7.2 BIOT–SAVART'S LAW

Biot–Savart's law states that the magnetic field intensity dH produced at a point P, as shown in Figure 7.1, by the differential current element $I\,dl$ is proportional to the product $I\,dl$ and the sine of the angle α between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element.

That is,

$$dH \propto \frac{I\,dl \sin \alpha}{R^2} \tag{7.1}$$

or

$$dH = \frac{kI\,dl \sin \alpha}{R^2} \tag{7.2}$$

where k is the constant of proportionality. In SI units, $k = 1/4\pi$, so eq. (7.2) becomes

$$dH = \frac{I\,dl \sin \alpha}{4\pi R^2} \tag{7.3}$$

From the definition of cross product in eq. (1.21), it is easy to notice that eq. (7.3) is better put in vector form as

$$\boxed{d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}} \tag{7.4}$$

where $R = |\mathbf{R}|$ and $\mathbf{a}_R = \mathbf{R}/R$. Thus the direction of $d\mathbf{H}$ can be determined by the right-hand rule with the right-hand thumb pointing in the direction of the current, the right-hand fingers encircling the wire in the direction of $d\mathbf{H}$ as shown in Figure 7.2(a). Alternatively, we can use the right-handed screw rule to determine the direction of $d\mathbf{H}$: with the screw placed along the wire and pointed in the direction of current flow, the direction of advance of the screw is the direction of $d\mathbf{H}$ as in Figure 7.2(b).

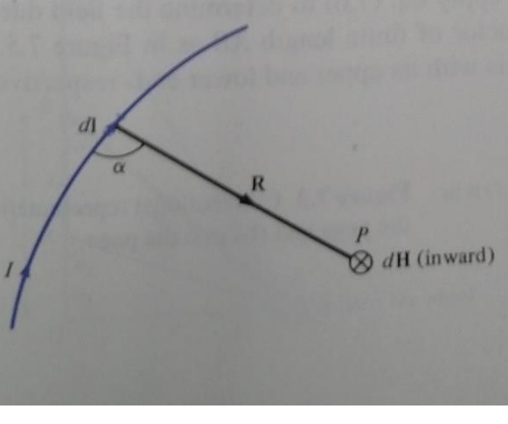


Figure 7.1 magnetic field $d\mathbf{H}$ at P due to current element $I\,d\mathbf{l}$.

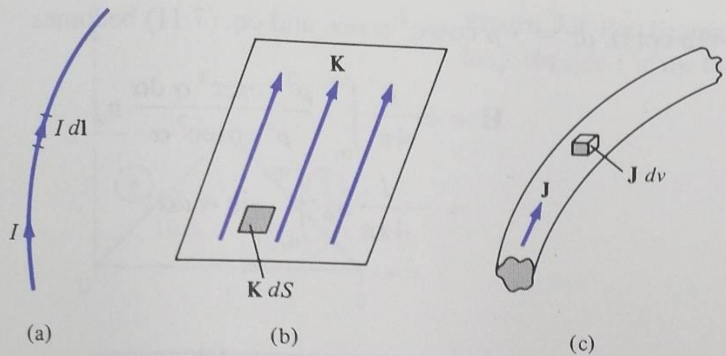


Figure 7.4 Current distributions: **(a)** line current, **(b)** surface current, **(c)** volume current.

α_2 and α_1 at P, the point at which **H** is to be determined. Particular note should be taken of this assumption as the formula to be derived will have to be applied accordingly. If we consider the contribution $d\mathbf{H}$ at P due to an element $d\mathbf{l}$ at $(0, 0, z)$,

$$d\mathbf{H} = \frac{I\, d\mathbf{l} \times \mathbf{R}}{4\pi R^3} \tag{7.9}$$

But $d\mathbf{l} = dz\, \mathbf{a}_z$ and $\mathbf{R} = \rho \mathbf{a}_\rho - z\mathbf{a}_z$, so

$$d\mathbf{l} \times \mathbf{R} = \rho\, dz\, \mathbf{a}_\phi \tag{7.10}$$

Hence,

$$\mathbf{H} = \int \frac{I\rho\, dz}{4\pi[\rho^2 + z^2]^{3/2}}\, \mathbf{a}_\phi \tag{7.11}$$

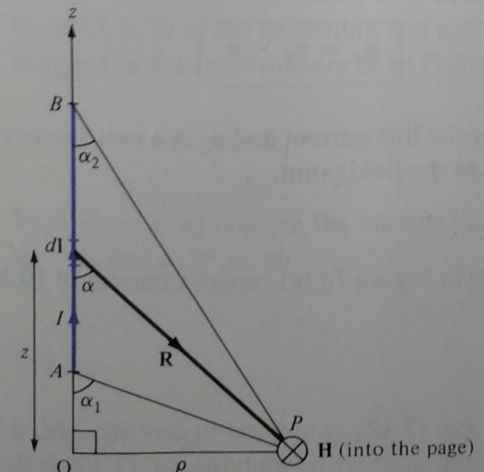


Figure 7.5 Field at point P due to a straight filamentary conductor.

