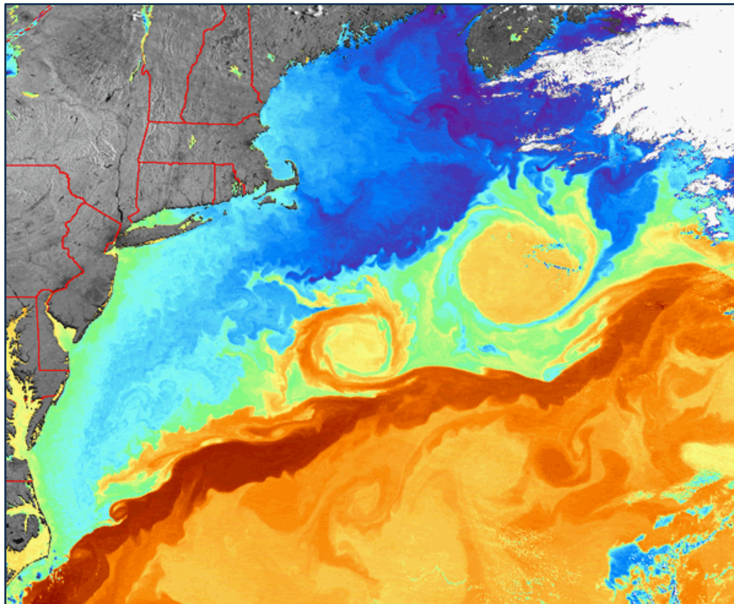
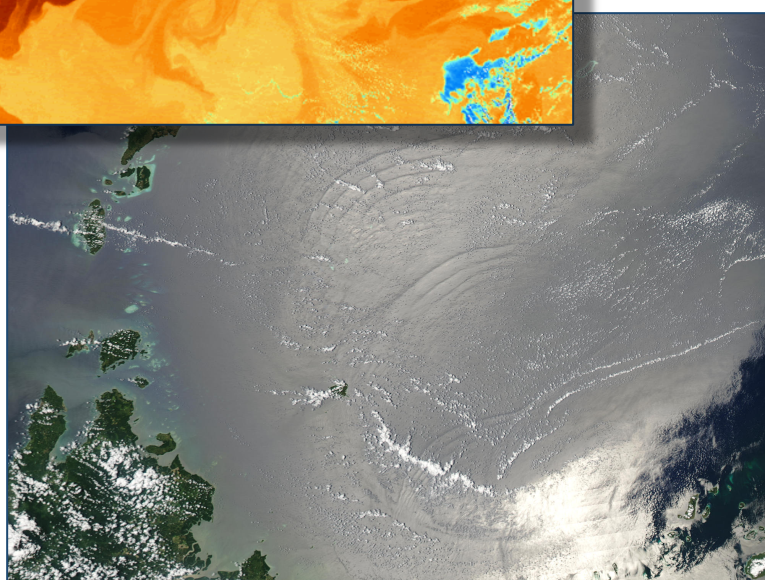
Οι νόμοι διατήρησης στη Φυσική Ωκεανογραφία





4. The conservation laws in the ocean dynamics

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- Fundamental dynamics and the equations for geophysical fluids
- Oceanic approximations
- Rotation
- Dominant scales in ocean dynamics
- Reduced equations – Scaling the ocean dynamics

Fundamental dynamics and the equations for geophysical fluids

i. Momentum Equation:

Pressure gradient

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nabla\Phi + \mathbf{F}$$

Material derivative

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}$$

local rate of change

advection

$$\Phi = -gz$$

Force potential

Non-conservative forces

e.g. $\mathbf{F} = \nu\nabla^2\mathbf{u}$

(see next transparency)

$$\boxed{\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p - g\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u}}$$

Non-linearity

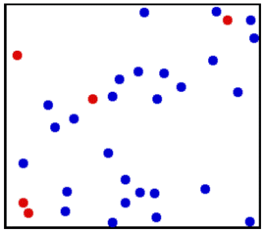
$$Re = \frac{\mathbf{u}\cdot\nabla\mathbf{u}}{\nu\nabla^2\mathbf{u}} = \frac{U^2L^2}{\nu UL} = \frac{UL}{\nu}$$

In the ocean, usually

$$Re >> 1$$

Mixing and diffusion

- Molecular random motions transport around “a property”

