προ Θ

ΣΥΝΟΨΗ ΧΑΜΙΛΤΟΝΙΑΝΩΝ

$$\hat{H}_{HM,m} = \hbar\omega_m\left(\hat{a}_m^{\dagger}\hat{a}_m + \frac{1}{2}\right) = \hbar\omega\left(\hat{N}_m + \frac{1}{2}\right)$$ m τρόπος HM πεδίου στην κοιλότητα

κι αγνοώντας τον όρο $\frac{\hbar\omega_m}{2}$

$$\hat{H}_{HM,m} = \hbar\omega_m \hat{a}_m^{\dagger}\hat{a}_m = \hbar\omega_m \hat{N}_m$$

$$\hat{H}_{\Delta\Sigma} = E_2\hat{S}_+\hat{S}_- + E_1\hat{S}_-\hat{S}_+ \qquad E_2 - E_1 = \hbar\Omega$$ διστάθμικό σύστημα

και θέτοντας $E_1 \equiv 0$

$$\hat{H}_{\Delta\Sigma} = \hbar\Omega\hat{S}_+\hat{S}_-$$

$$\hat{H}_{\varepsilon m} = \hbar g_m\left(\hat{S}_+ + \hat{S}_-\right)\left(\hat{a}_m^{\dagger} + \hat{a}_m\right) = \hat{H}_{\Delta\Sigma-HMm}$$

$\Delta\Sigma-HMm$ αλληλεπίδραση

αλληλεπίδραση ΔΣ – m τρόπου HM πεδίου στην κοιλότητα

($\hat{H}_{AF}$ ατομική φυσική)

atom field

$$\hbar g_m := e x_{12}\left(\frac{\hbar\omega_m}{\varepsilon_0 V}\right)^{1/2}\sin\left(\frac{m\pi z}{L}\right)$$

$$\hbar|g_m| = \underbrace{|e x_{12}|}_{|\wp|}\underbrace{\left(\frac{4\hbar\omega_m n_m}{\varepsilon_0 V}\right)^{1/2}\left|\sin\left(\frac{m\pi z}{L}\right)\right|}_{E_{0m}(z)}\frac{1}{2\sqrt{n_m}}$$

$$\Omega_{Rm} := \frac{|\wp|\, E_{0m}}{\hbar} \qquad \Omega_{Rm} := 2\sqrt{n_m}\, g_m$$

συχνότητα Rabi του m τρόπου

$|\uparrow n_m\rangle$, $|\downarrow n_m\rangle$ οι καταστάσεις της $\hat{H}_{HMm} + \hat{H}_{\Delta\Sigma}$

Ενώ, η συνολική Χαμιλτονιανή γράφεται

$$\hat{H}_{Rm} = \hbar\omega_m\hat{a}_m^{\dagger}\hat{a}_m + \hbar\Omega\hat{S}_+\hat{S}_- + \hbar g_m\left(\hat{S}_+ + \hat{S}_-\right)\left(\hat{a}_m^{\dagger} + \hat{a}_m\right)$$

ονομάζεται συχνά Χαμιλτονιανή Rabi

Αγνοώντας τους όρους $\hat{S}_+\hat{a}_m^{\dagger}$ κ $\hat{S}_-\hat{a}_m$, οι οποίοι φαίνονται "παράλογοι" όταν υπάρχει ένας μόνο τρόπος!

$$\hat{H}_{JCm} = \hbar\omega_m\hat{a}_m^{\dagger}\hat{a}_m + \hbar\Omega\hat{S}_+\hat{S}_- + \hbar g_m\left(\hat{S}_+\hat{a}_m + \hat{S}_-\hat{a}_m^{\dagger}\right)$$

ονομάζεται συχνά Χαμιλτονιανή Jaynes-Cummings

ΑΣΚΗΣΗ (Παραλείπουμε για απλότητα τον δείκτη του ΗΜ τρόπου m)

Ⓐ Βρείτε τι κάνουν οι όροι

$\hat{a}^{+}\hat{a},\ \hat{a}\hat{a}^{+},\ \hat{S}_{+}\hat{S}_{-},\ \hat{S}_{-}\hat{S}_{+},\ \hat{S}_{+}\hat{a}^{+},\ \hat{S}_{+}\hat{a},\ \hat{S}_{-}\hat{a}^{+},\ \hat{S}_{-}\hat{a}$ στις καταστάσεις $|\uparrow, n_m\rangle$ κ $|\downarrow, n_m\rangle$

Ⓑ Υπολογίστε τα

$\langle\hat{a}^{+}\hat{a}\rangle,\ \langle\hat{a}\hat{a}^{+}\rangle,\ \langle\hat{S}_{+}\hat{S}_{-}\rangle,\ \langle\hat{S}_{-}\hat{S}_{+}\rangle,\ \langle\hat{S}_{+}\hat{a}^{+}\rangle,\ \langle\hat{S}_{+}\hat{a}\rangle\ \langle\hat{S}_{-}\hat{a}^{+}\rangle,\ \langle\hat{S}_{-}\hat{a}\rangle$ στις $|\uparrow n_m\rangle$, $|\downarrow n_m\rangle$