Letting $z = \rho \cot \alpha$, $dz = -\rho \operatorname{cosec}^2 \alpha \, d\alpha$, and eq. (7.11) becomes

$$\mathbf{H} = -\frac{1}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \operatorname{cosec}^2 \alpha \, d\alpha}{\rho^3 \operatorname{cosec}^3 \alpha} \mathbf{a}_\phi$$
$$= -\frac{I}{4\pi\rho} \mathbf{a}_\phi \int_{\alpha_1}^{\alpha_2} \sin \alpha \, d\alpha$$

or

$$\boxed{\mathbf{H} = \frac{I}{4\pi\rho} (\cos \alpha_2 - \cos \alpha_1) \mathbf{a}_\phi} \tag{7.12}$$

This expression is generally applicable for any straight filamentary conductor of finite length. Notice from eq. (7.12) that **H** is always along the unit vector $\mathbf{a}_\phi$ (i.e., along concentric circular paths) irrespective of the length of the wire or the point of interest P. As a special case, when the conductor is *semiinfinite* (with respect to P) so that point A is now at $O(0, 0, 0)$ while B is at $(0, 0, \infty)$; $\alpha_1 = 90°$, $\alpha_2 = 0°$, and eq. (7.12) becomes

$$\mathbf{H} = \frac{I}{4\pi\rho} \mathbf{a}_\phi \tag{7.13}$$

Another special case is when the conductor is *infinite* in length. For this case, point A is at $(0, 0, -\infty)$ while B is at $(0, 0, \infty)$; $\alpha_1 = 180°$, $\alpha_2 = 0°$, so eq. (7.12) reduces to

$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi} \tag{7.14}$$

To find unit vector $\mathbf{a}_\phi$ in eqs. (7.12) to (7.14) is not always easy. A simple approach is to determine $\mathbf{a}_\phi$ from

$$\boxed{\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho} \tag{7.15}$$

where $\mathbf{a}_\ell$ is a unit vector along the line current and $\mathbf{a}_\rho$ is a unit vector along the perpendicular line from the line current to the field point.

EXAMPLE 7.1

The conducting triangular loop in Figure 7.6(a) carries a current of 10 A. Find **H** at (0, 0, 5) due to side 1 of the loop.

Solution:

This example illustrates how eq. (7.12) is applied to any straight, thin, current-carrying conductor. The key point to keep in mind in applying eq. (7.12) is figuring out α_1, α_2, ρ, and $\mathbf{a}_\phi$. To find **H** at (0, 0, 5) due to side 1 of the loop in Figure 7.6(a), consider Figure

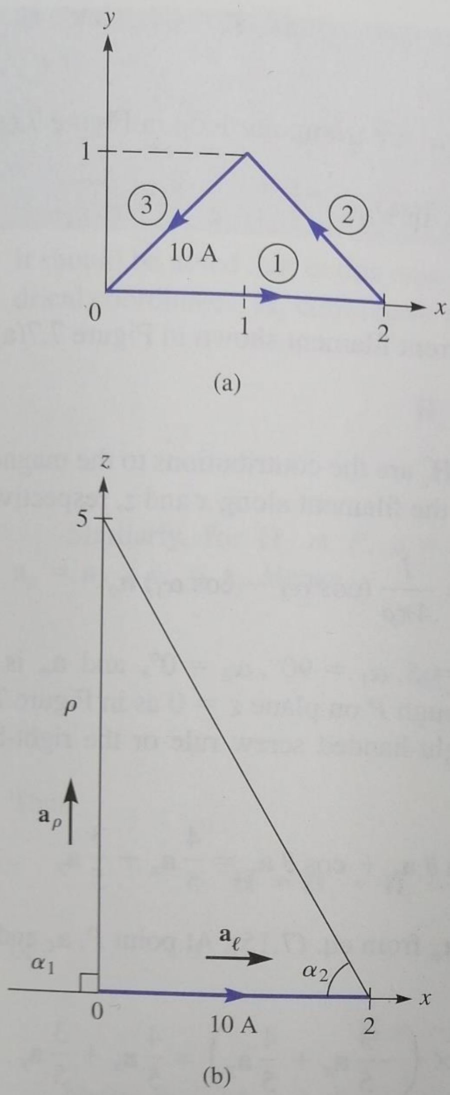


Figure 7.6 For Example 7.1: **(a)** conducting triangular loop, **(b)** side 1 of the loop.

7.6(b), where side 1 is treated as a straight conductor. Notice that we join the point of interest (0, 0, 5) to the beginning and end of the line current. Observe that α_1, α_2, and ρ are assigned in the same manner as in Figure 7.5 on which eq. (7.12) is based.

$$\cos \alpha_1 = \cos 90^\circ = 0, \qquad \cos \alpha_2 = \frac{2}{\sqrt{29}}, \qquad \rho = 5$$

To determine $\mathbf{a}_\phi$ is often the hardest part of applying eq. (7.12). According to eq. (7.15), $\mathbf{a}_\ell = \mathbf{a}_x$ and $\mathbf{a}_\rho = \mathbf{a}_z$, so

$$\mathbf{a}_\phi = \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y$$

Hence,

$$\begin{aligned}\mathbf{H}_1 &= \frac{I}{4\pi\rho}(\cos \alpha_2 - \cos \alpha_1)\mathbf{a}_\phi = \frac{10}{4\pi(5)}\left(\frac{2}{\sqrt{29}} - 0\right)(-\mathbf{a}_y) \\ &= -59.1\mathbf{a}_y \text{ mA/m}\end{aligned}$$

PRACTICE EXERCISE 7.1

Find **H** at (0, 0, 5) due to side 3 of the triangular loop in Figure 7.6(a).

Answer: $-30.63\mathbf{a}_x + 30.63\mathbf{a}_y$ mA/m.

EXAMPLE 7.2

Find **H** at (−3, 4, 0) due to the current filament shown in Figure 7.7(a).

