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TEXTS AND READINGS IN MATHEMATICS 51

Jacobi's Lectures on Dynamics

Second Edition

Edited by A. Clebsch

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Jacobi's Lectures On Dynamics

Texts and Readings in Mathematics

Jacobi's
Lectures on Dynamics
Second Revised Edition

Delivered at the
University of Königsberg in the winter semester 1842-1843
and according to the notes prepared by C. W. Brockardt

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Contents

Foreword

The present supplement to C.G.J. Jacobi's collected works contains the second revised edition of the "Lectures on Dynamics" edited by A. Clebsch in 1866 without the five treatises from Jacobi's literary estate added to them at that time. According to the plans drawn up for the publication of Jacobi's collected works the latter along with the major treatise "Nova methodus aequtionen differentiales partiellen primi ordinus inter numerum variabilium quemcunque propositas integrandi"' (New methods for the integration of first order partial differential equations of any number of variables), also edited by Clebsch, and a few other shorter works will form the contents of the fifth volume.

As has been remarked in the preface to the first edition of the "Lectures", they are based on the notes prepared with great care nad accuracy by C. W. Borckhardt who attended the lectures given by Jacobi at the University of Königsberg in the winter semester of 1842–43. The changes made by Clebsch in the edition of Brochardt's text are minor. Also Mr. E. Lottner, the publisher of the new edition, has only made slight stylistic changes in certain places where the expressions were not precise or sufficiently clear, and for the rest has confined himself to correcting a few printing and computational errors remaining in the first edition.

15 March 1884 Weierstrass

(Translated by Balgangadharan. Revised by B.Banerjee)

Note (B. Banerjee)

We have translated some of Jacobi's expressions as they were in his time to retain the flavour of the original. They are:

1. *vis viva*(lebendige Kraft) stands for twice the kinetic energy.

2. Force function (Kraftfunction) stands for potential or potential energy.

3. The principle of conservation of surface area (Das Princip der Erhaltung der Flächenräume) stands for the principle of conservation of angular momentum.

Lecture 1

Introduction

These lectures will be concerned with the advantages which, for integrating the differential equations of motion, one can derive from the special form of the equations. In 'Mécanique analytique' one finds everything related to the problems of setting up and transforming the differential equations, but very little on their integration. This problem is seldom posed; the only one that can be considered to be in that direction is the Method of Variation of Constants — a method of approximation which depends on the special form of the differential equations that occur in mechanics.

Among the large class of problems found in mechanics, we shall consider only those which relate to a system of n mass points, i.e., of n bodies whose spatial extension can be neglected and whose masses are assumed to be situated at their centres of gravity. We shall further consider only those problems in which the motion (of the system) depends only on the configuration of the points and not on their velocities. Thus all problems in which the resistance (of the medium) is to be taken into account, are excluded.

We shall first set up the differential equations for the motion of such a system and then list the principles which hold for the same. These principles are:

1. The principle of conservation of motion of the centre of gravity.
2. The principle of conservation of *vis viva*.[1]
3. The principle of conservation of surface area (angular momentum).

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[1] **Translator's notes:**
(a) Jacobi uses this term for twice the "kinetic energy"i.e. for the quantity mv^2 (instead of the presently accepted definition $\frac{1}{2}mv^2$ which appears to have been adopted in the post Jacobi era. We use *vis viva* in the translation)

4. The principle of least action, or as it should better be called, the principle of least expenditure of force.

The first three principles give integrals of the system of differential equations that have been set up. The last principle gives no integral, but only a symbolic formula into which the system of differential equations can be combined. However, it is not less important because of that. *Lagrange* had indeed originally derived all his results in mechanics from it. Later, when he wanted to derive them rigorously, he gave up the principle of least action and took (first in the Paris Academy prize-essay on the libration of the moon, and then, especially, in 'Mécanique analytique') the Principle of Virtual Velocities as the basis for his derivations. Thus the principle of least action, which was known as the mother of all new results, was treated as insignificant.

I have introduced a new principle[2] in mechanics, which agrees with the principles of conservation of *vis viva* and surface area in that it gives an integral, but for the rest is of an entirely different character. First, it is more general than the above principles; it holds as long as the differential equations depend only on the coordinates. Further, the above principles give a first integral in the form of a function of the coordinates and their derivatives equal to a constant. That is, integrals, from which differential equations are derived which on using the given differential equations, become identically zero.The new principle leads, on the basis of the earlier integrals, to the latter. According to this principle, one can, under the supposition that a problem of mechanics leads to a first order differential equation of two variables, in general obtain the multiplier (integrating factor) of the same.

In the cases where the other principles reduce the problem to a first order differential equation, it can be completely solved using the new principle. To these belong, the problem of attraction of a point by a fixed center[3], the law of attraction being arbitrary; that of attraction of

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(b) Jacobi uses the phrase "conservation of kinetic energy" for what is presently known as "conservation of total energy". The word 'conservation' in the context used by Jacobi should be understood in the sense of 'change in kinetic energy'= (−) change in potential energy'. Incidently, Jacobi never uses the term "potential energy". The quantity U (called *force function* by Jacobi) is minus the quantity which is presently called potential energy

[2]The principle of the last multiplier. Lecture 10 (Lecture 10-18 are devoted to a thorough discussion of this)

[3]Lecture 16

a point by two fixed centres[4], it being assumed that the attraction obeys *Newton*'s law and the rotation of a body about a point with no external forces acting on it. For the problem of attraction by two fixed centres besides the application of the older principles it is necessary to use an integral found by *Euler* by a special trick which reduces the problem to a first order differential equation of two variables. But this equation is very complicated and its integration is one of the great masterpieces of Euler. The new principle yields the integrating factor automatically.

The class of problems for which both the principles of conservation of *vis viva* and that of least action hold are to be specially noted. *Hamilton* has indeed remarked that in this case one can reduce the problem to a first order non-linear partial differential equation. If one finds a complete solution of the same, then one obtains all the integral equations[5]. *Hamilton* calls the function defined by the partial differential equation the characteristic function.

Hamilton has made this nice connection that he had discovered rather inaccessible and obscure, in that he makes the characteristic function depend at the same time on a second partial differential equation. The addition of this consideration makes the discovery unnecessarily complicated, since a more detailed investigation[6] shows that the second partial differential equation is completely superfluous.

We shall, for distinguishing, introduce the following terminology. We shall call the integrals of ordinary differential equations 'integrals' or 'integral equations' and the integrals of partial differential equation 'solutions'. Further, for a system of differential equations we shall distinguish between 'integrals' and 'integral equations'. 'Integrals'[7] are the first integrals which have the following form: a function of the coordinates and their derivatives equal to a constant and whose differential coefficient becomes identically zero on using the given system of differential equations, without the help of any other integrals. The rest of the 'integrals' are called integral equations. In this sense, the principles of conservation of *vis viva* and surface area give integrals and not integral equations.

Through *Hamilton*'s discovery the system of integral equations of mechanical problems has taken a very remarkable form. Namely, if

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[4]Lecture 29

[5]**Translator's note:** The term "integral equation" as used by Jacobi is defined in footnote 9. In English "integrated equations" would have been more appropriate.

[6]See last part of Lecture 19

[7]Integrals are a non-parameter family of invariant submanifolds of the vector fields, of codimension

one differentiates the characteristic function with respect to the arbitrary constants which it contains, it then gives the integral equations of the given system of differential equations. This is analogous to *Lagrange*'s theorem which states that the differential equations of a problem for which the principle of least action is valid, can be represented by the partial differential coefficients of a single function. Although *Hamilton* has given the form of the integral equations in question, which he obtains by means of the characteristic function, he has done nothing about actually finding them. We shall be concerned with this here, and by means of the results so obtained, handle the problems of attraction by a fixed centre, and by two fixed centres, and the motion of a point not subject to gravity on the tri-axial ellipsoid[8] (Its solution coincides with the finding of the shortest line on the ellipsoid).

The connection discovered by *Hamilton* also leads to new conclusions on the method of variation of constants. This method rests on the following: the integrals of a system of differential equations of dynamics contain a certain number of arbitrary constants, whose values are determined in each special case by the initial positions and initial velocities of the moving points. Now if the points collide during their motion, then only the values of the constants change, the form of the integral equations remain the same. For example, if a planet moves in an ellipse around the sun, and during the motion undergoes a collision, it will then move in a new ellipse, or perhaps in a hyperbola, in any case in a conic section, the form of the equation remains the same. If such collisions occur not momentarily but continuously, one can then look on the constants themselves as changing continuously, and indeed whether these changes precisely represent the action of the perturbing force. This theory of variation of constants will appear in a new light in the course of our investigations[9].

The principle of conservation of *vis viva* embraces a large class of problems to which, notably, the problem of three bodies belongs, or more generally, the problem of motion of n bodies with mutual attraction[10].

The more one enquires into the nature of forces, the more one reduces everything to mutual attractions and repulsions, and therefore the problem of determining the motion of n bodies with mutual attraction

[8]Lecture 28
[9]Lecture 36
[10]Lecture 2

becomes more important. This problem belongs to the category of those where our theory is applicable, that is, which reduces to the integration of a partial differential equation. Hence one recognises the necessity of studying partial differential equations but for 30 years[11] one has been concerned with only linear partial differential equations; while nothing has happened in the non-linear case. *Lagrange* had already solved the problem for three variables[12]. For more variables Pfaff has carried out a creditable but incomplete investigation. According to Pfaff, for the solution of partial differential equations one must first integrate a system of ordinary differential equations. After integrating these one has to pose a new system of differential equations which contains two variables less; these have to be further integrated, and so on, and then finally one arrives at the integration of the partial differential equation. *Hamilton* subsequently has, through his reduction of the differential equations of motion to a partial differential equation, transformed the problem to a more difficult one. Because, according to *Pfaff*, the integration of partial differential equations requires the integration of a series of systems of ordinary differential equations, while the problem of mechanics requires only the integration of one system of ordinary differential equations. Therefore the inverse reduction would be here of greater importance, whereby a partial differential equation is reduced to a system of ordinary differential equations. The first system of Pfaff agrees with that to which mechanics leads, and it can be shown that the rest of the system can be dispensed with. Thus as in this case the reduction of a problem to another frequently inverts itself–the progress of science transforms the first into the second and the other way round. In such transformations it is important that the connection between the two problems is demonstrated. The connection in question allows us to recognise that every progress in the theory of partial differential equations must lead to progress in mechanics.

A deeper study of the differential equations of mechanics shows that the number of integrations can always be reduced by half, while the other half can be dealt with by quadratures. There is a remarkable theorem which states that there is a qualitative difference between the integrals. Namely, while certain integrals have no more significance than quadratures, there are others which taken together hold for all the remaining.

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[11]The lectures were given in the winter of 1842–1843

[12]Lecture 22

This theorem can be stated as follows[13]: *if one knows, besides the integral given by the principle of conservation of 'vis viva', two more integrals of the dynamical equations then one can find a third from these two.* An example of this is the so-called area-theorems with regard to the three coordinate planes: if two of these hold,[14] then the third one can be derived from them.

If in accordance with the general theorem introduced, one finds from two integrals a third one, then one can find a fourth from this and one of the earlier ones and so on until one comes back to one of the given integrals. There are integrals which, with these operations, exhaust the whole system of integral equations, while for others the cycle breaks off earlier. This fundamental theorem has been found and lost for the last 30 years. It originates from *Poisson*, and was also known to *Lagrange*, who used it as a lemma in the second part of 'Mécanique analytique' published after his death.[15] But this theorem has always been taken to have a different significance; it was only meant to show that in an expansion certain terms are independent of time and it was no small difficulty to find in this fact its present significance. In this theorem lies the basis for the integration of first order partial differential equations.

[13]See end of Lecture 34

[14]See end of Lecture 34

[15]Mécanique Analytique, Section VII, 60, 61: Vol. II, pp. 70 of the third edition

Lecture 2

The Differential Equations of Motion

To begin with we shall consider a free[1] system of mass points. We call it a system because we assume that the points are subject to external forces not independently of one another, in which case one could consider each point by itself, but as they act mutually on one another one cannot consider any one without considering the others.Further, the system is a free one, i.e., one in which the points follow the action of the forces unhindered. Let any point of the system have a mass m, and its rectangular coordinates at time t be x, y, z and the components of the force acting on it X, Y, Z; then one has the well-known equation of motion:

$$m\frac{d^2x}{dt^2} = X, \quad m\frac{d^2y}{dt^2} = Y, \quad m\frac{d^2z}{dt^2} = Z;$$

Similar equations hold for all points of the system. X, Y, Z depend on the coordinates of all n points and can also contain their derivatives with respect to time t, which is always the case when the resistance is to be taken into account.

The above differential equations of motion can be brought into an extremely convenient symbolic form, if one multiplies each of them, after having making the right hand side zero, by arbitrary factors and adds the products. One then obtains the equation

$$\left(m\frac{d^2x}{dt^2} - X\right)\lambda + \left(m\frac{d^2y}{dt^2} - Y\right)\mu + \left(m\frac{d^2z}{dt^2} - Z\right)\nu + \cdots = 0,$$

where the $\cdots$ refer to similar terms which arise from the remaining points of the system. If one demands that this equation hold for all

[1]**Translator's note:** The term 'free' used by Jacobi should be understood in the sense of 'unconstrained'.

values of the quantities λ, μ, $\nu, \ldots$, then it represents the entire system of differential equations above. For the sake of clarity we shall denote the factors λ, μ, $\nu, \ldots$, with δx, δy, $\delta z, \ldots$, where the x, y, $z, \ldots$, are regarded purely as indices. Our symbolic equation thereby becomes

$$\sum \left\{ \left(m\frac{d^2x}{dt^2} - X \right)\delta x + \left(m\frac{d^2y}{dt^2} - Y \right)\delta y + \left(m\frac{d^2z}{dt^2} - Z \right)\delta z \right\} = 0,$$

where the summation refers to all points of the system. This equation must hold for all values of δx, δy, $\delta z, \ldots,$. The symbolic representation is in itself very important; it will frequently be the case that a symbol is considered as a quantity and computations and operations performed with it as is usual with quantities. We shall have examples of this later.

A special treatment is possible in the case in which only attractions by a fixed centre or the attractions of points among themselves are to be considered. In these cases the components X, Y, $Z, \ldots$, can be represented as partial derivatives of one and the same quantity. *Lagrange* has made the important remark that if one connects a fixed point with a moving one, the cosines of the angles which this line makes with the three coordinate axes are the partial derivatives of a quantity, the distance between the two points. Let the fixed point have coordinates a, b, c, the moving point x, y, z, and let the radius vector joining the points be r; one draws through the fixed point (a, b, c) three straight lines parallel to the coordinate axes towards their positive ends. Let the angles which the radius vector makes with these lines be α, β, γ. Then one has the following equations:

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2;$$
$$\frac{\partial r}{\partial x} = \frac{x-a}{r} = \cos\alpha, \qquad \frac{\partial r}{\partial y} = \frac{x-b}{r} = \cos\beta, \qquad \frac{\partial r}{\partial z} = \frac{x-c}{r} = \cos\gamma.$$

If now R is the force with which the point (x, y, z) is attracted by the point (a, b, c), then the components which act on the point (x, y, z) in the positive direction of the coordinates are:

$$-R\frac{\partial r}{\partial x}, \quad -R\frac{\partial r}{\partial y}, \quad -R\frac{\partial r}{\partial z},$$

of if we set $\int R\, dr = P$,

$$-\frac{\partial P}{\partial x}, -\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial z}.$$

These components are thus the partial differential coefficients of a quantity $-P$. This holds also for the mutual attraction of two points p and p_1. Let their coordinates be x, y, z and x_1, y_1, z_1, and their distance r; then

$$r^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2.$$

Let R be the force of attraction between p and p_1; then the components acting on p are $-R\frac{\partial r}{\partial x}$, $-R\frac{\partial r}{\partial y}$, $-R\frac{\partial r}{\partial z}$ and the components acting on p_1 are $-R\frac{\partial r}{\partial x_1}$, $-R\frac{\partial r}{\partial y_1}$, $-R\frac{\partial r}{\partial z_1}$, which are respectively equal and opposite since

$$\frac{\partial r}{\partial x} = \frac{x - x_1}{r}, \quad \frac{\partial r}{\partial x_1} = -\frac{x - x_1}{r},$$

so $\frac{\partial r}{\partial x} = -\frac{\partial r}{\partial x_1}$ and so also $\frac{\partial r}{\partial y_1} = -\frac{\partial r}{\partial y}$, $\frac{\partial r}{\partial z_1} = -\frac{\partial r}{\partial z}$. If one again introduces

$$P = \int R\, dr,$$

then the components acting on p are $-\frac{\partial P}{\partial x}, -\frac{\partial P}{\partial y}, -\frac{\partial P}{\partial z}$ and the components acting on p_1 are $-\frac{\partial P}{\partial x_1}, -\frac{\partial P}{\partial y_1}, -\frac{\partial P}{\partial z_1}$.

Let us now consider n mass points which attract one another. Let their masses be $m_1, m_2, \ldots, m_n$, their coordinates x_1, y_1, z_1; $x_2, y_2, z_2, \ldots, x_n, y_n, z_n$; let the distance between m_1 and m_2 be denoted by r_{12} and the integral of that function of r_{12} which expresses the attraction between the two points be denoted by P_{12}, where one has to consider the product of the masses m_1 and m_2 as a factor entering into it. (For *Newton*'s law, for example, $P_{12} = -\frac{m_1 m_2}{r_{12}}$.) These being assumed, the component of the force which acts on the point m_1 in the direction of the x-coordinates is $-\frac{\partial(P_{12}+P_{13}+\cdots+P_{1n})}{\partial x_1}$, and similarly for the two other components. So one has, for the point m_1,

$$m_1 \frac{d^2 x_1}{dt^2} = -\frac{\partial(P_{12} + \cdots + P_{1n})}{\partial x_1},$$
$$m_1 \frac{d^2 y_1}{dt^2} = -\frac{\partial(P_{12} + \cdots + P_{1n})}{\partial y_1},$$
$$m_1 \frac{d^2 z_1}{dt^2} = -\frac{\partial(P_{12} + \cdots + P_{1n})}{\partial z_1}.$$

Similar equations hold for the remaining points of the system; for the point m_2, for example, the quantity enclosed in brackets, whose differential coefficient is taken, equals $P_{21} + P_{23} + \cdots + P_{2n}$. These quantities P have however the property that each one of them depends only on the coordinates of the two points whose indices are attached to it. Hence, on differentiation with respect to x_1, y_1, or z_1, the differential coefficients of $P_{23}, P_{24}, \ldots, P_{2n}, P_{34}, \ldots, P_{n-1,n}$ vanish and only the differential coefficients of $P_{12}, P_{13}, \ldots, P_{1n}$ remain. Thus, the differential equation relating to the first point remains entirely unchanged if on the right side one introduces for the sum $P_{12} + P_{13} + \cdots + P_{1n}$ in brackets, all the remaining $P's$. A similar change can be made also in the quantity enclosed in brackets in the other differential equations, and then one has in the differential equations of the entire system the differential coefficients of one and the same quantity:

$$U = -(P_{12} + P_{13} + \cdots + P_{1n} + P_{23} + \cdots + P_{2n} + \cdots + P_{n-1,n}).$$

In this manner we have, for any point of the system, the equations

$$m_i \frac{d^2 x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \quad m_i \frac{d^2 y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \quad m_i \frac{d^2 z_i}{dt^2} = \frac{\partial U}{\partial z_i}.$$

This observation, that one can introduce the same quantity U in all the equations appears very simple; however overlooking this fact had prevented *Euler* from arriving at the generality of *Lagrange*'s results.*Euler* knew the principle of conservation of *vis viva* only for the attraction by fixed centres. At the end of *Nova methodus inveniendi curvas maximi minimive proprietate gaudentes*, *Euler* has in *Appendix de motu projectorum* contented himself with very incomplete expressions for the differential equations for mutual attraction. *Daniell Bernoulli* was the first to observe this in his paper communicated to the Philosophical section of the Berlin Academy[2] and thereby gave the principle of conservation of *vis viva* its true significance. *Lagrange* then used the observation for the problem which *Euler* had posed in the essay'*de motu projectorum* and thereby arrived at his principal result.

The expression U was retained by *Hamilton* under the name *force function.* The partial differential coefficient of this expression with respect to a coordinate of one of the n masses under consideration gives the force with which this mass is attracted by the other masses in the direction of that coordinate.

For the *Newton*'s law of attraction, the force function will be

$$U = \sum \frac{m_i m_{i_1}}{r_{i i_1}},$$

[2]Men. de l'acad. de Berlin, 1748

and for the case of three bodies,

$$U = \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}.$$

In the theory of transformation of the differential equations of motion into a partial differential equation of the first order, one has always to deal only with the force function; hence its introduction is of the greatest importance. For the time being we shall use it equally well for the concise representation of the equations.

It is of interest to clarify as to how much the limits of the mechanical problems can be extended, without giving up the introduction of the force function.

It is not necessary to assume that the law of attraction between any two mass points is the same for all pair of points. On the contrary one can make any arbitrary assumption about the force, provided the attraction depends only on the distance and any of the masses m_i is attracted by another mass m_{i_1} with the same force as m_{i_1} by m_i. This remark about the extension is not without use. For example,Bessel has raised the question whether in the universe the same law of attraction holds between any two bodies, not that the function of the distance between the two bodies changes but that a body in the solar system, e.g. the sun itself, attracts Saturn with a different mass from the one with which it attracts Uranus. This hypothesis will not disturb the introduction of the 'force function'. Besides the mutual attraction between the masses, attraction by fixed centres can also enter the problem. One can even assume a mathematical fiction, that each one of the fixed centres does not act on all the masses, but only on one or on a certain number of them. If, for example, the mass m_i is attracted by a fixed centre of mass k with coordinates a, b, c, then if *Newton*'s law holds the term

$$\frac{k m_1}{\sqrt{(x_1 - a)^2 + (y_1 - b)^2 + (z_1 - c)^2}}$$

appears in the force function, and one obtains similar terms for the other masses of the system if the fixed centre k acts on them. Finally, constant parallel forces, which again need not act on all the masses can appear. If, for example, a constant force (like gravity) acts on the mass m_1, whose components along the directions of the coordinate axes are A, B, Γ, then there appears in the 'force function' U the term $Ax_1 + By_1 + \Gamma z_1$, and similar terms for the other masses of the system, if the constant forces A, B, Γ or others act on them. It is also to be remarked in the case of fixed centres of attraction that if they act on all the masses occurring in the problem, obviously as it always happens in nature, one can look upon these as moving masses. Hereby several redundant terms occur in the force function, namely those which express the mutual attraction of the fixed centres; however these terms are pure constants and vanish by every differentiation.

The symbolic form into which the differential equations of motion have been brought was:

$$\sum\left\{\left(m_i\frac{d^2x_i}{dt^2}-X_i\right)\delta x_i+\left(m_i\frac{d^2y_i}{dt^2}-Y_i\right)\delta y_i\right.$$
$$\left.+\left(m_i\frac{d^2z_i}{dt^2}-Z_i\right)\delta z_i\right\}=0,$$

which we can write better as

$$\sum m_i\left(\frac{d^2x_i}{dt^2}\delta x_i+\frac{d^2y_i}{dt^2}\delta y_i+\frac{d^2z_i}{dt^2}\delta z_i\right)=\sum\left(X_i\delta x_i+Y_i\delta y_i+Z_i\delta z_i\right) \quad (2.1)$$

In the case where the force function can be introduced,

$$X_i=\frac{\partial U}{\partial x_i},\quad Y_i=\frac{\partial U}{\partial y_i},\quad Z_i=\frac{\partial U}{\partial z_i},$$

and therefore

$$\sum m_i\left(\frac{d^2x_i}{dt^2}\delta x_i+\frac{d^2y_i}{dt^2}\delta y_i+\frac{d^2z_i}{dt^2}\delta z_i\right)$$
$$=\sum\left(\frac{\partial U}{\partial x_i}\delta x_i+\frac{\partial U}{\partial y_i}\delta y_i+\frac{\partial U}{\partial z_i}\delta z_i\right).$$

In this equation here, as in the above, the $\delta x_i,\ldots$ are to be looked upon as arbitrary factors, which can take every value, and the x_i's as their indices. However, if one considers for a moment δx_i, δy_i, δz_i as infinitesimal increments of x_i, y_i, z_i, then by the rules of the differential calculus, the right side of the last equation would be

$$\sum\left(\frac{\partial U}{\partial x_i}\delta x_i+\frac{\partial U}{\partial y_i}\delta y_i+\frac{\partial U}{\partial z_i}\delta z_i\right)=\delta U, \quad (2.2)$$

and thus one has

$$\sum m_i\left\{\frac{d^2x_i}{dt^2}\delta x_i+\frac{d^2y_i}{dt^2}\delta y_i+\frac{d^2z_i}{dt^2}\delta z_i\right\}=\delta U \quad (2.3)$$

Here δU is provisionally to be seen only as an abbreviation for the sum (2.2) and agrees with the same only when one looks upon δ as an infinitesimally small increment. Although this notation has a meaning only when the force function exists, one has so far applied it profitably in many cases to the equation (2.1), in order to make the computations convenient. However, this can happen only under the proviso that one substitutes the partial differential coefficients $\frac{\partial U}{\partial x_i}$ in the expansion of δU by X_i. Hereby as a rule one arrives at the right results, when one has to do with only linear substitutions. This is the bold step that *Lagrange* has taken in his Turin memoir, indeed, without justifying it.

The notation δU is very advantageous if one introduces the $3n$ new variables $q_1, \ldots, q_{3n}$ for the coordinates x_1, y_1, z_1; x_2, y_2, $z_2, \ldots, x_n; y_n, z_n$. One needs only to introduce these new variables in U and expand according to the rules of the differential calculus:

$$\delta U = \frac{\partial U}{\partial q_1}\delta q_1 + \frac{\partial U}{\partial q_2}\delta q_2 + \cdots + \frac{\partial U}{\partial q_{3n}}\delta q_{3n},$$

However, simultaneously one must set

$$\frac{\partial x_i}{\partial q_1}\delta q_1 + \frac{\partial x_i}{\partial q_2}\delta q_2 + \cdots + \frac{\partial x_i}{\partial q_{3n}}\delta q_{3n} = \sum_s \frac{\partial x_i}{\partial q_s}\delta q_s$$

for δx_i. The correctness of this assertion can be seen in the following way.

The $3n$ differential equations of motion are

$$m_i\frac{d^2x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \quad m_i\frac{d^2y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \quad m_i\frac{d^2z_i}{dt^2} = \frac{\partial U}{\partial z_i},$$

where i takes all values from 1 to n, 1 and n included. If one multiplies these $3n$ equations by $\frac{\partial x_i}{\partial q_k}$, $\frac{\partial y_i}{\partial q_k}$, $\frac{\partial z_i}{\partial q_k}$, respectively and adds, then one has

$$\sum_i m_i \left\{ \frac{d^2x_i}{dt^2}\frac{\partial x_i}{\partial q_k} + \frac{d^2y_i}{dt^2}\frac{\partial y_i}{\partial q_k} + \frac{d^2z_i}{dt^2}\frac{\partial z_i}{\partial q_k} \right\} = \frac{\partial U}{\partial q_k}.$$

One obtains $3n$ such equations in which one inserts for q_k all the q one after another. These $3n$ equations represent completely the original system of equations, so that one can always substitute the one for the other. If we multiply the last system by arbitrary constants $\delta q_1, \delta q_2, \ldots, \delta q_s, \ldots, \delta q_{3n}$, and add, then we obtain a new symbolic equation, which replaces the last system of differential equations, and therefore the earlier one, completely.

This symbolic equation, then, is

$$\sum_s \sum_i m_i \left\{ \frac{d^2 x_i}{dt^2} \frac{\partial x_i}{\partial q_s} + \frac{d^2 y_i}{dt^2} \frac{\partial y_i}{\partial q_s} + \frac{d^2 z_i}{dt^2} \frac{\partial z_i}{\partial q_s} \right\} \delta q_s = \sum_s \frac{\partial U}{\partial q_s} \delta q_s,$$

or, if one carries out the summation on the left side in the reverse order,

$$\sum_i m_i \left\{ \frac{d^2 x_i}{dt^2} \sum_s \frac{\partial x_i}{\partial q_s} \delta q_s + \frac{d^2 y_i}{dt^2} \sum_s \frac{\partial y_i}{\partial q_s} \delta q_s \right.$$
$$\left. + \frac{d^2 z_i}{dt^2} \sum_s \frac{\partial z_i}{\partial q_s} \delta q_s \right\} = \sum_s \frac{\partial U}{\partial q_s} \delta q_s.$$

This equation is the same into which (2.3) goes over if one substitutes for δU, $\sum_s \frac{\partial U}{\partial q_s} \delta q_s$ and $\sum_s \frac{\partial x_i}{\partial q_s} \delta q_s$, $\sum_s \frac{\partial y_i}{\partial q_s} \delta q_s$,$\sum_s \frac{\partial z_i}{\partial q_s} \delta q_s$ respectively for δx_i, δy_i, δz_i. With this, the rule given above for the substitution of new variables is proved. In the transformed equations again the δq_s are to be considered further as independent quantities and the transformed symbolic equation decomposes also into the given second system of $3n$ equations.

But the importance of our symbolic equations (2.1) and (2.3) does not lie in these computational advantages. The true significance of this representation consists much more in that it can be preserved when the system is no longer free, but equations of constraint which express the connections between the points, enter. However, then, the variations are no more to be treated as entirely arbitrary and independent of one another, but as *virtual* variations, i.e., such as are consistent with constraints. If we take, for example, that there are three equations of constraint:

$$f = 0, \quad \phi = 0, \quad \psi = 0,$$

then the relations which should exist between the variations in order to make them virtual, are determined through the following equations:

$$\delta f = 0, \quad \delta \phi = 0, \quad \delta \psi = 0;$$

or, in expanded form:

$$\sum \left(\frac{\partial f}{\partial x_i} \delta x_i + \frac{\partial f}{\partial y_i} \delta y_i + \frac{\partial f}{\partial z_i} \delta z_i \right) = 0$$
$$\sum \left(\frac{\partial \phi}{\partial x_i} \delta x_i + \frac{\partial \phi}{\partial y_i} \delta y_i + \frac{\partial \phi}{\partial z_i} \delta z_i \right) = 0$$
$$\sum \left(\frac{\partial \psi}{\partial x_i} \delta x_i + \frac{\partial \psi}{\partial y_i} \delta y_i + \frac{\partial \psi}{\partial z_i} \delta z_i \right) = 0.$$

Every equation of constraint then gives a linear relation between the $3n$ variations $\ldots, \delta x_i, \delta y_i, \delta z_i, \ldots$. If one has m equations of constraint, and therefore m relations between the variations, then one can express all the variations through $3n - m$ of them and obtain, through substitution of these, our symbolic equation free of m variations. But the elimination of m variations will be extremely complicated. Lagrange has found an expedient for dealing with this difficulty by introducing a system of multipliers.

The extension contained in the above of our symbolic equation to a system limited by constraints is, as is self-evident, not proved, but is only an assertion, historically speaking. It seems necessary to say this explicitly, because although*Laplace* has, in "Mécanique céleste", proved this extension as little as it has been done here and has made only a historical claim, one has always taken this for a proof. *Poinsot* has written a paper[3] against this opinion and says there quite rightly, that mathematicians delude themselves often traversing a long route, but sometimes also on a very short one. On the long route they deceive themselves when after very elaborate calculations they arrive at an identity and call it a theorem. Our case is a counter-example.

It is in not all our intention to prove this extension. We would like to look upon it more as a principle which need not be proved. This is the point of view of many mathematicians, notably that of Gauss.[4]

[3] *Liouvilles* Journal Vol. 3, p 244

[4] Possibly Gauss had orally asserted this to Jacobi; no written statement on this appears to be found, at least according to the information kindly supplied by Professor Schering.

Lecture 3

The Principle of Conservation of Motion of the Centre of Gravity

We shall now proceed to the proofs of the general principles which hold for the mechanical problems considered so far. The first of these is (cf. Lecture 1) the principle of conservation of motion of the centre of gravity.

Let us first consider the simpler case in which a force function exists, so that we have

$$\sum m_i \left\{ \frac{d^2x_i}{dt^2}\delta x_i + \frac{d^2y_i}{dt^2}\delta y_i + \frac{d^2z_i}{dt^2}\delta z_i \right\} = \delta U.$$

We shall assume that U as well as the equations of constraint depend only on the differences of the coordinates, so that these remain the same when one increases all the x by one and the same quantity, and also this happens for all y and all z. Then the assumption

$$\begin{aligned} \delta x_1 &= \delta x_2 = \cdots = \delta x_n = \lambda, \\ \delta y_1 &= \delta y_2 = \cdots = \delta y_n = \mu, \\ \delta z_1 &= \delta z_2 = \cdots = \delta z_n = \nu, \end{aligned}$$

is one which is compatible with the equations of constraints. With these assumptions one obtains

$$\sum m_i \left\{ \frac{d^2x_i}{dt^2}\lambda + \frac{d^2y_i}{dt^2}\mu + \frac{d^2z_i}{dt^2}\nu \right\} = \sum \frac{\partial U}{\partial x_i}\lambda + \frac{\partial U}{\partial y_i}\mu + \frac{\partial U}{\partial z_i}\nu. \quad (3.1)$$

The right side, however, is equal to 0. In fact, since according to our assumption U depends only on the differences of the coordinates, one can, if one sets

$$x_1 - x_n = \xi_1, \quad x_2 - x_n = \xi_2, \ldots, x_{n-1} - x_n = \xi_{n-1},$$

give the quantity U, in so far as it depends on the x-coordinates, the form

$$U = F(\xi_1, \ldots, \xi_{n-1}).$$

Then simultaneously,

$$\frac{\partial U}{\partial x_1} = \frac{\partial F}{\partial \xi_1}, \quad \frac{\partial U}{\partial x_2} = \frac{\partial F}{\partial \xi_2}, \ldots, \frac{\partial U}{\partial x_{n-1}} = \frac{\partial F}{\partial \xi_{n-1}},$$
$$\frac{\partial U}{\partial x_n} = -\frac{\partial F}{\partial \xi_2} - \frac{\partial F}{\partial \xi_2} - \cdots - \frac{\partial F}{\partial \xi_{n-1}},$$

so that

$$\frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2} + \cdots + \frac{\partial U}{\partial x_n} = \sum \frac{\partial U}{\partial x_i} = 0,$$

and similarly

$$\sum \frac{\partial U}{\partial y_i} = 0, \quad \sum \frac{\partial U}{\partial z_i} = 0.$$

Accordingly our equation above reduces itself to

$$\sum m_i \left\{ \frac{d^2 x_i}{dt^2}\lambda + \frac{d^2 y_i}{dt^2}\mu + \frac{d^2 z_i}{dt^2}\nu \right\} = 0,$$

and since this equation must hold for all values of λ, μ and ν, we have,

$$\sum m_i \frac{d^2 x_i}{dt^2} = 0, \quad \sum m_i \frac{d^2 y_i}{dt^2} = 0, \quad \sum m_i \frac{d^2 z_i}{dt^2} = 0.$$

Let us now set

$$\sum m_i = M, \sum m_i x_i = MA, \sum m_i y_i = MB, \sum m_i z_i = MC,$$

so that A, B, C, as is well known, are the coordinates of the centre of gravity of the system; so one can write in place of the above equations the following:

$$\frac{d^2 A}{dt^2} = 0, \quad \frac{d^2 B}{dt^2} = 0, \quad \frac{d^2 C}{dt^2} = 0, \tag{3.2}$$

which on integration gives

$$A = \alpha^{(0)} + \alpha' t, B = \beta^{(0)} + \beta' t, C = \gamma^{(0)} + \gamma' t, \tag{3.3}$$

i.e., the centre of gravity moves in a straight line, whose equations in the running coordinates A, B, C are

$$\frac{A-\alpha^{(0)}}{\alpha'}=\frac{B-\beta^{(0)}}{\beta'}=\frac{C-\gamma^{(0)}}{\gamma'},$$

and it moves with a constant velocity $\sqrt{\alpha'^2+\beta'^2+\gamma'^2}$.

In the general case in which the force function does not exist, one has in place of equation (3.1) the following:

$$\sum m_i\left\{\frac{d^2x_i}{dt^2}\lambda+\frac{d^2y_i}{dt^2}\mu+\frac{d^2z_i}{dt^2}\nu\right\}=\sum X_i\lambda+\sum Y_i\mu+\sum Z_i\nu,$$

and since this same holds for all values of λ, μ and ν,

$$\sum m_i\frac{d^2x_i}{dt^2}=\sum X_i,\sum m_i\frac{d^2y_i}{dt^2}=\sum Y_i,\sum m_i\frac{d^2z_i}{dt^2}=\sum Z_i,$$

or, if one introduces the coordinates of the centre of gravity,

$$M\frac{d^2A}{dt^2}=\sum X_i,M\frac{d^2B}{dt^2}=\sum Y_i,M\frac{d^2C}{dt^2}=\sum Z_i, \tag{3.4}$$

i.e., the centre of gravity moves as though all the forces acting on the system can be brought to the centre of gravity by parallel translation of themselves, and as though the sum of all the masses are located at the centre of gravity.

If the forces parallely translated in this manner are in equilibrium in their new positions, then

$$\sum X_i=0,\sum Y_i=0,\sum Z_i=0,$$

so no accelerating force at all acts at the centre of gravity. This occurs when only mutual attractions act on the system, since then the action and reaction have the same point of application and cancel themselves out (this case has already been handled above, since, in this case a force function always exists); however, it ceases to hold as soon as fixed centres appear in the problem.

All that has been said up to now naturally holds only when the equations of constraint depend only on the differences of the x-coordinates, y-coordinates and z-coordinates. One such case is the Seilpolygon,[1] if one does not take the extension of the wires into account. In order that

[1] A polygon made up of wires with flexible joints.

in this case also the force function depends only on the differences of the coordinates, the end-points of the wires should not be considered as fixed, otherwise these points enter the problem as fixed centres. For an entirely free system the equations (3.4) naturally hold under all circumstances. If there exists a force function that depends not merely on the differences of the coordinates, which is the case when fixed centres or constant forces exist, then in this case equations (3.4)hold, but not the equations (3.2).

In the expression "Principle of conservation of motion of the centre of gravity", the word conservation derives from the fact that the motion of the centre of gravity is expressed by the same equations as if there were no equations of constraint. If, for example, in the Seilpolygon, the connection between the points is fully flexible, then the equations of motion of the centre of gravity are not altered, as they are independent of the equations of constraint. The modification is only that the sums $\sum X_i$, $\sum Y_i$, $\sum Z_i$, take other values, as soon as the coordinates of the individual points become different functions of time. If, moreover these sums are constants, which is the case for example when only gravity acts on the system, the motion of the centre of gravity is not changed at all by the equations of constraint.

Lecture 4

The Principle of Conservation of 'vis viva'

A hypothesis on the variations that under all circumstances is consistent with the equations of constraint is that one sets for each value of i

$$\delta x_i = \frac{dx_i}{dt}dt, \quad \delta y_i = \frac{dy_i}{dt}dt, \quad \delta z_i = \frac{dz_i}{dt}dt.$$

If we insert these values for the variations in the symbolic equation (2.2) of Lecture 2, which holds for the case of existence of a force function, then δU changes into dU and we obtain after division by dt,

$$\sum m_i \left\{ \frac{d^2x_i}{dt^2}\frac{dx_i}{dt} + \frac{d^2y_i}{dt^2}\frac{dy_i}{dt} + \frac{d^2z_i}{dt^2}\frac{dz_i}{dt} \right\} = \frac{dU}{dt}.$$

This equation admits of direct integration; its integral is

$$\frac{1}{2}\sum m_i \left\{ \left(\frac{dx_i}{dt}\right)^2 + \left(\frac{dy_i}{dt}\right)^2 + \left(\frac{dz_i}{dt}\right)^2 \right\} = U + h \qquad (4.1)$$

where h is the arbitrary constant of integration. If we denote by ds_i the path element covered in the time dt by the element of mass m_i, and its velocity by v_i, then we have

$$\left(\frac{dx_i}{dt}\right)^2 + \left(\frac{dy_i}{dt}\right)^2 + \left(\frac{dz_i}{dt}\right)^2 = \left(\frac{ds_i}{dt}\right)^2 = v_i^2,$$

and the equaiton above takes the form

$$\frac{1}{2}\sum m_i v_i^2 = U + h.$$

This is the theorem of *vis viva*. The *vis viva* of a point is the square of its velocity multiplied by its mass; the *vis viva* of a system is equal to the sum of the *vis viva* of the individual material points. Accordingly equation (4.1) can be thus expressed in words: *half the vis viva of a system is equal to the force-function plus a constant.*

The principle of conservation of *vis viva* is, as its derivation has shown, is independent of the equations of constraint, and herein lies the major part of its importance. It holds as long as the force function exists. In cases where the force-function can be introduced this principle can be extended. Hence according to our earlier remark it was *Daniel Bernoulli* who first elevated this principle to its present general significance, while before him one knew it only for attraction by fixed centres.

One can eliminate the arbitrary constant h by subtracting the two equations (4.1) for two different times, one from the other. Then one arrives at the theorem: *if a system moves from one position to another, then the difference of the vis viva of the system at the beginning and the end is equal to the difference between the values of the force function for the same instants.* The change in the 'vis viva' is thus dependent only on the initial and final values of the force function: the intermediate values have no influence. To make this clearer, we assume that a point moves on an arbitrary curve from a given initial point to a given end-point; if now the initial velocity is given, then the final velocity is one and the same, whatever be the shape of the curve lying in between. The velocity here must be measured naturally according to the actual motion followed, in the direction of the tangent to the curve; that part of the velocity is not to be taken into account here which is annihilated by the resistance when the push originally given to the point does not act in the direction of the tangent to the curve. This independence from the form of the path followed holds also for a system. As a corollary to this, one has the following theorem: *if the motion of a system is of such a sort that it can return to the same position, the 'vis viva' is the same after returning*, where it is assumed that the principle of *vis viva* holds generally. The word conservation in the name of the principle derives from this independence of the form of the path followed, or what is the same, of the equations of constraint (since the form of the path followed is determined by these).

The principle of conservation of *vis viva* has its origin in the theory of machines, whose basis, since Carnot, is the same principle. It has been laid down in this discipline that half the 'vis viva', that is $\frac{1}{2}m_i v_i^2$, is equal to the work done by the machine, or as one expresses in these practical matters, $\frac{1}{2}m_i v_i^2$ is that which is paid to the machine. This happens in the following way. In the theory of machines one assumes as a matter of principle that, if friction is disregarded, work is done only when a mass moves in the direction of the force acting on it (and indeed in the sense opposed to its action), while for a motion in a perpendicular direction no work is done. One assumes further that the work done by a machine is measured by the product of the acting force and the length of the path travelled by the mass set in motion by the force. Pushing a weight horizontally is not regarded as doing work, but only lifting it is and the work of lifting is measured by the product of the weight lifted and the height to which it has been lifted. For example, this is the work, which a crane does.

In a system of mass points, each one of them is the point of application of the force acting on it. In so far as this point of application is displaced through a motion of the system, the force acting on it must also be displaced. But the displacement of the point of application is not in general in the direction of the force which is acting on it, but at a certain angle with it. Therefore, to obtain the work of the system one must multiply the force not by the path described, but with the length of the projection of the path described in the direction of the the force. The forces $m_i \frac{d^2x_i}{dt^2}$, $m_i \frac{d^2y_i}{dt^2}$, $m_i \frac{d^2z_i}{dt^2}$, act at the point m_i, and indeed they act parallel to the coordinate axes. The displacement of m_i in the element of time dt is ds_i, the projections of the the same on the coordinate axes are respectively dx_i, dy_i, dz_i, therefore the work required for the forward motion of the point m_i in the time element dt is

$$m_i \left\{ \frac{d^2x_i}{dt^2} dx_i + \frac{d^2y_i}{dt^2} dy_i + \frac{d^2z_i}{dt^2} dz_i \right\},$$

and for the motion of the entire system in the element of time dt the work done is

$$m_i \left\{ \frac{d^2x_i}{dt^2} dx_i + \frac{d^2y_i}{dt^2} dy_i + \frac{d^2z_i}{dt^2} dz_i \right\} = \frac{1}{2} d \left(\sum m_i v_i^2 \right),$$

when one obtains for the work in the time elapsed from t_0 to t,

$$\frac{1}{2} \left\{ m_i v_{i(t=t_1)}^2 - \sum m_i v_{i(t=t_0)}^2 \right\}.$$

This half-difference of the initial and final values of the sum $\sum m_i v_i^2$ is thus the measure of the work of the system. This is the probable basis for the name (whose origin has been much disputed) *vis viva* given by *Leibnitz* for this sum.

In the case where the force function is a homogeneous function, and where one has to do with a free system, one can give a very interesting form to the theorem of *vis viva* which is contained in equation (4.1). Let U be a homogeneous function of dimension k; then it is well known that

$$\sum \left(x_i \frac{\partial U}{\partial x_i} + y_i \frac{\partial U}{\partial y_i} + z_i \frac{\partial U}{\partial z_i} \right) = kU.$$

If one has to do with a free system one can set

$$\delta x_i = x_i \omega, \quad \delta y_i = y_i \omega, \quad \delta z_i = z_i \omega,$$

where ω denotes an infinitely small constant, and one obtains on consideration of the equation for the homogeneity of U,

$$\delta U = kU\omega.$$

Hence our symbolic equation (equation (2.2) of Lecture 2)

$$\sum m_i \left(x_i \frac{d^2 x_i}{dt^2} + y_i \frac{d^2 y_i}{dt^2} + z_i \frac{d^2 z_i}{dt^2} \right) = kU,$$

where the common factor ω has been omitted. If we add to this the equation (4.1) multiplied by 2, we get

$$\sum \left\{ x_i \frac{d^2 x_i}{dt^2} + \left(\frac{dx_i}{dt} \right)^2 + y_i \frac{d^2 y_i}{dt^2} + \left(\frac{dy_i}{dt} \right)^2 + z_i \frac{d^2 z_i}{dt^2} + \left(\frac{dz_i}{dt} \right)^2 \right\}$$
$$= (k+2)U + 2h,$$

or

$$\sum m_i \frac{d}{dt} \left(x_i \frac{dx_i}{dt} + y_i \frac{dy_i}{dt} + z_i \frac{dz_i}{dt} \right) = (k+2)U + 2h,$$

or, also

$$\frac{1}{2} \sum m_i \frac{d^2}{dt^2} (x_i^2 + y_i^2 + z_i^2) = (k+2)U + 2h,$$

or, if we set $x_i^2 + y_i^2 + z_i^2 = r_i^2$ and multiply by 2,

$$\frac{d^2 (\sum m_i r_i^2)}{dt^2} = (2k+4)U + 4h. \tag{4.2}$$

The expression $\sum m_i r_i^2$ can be transformed in a remarkable way, namely so that the distances of all points from the origin of coordinates do not occur, but only the distances between the points and the distance of the centre of gravity from the origin. Transformations of this sort are the favourite formulas of *Lagrange.* The one in question one obtains in the following way.

As is easily seen,

$$\left(\sum m_i\right)\left(\sum m_i x_i^2\right) - \left(\sum m_i x_i\right)^2 = \sum m_i m_{i'}(x_i^2 + x_{i'}^2 - 2x_i x_{i'})$$

where the sum on the right hand side is extended only over different values of i, i', each combination reckoned once. Similar equations hold for y and z; if one adds all three, one has

$$\left(\sum m_i\right)\left(\sum m_i(x_i^2 + y_i^2 + z_i^2)\right) - \left(\sum m_i x_i\right)^2 - \left(\sum m_i y_i\right)^2 - \left(\sum m_i z_i\right)^2 = \sum m_i m_{i'}\{(x_i - x_{i'})^2 + (y_i - y_{i'})^2 + (z_i - z_{i'})^2\}.$$

Now one introduces, as before, the coordinates of the centre of gravity and sets

$$\sum m_i = M, \quad \sum m_i x_i = MA, \quad \sum m_i y_i = MB, \sum m_i z_i = MC;$$

and further denotes the distance of the points m_i, $m_{i'}$ from each other by $r_{ii'}$; then

$$M\sum m_i r_i^2 - M^2(A^2 + B^2 + C^2) = \sum m_i m_{i'} r_{ii'}^2. \tag{4.3}$$

Here one has to substitute in accordance with what we had earlier

$$A = \alpha^{(0)} + \alpha' t, \quad B = \beta^{(0)} + \beta' t, \quad C = \gamma^{(0)} + \gamma' t.$$

If one introduces these substitutions and differentiates twice with respect to time, then

$$\frac{d^2(\sum m_i r_i^2)}{dt^2} = 2M(\alpha'^2 + \beta'^2 + r'^2) + \frac{d^2(\sum m_i m_{i'} r_{i,i'}^2)}{Mdt^2},$$

and when one inserts this in the equation (4.2), we get

$$\frac{d^2(\sum m_i m_{i'} r_{ii'}^2)}{dt^2} = (2k+4)U + 4h - 2M(\alpha'^2 + \beta'^2 + \gamma'^2),$$

or finally, when one sets $4h - 2M(\alpha'^2 + \beta'^2 + \gamma'^2) = 4h'$, we have

$$\frac{d^2(\sum m_i m_{i'} r_{i,i'}^2)}{dt^2} = (2k+4)U + 4h'. \tag{4.4}$$

In the equation (4.3) the quantities r_i are the radius vectors of the material points of the system drawn from the origin of the coordinate system, $\sqrt{A^2 + B^2 + C^2}$ is the radius vector of the centre of gravity also measured from the origin. Therefore these quantities change as soon as the origin is changed. The quantities $r_{ii'}$ on the other hand are independent of the choice of the origin, since they are the relative distances of any two points of the system. Let us now choose the centre of gravity to be the origin of coordinates, so that $A^2 + B^2 + C^2 = 0$; if at the same time we denote the radius vectors measured by ρ_i from the centre of gravity reckoned outwards, then equation (4.3) changes to

$$M \sum m_i \rho_i^2 = \sum m_i m_{i'} r_{ii'}^2. \tag{4.5}$$

If one eliminates $\sum m_i m_{i'} r_{ii'}^2$ from this equation and equation (4.3), then it gives

$$\sum m_i r_i^2 = \sum m_i \rho_i^2 + M(A^2 + B^2 + C^2); \tag{4.6}$$

i.e., the sum $\sum m_i r_i^2$ taken for any one of the points (when it is considered as the origin) is equal to the same sum for the centre of gravity increased by the sum of the squares of the distances of these points from the centre of gravity multiplied by the mass of the point. Hence one sees that $\sum m_i r_i^2$ is a minimum for the centre of gravity and that this quantity increases proportionally to the square of the distance from the centre of gravity. $\sum m_i r_i^2$ will therefore take a constant value for all points which lie on the surface of a sphere with the centre of gravity as centre. An analogous result holds for the plane, where the geometric locus of the points for which $\sum m_i r_i^2$ remains constant is a circle.

The formula (4.6) can also be independently proved. In fact if we displace our earlier entirely arbitrary system of coordinates parallel to itself, so that the new origin of coordinates falls at the centre of gravity, and denote the coordinates in the new coordinate system of our n material points with ξ_1, η_1, ζ_1; ξ_2, η_2, ζ_2; ..., ξ_n, η_n, ζ_n, then we have, for any i,

$$x_i = \xi_i + A, \quad y_i = \eta_i + B, \quad z_i = \zeta_i + C,$$

where A, B, C are defined as coordinates of the centre of gravity through the equations

$$\sum m_i = M, \sum m_i x_i = MA, \sum m_i y_i = MB, \sum m_i z_i = MC.$$

Therefore

$$\sum m_i r_i^2 = \sum m_i x_i^2 + \sum m_i y_i^2 + \sum m_i z_i^2$$
$$= \sum m_i \xi_i^2 + 2A \sum m_i \xi_i + A^2 \sum m_i + \sum m_i \eta_i^2 +$$
$$(2B \sum m_i \eta_i) + B^2 \sum m_i + \sum m_i \zeta_i^2 + 2C \sum m_i \zeta_i$$
$$+ C^2 \sum m_i$$

Now, however,

$$MA = \sum m_i x_i = \sum m_i \xi_i + \sum m_i A = \sum m_i \xi_i + MA.$$

Therefore

$$\sum m_i \xi_i = 0,$$

and even so

$$\sum m_i \eta_i = 0, \quad \sum m_i \zeta_i = 0.$$

From this we obtain

$$\sum m_i r_i^2 = \sum m_i (\xi_i^2 + \eta_i^2 + \zeta_i^2) + M(A^2 + B^2 + C^2),$$

in agreement with formula (4.6).

A similar formula holds for the differentials. From our present formulae, we find the differentials

$$dx_i = d\xi_i + dA, dy_i = d\eta_i + dB, dz_i = d\zeta_i + dC,$$
$$\sum m_i d\xi_i = 0, \quad \sum m_i d\eta_i = 0, \quad \sum m_i d\zeta_i = 0,$$

and from them one obtains

$$\begin{aligned} \sum m_i (dx_i^2 + dy_i^2 + dz_i^2) &= \sum m_i (d\xi_i^2 + d\eta_i^2 + d\zeta_i^2) \\ &\quad + M(dA^2 + dB^2 + dC^2), \end{aligned}$$

or, if we divide by dt^2,

$$\sum m_i \left\{ \left(\frac{dx_i}{dt}\right)^2 + \left(\frac{dy_i}{dt}\right)^2 + \left(\frac{dz_i}{dt}\right)^2 \right\} =$$
$$\sum m_i \left\{ \left(\frac{d\xi_i}{dt}\right)^2 + \left(\frac{d\eta_i}{dt}\right)^2 + \left(\frac{d\zeta_i}{dt}\right)^2 \right\} +$$
$$M \left\{ \left(\frac{dA}{dt}\right)^2 + \left(\frac{dB}{dt}\right)^2 + \left(\frac{dC}{dt}\right)^2 \right\} \quad (4.7)$$

i.e., the absolute *vis viva* of the system is equal to the relative *vis viva* of the same with respect to the centre of gravity (or, as one expresses it, about the centre of gravity), increased by the absolute *vis viva* of the centre of gravity. Therefore, the absolute *vis viva* of the system is always greater than its relative *vis viva* about the centre of gravity.

One can introduce the relative *vis viva* about the centre of gravity into the theorem of conservation of *vis viva*. This theorem was contained in the equation

$$\frac{1}{2}\sum m_i \left\{\left(\frac{dx_i}{dt}\right)^2 + \left(\frac{dy_i}{dt}\right)^2 + \left(\frac{dz_i}{dt}\right)^2\right\} = U + h.$$

If one transforms the left side of this equation by means of equation (4.7), then one obtains

$$\frac{1}{2}\sum m_i \left\{\left(\frac{d\xi_i}{dt}\right)^2 + \left(\frac{d\eta_i}{dt}\right)^2 + \left(\frac{d\zeta_i}{dt}\right)^2\right\} = U + h - \frac{1}{2}M\left\{\left(\frac{dA}{dt}\right)^2 + \left(\frac{dB}{dt}\right)^2 + \left(\frac{dC}{dt}\right)^2\right\}.$$

However,

$$h - \frac{1}{2}M\left\{\left(\frac{dA}{dt}\right)^2 + \left(\frac{dB}{dt}\right)^2 + \left(\frac{dC}{dt}\right)^2\right\} = h - \frac{1}{2}M(\alpha'^2 + \beta'^2 + \gamma'^2),$$

which is the same as what we denoted earlier by h'. And then, we have

$$\frac{1}{2}\sum m_i \left\{\left(\frac{d\xi_i}{dt}\right)^2 + \left(\frac{d\eta_i}{dt}\right)^2 + \left(\frac{d\zeta_i}{dt}\right)^2\right\} = U + h'. \tag{4.8}$$

Thus the theorem of *vis viva* holds for the relative *vis viva* about the centre of gravity just as for the absolute, only the constant changes from h to h'. One should however not forget that it has been assumed here that the principle of conservation of motion of the centre of gravity holds. Because of this assumption we can substitute of $\alpha'^2 + \beta'^2 + \gamma'^2$ for

$$\left(\frac{dA}{dt}\right)^2 + \left(\frac{dB}{dt}\right)^2 + \left(\frac{dC}{dt}\right)^2.$$

Moreover one could have anticipated the result (4.8). In fact, in case the principle of conservation of motion of the centre of gravity holds, U and the equations of constraint depend only on the differences of the coordinates. So these expressions remain unchanged if one sets ξ_i, η_i, ζ_i in place of x_i, y_i, z_i, where

$$x_i = \xi_i + A, \quad y_i = \eta_i + B, \quad z_i = \zeta_i + C;$$

one has further

$$\frac{d^2A}{dt^2} = 0, \quad \frac{d^2B}{dt^2} = 0, \quad \frac{d^2C}{dt^2} = 0;$$

therefore

$$\frac{d^2x_i}{dt^2} = \frac{d^2\xi_i}{dt^2}, \quad \frac{d^2y_i}{dt^2} = \frac{d^2\eta_i}{dt^2}, \quad \frac{d^2z_i}{dt^2} = \frac{d^2\zeta_i}{dt^2}.$$

The symbolic equation

$$\sum m_i\Big(\frac{d^2x_i}{dt^2}\delta x_i + \frac{d^2y_i}{dt^2}\delta y_i + \frac{d^2z_i}{dt^2}\delta z_i\Big) = \delta U$$

and the equations of constraint of the problem also hold if one substitutes ξ_i, η_i, ζ_i for x_i, y_i, z_i, i.e., these equations hold for the relative motion about the centre of gravity just as for the absolute. The same must therefore be the case with the derived result- the theorem of *vis viva*, where the constant of integration can change, which actually happens.

From the above discussion one sees that, if the principle of conservation of motion of the centre of gravity holds, one has to determine only the relative motion of the system about the centre of gravity. Then one finds the motion of the centre of gravity and one obtains the absolute motion of the system through by a simple addition of the two motions.

The solar system provides an example of this category of problems. But we know only its relative motion. We do not have any data to determine the motion of the centre of gravity, since for this there must actually exist fixed stars, which is very doubtful, and these must be so near to us that they have, in respect of the 40 million mile long line (the major axis of the earth's orbit), a measurable parallax. *Argelander* has in recent times sought to determine the ratio $\alpha' : \beta' : \gamma'$ (see equation (3.3) of Lecture 3) i.e., the direction of motion of the centre of gravity, following an idea suggested by the elder *Herschel.* However this determination is based only on probabilistic grounds.

We now return to equation (4.4) which, in the case where U is a homogeneous function of order k, contains the principle of conservation of *vis viva* in the interesting form

$$\frac{d^2(\sum m_i m_{i'} r_{ii'}^2)}{M dt^2} = (2k+4)U + 4h'.$$

Keeping in mind equation (4.5) one can write for this

$$\frac{d^2(\sum m_i\rho_i^2)}{dt^2} = (2k+4)U + 4h',$$

where ρ_i is the radius vectors drawn from the centre of gravity. For the solar system, $k = -1$, so one has

$$\frac{d^2(\sum m_i\rho_i^2)}{dt^2} = 2U + 4h',$$

where

$$U = \sum \frac{m_i m_{i'}}{r_{ii'}}.$$

Several remarks can be made on this equation. If the attraction were inversely proportional, not to the square of the distance, but to the cube of the same, one could integrate the above equation. Since in this case k would be $= -2$, $2k + 4 = 0$, then if one abbreviates $\sum m_i\rho_i^2$ by R,

$$\frac{d^2R}{dt^2} = 4h'.$$

But then the solar system would break up, because a double integration gives

$$R = 2h't^2 + h''t + h''',$$

so R tends to infinity with increasing time. Since $R = \sum m_i\rho_i^2$, then at least one body of the solar system must move to an infinite distance from the centre of gravity.

Similar considerations show that for the actual case of the solar system i.e., for attraction inversely proportional to the square of the distance, the constant h' must be negative if the solar system were to be stable. In fact so far as only attractive forces act in the solar system, the force-function U by its nature is a positive quantity. Now *Bessel* has indeed made the hypothesis that the sun exercises a repulsive force against the comets and has related this to the phenomenon that all comet tails are turned away from the sun. As this is not yet certain, we will disregard this repulsive force for general considerations. Accordingly U is definitely a positive quantity. Assuming this, we obtain through integration of the equation

$$\frac{d^2R}{dt^2} = 2U + 4h'$$

between the limits 0 and t

$$\frac{dR}{dt} - R_0' = \int_0^t (2U + 4h')dt,$$

or, if α denotes the smallest value of U between the limits 0 and t,

$$\frac{dR}{dt} - R_0' > (2\alpha + 4h')t,$$

where R_0' is the value of $\frac{dR}{dt}$ for $t = 0$. A second integration of this equation between the limits 0 and t gives, if R_0 is the value of R for $t = 0$,

$$R - R_0 - R_0't > (\alpha + 2h')t^2,$$

or

$$R > R_0 + R_0't + (\alpha + 2h')t^2.$$

Here α is a positive definite quantity, since U by its nature is positive. Now if $2h'$ were positive, so also would $\alpha + 2h'$; then with increasing t, R would increase to infinity, i.e., the solar system would not be stable; so $2h'$ must be negative. But its numerical value cannot be greater than the largest value that U takes between 0 and t; if it were otherwise, all the elements of the integrals $2\int_0^t (U + 2h')dt$ would be negative, and one could therefore set

$$\frac{dR}{dt} - R_0' < -2\beta t,$$

where β is a positive quantity, namely the smallest numerical value that $U + 2h'$ takes between 0 and t; integration gives

$$R < R_0 + R_0't - \beta t^2,$$

i.e., with increasing t, R approaches minus infinity, which is absurd since R denotes the sum of squares. One can combine all these considerations into the assertion that between the limits of integration $U + 2h'$ can have neither purely positive nor purely negative values, assuming the stability of the solar system. $U + 2h'$ must then oscillate back and forth from positive to negative, i.e., U must oscillate around $-2h'$. However these oscillations of U must be contained between definite finite limits, for if it be assumed that U becomes infinite quantity, since $U = \sum \frac{m_i m_{i'}}{r_{ii'}}$, this can happen only if two bodies come infinitely close. Then their attraction would become infinitely great, they would not be able to separate; so from that time on a definite $r_{ii'} = 0$, and thereby $U = \infty$, so that if one integrates beyond this time, $\int\int(U+2h')dt^2$, and with it R, takes an infinitely large positive value, whatever the value h' has. So the other bodies of the solar system must themselves be infinitely further distant, and thereby the stability the system must be lost. U must then make oscillations about $-2h'$, which are contained between two definite finite limits, for which behavior periodic functions whose constant term $= -2h'$ give an example. This will be satisfied by the formula for elliptic motion. Here $U = \frac{1}{r}$, $-2h' = \frac{1}{a}$ (except for a common constant factor of both quantities), r must also oscillate about a, which is in fact the case; further the expansion of $\frac{1}{r}$ in terms of the mean anomaly must contain the constant term $\frac{1}{a}$, and this too actually happens. For the mutual attraction of two bodies negative values of h' give the elliptic motion, $h' = 0$ corresponds to a parabolic and positive values of h' to a hyperbolic motion, which are also in agreement with our results.

