}n_0(\nu_2 - \nu_1)\right\} h\nu F(\nu)\,Bt_2^2$$

$$\frac{d\wp}{d\tau} = -\frac{\wp}{\tau_0} + \left\{\frac{A'}{A}\nu_2 + \wp(\nu_2 - \nu_1)\right\} n_0 h\nu F(\nu) Bt_2$$

$$t_2 r_c h\nu F(\nu) Bt_2 \overset{(5)}{=} \frac{\cancel{h\nu F(\nu)}Bt_2^2}{Bt_0(t_2-t_1)\cancel{h\nu F(\nu)}} = \frac{1}{\tau_0(1-\tau_1)}$$

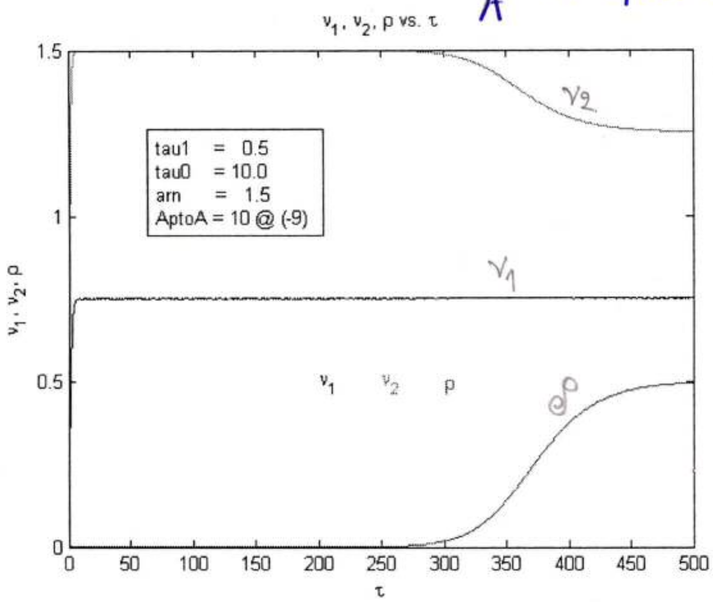
οπότε $\boxed{\frac{d\wp}{d\tau} = -\frac{\wp}{\tau_0} + \left\{\frac{A'}{A}\nu_2 + \wp(\nu_2 - \nu_1)\right\}\cdot\frac{1}{\tau_0(1-\tau_1)}}$ (ε3)

Τώρα η λύση των διαφορικών εξισώσεων (ε1)(ε2)(ε3) για τα $\nu_1, \nu_2, \wp$ εξαρτάται μόνο από τα $\tau_1, r_N, \tau_0, \frac{A'}{A}$.

Τις λύνουμε στο matlab με τη βοήθεια των αρχείων laser.m και call_laser_commands.m

Τρέξιματα laser.m κ call_laser_commands.m

για $\tau_1=0.5$ $\tau_0=10.0$ $r_N=1.5$ $\frac{A'}{A}=10^{-9}$ ή 10^{-4} ή 10^{-1}



$\boxed{\frac{A'}{A}=10^{-9}}$

• σε μερικά $\tau=\frac{t}{t_2}$ π.χ. $\tau=10$ $\Big\}$ $\nu_1=1.5=r_N$, $\nu_2=0.75$ $\left(\tau_1=\frac{t_1}{t_2}=0.5\right)$

αλλά $\rho=0$

διότι σε τ πολύ μικρά...

$$\frac{d\nu_1}{d\tau}=\overset{\text{μικρο}}{\nu_2}+\rho(\nu_2-\nu_1)-\overset{\text{αρνητ.}}{\frac{\nu_1}{\tau_1}} \quad (\text{Ε}1)$$

$$\frac{d\nu_2}{d\tau}=r_N+\rho(\nu_1-\nu_2)-\overset{\text{μικρο}}{\nu_2} \quad (\text{Ε}2)$$

$$\frac{d\rho}{d\tau}=-\frac{\rho}{\tau_0}+\left\{\frac{A'}{A}\nu_2+\rho(\nu_2-\nu_1)\right\}\frac{1}{\tau_0(1-\tau_1)} \quad (\text{Ε}3)$$

↑ η άντληση σχεδόν αμέσως δημιουργεί $\nu_2\neq0 \Rightarrow \nu_1\neq0$

και μάλιστα $\nu_2=1.5=r_N$
$\nu_1=0.75=\frac{r_N}{2}$ $\left(\tau_1=\frac{t_1}{t_2}=0.5\right)$