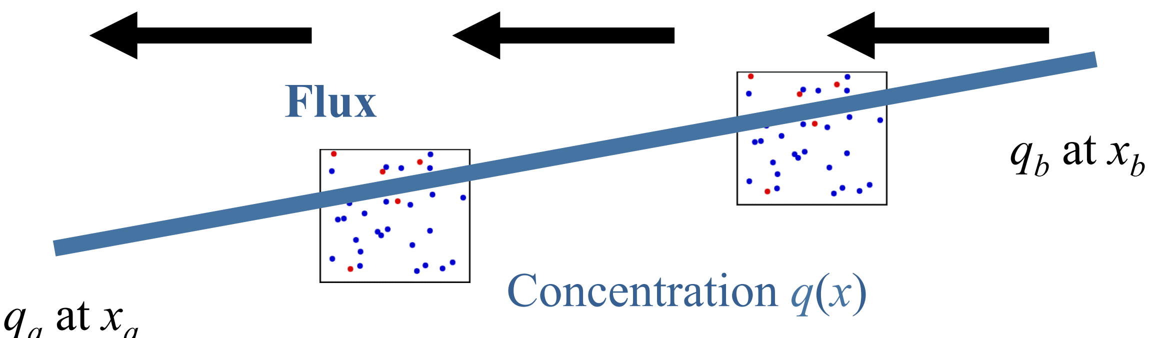


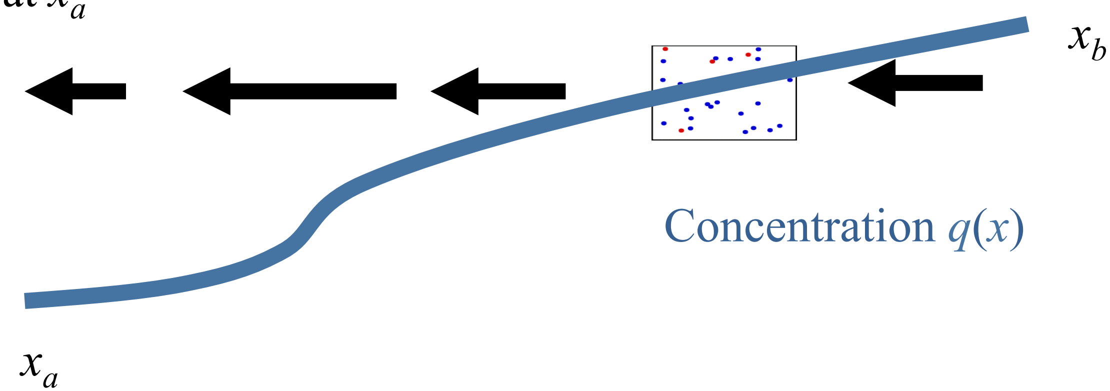
- **Flux=Force/Resistance** (De Groot, 1963)

$$F_q = \kappa_q \frac{\partial q}{\partial x}$$

Flux of q — medium Resistance — Force (gradient)

If the concentration of q changes linearly along a path, there should be a constant flux of q along this path (and the concentration of q is constant in time along the path) – Or the opposite: Constant concentration in any point in a path, means constant flux along the path.



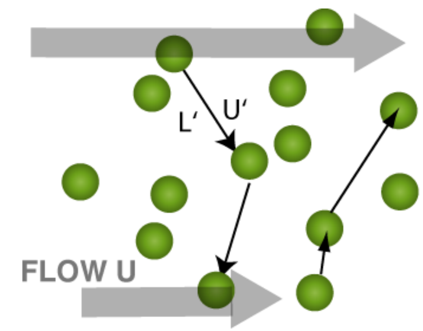


If there is a change of flux of q along the path $\left(\partial F/\partial x\right)$ it will produce a change of q in time $\left(\partial q/\partial t\right)$.

$$\frac{\partial q}{\partial t} = \frac{\partial F}{\partial x} = -\kappa_q \frac{\partial^2 q}{\partial x^2}$$

For velocity: resistance is μ (molecular viscosity) and $\nu=\mu/\rho$ (kinematic viscosity)

Why is viscosity a non-conservative force?



Ignoring any other force and density variations (and working only in one dimension for simplicity):

$$\rho \frac{\partial u}{\partial t} = \frac{\partial \rho u}{\partial t} = \rho \nu \frac{\partial^2 u}{\partial z^2}$$

Multiplying by u

$$u \frac{\partial \rho u}{\partial t} = \rho \nu u \frac{\partial^2 u}{\partial z^2} \Rightarrow$$

$$\frac{\partial \left(\rho u^2 / 2\right)}{\partial t} = \rho \nu u \frac{\partial}{\partial z} \frac{\partial u}{\partial z} = \rho \nu \frac{\partial}{\partial z} u \frac{\partial u}{\partial z} - \rho \nu \left(\frac{\partial u}{\partial z}\right)^2 =$$

$$= \rho \nu \frac{\partial}{\partial z} \frac{\partial \left(u^2 / 2\right)}{\partial z} - \rho \nu \left(\frac{\partial u}{\partial z}\right)^2 = \frac{\partial}{\partial z}\left(\nu \frac{\partial}{\partial z}\left(\rho u^2 / 2\right)\right) - \rho \nu \left(\frac{\partial u}{\partial z}\right)^2 \Rightarrow$$

$$\frac{\partial \left(\rho u^2 / 2\right)}{\partial t} = \frac{\partial}{\partial z}\left(\nu \frac{\partial}{\partial z}\left(\rho u^2 / 2\right)\right) - \rho \nu \left(\frac{\partial u}{\partial z}\right)^2$$

In three-dimensions

$$\frac{\partial \left(\rho u^2 / 2\right)}{\partial t} = \nabla\left(\nu \nabla\left(\rho u^2 / 2\right)\right) - \rho \nu \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right]$$

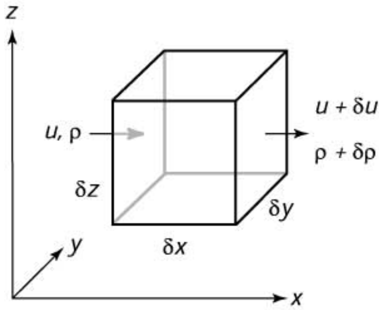
The heat added to the water increases its entropy at the rate of heat generation divided by absolute temperature (entropy source term).

Local rate of change of energy concentration

Energy Flux (Divergence of viscosity times the gradient of energy)

Loss of Mechanical Energy (Transformation to heat – always negative)

ii. Conservation of mass (continuity equation):



Mass flux to the volume (x=direction) $= \rho u \delta y \delta z$

Mass flux from the volume (x=direction) $= \left(\rho + \frac{\partial \rho}{\partial x}\delta x\right)\left(u + \frac{\partial u}{\partial x}\delta x\right)\delta y \delta z$

Accumulation $\left[u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial x}\frac{\partial u}{\partial x}\delta x\right]\delta x \delta y \delta z = \left[\frac{\partial(\rho u)}{\partial x} + O(\delta x)\right]\delta x \delta y \delta z$

small terms

In 3-D: $\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho \upsilon)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\delta x \delta y \delta z$

This accumulation must be accompanied by increase of mass in the volume

$$\frac{\partial}{\partial t}(\text{density } \times \text{volume}) = \left(\frac{\partial \rho}{\partial t}\right)\delta x \delta y \delta z = -\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho \upsilon)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)\delta x \delta y \delta z$$

$$\left(\frac{\partial \rho}{\partial t}\right) = -\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho \upsilon)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right) \Rightarrow$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{u}) = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \mathbf{u}\nabla\rho + \rho\nabla\mathbf{u} = 0$$

$$\boxed{\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u}}$$

iii. Conservation of dissolved material concentration:

$$\frac{\partial(\rho q)}{\partial t} + \nabla(\rho q \mathbf{u}) = S^q$$

→ non conservative sources and sinks of q

e.g. $\kappa \nabla^2 q$

and using the continuity equation

$$\boxed{\frac{Dq}{Dt} = \kappa \nabla^2 q + S^q}$$

For the ocean (and the atmosphere) κ is of the order of ν, i.e. the ***Prandtl number*** $\nu / \kappa = O(1)$

thus advection dominates diffusion (as Re is very large).