The theorem that U oscillates about $-2h'$ or $U + 2h'$ about 0 can be also expressed as follows: $2U + 2h'$ oscillates about U; $2U + 2h'$ is according to equation (4.8), the 'vis viva' (about the centre of gravity); so the value of the 'vis viva' must oscillate about the value of the force function. If all the distances in the system became very large, then the force-function becomes very small, and also the *vis viva*, according to the theorem of *vis viva*. With this the velocities too become very small, or the more the distances increase the smaller become the velocities. Stability rests on this.

In this and similar considerations lies the kernel of the celebrated investigations of *Laplace, Lagrange* and *Poisson* on the stability of the planetary system. There exists, namely, the theorem:

Theorem 4.1 *If one assumes the elements of a planetary orbit variable and expands the major axis in terms of time, then it occurs only as an argument of periodic functions, no term proportional to time ever occurs.*

This theorem was for the first time proved by *Laplace* only for small eccentricities and the first power of the masses.*Lagrange* extended this to arbitrary eccentricities with one stroke of the pen[1]. *Poisson* finally proved[2] that it also holds when one considers the second power of the masses. This work is one of his finest. With the consideration of the third power of the masses already time occurs outside periodic functions, but is still multiplied by these; if the fourth power are taken into consideration,

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[1] "Mem. de l'Institut, 1808

[2] Journal de l'école polytechnique, cat. 15

then t occurs even without being multiplied by periodic functions. The result for the third power still gives oscillations about a mean value, but infinitely large for $t = \infty$; when the fourth power is considered such oscillations do not occur at all. One arrives at similar results for small oscillations; on consideration of the higher powers of the displacement, one arrives here at the result that a small impulse leads, with increasing time, always to large oscillations.

But all these results, to be precise, do not prove anything. For, if one neglects the higher powers of the displacement, one assumes that time is small, and one cannot derive any conclusions for large values of t. Therefore one does not have to wonder — if for the first and second powers of the masses time already occurs ouside the periodic functions. For the justification for neglecting higher powers of the mass in the expansion in terms of the mass lies in the assumption that t does not exceed a certain limit. One therefore moves in a circle.

The pendulum provides an intuitive example for this. The position in which the sphere finds itself directly above the point of suspension is that of an unstable equilibrium of the pendulum. One obtains here the time outside the sine and cosine functions, and concludes from this rightly that an infinitely small impulse gives a finite motion. But it would be false to conclude from the circumstance that time appears outside of periodic functions that the motion of the pendulum is not periodic, since in the present case the sphere rotates periodically about its point of suspension. Similarly it would be false to conclude from the results which one obtains taking into account the higher powers of the masses in the solar system, that it is unstable.

Lecture 5

The Principle of Conservation of Surface Area

We had made the assumption that the force function U and the equations of constraint remain unchanged if one changes all the x-coordinates through one and the same constant, likewise all the y-coordinates through a second and all the z-coordinates through a third, and obtained the principle of conservation of motion of the centre of gravity. This given change of coordinates comes about when one displaces the origin, but allows the coordinate axes to remain parallel.

We shall now make another assumption: the equations of constraint shall remain unchanged if, with the x-axis unchanged, we rotate the y and z-axes in their plane through an arbitrary angle. If we set

$$y = r\cos v, \quad z = r\sin v,$$

these do not change with an increase of the angle v through an arbitrary angle δv. If we denote the angle v for different points of the system with $v_1, v_2, \ldots, v_i, \ldots$ respectively, then U and the equations of constraint must remain unchanged when al the v are changed through the same angle δv, i.e., they must depend only on the differences $v_i - v_{i'}$. This is the case of an entirely free system and above all, a case in which only the distances between two mass points of the system occur. Through introduction of r and v, the expression for any such distance becomes

$$\begin{aligned} r_{12}^2 &= (x_1 - x_2)^2 + (r_1\cos v_1 - r_2\cos v_2)^2 + (r_1\sin v_1 - r_2\sin v_2)^2 \\ &= (x_1 - x_2)^2 + (r_1^2 + r_2^2 - 2r_1r_2\cos(v_1 - v_2), \end{aligned}$$

Thus it depends only on the difference $v_1 - v_2$. Also belongs here the case where the points of the system are constrained to move on a surface of revolution whose axis of rotation is the x-axis; then the v do not occur at all in the equations of constraint. It is further to be remarked that when fixed points appear in the problem, they must lie on the x-axis.

With these assumptions on U and the equations of constraint, all the v can be increased simultaneously by δv. The x_i remain unchanged hereby, but the y_i and z_i will change since

$$y_i = r_i \cos v_i, \quad z_i = r_i \sin v_i;$$

thus one has

$$\delta x_i = 0, \quad \delta y_i = -r_i \sin v_i \delta v = -z_i \delta v, \quad \delta z_i = r_i \cos v_i \delta v = y_i \delta v$$

as the virtual variations of coordinates that hold for our problem. Substitution of these values in the symbolic equation (2.2) of the Lecture 2 leads to the equations:

$$\delta v \sum m_i \left\{ -z_i \frac{d^2 y_i}{dt^2} + y_i \frac{d^2 z_i}{dt^2} \right\} = \delta U;$$

U remains unchanged for the given displacement and so $\delta U = 0$ and one has

$$\sum m_i \left(y_i \frac{d^2 z_i}{dt^2} - z_i \frac{d^2 y_i}{dt^2} \right) = 0. \tag{5.1}$$

We want to remark here that this equation remains valid in the more general case where instead of δU on the right side the expression

$$\sum (X_i \delta x_i + Y_i \delta y_i + Z_i \delta z_i),$$

appears if only

$$\sum (Y_i z_i - Z_i y_i) = 0. \tag{5.2}$$

If this expression is not equal to zero, then it occurs on the right side of equation (5.1) instead of 0. Let us therefore assume that either the force function U with the given properties exists or that, in the more general case where it does not exist, the equation (5.2) is satisfied. Then equation (5.1) holds in the form given above; its left side is however integrable and one obtains by integration

$$\sum m_i \left\{ y_i \frac{dz_i}{dt} - z_i \frac{dy_i}{dt} \right\} = \alpha, \tag{5.3}$$

where α denotés the constant of integration.Further, if one introduces polar coordinates r_i and v_i, then 5.3 takes the form

$$\sum m_i r_i^2 \frac{dv_i}{dt} = \alpha. \tag{5.4}$$

The principle of conservation of surface area is contained in this equation. It is indeed well-known that $r^2 dv$ is equal to twice the surface element in polar coordinates and a subsequent integration of equation (5.4) from 0 to t gives the theorem:

Theorem 5.1 *If one multiplies every one of the surface areas described in the yz-plane by the radius vectors projected in this plane by the mass of the point belonging to it, then the sum of the products is proportional to time.*

This is the celebrated principle of conservation of surface areas. It holds, as stated, when U and the equations of constraint remain unchanged when one rotates the y and z axes in their plane around the x axis, a hypothesis which one can express analytically for the equations of constraint thus: that for any equation of constraint $f = 0$, the equation

$$\sum \left(z_i \frac{\partial f}{\partial y_i} - y_i \frac{\partial f}{\partial z_i} \right) = 0$$

must be satisfied identically.

That in the transformation $y\ dz - z\ dy = r^2\ dv$ used earlier, only the differential of the quantity v occurs is, in many cases, a very important circumstance; it follows from this transformation that $y\ dz - z\ dy$ multiplied by a homogeneous function of order 2 in y and z is a total differential, since it can be represented as a product of dv and a function of v alone.

In the case in which U and the equations of constraint remain unchanged also when one rotates the x and z axes around the y axis and the x and y axes around the z axis, one has besides equation (5.3) two more similar ones, namely

$$\sum m_i \left(z_i \frac{dx_i}{dt} - x_i \frac{dz_i}{dt} \right) = \beta \tag{5.5}$$

$$\sum m_i \left(x_i \frac{dy_i}{dt} - y_i \frac{dx_i}{dt} \right) = \gamma. \tag{5.6}$$

This holds, for example, for n bodies moving freely in space. In this case one therefore always has four integrals, the three conservation of surface area theorems and the theorem of conservation of 'vis viva'.

It is a very remarkable circumstance, to which we have already drawn attention in the introduction, that of these surface area theorems, either only one holds or all three. We shall see it proven as a result of pure computation, as a mere consequence of a mathematical identity, that the third surface area theorem always follows from the other two. When all the three surface area theorems hold, one can, without affecting the generality of the solution, take two of the constants, α, β, γ to be zero. In every problem these constants are determined by the equations of constraint. Whichever way these may be constituted it is always possible to change the coordinate axes so that in the new coordinate system two of the constants vanish. In fact, the new coordinates being ξ_i, η_i, ζ_i, the general transformation formulae for coordinates are

$$\xi_i = ax_i + by_i + cz_i, \quad \eta_i = a'x_i + b'y_i + c'z_i, \quad \zeta_i = a''x_i + b''y_i + c''z_i.$$

The constants $a, b, c, a', b', c', a'', b''c''$ satisfy among others the nine equations

$$\begin{array}{lll} b'c'' - b''c' = a, & c'a'' - c''a' = b, & a'b'' - a''b' = c, \\ b''c - bc'' = a', & c''a - ca'' = b', & a''b - ab'' = c', \\ bc' - b'c = a'', & ca' - c'a = b'', & ab' - a'b = c''. \end{array}$$

So, on considering these equations,

$$\eta_i \frac{d\zeta_i}{dt} - \zeta_i \frac{d\eta_i}{dt} = a\left(y_i \frac{dz_i}{dt} - z_i \frac{dy_i}{dt}\right) + b\left(z_i \frac{dx_i}{dt} - x_i \frac{dz_i}{dt}\right) + c\left(x_i \frac{dy_i}{dt} - y_i \frac{dx_i}{dt}\right),$$

and therefore,

$$\sum m_i \left(\eta_i \frac{d\zeta_i}{dt} - \zeta_i \frac{d\eta_i}{dt}\right) = a\alpha + b\beta + c\gamma. \tag{5.7}$$

From this one sees that if the surface theorems hold for all three coordinate planes in one coordinate system, then they hold in every coordinate system.[1] We shall represent the new constant $a\alpha + b\beta + c\gamma$ in another

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[1]The surface area theorems considered so far, which refer to a fixed origin of coordinates, cannot be applied to the solar system, because in space we do not have a fixed point. But one can easily convince oneself that if one sets

$$x_i = \xi + A, \quad y_i = \eta_i + B, \quad , z_i = \zeta_i + C,$$

where A, B, C are the coordinates of the centre of gravity (Lecture 3), that the syrface area theorems (5.3), (5.5),(5.6) also hold when one substitutes ξ_i, η_i, ζ_i for x_i, y_i, z_i respectively and at the same time changes α, β, γ into

$$M(\beta^{(0)}\gamma' - \gamma^{(0)}\beta'), M(\gamma^{(0)}\alpha' - \alpha^{(0)}\gamma'), M(\alpha^{(0)}\beta' - \beta^{(0)}\alpha'),$$

i.e., those surface area theorems also hold for the case when the centre of gravity with uniform rectilinear motion is chosen as the origin of coordinates.

form. If one denotes the angles which the ξ-axis makes with the x, y, z axes by l, m, n, then

$$a = \cos l, \qquad b = \cos m, \qquad c = \cos n.$$

If one further sets

$$\frac{\alpha}{\sqrt{\alpha^2+\beta^2+\gamma^2}} = \cos\lambda, \frac{\beta}{\sqrt{\alpha^2+\beta^2+\gamma^2}} = \cos\mu, \frac{\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}} = \cos\nu,$$

then one has

$$a\alpha + b\beta + c\gamma = \sqrt{\alpha^2+\beta^2+\gamma^2}(\cos l\cos\lambda + \cos m\cos\mu + \cos n\cos\nu).$$

But since $\cos^2\lambda + \cos^2\mu + \cos^2\nu = 1$, then λ, μ, ν can be looked upon as the angles which a certain line L makes with the x, y and z axes. If one denotes the angle which this line makes with the ξ-axis by V, then one has

$$\cos l\cos\lambda + \cos m\cos\mu + \cos n\cos\nu = \cos V,$$

and therefore

$$a\alpha + b\beta + c\gamma = \sqrt{\alpha^2+\beta^2+\gamma^2}\cdot\cos V.$$

The constant of the surface area theorem for the $\eta-\zeta$-plane is therefore

$$\sqrt{\alpha^2+\beta^2+\gamma^2}$$

multiplied by the cosine of the angle which the ξ-axis makes with the line L given by the above construction. The same holds naturally for the two other surface area theorems in the new coordinate system, only we have to take in place of V the angles V', V'' which the line L makes with η and ζ axes. If one now allows the ξ-axis to coincide with the line L, then the angle $V = 0$ and at the same time $V' = 90°$ and $V'' = 90°$, so that $\cos V = 1$, $\cos V' = 0$, $\cos V'' = 0$. Hence one sees that the constants of the surface area theorem for the $\xi\eta$- and $\xi\zeta$-planes actually become zero and at the same time the constant of the surface area theorem for the $\eta\zeta$-plane is $\sqrt{\alpha^2+\beta^2+\gamma^2}$, i.e., equal to the maximum which it can attain, since its value in the general form is $\sqrt{\alpha^2+\beta^2+\gamma^2}\cos V$.

Laplace has named the $\eta\zeta$-plane determined in this manner–the invariant plane. He believed that it could be used to find out whether in the course of thousands of years collisions have occurred in the solar system, since they would change the plane's location. Conversely, if two measurements at two different times give different positions of this plane then collisions must have occurred during that time. This is, however, the simplest use of the invariable plane. If we write again for the new coordinates the letters x, y, z introduced earlier, so that the plane yz is the invariable plane, then we have the three surface area theorems:

$$\sum m_i\left(y_i\frac{dz_i}{dt} - z_i\frac{dy_i}{dt}\right) = \epsilon,$$
$$\sum m_i\left(z_i\frac{dx_i}{dt} - x_i\frac{dz_i}{dt}\right) = 0,$$
$$\sum m_i\left(x_i\frac{dy_i}{dt} - y_i\frac{dx_i}{dt}\right) = 0,$$

where $\epsilon = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$. For the case of two bodies one can give an interesting geometrical interpretation to these surface area theorems. In this case one has

$$m_1\left(y_1\frac{dz_1}{dt} - z_1\frac{dy_1}{dt}\right) + m_2\left(y_2\frac{dz_2}{dt} - z_2\frac{dy_2}{dt}\right) = \epsilon,$$
$$m_1\left(z_1\frac{dx_1}{dt} - x_1\frac{dz_1}{dt}\right) + m_2\left(z_2\frac{dx_2}{dt} - x_2\frac{dz_2}{dt}\right) = 0,$$
$$m_1\left(x_1\frac{dy_1}{dt} - y_1\frac{dx_1}{dt}\right) + m_2\left(x_2\frac{dy_2}{dt} - y_2\frac{dx_2}{dt}\right) = 0.$$

By elimination of m_1 and m_2 from the last two equations, it follows that

$$\begin{aligned}\left(z_1\frac{dx_1}{dt} - x_1\frac{dz_1}{dt}\right) : \left(x_1\frac{dy_1}{dt} - y_1\frac{dx_1}{dt}\right) \quad &= \quad \left(z_2\frac{dx_2}{dt} - x_2\frac{dz_2}{dt}\right) \\ &\quad : \left(x_2\frac{dy_2}{dt} - y_2\frac{dx_2}{dt}\right).\end{aligned} \tag{5.8}$$

This proposition has a simple geometrical meaning. In fact one imagines at m_1 a tangent drawn to the curve described by m_1, and considers a plane E_1 laid through this tangent and the origin of coordinates and a normal N_1 drawn to this plane at the origin. Let the cosines of the angle which N_1 makes with the coordinate axes be p_1, q_1, r_1; then one has for the point m_1 the two equations

$$p_1x_1 + q_1y_1 + r_1z_1 = 0,$$
$$p_1\,dx_1 + q_1\,dy_1 + r_1\,dz_1 = 0,$$

which can also be written in the form of a double proportion, namely

$$p_1 : q_1 : r_1 = (y_1\, dz_1 - z_1\, dy_1) : (z_1\, dx_1 - x_1\, dz_1) : (x_1\, dy_1 - y_1\, dx_1).$$

When one makes a similar construction for the point m_2, one obtains, if one constructs the plane E_2 corresponding to E_1 and the normal N_2 corresponding to N_1 and determines the cosines p_2, q_2, r_2,

$$p_2 : q_2 : r_2 = (y_2\, dz_2 - z_2\, dy_2) : (z_2\, dx_2 - x_2\, dz_2) : (x_2\, dy_2 - y_2\, dx_2).$$

From this it follows that one can write the equation (5.8) in terms of the quantities p_1, q_1, r_1, p_2, q_2, r_2:

$$q_1 : r_1 = q_2 : r_2.$$

It is easy to find the geometric meaning of this equation. The equations of the lines N_1 and N_2 are

$$\frac{x}{p_1} = \frac{y}{q_1} = \frac{z}{r_1} \quad \text{and} \quad \frac{x}{p_2} = \frac{y}{q_2} = \frac{z}{r_2}.$$

Therefore, one has, the equations of their projections on the yz-plane,

$$\frac{y}{q_1} = \frac{z}{r_1} \quad \text{and} \quad \frac{y}{q_2} = \frac{z}{r_2}.$$

However, since $q_1 : r_1 = q_2 : r_2$, these two equations are identical, i.e., N_1 and N_2 have the same projections on the yz-plane, or again, N_1 and N_2 lie in a plane which is perpendicular to the yz-plane and which, since N_1 and N_2 go through the origin, contains the x axis. From this it follows that the planes E_1 and E_2 cut the yz-plane along the same line. So for the free motion of two masses m_1 and m_2 the following theorem holds:

Theorem 5.2 *If one draws tangents at m_1 and m_2 to the paths of the two points and considers planes laid through these tangents and the centre of gravity of the system (this is the origin of coordinates), then these cut the invariant plane (the yz-plane) along one and the same straight line.*

This geometric interpretation goes back to Poinsot. I have made an interesting application of this to the problem of three bodies (Crelle Journal, Bd. 26, p. 115, Math. Werke, Bd I, p. 30).

Just as the stability of the planetary system with respect to its dimensions can be derived from the theorem of conservation of *vis viva*, so too the principle of conservation of surface areas can be used to prove the stability with respect to to the form of the orbit. The proof given earlier will show that the major axes of the ellipses in which the planets move cannot increase beyond a certain limit. Similarly one can prove from the surface area theorem that the eccentricities can vary only between certain limits and on this depends the form of the orbits. However, apart from the drawback of the earlier proof that on consideration of higher powers, secular terms, i.e., those which contain the time outside the periodic functions sine and cosine, occur, this proof is incomplete in that it holds only for massive celestial bodies. In the equation from which one derives the result in question, the individual terms are multiplied by the masses of the celestial bodies, and therefore the bodies with small masses influence the entire equation so little that one can draw no conclusion about their eccentricities. In fact the stability of the form of the orbit does not hold for comets; it also does not hold for small planets, e.g. Mercury, whose mass is so small that up to now it could only be estimated by guesswork.The investigations first carried out by Encke, to obtain the mass by observation, were possible because the comet named after him came extraordinarily close to Mercury.

If, to the mutual attraction of material points, attractions by fixed centres are added, then the principle of conservation of surface areas ceases to hold except when these centres lie in a straight line. Let us take this line as the x-axis; then the surface area theorem holds, for the yz-plane, but for the other two planes it does not hold. In fact, let us consider a material point m_i and imagine through it a plane E_i parallel to the yz-plane. The resultant of all the attractions which the point m_i experiences from all the fixed centres lying on the x-axis will be directed from it towards a certain point of the x-axis; one can then resolve this force into two, one of which goes along the line through m_i parallel to the x-axis, and the other from the point m_i towards the point of intersection of the plane E_i with the x axis, and therefore lies in this plane. We shall denote the latter by Q_i and resolve it into two components parallel to the axes of y and z. If we shift to the earlier notation, the component parallel to the y-axis is $Q_i \cos v_i$, and the component parallel to the z-axis is $Q_i \sin v_i$. Hence there comes now, in the symbolic equation for the motion, in addition to the earlier δU, the expression $\sum Q_i(\cos v_i \delta y_i + \sin v_i \delta z_i)$. We also have, if we understand by U only that part of the force-function which comes from the mutual attraction of the points,

$$\sum m_i \left\{ \frac{d^2 x_i}{dt^2} \delta x_i + \frac{d^2 y_i}{dt^2} \delta y_i + \frac{d^2 z_i}{dt^2} \delta z_i \right\}$$
$$= \delta U + \sum Q_i (\cos v_i \delta y_i + \sin v_i \delta z_i),$$

or if we set in the above,

$$\delta x_i = 0, \delta y_i = -r_i \sin v_i \delta v = -z_i \delta v, \delta z_i = r_i \cos v_i \delta v = y_i \delta v,$$

whereby δU vanishes,

$$\sum m_i \left(y_i \frac{d^2 z_i}{dt^2} - z_i \frac{d^2 y_i}{dt^2} \right) = 0,$$

and therefore by integration,

$$\sum m_i \left(y_i \frac{dz_i}{dt} - z_i \frac{dy_i}{dt} \right) = \alpha,$$

i.e., the principle of conservation of surface areas holds for the plane to which the line which contains all the fixed centres is perpendicular. In this case one has two integrals, the theorem of *vis viva* and the surface area theorem. However, if there are fixed centres which do not lie in a straight line occurring in the problem, then the surface area theorem does not hold and one has only one integral from the theorem of *vis viva*.

If we assume further that the centres are not fixed, but one of these has a certain motion independent of the other material points of the system, so that this motion is a given function of time, then the principle of *vis viva* also ceases to hold. Such cases occur in nature. Here belongs, for example, the attraction of a comet by Jupiter and the Sun, where the orbits of the Sun and Jupiter are to be seen as given, and the comet as a material point which has no influence on their orbits. Here, as was mentioned, the principle of *vis viva* ceases to hold; and this rests essentially on this, that one has for the distance r of a point mass (x, y, z) from a centre (a, b, c), the differential equation

$$dr = \frac{x-a}{r} dx + \frac{y-b}{r} dy + \frac{z-x}{r} dz.$$

But this differential equation assumes that a, b, c are constant; this ceases to hold in our case, and with it the principle of *vis viva*. One can indeed always represent the force acting on an individual point as the partial differential coefficient of a function U, but this function now contains the time explicitly, besides the coordinates; it is therefore no more so that

$$\frac{dU}{dt} = \sum \left(\frac{\partial U}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial U}{\partial y_i} \frac{dy_i}{dt} + \frac{\partial U}{\partial z_i} \frac{dz_i}{dt} \right),$$

but on the right side now comes also the partial differential coefficient $\frac{\partial U}{\partial t}$, so that

$$\sum \left(\frac{\partial U}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial U}{\partial y_i} \frac{dy_i}{dt} + \frac{\partial U}{\partial z_i} \frac{dz_i}{dt} \right) = \frac{dU}{dt} - \frac{\partial U}{\partial t}.$$

Now the differential equation of the theorem of *vis viva* was

$$\begin{aligned} \sum m_i \left(\frac{dx_i}{dt} \frac{d^2 x_i}{dt^2} + \frac{dy_i}{dt} \frac{d^2 y_i}{dt^2} + \frac{dz_i}{dt} \frac{d^2 z_i}{dt^2} \right) \\ = \sum \left(\frac{\partial U}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial U}{\partial y_i} \frac{dy_i}{dt} + \frac{\partial U}{\partial z_i} \frac{dz_i}{dt} \right). \end{aligned}$$

This would be integrable if one could set $\frac{dU}{dt}$ for the right hand side. Now, however, one must set it to be $\frac{dU}{dt} - \frac{\partial U}{\partial t}$ and therefore one cannot integrate it anymore. If in the equation

$$\sum m_i \left(\frac{dx_i}{dt} \frac{d^2 x_i}{dt^2} + \frac{dy_i}{dt} \frac{d^2 y_i}{dt^2} + \frac{dz_i}{dt} \frac{d^2 z_i}{dt^2} \right) = \frac{dU}{dt} - \frac{\partial U}{\partial t}$$

one thinks U as the sum $U + V$, where V contains the time explicitly, but U does not, then one has

$$\sum m_i \left(\frac{dx_i}{dt} \frac{d^2 x_i}{dt^2} + \frac{dy_i}{dt} \frac{d^2 y_i}{dt^2} + \frac{dz_i}{dt} \frac{d^2 z_i}{dt^2} \right) = \frac{dU}{dt} + \frac{dV}{dt} - \frac{\partial V}{\partial t}. \tag{5.9}$$

This is the equation which holds in place of the differential equations for the principle of *vis viva*, which, however does not give an integral. The principle of surface area also does not hold. One has therefore no single principle which gives an integral. Nevertheless I have remarked that there exists a hypothesis on the motion of the fixed centres, and indeed a hypothesis which comes very close to the just mentioned case occurring in nature. If one assumes this hypothesis one can, from a combination of both the principles, obtain an integral. This hypothesis consists in assuming that the fixed centres move in circles with the same angular velocity about one and the same axis, so that one has, for the coordinates of any one centre (a, b, c),

$$a = \text{ constant}, \quad b = \beta \cos nt, \quad c = \beta \sin nt,$$

where n has the same value for all centres and the x axis is the common axis of rotation. This comes very near to the case found in nature, since the Sun and Jupiter move in the ecliptic around their common centre of gravity in ellipses of very small eccentricities (nearly $= \frac{1}{20}$), and consequently they can be considered as circles. Their periods of revolution are the same and if one sets this to be T, then one has the equation $nT = 2\pi$ for the determination of n.

We shall now investigate, using the differential equations, what happens to the surface area theorem in this case. We shall, for generality take besides the centres, not one single material point, but a whole system of points, and then in our case the force function will consist of two complexes of terms. The first complex arises from the mutual attraction of the material points and consists of terms of the form

$$\frac{m_i m_i'}{\sqrt{(x_i - x_{i'})^2 + (y_i - y_{i'})^2 + (z_i - z_{i'})^2}},$$

or, if as before we introduce r_i and v_i,

$$\frac{m_i m_i'}{\sqrt{(x_i - x_{i'})^2 + r_i^2 + r_{i'}^2 - 2rr_{i'} \cos(v_i - v_{i'})}}.$$

The second complex arises from the attraction of the centres and consists of terms of the form

$$\frac{m_i \mu}{\sqrt{(x_i - a)^2 + (y_i - b)^2 + (z_i - c)^2}},$$

or, if here also we introduce r_i and v_i and at the same time $b = \beta \cos nt$, $c = \beta \sin nt$,

$$\frac{m_i \mu}{\sqrt{(x_i - a)^2 + r_i^2 + \beta^2 - 2r_i\beta \cos(v_i - nt)}}. \tag{5.10}$$

Both complexes remain unchanged when one increases all the v_i by the same quantity and at the same time t by its nth fraction,that is, if one sets for any value of i

$$\delta v_i = n\delta t,$$

which variations are virtual in our case. We shall call the first complex of terms U, the second V.

In the general symbolic equation

$$\sum m_i\Big(\frac{d^2x_i}{dt^2}\delta x_i+\frac{d^2y_i}{dt^2}\delta y_i+\frac{d^2z_i}{dt^2}\delta z_i\Big)=\sum\Big(\frac{\partial U}{\partial x_i}\delta x_i+\frac{\partial U}{\partial y_i}\delta y_i+\frac{\partial U}{\partial z_i}\delta z_i\Big),$$

$U+V$ appears in the place of U in this case, so that the right side becomes

$$\sum\Big(\frac{\partial U}{\partial x_i}\delta x_i+\frac{\partial U}{\partial y_i}\delta y_i+\frac{\partial U}{\partial z_i}\delta z_i\Big)+\sum\Big(\frac{\partial V}{\partial x_i}\delta x_i+\frac{\partial V}{\partial y_i}\delta y_i+\frac{\partial V}{\partial z_i}\delta z_i\Big).$$

U does not contain t explicitly, the first sum is therefore equal to δU; but in V, t is indeed contained explicitly and the second sum lacks $\frac{\partial V}{\partial t}\delta t$ to give the complete δV, i.e., it is equal to $\delta V-\frac{\partial V}{\partial t}\delta t$, and one has

$$\sum m_i\Big(\frac{d^2x_i}{dt^2}\delta x_i+\frac{d^2y_i}{dt^2}\delta y_i+\frac{d^2z_i}{dt^2}\delta z_i\Big)=\delta U+\delta V-\frac{\partial V}{\partial t}\delta t.$$

The above variations, however, are so arranged that U and V remain unchanged by them, and therefore $\delta U=0$ and $\delta V=0$; further,

$$\delta x_i=0,\delta y_i=-r_i\sin v_i\delta v_i=-nz_i\delta t,\delta z_i=r_i\cos v_i\delta v_i=ny_i\delta t,$$

so

$$n\sum m_i\Big(y_i\frac{d^2z_i}{dt^2}-z_i\frac{d^2y_i}{dt^2}\Big)=-\frac{\partial V}{\partial t} \tag{5.11}$$

This is the equation which holds in our case in place of the differential equation for the principle of conservation of surface areas; V is our aggregate of terms of the form (5.10), where n must be the same in all the terms, but all other quantities can take values which vary from one term to another. Now equation (5.9) was

$$\sum m_i\Big(\frac{dx_i}{dt}\frac{d^2x_i}{dt^2}+\frac{dy_i}{dt}\frac{d^2y_i}{dt^2}+\frac{dz_i}{dt}\frac{d^2z_i}{dt^2}\Big)=\frac{dU}{dt}+\frac{dV}{dt}-\frac{\partial V}{\partial t},$$

or

$$\frac{1}{2}\sum m_i\frac{d}{dt}\Big\{\Big(\frac{dx_i}{dt}\Big)^2+\Big(\frac{dy_i}{dt}\Big)^2+\Big(\frac{dz_i}{dt}\Big)^2\Big\}=\frac{dU}{dt}+\frac{dV}{dt}-\frac{\partial V}{\partial t}.$$

If one subtracts equation (5.11) from this, then one has on integration

$$\frac{1}{2}\sum m_i\Big\{\Big(\frac{dx_i}{dt}\Big)^2+\frac{dy_i}{dt}\Big)^2+\frac{dz_i}{dt}\Big)^2\Big\}$$
$$-n\sum m_i\Big(y_i\frac{dz_i}{dt}-z_i\frac{dy_i}{dt}\Big)=U+V+h''. \tag{5.12}$$

This is the principle which arises from the combination of the principles of *vis viva* and of surface areas, and holds if the centres of attraction move with uniform velocity about an axis of rotation. To this category belongs, for example, the motion on the surface of the earth or in its neighborhood, since the earth is an aggregate of such centres of attraction. In fact, if the density of the earth varied considerably from meridian to meridian, the problem must be considered from this point of view. Under this assumption, if at the same time the moon were near the earth which itself moved more slowly, the attraction of the moon by the earth would be a function also of the hour-angle. Then the moments of inertia with respect to different meridian planes would be different, and this could be discovered by observation.

Lecture 6

The Principle of Least Action

We come now to a new principle which does not give an integral, as the earlier ones did. This is the *principe de la moindre action*, wrongly called the principle of least action. Its importance lies, first, in the form in which it represents the differential equations of motion and secondly in that it gives a function which will be a minimum when the differential equations are satisfied. Such a minimum exists, indeed, in all problems, but one does not as a rule know where. While, therefore, the interest of this principle consists precisely in that in general the minimum can be *given*, in earlier times one gave an exaggerated importance to the fact that such a minimum exists at all. An example of the principle in question appears in *Euler*'s treatise *de motu projectorum* cited earlier. After *Euler* himself proved it for the attraction by fixed centres, he did not succeed in doing so for mutual attractions (between point masses) for which the validity of the principle of *vis viva* was unknown. He contents himself, therefore, with saying that, for mutual attractions the computation would be very long and the the principle of least action must hold because the foundations of a sound Metaphysics showed that in Nature the forces must necessarily always produce the least action (because of the inherent inertia of bodies, according to him). But this shows neither a sound nor any Metaphysics at all and, in fact, *Euler* made this statement because of a misunderstanding of the name *least action*. *Maupertuis* wanted to express by this name that nature achieves its work with the least expenditure of force and this is the real significance of the name *principe de la moindre action*.

In almost all textbooks, even in the best, those of *Poisson, Lagrange* and *Laplace*, the principle has been so presented that, in my view, it is impossible to understand. Namely, it is stated that the integral

$$\int \sum m_i v_i ds_i$$

(where $v_i = \frac{ds_i}{dt}$ denotes the velocity of the point m_i) will be a minimum if the integral is extended from one configuration of the system to another. It is indeed said that this theorem holds only as long as the theorem of *vis viva* holds, but one forgot to say that one must eliminate the time from the above integral using the theorem of *vis viva* and reduce everything to space elements. The minimum of the above integral is further to be understood in the sense that when the initial and final positions are given, the integral, among all possible paths from one position to another, would be a minimum for the one actually described.

Let us eliminate the time from the integral above. If we set $v_i = \frac{ds_i}{dt}$, then

$$\int \sum m_i v_i ds_i = \int \frac{\sum m_i ds_i^2}{dt}.$$

But, according to the theorem of 'vis viva',

$$\frac{1}{2} \sum m_i v_i^2 = U + h,$$

or

$$\sum \frac{m_i ds_i^2}{dt^2} = 2(U + h),$$

or

$$\frac{1}{dt} = \sqrt{\frac{2(U + h)}{\sum m_i ds_i^2}}.$$

If one inserts this value of $\frac{1}{dt}$, then it follows that

$$\int \sum m_i v_i ds_i = \int \sqrt{2(U + h)} \sqrt{\sum m_i ds_i^2}.$$

The differential equations of motion, integrated, express the $3n$ coordinates of the problem as a function of time; between any two coordinates, one can, however, eliminate the time and give, if one wishes, $3n - 1$ coordinates expressed by means of one, for example x_1. Under this assumption, one can substitute for $\sum m_i ds_i^2$ the expression $\sum m_i \left(\frac{ds_i}{dx_1}\right)^2 dx_1^2$ and then obtain the integral in the form

$$\int \sqrt{2(U + h)} \sqrt{\sum m_i \left(\frac{ds_i}{dx_1}\right)^2} \, dx_1,$$

with which only an entirely definite concept is associated. Let us, in order to give no preference to any one coordinate, express the integral in its earlier form

$$\int \sqrt{2(U+h)}\sqrt{\sum m_i ds_i^2}.$$

We can then express the principle of least action thus: *If two positions of the system are given (i.e., if one knows the values which the other* $3n-1$ *coordinates take for* $x_1 = a$ *and* $x_1 = b$*), and extends the integral*

$$\int \sqrt{2(U+h)}\sqrt{\sum m_i ds_i^2}$$

over the whole path of the system from the first position to the second, then its value is a minimum for the actual path among all possible paths), i.e., those which are consistent with the conditions of the system (if they are given). Then

$$\int \sqrt{2(U+h)}\sqrt{\sum m_i ds_i^2}$$

will be a minimum, or,

$$\delta \int \sqrt{2(U+h)}\sqrt{\sum m_i ds_i^2} = 0. \tag{6.1}$$

It is difficult to find a metaphysical reason for the principle of least action if it is expressed in this correct form, as is necessary. There exist minima of an entirely different kind, from which also one can derive the differential equations of motion–a method which has many advocates.

We must impose a limitation on the principle of least action. Namely, the minimum of the integral exists not between any two arbitrary positions of the system, but only when the initial and final positions are sufficiently close. We shall presently discuss which limits should not be crossed.

Let us consider first a special case. A single material point moves on a given surface driven forward by an initial impulse, without attractive forces acting on it. In this case $U = 0$ and the sum $\sum m_i ds_i^2$ is just mds^2 and then $\int ds$, or s, will be a minimum, i.e., the point describes the shortest line on the surface. But the shortest lines have their property of being a minimum only between certain limits. For example, on the sphere, where the great circles are the shortest lines, this property does not hold when one considers a longitude greater than 180°. In order to see this one should not take help of completion to 360°, which will prove nothing. Since the minima need exist only with respect to lines lying infinitely close. One can convince oneself of this in another way. Let B be the pole from A.Let the great circle $A\alpha B$ be extended beyond B to C and let the great circle $A\beta B$ lie infinitely near $A\alpha B$;

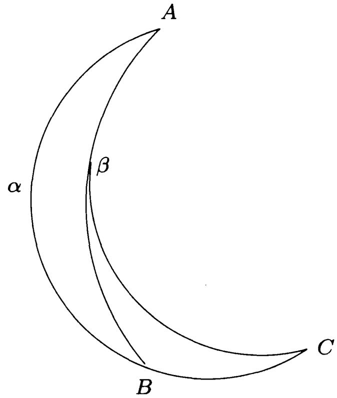

Figure 6.1

then

$$A\alpha BC = A\beta B + BC = A\beta + \beta B + BC.$$

Further, let β be infinitely near B and βC the arc of a great circle, then $\beta C < \beta B + BC$, so the broken line $A\beta + \beta C$ is shorter than the great circle $A\alpha BC$. On the sphere then, 180° is the limit for the minimum property. In order to determine this limit in general, I have established the following theorem which I arrived at through a deeper investigation:

Theorem 6.1 *When one draws the shortest lines from a point on a surface in various directions, two cases can arise: two infinitely near shortest lines either run near each other without intersecting, or they intersect and then the successive points of intersection form their enveloping curve. In the first case the shortest lines never cease to be the shortest; in the second case they are, only up to the point of contact with the enveloping curve.*

The first case holds, as is obvious, for all developable surfaces, since in a plane two straight lines passing through a point never intersect again; further it holds, as I have found, for all concavo-convex surfaces, i.e., those in which two mutually perpendicular normal sections have their radii of curvature on opposite sides; for example, the one-sheeted hyperboloid and the hyperbolic paraboloid. Moreover it should not be said that there are no concavo-concave surfaces which belong to this category, at least the impossibility of this has not been proved. The ellipsoid of revolution gives an example of the second kind. If we take it slightly differing from the sphere, then the shortest lines going through an arbitrary point on the surface do not indeed, as on the sphere, all intersect at the pole, but they form a small enveloping curve in the region of the pole. Under these circumstances there appears to be a paradox in the considerations on surfaces; for the enveloping curves have in general the property that the systems of curves which are enveloped by them can never enter the interior of the enveloping curve. So there would be a piece of the surface with the property that no shortest line can be drawn from a given point to any point in its interior, which is impossible. The paradox disappears however on a more precise consideration of the enveloping curve as can be seen from Figure 6.2, in which $ABCD$ is the enveloping curve which has approximately the form of the evolute of an ellipse and EFG represents the shortest line. Here it enters from

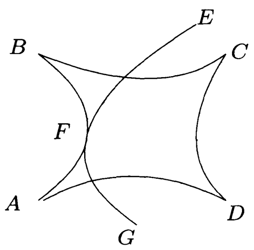

Figure 6.2

E the piece of surface bounded by the enveloping curve, touches the curve at a point F and ceases thence to be the shortest line. This property of shortest lines, that they cease to be so, if they touch their common enveloping curve has, as has been said, been found through deep consideration, but in hindsight it can be understood easily. For, if two shortest lines intersect, at the point of intersection, not only the first variation, but also the second variation, will be zero; the difference reduces to an infinitesimal quantity of the third order, i.e., a minimum does not exists any more.

We now return to the general consideration of the minimum for the principle of least action. The arbitrary constants which remain after the integration of the differential equations of motion can most easily be determined by the initial positions and initial velocities of the motion. If these are given all the integration constants are determined through these, and there cannot be any ambiguities. But for the principle of least action one takes as given not the initial positions and the initial velocities, but the initial and final positions. So in order to determine the actual motion, one must derive the initial velocities from the final position by solving the equations. These equations need not be linear, so one can obtain many systems of values of the initial velocities and to these correspond the many motions of the system from the given initial position to the given final position, all of which give minima in respect of motions lying infinitely close. Now, in so far as one can allow the interval between the initial and end positions to increase continuously from zero, the different systems of values which one obtains from the solution of equations for the initial velocities also alter. So now with this alteration in the system of values there occurs the case that two systems of values become equal; so this is the limit beyond which no minimum occurs.

This theorem which is of little importance for mechanics in the narrower sense, I have made known in *Crelles*' Journal[1] but only as a note without proof. As an example of the same we shall choose the motion of planets around the sun.

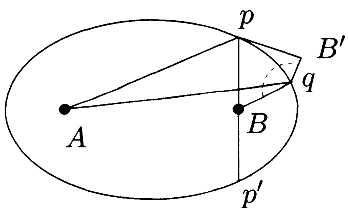

Figure 6.3

Let a focus A of the ellipse be the location of the Sun, a the major axis of the ellipse and, in addition, p, q two positions of the planet. If we denote position, for the moment unknown, of the second focus by B, then the distances of the point B from the two planetary positions p and q are known; namely the distances are equal to $a - Ap$ and $a - Aq$, from the well-known property of the ellipse. But this gives for B two positions B and B', one above and the other below the line joining p and

[1]Vol. 17, p. 68 ff

q. So there exist two ellipses and accordingly two motions of the planet which are possible for the given pieces of the planetary orbit. In order that the two solutions coincide, the points B and B' must lie on the line connecting p and q, i.e., p, B and q must lie on the same straight line, and consequently q at p'. The point p' then denotes the limit beyond which one may not extend the integral from p without its ceasing to be a minimum.

We now return to the proper mechanical significance of the principle of least action. This consists in that the equation (6.1) contains the fundamental equations of dynamics for the cases in which the principle of 'vis viva' is valid. In fact equation (6.1) was

$$\delta \int \sqrt{2(U+h)}\sqrt{\sum m_i ds_i^2} = 0.$$

After elimination of the time here we can look upon all coordinates as functions of one, e.g., x_1, and write accordingly

$$\delta \int \sqrt{2(U+h)}\sqrt{\sum m_i \left(\frac{dx_i}{dx_1}\right)^2}\, dx_1 = 0,$$

or,

$$\delta \int \sqrt{2(U+h)}\sqrt{\sum m_i \left\{\left(\frac{dx_i}{dx_1}\right)^2 + \left(\frac{dy_i}{dx_1}\right)^2 + \left(\frac{dz_i}{dx_1}\right)^2\right\}}\, dx_1 = 0,$$

or, if we set,

$$\frac{dx_i}{dx_1} = x_i', \quad \frac{dy_i}{dx_1} = y_i', \quad \frac{dz_i}{dx_1} = z_i',$$

$$\delta \int \sqrt{2(U+h)}\sqrt{\sum m_i (x_i'^2 + y_i'^2 + z_i'^2)dx_1} = 0.$$

Introducing the notation

$$2(U+h) = A, \quad \sum m_i (x_i'^2 + y_i'^2 + z_i'^2) = B, \quad \sqrt{A}\sqrt{B} = P,$$

we have finally

$$\delta \int P\, dx_1 = 0;$$

in other words, it gives the following rule: one substitutes in $\int P\, dx_1$, $x_i + \delta x_i$, $y_i + \delta y_i$, $z_i + \delta z_i$, in place of x_i, y_i z_i respectively, where δx_i, δy_i, δz_i, are arbitrary functions (which do not become infinite inside the limits of integration) multiplied by an infinitely small factor α, expands in powers of α, and sets the term which is multiplied by the first power of α equal to zero. Here it is to be remarked that, first, since the limits of integration are given, no variation can affect them, that further, on the same grounds, all variations must vanish at the limits, and finally, δx_1 is moreover zero since x_1 is the independent variable. Hence, one obtains according to the rules of calculus of variations,

$$\begin{aligned}\delta \int P\,dx_1 &= \int \delta P\,dx_1 \\ &= \int \sum \left\{ \frac{\partial P}{\partial x_i}\delta x_i + \frac{\partial P}{\partial y_i}\delta y_i + \frac{\partial P}{\partial z_i}\delta z_i + \frac{\partial P}{\partial x_i'}\delta x_i' \right. \\ &\qquad \left. + \frac{\partial P}{\partial y_i'}\delta y_i' + \frac{\partial P}{\partial z_i'}\delta z_i' \right\} dx_1.\end{aligned}$$

Now,

$$\int \frac{\partial P}{\partial x_i'}\delta x_i' dx_1 = \int \frac{\partial P}{\partial x_i'}\frac{d\delta x_i}{dx_1}dx_1 = \frac{\partial P}{\partial x_i'}\delta x_i - \int \frac{d\frac{\partial P}{\partial x_i'}}{dx_1}\delta x_i dx_1,$$

or, since δx_i vanishes at the limits of integration,

$$\int \frac{\partial P}{\partial x_i'}\delta x_i' dx_1 = -\int \frac{d\frac{\partial P}{\partial x_i'}}{dx_1}\delta x_i dx_1.$$

Similar equations hold for y_i and z_i. The use of these gives

$$\begin{aligned}\delta \int P\,dx_1 &= \\ &\int \sum \left[\left(\frac{\partial P}{\partial x_i} - \frac{d\frac{\partial P}{\partial x_i'}}{dx_1}\right)\delta x_i + \left(\frac{\partial P}{\partial y_i} - \frac{d\frac{\partial P}{\partial y_i'}}{dx_1}\right)\delta y_i \right. \\ &\qquad \left. + \left(\frac{\partial P}{\partial z_i} - \frac{d\frac{\partial P}{\partial z_i'}}{dx_1}\right)\delta z_i \right] dx_1.\end{aligned}$$

However, $P = \sqrt{A}\sqrt{B}$, $A = 2(U+h)$, $B = \sum m_i(x_i'^2 + y_i'^2 + z_i'^2)$,

$$\frac{\partial P}{\partial x_i} = \frac{1}{2}\sqrt{\frac{B}{A}}\frac{\partial A}{\partial x_i} = \sqrt{\frac{B}{A}}\frac{\partial U}{\partial x_i}, \quad \frac{\partial P}{\partial x_i'} = \frac{1}{2}\sqrt{\frac{A}{B}}\frac{\partial B}{\partial x_i'} = \sqrt{\frac{A}{B}}m_i x_i';$$

then one has

$$\frac{\partial P}{\partial x_i} - \frac{d\frac{\partial P}{\partial dx_i'}}{dx_1} = \sqrt{\frac{B}{A}}\frac{\partial U}{\partial x_i} - \frac{d\left(m_i\sqrt{\frac{A}{B}}\frac{dx_i}{dx_1}\right)}{dx_1}.$$

If one now sets (see pg. 47)

$$\sqrt{\frac{B}{A}}\, dx_1 = dt, \tag{6.2}$$

then one obtains

$$\frac{\partial P}{\partial x_i} - \frac{d\frac{\partial P}{\partial x_i'}}{dx_1} = \sqrt{\frac{B}{A}}\left(\frac{\partial U}{\partial x_i} - m_i \frac{d^2 x_i}{dt^2}\right),$$

and similarly for y and z. If one introduces these expressions one gets,

$$\delta \int P\, dx_1 =$$
$$\int \sqrt{\frac{B}{A}} \sum \left\{ \left(\frac{\partial U}{\partial x_i} - m_i \frac{d^2 x_i}{dt^2}\right)\delta x_i + \left(\frac{\partial U}{\partial y_i} - m_i \frac{d^2 y_i}{dt^2}\right)\delta y_i + \left(\frac{\partial U}{\partial z_i} - m_i \frac{d^2 z_i}{dt^2}\right)\delta z_i \right\} dx_1.$$

Since these variations should vanish, according to our principle,

$$0 = \sum \left\{ \left(\frac{\partial U}{\partial x_i} - m_i \frac{d^2 x_i}{dt^2}\right)\delta x_i + \left(\frac{\partial U}{\partial y_i} - m_i \frac{d^2 y_i}{dt^2}\right)\delta y_i \cdot + \cdot \left(\frac{\partial U}{\partial z_i} - m_i \frac{d^2 z_i}{dt^2}\right)\delta z_i \right\},$$

or

$$\begin{aligned} \sum m_i \left(\frac{d^2 x_i}{dt^2}\delta x_i + \frac{d^2 y_i}{dt^2}\delta y_i + \frac{d^2 z_i}{dt^2}\delta z_i\right) &= \sum \left(\frac{\partial U}{\partial x_i}\delta x_i + \frac{\partial U}{\partial y_i}\delta y_i + \frac{\partial U}{\partial z_i}\delta z_i\right) \\ &= \delta U, \end{aligned} \tag{6.3}$$

which is the earlier symbolic equation.

Equation (6.2) is none other than the theorem of *vis viva*. For, squaring it one finds

$$B dx_1^2 = A dt^2,$$

or,

$$\sum m_i \left\{ \left(\frac{dx_i}{dt}\right)^2 + \left(\frac{dy_i}{dt}\right)^2 + \left(\frac{dz_i}{dt}\right)^2 \right\} = 2(U + h).$$

This was to be expected since we had eliminated time from the principle of least action through the principle of *vis viva.*

Lecture 7

Further considerations on the principle of least action—The *Lagrange* multipliers

Apart from the drawback of the usual way of expressing the principle of least action in that one does not introduce the theorem of *vis viva* in the integral, there comes another. This one is that one says that the integral shall be the largest or the smallest, instead of saying that its first variation should vanish. The confounding of these, by no means identical, requirements has become so much of a custom that one can hardly ascribe it to the authors as mistakes. One finds in this respect a strange quid pro quo between *Lagrange* and *Poisson* which refers to the shortest line. *Lagrange* says entirely correctly that in this case the integral can never be a maximum, since however long a curve may be between two points on a given surface, one can always find a longer one, and hence concludes that the integral must *always* be a minimum. On the other hand, *Poisson*, who knew that the integral, in certain cases, namely on closed surfaces, ceases to be a minimum beyond certain limits, concludes from this that in those case it must be a maximum. Both conclusions are false; in the case of the shortest line the integral, to be sure, can never be a maximum; rather it is either a minimum or neither of the two, maximum or minimum.

The elimination of time from the integral which comes into consideration for the principle of least action, should happen directly using the principle of *vis viva* and not through the principle of surface areas or any other integral equation of the problem. Only then one can arrive at the principle of least action. *Lagrange* says in one place that he has in the Turin memoirs derived the differential equations of motion from the principle of least action in conjuction with the principle of *vis viva.* Such a way of expression is not admissible according to the remarks made above. *Lagrange* applied the variational calculus just then discovered by him to the principle of least action already used by *Euler*, but needed here the principle of *vis viva* in the extension which *Daniel Bernoulli* had given it, and in this manner came to the symbolic equation of dynamics from which we started, and which we want to write down here once again; it was

$$\sum m_i \left\{ \frac{d^2 x_i}{dt^2}\delta x_i + \frac{d^2 y_i}{dt^2}\delta y_i + \frac{d^2 z_i}{dt^2}\delta z_i \right\}$$
$$= \sum \{X_i \delta x_i + Y_i \delta y_i + Z_i \delta z_i\}, \tag{7.1}$$

where δU is to be put on the right side if the principle of *vis viva* holds. If one abstracts from the fact that δU can be set on right side of the above equation in the usual sense of the variational calculus only if the quantities X_i, Y_i, Z_i are the partial differential coefficients of a single function U, and if one considers it purely as a symbolic abbreviated notation, one has

$$\sum m_i \left\{ \frac{d^2 x_i}{dt^2}\delta x_i + \frac{d^2 y_i}{dt^2}\delta y_i + \frac{d^2 z_i}{dt^2}\delta z_i \right\} = \delta U, \tag{7.2}$$

also when the theorem of *vis viva* does not hold. Now this equation, as already explained earlier, is still correct when there are equations of constraint, but then the variations are no longer independent of one another. If one has m equations of constraint

$$f = 0,\ \phi = 0, \ldots \tag{7.3}$$

then there is, between the variations, the m relations

$$\begin{aligned} \sum \left(\frac{\partial f}{\partial x_i}\delta x_i + \frac{\partial f}{\partial y_i}\delta y_i + \frac{\partial f}{\partial z_i}\delta z_i \right) &= 0, \\ \sum \left(\frac{\partial \phi}{\partial x_i}\delta x_i + \frac{\partial \phi}{\partial y_i}\delta y_i + \frac{\partial \phi}{\partial z_i}\delta z_i \right) &= 0, \\ \vdots &= \vdots \end{aligned} \tag{7.4}$$

By means of these m equations one can eliminate m of the $3n$ variations δx_i, δy_i, $\delta z_i, \ldots$ from the equation (7.1), and when one sets the remaining as independent of one another the symbolic equation (7.1) breaks up into the differential equations of motion. But this elimination would be very laborious and has moreover some drawbacks; first, one must prefer certain coordinates to the others and one does not obtain symmetric formulae, and besides this, the form of the elimination equations will be different for different number of equations of constraint. Because of this circumstance the generality of the investigation would be rendered very difficult. *Lagrange* has overcome all these difficulties through the introduction of multipliers, a method which *Euler* had already frequently applied for the problems "de maximis et minimis". Since the variations δx_i, δy_i, $\delta z_i, \ldots$ occur in the equations (7.1) and (7.4) linearly, one can carry out the elimination of m of them in the following way: one multiplies the equations (7.4) by λ, $\mu, \ldots$, and adds them to (7.1). Let the resulting equation be called (L). Now one determines the factors $\lambda, \mu, \ldots$, so that in the equation denoted by (L), m of the expressions multiplied by the variations δx_i, δy_i, δz_i vanish identically; then the expression multiplied by the remaining $3n - m$ variations set equal to zero give the differential equations of the problem. In this manner one sees that all the expressions multiplied by the $3n$ variations $\delta x_i, \delta y_i, \delta z_i$ are to be set equal to zero in the equation (L), and then these equations are to be so looked upon that m of them define the multipliers $\lambda, \mu, \ldots$, the remaining in which the multipliers so determined are substituted give the differential equations of the problem. In other words, if one looks upon all variations as independent, one has to eliminate the multipliers $\lambda, \mu, \ldots$ from the $3n$ equations into which the equation (L) breaks up, and the remaining $3n - m$ give the differential equations of the problem. However, instead of carrying out this elimination, one does better by letting the unknown multipliers in the $3n$ equations to remain and base further investigations on these. These $3n$ equations will then be of the form

$$\begin{aligned} m_i \frac{d^2 x_i}{dt^2} &= X_i + \lambda \frac{\partial f}{\partial x_i} + \mu \frac{\partial \phi}{\partial x_i} + \cdots \\ m_i \frac{d^2 y_i}{dt^2} &= Y_i + \lambda \frac{\partial f}{\partial y_i} + \mu \frac{\partial \phi}{\partial y_i} + \cdots \\ m_i \frac{d^2 x_i}{dt^2} &= Z_i + \lambda \frac{\partial f}{\partial z_i} + \mu \frac{\partial \phi}{\partial z_i} + \cdots , \end{aligned} \tag{7.5}$$

where the same multipliers $\lambda, \mu, \ldots$,occur for all the n values of i. This is the form that *Lagrange* has given to the equations of motion of a system with arbitrary constraints.

The quantities which are added to the forces X_i, Y_i, Z_i express the effect of the system, i.e. the modification which the acting forces X_i,

Y_i, Z_i undergo on account of the connectioons between the mass points. One arrives at these results also in statics where one proves that if in the n points of the system the forces

$$\lambda\frac{\partial f}{\partial x_i}+\mu\frac{\partial\phi}{\partial x_i}+\cdots,\lambda\frac{\partial f}{\partial y_i}+\mu\frac{\partial\phi}{\partial y_i}+\cdots,\lambda\frac{\partial f}{\partial z_i}+\mu\frac{\partial\phi}{\partial z_i}+\cdots$$

are brought parallel to the coordinate axes they are cancelled by the constraints of the system, whence it follows that the forces annulled by the constraints of the system are not determined, but contain indeterminate quantities $\lambda, \mu, \ldots$. The introduction of the multipliers $\lambda, \mu, \ldots$ is therefore not a mere artifice of computation, but these quantities have their well-defined significance in Statics. One can also arrive at the equation of motion (7.5) from the theorem in Statics just stated, where the passage from Statics to Mechanics is based on the following consideration:

The mass points of the system cannot follow the impulses imparted to them because of the binding of the system. In order to find out the actual motion, one must therefore introduce such forces that will be annulled by the constraints of the system. Their introduction is to be regarded as those which allow the points to follow the forces applied to them without hindrance; in other words, by introducing forces through which the constraints of the system are cancelled, one can regard the system as free. This is to be seen as a principle and equation (7.5) is obtained directly from it.

This principle which has given us the modifications of the accelerating forces because of the binding of the system also allows us to find the modifications of the instantaneous forces through the binding of the system. The formulae which one has to apply here are absolutely the same. If instantaneous impulses a_i, b_i, c_i act at the point m_i, then the impulses modified in respect to the constraints of the system are the following:

$$\begin{aligned}&a_i+\lambda_1\frac{\partial f}{\partial x_i}+\mu_1\frac{\partial\phi}{\partial x_i}+\cdots\\&b_i+\lambda_1\frac{\partial f}{\partial y_i}+\mu_1\frac{\partial\phi}{\partial y_i}+\cdots\\&c_i+\lambda_1\frac{\partial f}{\partial z_i}+\mu_1\frac{\partial\phi}{\partial z_i}+\cdots\end{aligned}\tag{7.6}$$

where the quantities $\lambda_1, \mu_1, \ldots$ remain the same for all points.

If one wants to determine the quantities $\lambda, \mu, \ldots$ and $\lambda_1, \mu_1, \ldots$, then one must differentiate the equations $f = 0, \phi = 0, \ldots$. For determining the quantities $\lambda, \mu, \ldots$ one must differentiate twice and substitute from equations (7.5) for the second derivatives of the coordinates; for determining the quantities $\lambda_1, \mu_1, \ldots$, one has to differentiate only once, since the instantaneous impulses are proportional to the velocities, i.e., the first derivatives. We want to actually develop the equations for determining $\lambda_1, \mu_1, \ldots$, assuming that the instantaneous impulses ensue at the beginning of the motion and that the system is at this instant is completely at rest. Under these circumstances we can leave the accelerating forces entirely out of consideration at the beginning of the motion, since these give only infinitely small velocities, and when we construct the differential equations

$$\sum \left\{ \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial f}{\partial y_i}\frac{dy_i}{dt} + \frac{\partial f}{\partial z_i}\frac{dz_i}{dt} \right\} = 0,$$
$$\sum \left\{ \frac{\partial \phi}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial \phi}{\partial y_i}\frac{dy_i}{dt} + \frac{\partial \phi}{\partial z_i}\frac{dz_i}{dt} \right\} = 0,$$

and so on for the determination of $\lambda_1, \mu_1, \ldots$, we have therefore to set the quantities (7.6) for $\frac{dx_i}{dt}, \frac{dy_i}{dt}, \frac{dz_i}{dt}$ after they have been divided by m_i. This gives the following result; if one puts

$$A = \sum \frac{1}{m_i} \left\{ \frac{\partial f}{\partial x_i} a_i + \frac{\partial f}{\partial y_i} b_i + \frac{\partial f}{\partial z_i} c_i \right\},$$
$$B = \sum \frac{1}{m_i} \left\{ \frac{\partial \phi}{\partial x_i} a_i + \frac{\partial \phi}{\partial y_i} b_i + \frac{\partial \phi}{\partial z_i} c_i \right\},$$
$$(f, f) = \sum \frac{1}{m_i} \left(\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial y_i}\frac{\partial f}{\partial y_i} + \frac{\partial f}{\partial z_i}\frac{\partial f}{\partial z_i} \right),$$
$$(f, \phi) = \sum \frac{1}{m_i} \left(\frac{\partial f}{\partial x_i}\frac{\partial \phi}{\partial x_i} + \frac{\partial f}{\partial y_i}\frac{\partial \phi}{\partial y_i} + \frac{\partial f}{\partial z_i}\frac{\partial \phi}{\partial z_i} \right), \ldots$$

then one has for the determination of $\lambda_1, \mu_1, \ldots$ the equations

$$\begin{aligned} 0 &= A + (f, f)\lambda_1 + (f, \phi)\mu_1 + (f, \psi)\nu_1 + \cdots \\ 0 &= B + (\phi, f)\lambda_1 + (\phi, \phi)\mu_1 + (\phi, \psi)\nu_1 + \cdots \\ 0 &= C + (\psi, f)\lambda_1 + (\psi, \phi)\mu_1 + (\psi, \psi)\nu_1 + \cdots \end{aligned} \tag{7.7}$$

and so on.

The equations for determining $\lambda, \mu, \ldots$ have the same form only A, B, C take different values. We now go back to the differential equations (7.5). If we multiply them in order by $\delta x_i, \delta y_i, \delta z_i$ and add all the $3n$ products, then we again obtain the symbolic equations which we had denoted by (L), namely

$$\sum m_i \left(\frac{d^2x_i}{dt^2}\delta x_i + \frac{d^2y_i}{dt^2}\delta y_i + \frac{d^2z_i}{dt^2}\delta z_i\right) = \delta U + \lambda\,\delta f + \mu\,\delta\phi + \cdots\,, \quad (7.8)$$

which is equivalent to the system (7.5).

For considering the entire extent of the problem which is contained in (7.5), we must take into account the case in which time enters explicitly into the constraints. Even then equations (7.5) hold. In order to visualise how time can be involved in the constraints, we consider for example mass points connected to moving centres whose motions are given, so that the centres act on the mass points without there being any reaction. For this assumption it is, however, necessary to assign infinitely large masses to the moving centres in relation to the masses of the points. In this case equations (7.5) hold at once for the mass points. But the moving centres maintain the given motions unaltered. In fact let M, the mass of a centre, be infinitely large and let p one of its coordinates; then the force acting in the direction of the coordinate p is proportional to M; if we call this MP, then taking into account the binding of the system we have

$$M\,\frac{d^2p}{dt^2} = MP + \lambda\frac{\partial f}{\partial p} + \mu\frac{\partial \phi}{\partial p} + \cdots$$

But after division by the infinitely large mass M, all the rest of the terms vanish and we have

$$\frac{d^2p}{dt^2} = P.$$

The same holds for the other coordinates, i.e., the centres follow their given motions without regard for binding. The values of $\lambda, \mu, \ldots$ and $\lambda_1, \mu_1, \ldots$ here will be of course different from the earlier ones, because on differentiation, their partial differential coefficients with respect to time will also be added. Thus, for example, to A (equation (7.7)) comes the term $\frac{\partial f}{\partial t}$, similarly to B, $\frac{\partial \phi}{\partial t}$ and so on.