$\Delta\nu=0.75$

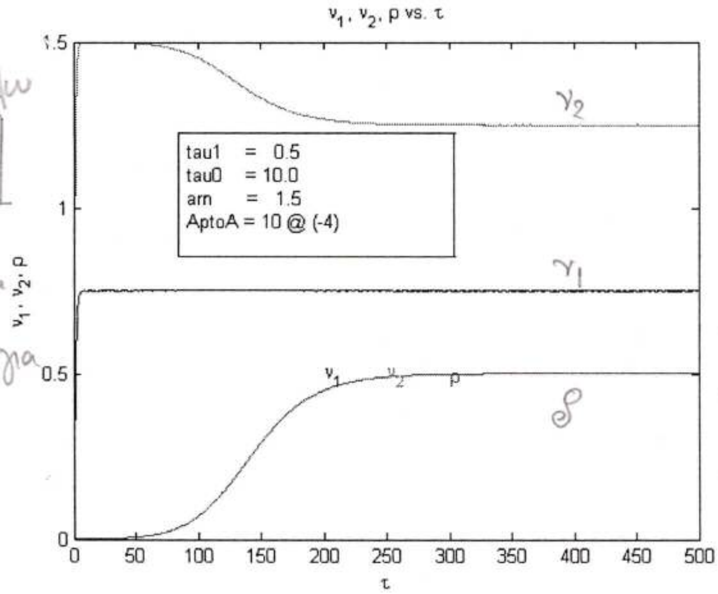
Όμως στην (Ε3) ο μόνος όρος που θα κινήσει αρχικά την $d\rho/d\tau$ είναι ο

$\frac{A'}{A}\nu_2\frac{1}{\tau_0(1-\tau_1)}$ αλλά $\frac{A'}{A}=10^{-9}$ $\left(\frac{\text{ΠΟΛΥ}}{\text{ΜΙΚΡΟ}}\right)$

οπότε αργεί να κινηθεί ο μηχανισμός παραγωγής ακτινοβολίας

Αν όμως βάλω $\boxed{\frac{A'}{A}=10^{-4}}$ όλα γίνονται γρηγορότερα

ρ αισθητή για $\tau\simeq40$

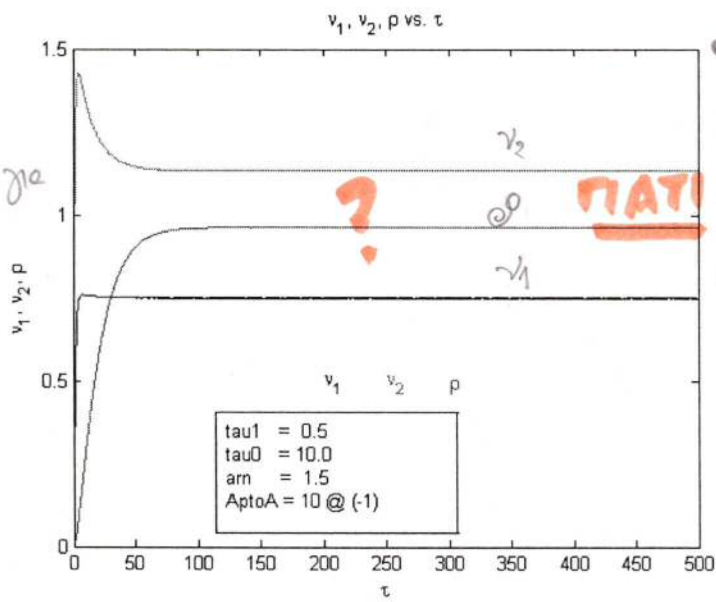


• Βλέπουμε ότι μετά το $\tau\simeq250$ αρχίζει να γίνεται αισθητή η ρ καθώς η ν_2 μικραίνει *
η ν_1 παραμένει ~"σταθερή" ήδη από $t>$ μερικά t_2

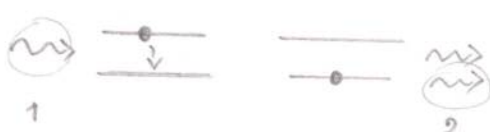
Και πάμε προς τη στάσιμη κατάσταση
$\nu_1=0.75$ $\nu_2=1.25$ $\rho=0.5$
όπως άλλωστε προβλέπουν οι εξ (Λ1'')(Λ2'')(Λ3'')

$\boxed{\frac{A'}{A}=10^{-1}}$

ρ αισθητή για $\tau\simeq0$



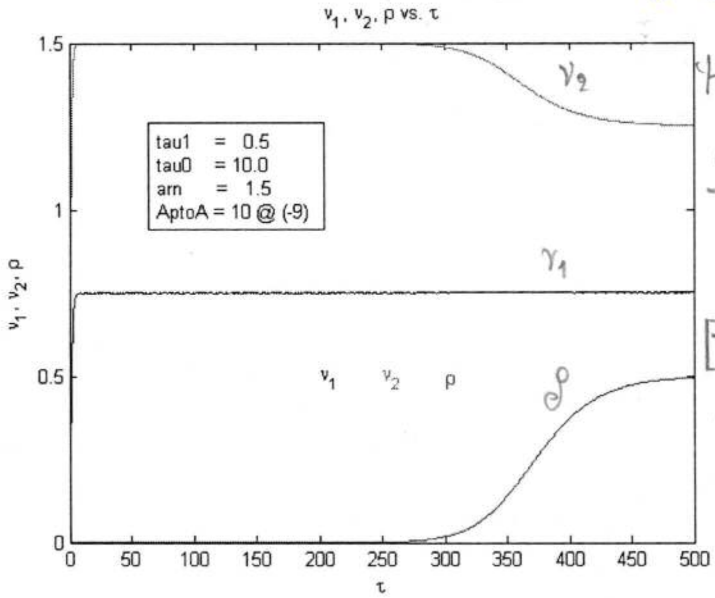
* θεριεύει ο μηχανισμός εξαναγκασμένης εκπομπής