Solution:

Let $\mathbf{H} = \mathbf{H}_x + \mathbf{H}_z$, where $\mathbf{H}_x$ and $\mathbf{H}_z$ are the contributions to the magnetic field intensity at $P(-3, 4, 0)$ due to the portions of the filament along x and z, respectively.

$$\mathbf{H}_z = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\,\mathbf{a}_\phi$$

At $P(-3, 4, 0)$, $\rho = (9 + 16)^{1/2} = 5$, $\alpha_1 = 90°$, $\alpha_2 = 0°$, and $\mathbf{a}_\phi$ is obtained as a unit vector along the circular path through P on plane $z = 0$ as in Figure 7.7(b). The direction of $\mathbf{a}_\phi$ is determined using the right-handed screw rule or the right-hand rule. From the geometry in Figure 7.7(b),

$$\mathbf{a}_\phi = \sin\theta\,\mathbf{a}_x + \cos\theta\,\mathbf{a}_y = \frac{4}{5}\mathbf{a}_x + \frac{3}{5}\mathbf{a}_y$$

Alternatively, we can determine $\mathbf{a}_\phi$ from eq. (7.15). At point P, $\mathbf{a}_\ell$ and $\mathbf{a}_\rho$ are as illustrated in Figure 7.7(a) for $\mathbf{H}_z$. Hence,

$$\mathbf{a}_\phi = -\mathbf{a}_z \times \left(-\frac{3}{5}\mathbf{a}_x + \frac{4}{5}\mathbf{a}_y\right) = \frac{4}{5}\mathbf{a}_x + \frac{3}{5}\mathbf{a}_y$$

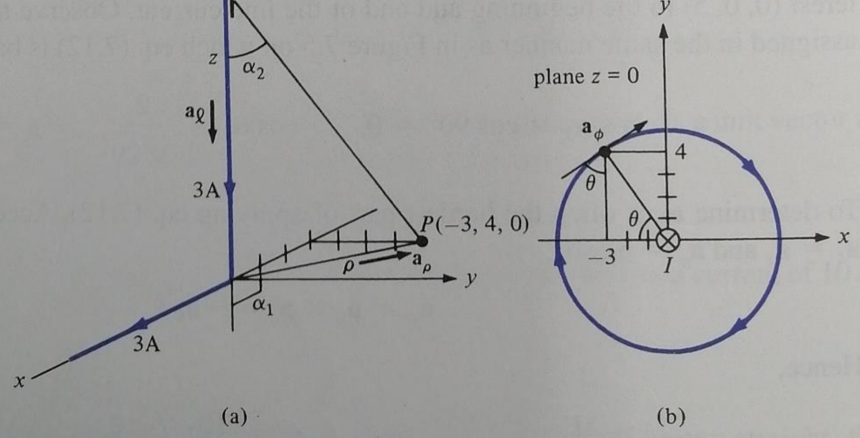


Figure 7.7 For Example 7.2: **(a)** current filament along semiinfinite x- and z-axes; $\mathbf{a}_\ell$ and $\mathbf{a}_\rho$ for $\mathbf{H}_z$ only; **(b)** determining $\mathbf{a}_\rho$ for $\mathbf{H}_z$.

as obtained before. Thus

$$\mathbf{H}_z = \frac{3}{4\pi(5)}(1 - 0)\frac{(4\mathbf{a}_x + 3\mathbf{a}_y)}{5}$$
$$= 38.2\mathbf{a}_x + 28.65\mathbf{a}_y \text{ mA/m}$$

It should be noted that in this case $\mathbf{a}_\phi$ happens to be the negative of the regular $\mathbf{a}_\phi$ of cylindrical coordinates. $\mathbf{H}_z$ could have also been obtained in cylindrical coordinates as

$$\mathbf{H}_z = \frac{3}{4\pi(5)}(1 - 0)(-\mathbf{a}_\phi)$$
$$= -47.75\mathbf{a}_\phi \text{ mA/m}$$

Similarly, for $\mathbf{H}_x$ at P, $\rho = 4$, $\alpha_2 = 0°$, $\cos\alpha_1 = 3/5$, and $\mathbf{a}_\phi = \mathbf{a}_z$ or $\mathbf{a}_\phi = \mathbf{a}_\ell \times \mathbf{a}_\rho = \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$. Hence,

$$\mathbf{H}_x = \frac{3}{4\pi(4)}\left(1 - \frac{3}{5}\right)\mathbf{a}_z$$
$$= 23.88 \mathbf{a}_z \text{ mA/m}$$

Thus

$$\mathbf{H} = \mathbf{H}_x + \mathbf{H}_z = 38.2\mathbf{a}_x + 28.65\mathbf{a}_y + 23.88\mathbf{a}_z \text{ mA/m}$$

or

$$\mathbf{H} = -47.75\mathbf{a}_\phi + 23.88\mathbf{a}_z \text{ mA/m}$$

Notice that although the current filaments appear semiinfinite (they occupy the positive z- and x-axes), it is only the filament along the z-axis that is semiinfinite with respect to point P. Thus $\mathbf{H}_z$ could have been found by using eq. (7.13), but the equation could not have been used to find $\mathbf{H}_x$ because the filament along the x-axis is not semiinfinite with respect to P.

PRACTICE EXERCISE 7.2

The positive y-axis (semiinfinite line with respect to the origin) carries a filamentary current of 2 A in the $-\mathbf{a}_y$ direction. Assume it is part of a large circuit. Find $\mathbf{H}$ at

(a) $A(2, 3, 0)$

(b) $B(3, 12, -4)$

Answer: (a) $145.8\mathbf{a}_z$ mA/m, (b) $48.97\mathbf{a}_x + 36.73\mathbf{a}_z$ mA/m.