Time can also enter the constraints in an entirely different way; for example, when the binding between two points becomes slack or is reduced, perhaps through a rise in temperature; in which case one can attribute all constraints of this sort to moving centres, if one holds fast to the basic theorem that two constraints which lead to the same equation can be replaced by each other.

Moreover, time can make the problem more difficult, for example, if the masses vary with time. However, up to now, one has not found it necessary to make this assumption in the planetary system, since the observations for deciding whether this actually occurs have not been sufficiently precise.

Lecture 8

Hamilton's Integral and *Lagrange*'s Second Form of Dynamical Equations

One can substitute another principle in place of the principle of least action where also the first variation of an integral vanishes, and from which one can derive the differential equations of motion in a still simpler way than from the principle of least action. It appears that this principle had not been noticed earlier, because here in general one does not obtain a minimum with the vanishing of the variation, as it happens in the case of the principle of least action. *Hamilton* is the first to have started out from this principle. We shall use it to formulate the equations of motion in the form Lagrange has given them in Mecanique Analytique. Let, first, the forces X_i, Y_i, Z_i be the partial derivatives of a function U; further let T be half the 'vis viva', i.e.,

$$T = \frac{1}{2}\sum m_i v_i^2 = \frac{1}{2}\sum m_i \left\{ \left(\frac{dx_i}{dt}\right)^2 + \left(\frac{dy_i}{dt}\right)^2 + \left(\frac{dz_i}{dt}\right)^2 \right\};$$

then the new principle is contained in the equation

$$\delta \int (T+U)\, dt = 0. \tag{8.1}$$

This principle is more general in comparision with that of least action in so far as here U can depend on t explicitly, which was excluded in the earlier principle. There the time had to be eliminated through the principle of *vis viva*, which holds only when U does not contain t explicitly.

We shall use equation (8.1) for proving the reduction of the differential equations of motion to a first order partial differential equation. As Hamilton has shown, one can decompose the variation (8.1) by partial integration into two parts, so that one stands outside and the other inside the integral sign and both of them must vanish separately. In this way the expression under the integral sign equated to zero gives the differential equations of the problem, and the expression outside the integral sign its integral equations.

The new principle can be stated completely in the following way: *Let the positions of the system be given at a given initial time* t_0 *and a given final time* t_1*; then for the determination of the actually ensuing motion, one has the equation*

$$\delta \int (T+U)\, dt = 0. \tag{8.2}$$

Here the integration extends from t_0 to t_1, U is the force-function which can contain the time explicitly and T is half the 'vis viva'; so one has

$$T = \frac{1}{2}\sum m_i(x_i'^2 + y_i'^2 + z_i'^2), \qquad x_i' = \frac{dx_i}{dt},\ y_i' = \frac{dy_i}{dt},\ z_i' = \frac{dz_i}{dt},$$

If one carries out the variation prescribed by this principle so that one adds the variations δx_i, δy_i, δz_i to the coordinates according to the rules of the calculus of variations and does not vary the independent variable t, one gets

$$\delta \int T\, dt = \int \delta T dt = \int \sum m_i(x_i'\delta x_i' + y_i'\delta y_i' + z_i'\delta z_i') \tag{8.3}$$

or if one introduces the expressions $\frac{d\delta x_i}{dt}$, $\frac{d\delta y_i}{dt}$, $\frac{d\delta z_i}{dt}$, for $\delta x_i'$, $\delta y_i'$, $\delta z_i'$, and integrates by parts,

$$\begin{aligned}\delta \int T\, dt &= \int \sum m_i \left(x_i' \frac{d\delta x_i}{dt} + y_i' \frac{d\delta y_i}{dt} + z_i' \frac{d\delta z_i}{dt} \right) \\ &= \sum m_i(x_i'\delta x_i + y_i'\delta y_i + z_i'\delta z_i) \\ &\quad - \int \sum m_i(x_i''\delta x_i + y_i''\delta y_i + z_i''\delta z_i)\, dt\end{aligned}$$

where x_i'', y_i'', z_i'' are the second differential coefficients of x_i, y_i, z_i with respect to t. Since, however, the initial and final positions are given, δx_i, δy_i, δz_i vanish at the limits of integration and the term standing outside the integral sign is equal to zero, so that

$$\delta \int T\, dt = -\int \left\{ \sum m_i(x_i''\delta x_i + y_i''\delta y_i + z_i''\delta z_i) \right\}\, dt.$$

Then one has

$$\delta \int (T+U)\, dt = -\int \left\{ \sum m_i(x_i''\delta x_i + y_i''\delta y_i + z_i''\delta z_i) - \delta U \right\}\, dt,$$

where

$$\delta U = \sum \left(\frac{\partial U}{\partial x_i}\delta x_i + \frac{\partial U}{\partial y_i}\delta y_i + \frac{\partial U}{\partial z_i}\delta z_i \right),$$

an equation from which in fact the basic symbolic equation (2.2) of dynamics given earlier in the second lecture (page 12) follows.

The principle contained in equation (8.1) is very useful in the transformation of coordinates. It holds for any coordinate system. Therefore, in a new system one has to vary with respect to the new coordinates as earlier in the old one, and the entire substitution which is to be carried out is limited to the two expressions T and U.

We shall first apply this to polar coordinates; the transformation formulae in this case are

$$x_i = r_i \cos\phi_i, \qquad y_i = r_i \sin\phi_i \cos\psi_i, \qquad z_i = r_i \sin\phi_i \sin\psi_i.$$

From these follow, by differentiation,

$$\begin{aligned} dx_i &= \cos\phi_i \; dr_i - r \sin\phi_i \; d\phi_i, \\ dy_i &= \sin\phi_i \cos\psi_i \; dr_i + r_i \cos\phi_i \sin\psi_i \; d\phi_i - r_i \sin\phi_i \sin\psi_i \; d\psi_i \\ dz_i &= \sin\phi_i \sin\psi_i \; dr_i + r_i \cos\phi_i \sin\psi_i \; d\phi_i + r_i \sin\phi_i \cos\psi_i \; d\psi_i; \end{aligned}$$

and so

$$dx_i^2 + dy_i^2 + dz_i^2 = dr_i^2 + r_i^2 d\phi_i^2 + r_i^2 \sin^2\phi_i d\psi_i^2,$$

or

$$x_i'^2 + y_i'^2 + z_i'^2 = r_i'^2 + r_i^2 \phi_i'^2 + r_i^2 \sin^2\phi_i \cdot \psi_i'^2,$$

where

$$r_i' = \frac{dr_i}{dt}, \quad \phi_i' = \frac{d\phi_i}{dt}, \quad \psi_i' = \frac{d\psi_i}{dt}.$$

Then one has at once:

$$\begin{aligned} T &= \frac{1}{2} \sum m_i (x_i'^2 + y_i'^2 + z_i'^2) \\ &= \frac{1}{2} \sum m_i (r_i'^2 + r_i^2 \phi_i'^2 + r_i^2 \sin^2\phi_i \cdot \psi_i'^2). \end{aligned} \tag{8.4}$$

Under these assumptions and also taking U as expressed in the new coordinates, we shall find the equation which proceeds from $\delta \int (T+U)\, dt = 0$, according to the general rules of variational calculus.

If P is a function of several variables $\ldots p \ldots$ and their differential coefficients $\ldots p' \ldots$, where it is assumed that all the p depend on one independent variable t, and if the first variation of $\int P\, dt$ vanishes:

$$\delta \int P\, dt = 0,$$

where the integral is to be taken from t_0 to t, and the p are given corresponding to these values of t, then this leads, as the derivations carried out in the sixth lecture (page 54) have shown, to the equation,

$$0 = \sum \left[\frac{d\frac{\partial P}{\partial p'}}{dt} - \frac{\partial P}{\partial p} \right] \delta p. \tag{8.5}$$

In our case the quantities p are r_i, ϕ_i, ψ_i, and $P = T + U$; further U does not contain the derivatives r_i', ϕ_i', ψ_i'; hence we obtain

$$0 = \sum \left[\frac{d\frac{\partial T}{\partial r'}}{dt} - \frac{\partial T}{\partial r_i} - \frac{\partial U}{\partial r_i} \right] \delta r_i + \sum \left[\frac{d\frac{\partial T}{\partial \phi_i'}}{dt} - \frac{\partial T}{\partial \phi_i} - \frac{\partial U}{\partial \phi_i} \right] \delta \phi_i +$$
$$\sum \left[\frac{d\frac{\partial T}{\partial \psi_i'}}{dt} - \frac{\partial T}{\partial \psi_i} - \frac{\partial U}{\partial \psi_i} \right] \delta \psi_i.$$

Now, according to (8.3),

$$\frac{\partial T}{\partial r_i'} = m_i r_i', \qquad \frac{\partial T}{\partial \phi_i'} = m_i r_i^2 \phi_i', \qquad \frac{\partial T}{\partial \psi_i'} = m_i r_i^2 \sin^2 \phi_i \psi_i',$$
$$\frac{\partial T}{\partial r_i} = m_i (r_i {\phi_i'}^2 + r_i \sin^2 \phi_i {\psi_i'}^2), \qquad \frac{\partial T}{\partial \phi_i} = \frac{1}{2} m_i r_i^2 \sin(2\phi_i) {\psi_i'}^2,$$
$$\frac{\partial T}{\partial \psi_i} = 0;$$

so one has

$$0 = \sum \left\{ m_i \left(\frac{dr_i'}{dt} - r_i {\phi_i'}^2 - r_i \sin^2 \phi_i {\psi_i'}^2 \right) - \frac{\partial U}{\partial r_i} \right\} \delta r_i +$$
$$\sum \left\{ m_i \left(\frac{d(r_i^2 \phi_i')}{dt} - \frac{1}{2} r_i^2 \sin(2\phi_i) \psi_i' \right) - \frac{\partial U}{\partial \phi_i} \right\} \delta \phi_i +$$
$$\sum \left\{ m_i \frac{d(r_i^2 \sin^2 \phi_i \psi_i')}{dt} - \frac{\partial U}{\partial \psi_i} \right\} \delta \psi_i,$$

or

$$\sum m_i \left\{ \left(\frac{d^2 r_i}{dt^2} - r_i {\phi_i'}^2 - r_i \sin^2 \phi_i {\psi_i'}^2 \right) \delta r_i + \right.$$
$$\left. \left(\frac{d(r_i^2 \phi_i')}{dt} - \frac{1}{2} r_i^2 \sin(2\phi_i) {\psi_i'}^2 \right) \delta \phi_i + \frac{d(r_i^2 \sin^2 \phi_i \psi_i')}{dt} \delta \psi_i \right\} =$$
$$\sum \left(\frac{\partial U}{\partial r_i} \delta r_i + \frac{\partial U}{\partial \phi_i} \delta \phi_i + \frac{\partial U}{\partial \psi_i} \delta \psi_i \right) = \delta U.$$

If equations of constraint $f = 0, \widetilde{\omega} = 0 \dots$ hold, then the sum $\lambda\delta f + \mu\delta\widetilde{\omega} + \cdots$ comes in addition to δU on the right side of this equation, and one has, in this case

$$\sum m_i \left\{ \left(\frac{d^2 r_i}{dt^2} - r_i {\phi_i}'^2 - r_i \sin^2 \phi_i {\psi_i}'^2 \right) \delta r_i + \left(\frac{d(r_i^2 \phi_i')}{dt} - \frac{1}{2} r_i^2 \sin(2\phi_i) {\psi_i}'^2 \right) \delta\phi_i + \frac{d(r_i^2 \sin^2 \phi_i \psi_i')}{dt} \delta\psi_i \right\} = \delta U + \lambda\delta f + \mu\delta\widetilde{\omega} + \cdots, \quad (8.6)$$

an equation which breaks up in $3n$ equations of the following form:

$$\begin{aligned} m_i \left\{ \frac{d^2 r_i}{dt^2} - r_i {\phi_i}'^2 - r_i \sin^2 \phi_i {\psi_i}'^2 \right\} &= \frac{\partial U}{\partial r_i} + \lambda \frac{\partial f}{\partial r_i} + \mu \frac{\partial \widetilde{\omega}}{\partial r_i} + \cdots \\ m_i \left\{ \frac{d(r_i^2 \phi_i'')}{dt} - \frac{1}{2} r_i^2 \sin(2\phi_i) {\psi_i}'^2 \right\} &= \frac{\partial U}{\partial \phi_i} + \lambda \frac{\partial f}{\partial \phi_i} + \mu \frac{\partial \widetilde{\omega}}{\partial \phi_i} + \cdots \quad (8.7) \\ m_i \frac{d(r_i^2 \sin^2 \phi_i \psi_i')}{dt} &= \frac{\partial U}{\partial \psi_i} + \lambda \frac{\partial f}{\partial \psi_i} + \mu \frac{\partial \widetilde{\omega}}{\partial \psi_i} + \cdots \end{aligned}$$

Of extreme importance is the transformation of the original coordinates into new ones which are so chosen that, when everything is expressed in terms of those, the equations of constraint are satisfied automatically. Namely, if there are m equations of constraint, then all the $3n$ coordinates admit of expression in terms of $3n - m$ of them, or through $3n - m$ functions of those. In most cases it is very important to introduce not the coordinates themselves but new quantities, in order to avoid irrational quantities. For example, for the motion of a point on an ellipsoid, the formulae

$$x = a \cos\eta, \qquad y = b \sin\eta \cos\zeta, \qquad z = c \sin\eta \sin\zeta,$$

which satisfy the equation of the ellipsoid identically are of the greatest importance. We shall call these new $3n - m = k$ coordinates $q_1, \dots q_k$; they shall be so constituted that when one expresses $x_1, y_1, z_1, x_2, y_2, z_2, \dots$ through them and inserts these expressions in the m equations of constraint $f = 0, \widetilde{\omega} = 0, \dots$ the left sides of these equations vanish identically, that is,

$$f(q_1, \dots, q_k) = 0, \widetilde{\omega}(q_1, \dots, q_k) = 0, \dots \quad (8.8)$$

without any relation existing between the q's. Through this the differential equations of motion will be significantly simplified. Namely, for any coordinate system whatsoever, according to equation (8.5), the general basic symbolic equation of dynamics, when equations of constraint hold, is

$$\sum \left[\frac{d \frac{\partial T}{\partial q_s'}}{dt} - \frac{\partial T}{\partial q_s} \right] \delta q_s = \delta U + \lambda \delta f + \mu \delta \widetilde{\omega} + \cdots$$

where the summation sign extends over all q. But the equations (8.8) hold identically for all q's; hence, on introducing these quantities, one has $\delta f = 0, \delta \widetilde{\omega} = 0, \ldots$, etc, and the above equation reduces to

$$\sum \left[\frac{d \frac{\partial T}{\partial q_s'}}{dt} - \frac{\partial T}{\partial q_s} \right] \delta q_s = \delta U,$$

which breaks up into k differential equations of the form

$$\frac{d \frac{\partial T}{\partial q_s'}}{dt} - \frac{\partial T}{\partial q_s} = \frac{\partial U}{\partial q_s}. \tag{8.9}$$

This is the form in which *Lagrange* had expressed the differential equations of mechanics already in the old edition of Mécanique analytique.

If one considers all coordinates expressed through the quantities q, one obtains by differentiation

$$\begin{aligned} x_i' &= \frac{\partial x_i}{\partial q_1} q_1' + \frac{\partial x_i}{\partial q_2} q_2' + \cdots + \frac{\partial x_i}{\partial q_k} q_k' \\ y_i' &= \frac{\partial y_i}{\partial q_1} q_1' + \frac{\partial y_i}{\partial q_2} q_2' + \cdots + \frac{\partial y_i}{\partial q_k} q_k' \\ z_i' &= \frac{\partial z_i}{\partial q_1} q_1' + \frac{\partial z_i}{\partial q_2} q_2' + \cdots + \frac{\partial z_i}{\partial q_k} q_k'. \end{aligned}$$

If one inserts these values in $T = \frac{1}{2} \sum m_i ({x_i'}^2 + {y_i'}^2 + {z_i'}^2)$, one obtains an expression which, in relation to the quantities $q_1', \ldots, q_k'$ is a homogeneous function of the second degree whose coefficients are known functions of $q_1, \ldots, q_k$. If we set

$$\frac{\partial T}{\partial q_s'} = p_s,$$

then we can also write the equation in (8.9) as

$$\frac{dp_s}{dt} = \frac{\partial (T + U)}{\partial q_s}. \tag{8.10}$$

This is however not yet the final form of the equations of motion, rather it requires a further transformation; but before we go over to that, we shall extend what we did so far to the case in which no force function exists, but where in place of δU in the original symbolic equation of motion $\sum(X_i\delta x_i + Y_i\delta y_i + Z_i\delta z_i))$ occurs. If everything is expressed in terms of the q's, then

$$\delta U = \sum_s \frac{\partial U}{\partial q_s}\delta q_s.$$

If one compares this with the expression $\sum(X_i\delta x_i + Y_i\delta y_i + Z_i\delta z_i))$ just mentioned and remembers the rule given in Lecture 2 (page 14) according to which, for a transformation of coordinates,

$$\sum_s \frac{\partial x_i}{\partial q_s}\delta q_s, \quad \sum_s \frac{\partial y_i}{\partial q_s}\delta q_s, \quad \sum_s \frac{\partial z_i}{\partial q_s}\delta q_s,$$

are to be substituted for $\delta x_i, \delta y_i, \delta z_i$ respectively, then one sees that in place of $\sum_s \frac{\partial U}{\partial q_s}\delta q_s$ the expression

$$\sum_i \sum_s \left(X_i\frac{\partial x_i}{\partial q_s} + Y_i\frac{\partial y_i}{\partial q_s} + Z_i\frac{\partial z_i}{\partial q_s}\right)\delta q_s$$

enters, and in place of $\frac{\partial U}{\partial q_s}$ the expression

$$Q_s = \sum_i \left(X_i\frac{\partial x_i}{\partial q_s} + Y_i\frac{\partial y_i}{\partial q_s} + Z_i\frac{\partial z_i}{\partial q_s}\right). \tag{8.11}$$

Because of this change, equation (8.9) is replaced by the following:

$$\frac{d\frac{\partial T}{\partial q'_s}}{dt} - \frac{\partial T}{\partial q_s} = Q_s. \tag{8.12}$$

Here one sets for s the values from 1 to k and thus obtains the equations of motion in the present case expressed in terms of the quantities q.

We shall verify equation (8.12) in yet another way, and indeed shall start from equation (7.5)

$$m_i\frac{d^2x_i}{dt^2} = X_i + \lambda\frac{\partial f}{\partial x_i} + \mu\frac{\partial \tilde{\omega}}{\partial x_i} + \cdots$$
$$m_i\frac{d^2y_i}{dt^2} = Y_i + \lambda\frac{\partial f}{\partial y_i} + \mu\frac{\partial \tilde{\omega}}{\partial y_i} + \cdots$$
$$m_i\frac{d^2z_i}{dt^2} = Z_i + \lambda\frac{\partial f}{\partial z_i} + \mu\frac{\partial \tilde{\omega}}{\partial z_i} + \cdots$$

If one multiplies these equations by $\frac{\partial x_i}{\partial q_s}, \frac{\partial y_i}{\partial q_s}, \frac{\partial z_i}{\partial q_s}$ and sums over i, one obtains, as the multiplier of λ,

$$\sum_i \left(\frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial q_s} + \frac{\partial f}{\partial y_i}\frac{\partial y_i}{\partial q_s} + \frac{\partial f}{\partial y_i}\frac{\partial y_i}{\partial q_s} \right) = \frac{\partial f(q_1, \ldots, q_k)}{\partial q_s}.$$

However, the expression on the right vanishes according to (8.8), and the same holds for the coefficients of μ and ν; hence one obtains, taking into account equation (8.11):

$$\sum_i m_i \left\{ \frac{d^2 x_i}{dt^2}\frac{\partial x_i}{\partial q_s} + \frac{d^2 y_i}{dt^2}\frac{\partial y_i}{\partial q_s} + \frac{d^2 z_i}{dt^2}\frac{\partial z_i}{\partial q_s} \right\} = Q_s. \qquad (8.13)$$

In order to verify equation (8.12) we must also show that its left side is identical with the left side of this equation. This will be proved in the following way. One has

$$T = \frac{1}{2}\sum m_i (x_i'^2 + y_i'^2 + z_i'^2),$$

and hence

$$\frac{\partial T}{\partial q_s'} = \sum_i m_i \left(x_i' \frac{\partial x_i'}{\partial q_s'} + y_i' \frac{\partial y_i'}{\partial q_s'} + z_i' \frac{\partial z_i'}{\partial q_s'} \right),$$
$$\frac{\partial T}{\partial q_s} = \sum_i m_i \left(x_i' \frac{\partial x_i'}{\partial q_s} + y_i' \frac{\partial y_i'}{\partial q_s} + z_i' \frac{\partial z_i'}{\partial q_s} \right).$$

One has, however, the differential equations

$$x_i' = \frac{\partial x_i}{\partial q_1} q_1' + \frac{\partial x_i}{\partial q_2} q_2' + \cdots + \frac{\partial x_i}{\partial q_k} q_k',$$
$$y_i' = \frac{\partial y_i}{\partial q_1} q_1' + \frac{\partial y_i}{\partial q_2} q_2' + \cdots + \frac{\partial y_i}{\partial q_k} q_k',$$
$$z_i' = \frac{\partial z_i}{\partial q_1} q_1' + \frac{\partial z_i}{\partial q_2} q_2' + \cdots + \frac{\partial z_i}{\partial q_k} q_k'.$$

It follows from this

$$\frac{\partial x_i'}{\partial q_s'} = \frac{\partial x_i}{\partial q_s}, \quad \frac{\partial y_i'}{\partial q_s'} = \frac{\partial y_i}{\partial q_s}, \quad \frac{\partial z_i'}{\partial q_s'} = \frac{\partial z_i}{\partial q_s};$$

further,

$$\frac{\partial x_i'}{\partial q_s} = \frac{\partial^2 x_i}{\partial q_s \partial q_1} q_1' + \frac{\partial^2 x_i}{\partial q_s \partial q_2} q_2' + \cdots + \frac{\partial^2 x_i}{\partial q_s \partial q_k} q_k' = \frac{d\frac{\partial x_i}{\partial q_s}}{dt},$$
$$\frac{\partial y_i'}{\partial q_s} = \frac{\partial^2 y_i}{\partial q_s \partial q_1} q_1' + \frac{\partial^2 y_i}{\partial q_s \partial q_2} q_2' + \cdots + \frac{\partial^2 y_i}{\partial q_s \partial q_k} q_k' = \frac{d\frac{\partial y_i}{\partial q_s}}{dt},$$
$$\frac{\partial z_i'}{\partial q_s} = \frac{\partial^2 z_i}{\partial q_s \partial q_1} q_1' + \frac{\partial^2 z_i}{\partial q_s \partial q_2} q_2' + \cdots + \frac{\partial^2 z_i}{\partial q_s \partial q_k} q_k' = \frac{d\frac{\partial z_i}{\partial q_s}}{dt}.$$

Substitution of these values in $\frac{\partial T}{\partial q_s'}$ and $\frac{\partial T}{\partial q_s}$ gives

$$\frac{\partial T}{\partial q_s'} = \sum_i m_i \left(x_i' \frac{\partial x_i}{\partial q_s} + y_i' \frac{\partial y_i}{\partial q_s} + z_i' \frac{\partial z_i}{\partial q_s} \right)$$
$$\frac{\partial T}{\partial q_s} = \sum_i m_i \left(x_i' \frac{d\frac{\partial x_i}{\partial q_s}}{dt} + y_i' \frac{d\frac{\partial y_i}{\partial q_s}}{dt} + z_i' \frac{d\frac{\partial z_i}{\partial q_s}}{dt} \right);$$

hence

$$\frac{d\frac{\partial T}{\partial q_s'}}{dt} - \frac{\partial T}{\partial q_s} = \sum_i m_i \left(\frac{dx_i'}{dt}\frac{\partial x_i}{\partial q_s} + \frac{dy_i'}{dt}\frac{\partial y_i}{\partial q_s} + \frac{dz_i'}{dt}\frac{\partial z_i}{\partial q_s} \right)$$
$$= \sum_i m_i \left(\frac{d^2 x_i}{dt^2}\frac{\partial x_i}{\partial q_s} + \frac{d^2 y_i}{dt^2}\frac{\partial y_i}{\partial q_s} + \frac{d^2 z_i}{dt^2}\frac{\partial z_i}{\partial q_s} \right),$$

whence the identity of equations (8.12) and (8.13) is proved, and at the same time the first is verified.

So, if no force function exists, one has equations of the form (8.12) as the equations of motion, but when one such exists, equations of the form (8.9), or what is the same, of the form (8.10), namely

$$\frac{dp_s}{dt} = \frac{\partial (T+U)}{\partial q_s}, \qquad p_s = \frac{\partial T}{\partial q_s'}.$$

One gets a noteworthy result from this form of the equations: if one can so choose the new variables that one of the q_s does not enter into the force-function, and in the representation for T, the variables q_s do not come in, but only their differential coefficients q_s', then in this circumstance, there always exists an integral of the given system of differential equations; in fact p_s = constant, or what is the same, $\frac{\partial T}{\partial q_s'}$ = constant. Since $\frac{\partial (T+U)}{\partial q_s} = 0$ under the assumptions made, one has therefore $\frac{dp_s}{dt} = 0$, $p_s =$ constant. This case occurs, for example, in the attraction of a point by a fixed centre. If the centre is at the origin of coordinates, one has, in polar coordinates (see equation (8.2)),

$$U = \frac{a}{r}, \quad T = \frac{1}{2}m(r'^2 + r^2\phi'^2 + r' \sin^2\phi\ \psi'^2);$$

ψ does not come into U, and into T also no ψ, but only its derivative ψ', so one has

$$\frac{\partial T}{\partial \psi'} = mr^2 \sin^2\phi\ \psi' = \text{ constant},$$

or, when one allows the factor m to go into the constant,

$$r^2 \sin^2\phi \cdot \psi' = \text{ constant},$$

which one can also derive from the third equation (8.7). This is the principle of surface area in relation to the yz-plane. In fact,

$$x = r\cos\phi, \qquad y = r\sin\phi\cos\psi, \qquad z = r\sin\phi\sin\psi,$$

and

$$\tan\psi = \frac{z}{y}, \qquad \frac{1}{\cos^2\psi} \cdot \psi' = \frac{yz' - zy'}{y^2},$$

or on multiplication by $y^2 = r^2\sin^2\phi\cos^2\psi$,

$$r^2\sin^2\phi \cdot \psi' = y\frac{dz}{dt} - z\frac{dy}{dt},$$

and therefore,

$$r^2\sin^2\phi \cdot \psi' = y\frac{dz}{dt} - z\frac{dy}{dt} = \text{constant},$$

the principle of surface area for the yz-plane.

Lecture 9

Hamilton's Form of the Equations of Motion

After the publication of the first edition of Mécanique analytique, the most important step forward in the transformation of the differential equation of motion was made by *Poisson* in a paper which deals with the method of variation of constants and which appears in Volume 15 of the Polytechnique Journal. Here Poisson introduces the quantity $p = \frac{\partial T}{\partial q'}$, in place of the quantity q'; now since, as already remarked, T is a homogeneous function of the second degree in the quantities q' whose coefficients depend on q, p is a linear function of the quantities q'; for the definition of p one has the k equations of the form $p_i = \widetilde{\omega}_i$, where $\widetilde{\omega}_i$ is linear with respect to $q'_1, \ldots, q'_k$. If one solves these linear equations for the quantities q', one then obtains equations of the form $q'_i = K_i$ where the K_i's are linear expressions in p whose coefficients depend on the q. We shall insert these expressions for q'_i in the equation (8.10) of Lecture 8, i.e.,in the equation

$$\frac{dp_i}{dt} = \frac{\partial(T+U)}{\partial q_i} = \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i},$$

where $\frac{\partial U}{\partial q_i}$ contains only q, while $\frac{\partial T}{\partial q_i}$ is, besides, a function of the quantities q', indeed, a homogeneous function of the second degree of these quantities. If we set $q'_i = K_i$, then $\frac{\partial T}{\partial q_i}$ is a homogeneous function of second degree in the quantities p_i. Hence the above equations will be of the form

$$\frac{dp_i}{dt} = P_i,$$

where P_i is an expression in p and q and in fact of the second degree with respect to p. These equations combined with the equation $q'_i = \frac{dq_i}{dt} = K_i$ give

$$\frac{dq_i}{dt} = K_i, \quad \frac{dp_i}{dt} = P_i. \tag{9.1}$$

This is the form to which Poisson brings the equations of motion where K_i and P_i contain no variables other than the p's and q's. From this system of $2k$ equations, one obtains the remarkable theorem that

$$\frac{\partial K_i}{\partial p_{i'}} = \frac{\partial K_{i'}}{\partial p_i}, \qquad \frac{\partial K_i}{\partial q_{i'}} = -\frac{\partial P_{i'}}{\partial p_i}, \qquad \frac{\partial P_i}{\partial q_{i'}} = \frac{\partial P_{i'}}{\partial q_i}, \tag{9.2}$$

of which Poisson obtains the first group exactly in the way described, while the remaining can be written down directly from his results.

The equations (9.2) show that the quantities K_i and P_i can be looked upon as the partial differential coefficients of a *single* function with respect to the variables p_i and $-q_i$. Poisson does not make this remark which follows without anything further from the equation (9.2); still less does he try to find out that function. It is Hamilton, rather, who has first made this determination, and has greatly simplified the entire transformation through the introduction of his characteristic function. One would arrive at this almost immediately if one wished to derive the theorem of conservation of 'vis viva' from Lagrange's second form of the differential equations given in the preceding Lecture, a derivation which is not quite obvious. The theorem of kinetic energy is, when one considers the case in which time comes in explicitly in the force-function,

$$T = U - \int \frac{\partial U}{\partial t}\, dt + \text{ constant},$$

or on differentiation,

$$\frac{d(T-U)}{dt} + \frac{\partial U}{\partial t} = 0$$

(Lecture 5, p.44). For deriving this result from Lagrange's second form of the differential equations

$$\frac{dp_i}{dt} = \frac{\partial (T+U)}{\partial q_i}, \qquad p_i = \frac{\partial T}{\partial q_i'}$$

(contained in equation (8.10)), one proceeds in the following way. T is a homogeneous function of the second degree in the quantities q', so that one has, as is well known,

$$2T = q_1'\frac{\partial T}{\partial q_1'} + q_2'\frac{\partial T}{\partial q_2'} + \cdots + q_k'\frac{\partial T}{\partial q_k'} = \sum q_i' p_i,$$

or

$$T = \sum q_i' \frac{\partial T}{\partial q_i'} - T,$$

and hence, one obtains through total differentiation

$$dT = \sum q_i' d\frac{\partial T}{\partial q_i'} + \sum \frac{\partial T}{\partial q_i'} dq_i' - \sum \frac{\partial T}{\partial q_i'} dq_i' - \sum \frac{\partial T}{\partial q_i} dq_i,$$

or, since the second and third terms mutually cancel,

$$dT = \sum q_i' d\frac{\partial T}{\partial q_i'} - \sum \frac{\partial T}{\partial q_i} dq_i = \sum q_i' dp_i - \sum \frac{\partial T}{\partial q_i} dq_i, \tag{9.3}$$

which is an identity. If one introduces here for $d\frac{\partial T}{\partial q_i'} = dp_i$ its value from (8.10) of the previous lecture, and divides by dt, it gives

$$\begin{aligned} \frac{dT}{dt} &= \sum \frac{\partial (T+U)}{\partial q_i} q_i' - \sum \frac{\partial T}{\partial q_i}\frac{dq_i}{dt} \\ &= \sum \frac{\partial U}{\partial q_i} q_i' = \frac{dU}{dt} - \frac{\partial U}{\partial t}, \end{aligned}$$

and so we have

$$\frac{d(T-U)}{dt} + \frac{\partial U}{\partial t} = 0. \text{ q.e.d.}$$

The identity (9.3) leads easily to *Hamilton*'s characteristic function. Namely, the partial differential coefficients $\frac{\partial T}{\partial q_i}$ and $\frac{\partial T}{\partial q_i'} = p_i$ which appear on the right side of equation (9.3) (the differentials of the latter) are constructed when T is looked upon as a function of q and q'. But if we introduce the quantities p_i instead for the q_i' through the linear equations $q_i' = K_i$ already mentioned, then T will thereby be a function of the q and p and we shall for the sake of distinction denote by $\left(\frac{\partial U}{\partial p_i}\right)$ and $\left(\frac{\partial T}{\partial q_i}\right)$ the differential coefficients of T with respect to p_i and q_i constructed on this hypothesis. Then

$$dT = \sum \left(\frac{\partial T}{\partial p_i}\right) dp_i + \sum \left(\frac{\partial T}{\partial q_i}\right) dq_i,$$

and so from equation (9.3),

$$\sum \left(\frac{\partial T}{\partial p_i}\right) dp_i + \sum \left(\frac{\partial T}{\partial q_i}\right) dq_i = \sum q_i' dp_i - \sum \frac{\partial T}{\partial q_i} dq_i.$$

Since this equation is satisfied identically, it follows

$$\left(\frac{\partial T}{\partial p_i}\right) = q_i', \tag{9.4}$$

$$\left(\frac{\partial T}{\partial q_i}\right) = -\frac{\partial T}{\partial q_i}. \tag{9.5}$$

Equation (9.4) shows that there exists a sort of reciprocity between the quantities p and q'.For, in combination with the earlier relation $\frac{\partial T}{\partial q_i'} = p_i$, we obtain the equations

$$\frac{\partial T}{\partial q_i'} = p_i, \qquad \left(\frac{\partial T}{\partial p_i}\right) = q_i',$$

a correlation which is analogous to what comes in the theory of surfaces of second order. If we set the value of $\frac{\partial T}{\partial q_i}$ found in (9.5) in equation (8.10) of the previous lecture, then we have

$$\frac{dp_i}{dt} = -\left(\frac{\partial T}{\partial q_i}\right) + \frac{\partial U}{\partial q_i}.$$

Since U does not at all contain p, and q', so

$$\frac{\partial U}{\partial q_i} = \left(\frac{\partial U}{\partial q_i}\right), \text{ so } \frac{dp_i}{dt} = -\left(\frac{\partial (T-U)}{\partial q_i}\right).$$

Further, since U contains no p, one can write equation (9.4) also as

$$\frac{dq_i}{dt} = \left(\frac{\partial (T-U)}{\partial p_i}\right).$$

Then, if we set

$$T - U = H \tag{9.6}$$

we have

$$\frac{dq_i}{dt} = \left(\frac{\partial H}{\partial p_i}\right), \qquad \frac{dp_i}{dt} = -\left(\frac{\partial H}{\partial q_i}\right), \tag{9.7}$$

from which one sees that $H = T - U$ is the characteristic function. The theorem of *vis viva* is obtained automatically from these equations; for from the two equations (9.7) follows

$$\left(\frac{\partial H}{\partial p_i}\right)\frac{dp_i}{dt} + \left(\frac{\partial H}{\partial q_i}\right)\frac{dq_i}{dt} = 0,$$

and this summed over all i gives

$$\frac{dH}{dt} - \frac{\partial H}{\partial t} = 0,$$

i.e., the theorem of kinetic energy.

Since it is self-evident that in the equations (9.7) the quantities p and q are to be looked upon as independent variables, one can delete the brackets around the differential coefficients and obtain

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad H = T - U. \tag{9.8}$$

In the more general case where no force-function exists, in place of $\frac{\partial U}{\partial q_i}$ there enters the expression

$$Q = \sum \left(X\frac{\partial x}{\partial q_i} + Y\frac{\partial y}{\partial q_i} + Z\frac{\partial z}{\partial q_i} \right),$$

where the sum is extended over all x, y, z, and there occurs in place of (9.8) the following:

$$\frac{dq_i}{dt} = \frac{\partial T}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial T}{\partial q_i} + Q_i.$$

When there are no equations of constraint, the quantities q coincide with the coordinates; the first of the equation (9.8) is an identity, the second goes over into the system

$$m_i\frac{d^2x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \qquad m_i\frac{d^2y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \qquad m_i\frac{d^2z_i}{dt^2} = \frac{\partial U}{\partial z_i},$$

which is the original form of the equations of motion.

Lecture 10

The Principle of the Last Multiplier

Extension of Euler's multipliers to three variables. Setting up the last multiplier for this case.

The principle of the last multiplier accomplishes, in all cases where the integration of a system of differential equations of motion is reduced to a first order differential equation of two variables, the integration of this last equation by giving its multipliers. Here it is assumed that the applied forces X_i, Y_i, Z_i depend only on the coordinates and the time.

If we introduce the derivatives $\frac{dx_i}{dt}, \frac{dy_i}{dt}, \frac{dz_i}{dt}$, as new variables x_i', y_i', z_i', in the original system of differential equations of motion, then they take the following form:

$$\begin{aligned}
m_i \frac{dx_i'}{dt} &= X_i + \lambda \frac{\partial f}{\partial x_i} + \mu \frac{\partial \widetilde{\omega}}{\partial x_i} + \cdots, & \frac{dx_i}{dt} &= x_i' \\
m_i \frac{dy_i'}{dt} &= Y_i + \lambda \frac{\partial f}{\partial y_i} + \mu \frac{\partial \widetilde{\omega}}{\partial y_i} + \cdots, & \frac{dy_i}{dt} &= y_i' \\
m_i \frac{dz_i'}{dt} &= Z_i + \lambda \frac{\partial f}{\partial z_i} + \mu \frac{\partial \widetilde{\omega}}{\partial z_i} + \cdots, & \frac{dz_i}{dt} &= z_i'.
\end{aligned}$$

These are $6n$ differential equations, but between the $6n$ variables $x_i, y_i, z_i,\ x_i', y_i', z_i'$ depending on t ocurring in them, there already exist $2m$ relations, namely

$$\begin{gathered}
f = 0, \qquad \widetilde{\omega} = 0, \ldots, \\
\sum \left(\frac{\partial f}{\partial x_i} x_i' + \frac{\partial f}{\partial y_i} y_i' + \frac{\partial f}{\partial z_i} z_i' \right) = 0, \\
\sum \left(\frac{\partial \widetilde{\omega}}{\partial x_i} x_i' + \frac{\partial \widetilde{\omega}}{\partial y_i} y_i' + \frac{\partial \widetilde{\omega}}{\partial z_i} z_i' \right) = 0, \ldots
\end{gathered}$$

When t occurs explicitly in $f, \widetilde{\omega}, \ldots$, the terms $\frac{\partial f}{\partial t}, \frac{\partial \widetilde{\omega}}{\partial t}, \ldots$, are to be introduced respectively in addition on the left sides of the last m equations. Therefore one has to still find $6n - 2m$ integral equations.

We first assume that t occurs explicitly neither in X_i, Y_i, Z_i nor in $f, \widetilde{\omega}, \ldots$. Then by means of one of the $6n$ equations, say, the equation $\frac{dx_1}{dt} = x_1'$ or $dt = \frac{dx_1}{x_1'}$, one can eliminate the time from the remaining, and then one has a system of $6n-1$ differential equations, the complete integration of which requires $6n-2m-1$ integrals. If these integrations are assumed to be carried out, one can express the $6n$ quantities $x_i, y_i, z_i, x_i', y_i', z_i', \ldots$, through one of them, for example x_1. If in this way we think of x_1' as expressed as a function of x_1, then the equation $dt = \frac{dx_1}{x_1'}$ gives an integration

$$t + \text{ constant } = \int \frac{dx_1}{x_1'};$$

when the time does not occur explicitly, then the last integration is reduced to a simple quadrature, and the time is then always associated with an arbitrary constant through addition. This occurs, for instance, in the elliptic motion of planets. However, if we assume that the system of $6n - 1$ differential equations, which in obtained on elimination of the time, is not completely integrated, but one integration is missed, then one has not found $6n-2m-1$ integrals, but only $6n-2m-2$; then one cannot express all the variables through a single one, x_1 for example, but can through two, x_1 and y_1 for example. In this case there remains one differential equation between x_1 and y_1 to be integrated; namely, if one eliminates the differential of the time from $\frac{dy_1}{dt} = y_1'$ through $dt = \frac{dx_1}{x_1'}$, then one has

$$dx_1 : dy_1 = x_1' : y_1',$$

where x_1' and y_1' are, according to our assumption, functions of x_1 and y_1. Now, for this differential equation, the principle set up by me gives the multiplier. After one has integrated it with its help one finds, as remarked above, the time through a simple quadrature. So, when time does not occur explicitly, one needs to perform only $6n - 2m - 2$ integrations in order to obtain the last two without any further device.

When, however, the time occurs explicitly, not merely as its differential, then it cannot be eliminated from the differential equations. If, however, then $6n - 2m - 1$ integrations can be carried out, by which everything is reduced to the integration of a differential equation of the form

$$dx_1 - x_1'dt = 0,$$

where x_1' is a function of x_1 and t; then again one obtains the last integral through the principle of the last multiplier.

After we have seen what the principle under discussion achieves we proceed to its derivation.

Though *Euler* had already seen in many examples that one can transform first order differential equations of two variables into perfect differentials through multipliers and then integrate them, it took him very long to arrive at the insight that this must be a general property of these differential equations. This was because, at that time, for him the idea of solving the integral equation for the arbitrary constants lay far off. If he had been familiar with this idea, he would not have despaired of reducing a linear partial differential equation to an ordinary one, a problem which he held to be more difficult than that of integrating a second order differential equation of two variables, which has not been solved even today. On the other hand the reduction of linear partial differential equations to ordinary ones is now considered elementary. *Euler* had also never extended the theory of multipliers to a system of differential equations, although the procedure for this case is just as simple, if one thinks of the integral equations for the arbitrary constants as solved.

Let us first consider a differential equation of two variables x and y, and indeed let it be given in the form of a proportion

$$dx : dy = X : Y,$$

which, is identical with the equation

$$X\,dy - Y\,dx = 0.$$

If one considers the integral brought to the form F = constant, one obtains by differentiation the equation

$$\frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial x}\,dx = 0,$$

of which the left side can differ from the left side of the previous differential equation only by a factor M; so one has

$$MX = \frac{\partial F}{\partial y}, \qquad -MY = \frac{\partial F}{\partial x},$$

and this gives the equation for determining M

$$\frac{\partial(MX)}{\partial x} + \frac{\partial(MY)}{\partial y} = 0. \tag{10.1}$$

Let us extend the theory of these multipliers M to a system of two simultaneous differential equations of three variables. Let it be displayed the following form:

$$dx : dy : dz = X : Y : Z; \tag{10.2}$$

Let the integral equations, solved for the arbitrary constants, be

$$f = \alpha, \qquad \phi = \beta; \tag{10.3}$$

then one has

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0, \quad \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0,$$

and hence it follows

$$\begin{aligned} dx : dy : dz = \left(\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial \phi}{\partial y}\right) \ &: \ \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial z}\right) \\ &: \ \left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x}\right). \end{aligned}$$

If one sets

$$\begin{aligned} A &= \left(\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial \phi}{\partial y}\right), \\ B &= \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial z}\right), \\ C &= \left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x}\right), \end{aligned}$$

then

$$dx : dy : dz = A : B : C,$$

which, with the given system of equations(10.2) leads to the proportion

$$A : B : C = X : Y : Z.$$

Therefore, there exists a multiplier M with the property

$$A = MX, \quad B = MY, \quad C = MZ.$$

But A, B, C satisfy the relation

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0;$$

one has, therefore, for M the equation

$$\frac{\partial(MX)}{\partial x} + \frac{\partial(MY)}{\partial y} + \frac{\partial(MZ)}{\partial z} = 0,$$

or

$$X\frac{\partial M}{\partial x} + Y\frac{\partial M}{\partial y} + Z\frac{\partial M}{\partial z} + \left\{\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right\} M = 0. \tag{10.4}$$

Since $f = \alpha$ and $\phi = \beta$ are integrals of the system (10.2), so by virtue of this df and $d\phi$ must vanish identically, without the help of the integral equations. However,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz, \quad d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz,$$

and consequently one obtains by means of the system of equations (10.2),

$$\begin{aligned} X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z} &= 0, \\ X\frac{\partial \phi}{\partial x} + Y\frac{\partial \phi}{\partial y} + Z\frac{\partial \phi}{\partial z} &= 0, \end{aligned} \tag{10.5}$$

which are to be looked upon as the defining equaitons of the integrals of the system (10.2).

One can hence prove that any function of f and ϕ set equal to a constant is indeed an integral of the system (10.2). In fact, if $\widetilde{\omega}$ is any function of f and ϕ, one multiplies the equations (10.5) by $\frac{\partial \widetilde{\omega}}{\partial f}$ and $\frac{\partial \widetilde{\omega}}{\partial \phi}$ and adds, then one obtains

$$\begin{aligned} X\left(\frac{\partial \widetilde{\omega}}{\partial f}\frac{\partial f}{\partial x} + \frac{\partial \widetilde{\omega}}{\partial \phi}\frac{\partial \phi}{\partial x}\right) &+ Y\left(\frac{\partial \widetilde{\omega}}{\partial f}\frac{\partial f}{\partial y} + \frac{\partial \widetilde{\omega}}{\partial \phi}\frac{\partial \phi}{\partial y}\right) \\ &+ Z\left(\frac{\partial \widetilde{\omega}}{\partial f}\frac{\partial f}{\partial z} + \frac{\partial \widetilde{\omega}}{\partial \phi}\frac{\partial \phi}{\partial z}\right) = 0, \end{aligned}$$

or

$$X\frac{\partial \widetilde{\omega}}{\partial x} + Y\frac{\partial \widetilde{\omega}}{\partial y} + Z\frac{\partial \widetilde{\omega}}{\partial z} = 0; \tag{10.6}$$

so $\widetilde{\omega}$ is an integral of (10.2). Conversely, any integral of (10.2) is a function of f and ϕ. For, if it be assumed that there exist an integral $\widetilde{\omega} = r$ which is not a function of f and ϕ, then equation (10.6) holds for $\widetilde{\omega}$. Now let ω be an arbitrary function of f, ϕ and $\widetilde{\omega}$. Then one multiplies the equaiton (10.5) and (10.6) by $\frac{\partial\omega}{\partial f}$, $\frac{\partial\omega}{\partial\phi}$ and $\frac{\partial\omega}{\partial\widetilde{\omega}}$ respectively and adds and so obtains

$$X\frac{\partial\omega}{\partial x}+Y\frac{\partial\omega}{\partial y}+Z\frac{\partial\omega}{\partial z}=0;$$

consequently, ω is also an integral of equation (10.2). However ω is an entirely arbitrary function of f, ϕ and $\widetilde{\omega}$ and these are mutually independent. Therefore, one can introduce f, ϕ, $\widetilde{\omega}$ as new variables in place of the original variables x, y, z and express these original variables through f, ϕ, $\widetilde{\omega}$. So one can represent any function of x, y, z as a function of f, ϕ, $\widetilde{\omega}$, and an arbitrary function of f, ϕ, $\widetilde{\omega}$ is equivalent to an arbitrary function of x, y, z. So one can set any function of x, y, z for ω, i.e., any function of x, y, z set equal to a constant is an integral of the system (10.2), which is impossible. So there can be only two mutually independent integrals of the system (10.2), and any third is a function of the two mutually independent f and ϕ.

One can use this result to derive from *one* value of the multiplier M all others. Let N be a second value of this multiplier, so

$$X\frac{\partial M}{\partial x}+Y\frac{\partial M}{\partial y}+Z\frac{\partial M}{\partial z}+\left\{\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right\}M=0,$$
$$X\frac{\partial N}{\partial x}+Y\frac{\partial N}{\partial y}+Z\frac{\partial N}{\partial z}+\left\{\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right\}N=0.$$

If one multiplies the second of these equations by M, the first by N and takes the difference of the results, one obtains

$$\begin{aligned}0 \;=\;& X\left\{M\frac{\partial N}{\partial x}-N\frac{\partial M}{\partial x}\right\}+Y\left\{M\frac{\partial N}{\partial y}-N\frac{\partial M}{\partial y}\right\}\\ &+Z\left\{M\frac{\partial N}{\partial z}-N\frac{\partial M}{\partial z}\right\},\end{aligned}$$

or, if one divides by M^2,

$$0=X\frac{\partial(N/M)}{\partial x}+Y\frac{\partial(N/M)}{\partial y}+Z\frac{\partial(N/M)}{\partial z}.$$

Then $\frac{N}{M}=$ constant is an integral of the system (10.2), and thereby $\frac{N}{M}$ is a function of f and ϕ, or

$$N=MF(f,\phi), \qquad (10.7)$$

i.e., *if M is one value of the multiplier, all other values are of the form* $MF(f,\phi)$. However, as has been assumed, $f=\alpha$ and $\phi=\beta$ are integrals of (10.2) and so will $F(f,\phi)=$ constant, i.e., if one takes help of the integral equations, all the different values of the multiplier differ from one another by a constant factor.

We shall now see what advantage the knowledge of one value of M affords; hereby one does not find the integral itself, as in the case of a differential equation of two variables, but one finds by means of the equations $A=MX, B=MY, C=MZ$, the values of the quantities

$$A=\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial y}, B=\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial x}-\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial z}, C=\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial x}.$$

The advantage one can derive from these arises only when one already knows one integral, e.g., ϕ, and seeks another, f. One introduces in place of one of the variables, e.g., z, the expression ϕ, so that z is represented as a function of ϕ, x and y. We shall accordingly think of the required integral f expressed through x, y, ϕ, and shall denote the partial differential coefficients constructed on this hypothesis by $\left(\frac{\partial f}{\partial x}\right), \left(\frac{\partial f}{\partial y}\right), \left(\frac{\partial f}{\partial z}\right)$; then we have

$$\frac{\partial f}{\partial x}=\left(\frac{\partial f}{\partial x}\right)+\left(\frac{\partial f}{\partial \phi}\right)\left(\frac{\partial \phi}{\partial x}\right),$$
$$\frac{\partial f}{\partial y}=\left(\frac{\partial f}{\partial y}\right)+\left(\frac{\partial f}{\partial \phi}\right)\left(\frac{\partial \phi}{\partial y}\right),$$
$$\frac{\partial f}{\partial z}=\left(\frac{\partial f}{\partial z}\right)+\left(\frac{\partial f}{\partial \phi}\right)\left(\frac{\partial \phi}{\partial z}\right),$$

and obtain for the quantities A, B, C, the expressions

$$A=\left(\frac{\partial f}{\partial y}\right)\frac{\partial \phi}{\partial z}, \quad B=-\left(\frac{\partial f}{\partial x}\right)\frac{\partial \phi}{\partial z}, \quad C=\left(\frac{\partial f}{\partial x}\right)\frac{\partial \phi}{\partial y}-\left(\frac{\partial f}{\partial y}\right)\frac{\partial \phi}{\partial x}.$$

From these it follows that if one knows the integral $\phi=\beta$ and one value of the multiplicator M, one can determine f. Indeed, if one thinks of f expressed through x, y and $\phi=\beta$, then

$$df=\left(\frac{\partial f}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}\right)dy+\left(\frac{\partial f}{\partial \phi}\right)d\phi$$

or, since $d\phi=0$,

$$df=\left(\frac{\partial f}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}\right)dy.$$

But one has, from the above equations for A and B,

$$\left(\frac{\partial f}{\partial y}\right) = \frac{A}{\frac{\partial \phi}{\partial z}}, \quad \left(\frac{\partial f}{\partial x}\right) = -\frac{B}{\frac{\partial \phi}{\partial z}},$$

and therefore,

$$df = \frac{A\,dy - B\,dx}{\frac{\partial \phi}{\partial z}}.$$

Since now $A = MX$ and $B = MY$, so

$$df = \frac{M}{\frac{\partial \phi}{\partial z}}(X\,dy - Y\,dx), \tag{10.8}$$

and this gives

$$\int \frac{M}{\frac{\partial \phi}{\partial z}}(X\,dy - Y\,dy) = f = \alpha$$

as the second integral of the system (10.2). Here one must assume X, Y as functions of x, y, z, expressed through x, y and $\phi = \beta$ as given. Under this assumption, as we see from (10.8), $M/\frac{\partial \phi}{\partial z}$ is the integrating factor of the differential equation $X\,dy - Y\,dx = 0$. Hence, we have the following theorem:

Theorem 10.1 *If the system of differential equations*

$$dx : dy : dz = X : Y : Z$$

is given and one knows, first of all, an integral $\phi = \beta$ of the same, and, secondly, a value of the multiplier M of the system, which satisfies the partial differential equation

$$X\frac{\partial M}{\partial x} + Y\frac{\partial M}{\partial y} + Z\frac{\partial M}{\partial z} + \left\{\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right\} M = 0,$$

then $M/\frac{\partial \phi}{\partial z}$ is an integrating factor of the differential equation

$$X\,dy - Y\,dy = 0,$$

it being assumed that from the given factor as well as from X and Y, the variable z has been eliminated by virtue of the integral $\phi = \beta$ already found.

One might regard this theorem as fruitless. Because, while for the knowledge of the second integral f the solution of the partial differential equation

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z} = 0$$

is required, to determine M and thence to find the second integral f, we have to solve the very complicated differential equation

$$X\frac{\partial M}{\partial x} + Y\frac{\partial M}{\partial y} + Z\frac{\partial M}{\partial z} + \left\{\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right\} M = 0. \qquad (10.9)$$

It appears that an easier problem has been turned into a more difficult one. However, here a peculiar situation arises. The partial differential equation which defines f,

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z} = 0,$$

also has the solution f = constant, but this obvious solution does not give an integral of the given system and must therefore be excluded. Such an exclusion of a solution is not necessary with the multiplier M; and if, for example, M set equal to a constant gives a solution of the equation (10.9), this value of M can be thought as much a multiplier as any other. The case where one can set M = constant occurs if

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0; \qquad (10.10)$$

then equation (10.9) reduces to

$$X\frac{\partial M}{\partial x} + Y\frac{\partial M}{\partial y} + Z\frac{\partial M}{\partial z} = 0;$$

one can then set M = constant, 1, for example, and we then have the following theorem:

Theorem 10.2 *If, in the system of differential equations*

$$dx : dy : dz = X : Y : Z,$$

X, Y, Z *are functions of* x, y, z, *which satisfy the condition*

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0;$$

if, further, one knows an integral $\phi = \beta$ of the system, by means of this equation expresses z through the quantities x, y, β and substitutes the value found in X, Y, $\frac{\partial\phi}{\partial z}$; then

$$\frac{1}{\frac{\partial\phi}{\partial z}}(X\,dy - Y\,dx) = f$$

is a perfect differential, and one finds the second integral $f = \alpha$ of the system through a mere quadrature.

There is yet a second and more general case to be mentioned, which includes the one just stated, and in which M can be similarly determined in general. If one introduces into the equation (10.4) holding for M, after one has brought it by division by MX to the form

$$\frac{1}{M}\left(\frac{\partial M}{\partial x} + \frac{Y}{X}\frac{\partial M}{\partial y} + \frac{Z}{X}\frac{\partial M}{\partial z}\right) + \frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) = 0,$$

the values following from the given system (10.2):

$$\frac{Y}{X} = \frac{dy}{dx}, \qquad \frac{Z}{X} = \frac{dz}{dx},$$

then one obtains

$$\frac{1}{M}\left(\frac{\partial M}{\partial x} + \frac{\partial M}{\partial y}\frac{dy}{dx} + \frac{\partial M}{\partial z}\frac{dz}{dx}\right) + \frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) = 0,$$

or

$$\frac{1}{M}\frac{dM}{dx} + \frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) = 0,$$

or, finally,

$$\frac{d\log M}{dx} + \frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) = 0. \qquad (10.11)$$

If now $\frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)$ is a complete differential coefficient with respect to x, so of the form $\frac{d\xi}{dx}$, then one has

$$\frac{d\log M}{dx} + \frac{d\xi}{dx} = 0,$$
$$M = Ce^{-\xi}.$$

Hence, one obtains the following theorem:

Theorem 10.3 *Let the given system be* $dx : dy : dz = X : Y : Z$ *and let, further, the expression*

$$\frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)$$

be equal to $\frac{d\xi}{dx}$, *i.e., equal to some complete differential coefficient with respect to* x; *finally, let* $\phi = \beta$ *be a known integral of the system; then*

$$\frac{e^{-\xi}}{\frac{\partial \phi}{\partial z}}(X\,dy - Y\,dx)$$

is a complete differential, it being assumed here that by virtue of the integral $\phi = \beta$ *everything is expressed in terms of* x *and* y.

One can also express the result as follows. Both the variables of the differential expression whose integrating factor is given, are not x and y, but x and z or y and z.

We shall give an example of these theorems. First, let there be an ordinary differential equation of the second order to be integrated, viz,

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) = u.$$

If one introduces a new variable $z = \frac{dy}{dx}$, then one has two equations

$$\frac{dy}{dx} = z, \qquad \frac{dz}{dx} = u,$$

so

$$dx : dy : dz = 1 : z : u;$$

then in our earlier notation,

$$X = 1, \qquad Y = z, \qquad Z = u.$$

In order to be able to apply the first of the two theorems stated, one must have

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0;$$

in the present case, $\frac{\partial X}{\partial x} = 0$, $\frac{\partial Y}{\partial y} = 0$, $\frac{\partial Z}{\partial z} = \frac{\partial u}{\partial z}$; so one has the condition

$$\frac{\partial u}{\partial z} = 0;$$

i.e., in u, z, or, what is the same, $\frac{dy}{dx}$, should not occur. If one makes this assumption, one has the theorem:

Theorem 10.4 *Let the differential equation to be integrated be*

$$\frac{d^2y}{dx^2} = f(x, y),$$

where f does not contain $\frac{dy}{dx}$; if one knows a first integral

$$\phi\left(x, y, \frac{dy}{dx}\right) = \alpha,$$

which is solved for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \psi(x, y, \alpha),$$

or

$$dy - \psi(x, y, \alpha)\ dx = 0;$$

then

$$\frac{1}{\frac{\partial\phi}{\partial\frac{dy}{dx}}}$$

expressed in x, y and α is the integrating factor of this differential equation.

The calculus of variation gives an example of the second theorem. The simplest problem of that calculus is that in which the integral

$$\int \psi\left(x, y, \frac{dy}{dx}\right)\ dx$$

is to be a *minimum* or a *maximum*. This exercise leads to the differential equation

$$\frac{d\frac{\partial\psi}{\partial y'}}{dx} = \frac{\partial\psi}{\partial y}, \qquad y' = \frac{dy}{dx}.$$

The first of these gives on expansion

$$\frac{\partial^2\psi}{\partial x\partial y'} + \frac{\partial^2\psi}{\partial y\partial y'}y' + \frac{\partial^2\psi}{\partial y'^2}\frac{dy'}{dx} = \frac{\partial\psi}{\partial y};$$

one has then

$$\frac{dy'}{dx} = \frac{\frac{\partial\psi}{\partial y} - \frac{\partial^2\psi}{\partial x\partial y'} - \frac{\partial^2\psi}{\partial y\partial y'}y'}{\frac{\partial^2\psi}{\partial y'^2}} = u;$$

or if one sets for brevity

$$v = \frac{\partial \psi}{\partial y} - \frac{\partial^2 \psi}{\partial x \partial y'} - \frac{\partial^2 \psi}{\partial y \partial y'} y' = \frac{\partial^2 \psi}{\partial y'^2} \frac{dy'}{dx},$$

then

$$\frac{dy'}{dx} = \frac{v}{\frac{\partial^2 \psi}{\partial y'^2}} = u.$$

Besides, now $\frac{dy}{dx} = y'$, so that one has

$$dx : dy : dy' = 1 : y' : u.$$

Here y' enters in place of the variable which was denoted above by z, and also

$$X = 1, Y = y', Z = u.$$

In order that the second theorem may find an application, the expression

$$\frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)$$

must be a complete differential in x; in the present case this is equal to $\frac{\partial u}{\partial y'}$, and the question is whether $\frac{\partial u}{\partial y'}$ admits representation as a complete differential coefficient. We have

$$u = \frac{v}{\frac{\partial^2 \psi}{\partial y'^2}},$$

and so

$$\frac{\partial u}{\partial y'} = \frac{\frac{\partial^2 \psi}{\partial y'^2} \cdot \frac{\partial v}{\partial y'} - \frac{\partial^3 \psi}{\partial y'^3} v}{\left(\frac{\partial^2 \psi}{\partial y'^2}\right)^2}.$$

But at the same time

$$\begin{aligned} \frac{\partial v}{\partial y'} &= \frac{\partial^2 \psi}{\partial y \partial y'} - \frac{\partial^3 \psi}{\partial x \partial y'^2} - \frac{\partial^3 \psi}{\partial y \partial y'^2} y' - \frac{\partial^2 \psi}{\partial y \partial y'} \\ &= -\left(\frac{\partial}{\partial x}\frac{\partial^2 \psi}{\partial y'^2} + \frac{\partial}{\partial y}\frac{\partial^2 \psi}{\partial y'^2}\frac{dy}{dx}\right); \end{aligned}$$

and as a consequence of the equation

$$v = \frac{\partial^2 \psi}{\partial y'^2} \frac{dy'}{dx},$$

$\frac{\partial^2\psi}{\partial y'^2}$ cancels out in the numerator and denominator of $\frac{\partial u}{\partial y'}$, and so one has

$$\frac{\partial u}{\partial y'} = -\frac{\frac{\partial}{\partial x}\frac{\partial^2\psi}{\partial y'^2} + \frac{\partial}{\partial y}\frac{\partial^2\psi}{\partial y'^2}\frac{dy}{dx} + \frac{\partial}{\partial y'}\frac{\partial^2\psi}{\partial y'^2}\frac{dy'}{dx}}{\frac{\partial^2\psi}{\partial y'^2}},$$

or

$$\frac{\partial u}{\partial y'} = -\frac{\frac{d}{dx}\frac{\partial^2\psi}{\partial y'^2}}{\frac{\partial^2\psi}{\partial y'^2}} = -\frac{d\log\frac{\partial^2\psi}{\partial y'^2}}{dx}.$$

Then $\frac{\partial u}{\partial y'}$ is in fact a complete differential coefficient in x, and according to (10.11),

$$\frac{d\log M}{dx} = \frac{d\log\frac{\partial^2\psi}{\partial y'^2}}{dx},$$

$$M = C\frac{\partial^2\psi}{\partial y'^2}.$$

One has accordingly a theorem which holds for all problems in the calculus of variations in which the integral $\int \psi(x,y,y')\,dx$ must be a maximum or a minimum. In order that this condition must be fulfilled there should hold between x and y a differential equation of the second order

$$\frac{d\frac{\partial\psi}{\partial y'}}{dx} = \frac{\partial\psi}{\partial y}$$

which possesses the following property: *if one knows a first integral, $\phi(x,y,\frac{dy}{dx}) = \alpha$, and brings it to the form $dy - F(x,y,\alpha)\,dx = 0$, then*

$$\frac{1}{\frac{\partial\psi}{\partial\frac{dy}{dx}}}\cdot\frac{\partial^2\psi}{\partial y'^2},$$

expressed in terms of x, y and α is an integrating factor of this differential equation.