iv. Internal energy conservation:

$$\rho \frac{De}{Dt} = -p \nabla \mathbf{u} + \rho Q$$

Change of internal energy due to pressure variations ($-p \nabla \mathbf{u}$)

cooling/heating (ρQ)

and using the continuity equation

$$\boxed{\frac{De}{Dt} = -p \frac{D}{Dt}\left(\frac{1}{\rho}\right) + Q}$$

1st Law of Thermodynamics

v. Entropy conservation:

$$\boxed{T \frac{D\eta}{Dt} = Q - \sum_k \mu_k S^{(q_k)}}$$

Temperature $T(K)$

cooling/heating

μ_k chemical potential

2nd Law of Thermodynamics

vi. Equation of state:

$$\boxed{\rho = \rho(T, p, q)} \longrightarrow \text{ocean } S(psu)$$

$$\mathrm{d}\rho = \left(\frac{\partial \rho}{\partial T}\right)_{S,p} \mathrm{d}T + \left(\frac{\partial \rho}{\partial S}\right)_{T,p} \mathrm{d}S + \left(\frac{\partial \rho}{\partial p}\right)_{S,T} \mathrm{d}p = \rho\left(\alpha \mathrm{d}T + \beta \mathrm{d}S + \gamma \mathrm{d}p\right)$$

$$\alpha = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{S,p} \quad \text{thermal expansion coefficient } \left(\mathrm{K}^{-1}\right)$$

$$\beta = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial S}\right)_{T,p} \quad \text{haline contraction coefficient } \left(\mathrm{psu}^{-1}\right)$$

$$\gamma = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_{S,T} \quad \text{compressibility coefficient } \left(\mathrm{Pa}^{-1}\right)$$

The equation for ρ is obtained in a sequence of steps. First, the density ρ_w of pure water ($S = 0$) is given by

$$\rho_w = 999.842594 + 6.793952 \times 10^{-2} t - 9.095290 \times 10^{-3} t^2 + 1.001685 \times 10^{-4} t^3 - 1.120083 \times 10^{-6} t^4 + 6.536332 \times 10^{-9} t^5. \quad \text{(A3.1)}$$

Second, the density at one standard atmosphere (effectively $p = 0$) is given by

$$\rho(S, t, 0) = \rho_w + S(0.824493 - 4.0899 \times 10^{-3} t + 7.6438 \times 10^{-5} t^2 - 8.2467 \times 10^{-7} t^3 + 5.3875 \times 10^{-9} t^4) + S^{3/2}(-5.72466 \times 10^{-3} + 1.0227 \times 10^{-4} t - 1.6546 \times 10^{-6} t^2) + 4.8314 \times 10^{-4} S^2. \quad \text{(A3.2)}$$

Finally, the density at pressure p is given by

$$\rho(S, t, p) = \rho(S, t, 0)/(1 - p/K(S, t, p)). \quad \text{(A3.3)}$$

where K is the secant bulk modulus. The pure water value K_w is given by

$$K_w = 19652.21 + 148.4206 t - 2.327105 t^2 + 1.360477 \times 10^{-2} t^3 - 5.155288 \times 10^{-5} t^4. \quad \text{(A3.4)}$$

The value at one standard atmosphere ($p = 0$) is given by

$$K(S, t, 0) = K_w + S(54.6746 - 0.603459 t + 1.09987 \times 10^{-2} t^2 - 6.1670 \times 10^{-5} t^3) + S^{3/2}(7.944 \times 10^{-2} + 1.6483 \times 10^{-2} t - 5.3009 \times 10^{-4} t^2) \quad \text{(A3.5)}$$

and the value at pressure p by

$$K(S, t, p) = K(S, t, 0) + p(3.239908 + 1.43713 \times 10^{-3} t + 1.16092 \times 10^{-4} t^2 - 5.77905 \times 10^{-7} t^3) + pS(2.2838 \times 10^{-3} - 1.0981 \times 10^{-5} t - 1.6078 \times 10^{-6} t^2) + 1.91075 \times 10^{-4} p S^{3/2} + p^2(8.50935 \times 10^{-5} - 6.12293 \times 10^{-6} t + 5.2787 \times 10^{-8} t^2) + p^2 S(-9.9348 \times 10^{-7} + 2.0816 \times 10^{-8} t + 9.1697 \times 10^{-10} t^2). \quad \text{(A3.6)}$$

For small variations (and assuming α, β, γ independent of T, S, p)

$$\rho = \rho_0\left[1 + \alpha\left(T - T_0\right) + \beta\left(S - S_0\right) + \gamma\left(p - p_0\right)\right]$$

In reality the thermal expansion coefficient depends on temperature (**cabelling**) and the non-linearity induced has the result that two water masses characterized by a different temperature and salinity but same density when mix together, the resulting mixed water can become denser and eventually sinks. Thermal expansion coefficient is also dependent on pressure (**thermobaricity**). For this reason two water masses with different salinity and temperature but same density at the surface, don't have the same density below the surface.