Ⓐ $\hat{a}^{+}\hat{a}|\uparrow, n\rangle = n|\uparrow, n\rangle \qquad \hat{a}^{+}\hat{a}|\downarrow, n\rangle = n|\downarrow, n\rangle$

$[\hat{a}, \hat{a}^{+}] = 1 \iff \hat{a}\hat{a}^{+} = 1 + \hat{a}^{+}\hat{a}$

$\hat{a}\hat{a}^{+}|\uparrow, n\rangle = (1+\hat{a}^{+}\hat{a})|\uparrow, n\rangle = (n+1)|\uparrow, n\rangle$

$\hat{a}\hat{a}^{+}|\downarrow, n\rangle = (1+\hat{a}^{+}\hat{a})|\downarrow, n\rangle = (n+1)|\downarrow, n\rangle$

$\hat{S}_{+}\hat{S}_{-}|\uparrow, n\rangle = |\uparrow, n\rangle \qquad \hat{S}_{+}\hat{S}_{-}|\downarrow, n\rangle = |0, n\rangle$

$\hat{S}_{-}\hat{S}_{+}|\uparrow, n\rangle = |0, n\rangle \qquad \hat{S}_{-}\hat{S}_{+}|\downarrow, n\rangle = |\downarrow, n\rangle$

$\hat{S}_{+}\hat{a}^{+}|\uparrow, n\rangle = \sqrt{n+1}\,|0, n+1\rangle \qquad \hat{S}_{+}\hat{a}^{+}|\downarrow n\rangle = \sqrt{n+1}\,|\uparrow, n+1\rangle$

$\hat{S}_{+}\hat{a}|\uparrow n\rangle = \sqrt{n}\,|0, n-1\rangle \qquad \hat{S}_{+}\hat{a}|\downarrow n\rangle = \sqrt{n}\,|\uparrow, n-1\rangle$

$\hat{S}_{-}\hat{a}^{+}|\uparrow n\rangle = \sqrt{n+1}\,|\downarrow, n+1\rangle \qquad \hat{S}_{-}\hat{a}^{+}|\downarrow n\rangle = \sqrt{n+1}\,|0, n+1\rangle$

$\hat{S}_{-}\hat{a}|\uparrow, n\rangle = \sqrt{n}\,|\downarrow, n-1\rangle \qquad \hat{S}_{-}\hat{a}|\downarrow, n\rangle = \sqrt{n}\,|0, n-1\rangle$

Ⓑ $\langle\uparrow n|\hat{a}^{+}\hat{a}|\uparrow n\rangle = n$

$\langle\downarrow n|\hat{a}^{+}\hat{a}|\downarrow n\rangle = n$

$\langle\uparrow n|\hat{a}\hat{a}^{+}|\uparrow n\rangle = n+1$

$\langle\downarrow n|\hat{a}\hat{a}^{+}|\downarrow n\rangle = n+1$

$\langle\uparrow n|\hat{S}_{+}\hat{S}_{-}|\uparrow n\rangle = 1$

$\langle\downarrow n|\hat{S}_{+}\hat{S}_{-}|\downarrow n\rangle = \langle\downarrow n|0\, n\rangle = 0$

$\langle\uparrow n|\hat{S}_{-}\hat{S}_{+}|\uparrow n\rangle = \langle\uparrow n|0\, n\rangle = 0$

$\langle\downarrow n|\hat{S}_{-}\hat{S}_{+}|\downarrow n\rangle = \langle\downarrow n|\downarrow n\rangle = 1$

συνεχίζεται ...

$\langle \hat{a}^{+}\hat{a} \rangle_{\|\uparrow n\rangle} = n$	$\langle \hat{a}^{+}\hat{a} \rangle_{\|\downarrow n\rangle} = n$
$\langle \hat{a}\hat{a}^{+} \rangle_{\|\uparrow n\rangle} = n+1$	$\langle \hat{a}\hat{a}^{+} \rangle_{\|\downarrow n\rangle} = n+1$
$\langle \hat{S}_{+}\hat{S}_{-} \rangle_{\|\uparrow n\rangle} = 1$	$\langle \hat{S}_{+}\hat{S}_{-} \rangle_{\|\downarrow n\rangle} = 0$
$\langle \hat{S}_{-}\hat{S}_{+} \rangle_{\|\uparrow n\rangle} = 0$	$\langle \hat{S}_{-}\hat{S}_{+} \rangle_{\|\downarrow n\rangle} = 1$

$$\langle \hat{S}_{+}\hat{a}^{+} \rangle_{|\uparrow n\rangle} = 0 \qquad \langle \hat{S}_{+}\hat{a}^{+} \rangle_{|\downarrow n\rangle} = 0$$

