$$A=\begin{bmatrix} \frac{\Delta}{2} & -\frac{\Omega_R}{2} & 0 \\ -\frac{\Omega_R}{2} & -\frac{\Delta}{2} & -\frac{\Omega_R'}{2} \\ 0 & -\frac{\Omega_R'}{2} & -\frac{3\Delta}{2} \end{bmatrix} \qquad A\vec{u}=\lambda\vec{u} \qquad \boxed{\text{ΤΣ}}$$

Άς το λύσουμε για $\boxed{\Delta=\Omega_R=\Omega_R'=\alpha=2\beta}$

$$A=\beta\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -3 \end{bmatrix} \qquad A\vec{u}=\lambda\vec{u} \Leftrightarrow (A-\lambda I)\vec{u}=\vec{0}$$

$$\begin{bmatrix} \beta-\lambda & -\beta & 0 \\ -\beta & -\beta-\lambda & -\beta \\ 0 & -\beta & -3\beta-\lambda \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (\alpha-\beta)^3=\alpha^3-3\alpha^2\beta+3\alpha\beta^2-\beta^3$$

det $=0$ (για μη τετριμμένη λύση) $\Rightarrow$

$$\beta-\lambda\begin{vmatrix} -\beta-\lambda & -\beta \\ -\beta & -3\beta-\lambda \end{vmatrix}+\beta\begin{vmatrix} -\beta & 0 \\ -\beta & -3\beta-\lambda \end{vmatrix}=0 \Rightarrow$$

$$(\beta-\lambda)\left[(\beta+\lambda)(3\beta+\lambda)-\beta^2\right]+\beta\left[\beta(3\beta+\lambda)\right]=0 \Rightarrow$$

$$(\beta-\lambda)\left(3\beta^2+\beta\lambda+3\beta\lambda+\lambda^2-\beta^2\right)+\beta^2(3\beta+\lambda)=0 \Rightarrow$$

$$(\beta-\lambda)\left(2\beta^2+4\beta\lambda+\lambda^2\right)+\beta^2(3\beta+\lambda)=0 \Rightarrow$$

$$2\beta^3+4\beta^2\lambda+\beta\lambda^2-2\beta^2\lambda-4\beta\lambda^2-\lambda^3+3\beta^3+\beta^2\lambda=0 \Rightarrow$$

$$-\lambda^3-3\beta\lambda^2+3\beta^2\lambda+5\beta^3=0 \Rightarrow \lambda^3+3\beta\lambda^2-3\beta^2\lambda-5\beta^3=0$$

$\boxed{\lambda_3=-\beta}$ λύση διότι $-(-\beta)^3-3\beta(-\beta)^2+3\beta^2(-\beta)+5\beta^3=$

$$\beta^3-3\beta^3-3\beta^3+5\beta^3=0$$

$$
\begin{array}{l|l}
-\lambda^3 - 3\beta\lambda^2 + 3\beta^2\lambda + 5\beta^3 & (\lambda+\beta) \\ \hline
\ominus\ -\lambda^3 - \beta\lambda^2 & -\lambda^2 - 2\beta\lambda + 5\beta^2 \\
\quad -2\beta\lambda^2 + 3\beta^2\lambda + 5\beta^3 & \\
\ominus\ -2\beta\lambda^2 - 2\beta^2\lambda & \\
\qquad 5\beta^2\lambda + 5\beta^3 & \\
\ominus\ 5\beta^2\lambda + 5\beta^3 & \\
\qquad 0 &
\end{array}
$$

$$(-\lambda^3 - 3\beta\lambda^2 + 3\beta^2\lambda + 5\beta^3) = (\lambda+\beta)\underbrace{(-\lambda^2 - 2\beta\lambda + 5\beta^2)}$$

$$\Delta_2 = 4\beta^2 + 4 \cdot 5\beta^2 = 24\beta^2$$

$$24 = 4 \cdot 6$$

$$\lambda_{1,2} = \frac{2\beta \pm \sqrt{24\beta^2}}{-2} = \frac{2\beta \pm 2\beta\sqrt{6}}{-2}$$

$$\lambda_{1,2} = -\beta \mp \beta\sqrt{6}$$

$$\lambda_3 = -\beta \qquad \lambda_2 = -\beta + \beta\sqrt{6} \qquad \lambda_1 = -\beta - \beta\sqrt{6}$$