Oceanic approximations

Incompressibility

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla\cdot\mathbf{u} \longrightarrow \text{Compressibility} \quad \beta = -\frac{1}{V}\frac{DV}{DP} = -\left(\frac{1}{V}\frac{DV}{Dt}\right)\left(\frac{Dt}{DP}\right) = -\left(\frac{1}{V}\frac{DV}{Dt}\right)\bigg/\frac{DP}{Dt}$$

If the volume does not change under changes in pressure

$$\beta = 0; \quad \frac{1}{V}\frac{DV}{Dt} = 0$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} \sim \frac{1}{T}\frac{\delta\rho}{\rho} = \frac{U}{L}\frac{\delta\rho}{\rho} << \frac{U}{L} \sim \nabla\cdot\mathbf{u}$$

$$\frac{\delta\rho}{\rho} = O\left(10^{-3}\right)$$

Using

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{V}{m}\frac{D}{Dt}\left(\frac{m}{V}\right) = -\frac{1}{V}\frac{DV}{Dt}$$

Boussinesq Continuity Equation

$$\boxed{\nabla\cdot\mathbf{u} = 0 \quad \text{and} \quad \frac{D\rho}{Dt} = 0}$$

So, for incompressible fluid $\frac{1}{\rho}\frac{D\rho}{Dt} = 0$

Linearized equation of state under incompressibility

Typical values ρ_0=1028 kg/m^3, T_0=10 ^oC=283 ^oK, S_0=35 ‰

$$\boxed{\rho = \rho_0\left[1-\alpha\left(T-T_0\right)+\beta\left(S-S_0\right)\right]}$$

Boussinesq Equation of State

$$\alpha = -\frac{1}{\rho}\frac{\partial\rho}{\partial T} \approx 2\times10^{-4}\,\mathrm{K}^{-1}$$

Thermal expansion coefficient

$$\beta = +\frac{1}{\rho}\frac{\partial\rho}{\partial S} \approx 8\times10^{-4}\,\mathrm{ppt}^{-1}$$

Haline contraction coefficient

Neglecting possible ***thermobaric instability*** and ***cabelling instability***

Oceanic approximations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla\mathbf{u} = -\frac{1}{\rho_0}\nabla p - g\frac{\rho}{\rho_0}\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u} \quad (1)$$

$$\nabla\cdot\mathbf{u} = 0 \quad (2)$$

$$\frac{\partial S}{\partial t} + \mathbf{u}\cdot\nabla S = \kappa_S\nabla^2 S + S^S \quad (3)$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \kappa_T\nabla^2 T + \frac{Q}{c_p\rho_0} \quad (4)$$

$$\rho = \rho_0\left[1 - \alpha\left(T - T_0\right) + \beta\left(S - S_0\right)\right] \quad (5)$$

$$c_p = 4\times 10^{-3}\ J\,kg^{-1}\ \mathrm{K}^{-1}$$

Boussinesq Equations

- Entropy and density conservation equations become redundant due to the thermodynamic approximations

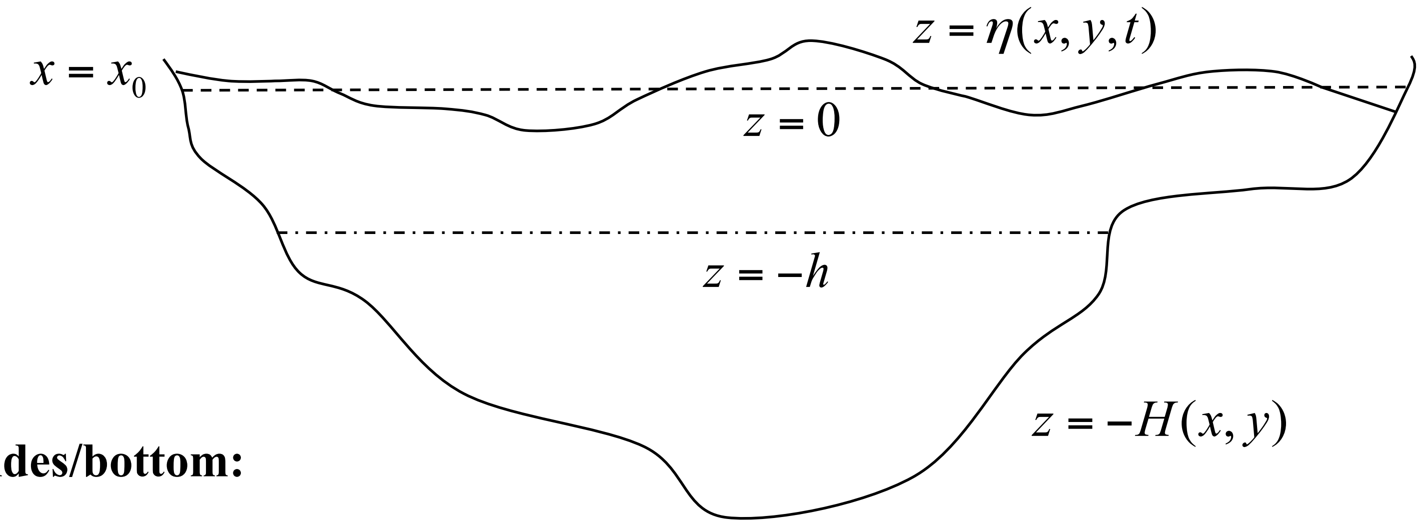
- Boussinesq approximations cancel acoustic and shock waves, it is a good approximation since

$$\text{Mach number}: \ M = \frac{U}{C_s} << 1$$

$$C_s \approx 1500\,m^{-1}\ s^{-1}, \quad U \approx 1\,m^{-1}\ s^{-1}$$

- When the right hand side of equations (3) and (4) is zero, the processes are called **adiabatic**.

Boundary conditions



Sides/bottom:

$$\mathbf{u} \cdot \hat{\mathbf{n}} = 0 \text{ at } z = -H(x, y) \;\&\; x = x_0$$

Top:

$$w = \frac{D\eta}{Dt} \text{ at } z = \eta(x, y, t) \qquad \frac{D\eta}{Dt} = 0 \;\&\; p = 0 \text{ at } z = 0 \text{ Rigid-lid}$$

$$p = p_{\text{atm}}(x, y, t)$$

↓

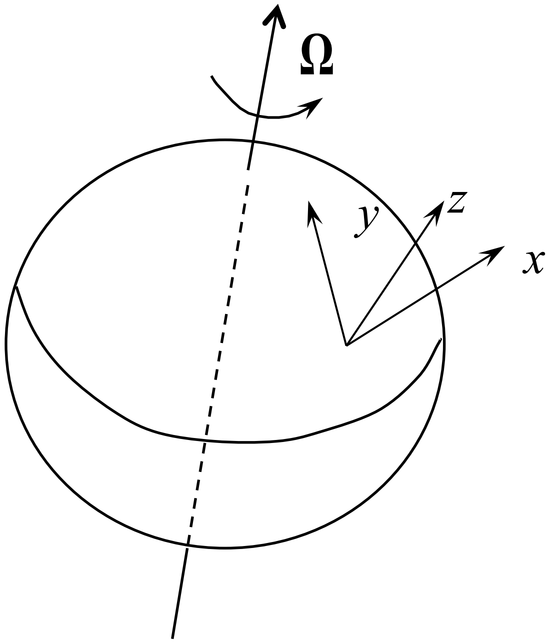
$$h \approx -\delta p_{\text{atm}} / g\rho_0$$