$$\langle \hat{S}_{+}\hat{a} \rangle_{|\uparrow n\rangle} = 0 \qquad \langle \hat{S}_{+}\hat{a} \rangle_{|\downarrow n\rangle} = 0$$

$$\langle \hat{S}_{-}\hat{a}_{m}^{+} \rangle_{|\uparrow n\rangle} = 0 \qquad \langle \hat{S}_{-}\hat{a}_{m}^{+} \rangle_{|\downarrow n\rangle} = 0$$

$$\langle \hat{S}_{-}\hat{a} \rangle_{|\uparrow n\rangle} = 0 \qquad \langle \hat{S}_{-}\hat{a} \rangle_{|\downarrow n\rangle} = 0$$

... συνεχίζεται

$$\langle \uparrow n | \hat{S}_{+}\hat{a}^{+} | \uparrow n \rangle = \sqrt{n+1}\, \langle \uparrow n | 0, n+1 \rangle = 0$$

$$\langle \downarrow n | \hat{S}_{+}\hat{a}^{+} | \downarrow n \rangle = \sqrt{n+1}\, \langle \downarrow n | \uparrow n+1 \rangle = 0$$

$$\langle \uparrow n | \hat{S}_{+}\hat{a} | \uparrow n \rangle = \sqrt{n}\, \langle \uparrow n | 0, n-1 \rangle = 0$$

$$\langle \downarrow n | \hat{S}_{+}\hat{a} | \downarrow n \rangle = \sqrt{n}\, \langle \downarrow n | \uparrow n-1 \rangle = 0$$

$$\langle \uparrow n | \hat{S}_{-}\hat{a}^{+} | \uparrow n \rangle = \sqrt{n+1}\, \langle \uparrow n | \downarrow n+1 \rangle = 0$$

$$\langle \downarrow n | \hat{S}_{-}\hat{a}^{+} | \downarrow n \rangle = \sqrt{n+1}\, \langle \downarrow n | 0\, n+1 \rangle = 0$$

$$\langle \uparrow n | \hat{S}_{-}\hat{a} | \uparrow n \rangle = \sqrt{n}\, \langle \uparrow n | \downarrow n-1 \rangle = 0$$

$$\langle \downarrow n | \hat{S}_{-}\hat{a} | \downarrow n \rangle = \sqrt{n}\, \langle \downarrow n | 0, n-1 \rangle = 0$$

ΜΕΣΕΣ (ΑΝΑΜΕΝΟΜΕΝΕΣ) ΤΙΜΕΣ ΜΕΓΕΘΩΝ για τη Χαμιλτονιανή Jaynes – Cummings

Για την κατάσταση: Ⓐ $|\psi_A(t)\rangle = c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow, n-1\rangle = |A\rangle$

$$\langle \hat{a}^+\hat{a}\rangle_{|A\rangle} = \left(c_1^*(t)\langle\downarrow n| + c_2^*(t)\langle\uparrow n-1|\right)\hat{a}^+\hat{a}\left(c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow n-1\rangle\right)$$

$$= |c_1(t)|^2\langle\downarrow n|\hat{a}^+\hat{a}|\downarrow n\rangle + c_1^*(t)c_2(t)\langle\downarrow n|\hat{a}^+\hat{a}|\uparrow n-1\rangle$$

$$+ c_2^*(t)c_1(t)\langle\uparrow n-1|\hat{a}^+\hat{a}|\downarrow n\rangle + |c_2(t)|^2\langle\uparrow n-1|\hat{a}^+\hat{a}|\uparrow n-1\rangle$$

$$= |c_1(t)|^2 n \overbrace{\langle\downarrow n|\downarrow n\rangle}^{1} + c_1^*(t)c_2(t)(n-1)\overbrace{\langle\downarrow n|\uparrow n-1\rangle}^{0}$$

$$+ c_2^*(t)c_1(t)\,n\overbrace{\langle\uparrow n-1|\downarrow n\rangle}^{0} + |c_2(t)|^2(n-1)\overbrace{\langle\uparrow n-1|\uparrow n-1\rangle}^{1}$$

$$= |c_1(t)|^2 n + |c_2(t)|^2 n - |c_2(t)|^2 \Rightarrow \langle\hat{a}^+\hat{a}\rangle_{|A\rangle} = n - |c_2(t)|^2$$

$$\langle\hat{a}\hat{a}^+\rangle_{|A\rangle} = \left(c_1^*(t)\langle\downarrow n| + c_2^*(t)\langle\uparrow n-1|\right)\hat{a}\hat{a}^+\left(c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow n-1\rangle\right)$$

$$= |c_1(t)|^2\langle\downarrow n|\hat{a}\hat{a}^+|\downarrow n\rangle + c_1^*(t)c_2(t)\langle\downarrow n|\hat{a}\hat{a}^+|\uparrow n-1\rangle$$