EXAMPLE 7.3

A circular loop located on $x^2 + y^2 = 9$, $z = 0$ carries a direct current of 10 A along $\mathbf{a}_\phi$. Determine $\mathbf{H}$ at (0, 0, 4) and (0, 0, −4).

Solution:

Consider the circular loop shown in Figure 7.8(a). The magnetic field intensity $d\mathbf{H}$ at point $P(0, 0, h)$ contributed by current element $I\,d\mathbf{l}$ is given by Biot–Savart's law:

$$d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

where $d\mathbf{l} = \rho\,d\phi\,\mathbf{a}_\phi$, $\mathbf{R} = (0, 0, h) - (x, y, 0) = -\rho\mathbf{a}_\rho + h\mathbf{a}_z$, and

$$d\mathbf{l} \times \mathbf{R} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho\,d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\,d\phi\,\mathbf{a}_\rho + \rho^2\,d\phi\,\mathbf{a}_z$$

Hence,

$$d\mathbf{H} = \frac{I}{4\pi[\rho^2 + h^2]^{3/2}}(\rho h\,d\phi\,\mathbf{a}_\rho + \rho^2\,d\phi\,\mathbf{a}_z) = dH_\rho\,\mathbf{a}_\rho + dH_z\,\mathbf{a}_z$$

By symmetry, the contributions along $\mathbf{a}_\rho$ add up to zero because the radial components produced by pairs of current element 180° apart cancel. This may also be shown mathematically by writing $\mathbf{a}_\rho$ in rectangular coordinate systems (i.e., $\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$).

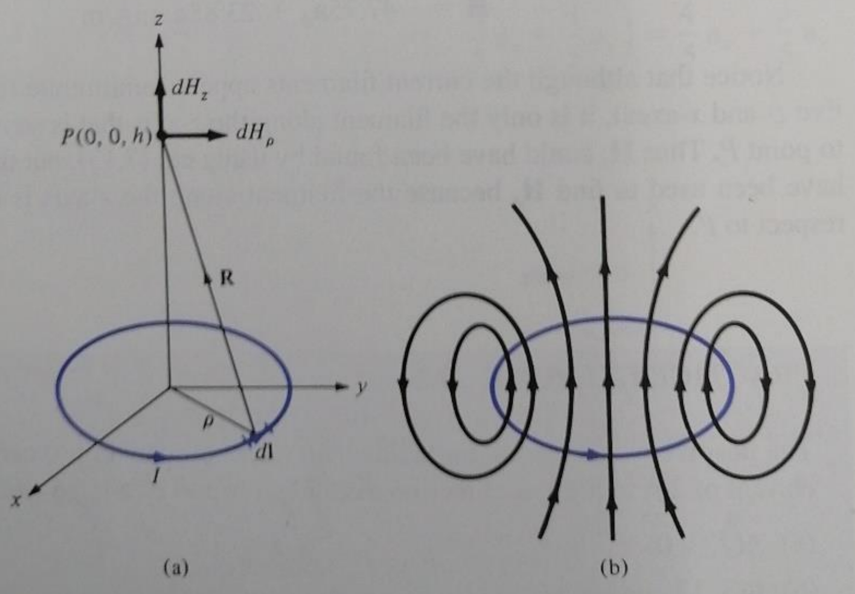


Figure 7.8 For Example 7.3: **(a)** circular current loop, **(b)** flux lines due to the current loop.

Integrating $\cos\phi$ or $\sin\phi$ over $0 \le \phi \le 2\pi$ gives zero, thereby showing that $\mathbf{H}_\rho = 0$. Thus

$$\mathbf{H} = \int dH_z\,\mathbf{a}_z = \int_0^{2\pi} \frac{I\rho^2\,d\phi\,\mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}} = \frac{I\rho^2 2\pi\mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}}$$

or

$$\mathbf{H} = \frac{I\rho^2\mathbf{a}_z}{2[\rho^2 + h^2]^{3/2}}$$

(a) Substituting $I = 10\,A$, $\rho = 3$, $h = 4$ gives

$$\mathbf{H}(0, 0, 4) = \frac{10\,(3)^2\,\mathbf{a}_z}{2[9 + 16]^{3/2}} = 0.36\mathbf{a}_z \text{ A/m}$$

(b) Notice from $d\mathbf{l} \times \mathbf{R}$ above that if h is replaced by $-h$, the z-component of $d\mathbf{H}$ remains the same while the ρ-component still adds up to zero due to the axial symmetry of the loop. Hence

$$\mathbf{H}(0, 0, -4) = \mathbf{H}(0, 0, 4) = 0.36\mathbf{a}_z \text{ A/m}$$

The flux lines due to the circular current loop are sketched in Figure 7.8(b).

PRACTICE EXERCISE 7.3

A thin ring of radius 5 cm is placed on plane $z = 1$ cm so that its center is at (0, 0, 1 cm). If the ring carries 50 mA along $\mathbf{a}_\phi$, find $\mathbf{H}$ at

(a) (0, 0, −1 cm)

(b) (0, 0, 10 cm)

Answer: (a) $400\mathbf{a}_z$ mA/m, (b) $57.3\mathbf{a}_z$ mA/m.