4 8

Τρέξιματα laser.m κ call_laser_commands.m

για $\tau_1 = 0.5$ $\tau_0 = 10.0$ ή 5.0 ή 1.0 ή 0.5 $r_N = 1.5$ $\frac{A'}{A} = 10^{-9}$



Η επίδραση του τ_0 υπάρχει σε δύο όρους της (Ε3)

$$\frac{d\rho}{d\tau} = \underbrace{-\frac{\rho}{\tau_0}}_{1os} + \left\{\frac{A'}{A}\nu_2 + \underbrace{\rho(\nu_2-\nu_1)\right\}\frac{1}{\tau_0(1-\tau_1)}}_{2os}$$

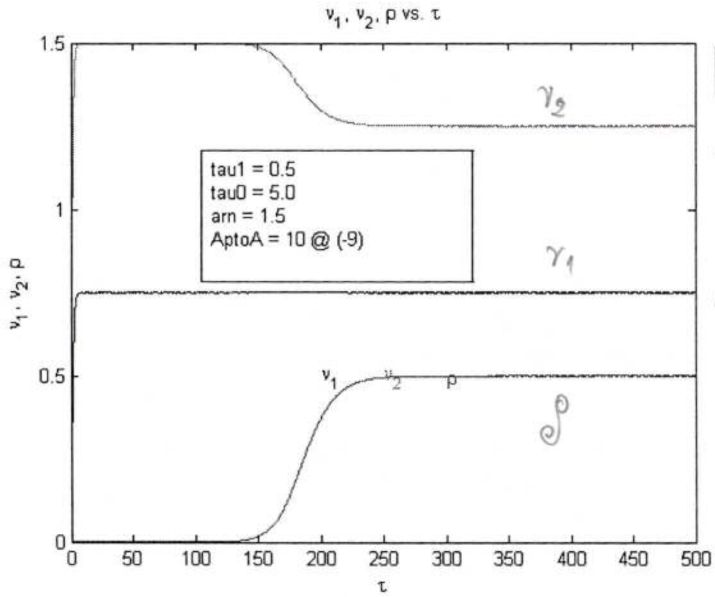
$\boxed{\tau_0 = 10.0}$ $\tau_0(1-\tau_1) = 10\cdot(1-0.5) = 5$

$\frac{1}{\tau_0} = 0.1$ $\frac{1}{\tau_0(1-\tau_1)} = \frac{1}{5} = 0.2$

$(\nu_2-\nu_1)_{\alpha\rho\chi} = 0.75 = \frac{3}{4}$

$\frac{1}{5}\cdot\frac{3}{4} = \frac{3}{20} = 0.15$

ΑΡΑ $2os - 1os = \underline{0.05}$



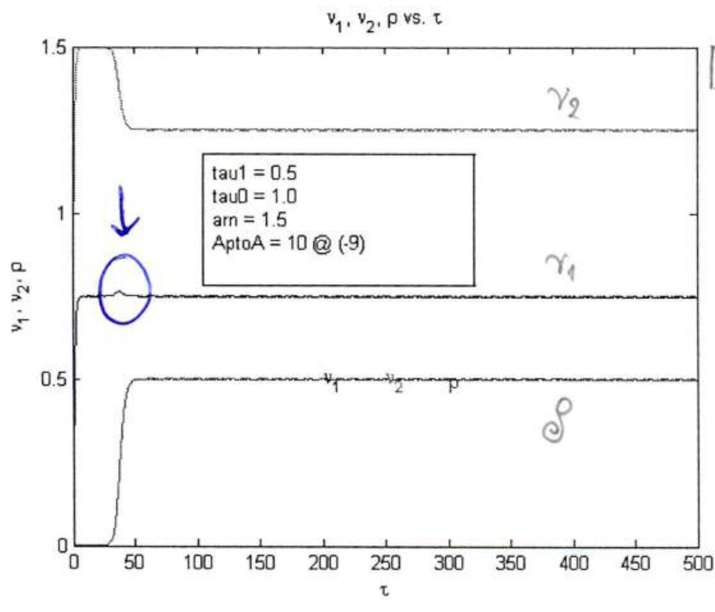
$\boxed{\tau_0 = 5.0}$ $\tau_0(1-\tau_1) = 5\cdot(1-0.5) = 2.5$

$\frac{1}{\tau_0} = \frac{1}{5} = 0.2$ $\frac{1}{\tau_0(1-\tau_1)} = \frac{2}{5}$