To this category of problems of maxima and minima belongs, for example, the determination of the shortest line on a given surface. This problem leads to a differential equation of the second order; if one knows an integral of the same, then the multiplier of the differential equation of the first order yet to be integrated is determined.

What has up to now been said about the simplest case of the calculus of variations admits the most general extension in which stands under the integral sign a function which contains arbitrarily many variables

$y, z, u, \ldots$ depending on a variable x and the differential coefficients of each up to an arbitrary high order. If such a problem can be reduced to a first order differential equation of two variables, the final integration can always be carried out. But in order to arrive at this result, it is necessary to introduce a theorem on the expressions which occurs in the solution of linear equations, and which have been called resultants by *Laplace*, determinants by *Gauss* and alternating functions by *Cauchy*.

Lecture 11

Survey of those properties of determinants that are used in the theory of the last multiplier

If one sets

$$\begin{aligned}P = {} & (a_2 - a_1)(a_3 - a_1)\cdots(a_s - a_1)\cdots(a_n - a_1)\\ & (a_3 - a_2)\cdots(a_s - a_2)\cdots(a_n - a_2)\\ & \cdots\\ & \cdots\\ & (a_n - a_{n-1}),\end{aligned}$$

then the product P so defined has the property that through a permutation of the quantities $a_1, a_2, \ldots, a_n$, or what is the same, of the indices $1, 2, \ldots, n$, it changes only its sign and not its absolute value. Regarding these permutations, only the following will be referred to.

Let us denote the indices $1, 2, \ldots, n$ after changing their order in an entirely arbitrary manner by $i_1, i_2, \ldots, i_n$, and the permutation by which

$$1, 2, 3, \ldots, s, \ldots, n$$

goes over to

$$i_1, i_2, \ldots, i_s, \ldots, i_n$$

by J. Howsoever may the permutation J be carried out, one can always separate the indices $1, \ldots, n$ into certain groups of such a nature that, through the permutation J, all the indices which belong to a group permute among themselves, or go over as a whole to another group, so that in any case the indices which belong to a group remain together. With respect to these groups one can classify the permutations, so that for certain of these all groups go over into themselves, for others a definite group of indices goes over into a second, and so on. This by no means exhausted subject is one of the most important in algebra. In all cases where the solution of equations has been possible, the reason is to be sought here.

The most important of these classifications of permutations is that into positive and negative permutations,in the former P remains unchanged,in the latter P changes into $-P$. To the second class, for example, belongs the simplest case in which one exchanges only two indices i, i'. One sees this immediately if one brings P to the form

$$P = \pm(a_i - a_{i'}) \prod (a_i - a_k) \prod (a_{i'} - a_k) \prod (a_k - a_{k'}),$$

where k denotes all indices different from i and i', and k and k' all combinations of pairs of indices different from i and i', whereby the exchange of the two occuring in the same difference is excluded. To decide whether a permutation

$$\left.\begin{matrix} 1, 2, \ldots, n \\ i_1, i_2, \ldots, i_n \end{matrix}\right\} \tag{J}$$

is positive or negative, one compares the series at each i with the succeeding members. If μ is the number of those cases in which a larger i stands before a succeeding smaller, then J is a positive or negative permutation according as μ is even or odd; or, simply, J is positive or negative according as one obtains the permutations $i_1, \ldots, i_n$ from $1, \ldots, n$, by an even or an odd number of interchanges of two elements.

In order to pass to determinants from what has gone so far, one considers the n^2 quantities

$$\begin{matrix} a_1, b_1, c_1, \ldots, p_1, \\ a_2, b_2, c_2, \ldots, p_2, \\ \ldots \\ a_n, b_n, c_n, \ldots, p_n. \end{matrix}$$

One forms the product $a_1 b_2 c_3 \cdots p_n$ and permutes the indices in it in all possible ways, gives each of the resulting products a plus or a minus sign according as the permutation is positive or negative, and sums all these products with the sign associated with them. The expression resulting thereby:

$$R = \sum \pm a_1 b_2 c_3 \cdots p_n,$$

where the double sign must have the meaning given above, is the determinant of the n^2 quantities $a_1, \ldots, p_n$, and these n^2 quantities will be called the elements of the determinant$R$. One can think of R as arising from the development of P in such a way that in any term, that particular a which does not come into it may be introduced as a factor raised to the 0th power, and then for any value of the index i, places $a_i, b_i, c_i, \ldots, p_i$ respectively in place of the powers $a_i^0, a_i^1, a_i^2, \ldots, a_i^{n-1}$. The determinant R has the following fundamental properties:

1. If one permutes any two indices i and k, or any two letters, for example, a and b, with each other, then R changes to $-R$. It follows then that whenever two rows of quantities coincide with each other, so

$$a_i = a_k, b_i = b_k, \ldots, p_i = p_k,$$

or

$$g_1 = h_1, g_2 = h_2, \ldots, g_n = h_n,$$

then the determinant vanishes.

2. The determinant is homogeneous and linear with respect to the quantities standing in a row , so with respect to the quantities $a_i, b_i, \ldots, p_i$, and also the quantities $g_1, g_2, \ldots, g_n$. Therefore one has

$$R = \frac{\partial R}{\partial a_i} a_i + \frac{\partial R}{\partial b_i} b_i + \cdots + \frac{\partial R}{\partial p_i} p_i,$$
$$R = \frac{\partial R}{\partial g_1} g_1 + \frac{\partial R}{\partial g_2} b_2 + \cdots + \frac{\partial R}{\partial g_n} g_n.$$

If we set

$$\frac{\partial R}{\partial a_i} = A_i, \frac{\partial R}{\partial b_i} = B_i, \ldots, \frac{\partial R}{\partial p_i} = P_i,$$

then

$$R = A_i a_i + B_i b_i + \cdots + P_i p_i,$$

and even so

$$R = A_k a_k + B_k a_k + \cdots, + P_k p_k.$$

But R goes over to $-R$ through exchange of the indices i and k, so, as is evident from this, A_i into $-A_k$, B_i into $-B_k$, and so on; with it, the term in A_i multiplied by b_k into the term in $-A_k$ multiplied by b_i, i.e., in R, $a_i b_k$ and $a_k b_i$ have opposite factors, that is

$$\frac{\partial^2 R}{\partial a_i \partial b_k} = -\frac{\partial^2 R}{\partial a_k \partial b_i}.$$

Similarly, one has for the three indices i, k, l,

$$\frac{\partial^3 R}{\partial a_i \partial b_k \partial c_l} = \frac{\partial^3 R}{\partial a_k \partial b_l \partial c_i} = \frac{\partial^3 R}{\partial a_l \partial b_i \partial c_k} = -\frac{\partial^3 R}{\partial a_l \partial b_k \partial c_i}$$
$$= -\frac{\partial^3 R}{\partial a_k \partial b_i \partial c_l} = -\frac{\partial^3 R}{\partial a_i \partial b_l \partial c_k};$$

and from this follow the representations for R:

$$R = \sum\sum(a_i b_k - a_k b_i)\frac{\partial^2 R}{\partial a_i \partial b_k};$$
$$R = \sum\sum\sum \{a_i(b_k c_l - b_l c_k) + a_k(b_l c_i - b_i c_l) + a_l(b_i c_k - b_k c_i)\} \frac{\partial^3 R}{\partial a_i \partial b_k \partial c_l},$$

where the summation is to be extended over all combinations of the indices $1, 2 \ldots, n$ in pairs and triples. This representation of a determinant through products of determinants of lower order appears first in a treatise of *Laplace* on the planetary system in the Paris Memoir of 1772. *Laplace* and *Cramer* in Geneva above all are the first who have investigated the properties of determinants properly.

3. The equation introduced above,

$$R = g_1\frac{\partial R}{\partial g_1} + g_2\frac{\partial R}{\partial g_2} + \cdots + g_n\frac{\partial R}{\partial g_n},$$

gives, when one writes a for g,

$$R = a_1\frac{\partial R}{\partial a_1} + a_2\frac{\partial R}{\partial a_2} + \cdots + a_n\frac{\partial R}{\partial a_n}.$$

To these equations are added $n-1$ others, which can be proved through this, that R must vanish identically when one sets two rows of quantities mutually equal; they give

$$0 = b_1\frac{\partial R}{\partial a_1} + b_2\frac{\partial R}{\partial a_2} + \cdots + b_n\frac{\partial R}{\partial a_n}$$
$$0 = c_1\frac{\partial R}{\partial a_1} + c_2\frac{\partial R}{\partial a_2} + \cdots + c_n\frac{\partial R}{\partial a_n}$$
$$\cdots$$
$$0 = p_1\frac{\partial R}{\partial a_1} + p_2\frac{\partial R}{\partial a_2} + \cdots + p_n\frac{\partial R}{\partial a_n}.$$

The solution of linear equations rests on these formulae. If one has the system

$$\begin{aligned} a_1x_1 + b_1x_2 + \cdots + p_1x_n &= y_1,\\ a_2x_1 + b_2x_2 + \cdots + p_2x_n &= y_2,\\ &\vdots\\ a_nx_1 + b_nx_2 + \cdots + p_nx_n &= y_n, \end{aligned}$$

and multiplies these equations respectively by $\frac{\partial R}{\partial a_1}, \frac{\partial R}{\partial a_2}, \ldots, \frac{\partial R}{\partial a_n}$, by $\frac{\partial R}{\partial b_1}, \frac{\partial R}{\partial b_2}, \ldots, \frac{\partial R}{\partial b_n}$ etc, where R has the meaning given above:

$$R = \sum \pm a_1 b_2 \cdots p_n,$$

then one has

$$\begin{aligned} Rx_1 &= \frac{\partial R}{\partial a_1}y_1 + \frac{\partial R}{\partial a_2}y_2 + \cdots + \frac{\partial R}{\partial a_n}y_n,\\ Rx_2 &= \frac{\partial R}{\partial b_1}y_1 + \frac{\partial R}{\partial b_2}y_2 + \cdots + \frac{\partial R}{\partial b_n}y_n,\\ &\vdots\\ Rx_n &= \frac{\partial R}{\partial p_1}y_1 + \frac{\partial R}{\partial p_2}y_2 + \cdots + \frac{\partial R}{\partial p_n}y_n. \end{aligned}$$

4. With the help of these formulae one proves a noteworthy theorem on the variation of the determinant R. One denotes the variations of the quantities $a_i, b_i, \ldots, p_i$ by $\delta a_i, \delta b_i, \ldots, \delta p_i$, and constructs the following n systems of linear equations:

$$\begin{aligned} 1)\quad a_1x_1' + b_1x_2' + \cdots + p_1x_n' &= \delta a_1,\\ a_2x_1' + b_2x_2' + \cdots + p_2x_n' &= \delta a_2,\\ &\cdots\\ a_nx_1' + b_nx_2' + \cdots + p_nx_n' &= \delta a_n; \end{aligned}$$

$$\begin{aligned} 2)\quad a_1x_1'' + b_1x_2'' + \cdots + p_1x_n'' &= \delta b_1,\\ a_2x_1'' + b_2x_2'' + \cdots + p_2x_n'' &= \delta b_2,\\ &\cdots\\ a_nx_1'' + b_nx_2'' + \cdots + p_nx_n'' &= \delta b_n; \end{aligned}$$

and so on; finally

$$\begin{aligned} n) \quad a_1x_1^{(n)} + b_1x_2^{(n)} + \cdots + p_1x_n^{(n)} &= \delta p_1, \\ a_2x_1^{(n)} + b_2x_2^{(n)} + \cdots + p_2x_n^{(n)} &= \delta p_2, \\ \cdots \\ a_nx_1^{(n)} + b_nx_2^{(n)} + \cdots + p_nx_n^{(n)} &= \delta p_n. \end{aligned}$$

Now,

$$\delta R = \sum_i \left\{ \frac{\partial R}{\partial a_i}\delta a_i + \frac{\partial R}{\partial b_i}\delta b_i + \cdots + \frac{\partial R}{\partial p_i}\delta p_i \right\}.$$

But according to the above formulae for the solution of equations, one has

$$Rx_1' = \frac{\partial R}{\partial a_1}\delta a_1 + \frac{\partial R}{\partial a_2}\delta a_2 + \cdots + \frac{\partial R}{\partial a_n}\delta a_n = \sum_i \frac{\partial R}{\partial a_i}\delta a_i,$$

and so too,

$$\begin{aligned} Rx_2'' &= \sum_i \frac{\partial R}{\partial b_i}\delta b_i, \quad Rx_3''' = \sum_i \frac{\partial R}{\partial c_i}\delta c_i, \quad \ldots, \quad Rx_n^{(n)} \\ &= \sum_i \frac{\partial R}{\partial p_i}\delta p_i; \end{aligned}$$

so

$$\delta R = R\left\{x_1' + x_2'' + x_3''' + \cdots + x_n^{(n)}\right\},$$

or,

$$\delta \log R = x_1' + x_2'' + x_3''' + \cdots + x_n^{(n)}.$$

Lecture 12

The multiplier for systems of differential equations with an arbitrary number of variables

We shall make an application of the theorem just given on the variation of determinants to a system of differential equations.

Let the following system be given:

$$\frac{dx_1}{dx} = X_1, \frac{dx_2}{dx} = X_2, \ldots, \frac{dx_i}{dx} = X_i, \ldots, \frac{dx_n}{dx} = X_n. \tag{12.1}$$

This system in which $X_1, X_2, \ldots, X_n$ are arbitrary functions of x,, $x_1, x_2, \ldots, x_n$, is integrated through the following system of equations:

$$\begin{aligned} x_1 &= f_1(x, \alpha_1, \alpha_2, \ldots, \alpha_n), \\ x_2 &= f_2(x, \alpha_1, \alpha_2, \ldots, \alpha_n), \\ &\ldots \\ x_n &= f_n(x, \alpha_1, \alpha_2, \ldots, \alpha_n). \end{aligned}$$

If one inserts here the values of $x_1, x_2, \ldots, x_n$ in $X_1, X_2, \ldots, X_n$ and determines also the differential coefficients $\frac{dx_1}{dx}, \frac{dx_2}{dx}, \ldots, \frac{dx_n}{dx}$ as functions of x and the n arbitrary constants $\alpha_1, \alpha_2, \ldots, \alpha_n$, then the system (12.1) is identically satisfied by these values, i.e., the equation (12.1) holds for all values of the variable x and the arbitrary constants $\alpha_1, \ldots, \alpha_n$. Therefore one can differentiate these with respect to each of these n constants. From each of the equations (12.1) arise in this manner n equations, in all n systems of such equations each with n equations i.e.

n^2 equations, all of the form

$$\frac{d\frac{\partial x_i}{\partial \alpha_k}}{dx} = \frac{\partial X_i}{\partial x_1}\frac{\partial x_1}{\partial \alpha_k} + \frac{\partial X_i}{\partial x_2}\frac{\partial x_2}{\partial \alpha_k} + \cdots + \frac{\partial X_i}{\partial x_n}\frac{\partial x_n}{\partial \alpha_k}.$$

From the first equation $\frac{dx_1}{dx} = X_1$ is the following system:

$$\frac{\partial x_1}{\partial \alpha_1}\frac{\partial X_1}{\partial x_1} + \frac{\partial x_2}{\partial \alpha_1}\frac{\partial X_1}{\partial x_2} + \cdots + \frac{\partial x_n}{\partial \alpha_1}\frac{\partial X_1}{\partial x_n} = \frac{d\frac{\partial x_1}{\partial \alpha_1}}{dx},$$
$$\frac{\partial x_1}{\partial \alpha_2}\frac{\partial X_1}{\partial x_1} + \frac{\partial x_2}{\partial \alpha_2}\frac{\partial X_1}{\partial x_2} + \cdots + \frac{\partial x_n}{\partial \alpha_2}\frac{\partial X_1}{\partial x_n} = \frac{d\frac{\partial x_1}{\partial \alpha_2}}{dx},$$
$$\frac{\partial x_1}{\partial \alpha_n}\frac{\partial X_1}{\partial x_1} + \frac{\partial x_2}{\partial \alpha_n}\frac{\partial X_1}{\partial x_2} + \cdots + \frac{\partial x_n}{\partial \alpha_n}\frac{\partial X_1}{\partial x_n} = \frac{d\frac{\partial x_1}{\partial \alpha_n}}{dx}.$$

Those from the rest of the equations (12.1) are the following systems:

$$\frac{\partial x_1}{\partial \alpha_1}\frac{\partial X_2}{\partial x_1} + \frac{\partial x_2}{\partial \alpha_1}\frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial x_n}{\partial \alpha_1}\frac{\partial X_2}{\partial x_n} = \frac{d\frac{\partial x_2}{\partial \alpha_1}}{dx},$$
$$\frac{\partial x_1}{\partial \alpha_2}\frac{\partial X_2}{\partial x_1} + \frac{\partial x_2}{\partial \alpha_2}\frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial x_n}{\partial \alpha_2}\frac{\partial X_2}{\partial x_n} = \frac{d\frac{\partial x_2}{\partial \alpha_2}}{dx},$$
$$\cdots$$
$$\frac{\partial x_1}{\partial \alpha_n}\frac{\partial X_2}{\partial x_1} + \frac{\partial x_2}{\partial \alpha_n}\frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial x_n}{\partial \alpha_n}\frac{\partial X_2}{\partial x_n} = \frac{d\frac{\partial x_2}{\partial \alpha_n}}{dx};$$

and so on; finally

$$\frac{\partial x_1}{\partial \alpha_1}\frac{\partial X_n}{\partial x_1} + \frac{\partial x_2}{\partial \alpha_1}\frac{\partial X_n}{\partial x_2} + \cdots + \frac{\partial x_n}{\partial \alpha_1}\frac{\partial X_n}{\partial x_n} = \frac{d\frac{\partial x_n}{\partial \alpha_1}}{dx},$$
$$\frac{\partial x_1}{\partial \alpha_2}\frac{\partial X_n}{\partial x_1} + \frac{\partial x_2}{\partial \alpha_2}\frac{\partial X_n}{\partial x_2} + \cdots + \frac{\partial x_n}{\partial \alpha_2}\frac{\partial X_n}{\partial x_n} = \frac{d\frac{\partial x_n}{\partial \alpha_2}}{dx},$$
$$\cdots$$
$$\frac{\partial x_1}{\partial \alpha_n}\frac{\partial X_n}{\partial x_1} + \frac{\partial x_2}{\partial \alpha_n}\frac{\partial X_n}{\partial x_2} + \cdots + \frac{\partial x_n}{\partial \alpha_n}\frac{\partial X_n}{\partial x_n} = \frac{d\frac{\partial x_n}{\partial \alpha_n}}{dx}.$$

If one compares these systems with those stated in item 4 of the previous lecture for establishing the theorem on the variation of determinants, then one finds that those go over into these with the following substitution:

$$\begin{aligned}
a_1 &= \frac{\partial x_1}{\partial \alpha_1}, b_1 = \frac{\partial x_2}{\partial \alpha_1}, \ldots, p_1 = \frac{\partial x_n}{\partial \alpha_1}; \\
a_2 &= \frac{\partial x_1}{\partial \alpha_2}, b_2 = \frac{\partial x_2}{\partial \alpha_2}, \ldots, p_2 = \frac{\partial x_n}{\partial \alpha_2}; \\
&\ldots \\
a_n &= \frac{\partial x_1}{\partial \alpha_n}, b_n = \frac{\partial x_2}{\partial \alpha_n}, \ldots, p_n = \frac{\partial x_n}{\partial \alpha_n}; \\
R &= \sum \pm a_1 b_2 \ldots p_n = \sum \pm \frac{\partial x_1}{\partial \alpha_1}\frac{\partial x_2}{\partial \alpha_2} \cdots \frac{\partial x_n}{\partial \alpha_n}; \\
x_1' &= \frac{\partial X_1}{\partial x_1}, x_2' = \frac{\partial X_1}{\partial x_2}, \ldots, x_n' = \frac{\partial X_1}{\partial x_n}, \\
x_1'' &= \frac{\partial X_2}{\partial x_1}, x_2'' = \frac{\partial X_2}{\partial x_2}, \ldots, x_n'' = \frac{\partial X_2}{\partial x_n}, \\
&\ldots \\
x_1^{(n)} &= \frac{\partial X_n}{\partial x_1}, x_2^{(n)} = \frac{\partial X_n}{\partial x_2}, \ldots, x_n^{(n)} = \frac{\partial X_n}{\partial x_n}, \\
\delta &= d/dx
\end{aligned}$$

Through this the total differential coefficient of $\log R$ with respect to x can be expressed in the remarkable form

$$\frac{d \log R}{dx} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}, \tag{12.2}$$

where

$$R = \sum \pm \frac{\partial x_1}{\partial \alpha_1}\frac{\partial x_2}{\partial \alpha_2} \cdots \frac{\partial x_n}{\partial \alpha_n}.$$

After a complete integration of the system of equations(12.1) one finds R from the equation (12.2) through an integration over x. But there exist cases in which the determinant R can be given before any integration, namely, when the sum $\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}$ can be transformed with the help of the system (12.1) into a total differential coefficient in x, or, what is a still simpler case, if X_1 does not contain x_1, X_2 does not contain x_2, etc. Then $\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n} = 0$, so

$$\frac{d \log R}{dx} = 0, \quad R = \text{ constant.}$$

The theorem contained in equation (12.2) has been established first by *Liouville*, and indeed in this form (*Liouville* Journal, Vol. 3, p. 348).

In another form, in which the arbitrary constants α are replaced by independent variables x and these by functions f of the variables x, in one of my papers (*Crelle* Journal, vol. 28, p. 336).*Liouville* has not used any help from this theorem which it affords for integration. Before we go into this application, we shall give a somewhat general form to the result obtained, wherein we bring in an alteration which indeed appears very inessential, without which nevertheless, its applicability would be very much limited.

If one writes the system (12.1) in the form of the proportion

$$dx : dx_1 : dx_2 : \cdots : dx_n = 1 : X_1 : X_2 \cdots : X_n,$$

multiplying the right hand side by an arbitrary quantity X one gets the form considered earlier:

$$dx : dx_1 : dx_2 : \cdots : dx_n = X : X_1 : X_2 \cdots : X_n, \tag{12.3}$$

if one replaces at the same time $X_1, X_2, \ldots X_n$ respectively by the quotients $\frac{X_1}{X}, \frac{X_2}{X}, \ldots, \frac{X_n}{X}$. Through this change, equation (12.2) becomes,

$$\begin{aligned}\frac{d\log R}{dx} &= \frac{\partial(\frac{X_1}{X})}{\partial x_1} + \frac{\partial(\frac{X_2}{X})}{\partial x_2} + \cdots + \frac{\partial(\frac{X_n}{X})}{\partial x_n}\\ &= \frac{1}{X}\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}\right)\\ &\quad - \frac{1}{X^2}\left(X_1\frac{\partial X}{\partial x_1} + X_2\frac{\partial X}{\partial x_2} + \cdots + X_n\frac{\partial X}{\partial x_n}\right).\end{aligned}$$

With the help of the equations

$$\frac{X_1}{X} = \frac{dx_1}{dx}, \frac{X_2}{X} = \frac{dx_2}{dx}, \ldots, \frac{X_n}{X} = \frac{dx_n}{dx},$$

the subtracted term on the right hand side of this equation can be brought into the form

$$-\frac{1}{X}\left(\frac{\partial X}{\partial x_1}\frac{dx_1}{dx} + \frac{\partial X}{\partial x_2}\frac{dx_2}{dx} + \cdots + \frac{\partial X}{\partial x_n}\frac{dx_n}{dx}\right)$$

or

$$-\frac{1}{X}\left(\frac{dX}{dx} - \frac{\partial X}{\partial x}\right).$$

If one inserts this into the expression for $\frac{d\log R}{dx}$, then we get

$$\frac{d\log R}{dx} = \frac{1}{X}\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}\right) - \frac{1}{X}\left(\frac{dX}{dx} - \frac{\partial X}{\partial x}\right),$$

or,

$$\frac{d\log(XR)}{dx} = \frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}\right). \tag{12.4}$$

So if $\frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}\right)$ can be transformed through the given system (12.3) into a total differential coefficient in x, or if $\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n} = 0$, then one can determine R before any integration. In the latter case, we have

$$XR = \text{ constant}, \quad R = \frac{\text{constant}}{X},$$

where, as earlier

$$R = \sum \pm \frac{\partial x_1}{\partial \alpha_1}\frac{\partial x_2}{\partial \alpha_2}\cdots\frac{\partial x_n}{\partial \alpha_n}.$$

We now assume that the system (12.1) is in fact of such a nature that R can be stated before carrying out any integration, and assume that one has found $n-1$ integrals, the nth still missing; then one can represent the $n-1$ integral equations in the form

$$\begin{aligned} x_2 &= \phi_2(x, x_1, \alpha_2, \alpha_3, \ldots, \alpha_n), \\ x_3 &= \phi_3(x, x_1, \alpha_2, \alpha_3, \ldots, \alpha_n), \\ &\cdots\cdots \\ x_n &= \phi_n(x, x_1, \alpha_2, \alpha_3, \ldots, \alpha_n), \end{aligned}$$

and then remains the differential equation

$$X dx_1 - X_1 dx = 0$$

to be integrated, the integral of which leads to an equation of the form

$$x_1 = \phi_1(x, \alpha_1, \alpha_2, \ldots, \alpha_n).$$

By comparison with the complete integration system above of the differential equations (12.1), it follows from this that the function now denoted by ϕ_1 is the same as that which was denoted by f_1 above, and that the functions $\phi_2, \phi_3, \ldots, \phi_n$ go over respectively into $f_2, f_3, \ldots, f_n$ when one substitutes for x_1 its value ϕ_1.

For distinguishing the differential coefficients of the quantities x_2, $x_3, \ldots x_n$, in so far as we look upon them as functions of x, x_1, α_2,

$\alpha_3, \ldots, \alpha_n$, from the differential coefficients considered hitherto we enclose them in brackets,

$$\frac{\partial x_i}{\partial \alpha_k} = \left(\frac{\partial x_i}{\partial \alpha_k}\right) + \left(\frac{\partial x_i}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_k},$$

where i and k can take values from 2 to n, inclusive. For $k = 1$, one has

$$\frac{\partial x_i}{\partial \alpha_1} = \left(\frac{\partial x_i}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_1},$$

an equation which one can handle under the general formulation when one takes into account that

$$\left(\frac{\partial x_2}{\partial \alpha_1}\right) = \left(\frac{\partial x_3}{\partial \alpha_1}\right) = \cdots = \left(\frac{\partial x_n}{\partial \alpha_1}\right) = 0. \tag{12.5}$$

This gives accordingly the formula

$$\frac{\partial x_i}{\partial \alpha_k} = \left(\frac{\partial x_i}{\partial \alpha_k}\right) + \left(\frac{\partial x_i}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_k},$$

for from $i = 2$ to $i = n$ and from $k = 1$ to $k = n$. Hence

$$\begin{aligned} R &= \sum \pm \frac{\partial x_1}{\partial \alpha_1}\left\{\left(\frac{\partial x_2}{\partial \alpha_2}\right) + \left(\frac{\partial x_2}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_2}\right\} \\ &\quad \cdots\left\{\left(\frac{\partial x_n}{\partial \alpha_n}\right) + \left(\frac{\partial x_n}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_n}\right\}, \end{aligned}$$

i.e., R is the determinant of the quantities

$$\begin{array}{llll}
\frac{\partial x_1}{\partial \alpha_1}, \left(\frac{\partial x_2}{\partial \alpha_1}\right) + \left(\frac{\partial x_3}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_1}, & \left(\frac{\partial x_3}{\partial \alpha_1}\right) + \left(\frac{\partial x_3}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_1}, \ldots & \left(\frac{\partial x_n}{\partial \alpha_1}\right) + \left(\frac{\partial x_n}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_1}, \\
\frac{\partial x_1}{\partial \alpha_2}, \left(\frac{\partial x_2}{\partial \alpha_2}\right) + \left(\frac{\partial x_2}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_2}, & \left(\frac{\partial x_3}{\partial \alpha_2}\right) + \left(\frac{\partial x_3}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_2}, \ldots & \left(\frac{\partial x_n}{\partial \alpha_2}\right) + \left(\frac{\partial x_n}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_2}, \\
\frac{\partial x_1}{\partial \alpha_3}, \left(\frac{\partial x_2}{\partial \alpha_3}\right) + \left(\frac{\partial x_2}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_3}, & \left(\frac{\partial x_3}{\partial \alpha_3}\right) + \left(\frac{\partial x_3}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_3}, \ldots & \left(\frac{\partial x_n}{\partial \alpha_3}\right) + \left(\frac{\partial x_n}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_3}, \\
\vdots \quad \vdots & \vdots & \vdots \\
\frac{\partial x_1}{\partial \alpha_n}, \left(\frac{\partial x_2}{\partial \alpha_n}\right) + \left(\frac{\partial x_2}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_n}, & \left(\frac{\partial x_3}{\partial \alpha_n}\right) + \left(\frac{\partial x_3}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_n}, \ldots & \left(\frac{\partial x_n}{\partial \alpha_n}\right) + \left(\frac{\partial x_n}{\partial x_1}\right)\frac{\partial x_1}{\partial \alpha_n}.
\end{array}$$

If one denotes by R_1 and R_2 the determinants into which R goes over when one reduces the n quantitites in the second column to their first term, for R_1, and to their second term, for R_2, then R as a linear homogeneous function of the former n quantities is equal to the sum of

R_1 and R_2. But R_2 has the common factor $\left(\frac{\partial x_2}{\partial x_1}\right)$ and after one has taken this out, the quantities in the first and second columns coincide, i.e., according to §1 of the preceding lecture R_2 is a vanishing determinant, and so R is equal to R_1, i.e., R is unchanged if the quantities in the second column are reduced to their first terms. The same holds for the quantities in the third, fourth and nth columns, and this gives R as the determinant of the quantities

$$\begin{array}{lllll}
\frac{\partial x_1}{\partial \alpha_1}, & \left(\frac{\partial x_2}{\partial \alpha_1}\right), & \left(\frac{\partial x_3}{\partial \alpha_1}\right), & \cdots & \left(\frac{\partial x_n}{\partial \alpha_1}\right), \\
\frac{\partial x_1}{\partial \alpha_2}, & \left(\frac{\partial x_2}{\partial \alpha_2}\right), & \left(\frac{\partial x_3}{\partial \alpha_2}\right), & \cdots & \left(\frac{\partial x_n}{\partial \alpha_2}\right), \\
\frac{\partial x_1}{\partial \alpha_3}, & \left(\frac{\partial x_2}{\partial \alpha_3}\right), & \left(\frac{\partial x_3}{\partial \alpha_3}\right), & \cdots & \left(\frac{\partial x_n}{\partial \alpha_3}\right), \\
\vdots & \vdots & \vdots & & \vdots \\
\frac{\partial x_1}{\partial \alpha_n}, & \left(\frac{\partial x_2}{\partial \alpha_n}\right), & \left(\frac{\partial x_3}{\partial \alpha_n}\right), & \cdots & \left(\frac{\partial x_n}{\partial \alpha_n}\right).
\end{array}$$

If one now represents this determinant as a linear homogeneous function of the quantities in the first row, and takes into account that according to (12.4), all these with the exception of $\frac{\partial x_1}{\partial \alpha_1}$ vanish, then we obtain R as the product of $\frac{\partial x_1}{\partial \alpha_1}$ and $\frac{\partial R}{\partial \frac{\partial x_1}{\partial \alpha_1}}$, i.e., the product of $\frac{\partial x_1}{\partial \alpha_1}$ and the determinant

$$Q = \sum \pm \left(\frac{\partial x_2}{\partial \alpha_2}\right)\left(\frac{\partial x_3}{\partial \alpha_3}\right) \cdots \left(\frac{\partial x_n}{\partial \alpha_n}\right), \tag{12.6}$$

whose elements are those which remain from the last scheme when one cancels the first horizontal and first vertical rows. One has, consequently,

$$R = \frac{\partial x_1}{\partial \alpha_1} Q. \tag{12.7}$$

This equation is of the highest importance. Since according to our assumption one can find R a priori from the given system (12.3) without having to perform any integration. Further, Q is known in terms of the $n-1$ integrations already performed, therefore the equation (12.7) leads, as we then see, to the remaining nth integration, where one determines the integrating factor of the differential equaiton

$$X dx_1 - X_1 dx = 0,$$

in which X and X_1 are expressed as functions of x and x_1. Let the complete integral of this equation be

$$F(x, x_1) = \alpha_1. \tag{12.8}$$

This gives, as solution for x_1, the same expression as what we designated above by

$$x_1 = \phi(x, \alpha_1, \alpha_2, \ldots, \alpha_n).$$

The substitution of this expression for x_1 makes (12.8) an identical equation; so one obtains, on differentiation with respect to α_1,

$$\frac{\partial F}{\partial x_1} \cdot \frac{\partial x_1}{\partial \alpha_1} = 1,$$

or, since

$$\frac{\partial x_1}{\partial \alpha_1} = \frac{R}{Q}$$

according to equation (12.7),

$$\frac{\partial F}{\partial x_1} = \frac{Q}{R}.$$

If we denote by N the integrating factor of $Xdx_1 - X_1 dx$, then we have

$$NX = \frac{\partial F}{\partial x_1}, \quad -NX_1 = \frac{\partial F}{\partial x},$$

and from the first of these equations,

$$N = \frac{1}{X}\frac{\partial F}{\partial x_1} = \frac{Q}{XR}. \tag{12.9}$$

$N = \frac{Q}{XR}$ is then the integrating factor of the equation $Xdx_1 - X_1 dx = 0$. Then one has the following theorem:

Theorem 12.1 *If in the system of differential equations*

$$dx : dx_1 : dx_2 : \cdots : dx_n = X : X_1 : X_2 : \cdots : X_n,$$

the expression

$$\frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right)$$

is a total differential of x, *one knows* $n-1$ *integrals of the system from which the variables* $x_2, x_3, \ldots, x_n$ *can be represented as functions of* x, x_1 *and* $n-1$ *arbitrary constants of integration through the equations*

$$\begin{aligned} x_2 &= \phi_2(x, x_1, \alpha_2, \ldots, \alpha_n), \\ x_3 &= \phi_3(x, x_1, \alpha_2, \ldots, \alpha_n), \ldots, \\ x_n &= \phi_n(x, x_1, \alpha_2, \ldots, \alpha_n), \end{aligned}$$

and there remains the only differential equation

$$X dx_1 - X dx = 0$$

to be integrated, then

$$N = \frac{Q}{XR}$$

is an integrating factor of this differential equation, where

$$XR = e^{\int \frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right) dx}$$

and

$$Q = \sum \pm \frac{\partial x_2}{\partial \alpha_2} \cdot \frac{\partial x_3}{\partial \alpha_3} \cdots \frac{\partial x_n}{\partial \alpha_n}.$$

If $\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0$, then $XR = \text{constant}$, and in this case the determinant Q itself is the integrating factor of the differential equation $X dx_1 - X_1 dx = 0$.

If one compares (12.4) of this lecture with (10.11) of Lecture 10, it is seen that the differential equation which $-\log XR$ satisfies, which is of $n+1$ variables, is that which we found at that time (for a system of two differential equations of three variables) for $\log M$. One can therefore set

$$\log M = -\log XR,$$

or,

$$M = \frac{1}{XR},$$

and then under the assumptions of the theorem first stated, MQ is the integrating factor of the last differential equation $X dx_1 - X_1 dx = 0$, where M is determined from the equation

$$X \frac{d \log M}{dx} + \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0.$$

One can construct the above determinant Q in another way. The simplest representation is that in the form of a product. Namely, just as we eliminated the constant α_1 from the variables $x_2, x_3, \ldots, x_n$ by means of x_1, and then represent the determinant R as the product of $\frac{\partial x_1}{\partial \alpha_1}$ and the determinant Q whose order is one less than the order of R, so too one can further eliminate the constant α_2 from the variables $x_3, x_4, \ldots, x_n$, by means of x_2 and then represent Q as the product of $\frac{\partial x_2}{\partial \alpha_2}$ and the determinant $P = \sum \pm \frac{\partial x_3}{\partial \alpha_3} \cdot \frac{\partial x_4}{\partial \alpha_4} \cdots \frac{\partial x_n}{\partial \alpha_n}$. One has to proceed in the following way; one eliminates the constant α_3 from $x_4, x_5, \ldots, x_n$ by means of x_3; the constant α_4 from $x_5, x_6, \ldots, x_n$ by means of x_4; and so on; so that one obtains the following representation for the integral equations:

$$\begin{aligned}
x_1 &= F_1(x, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n), \\
x_2 &= F_2(x, x_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \alpha_n), \\
x_3 &= F_3(x, x_1, x_2, \alpha_3, \alpha_4, \ldots, \alpha_{n-1}, \alpha_n), \qquad \text{(F)} \\
&\ldots \\
x_n &= F_n(x, x_1, x_2, \ldots, x_{n-1}, \alpha_n);
\end{aligned}$$

then

$$R = \frac{\partial x_1}{\partial \alpha_1} \cdot \frac{\partial x_2}{\partial \alpha_2} \cdots \frac{\partial x_n}{\partial \alpha_n}, \tag{12.10}$$

where the expressions F_1 to F_n are to be substituted for the quantities x_1 to x_n, and for the same mode of representation of the integral equations one has

$$Q = \frac{\partial x_2}{\partial \alpha_2} \cdot \frac{\partial x_3}{\partial \alpha_3} \cdots \frac{\partial x_n}{\partial \alpha_n}. \tag{12.11}$$

The transformation employed here consists of the following: If the n quantities $x_1, x_2, \ldots, x_n$ are functions of n others, $\alpha_1, \alpha_2, \ldots \alpha_n$, so that

$$\begin{aligned}
x_1 &= f_1(\alpha_1, \alpha_2, \ldots, \alpha_n), \\
x_2 &= f_2(\alpha_1, \alpha_2, \ldots, \alpha_n), \\
&\ldots \\
x_n &= f_n(\alpha_1, \alpha_2, \ldots, \alpha_n),
\end{aligned}$$

and one represents the quantities $x_1, \ldots, x_n$ in the following way through successive eliminations:

$$\text{(F.)} \qquad \begin{cases} x_1 = F_1(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \alpha_n), \\ x_2 = F_2(x_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \alpha_n), \\ x_3 = F_3(x_1, x_2, \alpha_3, \ldots, \alpha_{n-1}, \alpha_n), \\ \quad \ldots \\ x_n = F_n(x_1, x_2, \ldots, x_{n-1}, \alpha_n), \end{cases}$$

then

$$\sum \pm \frac{\partial f_1}{\partial \alpha_1} \cdot \frac{\partial f_2}{\partial \alpha_2} \cdots \frac{\partial f_n}{\partial \alpha_n} = \frac{\partial F_1}{\partial \alpha_1} \cdot \frac{\partial F_2}{\partial \alpha_2} \cdots \frac{\partial F_n}{\partial \alpha_n},$$

or, if we denote the differential coefficients of the quantities without brackets in the first representation and with brackets in the second, then

$$\sum \pm \frac{\partial x_1}{\partial \alpha_1} \cdot \frac{\partial x_2}{\partial \alpha_2} \cdots \frac{\partial x_n}{\partial \alpha_n} = \left(\frac{\partial x_1}{\partial \alpha_1}\right) \cdot \left(\frac{\partial x_2}{\partial \alpha_2}\right) \cdots \left(\frac{\partial x_n}{\partial \alpha_n}\right).$$

The form (F) of the integral equations is just the one it takes for the case of a single differential equation of higher order by successive integrations. The successive integrations of the equation

$$y^{(n+1)} = f(y^{(n)}, y^{(n-1)}, y^{(n-2)}, \ldots, y'', y', y, x),$$

give

$$\begin{aligned} y^{(n)} &= f_1(\alpha_n, y^{(n-1)}, y^{(n-2)}, \ldots, y'', y', y, x), \\ y^{(n-1)} &= f_2(\alpha_n, \alpha_{n-1}, y^{(n-2)}, \ldots, y'', y', y, x), \\ &\ldots \\ y'' &= f_{n-1}(\alpha_n, \alpha_{n-1}, \ldots, \alpha_2, y', y, x), \\ y' &= f_n(\alpha_n, \alpha_{n-1}, \ldots, \alpha_2, \alpha_1, y, x). \end{aligned}$$

If the present equation $y^{(n+1)} = f$ now belongs to the category for which the multiplier M can be determined a priori, then the integrating factor for the differential equation of the first order:

$$y' = f_n,$$

is MQ, where

$$Q = \frac{\partial y_n}{\partial \alpha_n} \cdot \frac{\partial y_{n-1}}{\partial \alpha_{n-1}} \cdots \frac{\partial y''}{\partial \alpha_2} \cdot \frac{\partial y'}{\partial \alpha_1}.$$

Lecture 13

Functional Determinants. Their application in setting up the Partial Differential Equation for the Multiplier

Determinants of the form

$$\sum \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}$$

have been called *functional determinants* by me, "alternating differential functions" by *Cauchy*, who has given some theorems on these in the 'Comptes Rendus' of the Paris Academy. Functional determinants are built up from the n^2 partial differential coefficients $\frac{\partial f_i}{\partial x_k}$ of n functions $f_1, f_2, \ldots, f_n$, each of which depends on n variables $x_1, x_2, \ldots, x_n$.

I have published an article in vol. 22 of *Crelle*'s Journal on functional determinants in which the analogy which exists between functional determinants in problems with many variables and differential coeffficients in problems with one variable has been demonstrated. The theorems proved therein express this analogy as follows:

1. If f is a function of φ and φ a function of x, then $\frac{df}{dx} = \frac{df}{d\varphi} \cdot \frac{d\varphi}{dx}$. To this corresponds the theorem for n variables: *if* $f_1, \ldots, f_n$ *are functions of* $\varphi_1, \varphi_2, \ldots, \varphi_n$, *and these are further functions of* $x_1, x_2, \ldots, x_n$, then

$$\sum \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} = \left(\sum \pm \frac{\partial f_1}{\partial \varphi_1} \cdot \frac{\partial f_2}{\partial \varphi_2} \cdots \frac{\partial f_n}{\partial \varphi_n}\right) \cdot \left(\sum \pm \frac{\partial \varphi_1}{\partial x_1} \frac{\partial \varphi_2}{\partial x_2} \cdots \frac{\partial \varphi_n}{\partial x_n}\right)$$

2. This can also be expressed in another form: if f and φ are functions of x, then

$$\frac{df}{d\varphi} = \frac{\frac{df}{dx}}{\frac{d\varphi}{dx}}.$$

To this one has the analogous theorem for n variables: if $f_1, f_2, \ldots, f_n$ and $\varphi_1, \varphi_2, \ldots, \varphi_n$ are functions of $x_1, x_2, \ldots, x_n$, then

$$\sum \pm \frac{\partial f_1}{\partial \varphi_1} \cdot \frac{\partial f_2}{\partial \varphi_2} \cdots \frac{\partial f_n}{\partial \varphi_n} = \left(\sum \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}\right) \Big/ \left(\sum \pm \frac{\partial \varphi_1}{\partial x_1} \cdot \frac{\partial \varphi_2}{\partial x_2} \cdots \frac{\partial \varphi_n}{\partial x_n}\right),$$

and therefore, when one sets $f_1 = x_1, f_2 = x_2, \ldots, f_n = x_n$,

$$\sum \pm \frac{\partial x_1}{\partial \varphi_1} \cdot \frac{\partial x_2}{\partial \varphi_2} \cdots \frac{\partial x_n}{\partial \varphi_n} = 1 \Big/ \sum \pm \frac{\partial \varphi_1}{\partial x_1} \cdots \frac{\partial \varphi_n}{\partial x_n}.$$

3. From the equation $\Pi(x, y) = 0$ one obtains

$$\frac{dy}{dx} = -\frac{\frac{\partial \Pi}{\partial x}}{\frac{\partial \Pi}{\partial y}}.$$

To this one has the analogous theorem; from the is equations between $2n$ variables:

$$\begin{aligned} \Pi_1(y_1, \ldots, y_n, x_1, \ldots, x_n) &= 0, \\ \Pi_2(y_1, \ldots, y_n, x_1, \ldots, x_n) &= 0, \\ &\ldots \\ \Pi_n(y_1, \ldots, y_n, x_1, \ldots, x_n) &= 0, \end{aligned}$$

one has

$$(-1)^n \sum \pm \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} \cdots \frac{\partial y_n}{\partial x_n} = \left(\sum \pm \frac{\partial \Pi_1}{\partial x_1} \frac{\partial \Pi_2}{\partial x_2} \cdots \frac{\partial \Pi_n}{\partial x_n}\right) \Big/ \left(\sum \pm \frac{\partial \Pi_1}{\partial y_1} \frac{\partial \Pi_2}{\partial y_2} \cdots \frac{\partial \Pi_n}{\partial y_n}\right).$$

4. In order that the equation $F(x) = 0$ has two equal roots we must have $F'(x) = 0$. To this the following is the analogy.*In order that*

the equations

$$\begin{aligned} F_1(x_1, \ldots, x_n) &= 0, \\ F_2(x_1, \ldots, x_n) &= 0, \\ &\ldots \\ F_n(x_1, \ldots, x_n) &= 0, \end{aligned}$$

have two coincident systems of roots, one must have at the same time

$$\sum \pm \frac{\partial F_1}{\partial x_1} \cdots \frac{\partial F_n}{\partial x_n} = 0.$$

5. If for all values of x, the differential coefficient $\frac{\partial F}{\partial x}$ vanishes, it follows that $F =$ constant. To this one has the analogy.*If*

$$\sum \pm \frac{\partial F_1}{\partial x_1} \cdots \frac{\partial F_n}{\partial x_n} = 0$$

for all values of $x_1, \ldots, x_n$, *there must exist between the n functions* $F_1, \ldots, F_n$ *an equation*

$$\pi(F_1, \ldots, F_n) = 0$$

in which the variables $x_1, x_2, \ldots, x_n$ *do not enter explicitly.* This gives in fact, for $n = 1$, $\pi(F) = 0$, so $F =$ constant, as it should be.

To these examples of the analogies many others can be added. These can be found partly in the article referred to, and partly in "de binis quibuslibet functionibus homongeneis etc."' published in vol. 12 of *Crelle*'s Journal.

Since we proceed from the considerations of functional determinants, we are led to formulate the theory of multipliers of a system of differential equations in the general case of n variables in a manner different from that given in Lecture 12, namely, in the same way in which we handled the case of three variables in Lecture 10.

Let the system

$$dx : dx_1 : \cdots : dx_n = X : X_1 : \cdots : X_n$$

be integrated through the equations

$$f_1 = \alpha_1, \ldots, f_n = \alpha_n,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote arbitrary constants. The direct differentials of these are

$$\begin{aligned}
&\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial x_1}dx_1 + \cdots + \frac{\partial f_1}{\partial x_n}dx_n = 0,\\
&\frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial x_1}dx_1 + \cdots + \frac{\partial f_2}{\partial x_n}dx_n = 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots\\
&\frac{\partial f_n}{\partial x}dx + \frac{\partial f_n}{\partial x_1}dx_1 + \cdots + \frac{\partial f_n}{\partial x_n}dx_n = 0,
\end{aligned}$$

which must be identical with the given system, since the arbitrary constants vanish on differentiation. If one adds to these n equations, linear with respect to $dx, dx_1, \ldots, dx_n$, the identity

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_n}dx_n = df$$

as the $n+1$th, where f denotes an arbitrary function of $x, x_1, \ldots, x_n$, and applies to these $n+1$ equations the method of solution for linear equations given in item 3 of Lecture 11, then this gives the values

$$R\,dx = A\,df, R\,dx_1 = A_1\,df, \ldots, R\,dx_n = A_n\,df$$

for $dx, dx_1, \ldots, dx_n$, where

$$\begin{gathered}
R = \sum \pm \frac{\partial f}{\partial x}\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n} = A\frac{\partial f}{\partial x} + A_1\frac{\partial f}{\partial x_1} + \cdots + A_n\frac{\partial f}{\partial x_n},\\
A = \frac{\partial R}{\partial \frac{\partial f}{\partial x}}, A_1 = \frac{\partial R}{\partial \frac{\partial f}{\partial x_1}}, \ldots, A_n = \frac{\partial R}{\partial \frac{\partial f}{\partial x_n}}.
\end{gathered}$$

This determination of the quantities $A, A_1, \ldots, A_n$ from the expansion of R in partial differential coefficients of f is precisely what will serve us in the sequel. Still it is of interest, namely, to follow the analogy with the case of three variables given in Lecture 10, to derive the quantities A in another way, without help from R. First,

$$A = \sum \pm \frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n}.$$

From A one obtains A_1, following item 2 of Lecture 11, if one exchanges the differentiation with respect to x with x_1 and changes the sign. One permutes cyclically the differentiations in all the $n+1$ variables: in place of those taken in $x, x_1, \ldots, x_n$ replace respectively those in $x_1, \ldots, x_n, x$, and then alter the sign or preserve it according as the number $n+1$ of variables is even or odd; then A changes to A_1. The last rule has the advantage that through simple repetition of the same operation A_1 changes to A_2, A_2 to A_3 and so on.

When one eliminates df from the values obtained for dx, $dx_1, \ldots,$ dx_n, this gives

$$dx : dx_1 : \cdots : dx_n = A : A_1 : \cdots : A_n,$$

which must agree with the given system

$$dx : dx_1 : \cdots : dx_n = X : X_1 : \cdots : X_n.$$

Therefore, the proportion

$$A : A_1 : \cdots : A_n = X : X_1 : \cdots : X_n$$

must hold, i.e., there must exist a multiplier M of the form

$$MX = A, MX_1 = A_1, \ldots, MX_n = A_n.$$

Now it comes to extending the identical equation satisfied by A proven for $n = 2$ in Lecture 10, to the general case, that is, to prove that the equation

$$\frac{\partial A}{\partial x} + \frac{\partial A_1}{\partial x_1} + \cdots + \frac{\partial A_n}{\partial x_n} = 0$$

is satisfied. If one takes into account the structure of the quantities A, $A_1, \ldots, A_n$, one sees easily that on the left side of this equation only the first and second differential coefficients of the quantities $f_1, \ldots, f_n$ can occur, and the latter indeed only linearly, i.e., never as a product of two differential coefficients of the second order. Further, as in A no differentiation with respect to x occur, in A_1 none with respect to x_1,etc., in A_n none with respect to x_n, the second differential coefficients which occur in the expression $\frac{\partial A}{\partial x} + \frac{\partial A_1}{\partial x_1} + \cdots + \frac{\partial A_n}{\partial x_n}$ cannot be of the form $\frac{\partial^2 f_s}{\partial x_i^2}$, but can only be of the form $\frac{\partial^2 f_s}{\partial x_i \partial x_k}$ where i and k are distinct. One can then represent the expression $\sum \frac{\partial A_i}{\partial x_i}$ under consideration as a sum of terms of the form

$$F_{ik}^{(s)} \frac{\partial^2 f_s}{\partial x_i \partial x_k}.$$

The values of $F_{ik}^{(s)}$ are found with the help of the formulae

$$R = \sum \pm \frac{\partial f}{\partial x}\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n} = A\frac{\partial f}{\partial x} + A_1\frac{\partial f_1}{\partial x_1} + \cdots + A_n\frac{\partial f_n}{\partial x_n},$$

$$A_i = \frac{\partial R}{\partial \frac{\partial f}{\partial x_i}}, A_k = \frac{\partial R}{\partial \frac{\partial f}{\partial x_k}},$$

and indeed for that, only the two differential coefficients $\frac{\partial A_i}{\partial x_i}$ and $\frac{\partial A_k}{\partial x_k}$ are to be investigated, since $\frac{\partial^2 f_s}{\partial x_i \partial x_k}$ manifestly does not occur in the remaining. Now, since the quantities A_i and A_k are themselves determinants, they can be represented in the following way:

$$A_i = \frac{\partial A_i}{\partial \frac{\partial f_1}{\partial x_k}}\frac{\partial f_1}{\partial x_k} + \frac{\partial A_i}{\partial \frac{\partial f_2}{\partial x_k}}\frac{\partial f_2}{\partial x_k} + \cdots + \frac{\partial A_i}{\partial \frac{\partial f_s}{\partial x_k}}\frac{\partial f_s}{\partial x_k} + \cdots + \frac{\partial A_i}{\partial \frac{\partial f_n}{\partial x_k}}\frac{\partial f_n}{\partial x_k},$$

$$A_k = \frac{\partial A_k}{\partial \frac{\partial f_1}{\partial x_i}}\frac{\partial f_1}{\partial x_i} + \frac{\partial A_k}{\partial \frac{\partial f_2}{\partial x_i}}\frac{\partial f_2}{\partial x_i} + \cdots + \frac{\partial A_k}{\partial \frac{\partial f_s}{\partial x_i}}\frac{\partial f_s}{\partial x_i} + \cdots + \frac{\partial A_k}{\partial \frac{\partial f_n}{\partial x_i}}\frac{\partial f_n}{\partial x_i}.$$

From this one obtains the two terms multiplied by $\frac{\partial^2 f_s}{\partial x_i \partial x_k}$ as contribution to the expression $\sum \frac{\partial A_i}{\partial x_i}$ under consideration. One of these arises from $\frac{\partial A_i}{\partial x_i}$ and is

$$\frac{\partial A_i}{\partial \frac{\partial f_s}{\partial x_k}}\frac{\partial^2 f_s}{\partial x_i \partial x_k},$$

and the other arises from $\frac{\partial A_k}{\partial x_k}$ and is

$$\frac{\partial A_k}{\partial \frac{\partial f_s}{\partial x_i}}\frac{\partial^2 f_s}{\partial x_i \partial x_k};$$

consequently,

$$F_{ik}^{(s)} = \frac{\partial A_i}{\partial \frac{\partial f_s}{\partial x_k}} + \frac{\partial A_k}{\partial \frac{\partial f_s}{\partial x_i}} = \frac{\partial^2 R}{\partial \frac{\partial f}{\partial x_i}\partial \frac{\partial f_s}{\partial x_k}} + \frac{\partial^2 R}{\partial \frac{\partial f}{\partial x_k}\partial \frac{\partial f_s}{\partial x_i}}.$$

The formula, contained in item 2 of Lecture 11:

$$\frac{\partial^2 R}{\partial a_i \partial b_k} = -\frac{\partial^2 R}{\partial a_k \partial b_i}, \text{ or } \frac{\partial^2 R}{\partial a_i \partial b_k} + \frac{\partial^2 R}{\partial a_k \partial b_i} = 0$$

gives in the present case

$$\frac{\partial^2 R}{\partial \frac{\partial f}{\partial x_i}\partial \frac{\partial f_s}{\partial x_k}} + \frac{\partial^2 R}{\partial \frac{\partial f}{\partial x_k}\partial \frac{\partial f_s}{\partial x_i}} = 0$$

therefore $F_{ik}^{(s)} = 0$. In this way, the identity

$$\frac{\partial A}{\partial x} + \frac{\partial A_1}{\partial x_1} + \cdots + \frac{\partial A_n}{\partial x_n} = 0$$

is proved in general. But we have

$$A = MX, A_1 = MX_1, \ldots, A_n = MX_n;$$

this gives

$$\frac{\partial(MX)}{\partial x} + \frac{\partial(MX_1)}{\partial x_1} + \cdots + \frac{\partial(MX_n)}{\partial x_n} = 0,$$

which is the partial differential equation for the multiplier M.

Lecture 14

The Second Form of the Equation Defining the Multiplier. The Multipliers of Step Wise Reduced Differential Equations. The Multiplier by the Use of Particular Integrals

We can now proceed to the further investigation for $n+1$ variables in the same way as for three variables in Lecture 10. When we expand the partial differential equation for the multiplier M, we get

$$X\frac{\partial M}{\partial x}+X_1\frac{\partial M}{\partial x_1}+\cdots+X_n\frac{\partial M}{\partial x_n}+\left\{\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}+\cdots+\frac{\partial X_n}{\partial x_n}\right\}M=0. \quad (14.1)$$

This differential equation will also be satisfied by another quantity N if one has also

$$X\frac{\partial N}{\partial x}+X_1\frac{\partial N}{\partial x_1}+\cdots+X_n\frac{\partial N}{\partial x_n}+\left\{\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}+\cdots+\frac{\partial X_n}{\partial x_n}\right\}N=0$$

If we multiply the second equation by $\frac{1}{M}$, the first by $\frac{N}{M^2}$ and subtract one from the other, then we have

$$X\frac{M\frac{\partial N}{\partial x}-N\frac{\partial M}{\partial x}}{M^2}+X_1\frac{M\frac{\partial N}{\partial x_1}-N\frac{\partial M}{\partial x_1}}{M^2}+\cdots+X_n\frac{M\frac{\partial N}{\partial x_n}-N\frac{\partial M}{\partial x_n}}{M^2}=0$$

or,

$$X\frac{\partial(N/M)}{\partial x}+X_1\frac{\partial(N/M)}{\partial x_1}+\cdots+X_n\frac{\partial(N/M)}{\partial x_n}=0$$

i.e., $\frac{N}{M}$ is a solution of the equation

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}+\cdots+X_n\frac{\partial f}{\partial x_n}=0. \quad (14.2)$$

For the complete integration of such an equation, the knowledge of n mutually independent solutions $f_1, f_2, \ldots, f_n$ is necessary, i.e., of n functions $f_1, f_2 \ldots, f_n$ which satisfy the equations

$$X\frac{\partial f_1}{\partial x} + X_1\frac{\partial f_1}{\partial x_1} + \cdots + X_n\frac{\partial f_1}{\partial x_n} = 0,$$
$$X\frac{\partial f_2}{\partial x} + X_1\frac{\partial f_2}{\partial x_1} + \cdots + X_n\frac{\partial f_2}{\partial x_n} = 0,$$
$$\cdots$$
$$X\frac{\partial f_n}{\partial x} + X_1\frac{\partial f_n}{\partial x_1} + \cdots + X_n\frac{\partial f_n}{\partial x_n} = 0,$$

without one of the n functions being a function of the rest. If one knows n such functions, the general solution is

$$F(f_1, \ldots, f_n).$$

One proves this by multiplying the n equations above by $\frac{\partial F}{\partial f_1}, \ldots, \frac{\partial F}{\partial f_n}$ respectively and adding. An $(n+1)$th solution f_{n+1} which does not depend on the other n solutions does not exist. If we assume that one such existed, then it would follow from the argument just applied that any function

$$\varphi(f_1, f_2 \ldots, f_{n+1})$$

of these $n+1$ solutions would again be a solution. Since, however, $f_1, f_2 \ldots, f_{n+1}$ have been taken to be independent of one another, one can introduce them as new variables $x, x_1, \ldots, x_n$, and then an arbitrary function of $f_1, f_2, \ldots, f_n, f_{n+1}$ is the same as an arbitrary function of $x, x_1, \ldots, x_n$. The differential equation in question for f will then be satisfied by any arbitrary function of $x, x_1, \ldots, x_n$, which is impossible. So there can exist only n mutually independent solutions $f_1, f_2 \ldots, f_n$.

These n solutions of the partial differential equation (14.2) have the property that they will be constants because of the integral equations of the system of ordinary differential equations,

$$dx : dx_1 : \cdots : x_n = X : X_1 : \cdots : X_n \tag{14.3}$$

For, these integral equations make the quantities X, $X_1, \ldots, X_n$ proportional to the differentials $dx, dx_1, \ldots, dx_n$, one can replace, in the partial differential equations satisfied by any function f,that is, in the equation

$$X\frac{\partial f_i}{\partial x} + X_1\frac{\partial f_i}{\partial x_1} + \cdots + X_n\frac{\partial f_i}{\partial x_n} = 0,$$

the quantities $X, X_1, \ldots, X_n$ by the differentials $dx, dx_1, \ldots, dx_n$ proportional to them, and obtain

$$\frac{\partial f_i}{\partial x}dx + \frac{\partial f_i}{\partial x_1}dx_1 + \cdots + \frac{\partial f_i}{\partial x_n}dx_n = 0,$$

or

$$df_i = 0.$$

and therefore f_i = constant.

If we accept that the $f_1, f_2 \ldots, f_n$ must be equal to n mutually independent arbitrary constants, $\alpha_1, \alpha_2 \ldots, \alpha_n$, we obtain the most general integral of the differential equations (14.3) possible, and therefore

$$f_1 = \alpha_1, f_2 = \alpha_2, \ldots, f_i = \alpha_i, \ldots, f_n = \alpha_n$$

constitute a complete system, solved for arbitrary constants, of integrals of the differential equation. Conversely, if the complete integration of the differential equations (14.3) is carried out by n equations with n mutually independent arbitrary constants, i.e., by n equations of such a nature that it is impossible to get from them a result free of all the n constants obtained by their elimination, and the solution of these n equations according to the constants gives them the values

$$f_1 = \alpha_1, f_2 = \alpha_2, \ldots, f_i = \alpha_i, \ldots, f_n = \alpha_n,$$

then one obtains through a differentiation

$$\frac{\partial f_i}{\partial x}dx + \frac{\partial f_i}{\partial x_1}dx_1 + \cdots + \frac{\partial f_i}{\partial x_n}dx_n = 0.$$

Since, however, $f_1 = \alpha_1$, $f_2 = \alpha_2, \ldots, f_n = \alpha_n$ form a complete system of integrals of the differential equation (14.3), the differentials dx, $dx_1, \ldots, dx_n$ are proportional to the quantities X, X_1, $\ldots, X_n$, so that,

$$X\frac{\partial f_i}{\partial x} + X_1\frac{\partial f_i}{\partial x_1} + \cdots + X_n\frac{\partial f_i}{\partial x_n} = 0,$$

i.e., $f_1, \ldots, f_n$ are solutions of (14.3).