LE 7.4

A solenoid of length ℓ and radius a consists of N turns of wire carrying current I. Show that at point P along its axis,

$$\mathbf{H} = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)\mathbf{a}_z$$

where $n = N/\ell$, θ_1 and θ_2 are the angles subtended at P by the end turns as illustrated in Figure 7.9. Also show that if $\ell \gg a$, at the center of the solenoid,

$$\mathbf{H} = nI\mathbf{a}_z$$

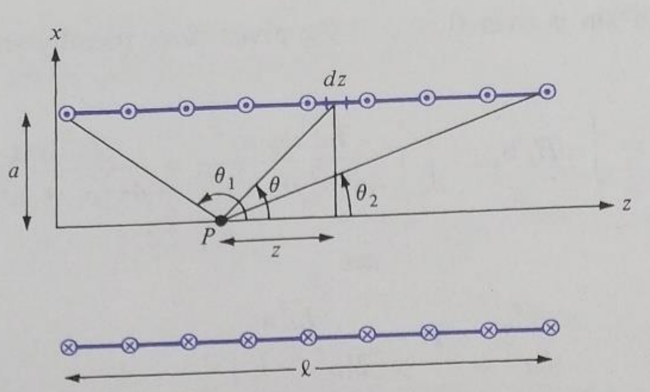


Figure 7.9 For Example 7.4; cross section of a solenoid.

Solution:

Consider the cross section of the solenoid as shown in Figure 7.9. Since the solenoid consists of circular loops, we apply the result of Example 7.3. The contribution to the magnetic field H at P by an element of the solenoid of length dz is

$$dH_z = \frac{I\,dl\,a^2}{2[a^2 + z^2]^{3/2}} = \frac{Ia^2 n\,dz}{2[a^2 + z^2]^{3/2}}$$

where $dl = n\,dz = (N/\ell)\,dz$. From Figure 7.9, $\tan\theta = a/z$; that is,

$$dz = -a\,\text{cosec}^2\,\theta\,d\theta = -\frac{[z^2 + a^2]^{3/2}}{a^2}\sin\theta\,d\theta$$

Hence,

$$dH_z = -\frac{nI}{2}\sin\theta\,d\theta$$

or

$$H_z = -\frac{nI}{2}\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta$$

Thus

$$\mathbf{H} = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)\,\mathbf{a}_z$$

as required. Substituting $n = N/\ell$ gives

$$\mathbf{H} = \frac{NI}{2\ell}(\cos\theta_2 - \cos\theta_1)\,\mathbf{a}_z$$

At the center of the solenoid,

$$\cos\theta_2 = \frac{\ell/2}{[a^2 + \ell^2/4]^{1/2}} = -\cos\theta_1$$

and

$$\mathbf{H} = \frac{In\ell}{2[a^2 + \ell^2/4]^{1/2}}\,\mathbf{a}_z$$

If $\ell \gg a$ or $\theta_2 \simeq 0°$, $\theta_1 \simeq 180°$,

$$\mathbf{H} = nI\mathbf{a}_z = \frac{NI}{\ell}\,\mathbf{a}_z$$

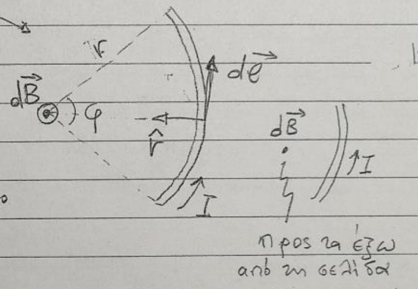
Μαγν. πεδίο από κυκλικό ρευματοφόρο αγωγό (και από κυκλικό βρόχο)

Θα χρησιμοποιήσουμε το νόμο Biot-Savart:

$\boxed{d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\hat{r}}{r^2}}$, μέτρο $\boxed{dB = \frac{\mu_0}{4\pi}\frac{I\,dl\,\sin\varphi}{r^2}}$

γωνία μεταξύ $d\vec{l}, \hat{r}$

Έστω αγωγός σχήματος κυκλικού τόξου όπως στο σχ.



Κάθε ρευματοφόρο στοιχείο $I\,d\vec{l}$ δημιουργεί μαγν. πεδίο $d\vec{B}$

Επειδή γωνία μεταξύ $d\vec{l}$ και $\hat{r}$ $= 90^\circ$ είναι $\sin\varphi = \sin 90^\circ = 1$ και

$\boxed{dB = \frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}}$ και κατεύθυνση $d\vec{B}$ προς τα έξω

Άρα συνολικό πεδίο: $B = \int dB$ και $dl = r\,d\varphi$.

οπότε $B = \int dB = \int_0^{\varphi} \frac{\mu_0}{4\pi}\frac{I r\,d\varphi}{r^2} =$

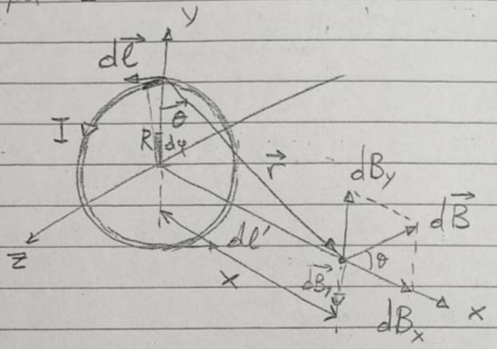
$= \frac{\mu_0 I}{4\pi r}\int_0^{\varphi} d\varphi \Rightarrow \boxed{B = \frac{\mu_0 I \varphi}{4\pi r}}$ σε ακτίνια (όχι μοίρες!)