$(\nu_2-\nu_1)_{\alpha\rho\chi} = 0.75 = \frac{3}{4}$

$\frac{2}{5}\cdot\frac{3}{4} = \frac{3}{10} = 0.3$

ΑΡΑ $2os - 1os = 0.3 - 0.2 = \underline{0.1}$



$\boxed{\tau_0 = 1.0}$ $\tau_0(1-\tau_1) = 1(1-0.5) = 0.5$

$\frac{1}{\tau_0} = 1$ $\frac{1}{\tau_0(1-\tau_1)} = 2$

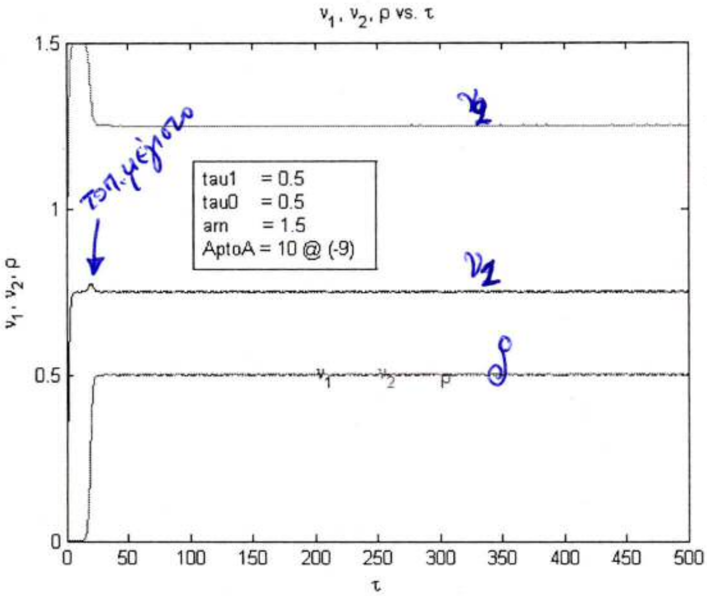
$(\nu_2-\nu_1)_{\alpha\rho\chi} = 0.75 = \frac{3}{4}$

$2\cdot\frac{3}{4} = \frac{3}{2} = 1.5$

ΑΡΑ $2os - 1os = 1.5 - 1 = \underline{0.5}$

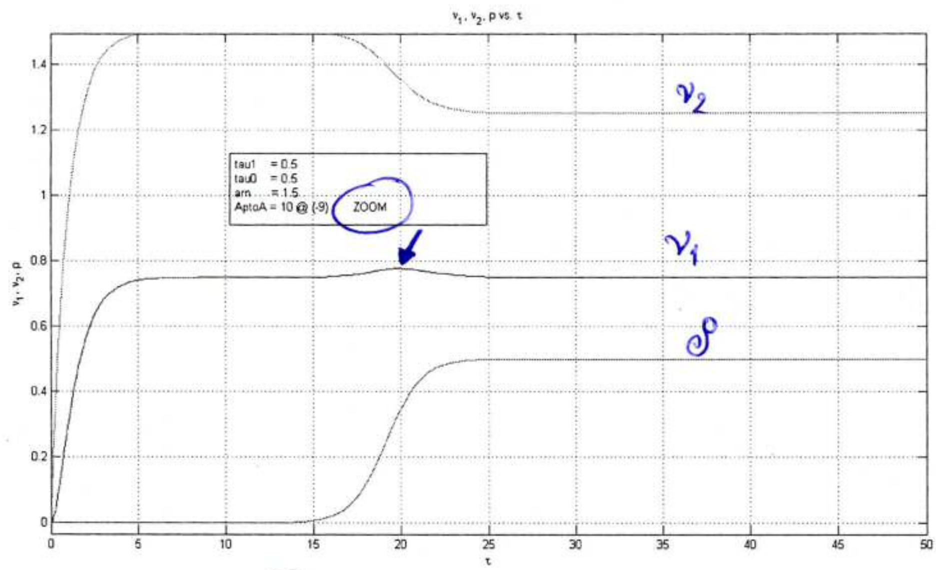
Δηλαδή αυξάνοντας το τ_0 η ρ εξελίσσεται γρηγορότερα

παρατηρούμε όμως μια κορυφούλα στην ν_1 ...



παρατηρούμε μιά πρόσκαιρη αύξηση της ν_1 κατά την απότομη πτώση της ν_2 απότομη αύξηση της ρ

δεν προλαβαίνουν τα ηλεκτρόνια που πέφτουν από την ② στην ① να φύγουν προς την βοηθητική.



Τοπικό max @ $\nu_1 \Rightarrow \dfrac{d\nu_1}{d\tau} = 0 \overset{(\varepsilon 1)}{\Rightarrow} \nu_2 + \rho(\nu_2 - \nu_1) - \dfrac{\nu_1}{\tau_1} = 0 \Rightarrow \rho(\nu_2 - \nu_1) - \dfrac{\nu_1}{\tau_1} = -\nu_2 \quad (*)$

$$(\varepsilon 2) \Rightarrow \frac{d\nu_2}{d\tau} = r_N + \rho(\nu_1 - \nu_2) - \nu_2 \overset{*}{=} r_N + \cancel{\rho(\nu_1 - \nu_2)} + \cancel{\rho(\nu_2 - \nu_1)} - \frac{\nu_1}{\tau_1}$$

$$\Rightarrow \frac{d\nu_2}{d\tau} = r_N - \frac{\nu_1}{\tau_1} \Rightarrow \frac{d^2\nu_2}{d\tau^2} = -\frac{1}{\tau_1}\frac{d\nu_1}{d\tau}$$

τότε 0 σημείο καμπής

όταν 0 τοπ. max

Phys. Bull. 36 (1985). Printed in Great Britain

PHYSICS AND COMPUTING

The value of computers in undergraduate physics teaching

D R Tilley

Ours is a well-established subject at degree level, and there is probably general agreement about the content and presentation of much that the student will meet. Courses on quantum mechanics, electromagnetism or solid state physics, for example, are likely to vary only marginally from one institution to another. By contrast, I believe there is still considerable uncertainty about how to make use of computers, and what to teach about them in the course of a physics degree. Mainframe computers, and more recently microprocessors, are familiar research tools, but even ten years ago the shortage of mainframe time for students greatly restricted their possible use in teaching. We are no longer trapped in this mainframe bottleneck, and accordingly most university and polytechnic departments have been increasing the computer-based content of physics degrees. Firstly, many students come to us with obvious enthusiasm for computers, and there is a desire to foster this enthusiasm and channel it partly towards learning physics; secondly, it is clear that the graduating physicist should have some competence in computer techniques.