- $\boxed{\lambda_3 = -\beta}$

$$\cancel{\beta}\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} U_{13} \\ U_{23} \\ U_{33} \end{bmatrix} = -\cancel{\beta}\begin{bmatrix} U_{13} \\ U_{23} \\ U_{33} \end{bmatrix} \Rightarrow \left\{\begin{array}{l} U_{13} - U_{23} = -U_{13} \\ -U_{13} - U_{23} - U_{33} = -U_{23} \\ -U_{23} - 3U_{33} = -U_{23} \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{l} U_{23} = 2U_{13} \\ U_{33} = -U_{13} \\ U_{23} = -2U_{33} \end{array}\right\}$$

π.χ. θέτουμε $U_{13} := \gamma \Rightarrow$ $\left.\begin{array}{l} U_{23} = 2\gamma \\ U_{33} = -\gamma \\ 2\gamma = -2(-\gamma) \end{array}\right\}$ $\boxed{\vec{V}_3 = \gamma\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}$

- $\boxed{\lambda_2 = -\beta + \beta\sqrt{6} = -\beta(1-\sqrt{6})}$

$$\beta\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} U_{12} \\ U_{22} \\ U_{32} \end{bmatrix} = -\beta(1-\sqrt{6})\begin{bmatrix} U_{12} \\ U_{22} \\ U_{32} \end{bmatrix} \Rightarrow \left\{\begin{array}{l} U_{12} - U_{22} = -(1-\sqrt{6})U_{12} \\ -U_{12} - U_{22} - U_{32} = -(1-\sqrt{6})U_{22} \\ -U_{22} - 3U_{32} = -(1-\sqrt{6})U_{32} \end{array}\right\}$$

$$\Rightarrow \begin{array}{l} U_{22} = (2-\sqrt{6})U_{12} \\ -U_{12} - U_{32} = \sqrt{6}U_{22} \\ -U_{22} = (2+\sqrt{6})U_{32} \end{array}$$

π.χ. θέτουμε $U_{12} := \delta$ $\begin{array}{l} U_{22} = (2-\sqrt{6})\delta \\ -\delta - U_{32} = \sqrt{6}U_{22} \\ -U_{22} = (2+\sqrt{6})U_{32} \end{array}$ $\rightsquigarrow$

$$\rightarrow -(2-\sqrt{6})\delta = (2+\sqrt{6})U_{32} \Rightarrow U_{32} = -\frac{(2-\sqrt{6})}{(2+\sqrt{6})}\delta$$

$$\boxed{\vec{V}_2 = \delta\begin{bmatrix} 1 \\ (2-\sqrt{6}) \\ -\frac{(2-\sqrt{6})}{(2+\sqrt{6})} \end{bmatrix}}$$

$$\rightsquigarrow -\delta + \frac{(2-\sqrt{6})}{(2+\sqrt{6})}\delta = \sqrt{6}(2-\sqrt{6})\delta$$

$$\frac{-(2+\sqrt{6})+(2-\sqrt{6})}{2+\sqrt{6}} = \sqrt{6}(2-\sqrt{6}) \Rightarrow -\cancel{2}-\sqrt{6}+\cancel{2}-\sqrt{6} = \sqrt{6}(4-6)$$
$$\Rightarrow -2\sqrt{6} = -2\sqrt{6}$$

- $\boxed{\lambda_1 = -\beta(1+\sqrt{6})}$

$$\beta\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} U_{13} \\ U_{23} \\ U_{33} \end{bmatrix} = -\beta(1+\sqrt{6})\begin{bmatrix} U_{13} \\ U_{23} \\ U_{33} \end{bmatrix} \Rightarrow \left\{\begin{array}{l} U_{13} - U_{23} = -(1+\sqrt{6})U_{13} \\ -U_{13} - U_{23} - U_{33} = -(1+\sqrt{6})U_{23} \\ -U_{23} - 3U_{33} = -(1+\sqrt{6})U_{33} \end{array}\right\} \Rightarrow$$

$$\begin{array}{l} -U_{23} = -(2+\sqrt{6})U_{13} \\ -U_{13} - U_{33} = -\sqrt{6}U_{23} \\ -U_{23} = (2-\sqrt{6})U_{33} \end{array}$$