Inverse Barometer Effect
(no oceanic acceleration)

Interior:

$$p_{\text{up}} = p_{\text{low}} \text{ at } z = -h$$

The effect of rotation



$$\left(\frac{D\mathbf{r}}{Dt}\right)_I = \left(\frac{D\mathbf{r}}{Dt}\right)_R + \mathbf{\Omega}\times\mathbf{r}$$

$$\mathbf{u}_I = \mathbf{u}_R + \mathbf{\Omega}\times\mathbf{r} \qquad (1)$$

$$\left(\frac{D\mathbf{u}_I}{Dt}\right)_I = \left(\frac{D\mathbf{u}_I}{Dt}\right)_R + \mathbf{\Omega}\times\mathbf{u}_I \qquad (2)$$

(1)&(2)

$$\left(\frac{D\mathbf{u}_I}{Dt}\right)_I = \left(\frac{D\mathbf{u}_R}{Dt}\right)_R + \frac{D\mathbf{\Omega}}{Dt}\times\mathbf{r} + \mathbf{\Omega}\times\left(\frac{D\mathbf{r}}{Dt}\right)_R + \mathbf{\Omega}\times\left(\mathbf{u}_R + \mathbf{\Omega}\times\mathbf{r}\right)$$

$$= \left(\frac{D\mathbf{u}_R}{Dt}\right)_R + 2\mathbf{\Omega}\times\mathbf{u}_R + \mathbf{\Omega}\times\left(\mathbf{\Omega}\times\mathbf{r}\right) + \frac{D\mathbf{\Omega}}{Dt}\times\mathbf{r}$$

Coriolis acceleration — small term (correction in g) — $= 0$

$x:\quad -2\Omega\sin\varphi\, v + 2\Omega\cos\varphi\, w$

$y:\quad +2\Omega\sin\varphi\, u$

$z:\quad -2\Omega\cos\varphi\, u$

f*=2Ωsin*φ

$$g_{effective} = -g + \mathbf{\Omega}\times\left(\mathbf{\Omega}\times\mathbf{r}\right)$$

- No Coriolis force on stationary bodies.
- Dominant horizontal deflection (right/left).
- Coriolis force does no work on a body because it is perpendicular to the velocity.

The β term

$$\beta = \frac{\partial f}{\partial y} = \frac{\partial f}{R_{earth}\partial\varphi} = $$

$$= \frac{\partial\left(2\Omega\sin\varphi\right)}{R_{earth}\partial\varphi} = \frac{2\Omega\cos\varphi}{R_{earth}}$$

Or expanding Ω around φ_0 or y_0

with $\varphi - \varphi_0 << 1,\ y - y_0 << R_{earth}$

$$\Omega = \Omega_e\left[\sin\varphi_0 + \cos\varphi_0\left(\varphi-\varphi_0\right)+...\right] = \frac{1}{2}\left[f_0 + \beta_0\left(y-y_0\right)+...\right]$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\frac{1}{\rho_0}\nabla p - g\frac{\rho}{\rho_0}\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u} \qquad \nabla\cdot\mathbf{u} = 0$$

$$\frac{\partial S}{\partial t} + \mathbf{u}\cdot\nabla S = \kappa_S\nabla^2 S + S^S \qquad \frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \kappa_T\nabla^2 T + \frac{Q}{c_{\mathrm{p}}\rho_0}$$

$$\rho = \rho_0\left[1-\alpha\left(T-T_0\right)+\beta\left(S-S_0\right)\right]$$

WORKING EQUATIONS

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\Omega\sin\varphi\upsilon + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2\Omega\sin\varphi u + \nu\frac{\partial^2 \upsilon}{\partial x^2} + \nu\frac{\partial^2 \upsilon}{\partial y^2} + \nu\frac{\partial^2 \upsilon}{\partial z^2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g + \nu\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial z^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + \upsilon\frac{\partial S}{\partial y} + w\frac{\partial S}{\partial z} = \kappa_S\frac{\partial^2 S}{\partial x^2} + \kappa_S\frac{\partial^2 S}{\partial y^2} + \kappa_S\frac{\partial^2 S}{\partial z^2} + S^S$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + \upsilon\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \kappa_T\frac{\partial^2 T}{\partial x^2} + \kappa_T\frac{\partial^2 T}{\partial y^2} + \kappa_T\frac{\partial^2 T}{\partial z^2} + \frac{Q}{c_{\mathrm{p}}\rho_0}$$

$$\rho = \rho_0\left[1-\alpha\left(T-T_0\right)+\beta\left(S-S_0\right)\right]$$

The "mean" state

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + \upsilon\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} = -\dfrac{1}{\rho_0}\dfrac{\partial P}{\partial x} + f\upsilon + \nu\dfrac{\partial^2 u}{\partial x^2} + \nu\dfrac{\partial^2 u}{\partial y^2} + \nu\dfrac{\partial^2 u}{\partial z^2} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial \upsilon}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \end{cases}$$