$$+ c_2^*(t)c_1(t)\langle\uparrow n-1|\hat{a}\hat{a}^+|\downarrow n\rangle + |c_2(t)|^2\langle\uparrow n-1|\hat{a}\hat{a}^+|\uparrow n-1\rangle =$$

$$= |c_1(t)|^2(n+1) + c_1^*(t)c_2(t)\,n\,\cancel{\langle\downarrow n|\uparrow n-1\rangle}$$

$$+ c_2^*(t)c_1(t)(n+1)\cancel{\langle\uparrow n-1|\downarrow n\rangle} + |c_2(t)|^2 n =$$

$$= |c_1(t)|^2 n + |c_1(t)|^2 + |c_2(t)|^2 n \Rightarrow \langle\hat{a}\hat{a}^+\rangle_{|A\rangle} = n + |c_1(t)|^2$$

$$\langle\hat{a}\hat{a}^+\rangle_{|A\rangle} - \langle\hat{a}^+\hat{a}\rangle_{|A\rangle} = 1$$

$\langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} = |c_2(t)|^2$

$\langle \hat{S}_- \hat{S}_+ \rangle_{|A\rangle} = |c_1(t)|^2$

ομοίως αποδεικνύονται ...

$\langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} + \langle \hat{S}_- \hat{S}_+ \rangle_{|A\rangle} = 1$

(IB)

$\langle \hat{a}^+ \hat{a} \rangle_{|A\rangle} + \langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} = n$

$\langle \hat{S}_+ \hat{a} \rangle_{|A\rangle} = \overset{*}{c_2}(t)\, c_1(t) \sqrt{n}$

$\langle \hat{S}_- \hat{a}^+ \rangle_{|A\rangle} = \overset{*}{c_1}(t)\, c_2(t) \sqrt{n}$

$\langle \hat{S}_+ \hat{a}^+ \rangle_{|A\rangle} = 0$

$\langle \hat{S}_- \hat{a} \rangle_{|A\rangle} = 0$

Για την κατάσταση (E) $|\psi_E(t)\rangle = c_1(t) |\uparrow n\rangle + c_2(t) |\downarrow n+1\rangle = |E\rangle$ ομοίως αποδεικνύονται

$\langle \hat{a}^+ \hat{a} \rangle_{|E\rangle} =$ $\quad \langle \hat{a} \hat{a}^+ \rangle_{|E\rangle} =$ $\quad \langle \hat{a} \hat{a}^+ \rangle_{|E\rangle} - \langle \hat{a}^+ \hat{a} \rangle_{|E\rangle} =$

$\langle \hat{S}_+ \hat{S}_- \rangle_{|E\rangle} =$ $\quad \langle \hat{S}_- \hat{S}_+ \rangle_{|E\rangle} =$ $\quad \langle \hat{S}_+ \hat{S}_- \rangle_{|E\rangle} + \langle \hat{S}_- \hat{S}_+ \rangle_{|E\rangle} =$

$\langle \hat{a}^+ \hat{a} \rangle_{|E\rangle} + \langle \hat{S}_+ \hat{S}_- \rangle_{|E\rangle} =$

$\langle \hat{S}_+ \hat{a} \rangle_{|E\rangle} =$ $\quad \langle \hat{S}_+ \hat{a}^+ \rangle_{|E\rangle} =$

$\langle \hat{S}_- \hat{a}^+ \rangle_{|E\rangle} =$ $\quad \langle \hat{S}_- \hat{a} \rangle_{|E\rangle} =$

IB'

$$\langle \hat{S}_+\hat{S}_-\rangle_{|A\rangle} = \left(c_1^*(t)\langle\downarrow n| + c_2^*(t)\langle\uparrow n-1|\right)\hat{S}_+\hat{S}_-\left(c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow n-1\rangle\right)$$

$$= |c_1(t)|^2\langle\downarrow n|\hat{S}_+\hat{S}_-|\downarrow n\rangle +$$
$$c_1^*(t)c_2(t)\langle\downarrow n|\hat{S}_+\hat{S}_-|\uparrow n-1\rangle +$$
$$c_2^*(t)c_1(t)\langle\uparrow n-1|\hat{S}_+\hat{S}_-|\downarrow n\rangle +$$
$$|c_2(t)|^2\langle\uparrow n-1|\hat{S}_+\hat{S}_-|\uparrow n-1\rangle =$$

$$= |c_1(t)|^2\cancel{\langle\downarrow n|0\, n\rangle} +$$
$$c_1^*(t)c_2(t)\cancel{\langle\downarrow n|\uparrow n-1\rangle} +$$
$$c_2^*(t)c_1(t)\cancel{\langle\uparrow n-1|0\, n\rangle} +$$
$$|c_2(t)|^2\langle\uparrow n-1|\uparrow n-1\rangle = |c_2(t)|^2$$