π.χ. θέτουμε $U_{13} = \varepsilon$ $\begin{array}{l} U_{23} = (2+\sqrt{6})\varepsilon \\ -\varepsilon - U_{33} = -\sqrt{6}(2+\sqrt{6})\varepsilon \\ -U_{23} = (2-\sqrt{6})U_{33} \\ -(2+\sqrt{6})\varepsilon = (2-\sqrt{6})U_{33} \end{array}$ $U_{33} = -\frac{(2+\sqrt{6})\varepsilon}{(2-\sqrt{6})}$

$$\boxed{\vec{V}_1 = \varepsilon\begin{bmatrix} 1 \\ (2+\sqrt{6}) \\ -\frac{(2+\sqrt{6})}{(2-\sqrt{6})} \end{bmatrix}}$$

$$\vec{V}_3\cdot\vec{V}_3 = 1 \Rightarrow \gamma^2(1+4+1)=1 \Rightarrow \gamma^2=\frac{1}{6} \Rightarrow \text{n.x } \gamma=\frac{1}{\sqrt{6}}$$

$$\vec{V}_2\cdot\vec{V}_2 = 1 \Rightarrow \delta^2\left\{1+(2-\sqrt{6})^2+\frac{(2-\sqrt{6})^2}{(2+\sqrt{6})^2}\right\}=1\Rightarrow$$

$$\delta^2\left\{\frac{(2+\sqrt{6})^2+(2-\sqrt{6})^2(2+\sqrt{6})^2+(2-\sqrt{6})^2}{(2+\sqrt{6})^2}\right\}=1\Rightarrow$$

$$\delta^2\left\{\frac{4+6+\cancel{4\sqrt{6}}+((4+6-4\sqrt{6})(4+6+4\sqrt{6}))+4+6-\cancel{4\sqrt{6}}}{(2+\sqrt{6})^2}\right\}=1$$

$$\delta^2\left\{\frac{10+(10-4\sqrt{6})(10+4\sqrt{6})+10}{(2+\sqrt{6})^2}\right\}=1$$

$$\delta^2\left\{\frac{10+100-16\cdot 6+10}{(2+\sqrt{6})^2}\right\}=1\Rightarrow\delta^2\,\frac{120-96}{(2+\sqrt{6})^2}=1$$

$$\delta^2=\frac{(2+\sqrt{6})^2}{24}\Rightarrow \text{n.x } \delta=\frac{2+\sqrt{6}}{2\sqrt{6}}$$

$$\vec{V}_1\cdot\vec{V}_1 = 1 \Rightarrow \varepsilon^2\left\{1+(2+\sqrt{6})^2+\frac{(2+\sqrt{6})^2}{(2-\sqrt{6})^2}\right\}=1\Rightarrow$$

$$\Rightarrow\varepsilon^2\,\frac{(2-\sqrt{6})^2+(2+\sqrt{6})^2(2-\sqrt{6})^2+(2+\sqrt{6})^2}{(2-\sqrt{6})^2}=1\Rightarrow$$

$$\Rightarrow\varepsilon^2\,\frac{4+6-4\sqrt{6}+(4+6+4\sqrt{6})\cdot(4+6-4\sqrt{6})+4+6+4\sqrt{6}}{(2-\sqrt{6})^2}=1\Rightarrow$$

$$\Rightarrow\varepsilon^2\,\frac{10-\cancel{4\sqrt{6}}+(10+4\sqrt{6})(10-4\sqrt{6})+10+\cancel{4\sqrt{6}}}{(2-\sqrt{6})^2}=1$$

$$\Rightarrow\varepsilon^2\,\frac{10+100-16\cdot 6+10}{(2-\sqrt{6})^2}=1\Rightarrow\varepsilon^2\,\frac{120-96}{(2-\sqrt{6})^2}=1\Rightarrow \text{n.x}$$

$$\varepsilon=\frac{2-\sqrt{6}}{2\sqrt{6}}$$

$$\lambda_3 = -\beta \qquad \vec{V}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\lambda_2 = -\beta(1-\sqrt{6}) \qquad \vec{V}_2 = \frac{2+\sqrt{6}}{2\sqrt{6}}\begin{bmatrix} 1 \\ (2-\sqrt{6}) \\ -\frac{(2-\sqrt{6})}{(2+\sqrt{6})} \end{bmatrix} = \frac{1}{2\sqrt{6}}\begin{bmatrix} (2+\sqrt{6}) \\ -2 \\ -(2-\sqrt{6}) \end{bmatrix}$$

$$\lambda_1 = -\beta(1+\sqrt{6}) \qquad \vec{V}_1 = \frac{(2-\sqrt{6})}{2\sqrt{6}}\begin{bmatrix} 1 \\ (2+\sqrt{6}) \\ -\frac{(2+\sqrt{6})}{(2-\sqrt{6})} \end{bmatrix} = \frac{1}{2\sqrt{6}}\begin{bmatrix} (2-\sqrt{6}) \\ -2 \\ -(2+\sqrt{6}) \end{bmatrix}$$