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial u\upsilon}{\partial y} + \frac{\partial uw}{\partial z} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + f\upsilon + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$\boxed{u = \bar{u} + u'; \quad \upsilon = \bar{\upsilon} + \upsilon'; \quad w = \bar{w} + w'; \quad P = \bar{P} + P'}$$

$$\overline{\frac{\partial \bar{u} + u'}{\partial t} + \frac{\partial(\bar{u}+u')(\bar{u}+u')}{\partial x} + \frac{\partial(\bar{u}+u')(\bar{\upsilon}+\upsilon')}{\partial y} + \frac{\partial(\bar{u}+u')(\bar{w}+w')}{\partial z}} =$$

$$\overline{-\frac{1}{\rho_0}\frac{\partial \bar{P} + P'}{\partial x} + f\left(\bar{\upsilon}+\upsilon'\right) + \nu\frac{\partial^2(\bar{u}+u')}{\partial x^2} + \nu\frac{\partial^2(\bar{u}+u')}{\partial y^2} + \nu\frac{\partial^2(\bar{u}+u')}{\partial z^2}}$$

$$\boxed{\begin{aligned} &\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{\upsilon}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} = \\ &-\frac{1}{\rho_0}\frac{\partial \bar{P}}{\partial x} + f\bar{\upsilon} + \nu\frac{\partial^2 \bar{u}}{\partial x^2} + \nu\frac{\partial^2 \bar{u}}{\partial y^2} + \nu\frac{\partial^2 \bar{u}}{\partial z^2} - \frac{\partial}{\partial x}\overline{(u'u')} - \frac{\partial}{\partial y}\overline{(u'v')} - \frac{\partial}{\partial z}\overline{(u'w')} \end{aligned}}$$

Similarly:

$$\frac{\partial \overline{v}}{\partial t} + \overline{u}\frac{\partial \overline{v}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y} + \overline{w}\frac{\partial \overline{v}}{\partial z} =$$

$$-\frac{1}{\rho_0}\frac{\partial \overline{P}}{\partial y} - f\overline{u} + \nu\frac{\partial^2 \overline{v}}{\partial x^2} + \nu\frac{\partial^2 \overline{v}}{\partial y^2} + \nu\frac{\partial^2 \overline{v}}{\partial z^2} - \frac{\partial}{\partial x}\overline{(u'v')} - \frac{\partial}{\partial y}\overline{(v'v')} - \frac{\partial}{\partial z}\overline{(v'w')}$$

$$\frac{\partial \overline{w}}{\partial t} + \overline{u}\frac{\partial \overline{w}}{\partial x} + \overline{v}\frac{\partial \overline{w}}{\partial y} + \overline{w}\frac{\partial \overline{w}}{\partial z} =$$

$$-\frac{1}{\rho_0}\frac{\partial \overline{P}}{\partial z} - g + \nu\frac{\partial^2 \overline{w}}{\partial x^2} + \nu\frac{\partial^2 \overline{w}}{\partial y^2} + \nu\frac{\partial^2 \overline{w}}{\partial z^2} - \frac{\partial}{\partial x}\overline{(u'w')} - \frac{\partial}{\partial y}\overline{(v'w')} - \frac{\partial}{\partial z}\overline{(w'w')}$$

The continuity equation:

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0$$

Removing the "mean" state: $\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$

First order closure scheme:

$$\nu\frac{\partial^2\overline{u}}{\partial x^2}-\frac{\partial}{\partial x}\overline{(u'u')}+\nu\frac{\partial^2\overline{u}}{\partial y^2}-\frac{\partial}{\partial y}\overline{(u'\upsilon')}+\nu\frac{\partial^2\overline{u}}{\partial z^2}-\frac{\partial}{\partial z}\overline{(u'w')}$$

$$A_H\frac{\partial^2\overline{u}}{\partial x^2} \qquad A_H\frac{\partial^2\overline{u}}{\partial y^2} \qquad A_V\frac{\partial^2\overline{u}}{\partial z^2}$$

- **Molecular diffusion and viscocity**
  κ_T = 0.0014 cm^2/sec (temperature)
  κ_S = 0.000013 cm^2/sec (salinity)
  ν = 0.018 cm^2/sec at 0°C (0.010 at 20°C)

Turbulence has the same coefficients for temeprature, salinity and momentum but is anisotropic (vertical/horizontal)

- **Eddy diffusivity and viscosity**
  A_H = (horizontal) = 1 to 10^4 m^2/sec
  A_V = (vertical) = 10^{-5} to 10^{-4} m^2/sec

SCALING

The basic equations include a very large number of distinct types of physical processes, ranging from molecular to global scales. Depending on the processes under investigation we can define appropriate scales that describe these processes.