Therefore it is completely the same, whether one says that $f_1, f_2, \ldots, f_n$ are mutually independent solutions of the partial differential (14.2), or one says that $f_1 = \alpha_1, f_2 = \alpha_2, \ldots, f_n = \alpha_n$ form a complete system of integrals of the differential equations (14.3). Now we have seen that

$$F(f_1, f_2, \ldots, f_n)$$

are the most general solutions of equation (14.2). Further $\frac{N}{M}$ satisfies just this equation. Hence it follows that if M is a definite solution of the equation (14.1) and N any other solution, then $\frac{N}{M}$ must be a function of $f_1, f_2 \dots, f_n$. This gives

$$N = MF(f_1, f_2, \dots, f_n);$$

if M is a multiplier, then

$$MF(f_1, f_2, \dots, f_n)$$

is the general form in which *all* multipliers are included. Through the integral equations of the system (14.3), however, we get $f_1 = \alpha_1, \dots, f_n = \alpha_n$; by use of the integral equations then the general form differs from M only through a constant factor. To avoid confusion we shall denote a definite value of the multiplier M by M_0, the general value by M, further the function of $f_1, f_2, \dots, f_n$ by which M_0 must be multiplied to give M by $\frac{1}{\tilde{\omega}}$, so that $M = M_0 \frac{1}{\tilde{\omega}}$. Then one can write the equations given at the end of the previous lecture:

$$MX = A, MX_1 = A_1, \dots, MX_n = A_n,$$

thus:

$$M_0 X = A\tilde{\omega}, M_0 X_1 = A_1 \tilde{\omega}, \dots, M_0 X_n = A_n \tilde{\omega}. \tag{14.4}$$

The partial differential equation (14.1) found for M can be transformed with the help of the differential equations (14.3). The equation

$$X\frac{\partial M}{\partial x} + X_1 \frac{\partial M}{\partial x_1} + \dots + X_n \frac{\partial M}{\partial x_n} + M\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \dots + \frac{\partial X_n}{\partial x_n}\right) = 0,$$

or, what is the same

$$X\left(\frac{\partial M}{\partial x} + \frac{X_1}{X}\frac{\partial M}{\partial x_1} + \dots + \frac{X_n}{X}\frac{\partial M}{\partial x_n}\right) + M\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \dots + \frac{\partial X_n}{\partial x_n}\right) = 0$$

taking into account (14.3), changes to

$$X\frac{dM}{dx} + M\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \dots + \frac{\partial X_n}{\partial x_n}\right) = 0,$$

or, to

$$X\frac{d\log M}{dx} \quad + \quad \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0. \qquad (14.5)$$

Since the differential equations (14.3) holds for x, $x_1, \ldots, x_n$, this equation is completely identical with equation (14.1). Using (14.3) one can go over from (14.1) to (14.5) and the other way around.

The multiplier M can often be determined from the equation (14.5). If $\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0$, then one finds M = constant. In other cases, by virtue of the differential equations (14.3), the expression

$$\frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right)$$

can be transformed into a total differential in x, a transformation which indeed frequently requires complex analytic devices. If one such is possible, then one obtains M from (14.5).

Now, if one has found in some way, a value M_0 of the multiplier, it can be used for the integration of the system (14.3). For, by means of M_0, one can give the integrating factor of that differential equation which remains to be integrated after finding $n-1$ integrals. From the first equation (14.4), one has

$$M_0 X = A\widetilde{\omega},$$

where $\widetilde{\omega}$is a function of the n solutions of the partial differential equation (14.2), or, as has been proved, a function of n integrals of the system (14.3). If we now assume that one knows $n-1$ of these integrals, namely, $f_2, f_3, \ldots, f_n$, so that only f_1 remains to be found, then we introduce in place of $n-1$ of the independent variables, namely, $x_2, x_3, \ldots, x_n$, the quantities $f_2, f_3, \ldots, f_n$, and express everything through $x, f_2, f_3, \ldots, f_n$. We investigate what changes are thereby brought about in the determinant

$$A = \sum \pm \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}.$$

If we write this as a linear function of the partial derivatives of f_1:

$$A = \frac{\partial f_1}{\partial x_1} B_1 + \cdots + \frac{\partial f_n}{\partial x_n} B_n,$$

then, according to the fundamental property of determinants, the following equations hold:

$$0 = \frac{\partial f_2}{\partial x_1} B_1 + \frac{\partial f_2}{\partial x_2} B_2 + \cdots + \frac{\partial f_2}{\partial x_n} B_n$$
$$0 = \frac{\partial f_3}{\partial x_1} B_1 + \frac{\partial f_3}{\partial x_2} B_2 + \cdots + \frac{\partial f_3}{\partial x_n} B_n$$
$$\cdots$$
$$0 = \frac{\partial f_n}{\partial x_1} B_1 + \frac{\partial f_n}{\partial x_2} B_2 + \cdots + \frac{\partial f_n}{\partial x_n} B_n.$$

If we now assume $f_2, f_3, \ldots, f_n$ introduced for $x_2, x_3, \ldots, x_n$, so that f_1 is expressed in the form

$$f_1 = \Phi(x, x_1, f_2, \ldots, f_n),$$

and enclose the differential coefficients of f_1 formed under this hypothesis in brackets, then

$$\frac{\partial f_1}{\partial x_1} = \left(\frac{\partial f_1}{\partial x_1}\right) + \left(\frac{\partial f_1}{\partial f_2}\right)\frac{\partial f_2}{\partial x_1} + \left(\frac{\partial f_1}{\partial f_3}\right)\frac{\partial f_3}{\partial x_1} + \cdots + \left(\frac{\partial f_1}{\partial f_n}\right)\frac{\partial f_n}{\partial x_1},$$
$$\frac{\partial f_1}{\partial x_2} = \left(\frac{\partial f_1}{\partial f_2}\right)\frac{\partial f_2}{\partial x_2} + \left(\frac{\partial f_1}{\partial f_3}\right)\frac{\partial f_3}{\partial x_2} + \cdots + \left(\frac{\partial f_1}{\partial f_n}\right)\frac{\partial f_n}{\partial x_2},$$
$$\cdots$$
$$\frac{\partial f_1}{\partial x_n} = \left(\frac{\partial f_1}{\partial f_2}\right)\frac{\partial f_2}{\partial x_n} + \left(\frac{\partial f_1}{\partial f_3}\right)\frac{\partial f_3}{\partial x_n} + \cdots + \left(\frac{\partial f_1}{\partial f_n}\right)\frac{\partial f_n}{\partial x_n};$$

and, the earlier equation

$$A = \left(\frac{\partial f_1}{\partial x_1}\right) B_1,$$

with

$$B_1 = \sum \pm \frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3} \cdots \frac{\partial f_n}{\partial x_n}.$$

If one substitutes this value of A in the equation

$$M_0 X = A\widetilde{\omega},$$

then one obtains

$$M_0 X = \left(\frac{\partial f_1}{\partial x_1}\right) B_1 \widetilde{\omega}. \tag{14.6}$$

Now, since f_1, which is sought, is an integral of the differential equation still remaining:

$$X\,dx_1 - X_1\,dx = 0,$$

in which the variables $x_2, x_3, \dots, x_n$ have been eliminated by means of the known $n-1$ integrals, so by determining the integrating factor, this differential equation must change to

$$df_1 = 0,$$

or,

$$\left(\frac{\partial f_1}{\partial x_1}\right)dx_1 + \frac{\partial f_1}{\partial x}dx = 0;$$

consequently, the integrating factor sought is

$$\frac{1}{X}\left(\frac{\partial f_1}{\partial x_1}\right),$$

or, according to (14.6),

$$\frac{M_0}{B_1\widetilde{\omega}};$$

i.e., one has, identically,

$$\frac{M_0}{B_1\widetilde{\omega}}(X\,dx_1 - X_1\,dx) = \left(\frac{\partial f_1}{\partial x_1}\right)dx_1 + \left(\frac{\partial f_1}{\partial x}\right)dx = df_1,$$

or,

$$\frac{M_0}{B_1}(X\,dx_1 - X_1\,dx) = \widetilde{\omega} df_1.$$

Here $\widetilde{\omega}$ is an arbitrary function of $f_1, f_2, \dots, f_n$. Meanwhile, with the help of the $n-1$ integrals found, $f_2, f_3, \dots, f_n$ will be equal to constants; then $\widetilde{\omega}$ will be a function merely of f_1 and, moreover, $\widetilde{\omega}\,df_1$ will itself be a total differential, just as df_1. One can therefore omit $\widetilde{\omega}$ in the divisor and obtain $\frac{M_0}{B_1}$ as the multiplier of the differential equation

$$X\,dx_1 - X_1\,dx = 0.$$

With this we are led to the following theorem:

Theorem 14.1 *Let the system of differential equations*

$$dx : dx_1 : dx_2 : \cdots : dx_n = X : X_1 : X_2 : \cdots : X_n$$

be given, of which we know $n-1$ *integrals:*

$$f_2 = \alpha_2, f_3 = \alpha_3, \ldots, f_n = \alpha_n;$$

further one knows a solution M *of the differential equation*

$$X\frac{d\log M}{ddx} + \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0;$$

if by virtue of the preceeding $n-1$ *integrals, the given system of differential equations is reduced to a first order differential equation*

$$X\, dx_1 - X_1\, dx = 0,$$

of two variables, then its integrating factor is

$$\frac{M}{\sum \pm \frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3}\cdots\frac{\partial f_n}{\partial x_n}}.$$

This is the same theorem as was given in Lecture 12. There we found for the multiplier the expression

$$M\sum \pm \frac{\partial x_2}{\partial \alpha_2}\frac{\partial x_3}{\partial \alpha_3}\cdots\frac{\partial x_n}{\partial \alpha_n};$$

since, however, $f_2 = \alpha_2, f_3 = \alpha_3, \ldots, f_n = \alpha_n$, one has, according to theorem as functional determinants introduced in Lecture 13, p.111,

$$\sum \pm \frac{\partial x_2}{\partial \alpha_2}\frac{\partial x_3}{\partial \alpha_3}\cdots\frac{\partial x_n}{\partial \alpha_n} = \frac{1}{\sum \pm \frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3}\cdots\frac{\partial f_n}{\partial x_n}},$$

so that both multipliers are identical.

The name 'multipliers' belonging to the system of differential equations (14.3) which we give to the quantity M defined by the equation (14.1) or (14.5) commends itself because they coincide with *Euler*'s multipliers or integrating factors for the case of two variables x and x_1

Till now we have shown that if the system can be reduced to a differential equation of two variables through $n-1$ integrals, the multiplier of this differential equation can be derived from the multiplier of the system. But this is only a special case of a more general theorem. Namely, if one does not know $n-1$ integrals, but only a smaller number, say $n-k$, so that one can reduce the given system of $n+1$ variables to a system of $k+1$ variables, then, as we have in fact seen, the multiplier of the reduced system can be determined from the multiplier of the given system. This generalisation will allow us to discuss a question concerning multipliers as yet remaining untouched. We have assumed up to now that with every integration of the given system of differential equations a new arbitrary constant occurs. It is, however, necessary to answer the question, whether and in what way the method of last multiplier can be extended to the case in which the arbitrary constants take special values, and in which therefore one cannot arrive at the complete integration of the given system of differential equations. In order to see how from the multipliers of a given system one can find the multiplier of the system reduced to any order, we proceed stepwise. We first take an integral equation $f_n = \alpha_n$ as given, whereby the order of the system can be reduced by one unit, and seek the multiplier of the system so reduced.

For the given system

$$dx : dx_1 : \cdots : dx_n = X : X_1 : \cdots : X_n, \tag{14.7}$$

the multiplier M is defined through the equation (14.1) or (14.5). If we however take all integrals of the system as known, then the solution of a differential equation is no more necessary and one can find M directly, indeed from any of the equations

$$MX = \tilde{\omega} A, MX_1 = \tilde{\omega} A_1, \ldots, MX_n = \tilde{\omega} A_n,$$

where

$$\begin{aligned} A &= \sum \pm \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}, \\ A_1 &= (-1)^n \sum \pm \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_3} \cdots \frac{\partial f_{n-1}}{\partial x_n} \frac{\partial f_n}{\partial x}, \end{aligned}$$

and so on, and $\tilde{\omega}$ is a function of $f_1, f_2 \ldots, f_n$. If we consider the first of the these equations, then

$$MX = \tilde{\omega}(f_1, f_2, \ldots, f_n) \sum \pm \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_2}, \cdots \frac{\partial f_n}{\partial x_n}.$$

If one assumes that the integral $f_n = \alpha_n$ has been found, and that x_n occurs in it, then x_n can be represented through f_n and the remaining variables x; if this expression for x_n is substituted in $f_1, \ldots, f_{n-1}$, then these quantities are functions of $x_1, x_2, \ldots, x_{n-1}$ and f_n. Enclosing within brackets the differential coefficients formed under this hypothesis, one obtains the following values for the elements of the determinant A:

$$\begin{aligned}
&\left(\frac{\partial f_1}{\partial x_1}\right) + \left(\frac{\partial f_1}{\partial f_n}\right)\frac{\partial f_n}{\partial x_1}, \left(\frac{\partial f_2}{\partial x_1}\right) + \left(\frac{\partial f_2}{\partial f_n}\right)\frac{\partial f_n}{\partial x_1}, \dots, \left(\frac{\partial f_{n-1}}{\partial x_1}\right) \\
&\qquad + \left(\frac{\partial f_{n-1}}{\partial f_n}\right)\frac{\partial f_n}{\partial x_1}, \frac{\partial f_n}{\partial x_1}, \\
&\left(\frac{\partial f_1}{\partial x_2}\right) + \left(\frac{\partial f_1}{\partial f_n}\right)\frac{\partial f_n}{\partial x_2}, \left(\frac{\partial f_2}{\partial x_2}\right) + \left(\frac{\partial f_2}{\partial f_n}\right)\frac{\partial f_n}{\partial x_2}, \dots, \left(\frac{\partial f_{n-1}}{\partial x_2}\right) \\
&\qquad + \left(\frac{\partial f_{n-1}}{\partial f_n}\right)\frac{\partial f_n}{\partial x_2}, \frac{\partial f_n}{\partial x_2}, \\
&\left(\frac{\partial f_1}{\partial x_{n-1}}\right) + \left(\frac{\partial f_1}{\partial f_n}\right)\frac{\partial f_n}{\partial x_{n-1}}, \left(\frac{\partial f_2}{\partial x_{n-1}}\right) + \left(\frac{\partial f_2}{\partial f_n}\right)\frac{\partial f_n}{\partial x_{n-1}}, \dots, \\
&\qquad \left(\frac{\partial f_{n-1}}{\partial x_{n-1}}\right) + \left(\frac{\partial f_{n-1}}{\partial f_n}\right)\frac{\partial f_n}{\partial x_{n-1}}, \frac{\partial f_n}{\partial x_{n-1}}, \\
&\qquad \left(\frac{\partial f_1}{\partial f_n}\right)\frac{\partial f_n}{\partial x_n}, \left(\frac{\partial f_2}{\partial f_n}\right)\frac{\partial f_n}{\partial x_n}, \dots, \left(\frac{\partial f_{n-1}}{\partial f_n}\right)\frac{\partial f_n}{\partial x_n}, \frac{\partial f_n}{\partial x_n}.
\end{aligned}$$

As shown on Lecture 12, page 105, one can omit here those terms of the first $n-1$ columns which are proportional to the elements of the last column; thereby the first $n-1$ elements of the last row now vanish, so that $\frac{\partial f_n}{\partial x_n}$ will be a factor of the determinant, and one obtains therefore

$$MX = \tilde{\omega}(f_1, \dots, f_{n-1}, f_n)\frac{\partial f_n}{\partial x_n}\sum \pm \left(\frac{\partial f_1}{\partial x_1}\right)\cdots\left(\frac{\partial f_{n-1}}{\partial x_{n-1}}\right),$$

or since $f_n = \alpha_n$,

$$MX = \tilde{\omega}(f_1, f_2, \dots, f_{n-1}, \alpha_n)\frac{\partial f_n}{\partial x_n}\sum \pm \left(\frac{\partial f_1}{\partial x_1}\right)\cdots\left(\frac{\partial f_{n-1}}{\partial x_{n-1}}\right). \quad (14.8)$$

Now, by virtue of the integral $f_n = \alpha_n$, x_n and dx_n have been eliminated from the given system (14.3), and one is thereby led to the reduced system

$$dx : dx_1 : \cdots : dx_{n-1} = X : X_1 : \cdots : X_{n-1}. \quad (14.9)$$

If μ is a multiplier of this system, one has for its determination the equation

$$\mu X = F\sum \pm \left(\frac{\partial f_1}{\partial x_1}\right)\left(\frac{\partial f_2}{\partial x_2}\right)\cdots\left(\frac{\partial f_{n-1}}{\partial x_{n-1}}\right),$$

where F is an arbitrary function of $f_1, f_2, \ldots, f_{n-1}$. A value of μ corresponds to the choice $F = \tilde{\omega}(f_1, f_2, \ldots, f_{n-1}, \alpha_n)$, which is determined by the equation

$$\mu X = \tilde{\omega}(f_1, f_2, \ldots, f_{n-1}, \alpha_n) \sum \pm \left(\frac{\partial f_1}{\partial x_1}\right)\left(\frac{\partial f_2}{\partial x_2}\right) \cdots \left(\frac{\partial f_{n-1}}{\partial x_{n-1}}\right).$$

From this last and (14.8) one obtains through division

$$M/\mu = \frac{\partial f_n}{\partial x_n}, \text{ or } \mu = \frac{M}{\frac{\partial f_n}{\partial x_n}}.$$

This expression is then the multiplier of the system (14.9).

One can proceed further in the same way. If one knows an integral $f_{n-1} = \alpha_{n-1}$ of the system (14.9), and reduces it thereby to the following:

$$dx : dx_1 : \cdots : dx_{n-2} = X : X_1 : \cdots : X_{n-2},$$

where x_{n-1} is eliminated, then the multiplier of this system is

$$\frac{M}{\frac{\partial f_n}{\partial x_n}\left(\frac{\partial f_{n-1}}{\partial x_{n-1}}\right)}.$$

If through a new integral $f_{n-1} = \alpha_{n-1}$ one eliminates the variable x_{n-2}, then one obtains as multiplier of the resulting system the expression

$$\frac{M}{\frac{\partial f_n}{\partial x_n}\left(\frac{\partial f_{n-1}}{\partial x_{n-1}}\right)\left(\left(\frac{\partial f_{n-2}}{\partial x_{n-2}}\right)\right)},$$

where the brackets signify that f_{n-1} to be expressed through f_n and $x_1, x_2, \ldots, x_{n-1}$ and f_{n-2} through f_n, f_{n-1} and $x_1, x_2, \ldots, x_{n-2}$. If one proceeds thus, one comes finally to the differential equation

$$dx : dx_1 = X : X_1,$$

or

$$X\, dx_1 - X_1\, dx = 0,$$

whose multiplier is

$$\frac{M}{\frac{\partial f_n}{\partial x_n}\frac{\partial f_{n-1}}{\partial x_{n-1}} \cdots \frac{\partial f_2}{\partial x_2}},$$

where the differentiations are to be so understood that the functions $f_n, f_{n-1}, \ldots, f_2$ have been taken as expressed in the form

$$\begin{aligned} f_n &= \varphi_n(x, x_1, x_2, x_3, \ldots, x_{n-2}, x_{n-1}, x_n), \\ f_{n-1} &= \varphi_{n-1}(x, x_1, x_2, x_3, \ldots, x_{n-2}, x_{n-1}, f_n), \\ f_{n-2} &= \varphi_{n-2}(x, x_1, x_2, x_3, \ldots, x_{n-2}, f_{n-1}, f_n), \\ &\ldots \\ f_2 &= \varphi_2(x, x_1, x_2, f_3, \ldots, f_{n-2}, f_{n-1}, f_n). \end{aligned}$$

In this step by step reduction, the integral equation appearing each time is used to eliminate one variable. The first integral $f_n = \alpha_n$ for example, is used to express x_n through $x, x_1, \ldots, x_{n-1}$ and α_n and to substitute the resulting values in $X, X_1, \ldots, X_{n-1}$. Here we have so far looked upon α_n as an arbitrary constant; however, it is easy to see that nothing is changed in this reasoning if one sets for α_n a definite value a_n. Only, in this case the reduced system is no longer equivalent to the given one, but corresponds only to the speical case in which in the integral equation $f_n = \alpha_n$, the arbitrary constant has the special value a_n. Although in the course of the integration one gives the arbitrary constant α_n a special value and thereby may introduce a special integral of the given system in the calculation, still one must know the complete integral $f_n = \alpha_n$, because the knowledge of f_n is necessary for the determination of the multiplier μ from M. It would then not suffice to know a particular integral $x_n = \Phi(x, x_1, \ldots, x_{n-1})$ without arbitrary constant, but one must know how the particular integral is arrived at from the complete integral, and what value one has given the arbitrary constant. Herein lies an extension of the principle of the last multiplier, which may be expressed in the following way:

Given the system of differential equations

$$dx : dx_1 : \cdots dx_n = X : X_1 : \cdots X_n,$$

let an integral of the same with an arbitrary constant be known and brought to the form $f_n = \alpha_n =$ constant. One gives the constant any particular value a_n whatever and solves $f_n = a_n$ for x_n and inserts its value thus obtained in $X, X_1, \ldots, X_{n-1}$. From this one obtains the first reduced system of differential equations

$$dx : dx_1 : \cdots dx_{n-1} = X : X_1 : \cdots X_{n-1},$$

which, however, has no more the generality of the given system, but represents only the case $\alpha_n = a_n$. *Further, for the first reduced system of differential equations let an integral with an arbitrary constant be known and brought to the form* $f_{n-1} = \alpha_{n-1} =$ *constant. One gives the constant* α_{n-1} *the special value* a_{n-1} *and solves* $f_{n-1} = a_{n-1}$ *for* x_{n-1} *and inserts this value thus obtained in the quantities* $X, X_1, \ldots, X_{n-1}$, *so that this gives the second reduced system of differential equations*

$$dx : dx_1 : \cdots dx_{n-2} = X : X_1 : \cdots X_{n-2};$$

and one proceeds in this way until one comes to the differential equation

$$dx : dx_1 = X : X_1;$$

then the multiplier of the last differential equation is

$$\frac{M}{\frac{\partial f_n}{\partial x_n} \frac{\partial f_{n-1}}{\partial x_{n-1}}, \cdots \frac{\partial f_2}{\partial x_2}}.$$

Here, however, $f_n, f_{n-1}, \ldots, f_2$ are no more $n-1$ integrals of the given system, but only $f_n = \alpha_n$ is one such; $f_{n-1} = \alpha_{n-1}$ is an integral of the first reduced system, which represents the special case $\alpha_n = a_n$ of the given system; $f_{n-2} = \alpha_{n-2}$ is an integral of the second reduced system which represents the special case $\alpha_{n-1} = a_{n-1}$ of the first reduced system, and so on.

With this the scope which can be given to the principle of last multiplier is exhausted. We now go to its applications.

Lecture 15

The Multiplier for Systems of Differential Equations with Higher Differential Coefficients. Applications to a System of Mass Points Without Constraints

All our considerations up to now concerned systems of differential equations with only first order differential coefficients. One can look upon systems of this sort as a special case of those in which differential coefficients of arbitrary order occur. But also, conversely, one can, by increasing the number of variables, reduce a system with higher order differential coefficients to the form of a system containing only first order differential coefficients, so that each becomes a special case of the other. We shall first concern ourselves with this reduction of an arbitrary system into another in which only differential coefficients of the first order occur. Let there be a system of i differential equations of $i+1$ variables $t, x, y, z, \ldots$; of which t is looked upon as the independent and $x, y, z, \ldots$ as the dependent variables. Let the highest differential coefficients which occur in these differential equations be mth in x, nth in y, pth in z, etc. If we further assume that we can solve for these highest differential coefficients, so that the differential equations take the following form:

$$\frac{d^m x}{dt^m} = A, \frac{d^n y}{dt^n} = B, \frac{d^p z}{dt^p} = C, \ldots \tag{15.1}$$

where the highest differential coefficients of x, y, z etc. which occur in $A, B, C, \ldots$ are of the $(m-1)$th, $(n-1)$th, $(p-1)$th order, then this is the canonical form of the differential equations that are to be studied. Any given system cannot always directly be reduced to this canonical form (15.1); for example, this will not go through if in one of the differential equations the highest differential coefficients $\frac{d^m x}{dt^m}, \frac{d^n y}{dt^n}, \frac{d^p z}{dt^p}, \ldots$ do not occur. One must then add the differentiation for elimination. For example, suppose in the equation under consideration the highest differential coefficients are $\frac{d^{m-\mu}x}{dt^{m-\mu}}, \frac{d^{n-\nu}y}{dt^{n-\nu}}, \frac{d^{p-\pi}z}{dt^{p-\pi}}, \ldots$, and if $\mu \leq \nu \leq \pi \cdots$, then one differentiates μ times with respect to t and uses the equation so obtained for eliminating $\frac{d^m x}{dt^m}$ from the remaining equations. If, among the equations arising from this eliminaton, there is again one in which none of the highest differential coefficients of $y, z, \ldots$,occur, one has to differentiate these anew, and so on. This consideration suffices to show that the reduction to the canonical form is possible in every case, but at the moment there is no general method for this reduction. To prove one such would be a very fine exercise[1]; it is the same as the problem of determining the number of arbitrary constants which arise in the integration of a given system of differential equations. This number arises directly from the canonical form, it is $m + n + p + \cdots$. The problem of determining the degree of the eliminating equation for a given system of algebraic equations has therefore a certain similarity with the problem in question.

A special case of the canonical form is that in which one eliminates all the variables $y, z, \ldots$ except two, t and x, and arranges them according to the order of the differential coefficient of x with respect to t. This elimination is, however, unnecessary for our consideration; we need only, as was remarked, to assume the differential equations reduced to the form (15.1), where the highest differential coefficients in $A, B, C, \ldots$ are the $(m-1)$th in x, $(n-1)$th in y, $(p-1)$th in $z, \ldots$.

This assumed, we shall introduce $m + n + p + \cdots - i$ new variables, namely,

$$\begin{aligned} x' &= \frac{dx}{dt}, x'' = \frac{dx'}{dt}, \ldots, x^{(m-1)} = \frac{d^{(m-2)}x}{dt}, \\ y' &= \frac{dy}{dt}, y'' = \frac{dy'}{dt}, \ldots, y^{(n-1)} = \frac{d^{(n-2)}y}{dt}, \\ z' &= \frac{dz}{dt}, z'' = \frac{dz'}{dt}, \ldots, z^{(p-1)} = \frac{d^{(p-2)}z}{dt} \\ &\cdots\cdots\cdots \end{aligned} \tag{15.2}$$

[1] Jacobi himself has solved this problem; one finds indication of this in his essay on the multiplier (*Crelles* Journal, vol. XXIX, p. 369) where a further paper concerned with this problem expected later is referred to. Of the two statements on the present problem found in the supplement was one contained in the first edition of these Lectures, which contains a complete exposition of the results, the other, which contains the proof, one finds in volume 64 of the Mathematical Journal (de investigande orderie the de investigando ordine systematis acquationum differentialum valgarium cujuseunque). Both articles now find their place in Volume 5. (Publisher's note)

One can then represent these equations along with the equations (15.1) as the following system

$$\left\{\begin{matrix} dt : dx : dx' : \cdots : dx^{(m-1)} \\ : dy : dy' : \cdots : dy^{(n-1)} \\ : dz : dz' : \cdots : dz^{(p-1)} \\ \cdots \end{matrix}\right\} = \left\{\begin{matrix} 1 : x' : x'' : \cdots : A \\ : y' : y'' : \cdots : B \\ : z' : z'' : \cdots : C \\ \cdots \end{matrix}\right\}. \tag{15.3}$$

If one applies the general theory to this system, then one obtains the differential equation for the multiplier:

$$0 = \frac{d\log M}{dt} + \frac{\partial A}{\partial x^{(m-1)}} + \frac{\partial B}{\partial y^{(n-1)}} + \frac{\partial C}{\partial z^{(p-1)}} + \cdots \tag{15.4}$$

One can then give M in all cases in which the sum

$$\frac{\partial A}{\partial x^{(m-1)}} + \frac{\partial B}{\partial y^{(n-1)}} + \frac{\partial C}{\partial z^{(p-1)}} + \cdots$$

is a total differential coefficient. If, for example,

$$\frac{\partial A}{\partial x^{(m-1)}} + \frac{\partial B}{\partial y^{(n-1)}} + \frac{\partial C}{\partial z^{(p-1)}} + \cdots = 0$$

which namely is the case when A contains no $\frac{d^{m-1}x}{dt^{m-1}}$, B no $\frac{d^{n-1}y}{dt^{n-1}}$, C no $\frac{d^{p-1}z}{dt^{p-1}}$, and so on, then one has

$$M = \text{ a constant}$$

and then according to our theory, when one has reduced the differential equations (15.1) to a first order differential equation of two variables,one can give its integrating factor.

This consideration would not be of very great interest if no such case occurred in practice. However, it does. Namely, so long as the motion of a free system of mass points depends solely on their configuration, so that the resistance of the medium does not come into consideration, the differential equations of motion are

$$m_i\frac{d^2x_i}{dt^2} = X_i, \quad m_i\frac{d^2y_i}{dt^2} = Y_i, \quad m_i\frac{d^2z_i}{dt^2} = Z_i, , \ldots \tag{15.5}$$

where X_i, Y_i, Z_i contain no first differential coefficients; therefore one has

$$\frac{\partial X_i}{\partial x_i'} = 0, \quad \frac{\partial Y_i}{\partial y_i'} = 0, \quad \frac{\partial Z_i}{\partial z_i'} = 0,$$

so

$$\frac{d\log M}{dt} = 0, \quad M = \text{ constant},$$

and the principle of last multiplier is applicable. However, as we shall conclude later, its application finds itself also in a system bound by constraints.

A special case that merits consideration is the one in which the quantities A, B, C in the canonical form of the differential equations

$$\frac{d^m x}{dt^m} = A, \quad \frac{d^n y}{dt^n} = B, \quad \frac{d^p z}{dt^p} = A, \tag{15.6}$$

do not depend on t. In this case one can eliminate t entirely, and indeed very simply by leaving out in the differential equations (15.3), dt on the left hand side and the corresponding term 1 on the right hand side. One obtains in this way a system whose order is one unit less,that is, equal to $m + n + p + \cdots - 1$. If one has integrated this system and thereby expressed all variables, and therefore also x', by one variable, for example x, then t is obtained, as already mentioned earlier, from the differential equation

$$dx - x'\,dt = 0.$$

Then one has

$$dt = \frac{dx}{x'}, \quad t = \int \frac{dx}{x'} + C.$$

So one finds t through a simple quadrature.

If now we have a multiplier M which is independent of t (here belongs the case in which $\frac{\partial A}{\partial x^{(m-1)}} + \frac{\partial B}{\partial y^{(n-1)}} + \frac{\partial C}{\partial z^{(p-1)}} + \cdots = 0$, therefore $M =$ constant), then this value of M gives the last multiplier of the system of order $m + n + p + \cdots - 1$, from which t has been eliminated. One can then carry out *both* the last integrations. On the other hand, if one has only one value of M which contains t, then one has no use for the $(m+n+p+\cdots-1)$th integration but only for the $(m+n+p+\cdots)$th, which leads to the value of t and is already reduced to a quadrature. And indeed this use lies in that one can save one quadrature and can determine t by solving an equation. In fact, according to the first of the equations (14.4) of the preceeding lecture, we have, for the multiplier M of the system of order n holding between the variables $x, x_1, x_2, \ldots, x_n$ and denoted there by (14.3), the formula

$$MX = \widetilde{\omega} \sum \pm \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}, \tag{15.7}$$

where $f_1 = \alpha_1$, $f_2 = \alpha_2$, ..., $f_n = \alpha_n$ represent the integrals of the former system and $\widetilde{\omega}$ is a function of $f_1, f_2, \ldots, f_n$, i.e., since these quantities become constant through the integrals of the system, represents a constant. This we shall apply to system (15.6). If

$$f_1 = \alpha_1, f_2 = \alpha_2, \ldots, f_{m+n+p+\cdots-1} = \alpha_{m+n+p+\cdots-1}$$

are integrals of the reduced system obtained from (15.6) on elimination of t, and if

$$f = t - \int \frac{dx}{x'} = \text{ constant}$$

is the last integral (15.6) leading to the value of t, then one obtains from the formula (15.7), in which t, x, $x', \ldots, x^{(m-1)}$, y, $y', \ldots, y^{(n-1)}$, z, $z', \ldots, z^{(p-1)}, \ldots$ have been inserted in place of x, $x_1, \ldots, x_n$ and at the same time 1 in place of X, the formula

$$\begin{aligned} M &= \widetilde{\omega} \sum \pm \frac{\partial f}{\partial x} \frac{\partial f_1}{\partial x'} \frac{\partial f_2}{\partial x''} \cdots \frac{\partial f_{m-1}}{\partial x^{(m-1)}} \frac{\partial f_m}{\partial y} \cdots \\ &\quad \frac{\partial f_{m+n-1}}{\partial y^{(n-1)}} \frac{\partial f_{m+n}}{\partial z} \cdots \frac{\partial f_{m+n+p-1}}{\partial z^{(p-1)}} \cdots \end{aligned}$$

for the multiplier M of the system (15.6). However, $f = t - \int \frac{dx}{x'}$, where x' is a given function of x and therefore

$$\frac{\partial f}{\partial x} = -\frac{1}{x'}, \frac{\partial f}{\partial x'} = 0, \frac{\partial f}{\partial x''} = 0, \ldots, \frac{\partial f}{\partial z^{(p-1)}} = 0, \text{ etc.}$$

and with this,

$$M = - \text{ constant} \frac{1}{x'} \sum \pm \frac{\partial f_1}{\partial x'} \frac{\partial f_2}{\partial x''} \cdots \frac{\partial f_{m+n+p-1}}{\partial z^{(p-1)}} \cdots .$$

The right side of this equation is also a multiplier of the system of order $(m + n + p + \cdots - 1)$, which is independent of t; so (15.7) gives for the multiplier of this system, which will be denoted by μ, the formula

$$\mu x' = \text{constant } \sum \pm \frac{\partial f_1}{\partial x'} \frac{\partial f_2}{\partial x''} \cdots \frac{\partial f_{m+n+p-1}}{\partial z^{(p-1)}} \cdots ,$$

where μ is an expression independent of t, as is self-evident. We then have

$$M = \text{ constant } \mu,$$

and since M depends on t by assumption, t is obtained by solving this equation. Meanwhile we know, by virtue of the determination of t, already known:

$$t = \int \frac{dx}{x'} + \text{ constant},$$

that the constant must be additively related to t; since the relation of t to the constant also goes over for the above equation for M, M must be of the form

$$e^{mt}N,$$

where N is independent of t. Then one obtains using logarithms

$$mt = \log \frac{\mu}{N} + \log \text{ constant}.$$

If $A, B, C, \ldots$ do not depend on t, then M, if it also does not depend on t, gives the last but one integration. On the other hand, if M depends on t, one can then avoid the quadrature using the knowledge of M, which would otherwise be necessary for the determination of t.

To the first case belong the differential equations (15.5)that hold for the motion of a system of n mass points, since the unknown value $M =$ constant of the multiplier of these is independent of t. The differential equations (15.5) from a system of order $6n$ which, according to our method, is represented through $6n + 1$ variables $x_i, x_i', y_i, y_i', z_i, z_i'$, and t. If one knows $6n - 2 = \nu$ integrals that do not depend on t,

$$f_1 = \alpha_1, f_2 = \alpha_2, \ldots, f_v = \alpha_v,$$

of this system, one can then express all the dependent variables through two, say x_1 and y_1, between which holds the first order differential equation, yet to be integrated:

$$x_1'\, dy_1 - y_1' dx_1 = 0.$$

So the integrating factor R of the last equation can be given. If one denotes the remaining $6n - 2 = \nu$ variables $x_i, x_i', y_i, y_i', z_i, z_i'$ of the $6n$ except x_1 and y_1 by $p_1, p_2, \ldots, p_\nu$, then

$$R = \sum \pm \frac{\partial p_1}{\partial \alpha_1} \cdot \frac{\partial p_2}{\partial \alpha_2} \cdots \frac{\partial p_\nu}{\partial \alpha_\nu},$$

where it is assumed that for the variables $p_1, \ldots, p_\nu$, the values given by the integrals $f_1 = \alpha_1, f_2 = \alpha_2, \ldots, f_\nu = \alpha_\nu$ have been substituted. If the given ν integral equations are solved neither for the variables $p_1, \ldots, p_\nu$ nor for the arbitrary constants $\alpha_1, \alpha_2, \ldots, \alpha_\nu$, and they are denoted by

$$\widetilde{\omega}_1 = 0, \widetilde{\omega}_2 = 0, \ldots, \widetilde{\omega}_\nu = 0,$$

then according to the theorem on functional determinants stated in Lecture 13, one obtains for the integrating factor R, the fraction

$$R = \frac{\sum \pm \frac{\partial \widetilde{\omega}_1}{\partial \alpha_1} \frac{\partial \widetilde{\omega}_2}{\partial \alpha_2} \cdots \frac{\partial \widetilde{\omega}_\nu}{\partial \alpha_\nu}}{\sum \pm \frac{\partial \widetilde{\omega}_1}{\partial p_1} \frac{\partial \widetilde{\omega}_2}{\partial p_2} \cdots \frac{\partial \widetilde{\omega}_\nu}{\partial p_\nu}}.$$

With the assumption made above that the integral equations are solved for the arbitrary constants, one has to set $\widetilde{\omega}_i = f_i - \alpha_i$; then the numerator of the fraction reduces to 1 and the integrating factor would be

$$R = \frac{1}{\sum \pm \frac{\partial f_1}{\partial p_1} \frac{\partial f_2}{\partial p_2} \cdots \frac{\partial f_\nu}{\partial p_\nu}}.$$

A comprehensive case in which the determinant formed by the numerator of the above fraction is significantly simplified is that in which $\widetilde{\omega}_1$ contains only α_1, $\widetilde{\omega}_2$ only α_1 and α_2 etc. and in general $\widetilde{\omega}_i$ depends only on $\alpha_1, \alpha_2, \ldots, \alpha_i$; then the determinant

$$\sum \pm \frac{\partial \widetilde{\omega}_1}{\partial \alpha_1} \frac{\partial \widetilde{\omega}_2}{\partial \alpha_2} \cdots \frac{\partial \widetilde{\omega}_\nu}{\partial \alpha_\nu}$$

reduces to the form

$$\frac{\partial \widetilde{\omega}_1}{\partial \alpha_1} \frac{\partial \widetilde{\omega}_2}{\partial \alpha_2} \cdots \frac{\partial \widetilde{\omega}_\nu}{\partial \alpha_\nu}.$$

Naturally, this form of the integral equations can always be realized through successive elimination. The analogous case for the denominator is the one in which $\widetilde{\omega}_1$ contains only p_1 of all the variables $p_1, \ldots, p_\nu$, $\widetilde{\omega}_2$ only p_1 and p_2 etc., $\widetilde{\omega}_i$ only $p_1, p_2, \ldots, p_i$. Then the determinant

$$\sum \pm \frac{\partial \widetilde{\omega}_1}{\partial \alpha_1} \frac{\partial \widetilde{\omega}_2}{\partial \alpha_2} \cdots \frac{\partial \widetilde{\omega}_\nu}{\partial \alpha_\nu}$$

reduces to the only term

$$\frac{\partial \widetilde{\omega}_1}{\partial p_1} \frac{\partial \widetilde{\omega}_2}{\partial p_2} \cdots \frac{\partial \widetilde{\omega}_\nu}{\partial p_\nu}.$$

If we do not know ν complete integrals but only ν special ones, i.e., those in which the constants $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ are given special values, then we can very well build the determinant in the denominator of R, but not the one in the numerator of R, and so for this purpose it is necessary to know under what form the constants enter in the integrals. However, if it is stipulated, without the arbitrary constants being assigned special values, that in $\widetilde{\omega}_1$ only α_1, in $\widetilde{\omega}_2$ only α_1, α_2 and so on, in $\widetilde{\omega}_i$ only $\alpha_1, \alpha_2, \ldots, \alpha_i$ come in, then we need only to know the form in which α_1 comes in $\widetilde{\omega}_1$, α_2 in $\widetilde{\omega}_2, \ldots$, α_i in $\widetilde{\omega}_i, \ldots$, α_ν in $\widetilde{\omega}_\nu$, in order to know how to build the determinant in the numerator of R. We do not on the contrary need to know how $\widetilde{\omega}_2$ depends on α_1, $\widetilde{\omega}_3$ on α_1 and $\alpha_2, \ldots$, $\widetilde{\omega}_i$ on $\alpha_1, \alpha_2, \ldots, \alpha_{i-1}$, for, as we have seen, the entire determinant reduces to $\frac{\partial \widetilde{\omega}_1}{\partial \alpha_1} \frac{\partial \widetilde{\omega}_2}{\partial \alpha_2} \cdots \frac{\partial \widetilde{\omega}_\nu}{\partial \alpha_\nu}$. This case occurs in the integration of an ordinary differential equation of higher order, if it is assumed that one can carry out the integration completely, but then to integrate further, the arbitrary constant must be given a special value.

Lecture 16

Examples of the Search for Multipliers. Attraction of a Point by a Fixed Centre in a Resisting Medium and in Empty Space

In order to show the applicability of the theory of multipliers, we shall first consider a case in which, deviating from all other examples to which these investigations relate, X_i, Y_i, Z_i will be functions not merely of the coordinates, but will also of the velocities, so that M is not a constant. This case is that of a planet which moves around the sun in a resisting medium. Without taking into account the resistance, it is well known that the equations for the motion of a planet are the following:

$$\frac{d^2x}{dt^2} = -k^2\frac{x}{r^3}, \quad \frac{d^2y}{dt^2} = -k^2\frac{y}{r^3}, \quad \frac{d^2z}{dt^2} = -k^2\frac{z}{r^3},$$

where x, y, z are the heliocentric coordinates of the planet, r its distance from the sun and k^2 the attraction which the sun exerts at unit distance. If $v = \sqrt{x'^2 + y'^2 + z'^2}$ is the velocity of the planet in the direction of the tangent to its trajectory and V the resistance in the same direction, then the components of the resistance along the axes of x, y, z are respectively

$$\frac{Vx'}{v}, \quad \frac{Vy'}{v}, \quad \frac{Vz'}{v}.$$

These quantities are to be added to the right sides of the differential equations, with the same sign as those the terms based on the attraction have. The equations of motion then become

$$\frac{d^2x}{dt^2} = -k^2\frac{x}{r^3} - \frac{Vx'}{v}, \quad \frac{d^2y}{dt^2} = -k^2\frac{y}{r^3} - \frac{Vy'}{v}, \quad \frac{d^2z}{dt^2} = -k^2\frac{z}{r^3} - \frac{Vz'}{v}.$$

If we take the resistance proportional to the nth power of the velocity

$$V = fv^n,$$

where f is a constant, one has the following differential equations

$$\begin{aligned}\frac{d^2x}{dt^2} &= -k^2\frac{x}{r^3} - fv^{n-1}x' = A,\\ \frac{d^2y}{dt^2} &= -k^2\frac{y}{r^3} - fv^{n-1}y' = B,\\ \frac{d^2z}{dt^2} &= -k^2\frac{z}{r^3} - fv^{n-1}z' = C.\end{aligned} \tag{16.1}$$

Comparison of this system with the general form (15.1) and (15.3) of the last lecture gives $m = n = p = 2$; then one obtains, according to the formula (15.4) of the same lecture, for the multiplier M of the system (16.1),

$$0 = \frac{d\log M}{dt} + \frac{\partial A}{\partial x'} + \frac{\partial B}{\partial y'} + \frac{\partial C}{\partial z'},$$

or, if the expressions for A, B, C are inserted,

$$\begin{aligned}\frac{d\log M}{dt} &= f\left\{\frac{\partial(v^{n-1}x')}{\partial x'} + \frac{\partial(v^{n-1}y')}{\partial y'} + \frac{\partial(v^{n-1}z')}{\partial z'}\right\}\\ &= f\left\{3v^{n-1} + (n-1)v^{n-2}\left(x'\frac{\partial v}{\partial x'} + y'\frac{\partial v}{\partial y'} + z'\frac{\partial v}{\partial z'}\right)\right\}.\end{aligned}$$

But

$$\frac{\partial v}{\partial x'} = \frac{x'}{v}, \quad \frac{\partial v}{\partial y'} = \frac{y'}{v}, \quad \frac{\partial v}{\partial z'} = \frac{z'}{v},$$

and so

$$x'\frac{\partial v}{\partial x'} + y'\frac{\partial v}{\partial y'} + z'\frac{\partial v}{\partial z'} = \frac{x'^2 + y'^2 + z'^2}{v} = v,$$

and hence

$$\frac{d\log M}{dt} = (n+2)fv^{n-1}. \tag{16.2}$$

For $n = -2$, then, one has $M =$ constant. This case, however, does not occur in nature, as otherwise the resistance must become smaller the faster the planet moves. We shall then investigate whether without this assumption for n, v^{n-1} can be changed into a total differential coefficient. The theorems of conservation of *vis viva* and surface area do not hold for this problem.Let us investigate instead what forms the equations corresponding to them take here. To obtain the equation analogous to the theorem of *vis viva*, one must multiply the three equations (16.1) by x', y', z' respectively and add; this gives

$$x'\frac{d^2x}{dt^2} + y'\frac{d^2y}{dt^2} + z'\frac{d^2z}{dt^2} = -\frac{k^2}{r^3}(xx' + yy' + zz') - fv^{n-1}(x'^2 + y'^2 + z'^2).$$

Now,

$$\begin{aligned} x'^2 + y'^2 + z'^2 &= v^2, x^2 + y^2 + z^2 = r^2, \\ x'\frac{d^2x}{dt^2} + y'\frac{d^2y}{dt^2} + z'\frac{d^2z}{dt^2} &= v\frac{dv}{dt}, \\ xx' + yy' + zz' &= r\frac{dr}{dt}; \end{aligned}$$

then

$$v\frac{dv}{dt} = -\frac{k^2}{r^2}\frac{dr}{dt} - fv^{n+1},$$

or

$$\frac{1}{2}\frac{d(v^2)}{dt} = k^2\frac{d(\frac{1}{r})}{dt} - fv^{n+1}.$$

and

$$f\int v^{n+1}dt = -\frac{1}{2}v^2 + \frac{k^2}{r}.$$

This is indeed a remarkable result; but we need, not $\int v^{n+1}dt$, but $\int v^{n-1}dt$.

To obtain the equations corresponding to the surface area theorem, we have to build from equation (16.1) the expressions $y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}$, $z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}$, and $x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}$; this gives

$$\begin{aligned} y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2} &= -fv^{n-1}(yz' - zy'), \\ z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2} &= -fv^{n-1}(zx' - z'x), \\ x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} &= -fv^{n-1}(xy' - x'y), \end{aligned}$$

and on integration,

$$\begin{aligned} -f\int v^{n-1}\,dt &= \log(yz' - y'z) - \log\alpha \\ &= \log(zx' - z'x) - \log\beta \\ &= \log(xy' - x'y) - \log\gamma; \end{aligned} \tag{16.3}$$

where $\log\alpha$, $\log\beta$, $\log\gamma$ are arbitrary constants of integration. One obtains from this first the sought for integral $\int v^{n-1}\,dt$, and secondly, two integral equations, namely

$$\frac{yz' - y'z}{\alpha} = \frac{zx' - z'x}{\beta} = \frac{xy' - x'y}{\gamma}, \tag{16.4}$$

which state that the ratios of the quantities $yz' - zy', zx' - xz', xy' - yx'$ are constants, a result which could have been foreseen. For, as the planet cannot cease to move in a plane in a resisting medium, the quantities in question, which, multiplied by dt represent the projections of the surface element described by the heliocentric radius vector, behave, according to a well-known theorem, like the cosines of the angles which the normal to the orbit of the planet makes with the three coordinate axes.

We deduce from equations (16.2) and (16.3) that

$$\log M = (n+2)f \int v^{n-1}\, dt = -(n+2) \log \frac{xy' - yx'}{\gamma},$$

therefore

$$M = \frac{\gamma^{n+2}}{(xy' - yx')^{n+2}},$$

or, leaving out the constant γ^{n+2} ,

$$M = \frac{1}{(xy' - yx')^{n+2}}$$

Thereby we can in fact apply the principle of the last multiplier to this problem. The given system (16.1) is of the sixth order and leads, after elimination of t, to a reduced system of order five. Meanwhile, since the motion takes place in a plane, we can let one coordinate plane, for example, the xy-plane, coincide with the plane of the orbit; then z is to be set equal to 0, and the last equation (16.1) vanishes and there remains a system of the fourth order and after elimination of t a reduced system of third order remains. However, we do not have a single integral of this last system. Because, of the three equations which hold in place of the surface-area theorem,now there exists only one, and it is not an integral equation. It leads only to $\int v^{n-1}\, dt$, the third expression given in (16.3). If now one has found for the system of the third order in question two integrals with two arbitrary constants α_1 and α_2, so that x' and y' can be represented as functions of x and y, and there remains then only the differential equation of the first order

$$x'\, dy - y'\, dx = 0$$

to be integrated. Its multiplier is

$$\frac{\frac{\partial x'}{\partial \alpha_1}\frac{\partial y'}{\partial \alpha_2} - \frac{\partial x'}{\partial \alpha_2}\frac{\partial y'}{\partial \alpha_1}}{(xy' - x'y)^{n+2}}.$$

As a second example of the last multiplier, we shall take one in which we obtain for the multiplier not an unknown differential equation, but one for which all integrations occurring can be carried out, namely the motion of a planet around the sun in a non-resisting medium. One observes easily that the motion must proceed in a plane and that therefore one obtains only a system of the fourth order, and after elimination of t, of the third order. Herein the principles of conservation of *vis viva* and that of surface area give two integrals and the principle of the last multiplier the last. For this problem then the integration can be carried out completely, as one sees a priori. The system of differential equations to be integrated is, as we have already seen above,

$$\frac{d^2x}{dt^2} = -k^2\frac{x}{r^3}, \quad \frac{d^2y}{dt^2} = -k^2\frac{y}{r^3}, \tag{16.5}$$

where k^2 denotes the attraction of the sun at unit distance. Let the two integrals that the principles of *vis viva* and surface area give be

$$f_1 = \alpha, f_2 = \beta,$$

where f_1 and f_2 are functions of x, y, x' and y' Then one finds as last multiplier for the remaining differential equation of x and y the expression

$$M\left(\frac{\partial x'}{\partial \alpha}\frac{\partial y'}{\partial \beta} - \frac{\partial x'}{\partial \beta}\frac{\partial y'}{\partial \alpha}\right) = \frac{M}{\frac{\partial f_1}{\partial x'}\frac{\partial f_2}{\partial y'} - \frac{\partial f_1}{\partial y'}\frac{\partial f_2}{\partial x'}},$$

where M is the multiplier of the system (16.5). But since we have to do with an entirely free motion, $M =$ constant, according to the preceding lecture. One can then set $M = 1$ and obtain as the last multiplier

$$\frac{1}{\frac{\partial f_1}{\partial x'}\frac{\partial f_2}{\partial y'} - \frac{\partial f_1}{\partial y'}\frac{\partial f_2}{\partial x'}}. \tag{16.6}$$

If we now imagine the quantities x' and y' expressed in terms of x and y through the equations $f_1 = \alpha$ and $f_2 = \beta$, and substitute in the differential equation

$$x'\,dy - y'\,dx = 0, \tag{16.7}$$

then this is the equation for which the expression (16.6) must be the multiplier. This we shall prove by carrying out the computation.

When we multiply the equations (16.5) by x' and y' respectively and add, we obtain the theorem of 'vis viva',

$$x'\frac{d^2x}{dt^2} + y'\frac{d^2y}{dt^2} = -k^2\frac{xx' + yy'}{r^3} = -k^2\frac{r'}{r^2},$$

and, on integration,

$$\frac{1}{2}(x'^2 + y'^2) = \frac{k^2}{r} + \alpha. \tag{16.8}$$

One obtains the principle of surface area when one derives from the equation $x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0$ by integration

$$xy' - x'y = \beta. \tag{16.9}$$

Our two integrals then are

$$f_1 = \frac{1}{2}(x'^2 + y'^2) - \frac{k^2}{r} = \alpha, f_2 = xy' - yx' = \beta.$$

From this, one obtains

$$\frac{\partial f_1}{\partial x'} = x', \quad \frac{\partial f_1}{\partial y'} = y', \quad \frac{\partial f_2}{\partial x'} = -y, \quad \frac{\partial f_2}{\partial y'} = x;$$

and by (16.6), the multiplier of (16.7) will be

$$\frac{1}{\frac{\partial f_1}{\partial x'}\frac{\partial f_2}{\partial y'} - \frac{\partial f_1}{\partial y'}\frac{\partial f_2}{\partial x'}} = \frac{1}{xx' + yy'},$$

i.e., the expression

$$\frac{x'\,dy - y'\,dx}{xx' + yy'} \tag{16.10}$$

is a total differential. This we have to prove by determining x' and y' from the equation (16.8) and (16.9). For abbreviation we write

$$\frac{k^2}{r} + \alpha = \lambda;$$

then we have the equations

$$x'^2 + y'^2 = 2\lambda, xy' - x'y = \beta$$

for the determination of x' and y'. The second of these equations is already linear in x' and y', and it now depends on deriving a second linear equation. This one can do best through the well-known identity

$$(x'^2 + y'^2)(x^2 + y^2) = (xx' + yy')^2 + (xy' - x'y)^2.$$

If one inserts in this the values for $x'^2 + y'^2$ and $xy' - x'y$, then one obtains

$$2\lambda r^2 = (xx' + yy')^2 + \beta^2, \qquad xx' + yy' = \sqrt{2\lambda r^2 - \beta^2}.$$

One has then the equation

$$yy' + xx' = \sqrt{2\lambda r^2 - \beta^2}, \quad xy' - yx' = \beta,$$

and these give

$$r^2 y' = \beta x + y\sqrt{2\lambda r^2 - \beta^2}, \quad r^2 x' = -\beta y + x\sqrt{2\lambda r^2 - \beta^2}.$$

If one divides both the equations by

$$r^2(yy' + xx') = r^2\sqrt{2\lambda r^2 - \beta^2},$$

one obtains

$$\frac{y'}{xx' + yy'} = \frac{\beta x}{r^2\sqrt{2\lambda r^2 - \beta^2}} + \frac{y}{r^2}, \frac{x'}{xx' + yy'} = -\frac{\beta y}{r^2\sqrt{2\lambda r^2 - \beta^2}} + \frac{x}{r^2},$$

and if one inserts these values in (16.10),

$$\frac{x'\,dy - y'\,dx}{xx' + yy'} = -\frac{\beta(x\,dx + y\,dy)}{r^2\sqrt{2\lambda r^2 - \beta^2}} + \frac{x\,dy - y\,dx}{r^2}.$$

Now $x\,dx + y\,dy = r\,dr$, further when we substitute for λ its value,

$$\sqrt{2\lambda r^2 - \beta^2} = \sqrt{2\alpha r^2 + 2k^2 r - \beta^2} = \sqrt{R},$$

where R is a function solely of r; then

$$\frac{x'\,dy - y'\,dx}{xx' + yy'} = -\frac{\beta}{\sqrt{R}}\frac{dr}{r} + \frac{x\,dy - y\,dx}{r^2}.$$

The first term on the right side is a total differential since it is equal to dr multiplied by a function of r. The second term has the form already introduced in Lecture 5 (p. 44) of a product of $x\,dy - y\,dx$ and a homogeneous function of order -2 in x and y, which can always be represented as a product of a function of the quotient $\frac{y}{x}$ and its differential, and is therefore a perfect differential. In the present case one has

$$\frac{x\,dy - y\,dx}{r^2} = \frac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} = d\arctan\frac{y}{x}.$$

The expression $\frac{x'\,dy - y'\,dx}{xx' + yy'}$ is then a perfect differential, which was to be proved.

We shall next go on to the differential equations of motion of a system that is not free.