$$\langle \hat{S}_-\hat{S}_+\rangle_{|A\rangle} = \left(c_1^*(t)\langle\downarrow n| + c_2^*(t)\langle\uparrow n-1|\right)\hat{S}_-\hat{S}_+\left(c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow n-1\rangle\right)$$

$$= |c_1(t)|^2\langle\downarrow n|\hat{S}_-\hat{S}_+|\downarrow n\rangle +$$
$$c_1^*(t)c_2(t)\langle\downarrow n|\hat{S}_-\hat{S}_+|\uparrow n-1\rangle +$$
$$c_2^*(t)c_1(t)\langle\uparrow n-1|\hat{S}_-\hat{S}_+|\downarrow n\rangle +$$
$$|c_2(t)|^2\langle\uparrow n-1|\hat{S}_-\hat{S}_+|\uparrow n-1\rangle =$$

$$= |c_1(t)|^2\langle\downarrow n|\downarrow n\rangle +$$
$$c_1^*(t)c_2(t)\cancel{\langle\downarrow n|0\, n-1\rangle} +$$
$$c_2^*(t)c_1(t)\cancel{\langle\uparrow n-1|\downarrow n\rangle} +$$
$$|c_2(t)|^2\cancel{\langle\uparrow n-1|0\, n-1\rangle} = |c_1(t)|^2$$

$$\langle \hat{S}_+ \hat{a} \rangle_{|A\rangle} = \left(c_1^*(t) \langle \downarrow n | + c_2^*(t) | \uparrow n-1 \rangle \right) \hat{S}_+ \hat{a} \left(c_1(t) | \downarrow n \rangle + c_2(t) | \uparrow n-1 \rangle \right) \quad \text{IB}''$$

$$= |c_1(t)|^2 \langle \downarrow n | \hat{S}_+ \hat{a} | \downarrow n \rangle +$$

$$c_1^*(t) c_2(t) \langle \downarrow n | \hat{S}_+ \hat{a} | \uparrow n-1 \rangle +$$

$$c_2^*(t) c_1(t) \langle \uparrow n-1 | \hat{S}_+ \hat{a} | \downarrow n \rangle +$$

$$|c_2(t)|^2 \langle \uparrow n-1 | \hat{S}_+ \hat{a} | \uparrow n-1 \rangle =$$

$$= |c_1(t)|^2 \sqrt{n} \cancel{\langle \downarrow n | \uparrow n-1 \rangle} +$$

$$c_1^*(t) c_2(t) \sqrt{n-1} \cancel{\langle \downarrow n | \emptyset\, n-2 \rangle} +$$

$$c_2^*(t) c_1(t) \sqrt{n} \langle \uparrow n-1 | \uparrow n-1 \rangle +$$

$$|c_2(t)|^2 \sqrt{n-1} \cancel{\langle \uparrow n-1 | \emptyset\, n-2 \rangle} = c_2^*(t) c_1(t) \sqrt{n}$$

$$\langle \hat{S}_- \hat{a}^+ \rangle_{|A\rangle} = \left(c_1^*(t) \langle \downarrow n | + c_2^*(t) \langle \uparrow n-1 | \right) \hat{S}_- \hat{a}^+ \left(c_1(t) | \downarrow n \rangle + c_2(t) | \uparrow n-1 \rangle \right)$$

$$= |c_1(t)|^2 \langle \downarrow n | \hat{S}_- \hat{a}^+ | \downarrow n \rangle +$$

$$c_1^*(t) c_2(t) \langle \downarrow n | \hat{S}_- \hat{a}^+ | \uparrow n-1 \rangle +$$

$$c_2^*(t) c_1(t) \langle \uparrow n-1 | \hat{S}_- \hat{a}^+ | \downarrow n \rangle +$$

$$|c_2(t)|^2 \langle \uparrow n-1 | \hat{S}_- \hat{a}^+ | \uparrow n-1 \rangle =$$

$$= |c_1(t)|^2 \sqrt{n+1} \cancel{\langle \downarrow n | \emptyset\, n+1 \rangle}$$

$$c_1^*(t) c_2(t) \sqrt{n} \langle \downarrow n | \downarrow n \rangle$$

$$c_2^*(t) c_1(t) \sqrt{n+1} \cancel{\langle \uparrow n-1 | \emptyset\, n+1 \rangle}$$

$$|c_2(t)|^2 \sqrt{n} \cancel{\langle \uparrow n-1 | \downarrow n \rangle} = c_1^*(t) c_2(t) \sqrt{n}$$

$$\hat{H}_R^m = \hbar\omega_m \hat{a}_m^{+}\hat{a}_m + \hbar\Omega \hat{S}_+\hat{S}_- + \hbar g^m (\hat{S}_+ + \hat{S}_-)(\hat{a}_m^{+} + \hat{a}_m)$$