$$u, \upsilon \rightarrow U \qquad w \rightarrow W$$

$$x, y \rightarrow L \qquad z \rightarrow H$$

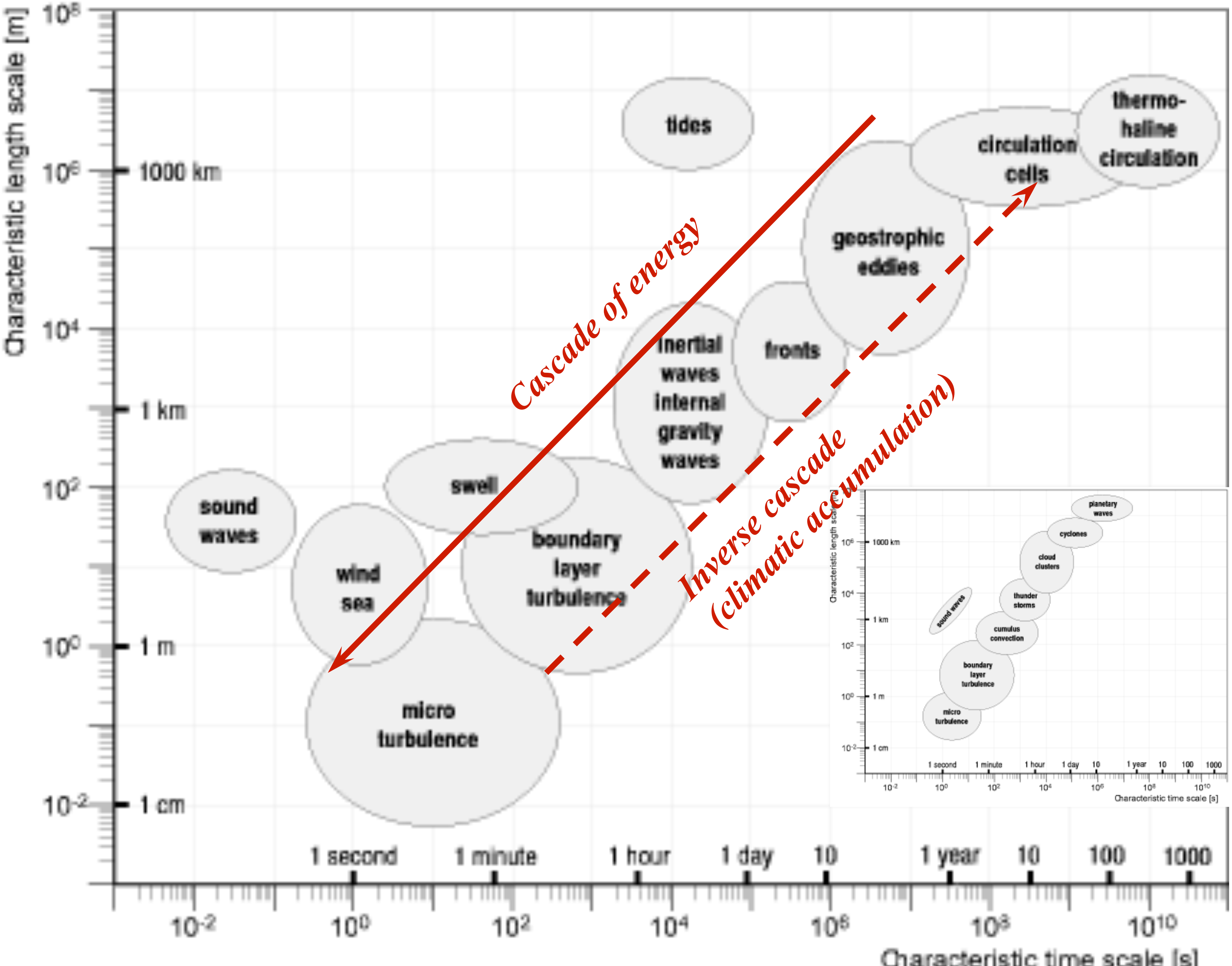
$$t \rightarrow T \approx \frac{L}{U} \approx \frac{H}{W}$$

<u>**Goal:**</u> Estimation of the relative importance of each term in the process under investigation and the possibility to simplify the basic equation by defining a dominant balance of the dynamics/thermodynamics involved.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + f\upsilon + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2} + \nu\frac{\partial^2 u}{\partial z^2}$$

SCALING

$$\frac{U^2}{L} \quad \frac{U^2}{L} \quad \frac{U^2}{L} \quad \frac{UW}{H} \quad \frac{\Delta P}{\rho L} \quad fU \quad \nu\frac{U}{L^2} \quad \nu\frac{U}{L^2} \quad \nu\frac{U}{H^2}$$


Characteristic length scale [m]
10^8
10^6
1000 km
10^4
1 km
10^2
10^0
1 m
10^-2
1 cm
tides
thermo-
haline
circulation
circulation
cells
geostrophic
eddies
inertial
waves
internal
gravity
waves
fronts
swell
sound
waves
wind
sea
boundary
layer
turbulence
micro
turbulence
Cascade of energy
Inverse cascade
(climatic accumulation)
1 second
1 minute
1 hour
1 day
10
1 year
10
100
1000
10^-2
10^0
10^2
10^4
10^6
10^8
10^10
Characteristic time scale [s]
planetary
waves
cyclones
cloud
clusters
thunder
storms
sound waves
cumulus
convection
boundary
layer
turbulence
micro
turbulence

Scaling numbers:

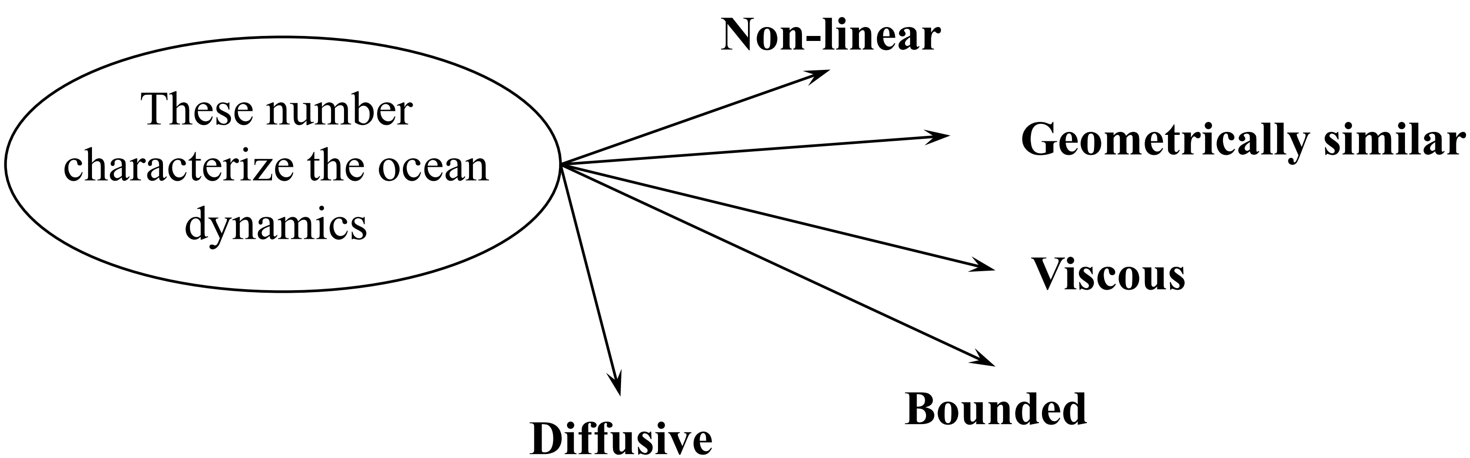
Aspect ratio: $\dfrac{H}{L}$

$$R_o = \frac{\text{Non-linear Term}}{\text{Coriolis Term}} = \frac{U^2}{L}\frac{1}{fU} = \frac{U}{fL}$$

$$E_k = \frac{\text{Viscosity Terms}}{\text{Coriolis Term}} = \nu\frac{U}{L^2}\frac{1}{fU} = \frac{\nu}{fL^2} \quad \left(\frac{\nu}{fH^2}\right)$$