One development that is not apparently occurring to any great extent in the UK is the replacement of conventional teaching by computer-assisted learning. This may simply reflect the relatively small size of departments and institutions, since the software development cost of an extensive CAL package such as has been produced for mechanics (Bork 1981) can only be justified for very large classes. Such classes, typically dominated by pre-medical and pre-engineering students, are common in the USA, but rare here. It is not my purpose, however, to go into the pros, cons and costs of CAL.

Example 1

A great deal of physics depends ultimately on random walks of various kinds. In this example, students start with a working program in BASIC which simulates a random walk and presents the results numerically. They extend the program so that samples of varying sizes can be studied undergoing a random walk with a fixed number of steps. The results are presented graphically and compared with the binomial distribution as in figure 1. In an extension, two-dimensional random walks are simulated, and students investigate the relationship between the overall distance travelled and the number of steps.

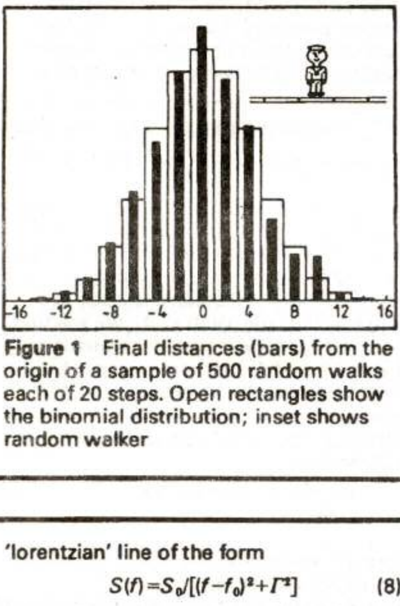


Figure 1 Final distances (bars) from the origin of a sample of 500 random walks each of 20 steps. Open rectangles show the binomial distribution; inset shows random walker

Example 2

Figure 5 shows experimental data for inelastic light scattering in the magnetic crystal FeF_2. A simple-minded theory of the line shape predicts a symmetric 'lorentzian' line of the form

$$S(f)=S_0/[(f-f_0)^2+\Gamma^2] \qquad (8)$$

where S is the intensity and f the frequency. f_0 is the frequency of maximum intensity, and Γ is related to the linewidth. However, the data of figure 5 are quite noticeably skewed to the right, and it is intuitively clear that in some sense the lorentzian curve does not fit the data, whereas an approximate skewed function (derived by Smith *et al* 1983) does. To put this intuitive view on a sound statistical basis is a worthwhile challenge for an able student. Broadly speaking, it has to be proved that the best-fit lorentzian has too large a χ^2, whereas the best-fit skew function has a small χ^2. The statistical basis, and hence the fitting procedures, need care since the data are photon counts, with a Poisson distribution, whereas a simple least-squares fit is justified only if the data obey normal statistics. Thus a project based on this example demands careful statistical and mathematical analysis, as well as competence in the use of subroutines for optimisation and graphics.

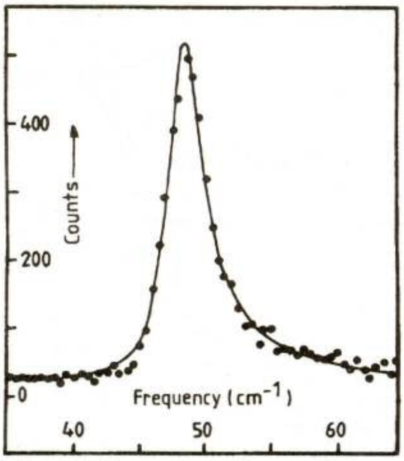


Figure 5 One-magnon Raman spectrum of FeF_2 at 36 K (dots), and theoretical curve of Smith *et al* (1983)

Computer-related content

Where a physics degree is being taught in a fairly conventional way, possible computer-orientated content may be crudely divided into four main areas. In commenting on these, I shall draw on our own experience at Essex, since it is probably fairly typical of elsewhere.

The first area, and the easiest to dispose of, is computer languages. Most physics departments probably either assume their students know BASIC or let them pick it up, while a higher-level language is often taught more formally. There is no doubt that FORTRAN is the majority choice – it has the same merits and demerits among computer languages that the VW Beetle has among cars – but it is unsuitable for various non-numerical applications. A good comparison of various languages has been given by Hamann (1983).

A more difficult matter is to employ effectively the enthusiasm for computing in reinforcing physics teaching. In the preface to his *Quantum Mechanics*, written nearly 60 years ago, Dirac says, in effect, that physics is abstract and is becoming more so. This trend has duly reached undergraduate physics, so that students meet far more mathematical and abstract material than they would have done even 30 years ago. The impact is strong in the second year, where long courses on topics like quantum mechanics and electromagnetism appear.