Lecture 17

The Multiplier of the Equations of Motion of a System Under Constraint in the first *Langrange* Form

We have shown in Lecture 7 (page 59) that the differential equations of a system which is bound through the equations of constraint

$$\varphi = 0, \quad \psi = 0, \quad \widetilde{\omega} = 0, \ldots$$

can be brought to the following form:

$$m_i \frac{d^2 x_i}{dt^2} = X_i + \lambda \frac{\partial \varphi}{\partial x_i} + \mu \frac{\partial \psi}{\partial x_i} + \nu \frac{\partial \widetilde{\omega}}{\partial x_i} + \cdots,$$
$$m_i \frac{d^2 y_i}{dt^2} = Y_i + \lambda \frac{\partial \varphi}{\partial y_i} + \mu \frac{\partial \psi}{\partial y_i} + \nu \frac{\partial \widetilde{\omega}}{\partial y_i} + \cdots,$$
$$m_i \frac{d^2 z_i}{dt^2} = Z_i + \lambda \frac{\partial \varphi}{\partial z_i} + \mu \frac{\partial \psi}{\partial z_i} + \nu \frac{\partial \widetilde{\omega}}{\partial z_i} + \cdots,$$

where the multipliers $\lambda, \mu, \nu, \ldots$ are to be determined, as already remarked there, by differentiating the equations $\varphi = 0, \psi = 0, \widetilde{\omega} = 0, \ldots$ twice. When we determine $\lambda, \mu, \nu, \ldots$, we find, as we shall show presently, that these quantities are not independent of x', y', z'. One cannot therefore set the multiplier M equal to 1 here, one must go back for this determination to equation (15.4) of Lecture 15, p. 132. According to this the multiplier M for the system of differential equations

$$\frac{d^m x}{dt^m} = A, \quad \frac{d^n y}{dt^n} = B, \quad \frac{d^p z}{dt^p} = C, \ldots,$$

is defined through the equation

$$0 = \frac{d \log M}{dt} + \frac{\partial A}{\partial x^{(m-1)}} + \frac{\partial B}{\partial y^{(n-1)}} + \frac{\partial C}{\partial z^{(p-1)}} + \cdots.$$

In the present case, this gives

$$-\frac{d\log M}{dt}=\sum_i \frac{1}{m_i}\left(\frac{\partial\varphi}{\partial x_i}\frac{\partial\lambda}{\partial x_i'}+\frac{\partial\varphi}{\partial y_i}\frac{\partial\lambda}{\partial y_i'}+\frac{\partial\varphi}{\partial z_i}\frac{\partial\lambda}{\partial z_i'}\right)$$
$$+\sum_i \frac{1}{m_i}\left(\frac{\partial\psi}{\partial x_i}\frac{\partial\mu}{\partial x_i'}+\frac{\partial\psi}{\partial y_i}\frac{\partial\mu}{\partial y_i'}+\frac{\partial\psi}{\partial z_i}\frac{\partial\mu}{\partial z_i'}\right)$$
$$+\cdots,$$

where on the right hand side, to each of the multipliers $\lambda, \mu, \nu, \ldots$ corresponds a sum. For the application of the theory of multipliers M, it is necessary that the right hand side of this equation be a total differential coefficient. In order to investigate whether this is the case, one must find out the values of $\lambda, \mu, \nu, \ldots$, or at least those of their differential coefficients with respect to x_i', y_i', z_i'. For determining these values, one differentiates twice in succession with respect to t one of these equations of constraint, $\varphi = 0$. The first differentiation gives

$$\sum\left(\frac{\partial\varphi}{\partial x_i}x_i'+\frac{\partial\varphi}{\partial y_i}y_i'+\frac{\partial\varphi}{\partial z_i}z_i'\right)=0;$$

the second differentiation leads to the equation

$$\sum\left(\frac{\partial\varphi}{\partial x_i}x_i''+\frac{\partial\varphi}{\partial y_i}y_i''+\frac{\partial\varphi}{\partial z_i}z_i''\right)+u=0,$$

where u represents the part of the result which arises from the differentiation of the factors $\frac{\partial\varphi}{\partial x_i}, \frac{\partial\varphi}{\partial y_i}, \frac{\partial\varphi}{\partial z_i}$, and is a homogeneous function of order 2 in the $3n$ quantities x_i', y_i', z_i'. If one denotes through the sequence $p_1, p_2, \ldots, p_{3n}$ the complex of all $3n$ coordinates x_i, y_i, z_i, then one can give the function u the form

$$u=\sum\frac{\partial^2\varphi}{\partial p_i^2}p_i'^2+2\sum\sum\frac{\partial^2\varphi}{\partial p_i\partial p_k}p_i'p_k',$$

where the last sum is to be extended over mutually distinct values of i and k. In the same way one is led through two-fold differentiations from the other equations of constraint to the equations

$$\sum\left(\frac{\partial\psi}{\partial x_i}x_i''+\frac{\partial\psi}{\partial y_i}y_i''+\frac{\partial\psi}{\partial z_i}z_i''\right)+v=0,$$
$$\sum\left(\frac{\partial\tilde\omega}{\partial x_i}x_i''+\frac{\partial\tilde\omega}{\partial y_i}y_i''+\frac{\partial\tilde\omega}{\partial z_i}z_i''\right)+w=0,$$

where, according to the notation for the coordinates introduced above, the functions $v, w, \ldots$, have the values

$$v = \sum \frac{\partial^2 \psi}{\partial p_i^2} p_i'^2 + 2 \sum \sum \frac{\partial^2 \psi}{\partial p_i \partial p_k} p_i' p_k',$$

$$w = \sum \frac{\partial^2 \widetilde{\omega}}{\partial p_i^2} p_i'^2 + 2 \sum \sum \frac{\partial^2 \widetilde{\omega}}{\partial p_i \partial p_k} p_i' p_k'.$$

Now in order to obtain $\lambda, \mu, \nu, \ldots$, one has to insert in these equations the values of x_i'', y_i'', z_i'' derived from the given system. Then the equation obtained from φ by differentiating twice gives

$$\begin{aligned} u + \sum \frac{\partial \varphi}{\partial x_i} \cdot \frac{1}{m_i} \left\{ X_i + \lambda \frac{\partial \varphi}{\partial x_i} + \mu \frac{\partial \psi}{\partial x_i} + \nu \frac{\partial \widetilde{\omega}}{\partial x_i} + \cdots \right\} \\ + \sum \frac{\partial \varphi}{\partial y_i} \cdot \frac{1}{m_i} \left\{ Y_i + \lambda \frac{\partial \varphi}{\partial y_i} + \mu \frac{\partial \psi}{\partial y_i} + \nu \frac{\partial \widetilde{\omega}}{\partial y_i} + \cdots \right\} \\ + \sum \frac{\partial \varphi}{\partial z_i} \cdot \frac{1}{m_i} \left\{ Z_i + \lambda \frac{\partial \varphi}{\partial z_i} + \mu \frac{\partial \psi}{\partial z_i} + \nu \frac{\partial \widetilde{\omega}}{\partial z_i} + \cdots \right\} = 0, \end{aligned}$$

or

$$u_1 + a\lambda + b\mu + c\nu + \cdots = 0,$$

if one sets

$$a = \sum \frac{1}{m_i} \left(\frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_i} + \frac{\partial \varphi}{\partial y_i} \frac{\partial \varphi}{\partial y_i} + \frac{\partial \varphi}{\partial z_i} \frac{\partial \varphi}{\partial z_i} \right),$$

$$b = \sum \frac{1}{m_i} \left(\frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_i} + \frac{\partial \varphi}{\partial y_i} \frac{\partial \psi}{\partial y_i} + \frac{\partial \varphi}{\partial z_i} \frac{\partial \psi}{\partial z_i} \right),$$

$$c = \sum \frac{1}{m_i} \left(\frac{\partial \varphi}{\partial x_i} \frac{\partial \widetilde{\omega}}{\partial x_i} + \frac{\partial \varphi}{\partial y_i} \frac{\partial \widetilde{\omega}}{\partial y_i} + \frac{\partial \varphi}{\partial z_i} \frac{\partial \widetilde{\omega}}{\partial z_i} \right),$$

$$\cdots$$

$$u_1 = u + \sum \frac{1}{m_i} \left(\frac{\partial \varphi}{\partial x_i} X_i + \frac{\partial \varphi}{\partial y_i} Y_i + \frac{\partial \varphi}{\partial z_i} Z_i \right).$$

One obtains for each single one of the equations of constraints $\varphi = 0, \psi = 0, \widetilde{\omega} = 0, \ldots$, one such linear equation between the quantities $\lambda, \mu, \nu, \ldots$. If one introduces as in Lecture 7, p.61, the notation

$$(F, \Phi) = \sum \frac{1}{m_i} \left(\frac{\partial F}{\partial x_i} \frac{\partial \Phi}{\partial x_i} + \frac{\partial F}{\partial y_i} \frac{\partial \Phi}{\partial y_i} + \frac{\partial F}{\partial z_i} \frac{\partial \Phi}{\partial z_i} \right),$$

so that

$$(F, \Phi) = (\Phi, F),$$

and sets

$$\begin{aligned} a = (\varphi, \varphi), b = (\varphi, \psi), c = (\varphi, \widetilde{\omega}), \ldots \\ a' = (\psi, \varphi), b' = (\psi, \psi), c' = (\psi, \widetilde{\omega}), \ldots \\ a'' = (\widetilde{\omega}, \varphi), b'' = (\widetilde{\omega}, \psi), c'' = (\widetilde{\omega}, \widetilde{\omega}), \ldots \end{aligned}$$

so that between these quantities the equations

$$a' = b, a'' = c, b'' = c', \ldots$$

hold.Further, if one sets

$$\begin{aligned} u_1 &= u + \sum \frac{1}{m_i}\left(\frac{\partial \varphi}{\partial x_i} X_i + \frac{\partial \varphi}{\partial y_i} Y_i + \frac{\partial \varphi}{\partial z_i} Z_i\right), \\ v_1 &= v + \sum \frac{1}{m_i}\left(\frac{\partial \psi}{\partial x_i} X_i + \frac{\partial \psi}{\partial y_i} Y_i + \frac{\partial \psi}{\partial z_i} Z_i\right), \\ w_1 &= w + \sum \frac{1}{m_i}\left(\frac{\partial \widetilde{\omega}}{\partial x_i} X_i + \frac{\partial \widetilde{\omega}}{\partial y_i} Y_i + \frac{\partial \widetilde{\omega}}{\partial z_i} Z_i\right), \end{aligned}$$

then one has the equations

$$\begin{aligned} u_1 + a\lambda + b\mu + c\nu + \cdots = 0, \\ v_1 + a'\lambda + b'\mu + c'\nu + \cdots = 0, \\ w_1 + a''\lambda + b''\mu + c''\nu + \cdots = 0, \end{aligned}$$

for the determination of $\lambda, \mu, \nu, \ldots$. Instead of solving these for $\lambda, \mu, \nu, \ldots$ and deriving $\frac{\partial \lambda}{\partial x_i'}, \frac{\partial \mu}{\partial x_i'}, \ldots$ by differentiation from the values so found, it is better to differentiate these linear equations directly, as the computation is then simplified considerably. The quantities $a, b, c, \ldots, a', b', c', \ldots$ do not at all contain the differential coefficients x_i', y_i', z_i', and are therefore to be looked upon as constants for these differentiations; further, the quantities $u_1, v_1, w_1, \ldots$ differ from $u, v, w, \ldots$ respectively only by expressions which also do not involve the differential coefficients x_i', y_i', z_i', and therefore $\frac{\partial u_1}{\partial x_i'} = \frac{\partial u}{\partial x_i}, \frac{\partial v_1}{\partial x_i'} = \frac{\partial v}{\partial x_i}$, and so on; then one obtains

$$\begin{aligned} \frac{\partial u}{\partial x_i'} + a\frac{\partial \lambda}{\partial x_i'} + b\frac{\partial \mu}{\partial x_i'} + c\frac{\partial \nu}{\partial x_i'} + \cdots = 0, \\ \frac{\partial v}{\partial x_i'} + a'\frac{\partial \lambda}{\partial x_i'} + b'\frac{\partial \mu}{\partial x_i'} + c'\frac{\partial \nu}{\partial x_i'} + \cdots = 0, \\ \frac{\partial w}{\partial x_i'} + a''\frac{\partial \lambda}{\partial x_i'} + b''\frac{\partial \mu}{\partial x_i'} + c''\frac{\partial \nu}{\partial x_i'} + \cdots = 0. \end{aligned}$$

The function u was defined through the equation

$$u = \sum \frac{\partial^2 \varphi}{\partial p_i^2} p_i'^2 + 2 \sum \sum \frac{\partial^2 \varphi}{\partial p_i \partial p_k} p_i' p_k',$$

where the quantities p denote the $3n$ coordinates x_i, y_i, z_i and in the second sum on the right side i is different from k. Through differentiation with respect to p_i' it gives

$$\frac{\partial u}{\partial p_i'} = 2 \sum_{k=1}^{3n} \frac{\partial^2 \varphi}{\partial p_i \partial p_k} p_k',$$

or if we set again x_i for p_i and x_k, y_k, z_k for the quantities p_k,

$$\frac{\partial u}{\partial x_i'} = 2 \sum_{k=1}^{n} \left(\frac{\partial \varphi}{\partial x_i \partial x_k} x_k' + \frac{\partial \varphi}{\partial x_i \partial y_k} y_k' + \frac{\partial \varphi}{\partial x_i \partial z_k} z_k' \right).$$

However, the sum on the right is the total differential coefficient of $\frac{\partial \varphi}{\partial x_i}$ with respect to t and so one has

$$\frac{\partial u}{\partial x_i'} = 2 \frac{d \frac{\partial \varphi}{\partial x_i}}{dt}.$$

In this equation one can write y or z for x, as is obvious. Futher one writes $v, w, \ldots$ for u and at the same time sets $\psi, \tilde{\omega}, \ldots$ for φ. One has then

$$\frac{\partial u}{\partial x_i'} = 2 \frac{d \frac{\partial \varphi}{\partial x_i}}{dt}, \frac{\partial v}{\partial x_i'} = 2 \frac{d \frac{\partial \psi}{\partial x_i}}{dt}, \frac{\partial w}{\partial x_i'} = 2 \frac{d \frac{\partial \tilde{\omega}}{\partial x_i}}{dt}, \ldots$$

and analogous equations for the partial differential coefficients with respect to y_i' and z_i'. Hereby the linear equations above for the quantities $\frac{\partial \lambda}{\partial x_i'}, \frac{\partial \mu}{\partial x_i'}, \frac{\partial \nu}{\partial x_i'}, \ldots$ change into the following:

$$2 \frac{d \frac{\partial \varphi}{\partial x_i}}{dt} + a \frac{\partial \lambda}{\partial x_i} + b \frac{\partial \mu}{\partial x_i} + c \frac{\partial \nu}{\partial x_i} + \cdots = 0,$$

$$2 \frac{d \frac{\partial \psi}{\partial x_i}}{dt} + a' \frac{\partial \lambda}{\partial x_i} + b' \frac{\partial \mu}{\partial x_i} + c' \frac{\partial \nu}{\partial x_i} + \cdots = 0,$$

$$2 \frac{d \frac{\partial \tilde{\omega}}{\partial x_i}}{dt} + a'' \frac{\partial \lambda}{\partial x_i} + b'' \frac{\partial \mu}{\partial x_i} + c'' \frac{\partial \nu}{\partial x_i} + \cdots = 0.$$

In order to solve these linear equations one must, as is well known, construct the determinant of the quantities

$$\begin{array}{l} a, b, c, \dots \\ a', b', c', \dots \\ a'', b'', c'', \dots \end{array}$$

or, in abbreviated notation, the determinant

$$R = \sum \pm ab'c'' \cdots ;$$

then in order to determine $\frac{\partial\lambda}{\partial x_i'}$ one has to multiply the above equations by $\frac{\partial R}{\partial a}, \frac{\partial R}{\partial a'}, \frac{\partial R}{\partial a''}, \dots$ and obtain by addition

$$0 = R\frac{\partial\lambda}{\partial x_i'} + 2\frac{\partial R}{\partial a}\frac{d\frac{\partial\varphi}{\partial x_i}}{dt} + 2\frac{\partial R}{\partial a'}\frac{d\frac{\partial\psi}{\partial x_i}}{dt} + 2\frac{\partial R}{\partial a''}\frac{d\frac{\partial\tilde{\omega}}{\partial x_i}}{dt} + \cdots$$

Similarly one obtains

$$0 = R\frac{\partial\mu}{\partial x_i'} + 2\frac{\partial R}{\partial b}\frac{d\frac{\partial\varphi}{\partial x_i}}{dt} + 2\frac{\partial R}{\partial b'}\frac{d\frac{\partial\psi}{\partial x_i}}{dt} + 2\frac{\partial R}{\partial b''}\frac{d\frac{\partial\tilde{\omega}}{\partial x_i}}{dt} + \cdots$$

$$0 = R\frac{\partial\nu}{\partial x_i'} + 2\frac{\partial R}{\partial c}\frac{d\frac{\partial\varphi}{\partial x_i}}{dt} + 2\frac{\partial R}{\partial c'}\frac{d\frac{\partial\tilde{\omega}}{\partial x_i}}{dt} + 2\frac{\partial R}{\partial c''}\frac{d\frac{\partial\tilde{\omega}}{\partial x_i}}{dt} + \cdots$$

Analogous equations hold for the differential coefficients of λ, μ, $\nu, \dots$ with respect to y_i', z_i'. The values of all these differential coefficients are to be inserted in the expression given above for $\frac{d\log M}{dt}$, which one can arrange in the following way:

$$\begin{aligned} \frac{d\log M}{dt} = &-\sum\frac{1}{m_i}\left(\frac{\partial\varphi}{\partial x_i}\frac{\partial\lambda}{\partial x_i'} + \frac{\partial\psi}{\partial x_i}\frac{\partial\mu}{\partial x_i'} + \frac{\partial\tilde{\omega}}{\partial x_i}\frac{\partial\nu}{\partial x_i'} + \cdots\right) \\ &-\sum\frac{1}{m_i}\left(\frac{\partial\varphi}{\partial y_i}\frac{\partial\lambda}{\partial y_i'} + \frac{\partial\psi}{\partial y_i}\frac{\partial\mu}{\partial y_i'} + \frac{\partial\tilde{\omega}}{\partial y_i}\frac{\partial\nu}{\partial y_i'} + \cdots\right) \\ &-\sum\frac{1}{m_i}\left(\frac{\partial\varphi}{\partial z_i}\frac{\partial\lambda}{\partial z_i'} + \frac{\partial\psi}{\partial z_i}\frac{\partial\mu}{\partial z_i'} + \frac{\partial\tilde{\omega}}{\partial z_i}\frac{\partial\nu}{\partial z_i'} + \cdots\right). \end{aligned}$$

Then one obtains for the product of R and the first of the three sums on the right side the result

$$-R\sum\frac{1}{m_i}\left(\frac{\partial\varphi}{\partial x_i}\frac{\partial\lambda}{\partial x_i'}+\frac{\partial\psi}{\partial x_i}\frac{\partial\mu}{\partial x_i'}+\frac{\partial\tilde{\omega}}{\partial x_i}\frac{\partial\nu}{\partial x_i'}+\cdots\right)$$

$$=2\frac{\partial R}{\partial a}\sum\frac{1}{m_i}\frac{\partial\varphi}{\partial x_i}\frac{d\frac{\partial\varphi}{\partial x_i}}{dt}+2\frac{\partial R}{\partial a'}\sum\frac{1}{m_i}\frac{\partial\varphi}{\partial x_i}\frac{d\frac{\partial\psi}{\partial x_i}}{dt}$$

$$+2\frac{\partial R}{\partial a''}\sum\frac{1}{m_i}\frac{\partial\psi}{\partial x_i}\frac{d\frac{\partial\tilde{\omega}}{\partial x_i}}{dt}+\cdots$$

$$+2\frac{\partial R}{\partial b}\sum\frac{1}{m_i}\frac{\partial\psi}{\partial x_i}\frac{d\frac{\partial\varphi}{\partial x_i}}{dt}+2\frac{\partial R}{\partial b'}\sum\frac{1}{m_i}\frac{\partial\psi}{\partial x_i}\frac{d\frac{\partial\psi}{\partial x_i}}{dt}$$

$$+2\frac{\partial R}{\partial b''}\sum\frac{1}{m_i}\frac{\partial\psi}{\partial x_i}\frac{d\frac{\partial\tilde{\omega}}{\partial x_i}}{dt}+\cdots$$

$$+2\frac{\partial R}{\partial c}\sum\frac{1}{m_i}\frac{\partial\tilde{\omega}}{\partial x_i}\frac{d\frac{\partial\varphi}{\partial x_i}}{dt}+2\frac{\partial R}{\partial c'}\sum\frac{1}{m_i}\frac{\partial\tilde{\omega}}{\partial x_i}\frac{d\frac{\partial\psi}{\partial x_i}}{dt}$$

$$+2\frac{\partial R}{\partial c''}\sum\frac{1}{m_i}\frac{\partial\tilde{\omega}}{\partial x_i}\frac{d\frac{\partial\tilde{\omega}}{\partial x_i}}{dt}+\cdots+\cdots$$

However, as we have seen, the elements of the determinant are related,

$$b=a',\quad c=a'',\quad ,c'=b'',\ldots,$$

and therefore from a well-known theorem on the solution of linear equations follow the relations

$$\frac{\partial R}{\partial b}=\frac{\partial R}{\partial a'},\quad\frac{\partial R}{\partial c}=\frac{\partial R}{\partial a''},\quad\frac{\partial R}{\partial c'}=\frac{\partial R}{\partial b''},\ldots.$$

From these considerations one can give to the right hand side of the equations above the form

$$\frac{\partial R}{\partial a}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\varphi}{\partial x_i}+2\frac{\partial R}{\partial a'}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\psi}{\partial x_i}+$$

$$2\frac{\partial R}{\partial a''}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\tilde{\omega}}{\partial x_i}+\cdots+2\frac{\partial R}{\partial b'}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\psi}{\partial x_i}\cdot\frac{\partial\psi}{\partial x_i}+$$

$$2\frac{\partial R}{\partial b''}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\psi}{\partial x_i}\cdot\frac{\partial\tilde{\omega}}{\partial x_i}+\cdots+2\frac{\partial R}{\partial c''}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\tilde{\omega}}{\partial x_i}\cdot\frac{\partial\tilde{\omega}}{\partial x_i}+\cdots$$

or, again if one writes $\frac{\partial R}{\partial a'} + \frac{\partial R}{\partial b}, \frac{\partial R}{\partial a''} + \frac{\partial R}{\partial c}, \frac{\partial R}{\partial b''} + \frac{\partial R}{\partial c'}, \ldots$ etc respectively for $2\frac{\partial R}{\partial a'}, 2\frac{\partial R}{\partial a''}, 2\frac{\partial R}{\partial b''}, \ldots,$

$$\begin{gathered}\frac{\partial R}{\partial a}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_i}+\frac{\partial R}{\partial a'}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\varphi}{\partial x_i}\frac{\partial\psi}{\partial x_i}+\\ \frac{\partial R}{\partial a''}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\varphi}{\partial x_i}\frac{\partial\tilde\omega}{\partial x_i}+\cdots+\frac{\partial R}{\partial b}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\psi}{\partial x_i}\frac{\partial\varphi}{\partial x_i}+\\ \frac{\partial R}{\partial b'}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_i}+\frac{\partial R}{\partial b''}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\psi}{\partial x_i}\frac{\partial\tilde\omega}{\partial x_i}+\cdots+\\ \frac{\partial R}{\partial c}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\tilde\omega}{\partial x_i}\frac{\partial\varphi}{\partial x_i}+\frac{\partial R}{\partial c'}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\tilde\omega}{\partial x_i}\frac{\partial\psi}{\partial x_i}+\\ \frac{\partial R}{\partial c''}\sum\frac{1}{m_i}\frac{d}{dt}\frac{\partial\tilde\omega}{\partial x_i}\frac{\partial\tilde\omega}{\partial x_i}+\cdots+\cdots\end{gathered}$$

If one sets the analogous values for the two sums ocurring in the expression for $\frac{d\log M}{dt}$ and reminds oneself of the values of

$$a, a', a'', \ldots, b, b', b'', \ldots, c, c', c'', \ldots,$$

then one obtains

$$\begin{aligned}R\frac{d\log M}{dt} &= \frac{\partial R}{\partial a}\frac{da}{dt}+\frac{\partial R}{\partial a'}\frac{da'}{dt}+\frac{\partial R}{\partial a''}\frac{da''}{dt}+\cdots\\ &\quad+\frac{\partial R}{\partial b}\frac{db}{dt}+\frac{\partial R}{\partial b'}\frac{db'}{dt}+\frac{\partial R}{\partial b''}\frac{db''}{dt}+\cdots\\ &\quad+\frac{\partial R}{\partial c}\frac{dc}{dt}+\frac{\partial R}{\partial c'}\frac{dc'}{dt}+\frac{\partial R}{\partial c''}\frac{dc''}{dt}+\cdots\\ &\quad+\cdots\\ &= \frac{dR}{dt},\end{aligned}$$

and so

$$R\frac{d\log M}{dt} = \frac{dR}{dt},$$

or, with the omission of the constant factor,

$$M = R.$$

One can also derive from the peculiar form of the quantities

$$a, a', a'', \ldots, b, b', b'', \ldots, c, c', c'', \ldots,$$

a remarkable representation for their determinant. We have set above

$$\begin{aligned} a &= (\varphi,\varphi), a' = (\varphi,\psi), a'' = (\varphi,\widetilde{\omega}), \dots \\ b &= (\psi,\varphi), b' = (\psi,\psi), b'' = (\psi,\widetilde{\omega}), \dots \\ c &= (\widetilde{\omega},\varphi), c' = (\widetilde{\omega},\psi), c'' = (\widetilde{\omega},\widetilde{\omega}), \dots \end{aligned}$$

where the quantities enclosed in brackets are similar to the expression

$$(\varphi,\psi) = \sum \frac{1}{m_i}\left(\frac{\partial\varphi}{\partial x_i}\frac{\partial\psi}{\partial x_i} + \frac{\partial\varphi}{\partial y_i}\frac{\partial\psi}{\partial y_i} + \frac{\partial\varphi}{\partial z_i}\frac{\partial\psi}{\partial z_i}\right).$$

These sums can be represented somewhat more simply if, as in the beginning of this Lecture, p.147, all the $3n$ coordinates are denoted by the same letter and $3n$ indices attached. If instead of the coordinates themselves we introduce quantities proportional to them and set

$$\sqrt{m_i}x_i = \xi_{3i-2}, \sqrt{m_i}y_i = \xi_{3i-1}, \sqrt{m_i}z_i = \xi_{3i},$$

so that the $3n$ quantities $\sqrt{m_i}x_i, \sqrt{m_i}y_i, \sqrt{m_i}z_i$ are identical with the $3n$ quantities $\xi_1, \xi_2, \dots, \xi_{3n}$, then the expansion for (φ,ψ) goes over into the form

$$(\varphi,\psi) = \sum \frac{\partial\varphi}{\partial\xi_i}\frac{\partial\psi}{\partial\xi_i},$$

in which the summation extends from $i = 1$ to $i = 3n$. Determinants whose elements are related in the way presented here can be represented as the sum of four squares (see my article "de formatione et proprietatibus determinantium", *Crelles* Journal, Vol. 22, p. 285). If m is the number of functions $\varphi, \psi, \widetilde{\omega}, \dots, \zeta$, or what is the same, of the equations of constraint holding for the mechanical system, and one constructs all possible determinants of the form

$$\sum \pm \frac{\partial\varphi}{\partial\xi_i}\frac{\partial\psi}{\partial\xi'_i}\frac{\partial\widetilde{\omega}}{\partial\xi''_i}\cdots\frac{\partial\zeta}{\partial\xi_i^{(m-1)}},$$

where $i, i', i'', \dots, i^{(m-1)}$ denote any m different numbers of the series $1, 2, \dots, 3n$, then the sum of the squares of these determinants is equal to R. In the article mentioned above,I have made a nice application of this theorem, first published by *Cauchy*[1], to the method of least squares. For the case of a point which moves on a given surface, the

[1] Journal de l'ecole polytechnique,cah. 17

equation to this surface $\varphi = 0$ is the single constraint; thereby the partial determinants from whose squares R can be put together reduce to $\frac{\partial\varphi}{\partial\xi_1} = \frac{1}{\sqrt{m_1}}\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial\xi_2} = \frac{1}{\sqrt{m_1}}\frac{\partial\varphi}{\partial y_1}, \frac{\partial\varphi}{\partial\xi_3} = \frac{1}{\sqrt{m_1}}\frac{\partial\varphi}{\partial z_i}$, so that

$$R = \frac{1}{m_1}\left\{\left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi}{\partial y_1}\right)^2 + \left(\frac{\partial\varphi}{\partial z_1}\right)^2\right\}.$$

The case $m = 3n$ which naturally does not occur in Mechanics (since the number of equations of constraint can at most be equal to $3n - 1$) is the simplest relating to the theorem on determinants. For, then the determinant R reduces to a single square.

Through the equation $M = R = \sum \pm ab'c'' \cdots$, we have found, for a system bound by any constraints whatsoever the multiplier of the system for the first *Lagrange* form of the differential equations and with it, under the assumption that all but one of the integrals are known, also the last multiplier.

Lecture 18

The Multiplier for the Equations of Motion of a Constrained System in Hamiltonian Form

We shall now seek the multiplier of the differential equation of a constrained system in the Hamiltonian form of the differential equations. Let T be half the *vis viva*, n the number of mass points, m the number of equations of constraint. Since along with i, k also will be used to denote a term in a series, the number $3n-m$ will not be denoted by k, but by μ. In Lecture 8, p. 67, we have so represented the $3n$ coordinates as functions of $3n-m$ new variables $q_1, q_2, \ldots, q_{3n-m}$ that the equations of constraint are satisfied identically on substitution of coordinates expressed in this way. We obtained T as a homogeneous function of second order of the quantities q_i' whose coefficients can contain the quantities q_i. We introduced further the quantities $p_i = \frac{\partial T}{\partial q_i'}$ in place of q_i and thus obtained in Lecture 9, p. 77, the differential equations of motion depending on the $2(3n-m)$ variables q_i and p_i in a form that holds also in the case in which no force function exists:

$$\frac{dq_i}{dt} = \frac{\partial T}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial T}{\partial q_i} + Q_i,$$

where

$$Q_i = \sum_{k=1}^{n} \left(X_k \frac{\partial x_k}{\partial q_i} + Y_k \frac{\partial y_k}{\partial q_i} + Z_k \frac{\partial z_k}{\partial q_i} \right).$$

These differential equations can also be written in the following way:

$$dt : dq_1 : dq_2 : \cdots : dq_\mu : dp_1 : \cdots : dp_\mu =$$
$$1 : \frac{\partial T}{\partial p_1} : \frac{\partial T}{\partial p_2} : \cdots : \frac{\partial T}{\partial p_\mu} : -\frac{\partial T}{\partial q_1} + Q_1 : \cdots : -\frac{\partial T}{\partial q_\mu} + Q_\mu.$$

If one applies the theory of multipliers to this system, then it gives

$$0 = \frac{d\log M}{dt} + \sum \frac{\partial \frac{\partial T}{\partial p_i}}{\partial q_i} + \sum \frac{\partial\left(- \frac{\partial T}{\partial q_i} + Q_i\right)}{\partial p_i}.$$

Since, in the problems we consider X_i, Y_i, Z_i depend only on the coordinates x_i, y_i, z_i and not on their differential coefficients, so the functions Q_i also contain only the variables q_i and not their differential coefficients, and therefore also none of the variables p_i. Therefore

$$\frac{\partial Q_i}{\partial p_i} = 0,$$

and

$$\frac{d\log M}{dt} = \sum \frac{\partial^2 T}{\partial p_i \partial q_i} - \frac{\partial^2 T}{\partial q_i \partial p_i} = 0,$$
$$M = \text{ constant.}$$

One can therefore set $M = 1$, so that the multiplier here has the same value as for an entirely free system. In order to obtain the last multiplier for this case, one must eliminate t, which we assume does not explicitly occur in Q_i, from the system of differential equations of order 2μ:

$$\frac{dq_i}{dt} = \frac{\partial T}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial T}{\partial q_i} + Q_i,$$

where i runs from 1 to μ. If one knows for the thereby reduced system of order $2\mu - 1$, $2\mu - 2$ integral equations

$$\widetilde{\omega}_1 = 0, \widetilde{\omega}_2 = 0, \ldots, \widetilde{\omega}_{2\mu-2} = 0,$$

with as many constants $\alpha_1, \alpha_2, \ldots, \alpha_{2\mu-2}$, then by virtue of these one can express all 2μ variables through two of those, say q_1 and q_2; then there is only one differential equation

$$\frac{\partial T}{\partial p_1}\, dq_2 - \frac{\partial T}{\partial p_2} dq_1 = 0$$

to be integrated, whose multiplier is

$$\frac{\sum \pm \frac{\partial \widetilde{\omega}_1}{\partial \alpha_1} \frac{\partial \widetilde{\omega}_2}{\partial \alpha_2} \cdots \frac{\partial \widetilde{\omega}_{2\mu-2}}{\partial \alpha_{2\mu-2}}}{\sum \pm \frac{\partial \widetilde{\omega}_1}{\partial q_3} \frac{\partial \widetilde{\omega}_2}{\partial q_4} \cdots \frac{\partial \widetilde{\omega}_{\mu-2}}{\partial q_\mu} \frac{\partial \widetilde{\omega}_{\mu-1}}{\partial p_1} \cdots \frac{\partial \widetilde{\omega}_{2\mu-2}}{\partial p_\mu}}.$$

If the forces X_i, Y_i, Z_i are the partial differential coefficients of one function U, which can also depend explicitly on t, so that

$$X_i = \frac{\partial U}{\partial x_i}, Y_i = \frac{\partial U}{\partial y_i}, Z_i = \frac{\partial U}{\partial z_i},$$

then $Q_i = \frac{\partial U}{\partial q_i}$, and the differential equation of motion go over (see p. 76?), if one sets

$$T - U = H,$$

into the simple form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

The further investigations, which form the kernel of these Lectures, will be tied to this Hamiltonian form of the differential equations. Everything up to now is to be looked upon as an introduction to it.

Lecture 19

Hamilton's Partial Differential Equation and its Extension to the Isoperimetric Problem

Hamilton's form of the differential equations of motion was derived in Lectures 8 and 9 from the principle that if the initial and final values of the coordinates are given, the variation of the integral $\int (T+U)\,dt$ must vanish. One can express this principle more generally so that it holds when not the initial and final values but other conditions which obtain at the limits are given. In this case, namely, it is not the entire variation of the integral $\int (T+U)\,dt$ which is to be set equal to zero, but only that part standing under the integral sign; the variation can then be expressed without the integral sign, or what is the same, the variation will be a total differential coefficient. In order to make this clear, we must go back to the derivation given in Lecture 8.

Let T be half the *vis viva* and U the force function which can depend, besides the coordinates, on t explicitly. One thinks of the $3n$ coordinates as functions of the $3n-m=\mu$ new variables $q_1, q_2, \ldots, q_\mu$, so represented that the m equations of constraint are fulfilled identically through these expressions; further, let

$$T+U=\varphi;$$

one has then, since φ is a function of the quantities $q_1, \ldots, q_\mu$ and $q_1', \ldots, q_\mu'$,

$$\delta\varphi = \sum \frac{\partial\varphi}{\partial q_i}\delta q_i + \sum \frac{\partial\varphi}{\partial q_i'}\delta q_i',$$

$$\delta\int\varphi\,dt = \int\delta\varphi\,dt = \int\left\{\sum\frac{\partial\varphi}{\partial q_i}\delta q_i\right\}dt + \int\left\{\sum\frac{\partial\varphi}{\partial q_i'}\delta q_i'\right\}dt.$$

However,

$$\int \frac{\partial\varphi}{\partial q_i'}\delta q_i'\,dt = \int \frac{\partial\varphi}{\partial q_i'}\frac{d\delta q_i}{dt}\,dt = \frac{\partial\varphi}{\partial q_i'}\delta q_i - \int \frac{d\frac{\partial\varphi}{\partial q_i'}}{dt}\delta q_i\,dt,$$

and if one integrates between the lower limit τ and the upper limit t, and denotes the value corresponding to the lower limit τ by an index $\circ$ attached above,

$$\int \frac{\partial\varphi}{q_i'}\delta q_i' dt = \frac{\partial\varphi}{\partial q_i'}\delta q_i - \frac{\partial\varphi^\circ}{\partial q_i'}\delta q_i^\circ - \int \frac{d\frac{\partial\varphi}{\partial q_i'}}{dt}\delta q_i dt.$$

Through substitution of this, one obtains

$$\delta\int\varphi dt = \sum\frac{\partial\varphi}{\partial q_i'}\delta q_i - \sum\frac{\partial\varphi^\circ}{\partial q_i'}\delta q_i^0 + \int\sum\left(\frac{\partial\varphi}{\partial q_i} - \frac{d\frac{\partial\varphi}{dq_i'}}{dt}\right)\delta q_i dt.$$

Now as q_i' does not occur in U,

$$\frac{\partial\varphi}{\partial q_i'} = \frac{\partial T}{\partial q_i'} = p_i, \tag{19.1}$$

and further, in consequence of the differential equation in *Lagrange*'s second form given in equation (8.9), the entire expression under the integral sign on the right hand side is

$$\frac{\partial\varphi}{\partial q_i} - \frac{d\frac{\partial\varphi}{\partial q_i'}}{dt} = \frac{\partial(T+U)}{\partial q_i} - \frac{d\frac{\partial T}{\partial q_i'}}{dt}.$$

Therefore, there remains for the variation sought for only the part free of the integral sign, and one has

$$\delta\int\varphi dt = \sum\frac{\partial\varphi}{\partial q_i'}\delta q_i - \sum\frac{\partial\varphi^\circ}{\partial q_i'}\delta q_i^\circ = \sum p_i\delta q_i - \sum p_i^0\delta q_i^0.$$

Under the earlier assumption the initial and final values of q were given, so $\delta q_i = 0$ and $\delta q_i^\circ = 0$, and then the right side of the last equation vanishes. This is not the case according to the present more general assumption. In order to understand correctly the sense in which the variation is taken, one must remind oneself that the part under the integral sign of the variation sought for vanishes only by virtue of the differential equations of motion which are assumed to be satisfied. The quantities q_i and q_i' as well as the quantities p_i must therefore be considered as given functions of t and 2μ constants, and the variations δq_i are merely the alterations of q_i, which rest on the alteration of the values of the 2μ arbitrary constants. The values of these variations which correspond to the lower limit τ of the integral are the quantities δq_i°. If we denote by V the integral whose variation is being considered, and so set

$$V = \int \varphi dt = \int (T+U)dt, \tag{19.2}$$

the last formula can be written as

$$\begin{aligned} \delta V \quad &= \quad p_1\delta q_1 + p_2\delta q_2 + \cdots + p_i\delta q_i + \cdots + p_\mu \delta q_\mu \\ &\quad -p_1^\circ \delta q_1^\circ - p_2^\circ \delta q_2^\circ - \cdots - p_i^\circ \delta q_i^\circ - \cdots - p_\mu^\circ \delta q_\mu^\circ, \end{aligned} \tag{19.3}$$

an expression to which the form $\frac{\partial V}{\partial t}\delta t$ is to be added when one does not consider t to be an independent variable.

This representation for the variation of V is very important. After integration of the differential equations of motion, one can represent all variables and therefore φ as a function of t and the 2μ integration constants, and from this representation of φ, one obtains also V, through quadrature, as a function of t and the previous 2μ constants. The choice of these quantitities which form the system of constants in the integral equations remains at our disposal. If we chose for these the 2μ initial values q_i°, p_i°, then the system of $2\mu + 1$ variables t, q_i, p_i and the 2μ constants q_i°, p_i° together form a system of $4\mu + 1$ quantities which, by virtue of the integral equations are bound to one another by 2μ relations, and of which any 2μ can be looked upon as functions of the remaining $2\mu + 1$. If, for example, we imagine the values of the 2μ quantities p_i, p_i° as expressed in terms of the $2\mu + 1$ quantities t, q_i, q_i° and substitute these values of p_i^0 in V, which is already known to us as a function of the $2\mu + 1$ quantities t, q_i°, p_i°, then $V = \int \varphi dt$ is given as a function of the $2\mu + 1$ quantities $t, q_1, q_2, \ldots, q_\mu, q_1^\circ, q_2^\circ, \ldots, q_\mu^\circ$. If one varies this expression of V, keeping t unchanged, then

$$\begin{aligned} \delta V \quad &= \quad \frac{\partial V}{\partial q_1}\delta q_1 + \frac{\partial V}{\partial q_2}\delta q_2 + \cdots + \frac{\partial V}{\partial q_\mu}\delta q_\mu \\ &\quad + \frac{\partial V}{\partial q_1^\circ}\delta q_1^\circ + \frac{\partial V}{\partial q_2^\circ}\delta q_2^\circ + \cdots + \frac{\partial V}{\partial q_\mu^\circ}\delta q_\mu^\circ. \end{aligned}$$

If one compares this with the expression (19.3) of δV, then one obtains

$$\frac{\partial V}{\partial q_i} = p_i, \frac{\partial V}{\partial q_i^\circ} = -p_i^\circ. \tag{19.4}$$

On the other hand, according to the definition of V given in (19.2),

$$\varphi = \frac{dV}{dt}.$$

But t is firstly contained in V explicitly and in addition, through the variables $q_1, q_2, \ldots, q_\mu$, implicitly; therefore one has

$$\varphi = \frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum \frac{\partial V}{\partial q_i}\frac{dq_i}{dt};$$

or, with the help of (19.4),

$$0 = \frac{\partial V}{\partial t} + \sum p_i q_i' - \varphi,$$

an equation, which, on introducing the function

$$\psi = \sum p_i q_i' - \varphi, \tag{19.5}$$

changes to the following:

$$\frac{\partial V}{\partial t} + \psi = 0. \tag{19.6}$$

The equation (19.6), when ψ is represented in the proper form, is a partial differential equation for V. In fact, the quantities q_i' and the quantities p_i introduced above through the equation

$$p_i = \frac{\partial \varphi}{\partial q_i'}, \tag{19.1}$$

form, as we know, two systems of quantities such that, with the help of q_i and t, the one can be substituted for the other. Therefore any given expression of $3\mu + 1$ variables t, q_i, q_i', p_i can be represented at the same time as a function of the $2\mu + 1$ variables t, q_i, q_i' and as a function of the $2\mu + 1$ variables t, q_i, p_i. One such expression is

$$\psi = \sum p_i q_i' - \varphi. \tag{19.5}$$

If we represented ψ as a function of the quantities t, q_i, p_i and substitute for p_i, according to the first of the equation (19.4), the partial differential coefficient $\frac{\partial V}{\partial q_i}$, then finally ψ is expressed through the quantities $t, q_1, q_2, \ldots, q_\mu, \frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \ldots, \frac{\partial V}{\partial q_\mu}$, and the equation (19.6) takes the form

$$\frac{\partial V}{\partial t} + \psi\left(t, q_1, q_2, \ldots, q_\mu, \frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \ldots, \frac{\partial V}{\partial q_\mu}\right) = 0.$$

This is *Hamilton*'s differential equation which $V = \int \varphi dt$ satisfies, if one looks upon it as a function of $t, q_1, q_2, \ldots, q_\mu$ and $q_1^\circ, q_2^\circ, \ldots, q_\mu^\circ$. The integration of the differential equations of motion gives a solution of this partial differential equation which contains μ arbitrary constants $q_1^\circ, q_2^\circ, \ldots, q_\mu^\circ$.

Everything that has been said up to now holds not only for problems of mechanics, but also when φ, instead of being equal to $T + U$, is an arbitrary function of $t, q_1, q_2, \ldots, q_\mu$, $q_1', q_2', \ldots, q_\mu'$. In problems of mechanics, however, ψ acquires a simple significance, as has already been shown in lecture 9. If then one substitutes for φ the value $\varphi = T + U$ in

$$\psi = \sum p_i q_i' - \varphi,$$

where U depends only on q_i and T is a homogeneous function of the quantities q_i', then

$$\begin{aligned} p_i &= \frac{\partial T}{\partial q_i'}, \\ \sum p_i q_i' &= \sum q_i' \frac{\partial T}{\partial q_i'} = 2T, \\ \psi &= T - U = H, \end{aligned}$$

and the partial differential equation goes over to

$$\frac{\partial V}{\partial t} + H = 0.$$

The results of the considerations up to now for problems of mechanics can be expressed the following way:

If

$$H = T - U, p_i = \frac{\partial T}{\partial q_i'},$$

and H is expressed through the quantities p_i, q_i, then

$$\frac{\partial q_i}{dt} = \frac{\partial H}{\partial p_i}, \frac{\partial p_i}{dt} = -\frac{\partial H}{\partial q_i}$$

are the differential equations of motion. One considers the motion in the interval from τ to t and introduce the initial values $q_1^\circ, q_2^\circ, \ldots, q_\mu^\circ$ and $p_1^\circ, p_2^\circ, \ldots, p_\mu^\circ$ as arbitrary constants in the integral equations. Further, one sets

$$p_i = \frac{\partial V}{\partial q_i}.$$

in H; *then*

$$\frac{\partial V}{\partial t} + H = 0$$

is a partial differential equation of the first order which defines V *as a function of the variables* $t, q_1, q_2, \ldots, q_\mu$. *Now one forms the integral*

$$\int_\tau^t (T + U)dt,$$

where $T + U$ *is, by virtue of the integral equations, a function merely of* t *and the* 2μ *constants* $q_1^\circ, q_2^\circ, \ldots, q_\mu^\circ$, $p_1^\circ, p_2^\circ, \ldots, p_\mu^\circ$, *and expresses the result of the quadrature through* $t, q_1, q_2, \ldots, q_\mu$ *and* $q_1^\circ, q_2^\circ, \ldots, q_\mu^\circ$; *then the value of the integral so represented,*

$$V = \int_\tau^t (T + U)dt$$

is a solution of the partial differential equation

$$\frac{\partial V}{\partial t} + H = 0.$$

If an arbitrary function φ of the quantities q_i, q_i', t, takes the place of $T + U$, one must at the same time substitute in place of the differential equations of motion those which let the variation of the part $\delta \int \varphi dt$ under the integral sign vanish. In order to make this analogy complete, one must bring these differential equations to the same form as that to *Hamilton*'s form of the differential equations of motion.One also replaces the differential coefficients q_i' by $p_i = \frac{\partial \varphi}{\partial q_i'}$, introduces the function $\psi = \sum p_i q_i' - \varphi$ and then proceeds similarly as in lecture 9. If one forms the variation of the function ψ,

$$\delta\psi = \sum q_i' \delta p_i + \sum p_i \delta q_i' - \delta\varphi$$

and one substitutes for $\delta\varphi$ its value

$$\delta\varphi = \sum \frac{\partial \varphi}{\partial q_i}\delta q_i + \sum p_i \delta q_i' + \frac{\partial \varphi}{\partial t}\delta t,$$

which also contains a term proportional to δt, if one does not choose the independent variable. Then one obtains

$$\delta\psi = \sum q_i' \delta p_i - \sum \frac{\partial \varphi}{\partial q_i}\delta q_i - \frac{\partial \varphi}{\partial t}\delta t.$$

If one compares this expression for $\delta\psi$ with that which one obtains when ψ is represented as a function of q_i, p_i and t, that is, with the expression

$$\delta\psi = \sum \left(\frac{\partial\psi}{\partial p_i}\right) \delta p_i + \sum \left(\frac{\partial\psi}{\partial q_i}\right) \delta q_i + \left(\frac{\partial\psi}{\partial t}\right) \delta t,$$

where the partial differential coefficients formed with the assumptions given above, are enclosed in brackets for clarity. Then from the comparison follows

$$q_i' = \left(\frac{\partial\psi}{\partial p_i}\right), \frac{\partial\varphi}{\partial q_i} = -\left(\frac{\partial\psi}{\partial q_i}\right), \frac{\partial\varphi}{\partial t} = -\left(\frac{\partial\psi}{\partial t}\right).$$

Through the second of these three equations, the differential equation

$$\frac{d\frac{\partial\varphi}{\partial q_i'}}{dt} = \frac{\partial\varphi}{\partial q_i},$$

which must be satisfied in order that the part $\delta\int\varphi dt$ of the variation under the integral sign vanishes, is transformed to

$$\frac{dp_i}{dt} = -\left(\frac{\partial\psi}{\partial q_i}\right)$$

while the first of the three equations is identical with

$$\frac{dq_i}{dt} = \left(\frac{\partial\psi}{\partial p_i}\right).$$

So the differential equations of all isoperimetric problem in which only the first differential coefficients appear under the given integral take the form

$$\frac{dq_i}{dt} = \left(\frac{\partial\psi}{\partial p_i}\right), \frac{dp_i}{dt} = -\left(\frac{\partial\psi}{\partial q_i}\right)$$

and the integration of these always leads to a solution of the partial differential equation of the first order

$$\frac{\partial V}{\partial t} + \psi = 0.$$

Omitting the brackets around the differential coefficients $\left(\frac{\partial\varphi}{\partial p_i}\right)$, $\left(\frac{\partial\varphi}{\partial q_i}\right)$ now no longer necessary for distinguishing them, one can express the result obtained for the general case thus: *Let φ be a given function* *of* $t, q_1, q_2, \dots, q_\mu$ *and* $q'_1, q'_2, \dots, q'_\mu$*; one introduces for the differential coeffcients* q'_i*, new variables*

$$p_i = \frac{\partial \varphi}{\partial q_i},$$

sets

$$\psi = \sum p_i q'_i - \varphi,$$

and expresses the function ψ *through the variables* p_i, q_i *and* t*; then the equations*

$$\frac{dq_i}{dt} = \frac{\partial \psi}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial \psi}{\partial q_i}$$

are the differential equations which must be fulfilled in order that the parts of the variation $\delta \int \varphi dt$ *under the integral sign must vanish. One denotes further the values of the* 2μ *variables for the lower limit* τ *by* $q_1^\circ, q_2^\circ, \dots, q_\mu^\circ, p_1^\circ, p_2^\circ, \dots, p_\mu^\circ$ *and introduces these quantities instead of the arbitrary constants in the integral equations of the system. Finally, if one sets*

$$p_i = \frac{\partial V}{\partial q_i},$$

then

$$\frac{\partial V}{\partial t} + \psi = 0$$

is a first order partial differential equation which defines V *as a function of the variables* $t, q_1, \dots, q_\mu$*. If one now forms the integral*

$$\int_\tau^t \varphi dt,$$

where, by virtue of the integral equations, φ *is a function merely of* t *and the* 2μ *constants* $q_1^\circ, q_2^\circ, \dots, q_\mu^\circ$ $p_1^\circ, p_2^\circ, \dots, p_\mu^\circ$*, and expresses the result of the quadrature as a function of* $t, q_1, q_2, \dots, q_\mu$ *and* $q_1^\circ, q_2^\circ, \dots, q_\mu^\circ$*, then the value of the integral so represented:*

$$V = \int_\tau^t \varphi dt,$$

is a solution of the partial differential equation

$$\frac{\partial V}{\partial t} + \psi = 0.$$

The connection between the functions φ and ψ contained in equation (19.5) produces a sort of reciprocity between them. Namely, if one sets

$$\psi = \sum q_i' \frac{\partial \varphi}{\partial q_i'} - \varphi = \sum p_i q_i' - \varphi$$

where

$$p_i = \frac{\partial \varphi}{\partial q_i},$$

and φ is looked upon as a function of q_i, q_i' and t, then at the same time

$$q_i' = \frac{\partial \psi}{\partial p_i},$$

it being assumed that ψ is looked upon as a function of q_i, p_i and t; therefore one has also

$$\varphi = \sum p_i \frac{\partial \psi}{\partial p_i} - \psi, \tag{19.7}$$

in which equation the quantities q_i' are to be introduced in place of the p_i by means of the equations

$$q_i' = \frac{\partial \psi}{\partial p_i}.$$

Then, through equation (19.7), to *any* given function ψ of t and the quantities q_i and p_i, one can find a corresponding function φ of t and the quantities q_i and q_i'; accordingly the equation $\frac{\partial V}{\partial t} + \psi = 0$ represents the most general partial differential equation of the first order which defines V as a function of $t, q_1, q_2, \ldots, q_\mu$, which does not contain V itself and which is solved for $\frac{\partial V}{\partial t}$. Herein this is a remarkable connection between two problems lying far from each other, the isoperimetric problem of the kind described and the integration of first order partial differential equations. This connection can be extended to other isoperimetric problems in which differential coefficients higher than the first appear under the integral.

The solution found for the partial differential equation $\frac{\partial V}{\partial t} + \varphi = 0$ contains, as we have seen, the μ arbitrary constants $q_1^\circ, q_2^\circ, \ldots q_\mu^\circ$, and since the quantity V itself does not occur, one can add to this solution another arbitrary constant and then one has a solution with $\mu + 1$ arbitrary constants. The solution V is therefore what one calls a complete solution of a first order partial differential equation, since one such must contain as many mutually independent constants as the mutually independent variables which occur the differential equation.

Just as the integration of the isoperimetric equations or the equations of motion give this complete solution of the partial differential equation $\frac{\partial V}{\partial t} + \psi = 0$, so also, conversely, from the complete solutions, assumed known, one can form the integral equations of the isoperimetric or differential equations of mechanics, and these are contained in the equation

$$\frac{\partial V}{\partial q_i} = p_i, \frac{\partial V}{\partial q_i^{\circ}} = -p_i^{\circ}, \tag{19.4}$$

already considered above (p.162), which also hold for the isoperimetric problem in question. We have then represented the integral equation under the same from as the differential equations earlier, namely by means of the partial differential coefficients of *one* function V. This is the discovery of *Hamilton* who has named the function V *principal function.* The second of the equations contained in (19.4), $\frac{\partial V}{q_i^{\circ}} = p_i^{\circ}$ gives the actual integral equation; the first system $\frac{\partial V}{q_i} = p_i$ gives the quantities p_i or q_i' as functions of t and q_i with μ constants q_i°. This is the system of the first integral equations, but it is of great importance that these also can be represented through the partial differential coefficients of V. We shall see later that the μ constants contained in V need not be the initial values of q_i°, but if one only knows generally a complete solution V of the partial differential equation $\frac{\partial V}{\partial t} + \psi = 0$, then the integral equations can always be expressed by the partial differential coefficients of this solution with respect to the constants contained in them.

Hamilton who has presented his discovery in two articles in the Philosophical Transactions,[1] defines V not only through the one partial differential equation $\frac{\partial V}{\partial t} + \psi = 0$ but he also gives also a second partial differential equation which V should satisfy. However, one can omit this because it can be derived from the one already given and because its addition only takes away the simplicity from the investigation. For the question of determining a function satisfying two simultaneous partial differential equations cannot in general be answered by the current methods of analysis.

In order to derive the second partial differential equation from the equation $\frac{\partial V}{\partial t} + \psi = 0$ already found, we need the following theorem, which is easy to prove.

[1] 1834, Part II and 1835, Part I

Let a system of n ordinary differential equations hold between $n+1$ *variables* $t, x_1, x_2, \ldots, x_n$*, and let the values of the remaining variables corresponding to the initial value* τ *of* t*, be* $x_1^\circ, x_2^\circ, \ldots, x_n^\circ$*, and suppose that the given system of differential equations are satisfied by the system of integral equations*

$$\begin{aligned} x_1 &= f_1(t, \tau, x_1^\circ, x_2^\circ, \ldots, x_n^\circ) \\ x_2 &= f_2(t, \tau, x_1^\circ, x_2^\circ, \ldots, x_n^\circ) \\ \ldots &\quad \ldots\ldots\ldots \\ x_n &= f_n(t, \tau, x_1^\circ, x_2^\circ, \ldots, x_n^\circ). \end{aligned} \tag{A}$$

Then one obtains an equivalent system of integral equations through interchange of the variables $t, x_1, x_2, \ldots, x_n$ *with their initial values* τ, x_1°, $x_2^\circ, \ldots, x_n^\circ$*, so that one saves entirely the troublesome task of eliminations and without further computation represents the integral equations solved for the arbitrary constants in the following way:*

$$\begin{aligned} x_1^\circ &= f_1(\tau, t, x_1, x_2, \ldots, x_n) \\ x_2^\circ &= f_2(\tau, t, x_1, x_2, \ldots, x_n) \\ \ldots &\quad \ldots\ldots\ldots \\ x_n^\circ &= f_n(\tau, t, x_1, x_2, \ldots, x_n). \end{aligned} \tag{B}$$

The proof of this theorem is the following. If the given system of differential equations satisfies the system of integral equations

$$\begin{aligned} x_1 &= F_1(t, \alpha_1, \alpha_2, \ldots, \alpha_n) \\ x_2 &= F_2(t, \alpha_1, \alpha_2, \ldots, \alpha_n) \\ \ldots &\quad \ldots\ldots\ldots \\ x_n &= F_n(t, \alpha_1, \alpha_2, \ldots, \alpha_n), \end{aligned} \tag{C}$$

then the same system of equations follows for the initial values, namely,

$$\begin{aligned} x_1^\circ &= F_1(\tau, \alpha_1, \alpha_2, \ldots, \alpha_n) \\ x_2^\circ &= F_2(\tau, \alpha_1, \alpha_2, \ldots, \alpha_n) \\ \ldots &\quad \ldots\ldots\ldots \\ x_n^\circ &= F_n(\tau, \alpha_1, \alpha_2, \ldots, \alpha_n). \end{aligned} \tag{D}$$

The system (A) must arise out of (C) and (D) through elimination of $\alpha_1, \alpha_2, \ldots, \alpha_n$. But the systems (C) and (D) transform into each other if one interchanges t with τ, and likewise x_1 with x_1°, x_2 with $x_2^\circ, \ldots, x_n$ with x_n°. Consequently one can make the same interchange in (A), and the system (B) given by this must be equivalent to (A).

A remarkable result can be derived from this theorem. The equations (B) are integrals, i.e. such integral equations which, if one differentiates them with the help of the differential equations give a result that vanishes identically. Any of the equations (A), on the contrary, contains n constants none of which is superfluous (supervacanea).[2] We obtain all the integral equations one after the other, if we differentiate one of them e.g. $x_1 = f_1(t, \tau, x_1^\circ, x_2^\circ, \dots, x_n^\circ)$, taking the help of the differential equations and continues the operations. Such an advantage one cannot in general follow from the knowledge of an integral: constant $= F(t, \tau, x_1, x_2, \dots, x_n)$ where τ denotes a special value of t. However, if it be the case that the constant is precisely the value of a variable, e.g. x_1, corresponding to the value τ of t, then one can derive all integral equations from one integral with only one constant. This case arises as soon as the function $F(t, \tau, x_1, \dots, x_n)$ reduces to x_1 for $t = \tau$; then according to the theorem above, one can interchange the variable with its initial value and obtain from one integral

$$\text{constant } = F(t, \tau, x_1, x_2, \dots, x_n),$$

the integral equation

$$x_1 = F(\tau, t, x_1^\circ, x_2^\circ, \dots, x_n^\circ),$$

from which all the others can be derived through successive differentiation.

We shall now see what becomes of V with the interchange of the variables with their initial values. Let the isoperimetric or dynamical differential equations be integrated using the system

$$\begin{aligned} q_1 &= \chi_1(t, \alpha_1, \alpha_2, \dots, \alpha_{2\mu}), p_1 = \tilde{\omega}_1(t, \alpha_1, \alpha_2, \dots, \alpha_{2\mu}), \\ q_2 &= \chi_2(t, \alpha_1, \alpha_2, \dots, \alpha_{2\mu}), p_2 = \tilde{\omega}_2(t, \alpha_1, \alpha_2, \dots, \alpha_{2\mu}), \\ &\dots\dots\dots \\ q_\mu &= \chi_\mu(t, \alpha_1, \alpha_2, \dots, \alpha_{2\mu}), p_\mu = \tilde{\omega}_\mu(t, \alpha_1, \alpha_2, \dots, \alpha_{2\mu}). \end{aligned}$$

If one sets the initial value τ for t, then one has at the same time

$$\begin{aligned} q_1^\circ &= \chi_1(\tau, \alpha_1, \alpha_2, \dots, \alpha_{2\mu}), p_1^\circ = \tilde{w}_1(\tau, \alpha_1, \alpha_2, \dots, \alpha_{2\mu}), \\ q_2^\circ &= \chi_2(\tau, \alpha_1, \alpha_2, \dots, \alpha_{2\mu}), p_2^\circ = \tilde{w}_1(\tau, \alpha_1, \alpha_2, \dots, \alpha_{2\mu}), \\ &\dots\dots\dots \\ q_\mu^\circ &= \chi_\mu(\tau, \alpha_1, \alpha_2, \dots, \alpha_{2\mu}), p_\mu^\circ = \tilde{w}_\mu(\tau, \alpha_1, \alpha_2, \dots, \alpha_{2\mu}). \end{aligned}$$

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[2]See the article " dilucidationes de aequatt. diff. vulg. systematis", *Crelles* Jour. Vol. 23.

In the integral

$$V = \int_\tau^t \varphi dt$$

φ is a function of $t, q_1, q_2, \ldots, q_\mu, p_1, p_2, \ldots, p_\mu$, and so, after inserting the values of $q_1, q_2, \ldots, q_\mu, p_1, p_2, \ldots, p_\mu$ from the integral equations, a function merely of $t, \alpha_1, \alpha_2, \ldots, \alpha_{2\mu}$. One can accordingly set

$$\int \varphi dt = \Phi(t, \alpha_1, \alpha_2, \ldots, \alpha_{2\mu})$$

and obtain

$$V = \int_\tau^t \varphi dt = \Phi(t, \alpha_1, \ldots, \alpha_\mu) - \Phi(\tau, \alpha_1, \ldots, \alpha_{2\mu}).$$

The quantity V determined in this way will be a complete solution of the partial differential equation $\frac{\partial V}{\partial t} + \psi = 0$ if the constants $\alpha_1, \alpha_2, \ldots, \alpha_{2\mu}$ are eliminated by virtue of the above 2μ equations for $q_1, q_2, \ldots, q_\mu$, $q_1^\circ, q_2^\circ, \ldots, q_\mu^\circ$. But, of these 2μ differential equations, one half goes over into the other half if one interchanges t with τ and the quantities q_i with the quantities q_i°. Therefore, each of the quantities $\alpha_1, \alpha_2, \ldots, \alpha_\mu$, expressed as a function of t, $q_1, q_2, \ldots, q_\mu$, $q_1^\circ, q_2^\circ, \ldots, q_\mu^\circ$, must be of such a nature that it remains unchanged when t is interchanged with τ, q_1 with q_1°, q_2 with $q_2^\circ, \ldots, q_\mu$ with q_μ°. From this it becomes clear that this interchange transforms,

$$V = \Phi(t, \alpha_1, \alpha_2, \ldots, \alpha_{2\mu}) - \Phi(\tau, \alpha_1, \alpha_2, \ldots, \alpha_{2\mu}),$$

into

$$V = \Phi(\tau, \alpha_1, \alpha_2, \ldots, \alpha_{2\mu}) - \Phi(t, \alpha_1, \alpha_2, \ldots, \alpha_{2\mu}),$$

i.e. into $-V$.

Till now we have not made any special assumption about the differential equations in all our discussions. In order to study the case considered by *Hamilton* we must now assume that the variable t does not occur in φ explicitly. This occurs in mechanics when the time t is not contained in the force function U and consequently also not in $\psi = H = T - U$. Then only the differential of t enters in the differential equations of motion:

$$dt : dq_1 : dq_2 : \ldots : dq_\mu : dp_1 : \ldots : dp_\mu = 1 : \frac{\partial\psi}{\partial p_1} : \frac{\partial\psi}{\partial p_2} : \ldots : \frac{\partial\psi}{\partial p_\mu} : -\frac{\partial\psi}{\partial q_i} : \ldots : -\frac{\partial\psi}{\partial q_\mu};$$

By omitting dt and 1, one eliminates the time entirely, expresses all variables, after integration of the remaining system, through one, e.g. q_1, and determines this last as a function of time if one solves for q_1 the equation

$$t - \tau = \int_{q_1^\circ}^{q_1} \frac{dq_1}{\frac{\partial \psi}{\partial p_1}}$$

arising by integration from the differential formula

$$dt = \frac{dq_1}{\frac{\partial \psi}{\partial p_1}}.$$

Then one obtains q_1 as a function of $t - \tau$, and since the remaining variables have already been expressed as function of q_1, all variables depend only on the difference $t - \tau$. This holds also for the function V which likewise contains both the quantities t and τ only in the combination $\theta = t - \tau$, and one has therefore

$$\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau} = \frac{\partial V}{\partial \theta}.$$

If now one interchanges $t, q_1, q_2, \dots q_\mu$ with the initial values $\tau, q_1^\circ, q_2^\circ, \dots q_\mu^\circ$, then V changes to $-V$ and θ to $-\theta$, and $\frac{\partial V}{\partial \theta}$ remains unchanged. If further, $\psi_\circ$ denotes the value into which ψ goes over when the quantities q_i and $p_i = \frac{\partial V}{\partial q_i}$ are interchanged with q_i° and $p_1^\circ = -\frac{\partial V}{\partial q_i^\circ}$, then the equation

$$0 = \frac{\partial V}{\partial t} + \psi = \frac{\partial V}{\partial \theta} + \psi$$

changes to

$$0 = \frac{\partial V}{\partial \theta} + \psi_\circ = -\frac{\partial V}{\partial \tau} + \psi_\circ.$$

This is *Hamilton*'s second partial differential equation which, we have now proved, can be derived from the first exchanging the variables with their initial values.

Lecture 20

Proof that the integral equations derived from a complete solution of *Hamilton*'s partial differential equation actually satisfy the system of ordinary differential equations. *Hamilton*'s equation for free motion

We shall now take the reverse step and prove how starting from the partial differential equations under consideration one is led to the dynamical or isoperimetric equations.*Let*

$$\frac{\partial V}{\partial t} + \psi = 0 \tag{20.1}$$

be an arbitrary partial differential equation of the first order, which does not contain V itself, so that ψ is any function of the quantities $t, q_1, q_2, \ldots, q_\mu, p_1, p_2, \ldots, p_\mu$, where $p_i = \frac{\partial V}{\partial q_i}$;. Suppose one knows a complete solution of the partial differential equation (20.1), i.e. a solution which contains only μ arbitrary constants $\alpha_1, \alpha_2, \ldots, \alpha_\mu$ except for those related to V through addition. If one sets now

$$\frac{\partial V}{\partial \alpha_1} = \beta_1, \frac{\partial V}{\partial \alpha_2} = \beta_2, \ldots, \frac{\partial V}{\partial \alpha_\mu} = \beta_\mu. \tag{20.2}$$

where $\beta_1, \beta_2, \ldots, \beta_\mu$ denote new arbitrary constants, then these equations, along with the equation

$$\frac{\partial V}{\partial q_1} = p_1, \frac{\partial V}{\partial q_2} = p_2, \ldots, \frac{\partial V}{\partial q_\mu} = p_\mu$$

are the integral equations of the differential equations

$$\frac{dq_i}{dt} = \frac{\partial \psi}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial \psi}{\partial q_i}, \tag{20.3}$$

where i takes the values $1, 2, \ldots, \mu$.

In order to prove this theorem, we have to observe that if the complete solution, assumed known, is substituted for V in the partial differential equation (20.1), the left hand side of the same must be an identically vanishing function of the quantities t, q_1, $q_2, \ldots, q_\mu$, $\alpha_1, \alpha_2, \ldots, \alpha_\mu$ and that accordingly, its partial differential coefficient with respect to any one of these quantities vanishes identically.

To derive the first half of the differential equation (20.3) from the equation (20.2), we proceed in the following way. If we differentiate the equation (20.2) with respect to t, we obtain the system of equations

$$\begin{aligned}
0 &= \frac{\partial^2 V}{\partial\alpha_1\partial t} + \frac{\partial^2 V}{\partial\alpha_1\partial q_1}\frac{dq_1}{dt} + \frac{\partial^2 V}{\partial\alpha_1\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial^2 V}{\partial\alpha_1\partial q_\mu}\frac{dq_\mu}{dt}, \\
0 &= \frac{\partial^2 V}{\partial\alpha_2\partial t} + \frac{\partial^2 V}{\partial\alpha_2\partial q_1}\frac{dq_1}{dt} + \frac{\partial^2 V}{\partial\alpha_2\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial^2 V}{\partial\alpha_2\partial q_\mu}\frac{dq_\mu}{dt}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0 &= \frac{\partial^2 V}{\partial\alpha_\mu\partial t} + \frac{\partial^2 V}{\partial\alpha_\mu\partial q_1}\frac{dq_1}{dt} + \frac{\partial^2 V}{\partial\alpha_\mu\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial^2 V}{\partial\alpha_\mu\partial q_\mu}\frac{dq_\mu}{dt}.
\end{aligned} \tag{20.4}$$

It would now amount to this: to solve these equations, linear with respect to $\frac{dq_1}{dt}, \frac{dq_2}{dt}, \cdots, \frac{dq_\mu}{dt}$, and to show that the values arising from the solution are identical with the quantities $\frac{\partial\psi}{\partial p_1}, \frac{\partial\psi}{\partial p_2}, \cdots, \frac{\partial\psi}{\partial p_\mu}$. But this identity can be obtained also without solving the equations if one proves that the quantities $\frac{dq_i}{dt}$ and the quantities $\frac{\partial\psi}{\partial p_i}$ satisfy the same system of linear equations. For this proof one must differentiate partially the partial differential equation $\frac{\partial V}{\partial t} + \psi = 0$ with respect to the constants $\alpha_1, \alpha_2, \ldots, \alpha_\mu$, and hereby observe that of the quantities t, q_i and p_i of which ψ is a function, only the last, p_i, contain the constants $\alpha_1, \alpha_2, \ldots, \alpha_\mu$. Differentiation with respect to α_i gives

$$0 = \frac{\partial^0 V}{\partial t\partial\alpha_i} + \frac{\partial\psi}{\partial p_1}\frac{\partial p_1}{\partial\alpha_i} + \frac{\partial\psi}{\partial p_2}\frac{\partial p_2}{\partial\alpha_i} + \cdots + \frac{\partial\psi}{\partial p_\mu}\frac{\partial p_\mu}{\partial\alpha_i},$$

and since

$$p_1 = \frac{\partial V}{\partial q_1}, p_2 = \frac{\partial V}{\partial q_2}, \ldots, p_\mu = \frac{\partial V}{\partial q_\mu},$$

so, $\frac{\partial p_k}{\partial\alpha_i} = \frac{\partial^2 V}{\partial\alpha_i\partial q_k}$, one obtains from these equations for $i = 1, 2, \ldots, \mu$, a system of differential equations which differs from the system (20.4) only in that, the quantities $\frac{\partial\psi}{\partial p_i}$ enter in the place of $\frac{dq_i}{dt}$. Hence we conclude that $\frac{dq_i}{dt} = \frac{\partial\psi}{\partial p_i}$ (See the remark on the following page).