$$R_E = \frac{\text{Non-linear Term}}{\text{Viscosity Term}} = \frac{U^2}{L}\Big/ A_H\frac{U}{L^2} = \frac{UL}{A_H}$$

$$F_R = \frac{U^2}{N^2H^2} \qquad B_u = \frac{NH}{fL}$$



Geostrophic – hydrostatic approximation (the large scale limit):

$$R_o = \frac{U}{fL} << 1 \quad E_k = \frac{\nu}{fL^2} << 1 \quad \frac{H}{L} << 1$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + f\upsilon + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - fu + \nu\frac{\partial^2 \upsilon}{\partial x^2} + \nu\frac{\partial^2 \upsilon}{\partial y^2} + \nu\frac{\partial^2 \upsilon}{\partial z^2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g + \nu\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial z^2}$$

$$f\upsilon = \frac{1}{\rho}\frac{\partial P}{\partial x}(1); \; fu = -\frac{1}{\rho}\frac{\partial P}{\partial y}(2); \frac{1}{\rho}\frac{\partial P}{\partial z} = -g\;(3)$$

The geostrophic – hydrostatic limit

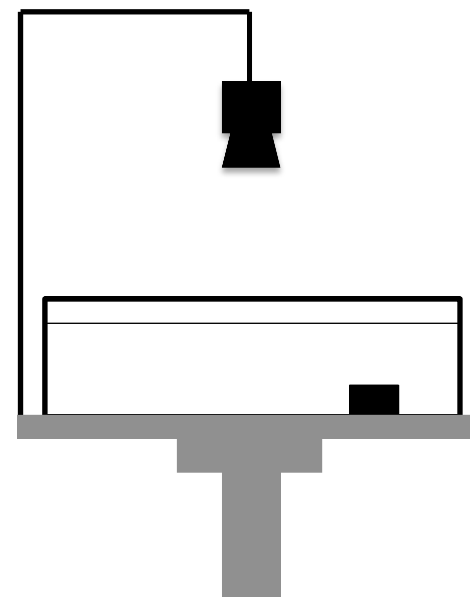
The "Taylor Columns"

$$\frac{\partial}{\partial y}(1)+\frac{\partial}{\partial x}(2) \qquad \frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}=\frac{1}{\rho_0 f}\left[\frac{\partial^2 P}{\partial x\partial y}-\frac{\partial^2 P}{\partial x\partial y}\right] = 0$$

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}=0}$$

Non-divergent horizontal flow (2D)

Boussinesq continuity $\Rightarrow \frac{\partial w}{\partial z}=0$ or $w=0$



$$f\upsilon = \frac{1}{\rho}\frac{\partial P}{\partial x} \quad (1)$$

$$fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} \quad (2)$$

$$\frac{1}{\rho}\frac{\partial P}{\partial z} + g = 0 \quad (3)$$

$$\frac{\partial}{\partial z}(1) \Rightarrow \frac{\partial(\rho f \upsilon)}{\partial z} = \frac{\partial}{\partial z}\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\frac{\partial P}{\partial z} = \frac{\partial}{\partial x}(-g\rho)$$

$$\frac{\partial}{\partial z}(2 \Rightarrow)\frac{\partial(\rho f u)}{\partial z} = -\frac{\partial}{\partial z}\frac{\partial P}{\partial y} = -\frac{\partial}{\partial y}\frac{\partial P}{\partial z} = -\frac{\partial}{\partial x}(-g\rho)$$

Using (3)

$$\Rightarrow f\frac{\partial(\rho\upsilon)}{\partial z} = -g\frac{\partial\rho}{\partial x}; f\frac{\partial(\rho u)}{\partial z} = g\frac{\partial\rho}{\partial y}$$

since $\rho\frac{\Delta U}{\Delta z} >> U\frac{\Delta\rho}{\Delta z}$ we can write

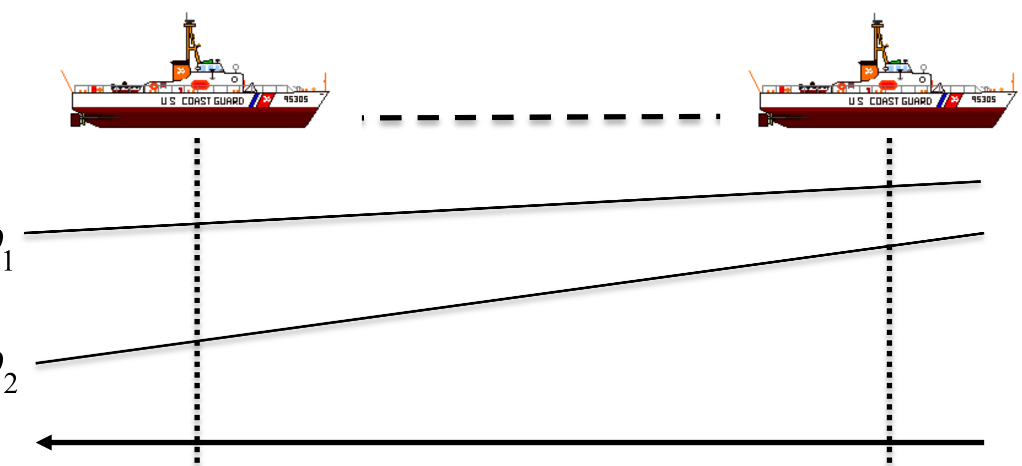
$$\boxed{\frac{\partial v}{\partial z} = -\frac{g}{\rho_0 f}\frac{\partial\rho}{\partial x}; \quad \frac{\partial u}{\partial z} = \frac{g}{\rho_0 f}\frac{\partial\rho}{\partial y}}$$

Thermal wind equations

$$\frac{\Delta\upsilon}{\Delta z} = -\frac{g}{\rho_0 f}\frac{\Delta\rho}{\Delta x}$$

$$\Rightarrow \upsilon_1 = \upsilon_2 - \frac{g}{\rho_0 f}\frac{\Delta\rho}{\Delta x}\Delta z$$

Geostrophy computes the velocity shear and not the absolute velocity



ρ_1

ρ_2

Level of no (known) motion