συνολικό πεδίο στο κέντρο του κυκλικού τόξου

Άρα:

- ▷ Πεδίο σε κυκλικό βρόχο (ακτίνας r): $\varphi = 2\pi$ και $B = \frac{\mu_0 I (2\pi)}{4\pi r} \Rightarrow \boxed{B = \frac{\mu_0 I}{2r}}$
- ▷ Πεδίο σε N κυκλικές σπείρες: $\boxed{B = \frac{\mu_0 N I}{2r}}$

Εφαρμογή 1: Να βρεθεί το μαγν. πεδίο σε απόσταση x από το κέντρο κυκλικού βρόχου ακτίνας R που διαρρέεται από ρεύμα I



Απ. Νόμος Biot-Savart:

$$\left.\begin{aligned} dB &= \frac{\mu_0}{4\pi}\frac{I\,dl\sin\varphi}{r^2} \\ &\text{Όμως } \sin\varphi = \sin 90^\circ = 1 \\ &\text{γιατί } d\vec{l} \perp \vec{r} \text{ και} \\ &r^2 = x^2 + R^2 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow dB = \frac{\mu_0 I}{4\pi}\frac{dl}{x^2+R^2}$$ → το αναλύουμε σε 2 συνιστώσες μία στην x, μία στην y

▷ Συνιστώσες του $d\vec{B}$:

∘ $$dB_x = dB\cos\theta = \frac{\mu_0 I}{4\pi}\frac{dl}{x^2+R^2}\frac{R}{r} = \frac{\mu_0 I}{4\pi}\frac{dl}{x^2+R^2}\frac{R}{\sqrt{x^2+R^2}}$$

∘ $$dB_y = dB\sin\theta = \frac{\mu_0 I}{4\pi}\frac{dl}{x^2+R^2}\frac{x}{r} = \frac{\mu_0 I}{4\pi}\frac{dl}{x^2+R^2}\frac{x}{\sqrt{x^2+R^2}}$$

Επειδή το πρόβλημα έχει κυλινδρική συμμετρία ως προς x πρέπει $B_y = 0$: $\forall\, d\vec{l}\ \exists\, d\vec{l}'$ αντιδιαμετρικά με αντίθετο dB_y → όλα αλληλοαναιρούνται

Άρα μόνο B_x. Αυτή υπολογίζεται ως:

$$\left.\begin{aligned} B_x &= \int dB_x = \int \frac{\mu_0 I}{4\pi}\frac{R\,dl}{(x^2+R^2)^{3/2}} \\ dl &= R\,d\varphi \end{aligned}\right\} \Rightarrow B_x = \frac{\mu_0 I}{4\pi}\frac{R^2}{(x^2+R^2)^{3/2}}\int_0^{2\pi} d\varphi$$

▷ Άρα, για τόξο 2π: $\boxed{B_x = \dfrac{\mu_0 I R^2}{2(x^2+R^2)^{3/2}}}$ (1)

▷▷ Για N κυκλικές σπείρες: (1) → $\boxed{B_x = \dfrac{\mu_0 N I R^2}{2(x^2+R^2)^{3/2}}}$ (2)

▷ Αν στις (1)·(2) θεωρήσουμε $x=0$ → στο κέντρο του βρόχου:

$$B_x = \frac{\mu_0 I}{2R} \text{ (για ένα βρόχο) ή } B_x = \frac{\mu_0 N I}{2R} \text{ (για N βρόχους)}$$

όπως πριν!

Εφαρμογή 2 [Young, p. 815]: Ένα πηνίο αποτελείται από 100 σπείρες ακτίνας 0.6 m και διαρρέεται από ρεύμα 5 A. (i) Να βρεθεί σε ποια απόσταση από το κέντρο, πάνω στον άξονα, το μέτρο του μαγν. πεδίου γίνεται $= \frac{1}{8}$ εκείνου του κέντρου.

Απ. Στο κέντρο: $B_{cent.} = \dfrac{\mu_0 N I}{2R}$

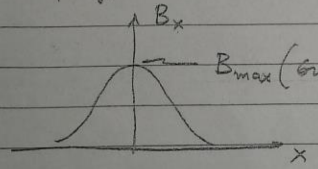
Σε απόσταση x από το κέντρο (στον άξονα): $B_{ax} = \dfrac{\mu_0 N I R^2}{2(x^2+R^2)^{3/2}}$

και $B_{ax} = \frac{1}{8} B_{cent} \Rightarrow \dfrac{R^2}{(x^2+R^2)^{3/2}} = \dfrac{1}{8}\dfrac{1}{R} \Rightarrow (x^2+R^2)^{3/2} = 8R^3 \Rightarrow$

$\Rightarrow (x^2+R^2)^{1/2} = 2R \Rightarrow x^2 + R^2 = 4R^2 \Rightarrow x^2 = 3R^2 \Rightarrow \boxed{x = \pm\sqrt{3}\,R}$

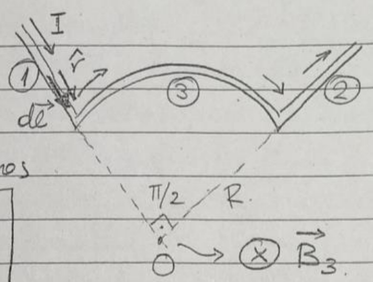
$\Rightarrow x \approx \pm 1.04\,m$

▷ Στην πραγματικότητα:



B_{max} (στο κέντρο) $= B_{cent} = \dfrac{\mu_0 N I}{2R}$

Εφαρμογή 3 [Halliday, p. 266]



Να βρεθεί το μαγν. πεδίο στο κέντρο O της διάταξης του σχήματος

[Είναι ένα κυκλικό τόξο ακτίνας R και γωνίας π/2 και δύο ευθύγραμμα τμήματα που οι προεκτάσεις τους τέμνονται στο O]

Απ.