0031-9112/85/110456 + 03 $02.25

Fortunately many of the formulae are suitable for numerical or graphical investigation, and for many students this can be very helpful. It has to be decided whether to provide a prewritten package into which numbers are fed, or whether the student should write at least part of the program. Our view is that impenetrable packages are rather boring, and the students will learn more if they do some programming. There is also the logistic question of how to timetable this kind of activity; we decided to treat computer 'experiments' as part of the second-year laboratory. Example 1 shows one type of computer-reinforcement of physics courses.

A third area is teaching the techniques of mathematical modelling and data analysis, and here challenging questions of aims, content and presentation arise. The uses a professional physicist makes of computers are varied, and many experienced physicists have developed considerable knowledge of numerical techniques. Although the older generation has acquired its knowledge informally, what is needed can naturally be defined and taught. Many physical systems are described, more or less accurately, by some kind of mathematical model. Analysing the predictions of a model often involves a mixture of analytical, approximate and numerical work, and qualitative judgments have to be made about the results. Nowadays the numerical processes commonly required are available in subroutine libraries, such as NAG.

After some initial uncertainty, I was persuaded by a friendly numerical analyst that it is not productive to try to turn physicists into amateur mathematicians; we should aim to ensure that they are discerning customers for the mathematical software available to them. For example, some knowledge of the ideas of ill-conditioning of matrices and stiffness of sets of differential equations is very useful. One way in which ill-conditioning can get into numerical work is through bad scaling, and a common preliminary step is to define suitable reduced variables. Thus the elementary problem of charging or discharging a capacitor is best worked in terms of the RC time.

Example 3 is less elementary; here the variables are scaled in terms of steady state values. The transition from equations (1)–(3) to equations (5)–(7) is more or less essential. The use of 'real-scale' numbers, e.g. $10^{23}\ \mathrm{m}^{-3}$ for n_1 and n_2, in equations (1)–(3) would almost certainly lead to overflow, whereas equations (5)–(7) are free from this complication. Example 3 also exhibits, I hope, a number of other instructive features. The analysis of the steady state solutions is entirely analytical, and knowledge of these analytical solutions is most valuable in studying the results of numerical work. Thus figure 4 shows, in fact, a switch from a metastable to a stable steady state solution.

The aspect of data analysis that involves the fitting of theoretical models to experimental results is easily accommodated in a

Example 3

The following represents a suitable set of rate equations for a two-level laser:

$$\mathrm{d}n_1/\mathrm{d}t = -n_1/t_1 + An_2 + B(n_2-n_1)W \qquad (1)$$
$$\mathrm{d}n_2/\mathrm{d}t = r_2 - An_2 - B(n_2-n_1)W \qquad (2)$$
$$\mathrm{d}W/\mathrm{d}t = hf\mathrm{F}(f)[A'n_2 + B(n_2-n_1)W] - W/t_0. \qquad (3)$$

Here, n_1, n_2 are the numbers of atoms in the lower and upper lasing levels and W is the electromagnetic energy per unit volume per unit frequency range. The terms in (1) and (2) are explained in figure 2. A and B are the coefficients of spontaneous and stimulated emission, satisfying the Einstein relation $A/B=4hf^3/c^3$. hf is the photon energy, $\mathrm{F}(f)$ is a lineshape function, and t_0 is the decay time of the cavity, due primarily to the partially transmitting mirrors. A' is not necessarily equal to A, because An_2 describes all transitions, while $A'n_2$ refers only to transitions involving the emission of a quantum into the lasing mode. Typically, $A'/A=0.15\times10^{-9}$.

The steady state equations, with the time derivatives zero, can be solved analytically. For this purpose, the term in A' can be neglected. The solutions are represented in figure 3 as graphs of n_1, n_2 and W versus r_2, with other parameters kept constant. The system undergoes a phase transition from nonlasing ($W=0$) to lasing ($W=0$) action at $r_2=r_c$, where

$$r_c = A/hfF(f)Bt_0(1-At_1) \qquad (4)$$

which shows that laser action requires $At_1<1$.

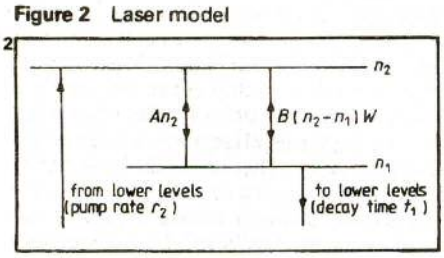
As a preliminary to numerical solution of the time-dependent equations it is essential to define dimensionless variables. The equations contain three time constants, A^{-1}, t_1 and t_0, and there is therefore latitude in the choice that is made. With $\tau=At$, we see that $\tau_1=At_1$ and $\tau_0=At_0$ will enter the equations as parameters. It is natural to scale r_2 as $\rho=r_2/r_c$, and n_1 and n_2 in terms of $n_0=r_c/A$, that is $\nu_1=n_1/n_0$, $\nu_2=n_2/n_0$. Finally, we define $\omega=BW/A$. In terms of these, equations (1)–(3) become

$$\mathrm{d}\nu_1/\mathrm{d}\tau = -\nu_1/\tau_1 + \nu_2 + \omega(\nu_2-\nu_1) \qquad (5)$$
$$\mathrm{d}\nu_2/\mathrm{d}\tau = \rho - \nu_2 - \omega(\nu_2-\nu_1) \qquad (6)$$
$$\mathrm{d}\omega/\mathrm{d}\tau = [\omega(\nu_2-\nu_1) + A'\nu_2/A]/\tau_0(1-\tau_1) - \omega/\tau_0. \qquad (7)$$

Typical values of the time-constant ratios are $\tau_1=0.5$, $\tau_0=10$.