ΓΕΝΙΚΗ ΛΥΣΗ $\quad \vec{x}(t) = \sum_{k=1}^{3} \sigma_k \vec{v}_k e^{-i\lambda_k t}$

$$\begin{bmatrix} C_1(t)\, e^{-\frac{i\Delta t}{2}} \\ C_2(t)\, e^{\frac{i\Delta t}{2}} \\ C_3(t)\, e^{3\frac{i\Delta t}{2}} \end{bmatrix} = \frac{\sigma_1}{2\sqrt{6}}\begin{bmatrix} (2-\sqrt{6}) \\ -2 \\ (2+\sqrt{6}) \end{bmatrix} e^{+i\beta(1+\sqrt{6})t} + \frac{\sigma_2}{2\sqrt{6}}\begin{bmatrix} (2+\sqrt{6}) \\ -2 \\ (2-\sqrt{6}) \end{bmatrix} e^{+i\beta(1-\sqrt{6})t}$$

$$+ \frac{\sigma_3}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} e^{+i\beta t}$$

Αρχική συνθήκη (ἔστωσαν) $\quad C_1(0)=1$

$C_2(0)=0$

$C_3(0)=0$

$$1 = \frac{\sigma_1}{\cancel{2\sqrt{6}}}(2-\sqrt{6}) + \frac{\sigma_2}{\cancel{2\sqrt{6}}}(2+\sqrt{6}) + \frac{\sigma_3}{\cancel{\sqrt{6}}}\cdot 1 \Rightarrow 2 = \sigma_1(2-\sqrt{6}) + \sigma_2(2+\sqrt{6}) + \sigma_3\cdot 2$$

$$0 = \frac{\sigma_1}{2\cancel{\sqrt{6}}}(-2) + \frac{\sigma_2}{2\cancel{\sqrt{6}}}(-2) + \frac{\sigma_3}{\cancel{\sqrt{6}}}\cdot 2 \Rightarrow 0 = -2\sigma_1 - 2\sigma_2 + 4\sigma_3$$

$$0 = \frac{\sigma_1}{2\cancel{\sqrt{6}}}(2+\sqrt{6}) + \frac{\sigma_2}{2\cancel{\sqrt{6}}}(2-\sqrt{6}) + \frac{\sigma_3}{\cancel{\sqrt{6}}}(-1) \Rightarrow 0 = \sigma_1(2+\sqrt{6}) + \sigma_2(2-\sqrt{6}) - 2\sigma_3$$

$$2 = (2-\sqrt{6})\sigma_1 + (2+\sqrt{6})\sigma_2 + 2\sigma_3$$

$$0 = -2\sigma_1 - 2\sigma_2 + 4\sigma_3 \Rightarrow \sigma_1 + \sigma_2 = 2\sigma_3 \quad (4)$$

$$0 = (2+\sqrt{6})\sigma_1 + (2-\sqrt{6})\sigma_2 - 2\sigma_3$$

$$2 = (2\cancel{-\sqrt{6}}+2\cancel{+\sqrt{6}})\sigma_1 + (2\cancel{+\sqrt{6}}+2\cancel{-\sqrt{6}})\sigma_2 \Rightarrow 2 = 4\sigma_1 + 4\sigma_2 \Rightarrow$$

$$1 = 2\sigma_1 + 2\sigma_2 \Rightarrow$$

$$(4) \Rightarrow 2\sigma_3 = \frac{1}{2} \Rightarrow \sigma_3 = \frac{1}{4} \qquad \sigma_1 + \sigma_2 = \frac{1}{2} \quad \oplus$$

$$2 = (2-\sqrt{6})\sigma_1 + (2+\sqrt{6})\sigma_2 + 2\frac{1}{4} \Rightarrow \frac{3}{2} = (2-\sqrt{6})\sigma_1 + (2+\sqrt{0})\sigma_2$$

$$0 = -2\sigma_1 - 2\sigma_2 + 4\cdot\frac{1}{4} \Rightarrow 2(\sigma_1+\sigma_2) = 1 \Rightarrow \sigma_1 + \sigma_2 = \frac{1}{2}$$

$$0 = (2+\sqrt{6})\sigma_1 + (2-\sqrt{6})\sigma_2 - 2\frac{1}{4} \Rightarrow \frac{1}{2} = (2+\sqrt{6})\sigma_1 + (2-\sqrt{6})\sigma_2$$