Rabi Hamiltonian describing the interaction of a two level atom with a single-mode harmonic field

$\hbar g^m (\hat{S}_+ \hat{a}_m^{+} + \hat{S}_- \hat{a}_m)$ the so-called counter-rotating terms (αγνοούμε)

$$\hat{H}_{JC}^m = \hbar\omega_m \hat{a}_m^{+}\hat{a}_m + \hbar\Omega \hat{S}_+\hat{S}_- + \hbar g^m (\hat{S}_+ \hat{a}_m + \hat{S}_- \hat{a}_m^{+})$$

Jaynes-Cummings Hamiltonian

Να υπολογιστούν τα $\langle \hat{a}_m^{+}\hat{a}_m \rangle$, $\langle \hat{S}_+\hat{S}_- \rangle$, $\langle \hat{S}_+\hat{a}_m \rangle$, $\langle \hat{S}_-\hat{a}_m^{+} \rangle$ για τις καταστάσεις

Ⓐ $|\psi_A(t)\rangle = c_1(t)|\downarrow, n\rangle + c_2(t)|\uparrow, n-1\rangle$ θα χρησιμοποιηθεί με αρχικές συνθήκες $c_1(0)=1, c_2(0)=0$

Ⓔ $|\psi_E(t)\rangle = c_1(t)|\downarrow, n+1\rangle + c_2(t)|\uparrow, n\rangle$ $c_1(0)=0, c_2(0)=1$

Ⓐ

$$\langle \hat{a}_m^{+}\hat{a}_m \rangle_A = \langle \psi_A(t)| \hat{a}_m^{+}\hat{a}_m |\psi_A(t)\rangle = \{c_1^*\langle\downarrow,n| + c_2^*\langle\uparrow,n-1|\} \hat{a}_m^{+}\hat{a}_m \{c_1|\downarrow,n\rangle + c_2|\uparrow,n-1\rangle\}$$
$$= |c_1|^2\langle\downarrow,n|\hat{a}_m^{+}\hat{a}_m|\downarrow,n\rangle + c_1^*c_2\langle\downarrow,n|\hat{a}_m^{+}\hat{a}_m|\uparrow,n-1\rangle$$
$$+ c_2^*c_1\langle\uparrow,n-1|\hat{a}_m^{+}\hat{a}_m|\downarrow,n\rangle + |c_2|^2\langle\uparrow,n-1|\hat{a}_m^{+}\hat{a}_m|\uparrow,n-1\rangle$$
$$= |c_1|^2\sqrt{n}\sqrt{n}\langle\downarrow,n|\downarrow,n\rangle + c_1^*c_2\sqrt{n-1}\sqrt{n-1}\langle\downarrow,n|\uparrow,n-1\rangle$$
$$+ c_2^*c_1\sqrt{n}\sqrt{n}\langle\uparrow,n-1|\downarrow,n\rangle + |c_2|^2\sqrt{n-1}\sqrt{n-1}\langle\uparrow,n-1|\uparrow,n-1\rangle =$$
$$= n\cdot|c_1|^2\cdot 1 + c_1^*c_2(n-1)\cdot 0 + c_2^*c_1 n\cdot 0 + (n-1)|c_2|^2\cdot 1 =$$
$$= n|c_1|^2 + n|c_2|^2 - |c_2|^2 = n(|c_1|^2+|c_2|^2) - |c_2|^2 = n - |c_2|^2 \Rightarrow$$

$$\boxed{\langle \hat{a}_m^{+}\hat{a}_m \rangle_A = n_m - |c_2(t)|^2} \checkmark$$

$$\langle \hat{S}_+\hat{S}_- \rangle_A = \langle \psi_A(t)| \hat{S}_+\hat{S}_- |\psi_A(t)\rangle = \{c_1^*\langle\downarrow,n| + c_2^*\langle\uparrow,n-1|\} \hat{S}_+\hat{S}_- \{c_1|\downarrow,n\rangle + c_2|\uparrow,n-1\rangle\}$$
$$= |c_1|^2\langle\downarrow,n|\hat{S}_+\hat{S}_-|\downarrow,n\rangle + c_1^*c_2\langle\downarrow,n|\hat{S}_+\hat{S}_-|\uparrow,n-1\rangle$$
$$+ c_2^*c_1\langle\uparrow,n-1|\hat{S}_+\hat{S}_-|\downarrow,n\rangle + |c_2|^2\langle\uparrow,n-1|\hat{S}_+\hat{S}_-|\uparrow,n-1\rangle =$$
$$= |c_1|^2\cdot 0 + c_1^*c_2\underbrace{\langle\downarrow,n|\uparrow,n-1\rangle}_{0} + c_2^*c_1\cdot 0 + |c_2|^2\underbrace{\langle\uparrow,n-1|\uparrow,n-1\rangle}_{1} \Rightarrow$$