The equations (5)–(7) are now in a suitable form for numerical work and can be programmed using library subroutines. Many 'numerical experiments' can be carried out; here there is room for just one example, shown in figure 4. r is increased from zero to $1.5r_c$, for which in the steady state $\nu_1=0.75$, $\nu_2=1.25$ and $\omega=0.5$. ν_1 and ν_2 reach values ρ and 0.5ρ, i.e. 1.5 and 0.75, within a time of order A^{-1}. Laser action, however, depends on the build-up of power ω through the nonlinear stimulated emission term. From equation (7) the solution with $\omega=0$ is near-metastable, since the switch from this to the lasing solution depends on the small ratio A'/A. This is why the build-up of the lasing solution takes a long time, of the order of $500A^{-1}$ in the example given.

Figure 2 Laser model

2


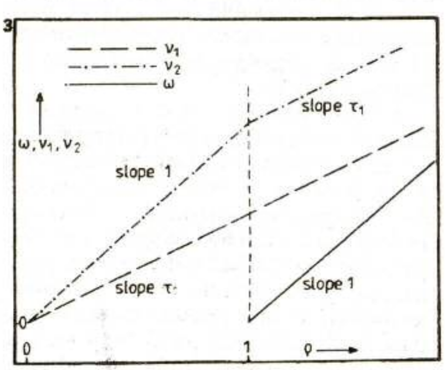
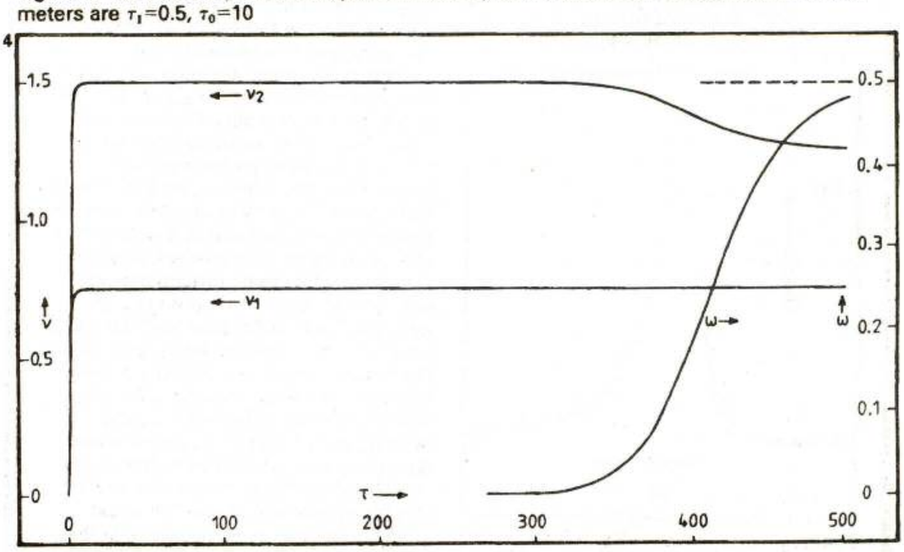
3


Figure 3 Steady-state solutions of the laser model. The graph uses the reduced variables defined in the text

Figure 4 Solution of laser rate equations when ρ is increased at $\tau=0$ from 0 to 1.5. Parameters are $\tau_1=0.5$, $\tau_0=10$

4


Example 4

Microprocessor timers are employed in many applications. In one of our experiments the student is asked to write a complete program to determine the frequency response of a commercially manufactured vibrating-beam apparatus. The beam is set into forced oscillation by a driving signal which is fed to the beam's actuator coils via an amplifier. Pick-up coils monitor the beam velocity to provide a signal which is repeatedly read by the micro via an A/D converter. Whenever the frequency is changed, the oscillation amplitude takes time to settle to a steady value and a preliminary experiment is needed to determine this settling time. In the main experiment the coils are driven in turn at each of a number of different frequencies and the steady state velocity is monitored to deduce an average amplitude. With a simple micro (KIM) the independent variable has to be the period and only a square-wave drive is readily available. With a BBC the frequency or its logarithm may be used instead and a range of driving waveforms is available. In either case the program compiles a table of amplitudes versus the appropriate variable, which can in due course be displayed graphically either on a 'scope or using screen graphics.

The velocity response to a sinusoidal driving force is shown in figure 6, the magnitude and phase both behaving essentially in accordance with the theory of the damped harmonic oscillator. Figure 7 shows the more complicated response to a square-wave drive. A square wave of fundamental frequency f is a sum of sinusoidal harmonics of frequencies $(2n+1)f$, where $n=0,1,2\ldots$. The beam resonates whenever one of these harmonic frequencies is equal to the frequency f_0=8.4 Hz of the peak in figure 6. Thus a 'subharmonic' series is seen at frequencies $f=f_0/(2n+1)$.