For the derivation of the second half of the differential equations (20.3), the equations $\frac{dp_i}{dt} = -\frac{\partial\psi}{\partial q_i}$, we take the help of the second half of the integral equations, i.e., the equations

$$\frac{\partial V}{\partial q_i} = p_i,$$

which form the first system of integral equations, since they represent relations between the quantities q_i and q_i' and the μ arbitrary constants. The equation $p_i = \frac{\partial V}{\partial q_i}$ gives, on total differentiation in t,

$$\frac{dp_i}{dt} = \frac{\partial^2 V}{\partial q_i \partial t} + \frac{\partial^2 V}{\partial q_i \partial q_1}\frac{dq_1}{dt} + \frac{\partial^2 V}{\partial q_1 \partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial^2 V}{\partial q_i \partial q_\mu}\frac{dq_\mu}{dt}.$$

If we write $\frac{\partial p_1}{\partial q_i}, \frac{\partial p_2}{\partial q_i}, \ldots, \frac{\partial p_\mu}{\partial q_i}$ respectively for $\frac{\partial^2 V}{\partial q_i \partial q_1}, \frac{\partial^2 V}{\partial q_i \partial q_2}, \ldots, \frac{\partial^2 V}{\partial q_i \partial q_\mu}$, and use the already derived equations $\frac{dq_1}{dt} = \frac{\partial\psi}{\partial p_1}, \frac{dq_2}{dt} = \frac{\partial\psi}{\partial p_2}, \ldots, \frac{dq_\mu}{dt} = \frac{\partial\psi}{\partial p_\mu}$, one obtains

$$\frac{dp_i}{dt} = \frac{\partial^2 V}{\partial q_i \partial t} + \frac{\partial p_1}{\partial q_i}\frac{\partial\psi}{\partial p_1} + \frac{\partial p_2}{\partial q_i}\frac{\partial\psi}{\partial p_2} + \cdots + \frac{\partial p_\mu}{\partial q_i}\frac{\partial\psi}{\partial p_\mu} \tag{20.5}$$

On the other hand, if we differentiate the equation $\frac{\partial V}{\partial t} + \psi = 0$ partially with respect to q_i, we find

$$0 = \frac{\partial^2 V}{\partial q_i \partial t} + \frac{\partial\psi}{\partial p_1}\frac{\partial p_1}{\partial q_i} + \frac{\partial\psi}{\partial p_2}\frac{\partial p_2}{\partial q_i} + \cdots + \frac{\partial\psi}{\partial p_\mu}\frac{\partial p_\mu}{\partial q_i} + \frac{\partial\psi}{\partial q_i};$$

and this equation substracted from (20.5) leads to the result

$$\frac{dp_i}{dt} = -\frac{\partial\psi}{\partial q_i}.$$

With this, the second half of the differential equations is also derived and the theorem stated above is completely proved. It is important that according to the result derived, the μ constants contained in V can be chosen arbitrarily, and need not be the initial values $q_1^\circ, q_2^\circ, \ldots, q_\mu^\circ$, since for introducing the initial values one has to solve equations or resort to elimination to carry out troublesome operations in most cases. These can now be avoided.

One point of the preceding proof deserves a special mention. Since we saw that the equation (20.4) set up for the quantities $\frac{dq_i}{dt}$, hold also for the quantities $\frac{\partial\psi}{\partial p_i}$ we concluded from this that the quantities $\frac{dq_i}{dt}$

and $\frac{\partial\psi}{\partial p_i}$ are equal to one another. However, we are justified in reaching this conclusion only when the quantities $\frac{dq_i}{dt}$ take finite and completely determined values through the system of linear equations (20.4). Now this holds always for a system of linear equations as long as the equations do not contradict one another or as long as one or more are not consequences of the remaining. In the first of these cases the values of the variables will be infinite, in the second case undetermined; the two cases differ only through the values of the entirely constant terms, for, assuming that the last equation of the system follows from the others, these, multiplied by suitable constants and added should give the last. Now if one alters in the last equation the constant term by an arbitrary amount, then it does not follow from the others, but contradicts them. Both cases therefore agree that if one takes the constant terms to the left hand side, the right hand side of one equation, say the last, must be represented by the sum of the right hand sides of the remaining equations multiplied by suitable factors. If one inserts for the coefficients standing in the last horizontal row the representation arising from this by virtue of the remaining, the determinant R of the equation in question reduces to a sum of determinants each of whom has two identical horizontal rows, and hence vanishes. Therefore R wil be also equal to zero, and so the exceptional case in which the above proof becomes invalid occurs (in so far as the coefficients of the linear equations remain finite, which we always assume) only if the determinant of the linear equations vanishes. The coefficients of the linear equatins (20.4) are

$$\begin{array}{c}
\frac{\partial^2 V}{\partial\alpha_1\partial q_1}, \frac{\partial^2 V}{\partial\alpha_1\partial q_2}, \ldots, \frac{\partial^2 V}{\partial\alpha_1\partial q_\mu}, \\
\frac{\partial^2 V}{\partial\alpha_2}\frac{\partial^2 V}{\partial q_1}, \frac{\partial^2 V}{\partial\alpha_2\partial q_2}, \ldots, \frac{\partial^2 V}{\partial\alpha_2\partial q_\mu}, \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial^2 V}{\partial\alpha_\mu\partial q_1}, \frac{\partial^2 V}{\partial\alpha_\mu\partial q_2}, \ldots, \frac{\partial^2 V}{\partial\alpha_\mu\partial q_\mu}.
\end{array}$$

Consequently, one can represent this determinant as a functional determinant in following two ways:

$$R = \sum \pm \frac{\partial\frac{\partial V}{\partial\alpha_1}}{\partial q_1}\frac{\partial\frac{\partial V}{\partial\alpha_2}}{\partial q_2}\cdots\frac{\partial\frac{\partial V}{\partial\alpha_\mu}}{\partial q_\mu} = \sum \pm \frac{\partial\frac{\partial V}{\partial\alpha_1}}{\partial\alpha_1}\frac{\partial\frac{\partial V}{\partial\alpha_2}}{\partial\alpha_2}\cdots\frac{\partial\frac{\partial V}{\partial\alpha_\mu}}{\partial\alpha_\mu}$$

From this two fold representation of R follows incidentally a general theorem on function of 2μ variables $q_1, q_2, \ldots, q_\mu, \alpha_1, \alpha_2, \ldots, \alpha_\mu$. If R were equal to zero, then according to §5 of lecture 13 (page 112) the quantities $\frac{\partial V}{\partial \alpha_1}, \frac{\partial V}{\partial \alpha_2}, \ldots, \frac{\partial V}{\partial \alpha_\mu}$, considered as functions of $q_1, q_2, \ldots, q_\mu$, would not be independent of one another, i.e., there must exist an equation involving $\frac{\partial V}{\partial \alpha_1}, \frac{\partial V}{\partial \alpha_2}, \ldots, \frac{\partial V}{\partial \alpha_\mu}, \alpha_1, \alpha_2, \alpha_\mu, t$ which does not contain $q_1, q_2, \ldots, q_\mu$. From the second representation for R, it follows equally that between $\frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \ldots, \frac{\partial V}{\partial q_\mu}, q_1, q_2, \ldots q_\mu, t$ there must exist an equation which does not contain $\alpha_1, \alpha_2, \ldots \alpha_\mu$. One has then a partial differential equation of the form

$$0 = F\Big(t, q_1, q_2, \ldots, q_\mu, \frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \ldots, \frac{\partial V}{\partial q_\mu}\Big),$$

i.e., a partial differential equation of the first order, which the assumed solution V must satisfy and which does not contain $\frac{\partial V}{\partial t}$. This is however impossible since V should actually be a *complete* solution of $\frac{\partial V}{\partial t} + \psi = 0$. Namely, in order that

$$V = f\Big(t, q_1, q_2, \ldots, q_\mu, \alpha_1, \alpha_2, \ldots, \alpha_\mu\Big) + C$$

satisfy the concept of a *complete* solution, it is necessary that one needs all the $\mu + 1$ differential coefficients

$$\frac{\partial V}{\partial t} = \frac{\partial f}{\partial t}, \frac{\partial V}{\partial q_1} = \frac{\partial f}{\partial q_1}, \frac{\partial V}{\partial q_2} = \frac{\partial f}{\partial q_2}, \ldots, \frac{\partial V}{\partial q_\mu} = \frac{\partial f}{\partial q_\mu} \tag{20.6}$$

for the elimination of the $\mu + 1$ constants $\alpha_1, \alpha_2, \ldots, \alpha, C$. If one can eliminate all the $\mu + 1$ constants without using the equation $\frac{\partial V}{\partial t} = \frac{\partial f}{\partial t}$, so that one arrives at an equation of the form

$$F\Big(t, q_1, q_2, \ldots, q_\mu, \frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \ldots, \frac{\partial V}{\partial q_\mu}\Big) = 0,$$

and we assume that for the elimination of the constants we can miss no more of the equations (20.6) than the one $\frac{\partial V}{\partial t} = \frac{\partial f}{\partial t}$, while each of the remaining equations is required for that. Then it must be possible for one of the constants $\alpha_1, \alpha_2, \ldots, \alpha_\mu$ to take a particular value without one of the equations $\frac{\partial V}{\partial q_i} = \frac{\partial f}{\partial q_i}$ ceasing to be required for the elimination of the constants. Then between μ equations one can in general eliminate only $\mu - 1$ quantities. The constant to which one gives a special value is thereby superfluous (supervacanea), and the function f is to be looked upon as involving only $\mu - 1$ constants. Hence $V = f + C$ is not a *complete* solution of the partial differential equation $\frac{\partial V}{\partial t} + \psi = 0$, but only of the equation $F = 0$, which contradicts our assumption. The determinant R can then never be zero, and with this, the conclusion that we drew in the proof of the equation (20.3) is valid.

In concluding this lecture we shall actually form the partial differential equation $\frac{\partial V}{\partial t} + \psi = 0$ for the free motion of n mass points. In this case, $\psi = T - U$, the $3n$ coordinators $x_i, y_i, \ldots, z_i$ are to be inserted for the quantities q_i and since $T = \frac{1}{2}\sum m_i(x_i'^2 + y_i'^2 + z_i'^2)$, it follows from the equations $p_i = \frac{\partial T}{\partial q_i'}$, that in place of the quantities p_i the quantities $m_i x_i', m_i y_i', m_i z_i'$ enter here. Since at the same time p is to be set equal to $\frac{\partial V}{\partial q}$, one has the equations

$$m_i x_i' = \frac{\partial V}{\partial x_i}, m_i y_i' = \frac{\partial V}{\partial y_i}, m_i z_i' = \frac{\partial V}{\partial z_i},$$

or $x_i' = \frac{1}{m_i}\frac{\partial V}{\partial x_i}, y_i' = \frac{1}{m_i}\frac{\partial V}{\partial y_i}, z_i' = \frac{1}{m_i}\frac{\partial V}{\partial z_i}$. The substitution of these values in T gives

$$T = \frac{1}{2}\sum \frac{1}{m_i}\left(\left(\frac{\partial V}{\partial x_i}\right)^2 + \left(\frac{\partial V}{\partial y_i}\right)^2 + \left(\frac{\partial V}{\partial z_i}\right)^2\right)$$

and since U is a function of time and the quantities q_1, i.e. the coordinates x_i, y_i, z_i, so one has

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum \frac{1}{m_i}\left(\left(\frac{\partial V}{\partial x_i}\right)^2 + \left(\frac{\partial V}{\partial y_i}\right)^2 + \left(\frac{\partial V}{\partial z_i}\right)^2\right) = U. \tag{20.7}$$

This is the first order partial differential equation on whose solution the integration of the differential equations of motion depends for the case where the motion is entirely free and a force function U that may contain besides the coordinates the time t explicitly, exists. If one has a complete solution of equation (20.7), i.e., a value of V, which contains $3n$ constants $\alpha_1, \alpha_2, \ldots, \alpha_{3n}$ in addition to the additive constants, then the equations

$$\frac{\partial V}{\partial \alpha_i} = \beta_i.$$

that hold for $i = 1, 2, \ldots 3n$, are the integral equations of the differential equations of motion

$$m_i \frac{d^2 x_i}{dt^2} = \frac{\partial U}{\partial x_i}, m_i \frac{d^2 y_i}{dt^2} = \frac{\partial U}{\partial y_i}, m_i \frac{d^2 z_i}{dt^2} = \frac{\partial U}{\partial z_i},$$

$(i = 1, 2, \ldots n)$ whose first integrals are contained in the system

$$\frac{\partial V}{\partial x_i} = m_i \frac{dx_i}{dt}, \frac{\partial V}{\partial y_i} = m_i \frac{dy_i}{dt}, \frac{\partial V}{\partial z_i} = m_i \frac{dz_i}{dt}.$$

Lecture 21

Investigation of the case in which *t* does not occur explicitly

The case where t does not occur explicitly in ψ., already introdudced above, requires special consideration. In this case the partial differential equation $\frac{\partial V}{\partial t} + \psi = 0$ can be reduced to another which contains one variable less. This rests on a very remarkable transformation of partial differential equations through which one of the independent variables and the partial differential coefficient belonging to it reverse their roles.

Let z be a function of n variables $x_1, x_2, \ldots x_n$ so that if $p_1, p_2, \ldots, p_n$ denote the partial differential coefficients of z with respect to x_1, $x_2, \ldots, x_n$,

$$dz = p_1 dx_1 + p_2 dx_2 + \cdots + p_n dx_n. \tag{21.1}$$

If one transfers the term $p_1 dx_1$ to the left hand side and further subtracts $x_1 dp_1$ from both sides, then equation (21.1) changes into

$$d(z - p_1 x_1) = -x_1 dp_1 + p_2 dx_2 + \cdots + p_n dx_n,$$

or, if we set

$$z - p_1 x_1 = y. \tag{21.2}$$

Then

$$dy = -x_1 dp_1 + p_2 dx_2 + \cdots + p_n dx_n.$$

Therefore, if $y = z - p_1 x_1$ is looked upon as a function of $p_1, x_2, x_3, \ldots, x_n$, one has

$$\frac{\partial y}{\partial p_1} = -x_1, \frac{\partial y}{\partial x_2} = p_2, \frac{\partial y}{\partial x_3} = p_3, \ldots, \frac{\partial y}{\partial x_n} = p_n.$$

If now z satisfies the first order partial differential equation:

$$\begin{aligned} 0 &= F\Big(x_1, x_2, \ldots, x_n,\ p_1, p_2, \ldots, p_n\Big) \\ &= F\Big(x_1, x_2, \ldots, x_n,\ \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \ldots, \frac{\partial z}{\partial x_n}\Big), \end{aligned} \tag{21.3}$$

and if we substitute in place of z the new variable $y = z - p_1 x_1$ and in place of x_1 the new variable $-\frac{\partial y}{\partial p_1}$, then the partial differential equation (21.3) transforms into

$$0 = F\Big(-\frac{\partial y}{\partial p_1}, x_2, x_3, \ldots, x_n, p_1, \frac{\partial y}{\partial x_2},, \frac{\partial y}{\partial x_3}, \ldots, \frac{\partial y}{\partial x_n},\Big). \tag{21.4}$$

This transformation which appears in the third volume of *Euler's* Integral Calculus is of special importance if x_1 does not occur in (21.3); for then, likewise $\frac{\partial y}{\partial p_1}$ does not occur in (21.4) and therefore p_1 can be looked upon as a constant while integrating. Let us apply this to the equation

$$\frac{\partial V}{\partial t} + \psi\Big(q_1, q_2, \ldots q_\mu,\ \frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \ldots, \frac{\partial V}{\partial q_\mu}\Big) = 0. \tag{21.5}$$

Since t does not occur in ψ,, it takes the place of x_1 in the formula given above. We introduce for t, the new independent variable

$$\alpha = \frac{\partial V}{\partial t}$$

and for V, the new dependent variable

$$W = V - t\frac{\partial V}{\partial t} = V - t\alpha,$$

so that

$$t = -\frac{\partial W}{\partial \alpha},$$

and

$$\frac{\partial V}{\partial q_1} = \frac{\partial W}{\partial q_1}, \frac{\partial V}{\partial q_2} = \frac{\partial W}{\partial q_2}, \ldots, \frac{\partial V}{\partial q_\mu} = \frac{\partial W}{\partial q_\mu}$$

We can also prove the formula for this transformation without using the differential equation

$$dV = p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n.$$

In fact V is a function of $t, q_1, q_2, \ldots, q_\mu$ and the arbitrary constants $\alpha_1, \alpha_2, \ldots, \alpha_n$. If we set

$$W = V - t\frac{\partial V}{\partial t},$$

and introduce in W a new variable α for t by means of the equation

$$\frac{\partial V}{\partial t} = \alpha.$$

then t is a function α and of the quantities other than t occurring in V, and

$$W = V - t\alpha$$

is a function of $\alpha, q_1, q_2, \ldots, q_\mu$ and of the constants $\alpha_1, \alpha_2, \ldots, \alpha_\mu$. Observing therefore the different significance of the differetiation for the functions V and W, one has

$$\begin{aligned}
\frac{\partial W}{\partial \alpha} &= \frac{\partial V}{\partial t}\frac{\partial t}{\partial \alpha} - \alpha\frac{\partial t}{\partial \alpha} - t = -t,\\
\frac{\partial W}{\partial q_i} &= \frac{\partial V}{\partial q_i} + \frac{\partial V}{\partial t}\frac{\partial t}{\partial q_i} - \alpha\frac{\partial t}{\partial q_i} = \frac{\partial V}{\partial q_i},\\
\frac{\partial W}{\partial \alpha_i} &= \frac{\partial V}{\partial \alpha_i} + \frac{\partial V}{\partial t}\frac{\partial t}{\partial \alpha_i} - \alpha\frac{\partial t}{\partial \alpha_i} = \frac{\partial V}{\partial \alpha_i}.
\end{aligned}$$

If then, according to our assumption, the time t does not occur explicitly in the function ψ of the equation (21.5), we introduce for t and V the new variables α and W through the equations

$$\frac{\partial V}{\partial t} = \alpha, \quad V - t\frac{\partial V}{\partial t} = W$$

and transform (21.5) into

$$\alpha + \psi\left(q_1, q_2, \ldots, q_\mu\ \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}, \ldots, \frac{\partial W}{\partial q_\mu}\right) = 0. \qquad (21.6)$$

After integration of this equation one finds V from the equation $V - t\frac{\partial V}{\partial t} = W$, which, after $\frac{\partial V}{\partial t} = \alpha$ and $t = -\frac{\partial W}{\partial \alpha}$ have been substituted it goes over into

$$V = W - \alpha\frac{\partial W}{\partial \alpha}.$$

Further in V, t is to be introduced again in place of α and indeed so by means of the equation

$$\frac{\partial W}{\partial \alpha} = -t,$$

which is to be solved for α.

It seems at first sight as though a complete solution V of equation (21.5) does not follow in this way from a complete solution W of equation (21.6). Since the number of constants in W is μ, so likewise μ constants occur in the derived solution V. But if V is to be a complete solution, it must contain $\mu + 1$ constants. One can easily include the missing constant in the following way. Since t itself does not occur in equation (21.5), but only $\frac{\partial V}{\partial t}$, so a solution V of equation (21.5) does not cease to be one such if one increases or decreases t bywe an arbitrary constant, so we write $t - \tau$ in place of t. Thereby the transformation formula $W = V - t\frac{\partial V}{\partial t}$ which holds between V and W changes into

$$W = V - (t - \tau)\frac{\partial V}{\partial t} = V - \alpha(t - \tau).$$

and t is no longer introduced through the equation $\frac{\partial W}{\partial \alpha} = -t$, but through the equation

$$\frac{\partial W}{\partial \alpha} = \tau - t.$$

Then V contains the required number $\mu + 1$ of constants, namely the $\mu - 1$ constants $\alpha_1, \alpha_2, \ldots, \alpha_{\mu-1}$ which occur in W besides the constant added to W, the additive constant itself, and the constant τ related to t. The integral equations of the isoperimetric equation are therefore

$$\frac{\partial V}{\partial \alpha_1} = \beta_1, \ \frac{\partial V}{\partial \alpha_2} = \beta_2, \ldots, \frac{\partial V}{\partial \alpha_{\mu-1}} = \beta_{\mu-1}, \text{ and } \frac{\partial V}{\partial \tau} = \text{ constant.}$$

Since τ occurs only in the combination $t - \tau$, so

$$\frac{\partial V}{\partial \tau} = -\frac{\partial V}{\partial t},$$

so one can replace the last of the μ integral equations by

$$\frac{\partial V}{\partial t} = \text{ constant.}$$

Hence, it follows that the equation $\frac{\partial V}{\partial t} = \alpha$ by means of which we introduce α for t is an integral and that α must be considered a constant.

As we have seen, the two equations $\frac{\partial V}{\partial t} = \alpha$ and $\frac{\partial W}{\partial \alpha} = \tau - t$ are equivalent; moreover, the partial differential coefficients $\frac{\partial V}{\partial \alpha_i}$ and $\frac{\partial W}{\partial \alpha_i}$, where i represents the number from 1 to $\mu - 1$, are equal; then one can represent the integral equation directly, without the help of V, through W and obtain them in the form

$$\frac{\partial W}{\partial \alpha_1} = \beta_1, \frac{\partial W}{\partial \alpha_2} = \beta_2, \ldots, \frac{\partial W}{\partial \alpha_{\mu-1}} = \beta_{\mu-1}, \frac{\partial W}{\partial \alpha} = \tau - t. \tag{21.7}$$

Likewise one can represent the first integral equation

$$\frac{\partial V}{\partial q_1} = p_1, \frac{\partial V}{\partial q_2} = p_2, \ldots, \frac{\partial V}{\partial q_\mu} = p_\mu$$

through W and since $\frac{\partial V}{\partial q_i} = \frac{\partial W}{\partial q_i}$, obtain them in the form

$$\frac{\partial W}{\partial q_1} = p_1, \frac{\partial W}{\partial q_2} = p_2, \ldots, \frac{\partial W}{\partial q_\mu} = p_\mu. \tag{21.8}$$

In the case of mechanics $\psi = T - U$, and therefore we have the theorem: *If the force function U does not contain the time t explicitly, so that the theorem of vis viva holds, one expresses half the 'vis viva' T through the quantities q_i and $p_i = \frac{\partial T}{\partial q_i'}$. Therefore one substitutes in the equation for 'vis viva'*

$$0 = \alpha + \psi = \alpha + T - U,$$

$\frac{\partial W}{\partial q_i}$ *in place of p_i, so that this equation goes over to a partial differential equation for W. If one knows a complete solution of the same which contains the $\mu - 1$ constants $\alpha_1, \alpha_2, \ldots, \alpha_{\mu-1}$ besides the constant additively related to W, then*

$$\frac{\partial W}{\partial \alpha_1} = \beta_1, \frac{\partial W}{\partial \alpha_2} = \beta_2, \ldots, \frac{\partial W}{\partial \alpha_{\mu-1}} = \beta_{\mu-1}, \frac{\partial W}{\partial \alpha} = \tau - t$$

are the integral equations of the differential equations of motion, to which one can also add the first system of integral equations

$$\frac{\partial W}{\partial q_1} = p_1, \frac{\partial W}{\partial q_2} = p_2, \ldots, \frac{\partial W}{\partial q_{\mu-1}} = p_{\mu-1}, \frac{\partial W}{\partial q_\mu} = p_\mu.$$

The 2μ constants contained in the integral equations are

$$\alpha_1, \alpha_2, \ldots, \alpha_{\mu-1}, \alpha,$$
$$\beta_1, \beta_2, \ldots, \beta_{\mu-1}, \tau.$$

In the case of an entirely free system, $\mu = 3n$, and further in place of the quantities p_i, enter the quantities

$$m_i x_i', m_i y_i', m_i z_i'.$$

then

$$T = \frac{1}{2} \sum \frac{1}{m_i} \left\{ (m_i x_i')^2 + (m_i y_i')^2 + (m_i z_i')^2 \right\}$$

and the partial differential equation takes the form

$$\frac{1}{2} \sum \frac{1}{m_i} \left\{ \left(\frac{\partial W}{\partial x_i} \right)^2 + \left(\frac{\partial W}{\partial y_i} \right)^2 + \left(\frac{\partial W}{\partial z_i} \right)^2 \right\} = U - \alpha.$$

Lecture 22

Lagrange's method of integration of first order partial differential equations in two independent variables. Application to problems of mechanics which depend only on two defining parameters. The free motion of a point on a plane and the shortest line on a surface

After we have reduced the problems of mechanics to the integration of a non-linear first order partial differential equation, we must concern ourselves with the integration of the same, i.e., with the search for a complete solution.

Very fine investigations of the integration of partial differential equations appear in the third part of *Euler's* Integral Calculus. Though he treats only special cases, he is so successful in describing them that his results, for the most part, require little or no addition of the general methods found later. *Euler's* works have, moreover, the great merit that the problems, considered as fully as possible, can be solved completely. His examples therefore always give the entire content of his methods according to the state of science existing then. It is, as a rule, an enrichment of the same if one can add anything new to *Euler*'s examples, since he would have seldom missed examples solvable by his methods.

Lagrange has given his general method of integration of first order partial differential equations, which is a completely new idea in Integral Calculus,for the first time in a paper which is in the proceedings of the Berlin Academy for the year 1772. This work contains the reduction of first order nonlinear partial differential equations to linear ones. The notions of complete and general solutions are introduced deriving the latter from the former and the method of finding complete solutions are given. Everything, however, is restricted to the case of three variables of which only two are mutually independent.*Lagrange's* method is the following. Suppose a first order partial differential equation is given;

$$\psi(x, y, z, p, q) = 0,$$

where x and y are the independent variables, z the dependent, and

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y},$$

so, that relation

$$dz = pdx + qdy$$

holds between the differentials of the three variables. If the given differential equation, solved for q gives

$$q = \chi(x, y, z, p)$$

then one has

$$dz = pdx + \chi(x, y, z, p)dy.$$

In order to find a complete solution of z, i.e., a solution which contains two arbitrary constants, it is obviously only necessary to find a value $p = \tilde{w}(x, y, z, a)$ which, substituted in the expression $dz = pdx + \chi dy$, makes it a total differential. Then it remains to determine z from the equation $dz = pdx+\chi dy$. The last requires the integration of a first order ordinary differential equation through which, besides a, a second constant b enters z. So it amounts to this: to determine p as a function of $\widetilde{\omega}$ of x, y, z and an arbitrary constant a so that the expression $pdx + \chi(x, y, z, p)dy$ is a complete differential. For this it is required that p differentiated with respect to y gives the same value as χ differentiated with respect to x, i.e., the equation

$$\frac{\partial p}{\partial y} + \frac{\partial p}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial \chi}{\partial x} + \frac{\partial \chi}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial \chi}{\partial p}\left(\frac{\partial p}{\partial x} + \frac{\partial p}{\partial z}\frac{\partial z}{\partial x}\right),$$

or

$$\frac{\partial \chi}{\partial x} + \frac{\partial \chi}{\partial z}p = -\frac{\partial \chi}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + \left(\chi - \frac{\partial \chi}{\partial p}p\right)\frac{\partial p}{\partial z}$$

must be satisfied. Since χ is a known function of x, y, z, p, this is a *linear* partial differential equation for p, which contains three independent variables x, y, z and the given problem is reduced to finding, for this linear partial differential equation for p, *one* solution $p = \tilde{w}(x, y, z, a)$ with one arbitrary constant a. This circumstance that one needs to know only one solution was brought out by *Lagrange* in an involved way.

Let us now consider the special case in which ψ and hence χ, does not depend on z. The given partial differential equation then takes the simpler form

$$\Psi(x, y, p, q) = 0. \tag{22.1}$$

In this case one can so determine p as a function of x, y, a, without z, that $pdx + \chi dy$ is a total differential. Since now $\frac{\partial \chi}{\partial z}$ as well as $\frac{\partial p}{\partial z}$ vanish, the linear partial differential equation for p reduces to

$$\frac{\partial \chi}{\partial p}\frac{\partial p}{\partial x} - \frac{\partial p}{\partial y} + \frac{\partial \chi}{\partial x} = 0.$$

Instead of assuming that the partial differential equation (22.1) is to be solved for q, we would rather solve it in its original form. If, further, we assume the equation $p = \widetilde{\omega}(x, y, a)$ is solved not for p, but for a and brought to the form $f(x, y, p) = a$, then we have the formulae

$$\frac{\partial \chi}{\partial x} = \frac{\partial q}{\partial x} = -\frac{\frac{\partial \Psi}{\partial x}}{\frac{\partial \Psi}{\partial q}}, \quad \frac{\partial \chi}{\partial p} = \frac{\partial q}{\partial p} = -\frac{\frac{\partial \Psi}{\partial p}}{\frac{\partial \Psi}{\partial q}},$$

$$\frac{\partial p}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial p}}, \quad \frac{\partial p}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial p}}$$

to use, and when we insert these values in the above partial differential equation for p, it goes over to the following partial differential equation for f:

$$\frac{\partial \Psi}{\partial p}\frac{\partial f}{\partial x} + \frac{\partial \Psi}{\partial q}\frac{\partial f}{\partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial f}{\partial p} = 0. \tag{22.2}$$

If one knows a solution of this equation without constants, then in the present case no further integration of a differential equation is required for the determination of a complete solution z of (22.1). So, if we set the preceding solution f equal to an arbitrary constant a, and determine p and q as functions of x and y from the equation

$$f(x, y, p) = a$$

combined with the given differential equation

$$\Psi(x, y, p, q) = 0,$$

then these will be of such a nature that $pdx + qdy$ is a total differential, since the condition (22.2) required for that is fulfilled, and one thereby obtains z from the formula

$$z = \int (pdx + qdy)$$

by simple quadrature, so that the second constant contained in the complete solution z is additively connected with z. This could have been foreseen, since z itself is missing in equation (22.1).

It now comes to finding *one* solution of the linear partial differential equation (22.2) in which, by virtue of the equation (22.1), the partial differential coefficients $\frac{\partial \Psi}{\partial p}, \frac{\partial \Psi}{\partial q}, \frac{\partial \Psi}{\partial x}$ are assumed to be functions of x, y and p, without q. But it is well known that this linear partial differential equation (22.2) is none other (see Lecture 10, p.92) than the defining equation of that function f of x, y and p which, set equal to a constant, gives an integral of the system of ordinary differential equation

$$dx : dy : dp : \frac{\partial \Psi}{\partial p} : \frac{\partial \Psi}{\partial q} : -\frac{\partial \Psi}{\partial x}. \tag{22.3}$$

The entire investigation is thereby reduced to finding *one* integral of the system (22.3) of ordinary differential equations.

We can complete this system further by seeking with the help of the equation $\Psi = 0$, the quantity to which dq is proportional. Differentiation of the equation $\Psi = 0$ gives

$$\frac{\partial \Psi}{\partial x}dx + \frac{\partial \Psi}{dy}dy + \frac{\partial \Psi}{\partial p}dp + \frac{\partial \Psi}{\partial q}dq = 0.$$

But according to the differential equation (22.3), one has the proportion

$$dx : dp = \frac{\partial \Psi}{\partial p} : -\frac{\partial \Psi}{\partial x},$$

so that $\frac{\partial \Psi}{\partial x}dx + \frac{\partial \Psi}{\partial p}dp$ itself vanishes; therefore $\frac{\partial \Psi}{\partial y}dy + \frac{\partial \Psi}{\partial q}dq$ must also vanish and one obtains

$$dy : dq = \frac{\partial \Psi}{\partial q} : -\frac{\partial \Psi}{dy}.$$

The system (22.3) is therefore complete:

$$dx : dy : dp : dq = \frac{\partial \Psi}{\partial p} : \frac{\partial \Psi}{\partial q} : -\frac{\partial \Psi}{\partial x} : -\frac{\partial \Psi}{\partial y}, \tag{22.4}$$

a result symmetrial with respect to x and p on the one hand and y and q on the other, from which the correctness of the computation follows. This system says that in place of (22.3) we may also allow q to occur in the function f, if we generalize the method of integration appropriately. Namely, we can look upon the equation $f(x, y, p) = a$ as the result of elimination of q between an equation

$$F(x, y, p, q) = a \tag{22.5}$$

and $\Psi(x, y, p, q) = 0$, so that if χ denotes as above the value of q arising from the solution of the equation $\Psi = 0$, we have identically

$$F(x, y, p, \chi) = f(x, y, p).$$

Therefore $F(x, y, p, \chi)$ must satisfy the partial differential equation (22.2), which leads to the differential equation

$$\frac{\partial \Psi}{\partial p}\frac{\partial F}{\partial x} + \frac{\partial \Psi}{\partial q}\frac{\partial F}{\partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial F}{\partial \chi}\left(\frac{\partial \Psi}{\partial p}\frac{\partial \chi}{\partial x} + \frac{\partial \Psi}{\partial q}\frac{\partial \chi}{\partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial \chi}{\partial p}\right) = 0$$

for F. But since χ satisfies the equation $\Psi(x, y, p, \chi) = 0$ identically, so one has

$$\frac{\partial \chi}{\partial x} = -\frac{\frac{\partial \Psi}{\partial x}}{\frac{\partial \Psi}{\partial \chi}}, \frac{\partial \chi}{\partial y} = -\frac{\frac{\partial \Psi}{\partial y}}{\frac{\partial \Psi}{\partial \chi}}, \frac{\partial \chi}{\partial p} = -\frac{\frac{\partial \Psi}{\partial p}}{\frac{\partial \Psi}{\partial \chi}},$$

Hereby the expression on the left hand side of the above equation multiplied by $\frac{\partial F}{\partial \chi}$ reduces to $-\frac{\partial \Psi}{\partial y}$ and one has

$$\frac{\partial \Psi}{\partial p}\frac{\partial F}{\partial x} + \frac{\partial \Psi}{\partial q}\frac{\partial F}{\partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial \Psi}{\partial y}\frac{\partial F}{\partial q} = 0. \tag{22.6}$$

Therefore, it follows that $F = a$ is in fact an integral of the system of differential equations (22.4). Since $f(x, y, p) = a$ is the result of elimination of q between $F(x, y, p, q) = a$ and $\Psi(x, y, p, q) = 0$, so the same values of p and q follow from the equations $F(x, y, p, q) = a$ and $\Psi(x, y, p, q) = 0$ as from $f(x, y, p) = a$ and $\Psi(x, y, p, q) = 0$. Moreover, if one observes that $\Psi = 0$ is an integral of the differential equation (22.4), and indeed a general one if the function Ψ contained a constant additively connected to it, otherwise a particular one, then one can collect together the results in the following theorem

If the partial differential equation

$$\Psi(x, y, p, q) = 0 \tag{22.1}$$

is given, where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$, *one forms the system of ordinary differential equations*

$$dx : dy : dp : dq = \frac{\partial \Psi}{\partial p} : \frac{\partial \Psi}{\partial q} : -\frac{\partial \Psi}{\partial x} : -\frac{\partial \Psi}{\partial y}. \tag{22.4}$$

If one knows apart from the integral $\Psi = 0$, *given a priori, another*

$$F(x, y, p, q) = 0, \tag{22.5}$$

one determines from (22.1) and (22.5) p *and* q *as functions of* x *and* y*; then one obtains* z *though the formula*

$$z = \int (pdx + qdy)$$

by means of a simple quadrature.

The equations (22.4) are of the same form as the differential equations of motion, only in place of the quantities q_1, q_2, p_1, p_2, $\psi + \alpha$, w, the quantities x, y, p, q, Ψ, z occur here. Consequently we obtain a new integral equation from (22.4) if we differentiate z with respect to the arbitrary constant contained therein and set the result equal to another arbitrary constant. Such a constant contained in z is a; we have then in the equation

$$\frac{\partial z}{\partial a} = \int \left(\frac{\partial p}{\partial a} dx + \frac{\partial q}{\partial a} dy \right) = b$$

the third integral of the system (22.4). That we are led to it by a simple quadrature is a significant use we have derived from the reduction of the system of ordinary differential equations (22.4) to the partial differential equation (22.1). In order to extend the analogy with the differential equations of motion completely, we add to the proportion (22.4) dt on the left hand side, 1 to the right, then, as we have seen in the previous lecture, t will be determined through the equation

$$\frac{\partial z}{\partial \alpha} = \int \left(\frac{\partial p}{\partial \alpha} dx + \frac{\partial q}{\partial \alpha} dy \right) = \tau - t,$$

where α is the constant contained in $\Psi = \psi + \alpha$.

After *Hamilton* had found the reduction of the differential equations of dynamics to a first order partial differential equation, one needed then to apply to them only the methods known for 65 years in order to obtain important results for all problems of mechanics which contain only two quantiities q_1 and q_2 to be determined.

If the theorem of *vis viva* holds for the problems of mechanics under consideration, then in the equation $0 = \Psi = \alpha + \psi$, the function ψ has the value

$$\psi = T - U,$$

The equation

$$T = U - \alpha,$$

which expresses here the theorem of *vis viva* and in which U is a function of q_1 and q_2 alone and T a function of q_1, q_2, p_1, p_2, goes over on substitution of the values $p_1 = \frac{\partial W}{\partial q_1}$, $p_2 = \frac{\partial W}{\partial q_2}$, to the partial differential equation for W, and the differential equations of motion will be

$$dt : dq_1 : dq_2 : dp_1 : dp_2 = 1 : \frac{\partial \psi}{\partial p_1} : \frac{\partial \psi}{\partial p_2} : -\frac{\partial \psi}{\partial q_1} : -\frac{\partial \psi}{\partial q_2}.$$

Let the second integral, free from t of this differential equation which is necessary for the determination of the complete solution W be

$$F(q_1, q_2, p_1, p_2) = a,$$

as one has then

$$W = \int (p_1 dq_1 + p_2 dq_2),$$

the third integral, free from t, of the differential equations of motion, is

$$\frac{\partial W}{\partial a} = b,$$

and t is introduced through the equation

$$\frac{\partial W}{\partial \alpha} = \tau - t.$$

This result can be expressed thus, independently of the theory of partial differential equations:

If, in a problem of mechanics which contains only two quantities q_1 *and* q_2 *to be determined, and in which the theorem of 'vis viva' holds and besides, if one knows another integral* $F(q_1, q_2, p_1, p_2) = a$, *where* $p_1 = \frac{\partial T}{\partial q_1'}$, $p_2 = \frac{\partial T}{\partial q_2'}$, *then one determines the quantities* p_1 *and* p_2 *from the equations* $\psi = T - U = -\alpha$ *and* $F = a$ *as functions of* q_1 *and* q_2, a *and* α; *then the two remaining integrals are given by the equations*

$$\begin{aligned} \int \left(\frac{\partial p_1}{\partial a} dq_1 + \frac{\partial p_2}{\partial a} dq_2 \right) &= b, \\ \int \left(\frac{\partial p_1}{\partial \alpha} dq_1 + \frac{\partial p_2}{\partial \alpha} dq_2 \right) &= \tau - t, \end{aligned}$$

so that the complete integration of the differential equations of motion, i.e. of the system

$$dt : dq_1 : dq_2 : dp_1 : dp_2 = 1 : \frac{\partial \psi}{\partial p_1} : \frac{\partial \psi}{\partial p_2} : -\frac{\partial \psi}{\partial q_1} : -\frac{\partial \psi}{\partial q_2}$$

is contained in these four integrals.

These are entirely new formulae; they hold for the motion of a point in a plane or on a curved surface, if the theorem of *vis viva* holds.

For free motion in the plane, if the mass of the point is set equal to unity, one has

$$\frac{d^2x}{dt^2} = \frac{\partial U}{\partial x}, \frac{d^2y}{dt^2} = \frac{\partial U}{\partial y}, T = \frac{1}{2}\left(x'^2 + y'^2\right),$$

and the theorem of *vis viva* is contained in the integral

$$\frac{1}{2}\left(x'^2 + y'^2\right) = U - \alpha.$$

If one knows a second integral i.e., a second equation according to which a function of x, y, x', y' will be equal to an arbitrary constant a and if from the two one determines x' and y' as functions of x, y, a and α, then the equation of the trajectory is

$$\int \left(\frac{\partial x'}{\partial a} dx + \frac{dy'}{\partial a} dy \right) = b,$$

and the time is expressed through the equation

$$\int \left(\frac{\partial x'}{\partial \alpha} dx + \frac{\partial y'}{\partial \alpha} dy \right) = \tau - t.$$

I had already communicated these formulae to the Paris Academy in the year 1836 as the simplest result of the reduction of problems of mechanics to partial differential equations. For the interest these formulae evoke and as they are derived for the most elementary example of mechanics, they deserve to find a place in text-books. They are already included in the teaching of students in polytechnics. *Poisson* has given a proof, or rather a verification of the same, in *Liovilles* Journal.[1]

A second case included in the above formulae is that in which a point given an initial push moves on a given surface. Such a point describes the shortest line whose determination depends on a second order differential equation. According to the earlier considerations, it follows that if one knows an integral of these differential equations, one can derive the equation holding between the coordinates along the trajectory by a simple quadrature. Since the force function vanishes in this case, the partial differential equation will be

$$T + \alpha = 0.$$

If $x, y, , z$ are the coordinates of the moving point, then

$$2T = \left(\frac{ds}{dt}\right)^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2}.$$

If one looks upon x and y as the quantities to be determined, denoted as above by q_1 and q_2, then one has to substitute the value arising from the equation of the surface

$$dz = pdx + qdy,$$

and obtain

$$2T = \frac{dx^2 + dy^2 + (pdx + qdy)^2}{dt^2},$$

or,

$$2T = x'^2 + y'^2 + (px' + qy')^2.$$

If ξ, η are the quantities denoted above by p_1 and p_2, then

$$\begin{aligned}
\xi &= \frac{\partial T}{\partial x'} = x' + p(px' + qy'), \\
\eta &= \frac{\partial T}{\partial y'} = y' + q(px' + qy'), \\
p\xi + q\eta &= (1 + p^2 + q^2)(px' + qy')
\end{aligned}$$

[1] Vol.2, p.335

If one sets

$$N = 1 + p^2 + q^2,$$

one finds, on solving for x' and y',

$$\begin{aligned} x' &= \xi - \frac{p}{N}(p\xi + q\eta), \\ y' &= \eta - \frac{q}{N}(p\xi + q\eta), \end{aligned}$$

and since for T, as a homogenous function of the second order in x' and y', one can apply the formula

$$2T = \frac{\partial T}{\partial x'}x' + \frac{\partial T}{\partial y'}y' = \xi x' + \eta y',$$

so this gives

$$\begin{aligned} 2T &= \xi^2 + \eta^2 - \frac{(p\xi + q\eta)^2}{1 + p^2 + q^2} \\ &= \frac{(1 + q^2)\xi^2 + (1 + p^2)\eta^2 - 2pq\xi\eta}{1 + p^2 + q^2}. \end{aligned}$$

The partial differential equation for W is therefore

$$0 = (1+q^2)\left(\frac{\partial W}{\partial x}\right)^2 + (1+p^2)\left(\frac{\partial W}{\partial y}\right)^2 - 2pq\frac{\partial W}{\partial x}\frac{\partial W}{\partial y} + 2\alpha(1+p^2+q^2).$$

This equation can be transformed in various ways by introducing two new variables in place of x and y. The following example of this will lead to a substitution with the help of which we shall determine the shortest lines on a tri-axial ellipsoid.

The cases introduced also belong to the applications of the principle of the last multiplier which performs the last integration for problems of mechanics with an arbitrary large number of pieces to be determined. We are then led to the same result through entirely different considerations.

Lecture 23

The reduction of the partial differential equation for those problems in which the principle of conservation of centre of gravity holds

We shall now investigate what use can be derived for the partial differential equation from the principle of conservation of centre of gravity.

As soon as the variables can be so chosen that one of them does not occur in the partial differential equation $T = U - \alpha$, but only the differential coefficient of W with respect to these variables, we can, by the same kind of transformation as that by which W was derived from V, omit this variable in question from the differential equation and so reduce the number of variables occurring in it.

If we consider the case of a free system of n mass points, where $T = \frac{1}{2}\sum m_i(x_i'^2 + y_i'^2 + z_i'^2)$, then we have (See Lecture 21, pp.185) the partial differential equation

$$\frac{1}{2}\sum \frac{1}{m_i}\left(\left[\frac{\partial W}{\partial x_i}\right]^2 + \left[\frac{\partial W}{\partial y_i}\right]^2 + \left[\frac{\partial W}{\partial z_i}\right]^2\right) = U - \alpha. \tag{23.1}$$

If the principle of conservation of centre of gravity holds, then U depends only on the difference of the coordinates, so that if one puts

$$\xi_1 = x_1 - x_n, \xi_2 = x_2 - x_n, \ldots, \xi_{n-1} = x_{n-1} - x_n$$

the function U, considered as a function of the x-coordinates, can be represented solely the quantities ξ_i. We denote the partial differential coefficients of W with square brackets when we consider W as a function of $x_1, x_2, \ldots, x_n$, and without them when considered a function of

$\xi_1, \xi_2, \dots, \xi_{n-1},$. We then obtain

$$\begin{aligned}
\left[\frac{\partial W}{\partial x_1}\right] &= \frac{\partial W}{\partial \xi_1}, \left[\frac{\partial W}{\partial x_2}\right] = \frac{\partial W}{\partial \xi_2}, \dots, \left[\frac{\partial W}{\partial x_{n-1}}\right] = \frac{\partial W}{\partial \xi_{n-1}}, \\
\left[\frac{\partial W}{\partial x_n}\right] &= -\left(\frac{\partial W}{\partial \xi_1} + \frac{\partial W}{\partial \xi_2} + \dots + \frac{\partial W}{\partial \xi_{n-1}}\right) + \frac{\partial W}{\partial x_n},
\end{aligned}$$

This formula, gives to the sum $\sum \frac{1}{m_i}\left[\frac{\partial W}{\partial x_i}\right]^2$ appearing in equation (23.1) the new form

$$\sum \frac{1}{m_i}\left[\frac{\partial W}{\partial x_i}\right]^2 = \sum \frac{1}{m_s}\left(\frac{\partial W}{\partial \xi_s}\right)^2 + \frac{1}{m_n}\left(\frac{\partial W}{\partial x_n} - \sum \frac{\partial W}{\partial \xi_s}\right)^2, \tag{23.2}$$

where i runs from 1 to n, and s from 1 to $n-1$.

After the introduction of this transformation in the partial differential equation (23.1), the original variables $x_1, x_2, \dots, x_{n-1}, x_n$ are completely replaced by $\xi_1, \xi_2, \dots, \xi_{n-1}, x_n$, and the variable x_n itself does not occur any more, but only the derivative of W with respect to it. Therefore, we introduce for x_n the new variable α' by means of the equation

$$\frac{\partial W}{\partial x_n} = \alpha',$$

and for W, the new variable to be looked upon as a function of ξ_1, $\xi_2, \dots, \xi_{n-1}$ and α:

$$W_1 = W + (\alpha_0 - x_n)\frac{\partial W}{\partial x_n},$$

where α_0 is an arbitrary constant. On using the equations

$$\frac{\partial W_1}{\partial \xi_1} = \frac{\partial W}{\partial \xi_1}; \frac{\partial W_1}{\partial \xi_2} = \frac{\partial W}{\partial \xi_2}, \dots, \frac{\partial W_1}{\partial \xi_{n-1}} = \frac{\partial W}{\partial \xi_{n-1}},$$

the expression (23.2) now goes over to

$$\sum \frac{1}{m_i}\left[\frac{\partial W}{\partial x_i}\right]^2 = \sum \frac{1}{m_s}\left(\frac{\partial W_1}{\partial \xi_s}\right)^2 + \frac{1}{m_n}\left(\alpha' - \sum \frac{\partial W_1}{\partial \xi_s}\right)^2, \tag{23.3}$$

and when we substitute the right hand side of (23.3) in (23.1) and we notice that for differentiation with respect to y_i or z_i, the derivatives of W and W_1 are the same, (23.1) changes into a partial differential equation for W_1, in which only the variable α' occurs but not the differential coefficient $\frac{\partial W_1}{\partial \alpha'}$. In order to make the passage back again from α' and W_1 to x_n and W, one uses the equations

$$\frac{\partial W_1}{\partial \alpha'} = \alpha_0 - x_n, \quad W = W_1 - \alpha' \frac{\partial W_1}{\partial \alpha'}.$$

One can simplify the expression (23.3) still further, if one transforms the linear terms in the partial differential coefficients of the dependent variable through a new transformation, which is analogous to the transformation of the equation of a section of a sphere with reference its centre. If one sets

$$W_1 = W_2 + \sum g_s \xi_s,$$

where $g_1, g_2, \ldots, g_{n-1}$ denote constants yet to be determined, so that

$$\frac{\partial W_1}{\partial \xi_s} = \frac{\partial W_2}{\partial \xi_s} + g_s,$$

then the expression (23.3) goes over to

$$\begin{aligned} \sum \frac{1}{m_i} \left[\frac{\partial W}{\partial x_i} \right]^2 &= \sum \frac{1}{m_s} \left\{ \frac{\partial W_2}{\partial \xi_s} + g_s \right\}^2 \\ &+ \frac{1}{m_n} \left\{ \alpha' - \sum g_s - \sum \frac{\partial W_2}{\partial \xi_s} \right\}^2 . \end{aligned} \tag{23.4}$$

Let s' be one of the indices s. If one looks on the right hand side of (23.4) for the term multiplied by the first power of $\frac{\partial W_2}{\partial \xi_{s'}}$ and sets its coefficients equal to zero, then one has

$$\frac{g_{s'}}{m_{s'}} - \frac{\alpha' - \sum g_s}{m_n} = 0. \tag{23.5}$$

This equation must hold for the $n-1$ values of s'. If one multiplies these by $m_{s'}$ and sums from $s' = 1$ to $s' = n-1$, then one obtains first of all the value of $\sum g_s$, namely

$$\left(1 + \frac{\sum m_s}{m_n}\right) \sum g_s = \frac{\alpha' \sum m_s}{m_n},$$

or, if one introduces as in Lecture 3 the notation

$$M = m_1 + m_2 + \cdots + m_n = \sum m_s + m_n,$$

then

$$\sum g_s = \alpha'\Big(1 - \frac{m_n}{M}\Big),$$
$$\alpha' - \sum g_s = \frac{\alpha'}{M} m_n.$$

If one substitutes these values in (23.5), one finds for $g_{s'}$, the simple value

$$g_{s'} = \frac{\alpha'}{M} m_{s'},$$

so that the formula for transformation of W_1 into W_2 is determined in the following way:

$$W_1 = W_2 + \frac{\alpha'}{M} \sum m_s \xi_s. \tag{23.6}$$

Through substitution of the values g_s in (23.4), the part independent of $\frac{\partial W_2}{\partial \xi_s}$ in this expression will be

$$\sum \frac{1}{m_s} g_s^2 + \frac{1}{m_n}\left\{\alpha' - \sum g_s\right\}^2 = \frac{\alpha'^2}{M},$$

and one obtains

$$\sum \frac{1}{m_i}\left[\frac{\partial W}{\partial x_i}\right]^2 = \sum \frac{1}{m_s}\left(\frac{\partial W_2}{\partial \xi_s}\right)^2 + \frac{1}{m_n}\Big(\sum \frac{\partial W_2}{\partial \xi_s}\Big)^2 + \frac{\alpha'^2}{M}. \tag{23.7}$$

If one inserts this expression in equation (23.1) and observes that W_1 differs from W_2 by a quantity which does not depend upon the variables y_i and z_i, then for differentiation with respect to y_i or z_i, not only the derivatives of W and W_1, but also those of W_1 and W_2 are equal. Then the equation (23.1) changes into a partial differential equation for the dependent variable W_2. This differential equation does not contain any more the $3n$ independent variables x_i, y_i, z_i, but only $3n-1$, since the n variables x are replaced by the $n-1$ variables ξ, and the newly introduced quantity α' is to be considered a constant since the differential coefficient of W_2 with respect to this does not appear. After one has integrated the partial differential equation for W_2 and determined W_1 from W_2 using the equation (23.6), there occurs, as already remarked above, the introduction of x_n by means of equation $\frac{\partial W_1}{\partial \alpha'} = \alpha_0 - x_n$,. This, on substitution of W_1 by W_2, changes into

$$\alpha_0 - x_n = \frac{\partial W_2}{\partial \alpha'} + \frac{1}{M} \sum m_s \xi_s.$$

This equation is likewise an integral of the differential equations of motion, which can be reduced to the partial differential equation (23.1), and indeed the one which is to be added after choosing the integral existing between the $3n-1$ variables ξ_s, y_i, z_i, entirely analogously as the equation $\tau - t = \frac{\partial W}{\partial \alpha} = \frac{\partial W_2}{\partial \alpha}$, through which t is introduced to form the last integral.

If one puts together both transformations

$$\begin{aligned} W &= W_1 - \alpha' \frac{\partial W}{\partial \alpha'} = W_1 - \alpha'(\alpha_0 - x_n) \\ W_1 &= W_2 + \frac{\alpha'}{M} \sum m_s \xi_s, \end{aligned}$$

then one obtains the formula

$$W_2 = W - \frac{\alpha'}{M} \sum_{i=1}^{n} m_i x_i + \alpha' \alpha_0,$$

in which, since W itself does not appear in the equation (23.1), one can omit the term $\alpha'\alpha_0$ because of the arbitrary constant connected with W.

Even as by this transformation the n variables x_i in the partial differential equation (23.1) can be reduced to $n-1$ variables $\xi_s = x_s - x_n$, so one can, by two new transformations of the same sort, reduce the $2n$ variables y_i and z_i to $2(n-1)$ variables $\eta_s = y_s - y_n$ and $\zeta_s = z_s - z_n$, and if finally one puts all the transformations together into one, then one has the following theorem:

In the case of a free system of n mass points for which the differential equations of motion can be reduced to the partial differential equation

$$\frac{1}{2} \sum \frac{1}{m_i} \left\{ \left[\frac{\partial W}{\partial x_i}\right]^2 + \left[\frac{\partial W}{\partial y_i}\right]^2 + \left[\frac{\partial W}{\partial z_i}\right]^2 \right\} = U - \alpha, \tag{23.1}$$

if one sets

$$\begin{aligned} \xi_1 &= x_1 - x_n, \xi_2 = x_2 - x_n, \ldots, \xi_{n-1} = x_{n-1} - x_n \\ \eta_1 &= y_1 - y_n, \eta_2 = y_2 - y_n, \ldots, \eta_{n-1} = y_{n-1} - y_n \\ \zeta_1 &= z_1 - z_n, \zeta_2 = z_2 - z_n, \ldots, \zeta_{n-1} = z_{n-1} - z_n. \end{aligned}$$

and introduces for W the new dependent variable

$$\Omega = W - \frac{\alpha'}{M} \sum m_i x_i - \frac{\beta'}{M} \sum m_i y_i - \frac{\gamma'}{M} \sum m_i z_i,$$

then the partial differential equation (23.1) is transformed into

$$\frac{1}{2}\sum\frac{1}{m_s}\left\{\left(\frac{\partial\Omega}{\partial\xi_s}\right)^2+\left(\frac{\partial\Omega}{\partial\eta_s}\right)^2+\left(\frac{\partial\Omega}{\partial\zeta_s}\right)^2\right\}$$
$$+\frac{1}{2m_n}\left\{\left(\sum\frac{\partial\Omega}{\partial\xi_s}\right)^2+\left(\sum\frac{\partial\Omega}{\partial\eta_s}\right)^2+\left(\sum\frac{\partial\Omega}{\partial\zeta_s}\right)^2\right\}=U-\beta, \tag{23.8}$$

where

$$\beta=\alpha+\frac{\alpha'^2+\beta'^2+\gamma'^2}{2M}.$$

After integration of this partial differential equation for Ω, the variables x_n, y_n, z_n are introduced through the equation

$$\begin{aligned}\alpha_0-x_n &= \frac{\partial\Omega}{\partial\alpha'}+\frac{1}{M}\sum m_s\xi_s,\\ \beta_0-y_n &= \frac{\partial\Omega}{\partial\beta'}+\frac{1}{M}\sum m_s\eta_s,\\ \gamma_0-z_n &= \frac{\partial\Omega}{\partial\gamma'}+\frac{1}{M}\sum m_s\zeta_s,\end{aligned}$$

and finally the variable t is determined by the equation

$$\tau-t=\frac{\partial\Omega}{\partial\alpha}.$$

However, as the four constants, α',β',γ' and α have been united into one constant β, so one has

$$\frac{\partial\Omega}{\partial\alpha'}=\frac{\alpha'}{M}\frac{\partial\Omega}{\partial\beta},\ \frac{\partial\Omega}{\partial\beta'}=\frac{\beta'}{M}\frac{\partial\Omega}{\partial\beta},\ \frac{\partial\Omega}{\partial\gamma'}=\frac{\gamma'}{M}\frac{\partial\Omega}{\partial\beta},\ \frac{\partial\Omega}{\partial\alpha}=\frac{\partial\Omega}{\partial\beta},$$

and hereby the above four equations go over to the following:

$$\frac{\partial\Omega}{\partial\beta}=\tau-t$$

$$\begin{aligned}\alpha_0-x_n &= \frac{\alpha'}{M}(\tau-t)+\frac{1}{M}\sum m_s\xi_s,\\ \beta_0-y_n &= \frac{\beta'}{M}(\tau-t)+\frac{1}{M}\sum m_s\eta_s\\ \gamma_0-z_n &= \frac{\gamma'}{M}(\tau-t)+\frac{1}{M}\sum m_s\zeta_s.\end{aligned}$$

The last three formulae agree with those given in Lecture 3 (p.18) for the rectlinear motion of the centre of gravity when one brings them to the form

$$\begin{aligned}
\alpha_0 + \frac{\alpha'}{M}(t-\tau) &= x_n + \frac{1}{M}\sum m_s\xi_s = \frac{1}{M}\sum m_i x_i,\\
\beta_0 + \frac{\beta'}{M}(t-\tau) &= y_n + \frac{1}{M}\sum m_s\eta_s = \frac{1}{M}\sum m_i y_i,\\
\gamma_0 + \frac{\gamma'}{M}(t-\tau) &= z_n + \frac{1}{M}\sum m_s\zeta_s = \frac{1}{M}\sum m_i z_i,
\end{aligned}$$

since the quantities on the right side are none other than the coordinates of the centre of gravity.

Lecture 24

Motion of a planet around the sun - Solution in polar coordinates

The treatment of some examples by *Hamilton's* method will lead to further general observations. The motion of a planet around the sun constitutes the first example.