▷ Για τα ευθ. τμήματα ① και ② είναι: γωνία μεταξύ $\hat{r}$ και $d\vec{l}$ $= 0$

Άρα $dB_{1,2} = \frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}\sin 0 = 0 \Rightarrow B_1 = B_2 = 0$

▷ Για το κυκλικό τόξο ③: $B_3 = \frac{\mu_0 I}{4\pi R}\cdot\frac{\pi}{2} \Rightarrow$

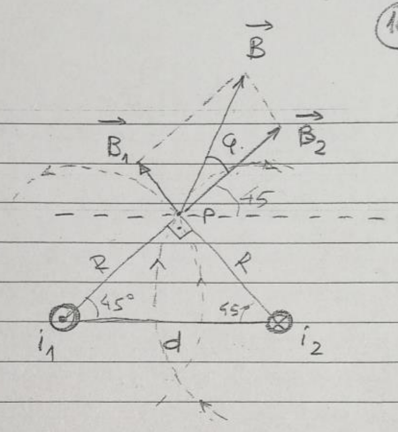
$$\Rightarrow \boxed{B_3 = \frac{\mu_0 I}{8R}} \to \text{μέτρο.}$$

Για την κατεύθυνση: κανόνας δεξιού χεριού → B_3 προς τα μέσα

Άρα τελικά: $B = \underbrace{B_1}_{0} + \underbrace{B_2}_{0} + B_3 = \frac{\mu_0 I}{8R}$.

** Γενικά προσθέτουμε διανυσματικά τα $\vec{B}_1, \vec{B}_2, \vec{B}_3$.

βλέπε επόμενο παράδειγμα / εφαρμογή



Εφαρμογή 4 [Halliday, p. 267]

Έστω δύο αγωγοί που διαρρέονται από τα αντίρροπα ρεύματα $i_1 = 15A$ και $i_2 = 32A$, και απέχουν κατά $d = 5.3$ cm. Να βρεθεί το μέτρο και η και η κατ/ση του μαγν. πεδίου στο σημείο P.

Απ. $B_1 = \frac{\mu_0 i_1}{2\pi R}$, $B_2 = \frac{\mu_0 i_2}{2\pi R}$ είναι τα μέτρα

με $R = d\cos 45° = d\frac{\sqrt{2}}{2} = \frac{d}{\sqrt{2}}$.

> Οι κατευθύνσεις είναι όπως στο σχ. (κανόνας δεξιού χεριού)

Συνολικό B (μέτρο): $B = \sqrt{B_1^2 + B_2^2} =$

$$= \frac{\mu_0}{\cancel{2}\pi d \frac{\sqrt{2}}{\cancel{2}}}\sqrt{i_1^2 + i_2^2} = \frac{4\pi \times 10^{-7}(T \cdot m/A)}{\pi \times 5.3 \times 10^{-2} m \sqrt{2}}\sqrt{15^2 + 32^2}\, A$$

$\Rightarrow B = 1.89 \times 10^{-4}$ T (ή $B = 1.89$ Gauss)

Για την κατεύθυνση του B:

$$\tan\varphi = \frac{B_1}{B_2} \Rightarrow \varphi = \tan^{-1}\frac{B_1}{B_2} = \tan^{-1}\frac{i_1}{i_2} \Rightarrow$$

$$\Rightarrow \varphi = \tan^{-1}\frac{15\cancel{A}}{32\cancel{A}} = 25°$$

οπότε $\theta = 45° + 25° = 70°$

ως προς τον οριζόντιο άξονα

Ο νόμος του Ampère

→ Εναλλακτικός τρόπος σύνδεσης μαγν. πεδίου και των πηγών του.

[~ αντίστοιχα με το νόμο του Gauss για το ηλ. πεδίο, που συνδέει το ηλ. πεδίο με τα φορτία που το παράγουν]

Νόμος του Ampère : $\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I}$

Εδώ:

$\oint \vec{B}\cdot d\vec{l}$: επικαμπύλιο ολοκλήρωμα (*)

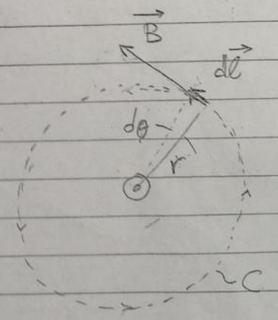
Όπως: $W = \int_a^b \vec{F}\cdot d\vec{l}$, $V_a - V_b = \int_a^b \vec{E}\cdot d\vec{l}$

(*) ↗ διαιρούμε το "δρόμο" σε απειροστά στοιχεία $d\vec{l}$,
↗ βρίσκουμε το εσωτ. γινόμενο $\vec{B}\cdot d\vec{l}$
↗ αθροίζουμε τα εσωτ. γινόμενα.