$$\oplus\ 2 = (2\cancel{-\sqrt{6}}+2\cancel{+\sqrt{6}})\sigma_1 + (2\cancel{+\sqrt{6}}+2\cancel{-\sqrt{6}})\sigma_2 \Rightarrow$$

$$2 = 4\sigma_1 + 4\sigma_2 \Rightarrow \sigma_1 + \sigma_2 = \frac{1}{2}$$

$$\ominus\ 1 = (\cancel{2}-\sqrt{6}\cancel{-2}-\sqrt{6})\sigma_1 + (\cancel{2}+\sqrt{6}\cancel{-2}+\sqrt{6})\sigma_2 \Rightarrow$$

$$1 = -2\sqrt{6}\,\sigma_1 + 2\sqrt{6}\,\sigma_2 \Rightarrow 1 = 2\sqrt{6}(\sigma_2 - \sigma_1) \Rightarrow \sigma_2 - \sigma_1 = \frac{1}{2\sqrt{6}}$$

$$\sigma_1 + \sigma_2 = \frac{1}{2} \qquad \oplus\ 2\sigma_2 = \frac{1}{2} + \frac{1}{2\sqrt{6}} = \frac{\sqrt{6}+1}{2\sqrt{6}} \Rightarrow$$

$$\sigma_2 - \sigma_1 = \frac{1}{2\sqrt{6}} \qquad \sigma_2 = \frac{1+\sqrt{6}}{4\sqrt{6}}$$

$$\ominus\ 2\sigma_1 = \frac{1}{2} - \frac{1}{2\sqrt{6}} = \frac{\sqrt{6}-1}{2\sqrt{6}} \Rightarrow \sigma_1 = -\frac{1-\sqrt{6}}{4\sqrt{6}}$$

$$\begin{bmatrix} C_1(t)e^{-i\beta t} \\ C_2(t)e^{i\beta t} \\ C_3(t)e^{3i\beta t}\end{bmatrix} = \frac{-1}{2\sqrt{6}}\,\frac{1-\sqrt{6}}{4\sqrt{6}}\begin{bmatrix}(2-\sqrt{6})\\ -2\\ (2+\sqrt{6})\end{bmatrix} e^{i\beta(1+\sqrt{6})t}$$

$$+\frac{1}{2\sqrt{6}}\,\frac{1+\sqrt{6}}{4\sqrt{6}}\begin{bmatrix}(2+\sqrt{6})\\ -2\\ (2-\sqrt{6})\end{bmatrix} e^{i\beta(1-\sqrt{6})t}$$

$$+\frac{1}{\sqrt{6}}\,\frac{1}{4}\begin{bmatrix}1\\ 2\\ -1\end{bmatrix} e^{i\beta t}$$

$$C_1(t)\cdot e^{-i\beta t} = \frac{(\sqrt{6}-1)}{48}(2-\sqrt{6})\,e^{i\beta(1+\sqrt{6})t} + \frac{(\sqrt{6}+1)}{48}(2+\sqrt{6})\,e^{i\beta(1-\sqrt{6})t}$$

$$+\frac{1}{4\sqrt{6}}\,e^{i\beta t}$$

$$C_1(t) = \frac{2\sqrt{6}-6-2+\sqrt{6}}{48}\cdot e^{i\beta(2+\sqrt{6})t} + \frac{2\sqrt{6}+6+2+\sqrt{6}}{48}\,e^{i\beta(2-\sqrt{6})t} + \frac{1}{4\sqrt{6}}$$

$$C_1(t) = \frac{3\sqrt{6}-8}{48}\,e^{i\beta(2+\sqrt{6})t} + \frac{3\sqrt{6}+8}{48}\,e^{i\beta(2-\sqrt{6})t} + \frac{1}{4\sqrt{6}}$$

$$C_2(t)\,e^{i\beta t} = \frac{(\sqrt{6}-1)}{48}(-2)\,e^{i\beta(1+\sqrt{6})t} + \frac{(\sqrt{6}+1)}{48}(-2)\,e^{i\beta(1-\sqrt{6})t} + \frac{1}{4\sqrt{6}}\cdot 2\,e^{i\beta t}$$

$$C_2(t) = \frac{(1-\sqrt{6})}{24}\,e^{i\beta\sqrt{6}t} - \frac{(1+\sqrt{6})}{24}\,e^{-i\beta\sqrt{6}t} + \frac{1}{2\sqrt{6}}$$