$$\boxed{\langle \hat{S}_+\hat{S}_- \rangle_A = |c_2(t)|^2} \checkmark$$

ΑΡΑ $$\boxed{\langle \hat{a}_m^{+}\hat{a}_m \rangle_A + \langle \hat{S}_+\hat{S}_- \rangle_A = n_m} \checkmark$$

$$\langle \hat{S}_+ \hat{a}_m \rangle_{A} = \langle \psi_A(t) | \hat{S}_+ \hat{a}_m | \psi_A(t) \rangle = \{ c_1^* \langle \downarrow, n | + c_2^* \langle \uparrow, n-1 | \} \hat{S}_+ \hat{a}_m \{ c_1 | \downarrow, n \rangle + c_2 | \uparrow, n-1 \rangle \}$$

$$= |c_1|^2 \langle \downarrow, n | \hat{S}_+ \hat{a}_m | \downarrow, n \rangle + c_1^* c_2 \langle \downarrow, n | \hat{S}_+ \hat{a}_m | \uparrow, n-1 \rangle +$$

$$+ c_2^* c_1 \langle \uparrow, n-1 | \hat{S}_+ \hat{a}_m | \downarrow, n \rangle + |c_2|^2 \langle \uparrow, n-1 | \hat{S}_+ \hat{a}_m | \uparrow, n-1 \rangle =$$

$$= |c_1|^2 \sqrt{n} \langle \downarrow, n | \uparrow, n-1 \rangle + c_1^* c_2 \cdot \sqrt{n-1} \cdot \langle \downarrow, n | \hat{S}_+ | \uparrow, n-2 \rangle$$

$$+ c_2^* c_1 \sqrt{n} \langle \uparrow, n-1 | \uparrow, n-1 \rangle + |c_2|^2 \sqrt{n-1} \langle \uparrow, n-1 | \hat{S}_+ | \uparrow, n-2 \rangle \Rightarrow$$

$$\boxed{\langle \hat{S}_+ \hat{a}_m \rangle_A = c_2^*(t) c_1(t) \cdot \sqrt{n}}$$

$$\langle \hat{S}_- \hat{a}_m^+ \rangle_{A} = \langle \psi_A(t) | \hat{S}_- \hat{a}_m^+ | \psi_A(t) \rangle = \{ c_1^* \langle \downarrow, n | + c_2^* | \uparrow, n-1 \rangle \} \hat{S}_- \hat{a}_m^+ \{ c_1 | \downarrow, n \rangle + c_2 | \uparrow, n-1 \rangle \}$$

$$= |c_1|^2 \langle \downarrow, n | \hat{S}_- \hat{a}_m^+ | \downarrow, n \rangle + c_1^* c_2 \langle \downarrow, n | \hat{S}_- \hat{a}_m^+ | \uparrow, n-1 \rangle +$$

$$c_2^* c_1 \langle \uparrow, n-1 | \hat{S}_- \hat{a}_m^+ | \downarrow, n \rangle + |c_2|^2 \langle \uparrow, n-1 | \hat{S}_- \hat{a}_m^+ | \uparrow, n-1 \rangle =$$

$$= |c_1|^2 \sqrt{n+1} \langle \downarrow, n | \hat{S}_- | \downarrow, n+1 \rangle + c_1^* c_2 \sqrt{n} \langle \downarrow, n | \downarrow, n \rangle$$

$$+ c_2^* c_1 \langle \uparrow, n-1 | \hat{S}_- | \downarrow, n+1 \rangle \sqrt{n+1} + |c_2|^2 \langle \uparrow, n-1 | \downarrow, n \rangle \sqrt{n} \Rightarrow$$

$$\boxed{\langle \hat{S}_- \hat{a}_m^+ \rangle_{A} = c_1^*(t) c_2(t) \sqrt{n}}$$

E

$$\langle \hat{a}_m^{\dagger}\hat{a}_m\rangle_{E} = \langle \psi_E(t)|\hat{a}_m^{\dagger}\hat{a}_m|\psi_E(t)\rangle = \{c_1^*\langle\downarrow,n+1| + c_2^*\langle\uparrow,n|\}\,\hat{a}_m^{\dagger}\hat{a}_m\,\{c_1|\downarrow,n+1\rangle + c_2|\uparrow,n\rangle\}$$

$$= |c_1|^2\langle\downarrow,n+1|\hat{a}_m^{\dagger}\hat{a}_m|\downarrow,n+1\rangle + c_1^*c_2\langle\downarrow,n+1|\hat{a}_m^{\dagger}\hat{a}_m|\uparrow,n\rangle$$

$$+ c_2^*c_1\langle\uparrow,n|\hat{a}_m^{\dagger}\hat{a}_m|\downarrow,n+1\rangle + |c_2|^2\langle\uparrow,n|\hat{a}_m^{\dagger}\hat{a}_m|\uparrow,n\rangle$$