Για να το καταλάβουμε:

Έστω το μαγν. πεδίο μακριού ευθ. αγωγού σε απόσταση r από τον αγωγό:

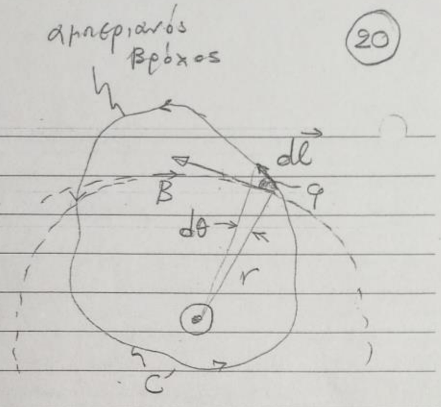
$B = \frac{\mu_0 I}{2\pi r}$; μαγν. δυν. γραμμές: κύκλοι με κέντρο τον αγωγό.



Στον κύκλο C: $\oint_C \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi r}\int_C dl$

$= \frac{\mu_0 I}{2\pi r}\int_0^{2\pi} r\,d\theta = \frac{\mu_0 I}{\cancel{2\pi r}}\cancel{2\pi r} = \mu_0 I$

Άρα ∀ κύκλο (οποιασδήποτε ακτίνας) : $\boxed{\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I}$



Για ένα πιο γενικό δρόμο:

$$\left.\begin{array}{l}\vec{B}\cdot d\vec{\ell} = B\,d\ell\cos\varphi \\ \text{Όμως } d\ell\cos\varphi \approx r\,d\theta\end{array}\right\} \Rightarrow$$

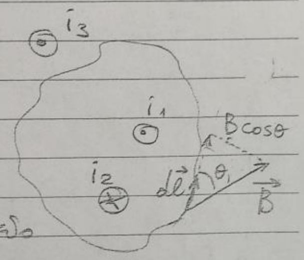
$$\oint_{C'} \vec{B}\cdot d\vec{\ell} = \int \frac{\mu_0 I}{2\pi \cancel{r}}\cdot \cancel{r}\,d\theta = \frac{\mu_0 I}{2\pi}\int d\theta$$

→ γωνία που σαρώνεται από το $\vec{r}$ και $d\vec{\ell}$ σε όλο τον κλειστό δρόμο C′

$$\Rightarrow \oint \vec{B}\cdot d\vec{\ell} = \frac{\mu_0 I}{\cancel{2\pi}}\cdot \cancel{2\pi} \Rightarrow \boxed{\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I}$$ και πάλι!

"έγκλειστο" ρεύμα στο βρόχο.

▷ Αν ο αγωγός δεν περιλαμβάνεται στο βρόχο ⇒ $\int d\theta = 0$ και $\oint \vec{B}\cdot d\vec{\ell} = 0$.



Άλλο παράδειγμα: έστω ο βρόχος που περικλείει τα i_1, i_2 αλλά όχι το i_3.

Έστω $\vec{B}$ το συνολικό πεδίο που οφείλεται στα τρία ρεύματα (ξέρουμε ότι ∃ στο επίπεδο, δεν ενδιαφέρει ο προσανατολισμός)

Τότε $\oint \underbrace{\vec{B}\cdot d\vec{\ell}}_{B\,d\ell\cos\theta} = \oint (B\cos\theta)\,d\ell = \mu_0 I_{enc}$ → (enclosed)

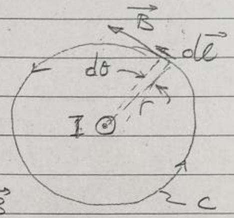
και $I_{enc} = \underbrace{i_1 - i_2}_{\text{πρόσημα με κανόνα δεξιού χεριού}}$

Εφαρμογή 1: Μαγν. πεδίο έξω από ευθ. αγωγό μεγάλου μήκους.

— Με το νόμο Biot-Savart είχαμε βρεί: $\boxed{B = \frac{\mu_0 I}{2\pi r}}$

Θα βρούμε το ίδιο με το νόμο του Ampère.

Θεωρούμε αμπεριανό βρόχο → κύκλο ακτίνας r



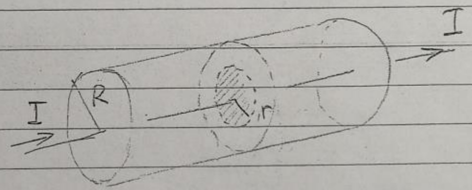
$$\oint_c \vec{B}\cdot d\vec{\ell} = \oint_c B\cos\theta \cdot d\ell$$

θ=0 και cosθ=1 γιατί $d\vec{\ell} \parallel \vec{B}$

$$= B\int_0^{2\pi} r\,d\theta = 2\pi r B \underbrace{=}_{\text{από Ampère}} \mu_0 I \Rightarrow \boxed{B = \frac{\mu_0 I}{2\pi r}}$$

Εφαρμογή 2 [Young, p. ~~819~~ 1083]. Μαγν. πεδίο μέσα & έξω από κυλ. αγωγό ακτίνας R.

Παίρνουμε ένα αμπεριανό βρόχο σχ. κύκλου, με ακτίνα r:



$$2\pi r \cdot B = \mu_0 I_{enc}$$

Όμως $\underbrace{J}_{\text{πυκνότητα ρεύματος}} = \frac{I_{enc}}{\pi r^2}$ και $J = \frac{I}{\pi R^2} \Rightarrow \frac{I_{enc}}{\pi r^2} = \frac{I}{\pi R^2} \Rightarrow I_{enc} = I\frac{r^2}{R^2}$

Άρα $2\pi r B = \mu_0 I \frac{r^2}{R^2} \Rightarrow$

$$\Rightarrow \boxed{B = \frac{\mu_0 I}{2\pi}\frac{r}{R^2}}$$