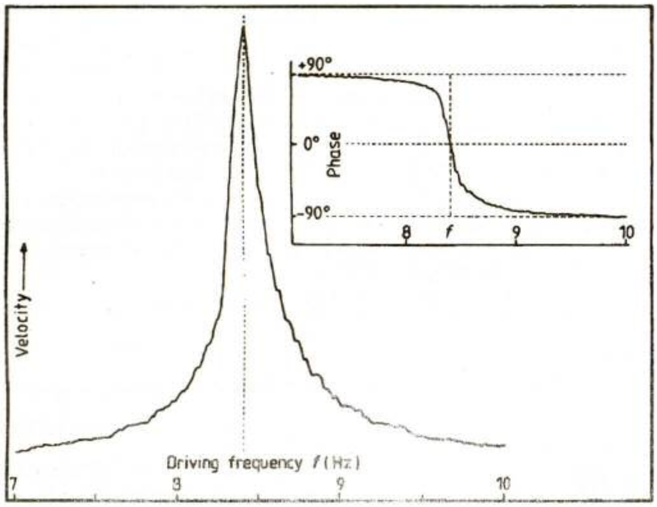


Figure 6 Magnitude (main curve) and phase (inset) of velocity response of vibrating-beam apparatus to sinusoidal driving force. The ripples are believed to be genuine features of the system response

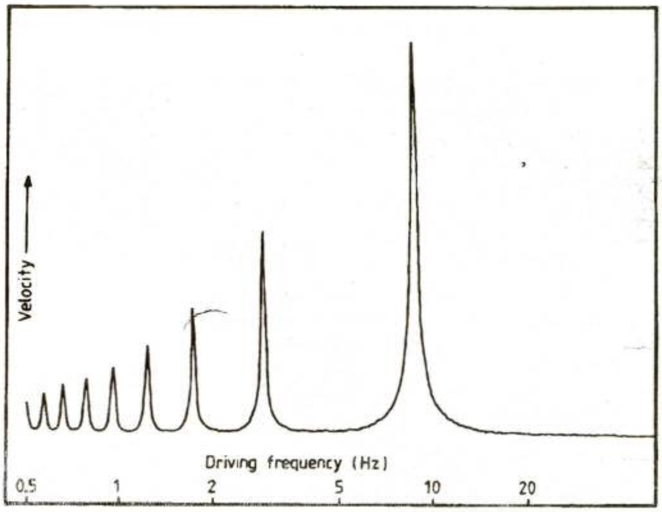


Figure 7 Magnitude of velocity response of vibrating-beam apparatus to square-wave driving force (note logarithmic frequency scale). The square wave is a sum of odd harmonics, so the beam resonates at frequencies $f_n=f_0/(2n+1)$, where f_0 is the resonant frequency of figure 6

course on mathematical software. Two questions (at least) arise: what are the parameters that give the best fit of this model to these results?; and, is the model, with these parameters, a valid description of the results? These points are highlighted in example 2. Like the previous example, this shows the importance and value of the non-numerical work that precedes numerical analysis: in this case some careful statistical mathematics is needed to define the numerical problem. It seems to me, in passing, that not all physicists are good at statistics, so that even in the heavyweight journals one can see the first question answered by means of a least-squares fit without justification, and the second question ignored.

A course on mathematical software can hardly be justified as a compulsory element of a physics degree, but it is certainly valuable as an option. Our decision was to offer it, together with microprocessor techniques, as a third-year option, and to place considerable emphasis on practical and project work. Access to an extensive subroutine library a few years ago meant using terminals on a mainframe machine, but the switch to distributed computing power will greatly increase flexibility.

I have left until last the topic I know least about, namely the application of computers in a laboratory and experimental context. Microprocessors are used in many ways for controlling apparatus and data handling, some of which will be familiar to students from school days. Excessive automation of teaching experiments to the point where, so to speak, only the microprocessor handles the apparatus is not necessarily desirable, but it is obstinate to resist automation where it is sensible. It will probably be some time before there is general agreement about where the proper balance lies. What does seem clear is that general microprocessor principles should be taught at least as an option. This means not only programming, probably in assembly code, but also familiarity with stepping motors, D/A and A/D converters and so on. Example 4 illustrates the kind of exercise that can be used.

Looking forward

The picture I have painted, which I believe is accurate, is of an evolutionary incorporation of computer-based techniques into physics courses. Individual enthusiasts have been developing their own ideas, not entirely in isolation but certainly without being able to draw on a fully developed general understanding of how to proceed. Some useful books have appeared, for example those by Killingbeck (1983) and Harding (1985), and occasional articles are published in journals such as the *European Journal of Physics*, *American Journal of Physics* and *Physics Education*. In an effort to aid the exchange of ideas, the Education and Computational Physics Groups of The Institute of Physics plan to hold occasional joint meetings; the first of these, on the second of the four areas defined above, will be at the University of Essex on 9 January 1986 ■

Acknowledgments

I have benefited from discussions with many colleagues, including in particular Mr W Hart and Dr M Davis. Thanks for the examples are due to Dr J Tilley and Mr I Gregorelli (example 1), Dr D Andrews and Professor R Loudon (example 2), Dr S Smith (example 3) and Mr G King (example 4). In connection with the planned meetings I have corresponded with members of many physics departments, and I hope they will forgive the occasional generalised and unattributed quotation in this article.

Further reading

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D R Tilley is in the Physics Department at the University of Essex