$$= |c_1|^2\sqrt{n+1}\sqrt{n+1}\,\langle\downarrow,n+1|\downarrow,n+1\rangle + c_1^*c_2\, n\,\langle\downarrow,n+1|\uparrow,n\rangle$$

$$+ c_2^*c_1\,(n+1)\,\langle\uparrow,n|\downarrow,n+1\rangle + |c_2|^2 n\,\langle\uparrow,n|\uparrow,n\rangle =$$

$$= |c_1|^2(n+1) + n|c_2|^2 = n(|c_1|^2+|c_2|^2) + |c_1|^2 \Rightarrow$$

$$\boxed{\langle \hat{a}_m^{\dagger}\hat{a}_m\rangle_{E} = n + |c_1(t)|^2} \checkmark \quad (4.60)$$

$$\langle \hat{S}_+\hat{S}_-\rangle_{E} = \langle \psi_E(t)|\hat{S}_+\hat{S}_-|\psi_E(t)\rangle = \{c_1^*\langle\downarrow,n+1| + c_2^*\langle\uparrow,n|\}\,\hat{S}_+\hat{S}_-\,\{c_1|\downarrow,n+1\rangle + c_2|\uparrow,n\rangle\}$$

$$= |c_1|^2\cdot 0 + c_1^*c_2\langle\downarrow,n+1|\uparrow,n\rangle + c_2^*c_1\cdot 0 + |c_2|^2\langle\uparrow,n|\uparrow,n\rangle \Rightarrow$$

$$\boxed{\langle \hat{S}_+\hat{S}_-\rangle_{E} = |c_2(t)|^2} \checkmark \quad (4.60)$$

$$\langle \hat{S}_+\hat{a}_m\rangle_{E} = \langle \psi_E(t)|\hat{S}_+\hat{a}_m|\psi_E(t)\rangle = \{c_1^*\langle\downarrow,n+1| + c_2^*\langle\uparrow,n|\}\,\hat{S}_+\hat{a}_m\,\{c_1|\downarrow,n+1\rangle + c_2|\uparrow,n\rangle\}$$

$$= |c_1|^2\langle\downarrow,n+1|\hat{S}_+\hat{a}_m|\downarrow,n+1\rangle + c_1^*c_2\langle\downarrow,n+1|\hat{S}_+\hat{a}_m|\uparrow,n\rangle +$$

$$c_2^*c_1\langle\uparrow,n|\hat{S}_+\hat{a}_m|\downarrow,n+1\rangle + |c_2|^2\langle\uparrow,n|\hat{S}_+\hat{a}_m|\uparrow,n\rangle$$

$$= |c_1|^2\langle\downarrow,n+1|\uparrow,n\rangle\sqrt{n+1} + c_1^*c_2\cdot 0 + c_2^*c_1\langle\uparrow,n|\uparrow,n\rangle\sqrt{n+1} + |c_2|^2 0$$

$$\boxed{\langle \hat{S}_+\hat{a}_m\rangle_{E} = c_2^*(t)\,c_1(t)\cdot\sqrt{n+1}}$$

$$\langle \hat{S}_-\hat{a}_m^{\dagger}\rangle_{E} = \langle \psi_E(t)|\hat{S}_-\hat{a}_m^{\dagger}|\psi_E(t)\rangle = \{c_1^*\langle\downarrow,n+1| + c_2^*\langle\uparrow,n|\}\,\hat{S}_-\hat{a}_m^{\dagger}\,\{c_1|\downarrow,n+1\rangle + c_2|\uparrow,n\rangle\}$$

$$= |c_1|^2\langle\downarrow,n+1|\hat{S}_-\hat{a}_m^{\dagger}|\downarrow,n+1\rangle + c_1^*c_2\langle\downarrow,n+1|\hat{S}_-\hat{a}_m^{\dagger}|\uparrow,n\rangle$$

$$+ c_2^*c_1\langle\uparrow,n|\hat{S}_-\hat{a}_m|\downarrow,n+1\rangle + |c_2|^2\langle\uparrow,n|\hat{S}_-\hat{a}_m^{\dagger}|\uparrow,n\rangle =$$

$$= |c_1|^2\langle\downarrow,n+1|\uparrow,n+2\rangle\sqrt{n+2} + c_1^*c_2\langle\downarrow,n+1|\downarrow,n+1\rangle\sqrt{n+1}$$

$$+ c_2^*c_1\cdot 0 + |c_2|^2\langle\uparrow,n|\downarrow,n+1\rangle\sqrt{n+1} \Rightarrow$$

$$\boxed{\langle \hat{S}_-\hat{a}_m^{\dagger}\rangle_{E} = c_1^*(t)\cdot c_2(t)\cdot\sqrt{n+1}}$$

APA $$\boxed{\langle \hat{a}_m^{\dagger}\hat{a}_m\rangle_{E} + \langle \hat{S}_+\hat{S}_-\rangle_{E} = n+1} \checkmark \quad (4.61)$$