In the case of a system of n free mass points, the partial differential equation to which the differential equations of motion can be reduced is the following (See p. 185):

$$T = \frac{1}{2}\sum \frac{1}{mi}\left\{\left(\frac{\partial W}{\partial x_i}\right)^2 + \left(\frac{\partial W}{\partial y_i}\right)^2 + \left(\frac{\partial W}{\partial y_i}\right)^2\right\} = U - \alpha.$$

For the motion of a planet whose heliocentric coordinates are x, y, z, this sum reduces to one term. Further if we set the mass of the planet equal to 1 and denote the attractive force of the sun at unit distance by k^2. then the force function is $U = \frac{k^2}{r}$, where $r^2 = x^2 + y^2 + z^2$, and one has

$$T = \frac{1}{2}\left\{\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right\} = \frac{k^2}{r} - \alpha. \qquad (24.1)$$

Since the radius vector occurs on the right side of this equation, it is appropriate to introduce polar coordinate, in place of rectangular coordinates, using the formulae

$$x = r\cos\varphi, y = r\sin\varphi\cos\psi, z = r\sin\varphi\sin\psi.$$

Then half the *vis viva* will be

$$T = \frac{1}{2}\left(x'^2 + y'^2 + z'^2\right) = \frac{1}{2}\left(r'^2 + r^2\varphi'^2 + r^2\sin^2\varphi.\psi'^2\right),$$

so

$$\frac{\partial T}{\partial r'} = r', \frac{\partial T}{\partial \varphi'} = r^2\varphi', \frac{\partial T}{\partial \psi'} = r^2 \sin^2\varphi\psi'.$$

These quantities are the earlier quantities p, so to be set equal to $\frac{\partial W}{\partial r}$, $\frac{\partial W}{\partial \varphi}$, $\frac{\partial W}{\partial \psi}$; so one has

$$r' = \frac{\partial W}{\partial r}, \varphi' = \frac{1}{r^2}\frac{\partial W}{\partial \varphi}, \psi' = \frac{1}{r^2\sin^2\varphi}\frac{\partial W}{\partial \psi},$$

and therefore becomes

$$T = \frac{1}{2}\left\{\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \varphi}\right)^2 + \frac{1}{r^2\sin^2\varphi}\left(\frac{\partial W}{\partial \psi}\right)^2\right\}.$$

The partial differential equation (24.1) is transforned in polar coordinates as follows:

$$\frac{1}{2}\left\{\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \varphi}\right)^2 + \frac{1}{r^2\sin^2\varphi}\left(\frac{\partial W}{\partial \psi}\right)^2\right\} = \frac{k^2}{r} - \alpha. \qquad (24.2)$$

We shall integrate this equation by splitting it into many equations, each of which contains only one independent variable. If we set alone the first term on the left equal to the one on the right hand side, we get

$$\frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2 = \frac{k^2}{r} - \alpha,$$

a differential equation which contains only one independent variable r, and then remains the equation

$$\left(\frac{\partial W}{\partial \varphi}\right)^2 + \frac{1}{\sin^2\varphi}\left(\frac{\partial W}{\partial \psi}\right)^2 = 0,$$

which does not contain r any more. We can carry out this splitting still more generally, in that we can add and subtract the term $\frac{\beta}{r^2}$ on the right hand side of the equation (24.2), and then split the equation into the two:

$$\frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2 = \frac{k^2}{r} - \alpha - \frac{\beta}{r^2}, \frac{1}{2}\left\{\left(\frac{\partial W}{\partial \varphi}\right)^2 + \frac{1}{\sin^2\varphi}\left(\frac{\partial W}{\partial \psi}\right)^2\right\} = \beta.$$

The integral of the first equation is

$$W = \int \sqrt{\frac{2k^2}{r} - 2\alpha - \frac{2\beta}{r^2}} dr + F(\varphi, \psi),$$

and if one substitutes this value in the second, one obtains for $F(\varphi, \psi)$ the differential equation

$$\frac{1}{2}\left\{\left(\frac{\partial F}{\partial \varphi}\right)^2 + \frac{1}{\sin^2\varphi}\left(\frac{\partial F}{\partial \psi}\right)^2\right\} = \beta.$$

This partial differential equations can be split furhter into two parts, each of which contains only one independent variable. Namely, one adds and subtracts on the right hand side the term $\frac{\gamma}{\sin^2\varphi}$ and splits the equation into

$$\frac{1}{2}\left(\frac{\partial F}{\partial \phi}\right)^2 = \beta - \frac{\gamma}{\sin^2\varphi} \text{ and } \frac{1}{2}\left(\frac{\partial F}{\partial \psi}\right)^2 = \gamma.$$

The integral of the first equation is

$$F(\varphi, \psi) = \int \sqrt{2\beta - \frac{2\gamma}{\sin^2\varphi}} d\varphi + f(\psi),$$

and it follows from the second, $f(\psi)$ must satisfy the equation

$$\frac{1}{2}\left(\frac{\partial f}{\partial \psi}\right)^2 = \gamma$$

i.e.,

$$f(\psi) = \sqrt{2\gamma}\psi,$$

then

$$F(\varphi, \psi) = \int \sqrt{2\beta - \frac{2\gamma}{\sin^2\varphi}} d\varphi + \sqrt{2\gamma}\psi,$$

and finally,

$$W = \int \sqrt{\frac{2k^2}{r} - 2\alpha - \frac{2\beta}{r^2}} dr + \int \sqrt{2\beta - \frac{2\gamma}{\sin^2\varphi}} d\varphi + \sqrt{2\gamma}\psi. \quad (24.3)$$

This is a complete solution of the differential equation (24.2) since it contains the required number of arbitrary constants. So one obtains the integral equations of the motion in the form

$$\frac{\partial W}{\partial \alpha} = \alpha' - t, \frac{\partial W}{\partial \beta} = \beta', \frac{\partial W}{\partial \gamma} = \gamma',$$

where α' is constant denoted earlier by τ. Carrying out the differentiation,

$$\begin{aligned} t-\alpha' &= \int \frac{dr}{\sqrt{2k^2/r-2\alpha-2\beta/r^2}}, \\ \beta' &= -\int \frac{dr}{r^2\sqrt{2k^2/r-2\alpha-2\beta/r^2}}+\int \frac{d\varphi}{\sqrt{2\beta-2\gamma/\sin^2\varphi}}, \\ \gamma' &= -\int \frac{d\varphi}{\sin^2\varphi\sqrt{2\beta-2\gamma/\sin^2\varphi}}+\frac{1}{\sqrt{2\gamma}}\psi. \end{aligned} \tag{24.4}$$

It is to be remarked that the method by which we have integrated the equation (24.2) can be extended to an arbitrary number of variables. This rests on the following. When one has n variables, if one sets

$$\begin{aligned} x_1 &= r\cos\varphi_1, \\ x_2 &= r\sin\varphi_1\cos\varphi_2, \\ x_3 &= r\sin\varphi_1\sin\varphi_2\cos\varphi_3, \\ &\ldots\ldots\ldots\ldots \\ x_{n-1} &= r\sin\varphi_1\sin\varphi_2\sin\varphi_3\ldots\sin\varphi_{n-2}\cos\varphi_{n-1}, \\ x_n &= r\sin\varphi_1\sin\varphi_2\sin\varphi_3\ldots\sin\varphi_{n-2}\sin\varphi_{n-1}, \end{aligned}$$

then

$$\begin{aligned} &dx_1^2+dx_2^2+\cdots+dx_4^2=dr^2+r^2d\varphi_1^2+r^2\sin^2\varphi_1 d\varphi_2^2+ \\ &\quad r^2\sin^2\varphi_1\sin^2\varphi_2 d\varphi_3^2+\cdots+r^2\sin^2\varphi_1\sin^2\varphi_2\ldots\sin^2\varphi_{n-2}d\varphi_{n-1}^2. \end{aligned}$$

The above method can therefore be used without any modification if the right hand side of the partial differential equation can be brought to the form

$$\begin{aligned} &f(r)+\frac{1}{r^2}f_1(\varphi_1)+\frac{1}{r^2\sin^2\varphi_1}f_2(\varphi_2)+\cdots+ \\ &\frac{1}{r^2\sin^2\varphi_1\sin^2\varphi_2\ldots\sin^2\varphi_{n-2}}f_{n-1}(\varphi_{n-1}) \end{aligned}$$

The arbitrary constants β,γ which occur in the integral equation (24.4) above have very remarkable properties which make their introduction very important in perturbation problems. It is therefore interesting to investigate the geometric significance of these constants. This can be done the following way.

If one sets the expression under the radical sign in the integral over r, equal to zero, one obtains an equation of the second degree in r, whose roots represent the largest and smallest values which the radius vector can take. The roots of the equation

$$\alpha r^2 - k^2 r + \beta = 0$$

are $a(1+e)$ and $a(1-e)$, where a is the semi major axis and e the eccentricity of the planetry orbit. These give the equation

$$\frac{k^2}{\alpha} = 2a, \frac{\beta}{\alpha} = a^2(1-e^2),$$

so

$$\alpha = \frac{k^2}{2a}, \beta = \frac{k^2}{2}a(1-e^2) = \frac{k^2}{2}\frac{p}{2} \tag{24.5}$$

where p is the parameter.

If one sets the expression under the square-root sign in the integral over φ, equal to zero, then one obtains the largest or the smallest value of $\sin\varphi$, namely $\sqrt{\frac{\gamma}{\beta}}$. Now $\cos\varphi = \frac{x}{r}$ where x denotes the distance of the planet from the ecliptic (the yz-plane), consequently $\cos\varphi$ can decrease up to zero. Thus there is no minimum but only a maximum for $\cos\varphi$, and it is $\varphi = 90° - J$, where J is the inclination of the planetary orbit to the ecliptic. To this value therefore corresponds the minimum value $\sqrt{\frac{\gamma}{\beta}}$ of $\sin\varphi$, i.e.,

$$\sqrt{\frac{\gamma}{\beta}} = \sin(90° - J) = \cos J, \tag{24.6}$$

$$\sqrt{\gamma} = \cos J.\sqrt{\beta} = \frac{k}{2}\cos J.\sqrt{p}. \tag{24.7}$$

In order to determine the geometric significance of the constants α', β', γ', one must first fix the limits in the integrals occurring in (24.4). Namely, one can take for the lower limit of one of these integrals either a given numerical value or such a value which makes the square-root contained in the integral vanish. If we make the latter assumption, which we do in the following, these limits depend on the arbitrary constants α, β, γ, and since the integral equations (24.4) arise from the equation (24.3) through differentiation with respect to these constants, one may think that new terms must be added to the equation (24.4), which arise from these limits. But the additional term are, according to the known rules of differentiation, multiplied by the values which the functions under the integral sign in equation (24.4) take for the lower limits of integration and since these values vanish, equations (24.4) remain unaltered.

Under these assumptions, we let the integral in r occurring in the first equation in (24.4) start from the value $a(1-e)$, which r takes at the perihelion, as the lower limit of integration. In case the upper limit falls on this value of r, then the first equation (24.4) gives $t - \alpha' = 0$, i.e.,

$$\alpha' = \text{The time at which the planet passes through the perihelion} \tag{24.8}$$

To find the significance of β', one determines first the value of the integral in φ occuring in the second equation (24.4):

$$\Phi = \int \frac{d\varphi}{\sqrt{2\beta - 2\gamma/\sin^2\varphi}} = \int \frac{\sin\varphi d\varphi}{\sqrt{2\beta - 2\gamma - 2\beta\cos^2\varphi}},$$

after taking as its lower limit as $\varphi = 90° - J$. Through the substitutions

$$\cos\varphi = \sqrt{\frac{\beta-\gamma}{\beta}}\cos\eta, \sin\varphi d\varphi = \sqrt{\frac{\beta-\gamma}{\beta}}\sin\eta d\eta.$$

this goes over to

$$\Phi = \sqrt{\frac{\beta-\gamma}{\beta}}\int \frac{\sin\eta d\eta}{\sqrt{2(\beta-\gamma)(1-\cos^2\eta)}},$$

i.e.,

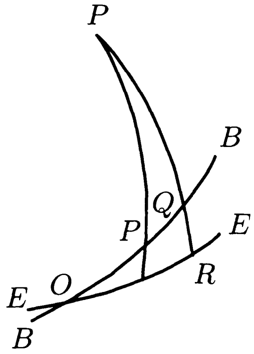
$$\Phi = \frac{1}{\sqrt{2\beta}}\int d\eta.$$

At the lower limit $\varphi = 90° - J$, according to equation (24.6), $\sin\varphi = \cos J = \sqrt{\frac{\gamma}{\beta}}$, so $\cos\varphi = \sqrt{\frac{\beta-\gamma}{\gamma}}$, therefore $\cos\eta = 1$, $\sin\eta = 0$. Accordingly, the integral over η is to be taken from the lower limit $\eta = 0$ and then

$$\Phi = \frac{1}{\sqrt{2\beta}}\eta,$$

so that the second equation in (24.4) goes over into

$$\beta' = -\int \frac{dr}{r^2\sqrt{2k^2/r - 2\alpha - 2\beta/r^2}} + \frac{1}{\sqrt{2\beta}}\eta.$$



From the relation between φ and η, one can recognise the geometrical significance of η. φ is the hypotenuse of a right angled spherical triangle whose sides are η and $90° - J$. Now let EE be the ecliptic, P its pole, BB the plane of the planetary orbit, O the ascending node. We draw through P, at right angles to BB, the great circle PQ which meets EE in R; then $QR = J$, so $PQ = 90° - J$. Further, if the radius vector taken from the centre of the sphere, the sun, to the planet meets the surface of the sphere in p, then $pP = \varphi$, and hence it follows that $\cos\varphi = \sin J. \cos(pQ)$, i.e.

$$\eta = pQ = 90° - Op.$$

Op is the distance of the planet from the ascending node O, which we shall denote by ζ. Accordingly,

$$\begin{aligned} \eta &= 90° - \zeta, \\ \beta' &= -\int \frac{dr}{r^2\sqrt{2k^2/r - 2\alpha - 2\beta/r^2}} + \frac{1}{\sqrt{2\beta}}(90° - \zeta). \end{aligned}$$

In order to determine β', one needs now only to take the point of time in which the planet passes through the perihelion. Then the integral over r will be zero and one obtains

$$\beta' = \frac{1}{\sqrt{2\beta}}(90° - \text{ distance of the perihelion from the ascending node}). \tag{24.9}$$

Finally, γ' is given by the third equation (24.4). For $\varphi = 90 - J$, i.e., when the radius vector of the planet meets the sphere in Q, the integral over φ will be zero and one obtains

$$\gamma' = \frac{1}{\sqrt{2\gamma}}\psi',$$

where ψ' represents the value of the angle ψ corresponding to the point Q. Since $\tan\psi = \frac{z}{y}$, ψ' denotes the angle which the y axis makes with the plane PQR, i.e., if the y axis passes through the equinoctial point V, $\psi' = VR = VO + OR =$ longitude of the ascending node $+90°$. One then has

$$\gamma' = \frac{1}{\sqrt{2\gamma}}(90° \ + \ \text{longitude of the ascending node}). \qquad (24.10)$$

With this all the constants occuring in equation (24.4) are determined.

For the integration of the partial differential equation (24.2) we could have also used the circumstance that ψ itself does not occur in (24.2), but only $\frac{\partial W}{\partial \psi}$. Consequently the transformation to be applied is:

$$W = W_1 + \varepsilon\psi, \frac{\partial W}{\partial \psi} = \varepsilon,$$

This would have then led us to the partial differential equation involving only two independent variables:

$$\frac{1}{2}\left\{\left(\frac{\partial W_1}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W_1}{\partial \varphi}\right)^2\right\} = \frac{k^2}{r} - \alpha - \frac{\varepsilon 2}{2r^2 \sin^2 \varphi}.$$

The integration of this will require a procedure which is not essentially different from the one applied above.

Lecture 25

Solution of the same problem by introducing the distances of the planet from two fixed points

A remarkable relation exists between two radius vectors of the planetary orbit and the arc connecting their end-points. One is led to it if one proceeds from the ordinary differential equations of elliptic motion,but through complicated calculations. We shall derive these relations without difficulty from the partial differential equation and for this we have only to assume that W can be represented by the heliocentric radius vector r and the distance ρ of the planet from one other point $M,$. The correctness of this hypothesis is not a priori evident[1] without further consideration. However it will be confirmed by calculations.

Let the coordinates of the point be (a, b, c) so that

$$\rho^2 = (x-a)^2 + (y-b)^2 + (z-c)^2.$$

With the hypothesis made above that W can be expressed in terms of r and ρ, one has

$$\begin{aligned}
\frac{\partial W}{\partial x} &= \frac{\partial W}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial W}{\partial \rho}\frac{\partial \rho}{\partial x} = \frac{\partial W}{\partial r}\frac{x}{r} + \frac{\partial W}{\partial \rho}\frac{x-a}{\rho}, \\
\frac{\partial W}{\partial y} &= \frac{\partial W}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial W}{\partial \rho}\frac{\partial \rho}{\partial y} = \frac{\partial W}{\partial r}\frac{y}{r} + \frac{\partial W}{\partial \rho}\frac{y-b}{\rho}, \\
\frac{\partial W}{\partial z} &= \frac{\partial W}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial W}{\partial \rho}\frac{\partial \rho}{\partial z} = \frac{\partial W}{\partial r}\frac{z}{r} + \frac{\partial W}{\partial \rho}\frac{z-c}{\rho}.
\end{aligned}$$

[1]The proof of this requires the result from the surface area theorem that the motion of the planet takes place in a plane, and the well-known fact that for a moving point on the plane, the distances from two fixed points can be looked upon as the determining elements.

These values are to be inserted in the partial differential equation

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2 = \frac{2k^2}{r} - 2\alpha, \qquad (25.1)$$

and then the left hand side changes to

$$\left(\frac{\partial W}{\partial r}\right)^2 + \left(\frac{\partial W}{\partial \rho}\right)^2 + \{2x(x-a) + 2y(y-b) + 2z(z-c)\}\frac{1}{r\rho}\frac{\partial W}{\partial r}\frac{\partial W}{\partial \rho}$$

The quantity within brackets is equal to $r^2 + \rho^2 - r_0^2$ where $r_0^2 = a^2 + b^2 - c^2$. Therefore (25.1) changes to

$$\left(\frac{\partial W}{\partial r}\right)^2 + \left(\frac{\partial W}{\partial \rho}\right)^2 + \frac{r^2+\rho^2-r_0^2}{r\rho}\frac{\partial W}{\partial r}\frac{\partial W}{\partial \rho} = \frac{2k^2}{r} - 2\alpha.$$

One can get rid of the product of the two partial differential coefficients if one introduces in place of r and ρ their sum and difference,

$$\sigma = r + \rho, \sigma' = r - \rho,$$

so that

$$\frac{\partial W}{\partial r} = \frac{\partial W}{\partial \sigma} + \frac{\partial W}{\partial \sigma'}, \frac{\partial W}{\partial \rho} = \frac{\partial W}{\partial \sigma} - \frac{\partial W}{\partial \sigma'}.$$

Then one has

$$2\left(\frac{\partial W}{\partial \sigma}\right)^2 + 2\left(\frac{\partial W}{\partial \sigma'}\right)^2 + \frac{r^2+\rho^2-r_0^2}{r\rho}\left\{\left(\frac{\partial W}{\partial \sigma}\right)^2 - \left(\frac{\partial W}{\partial \sigma'}\right)^2\right\}$$
$$= \frac{2k^2}{r} - 2\alpha,$$

and on multiplication by $r\rho$,

$$\{(r+\rho)^2 - r_0^2\}\left(\frac{\partial W}{\partial \sigma}\right)^2 - \{(r-\rho)^2 - r_0^2\}\left(\frac{\partial W}{\partial \sigma'}\right)^2 = 2\rho(k^2 - \alpha r),$$

or, after substituting for r, ρ their values

$$r = \frac{1}{2}(\sigma + \sigma'), \rho = \frac{1}{2}(\sigma - \sigma'),$$

finally

$$(\sigma^2 - r_0^2)\left(\frac{\partial W}{\partial \sigma}\right)^2 - (\sigma'^2 - r_0^2)\left(\frac{\partial W}{\partial \sigma'}\right)^2 = k^2(\sigma - \sigma') - \frac{1}{2}\alpha(\sigma^2 - \sigma'^2). \qquad (25.2)$$

This partial differential equation can be integrated by the method, introduced in the previous lecture, of splitting it into two ordinary differential equations, one of which contains only σ and $\frac{\partial W}{\partial \sigma}$ and the other only σ' and $\frac{\partial W}{\partial \sigma'}$. If one imagines an arbitrary constant β added and substracted at the same time on the right hand side, one is led to the two differential equations

$$\begin{aligned}(\sigma^2 - r_0^2)\left(\frac{\partial W}{\partial \sigma}\right)^2 &= -\frac{1}{2}\alpha\sigma^2 + k^2\sigma + \beta,\\ (\sigma'^2 - r_0^2)\left(\frac{\partial W}{\partial \sigma'}\right)^2 &= -\frac{1}{2}\alpha\sigma'^2 + k^2\sigma' + \beta,\end{aligned}$$

and hence for W we get

$$W = \pm\int\sqrt{\frac{-\frac{1}{2}\alpha\sigma^2 + k^2\sigma + \beta}{\sigma^2 - r_0^2}}d\sigma \pm \int\sqrt{\frac{-\frac{1}{2}\alpha\sigma'^2 + k^2\sigma' + \beta}{\sigma'^2 - r_0^2}}d\sigma'.$$

The signs of the two quantities under the square-root, or what is the same, of the integral,are arbitrary and independent of one another. One may therefore choose for W either the sum or the difference of the two integrals. One arrives at the right integral equations with either of the two assumptions, and one can therefore choose one or the other of the expressions on the grounds of greater or lesser simplicity of the resulting formulae. Let us decide for the difference and set for the sake of brevity

$$F(s) = \frac{-\frac{1}{2}\alpha s^2 + k^2 s + \beta}{s^2 - r_0^2}; \tag{25.3}$$

then we have as a solution of equation (25.2) the expression

$$W = \int d\sigma\sqrt{F(\sigma)} - \int d\sigma'\sqrt{F(\sigma')}, \tag{25.4}$$

to which we can also give the form

$$W = \int_{\sigma'}^{\sigma} ds\sqrt{F(s)}. \tag{4*}$$

Thus follows the formula, for example, for the introduction of time in the elliptic motion of a planet

$$\begin{aligned} t - \alpha' &= -\frac{\partial W}{\partial \alpha} = \frac{1}{4}\int \frac{\sigma^2 d\sigma}{\sqrt{(\sigma^2 - r_0^2)(-\frac{1}{2}\alpha\sigma^2 + k^2\sigma + \beta)}} \\ &\quad -\frac{1}{4}\int \frac{\sigma'^2 d\sigma'}{\sqrt{(\sigma'^2 - r_0^2)(-\frac{1}{2}\alpha\sigma'^2 + k^2\sigma' + \beta)}} \end{aligned}$$

where the right hand side in general consists of elliptic integrals. However, as is well-known, time can be expressed by arcs of a circle. Therefore there are consequences for elliptic integrals which lead to the fundamental theorem of addition.

The expression (25.4) is a complete solution of the partial differential equation (25.2) since one can, besides the arbitrary constant β, still add a second constant C to it. But the expression (25.4) is also a complete solution of the partial differential equation (25.1). For, in this connection, not only β and C, but also a, b, c are arbitrary constants, since they do not occur in(25.1) but in the expression (25.4). As a solution of (25.1) then, (25.4) contains more than the required number of constants i.e. there are superfluous constants among them. If one wants to use such complete solutions of a partial differential equation containing superfluous constants for the integration of the system of ordinary differential equations connected with it, one can always set the differential coefficients with respect to all the constants equal to new arbitrary constants, but these new constants are no longer independent of one another. On the other hand, one is free to use these superfluous constants at one's discretion. In the present case they can be used to transform the elliptic integral $\int ds\sqrt{F(s)}$ out of which the expression (4*) of W is constructed, into a circular one. This transformation takes place also for the elliptic integrals derived from this which occur in the partial derivatives of W with respect to the constants contained in $F(s)$.

This specialisation of the integral $\int ds\sqrt{F(s)}$ can happen in two ways. The first consists in making the numerator $-\frac{1}{2}\alpha s^2 + k^2 s + \beta$ of $F(s)$ into a complete square, the second is that this numerator has a common divisor $s - r_0$ with the denominator $s^2 - r_0^2$ of $F(s)$.

We choose the second way, and indeed on following grounds. If one derives the integral equations from (4*) without having to make any specialization of the constants, and from these the equation $a' = \frac{\partial W}{\partial s}$

which, since a is contained in σ, σ' and r_0 takes the form

$$a' = \sqrt{F(\sigma)}\frac{\partial\sigma}{\partial a} - \sqrt{F(\sigma')}\frac{\partial\sigma'}{\partial a} + a\int d\sigma\frac{F(\sigma)}{\sigma^2 - r_0^2} - a\int d\sigma'\frac{F(\sigma')}{\sigma'^2 - r_0^2}, \tag{25.5}$$

then one is not allowed to have the lower limits of integration of the elliptic integrals occurring here to be $x = a$, $y = b$, $z = c$, because then $\rho = 0$, $\sigma = \sigma' = r_0$ and the integrands would become infinite because of the $(-3/2)$th powers of $\sigma^2 - r_0^2$, $\sigma'^2 - r_0^2$ contained in them. The integrals in (25.5) becoming infinite is not prevented by the first way of specializing mentioned above, but by the second method. Since it is necessary to set $\rho = 0$, in the foirmulae to be derived so we choose the second method.

If we also assume that the numerator of $F(s)$ vanishes for $s = r_0$, so that we obtain the relation between β and r_0:

$$\beta = \frac{1}{2}\alpha r_0^2 - k^2 r_0. \tag{25.6}$$

Thereby

$$F(s) = \frac{-\frac{1}{2}\alpha(s^2 - r_0^2) + k^2(s - r_0)}{s^2 - r_0^2} = \frac{k^2}{s + r_0} - \frac{1}{2}\alpha, \tag{25.7}$$

so

$$W = \int_{\sigma'}^{\sigma} ds\sqrt{\frac{k^2}{s + r_0} - \frac{1}{2}\alpha}.$$

This is the formula for W which on differentiation gives the remarkable formulae for elliptic motion, discovered by *Euler* and *Lambert*, and used by *Olbers* and *Gauss* for determining the elements of the orbit.

The system of the first integral equations will be formed through the formulae

$$\frac{dx}{dt} = \frac{\partial W}{\partial x}, \frac{dy}{dt} = \frac{\partial W}{\partial y}, \frac{dz}{dt} = \frac{\partial W}{\partial z}.$$

We have already expressed above $\frac{\partial W}{\partial x}$, $\frac{\partial W}{\partial y}$ and $\frac{\partial W}{\partial z}$ through $\frac{\partial W}{\partial r}$ and $\frac{\partial W}{\partial \rho}$, and the latter quantities through $\frac{\partial W}{\partial \sigma}$ and $\frac{\partial W}{\partial \sigma'}$. If we use these relations and insert for $\frac{\partial W}{\partial \sigma}$ and $\frac{\partial W}{\partial \sigma'}$, then values $\sqrt{\frac{k^2}{\sigma + r_0} - \frac{1}{2}\alpha}$ and $-\sqrt{\frac{k^2}{\sigma' + r_0} - \frac{1}{2}\alpha}$

given by (25.7), we obtain the equations

$$\begin{aligned}
\frac{dx}{dt} &= \left(\frac{x}{r}+\frac{x-a}{\rho}\right)\sqrt{\frac{k^2}{\sigma+r_0}-\frac{1}{2}\alpha} \\
&\quad -\left(\frac{x}{r}-\frac{x-a}{\rho}\right)\sqrt{\frac{k^2}{\sigma'+r_0}-\frac{1}{2}\alpha}, \\
\frac{dy}{dt} &= \left(\frac{y}{r}+\frac{y-b}{\rho}\right)\sqrt{\frac{k^2}{\sigma+r_0}-\frac{1}{2}\alpha} \\
&\quad -\left(\frac{y}{r}-\frac{y-b}{\rho}\right)\sqrt{\frac{k^2}{\sigma'+r_0}-\frac{1}{2}\alpha}, \\
\frac{dz}{dt} &= \left(\frac{z}{r}+\frac{z-c}{\rho}\right)\sqrt{\frac{k^2}{\sigma+r_0}-\frac{1}{2}\alpha} \\
&\quad -\left(\frac{z}{r}-\frac{z-c}{\rho}\right)\sqrt{\frac{k^2}{\sigma'+r_0}-\frac{1}{2}\alpha},
\end{aligned} \tag{25.8}$$

whose correctness one can test if one squares them and adds and thereby, as one should, derives the theorem of *vis viva.*

The system of integral equations holding between the coordinates will be found through the formulae

$$a' = \frac{\partial W}{\partial a}, b' = \frac{\partial W}{\partial b}, c' = \frac{\partial W}{\partial c},$$

and a', b', c' denote new arbitrary constants. From equation (25.7) one obtains

$$\begin{aligned}
\frac{\partial W}{\partial a} &= -\frac{1}{2}k^2\frac{a}{r_0}\int_{\sigma'}^{\sigma}\frac{ds}{(s+r_0)^2\sqrt{\frac{k^2}{s+r_0}-\frac{1}{2}\alpha}} \\
&\quad +\frac{\partial\sigma}{\partial a}\sqrt{\frac{k^2}{\sigma+r_0}-\frac{1}{2}\alpha}-\frac{\partial\sigma'}{\partial a}\sqrt{\frac{k^2}{\sigma'+r_0}-\frac{1}{2}\alpha},
\end{aligned}$$

or, if one inserts for $\frac{\partial\sigma}{\partial a}$, $\frac{\partial\sigma'}{\partial a}$ their values $-\frac{x-a}{\rho}$, $\frac{x-a}{\rho}$ and takes into account that

$$-\frac{1}{2}k^2\int\frac{ds}{(s+r_0)^2\sqrt{\frac{k^2}{s+r_0}-\frac{1}{2}\alpha}} = \sqrt{\frac{k^2}{s+r_0}-\frac{1}{2}\alpha},$$

then

$$\frac{\partial W}{\partial a} = \left(\frac{a}{r_0} - \frac{x-a}{\rho}\right)\sqrt{\frac{k^2}{\sigma + r_0} - \frac{1}{2}\alpha}$$
$$- \left(\frac{a}{r_0} + \frac{x-a}{\rho}\right)\sqrt{\frac{k^2}{\sigma' + r_0} - \frac{1}{2}\alpha}.$$

On using these values and the corresponding values of $\frac{\partial W}{\partial b}$, $\frac{\partial W}{\partial c}$, one obtains the integral equations sought for in the following form:

$$\begin{aligned}
a' &= \left(\frac{a}{r_0} - \frac{x-a}{\rho}\right)\sqrt{\frac{k^2}{\sigma + r_0} - \frac{1}{2}\alpha} \\
&\quad - \left(\frac{a}{r_0} + \frac{x-a}{\rho}\right)\sqrt{\frac{k^2}{\sigma' + r_0} - \frac{1}{2}\alpha}, \\
b' &= \left(\frac{b}{r_0} - \frac{y-b}{\rho}\right)\sqrt{\frac{k^2}{\sigma + r_0} - \frac{1}{2}\alpha} \\
&\quad - \left(\frac{b}{r_0} + \frac{y-b}{\rho}\right)\sqrt{\frac{k^2}{\sigma' + r_0} - \frac{1}{2}\alpha}, \\
c' &= \left(\frac{c}{r_0} - \frac{z-c}{\rho}\right)\sqrt{\frac{k^2}{\sigma + r_0} - \frac{1}{2}\alpha} \\
&\quad - \left(\frac{c}{r_0} + \frac{z-c}{\rho}\right)\sqrt{\frac{k^2}{\sigma' + r_0} - \frac{1}{2}\alpha},
\end{aligned} \tag{25.9}$$

The constants a', b', c', can be determined by settting $\rho = 0$, which is an admissible value for ρ, since in consequence of the specialization of the constants contained in (25.6), the point (a, b, c) will be a point on the path of the planet[2].

[2]In order to prove this assertion, it is necessary to go back to the not yet specialized value (25.4) of W. It is a complete solution of the partial differential equation (25.2), and on introduction of the equation of the plane of the planet's orbit, the problem of the motion of the planet will be reduced to the last, if one seeks a solution in the variables, σ, σ' and looks upon a, b, c as arbitrary but given constants. Hence it follows that if one derives from (25.4) the new equation $\beta' = \frac{\partial W}{\partial \beta}$, where β' denotes an arbitrary constant which together with the equation of the plane of the planet's

orbit determines the orbit. Differentiation with respect to β gives

$$2\beta' = 2\frac{\partial W}{\partial \beta} = \int_{\sigma'}^{\sigma} \frac{ds}{\sqrt{f(s)}};$$

if one sets for abbreviation

$$f(s) = (s^2 - r_0^2)(-\frac{1}{2}\alpha s^2 + k^2 s + \beta)$$

This is the transcedental form of the integral of the differential equation

$$0 = \frac{d\sigma}{\sqrt{f(\sigma)}} - \frac{d\sigma'}{\sqrt{f(\sigma')}}.$$

Its integral equation in algebraic form, in consequence of *Euler*'s addition theorem for elliptic integrals, and indeed according to the form given it by *Lagrange* (Miscellanea Taurinensia, IV, p.110) is the following

$$\frac{\sqrt{f(\sigma)} + \sqrt{f(\sigma')}}{\sigma - \sigma'} = \sqrt{G^2 + k^2(\sigma + \sigma') - \frac{1}{2}\alpha(\sigma + \sigma')^2}, \qquad (I)$$

where G^2 denotes the constant of integration.

From the condition that the point (a, b, c) lies on the planetary orbit, i.e. that we can set $\rho = 0$, it follows $x = a$, $y = b$, $z = c$, $r = r_0$, $\sigma = \sigma' = r_0$. We first investigate the case in which ρ is infinitely small.

Let θ be the angle which the radius vector r_0 of the sun to the point (a, b, c) makes with the tangent to the planetary orbit at the point (a, b, c) directed to the infinitesimally close point (x, y, z), then one has, for infinitely small values of ρ.

$$r - r_0 = \rho \cos\theta$$

and in consequence of this,

$$\sigma - r_0 = r - r_0 + \rho = (1 + \cos\theta)\rho,$$

$$\sigma' - r_0 = r - r_0 - \rho = (1 - \cos\theta)\rho. \qquad (II)$$

Hence one obtains that, for infinitely small values of ρ, both the quantities $\sqrt{f(\sigma)}$ and $\sqrt{f(\sigma')}$ become proportional to $\sqrt{\rho}$, and that on the left hand side of equation (I) the numerator $\sqrt{f(\sigma)} + \sqrt{f(\sigma')}$ is proportional to $\sqrt{\rho}$, the denominator $\sigma - \sigma'$ proportional to ρ, and the entire function becomes infinite, while the right hand side has a finite value. The value $\rho = 0$ is admissible only if the function

$$f(s) = (s^2 - r_0^2)(-\frac{1}{2}\alpha s^2 + k^2 s + \beta)$$

has the factor $s - r_0$ twice, which for $s = \sigma$ and $s = \sigma'$ and for infinitely small values of ρ, becomes proportional to ρ, that is, when between β and r_0 one has the relation stated above:

$$\beta = \frac{1}{2}\alpha r_0^2 - k^2 r_0. \qquad (25.6)$$

Since we can let the moving point (x, y, z) coincide with the fixed point (a, b, c), the fractions $\frac{x-a}{\rho}$, $\frac{y-b}{\rho}$, $\frac{z-c}{\rho}$ appear under the form $\frac{0}{0}$. Their real values are $\cos\xi, \cos\eta, \cos\zeta$, if we denote by ξ, η, ζ the angles which the tangent to the planetary orbit at (a, b, c) makes with the x, y, z. axes. Moreover $\sigma = \sigma' = r_0$, then the relations

$$\begin{aligned} a' &= -2\cos\xi\sqrt{\frac{k^2}{2r_0} - \frac{1}{2}\alpha}, \\ b' &= -2\cos\eta\sqrt{\frac{k^2}{2r_0} - \frac{1}{2}\alpha}, \\ c' &= -\cos\zeta\sqrt{\frac{k^2}{2r_0} - \frac{1}{2}\alpha} \end{aligned} \tag{25.10}$$

follow from the equations (25.9). These same values arise with opposite signs from the equation (25.8) for the quantities $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ if one sets $\rho = 0$, and accordingly then $-a'$, $-b'$, $-c'$ are the components of the velocity of the planet at the point (a, b, c)[3].

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[3]If one squares the equations (25.9) and adds them, one obtains the relation between a', b', c':

$$a'^2 + b'^2 + c'^2 = 2\left(\frac{k^2}{r_0} - \alpha\right),$$

which is nothing other than the theorem of *vis viva* for the point (a, b, c). This relation between the constants a', b', c' confirms what was remarked above in the text about the behaviour of solutions with superfluous constants, and shows that the three equations (25.9) are satisfied by only two of them. These two to which they can be reduced can be obtained in the following way. If one eliminates from (25.9) both the redical signs contained in them, then one obtains

$$(bc' - b'c)x + (ca' - c'a)y + (ab' - a'b)z = 0 \tag{III}$$

as the equation of the plane of the planetary orbit, which satisfied by the values $x = a$, $y = b$, $z = c$. If one multiplies further the equations (25.9) by a, b, c in order and adds them, one obtains

$$-(aa' + bb' + cc')$$

$$= (\sigma + r_0)(\sigma' - r_0)\sqrt{\frac{k^2}{\sigma + r_0} - \frac{1}{2}\alpha} + (\sigma - r_0)(\sigma' + r_0)\sqrt{\frac{k^2}{\sigma' + r_0} - \frac{1}{2}\alpha} \tag{IV}$$

as the equation of the orbital curve in the plane of the orbit. It is easy to verify the identity of this result with that contained in equation (I) of the preceding remark. If one keeps the earlier definition of the angle θ, one has

$$aa' + bb' + cc' = -2r_0\cos\theta\sqrt{\frac{k^2}{2r_0} - \frac{1}{2}\alpha};$$

It only remains to introduce time, which can be done with the formula $\alpha' - t = \frac{\partial W}{\partial \alpha}$, or

$$t - \alpha' = \frac{1}{4}\int_{\sigma'}^{\sigma} \frac{ds}{\sqrt{\frac{k^2}{s+r_0} - \frac{1}{2}\alpha}}. \tag{25.11}$$

This integral leads to an arc of a circle ; if one brings it to the appropriate form, one obtains the formula given by *Gauss* in theoria motus[4]. The assumption $\alpha = 0$ corresponds to a parabolic motion. It gives the formulae which serve to determine the elements of a comet's orbit.

While the equations from (25.7) to (25.11) hold for two radius vectors r, r_0 drawn from the focus and the arc connecting them in the motion of a planet taking place in a conic section, they give more general formulae for this motion if the choice (25.6) is not made, so the point (a, b, c) does not lie on the planetary orbit. Likewise the equation (25.4) holds for W in the equation as well as in the integral equations derived from it appears the difference of two elliptic integrals which are of the same form and differ only in their arguments σ, σ'. According to the addition theorem for elliptic integrals, this difference can be transformed into *one* integral with a new argument σ'', increased by an algebraic and a circular or logarithmic function of σ and σ'. As we know, the integral equations do not contain any elliptic integrals, so the new argumentσ'' which depends algebraically on σ and σ' must be equal to a constant. The equation $\sigma'' =$ constant is therefore one of the integral equations[5], and is indeed the equation of the orbit, while the remaining algebraic and logarithmic parts supply the rest of the integral equations.

The general formulae following from (25.4) have also the remarkable property that, except for a modification to be mentioned,that they still hold when a second attracting force acts towards the point (a, b, c). Then a, b, c are no more arbitrary but given constants. We have besides α only one constant β and we are no longer free to dispose them off arbitrarily. The modification undergone by the present partial differential equation

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hence on consideration of the equation (II) it emerges that the equation (IV) leads to an identical result for infinitely small values of ρ, it being assumed that the radicals $\sqrt{\frac{k^2}{\sigma+r_0} - \frac{1}{2}\alpha}$, $\sqrt{\frac{k^2}{\sigma'+r_0} - \frac{1}{2}\alpha}$ both then approach the value $\sqrt{\frac{k^2}{2r_0} - \frac{1}{2}\alpha}$ taken with the same sign.

[4]See *Crelles* Journal vol.17, p.122

[5]See the remark on p.217

(25.2) of which the right hand side is

$$k^2(\sigma - \sigma') - \frac{1}{2}\alpha(\sigma^2 - \sigma'^2) = 2r\rho\left(\frac{k^2}{r} - \alpha\right),$$

consists in that a second term $\frac{k'^2}{\rho}$ arising from the attraction towards the point (a, b, c) is added to the force function $U = \frac{k^2}{r}$, which changes the right hand side into

$$2r\rho\left(\frac{k^2}{r} + \frac{k'^2}{\rho} - \alpha\right) = k^2(\sigma - \sigma') + k'^2(\sigma + \sigma') - \frac{1}{2}\alpha(\sigma^2 - \sigma'^2).$$

Accordingly the partial differential equation (25.2) goes over to the following

$$\begin{aligned}(\sigma^2 - r_0^2)\left(\frac{\partial W}{\partial \sigma}\right)^2 - (\sigma'^2 - r_0^2)\left(\frac{\partial W}{\partial \sigma'}\right)^2 &= (k^2 + k'^2)\sigma - \frac{1}{2}\alpha\sigma^2 \\ &\quad - \left\{(k^2 - k'^2)\sigma' - \frac{1}{2}\alpha\sigma'^2\right\}.\end{aligned}$$

Since one can resolve this equation into two ordinary differential equations

$$\begin{aligned}(\sigma^2 - r_0^2)\left(\frac{\partial W}{\partial \sigma}\right)^2 &= \beta + (k^2 + k'^2)\sigma - \frac{1}{2}\alpha\sigma^2, \\ (\sigma'^2 - r_0^2)\left(\frac{\partial W}{\partial \sigma'}\right)^2 &= \beta + (k^2 - k'^2)\sigma' - \frac{1}{2}\alpha\sigma'^2,\end{aligned}$$

one obtains for W the solution

$$\begin{aligned}W &= \int d\sigma\sqrt{\frac{\beta + (k^2 + k'^2)\sigma - \frac{1}{2}\alpha\sigma^2}{\sigma^2 - r_0^2}} \\ &\quad \pm \int d\sigma'\sqrt{\frac{\beta + (k^2 - k'^2)\sigma' - \frac{1}{2}\alpha\sigma'^2}{\sigma'^2 - r_0^2}}\end{aligned}$$

in which the two elliptic integrals differ not only in their arguments but also in their forms. For the problem of attraction towards two fixed centres in space the number of constants occurring here is not sufficient. For the problem in the plane, on the other hand (and the problem in space can be reduced to it), the above value of W is a complete solution. $\frac{\partial W}{\partial \beta} = \beta'$ gives the orbit of the point, $\frac{\partial W}{\partial \alpha} = \alpha' - t$, the time.

Lecture 26

Elliptic Coordinates

The main difficulty in the integration of a given differential equation appears to be in the choice of the right variables. There is no general rule for finding them. One must therefore adopt the reverse procedure: introduce a special substitution and investigate which problems can be adapted to the use of this substitution. I have communicated such a substitution to the Berlin Academy in a note, also published it in *Crelles* Journal[1], and quoted a series of problems, specially from mechanics, where it can be applied. Its applicability rests primarily on the fact that the expression

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2$$

takes a simple form in the new coordinates.We permit ourselves to go through problems one after another, the problem of attraction by two fixed centres considered in passing in the previous lecture being one of them. We begin by stating the remarkable substitution mentioned above, and for the sake of generality, for an arbitrary number of variables. Consider the equation

$$\frac{x_1^2}{a_1 + \lambda} + \frac{x_2^2}{a_2 + \lambda} + \ldots \frac{x_n^2}{a_n + \lambda} = 1 \tag{26.1}$$

The quantities $a_1, a_2, \ldots a_n$ can be ordered according to their magnitude, so that $a_1 < a_2 < a_3 < \ldots < a_n$.

Thus each of the differences $a_2 - a_1$, $a_3 - a_2, \ldots$ is positive. The numerators are all positive as is indicated by the fact that they are all

[1] Vol.XIX, p.309

squares. If one multiples the equation (26.1) by $(a_1+\lambda)(a_2+\lambda)\dots(a_n+\lambda)$, one obtains an equation of the n-th degree in λ, whose roots will be $\lambda_1, \lambda_2, \dots \lambda_n$. It is easy to prove that all these roots are positive. In fact, we let λ run through all values from $-\infty$ to ∞ and investigate what values the left hand side of the equation (26.1), which we denote by L, will assume thereby. For $\lambda = -\infty$, $L = 0$; with increasing λ, L will be negative and run through all negative values until it becomes infinite for $\lambda = -a_n$. Since a_n is the largest of the numbers $a_1, a_2, \dots a_n$, λ first reaches the value $-a_n$, i.e. $a_n+\lambda$ is the first denominator which vanishes. Until λ reaches the value $-a_n$, $a_n + \lambda$ is negative and as $a_n + \lambda$ nears 0, $\frac{X_n^2}{a_n+\lambda} = -\infty$. When λ increases further, $a_n+\lambda$ becomes positive, $\frac{X_n^2}{a_n+\lambda}$ makes a jump from $-\infty$ to ∞, and since the remaining factors are finite and indeed negative, what was shown for $\frac{X_n^2}{a_n+\lambda}$ holds also for L. If λ increases further and comes close to $-a_{n-1}$, then $L = -\infty$; therefore at least one root of this equation lies in this interval, and indeed only one because L stays continuous from $\lambda = -a_n$ to $\lambda = -a_{n-1}$. At $\lambda = -a_{n-1}$, L again makes a jump from $-\infty$ to ∞, and the same holds for the further progress of λ, so that in each of the intervals $-a_n$ to $-a_{n-1}$, $-a_{n-1}$ to $-a_{n-2}, \dots, -a_3$ to $-a_2$, $-a_2$ to $-a_1$ lies one and only one root of the equation. If now λ exceeds the value $-a_1$, then $L = +\infty$, and as λ increases further to $+\infty$ L takes the value below 0; in this interval $-a_1$ to ∞ must likewise lie one root. So we have concluded that the equation (26.1) has n real roots $\lambda_1, \lambda_2, \dots, \lambda_n$. We shall take these roots ordered according to magnitude, so that λ_1 lies between $+\infty$ and $-a_1$, λ_2 between $-a_1$ and $-a_2$, and so on, finally λ_n between $-a_{n-1}$ and $-a_n$. Thus

$$\lambda_1 > \lambda_2 > \lambda_3 > \dots > \lambda_n.$$

When one inserts these values for λ in equation (26.1), we get the following system of identical equations:

$$\begin{aligned}
\frac{x_1^2}{a_1+\lambda_1} + \frac{x_2^2}{a_2+\lambda_1} + \quad \dots \quad + \frac{x_{n-1}^2}{a_{n-1}+\lambda_1} + \frac{x_n^2}{a_n+\lambda_1} &= 1 \\
\frac{x_1^2}{a_1+\lambda_2} + \frac{x_2^2}{a_2+\lambda_2} + \quad \dots \quad + \frac{x_{n-1}^2}{a_{n-1}+\lambda_2} + \frac{x_n^2}{a_n+\lambda_2} &= 1 \qquad \text{(S)} \\
\frac{x_1^2}{a_1+\lambda_n} + \frac{x_2^2}{a_2+\lambda_n} + \quad \dots \quad + \frac{x_{n-1}^2}{a_{n-1}+\lambda_n} + \frac{x_n^2}{a_n+\lambda_n} &= 1.
\end{aligned}$$

If we look upon a's as constants, x and λ on the contrary as variables, their mutual dependence is of such a nature that, while $\lambda_1, \lambda_2, \dots, \lambda_n$

are found from quantities $x_1^2, x_2^2 \ldots, x_n^2$ through solution of the equation (26.1) of degree n, conversely $x_1^2, \ldots x_n^2$ are to be determined as functions of $\lambda_1, \ldots \lambda_n$ through a system of linear equations. So it comes to the solution of the system (S), for which we choose, among the different applicable methods, that of successive elimination. We first eliminate x_n^2. For example, we subtract the first equation multiplied by $a_n + \lambda_1$ from the second multiplied by $a_n + \lambda_2$ and obtain

$$x_1^2 \left\{ \frac{a_n+\lambda_2}{a_1+\lambda_2} - \frac{a_n+\lambda_1}{a_1+\lambda_1} \right\} + \quad \ldots \quad + x_{n-1}^2 \left\{ \frac{a_n+\lambda_2}{a_{n-1}+\lambda_2} - \frac{a_n+\lambda_1}{a_{n-1}+\lambda_1} \right\} = \lambda_2 - \lambda_1 .$$

On using the identity

$$\frac{a_n+\lambda_2}{a_1+\lambda_2} - \frac{a_n+\lambda_1}{a_1+\lambda_1} = \frac{(a_1-a_n)(\lambda_2-\lambda_1)}{(a_1+\lambda_2)(a_1+\lambda_1)},$$

and after cancelling the term $\lambda_2 - \lambda_1$ common to all terms, this equation goes over to

$$\frac{(a_1-a_n)x_1^2}{(a_1+\lambda_1)(a_1+\lambda_2)} + \frac{(a_2-a_n)x_2^2}{(a_2+\lambda_1)(a_2+\lambda_2)} + \cdots + \frac{(a_{n-1}-a_n)x_{n-1}^2}{(a_{n-1}+\lambda_1)(a_{n-1}+\lambda_2)} = 1.$$

One makes the same elimination between the first and the third equation, the first and the fourth equation and so on, finally the first and the n-th equation of the system (S), and thus obtains the following system of equations of order $n-1$:

$$\left.\begin{aligned}
&\frac{(a_1-a_n)x_1^2}{(a_1+\lambda_1)(a_1+\lambda_2)} + \frac{(a_2-a_n)x_2^2}{(a_2+\lambda_1)(a_2+\lambda_2)} + \cdots + \frac{(a_{n-1}-a_n)x_{n-1}^2}{(a_{n-1}+\lambda_1)(a_{n-1}+\lambda_2)} = 1 \\
&\frac{(a_1-a_n)}{(a_1+\lambda_1)(a_1+\lambda_3)}x_1^2 + \frac{(a_2-a_n)}{(a_2+\lambda_1)(a_2+\lambda_3)} + \cdots + \frac{(a_{n-1}-a_n)}{(a_{n-1}+\lambda_1)(a_{n-1}+\lambda_3)}x_{n-1}^2 = 1 \\
&\frac{(a_1-a_n)x_1^2}{(a_1+\lambda_1)(a_1+\lambda_n)} + \frac{(a_2-a_n)x_2^2}{(a_2+\lambda_1)(a_2+\lambda_n)} + \cdots + \frac{(a_{n-1}-a_n)x_{n-1}^2}{(a_{n-1}+\lambda_1)(a_{n-1}+\lambda_n)} = 1
\end{aligned}\right\} \quad (S_1.)$$

From this first reduced system of order $n-1$ one can go over in the same way to a second reduced system of order $n-2$, where one only need remark that if one looks upon

$$\frac{a_1-a_n}{a_1+\lambda_1}x_1^2, \frac{a_2-a_n}{a_2+\lambda_1}x_2^2, \ldots, \frac{a_{n-1}-a_n}{a_{n-1}+\lambda_1}x_{n-2}^2$$

as new variables the system (S_1) is reduced to the form of the sytem (S). So one obtains the second reduced system

$$\left.\begin{array}{l}
\left.\begin{array}{l}\dfrac{(a_1-a_n)(a_1-a_{n-1})}{(a_1+\lambda_1)(a_1+\lambda_2)(a_1+\lambda_3)}x_1^2 + \dfrac{(a_2-a_n)(a_2-a_{n-1})}{(a_2+\lambda_1)(a_2+\lambda_2)(a_2+\lambda_3)}x_2^2 + \ldots \\ \quad + \dfrac{(a_{n-2}-a_n)(a_{n-2}-a_{n-1})}{(a_{n-2}+\lambda_1)(a_{n-2}+\lambda_2)(a_{n-2}+\lambda_3)}x_{n-2}^2 = 1 \ldots\ldots \end{array}\right\} \\
\left.\begin{array}{l}\dfrac{(a_1-a_n)(a_1-a_{n-1})}{(a_1+\lambda_1)(a_1+\lambda_2)(a_1+\lambda_4)}x_1^2 + \dfrac{(a_2-a_n)(a_2-a_{n-1})}{(a_2+\lambda_1)(a_2+\lambda_2)(a_2+\lambda_4)}x_2^2 + \ldots \\ \quad + \dfrac{(a_{n-2}-a_n)(a_{n-2}-a_{n-1})}{(a_{n-2}+\lambda_1)(a_{n-2}+\lambda_2)(a_{n-2}+\lambda_4)}x_{n-2}^2 = 1 \end{array}\right\} \\
\dfrac{(a_1-a_n)(a_1-a_{n-1})}{(a_1+\lambda_1)(a_1+\lambda_2)(a_1+\lambda_3)}x_1^2 + \ldots \\
\quad + \dfrac{(a_{n-2}-a_n)(a_{n-2}-a_{n-1})}{(a_{n-2}+\lambda_1)(a_{n-2}+\lambda_2)(a_{n-2}+\lambda_3)}x_{n-2}^2 = 1.
\end{array}\right\} \tag{S$_2$.}$$

When one proceeds in this way, one comes finally to a system (S_{n-1}) which contains only one variable x_1^2 and consists of only one equation. This equation whose form is determined by the process of computation is

$$\frac{(a_1-a_n)(a_1-a_{n-1})\ldots(a_1-a_2)}{(a_1+\lambda_1)(a_1+\lambda_2)\ldots(a_1+\lambda_{n-1})(a_1+\lambda_n)}x_1^2 = 1.$$

and thus one obtains the following values arising from the solution (S):

$$\begin{aligned}
x_1^2 &= \frac{(a_1+\lambda_1)(a_1+\lambda_2)\ldots(a_1+\lambda_{n-1})(a_1+\lambda_n)}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)}, \\
x_2^2 &= \frac{(a_2+\lambda_1)(a_2+\lambda_2)\ldots(a_2+\lambda_{m-1})(a_2+\lambda_m)}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)},
\end{aligned}$$

$$\begin{aligned}
x_m^2 &= \frac{(a_m+\lambda_1)(a_m+\lambda_2)\ldots(a_m+\lambda_{n-1})(a_m+\lambda_n)}{(a_m-a_1)(a_m-a_2)\ldots(a_m-a_{m-1})(a_m-a_{m+1})\ldots(a_m-a_n)} \\
x_n^2 &= \frac{(a_n+\lambda_1)(a_n+\lambda_2)\ldots(a_n+\lambda_n)}{(a_n-a_1)(a_n-a_2)\ldots(a_n-a_{n-1})}.
\end{aligned} \tag{26.2}$$

Since these expressions become equal to squares they must be positivee, as can be easily shown. In the expression for x_1^2, for example, the first factor in the numerator is positive and remaining negative, so the numerator has the same sign as $(-1)^{n-1}$; in the denominator all $n-1$ factors are negative, so it has the same sign as the numerator; consequently the fraction is positive. A similar result holds for the values of the remaining quantities $x_2^2, x_3^2, \ldots, x_n^2$.

One can also verify the equations (26.2) if one substitutes them in the system (S.) and shows that it is identically satisfied. For this one needs a known lemma from the theory of decomposition into partial fractions, according to which the sum

$$\sum_{m=1}^{n} \frac{a_m^s}{(a_m - a_1)\ldots(a_m - a_{m-1})(a_m - a_{m+1})\ldots(a_m - a_n)}$$

vanishes for $s = 1, 2, \ldots, n-2$, and is equal to 1 for $s = n-1$, while for any higher value $n-1+r$ of s, it is equal to the sum of the combinations with repetitions of r of the elements $a_1, \ldots, a_n$, a theorem, whose consequences I have discussed in my inaugural dissertation.[2] The equation of the system (S) corresponding to λ_i is

$$1 = \frac{x_1^2}{a_1 + \lambda_i} + \frac{x_2^2}{a_2 + \lambda_i} + \cdots + \frac{x_n^2}{a_n + \lambda_i} = \sum_{m=1}^{n} \frac{x_m^2}{a_m + \lambda_i}.$$

In order that this be satisfied by the values (26.2) of $x_1^2, \ldots x_n^2$, the equation

$$1 = \sum_{m=1}^{n} \frac{(a_m + \lambda_1)\ldots(a_m + \lambda_{i-1})(a_m + \lambda_{i+1})\ldots(a_m + \lambda_n)}{(a_m - a_1)\ldots(a_m - a_{m-1})(a_m - a_{m+1})\ldots(a_m - a_n)} \qquad (26.3)$$

must be an identity, which is in fact verified by the theorem mentioned above, since in the numerator a_m^{n-1} is the highest power of a_m and this has coefficient 1.

The quantities $x_1^2, x_2^2, \ldots, x_n^2$ defined by the formula (26.2) satisfy yet another equation which is given immediately through the theorem stated. Namely, if one divides x_m^2 not only by $a_m + \lambda_i$ but also by the product of the factors $a_m + \lambda_i$, $a_m + \lambda_k$, where λ_i, λ_k denote two different roots of the equation (26.1), then one obtains a sum which differs from

[2] Disquisitiones analyticae de fractionibus simplicibus, Berelini 1825 (Coll. Works, Vol. III, p.3 f.f)

the right side of equation (26.3) only through this, that the numerator is raised, not up to the $(n-1)$th power with respect to a_m, but to the $(n-2)$th. Therefore, the sum will be zero and one has the equation

$$\frac{x_1^2}{(a_1+\lambda_i)(a_1+\lambda_k)} + \frac{x_2^2}{(a_2+\lambda_i)(a_2+\lambda_k)} + \cdots + \frac{x_n^2}{(a_n+\lambda_i)(a_n+\lambda_k)} = 0. \qquad (26.4)$$

Let us investigate what the left hand side of equation (26.4) becomes when λ_i, λ_k are no longer different roots, but one and the same root of equation (26.1). The question is then what value the expression

$$M_i = \frac{x_1^2}{(a_1+\lambda_i)^2} + \frac{x_2^2}{(a_2+\lambda_i)^2} + \cdots + \frac{x_n^2}{(a_n+\lambda_i)^2} \qquad (26.5)$$

has if it is expressed only through λ. The substitution of the values (26.2) for x_i^2 gives

$$M_i = \sum_{m=1}^{n} \frac{(a_m+\lambda_1)\dots(a_m+\lambda_{i-1})(a_m+\lambda_{i+1})\dots(a_m+\lambda_m)}{(a_m+\lambda_i)(a_m-a_1)\dots+(a_m-a_{m-1})(a_m-a_{m+1})\dots(a_m-a_n)}.$$

The numerator of the fraction under the summation sign is a function of degree $n-1$ in a_m. If we set in it for every $a_m+\lambda_s$ the expression $a_m+\lambda_i+\lambda_s-\lambda_i$ and develop the numerator in powers of $a_m+\lambda_i$, then the term free of $(a_m+\lambda_i)$ is

$$(\lambda_1-\lambda_i)\dots(\lambda_{i-1}-\lambda_i)(\lambda_{i-1}-\lambda_i)\dots(\lambda_n-\lambda_i).$$

All the remaining terms of the expression put together and divided by the factor $a_m+\lambda_i$ of the denominator form a function of degree $n-2$ in a_m, and therefore fall off on summation in consequence of the stated lemma. Accordingly this expression for M_i reduces to

$$M_i = \sum_{m=1}^{n} \frac{1}{(a_m+\lambda_i)} \frac{(\lambda_1-\lambda_i)\dots(\lambda_{i-1}-\lambda_i)(\lambda_{i+1}-\lambda_i)\dots(\lambda_n-\lambda_i)}{(a_m-a_1)\dots(a_m-a_{m-1})(a_m-a_{m+1})\dots(a_m-a_n)},$$

and since according to the theory of decomposition in partial fractions it is known that

$$\sum_{m=1}^{n} \frac{1}{a_m + \lambda_i} \cdot \frac{1}{(a_m - a_1)\dots(a_m - a_{m-1})(a_m - a_{m+1})\dots(a_m - a_n)}$$
$$= \frac{(-1)^{m-1}}{(a_1 + \lambda_i)\dots(a_n + \lambda_i)}$$

finally one has for M_i the value

$$M_i = \frac{(\lambda_1 - \lambda_i)\dots(\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1})\dots(\lambda_i - \lambda_n)}{(a_1 + \lambda_i)(a_2 + \lambda_i)\dots(a_n + \lambda_i)} \tag{26.6}$$

i.e. one has the equation

$$\frac{x_1^2}{(a_1 + \lambda_i)^2} + \dots + \frac{x_n^2}{(a_n + \lambda_i)^2}$$
$$= \frac{(\lambda_i - \lambda_1)\dots(\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1})\dots(\lambda_i - \lambda_n)}{(a_1 + \lambda_i)(a_2 + \lambda_i)\dots(a_n + \lambda_i)}. \tag{26.7}$$

This result can be derived in another way, which is somewhat simpler. One sets

$$u = 1 - \left\{ \frac{x_1^2}{a_1 + \lambda} + \dots + \frac{x_n^2}{a_n + \lambda} \right\}, \tag{26.8}$$

so that the equation $u = 0$ is identical with equation (26.1); then M_i defined by equation (26.5) can, with the help of u, be expressed in the form

$$M_i = \left(\frac{\partial u}{\partial \lambda} \right)_{\lambda = \lambda_i}$$

and one can therefore derive equation (26.6) for M_i if one replaces the variables $x_1^2, \dots x_n^2$ by $\lambda_1, \dots, \lambda_n$ on the right side of equation (S). In order to obtain this transformation one multiples u by the product of the denominators $(a_1 + \lambda)\dots(a_n + \lambda)$ and then obtains a rational integral function of order n in λ, which vanishes for the values $\lambda_1, \dots, \lambda_n$ of λ and in which the coefficient of λ^n is unity. One has then

$(a_1 + \lambda) \ldots (a_n + \lambda)u = (\lambda - \lambda_1) \ldots (\lambda - \lambda_n)$, or

$$u = \frac{(\lambda - \lambda_1) \ldots (\lambda - \lambda_n)}{(a_1 + \lambda) \ldots (a_n + \lambda)}, \tag{8*}$$

an equation by comparison of which with (26.8) one concludes, incidentally, that the values (26.2) of the quantities $x_1^2, \ldots x_n^2$ can be defined as the negatives of the numerators of the partial fractions $\frac{1}{(a_1+\lambda)}, \ldots, \frac{1}{(a_n+\lambda)}$ in the decomposition of the function (8*). If we differentiate the expression (8*) with respect to λ and set $\lambda = \lambda_i$, we obtain

$$\begin{aligned} M_i &= \left(\frac{\partial u}{\partial \lambda}\right)_{\lambda=\lambda_i} \\ &= \frac{(\lambda_i - \lambda_1) \ldots (\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1}) \ldots (\lambda_i - \lambda_n)}{(a_1 + \lambda_i)(a_2 + \lambda_i) \ldots (a_n + \lambda_i)}, \end{aligned}$$

in agreement with (26.6).

The results obtained up to now enable us to add to the above substitution, without further computation, the differential formulae following from the same. If one takes the logarithms of the values of x_m^2 contained in the equation (26.2) and then differentiates, one has

$$2\frac{dx_m}{x_m} = \frac{d\lambda_1}{a_m + \lambda_1} + \frac{d\lambda_2}{a_m + \lambda_2} + \cdots + \frac{d\lambda_n}{a_m + \lambda_n}.$$

Hence follows the formula for the sum of the squares of the differentials of $x_1, \ldots, x_n$:

$$\begin{aligned} &4(dx_1^2 + dx_2^2 + \cdots + dx_n^2) \\ &= \sum_{m=1}^{n} \frac{x_m^2}{(a_m + \lambda_1)^2} d\lambda_1^2 \\ &\quad + \cdots + \sum_{m=1}^{n} \frac{x_m^2}{(a_m + \lambda_n)^2} d\lambda_n^2 \\ &\quad +2\sum_{m=1}^{n} \frac{x_m^2}{(a_m + \lambda_1)(a_m + \lambda_2)} d\lambda_1 d\lambda_2 + \cdots . \end{aligned}$$

According to equation (24.4), the coefficient of $d\lambda_1.d\lambda_2$ vanishes and likewise the coefficients of all products of differentials of two distinct λ's would be zero. The coefficients of $d\lambda_1^2, \ldots, d\lambda_n^2$ are, on the contrary, according to equation (26.5), the quantities $M_1, M_2, \ldots, M_n$; so we have

$$4\left(dx_1^2 + \cdots + dx_n^2\right) = M_1 d\lambda_1^2 + \cdots + M_n d\lambda_n^2, \tag{26.9}$$

where the coefficients M_i are defined in equation (26.6). If one gives an extension to n dimensions of the notion of *vis viva* $\frac{1}{2}(x_1'^2 + x_2'^2 + x_3'^2)$ of a freely moving point of mass 1, and sets $T = \frac{1}{2}(x_1'^2 + \cdots + x_n'^2)$, then by virtue of equation (26.9) one can represent this expression T also through the variables λ and their different coefficients with respect to t and obtain

$$8T = 4\left(x_1'^2 + \cdots + x_n'^2\right) = M_1 \lambda_1'^2 + \cdots + M_n \lambda_n'^2. \tag{26.10}$$

The extension to n dimensions mentioned corresponds to the Hamiltonian partial differential equation whose left side is the expression

$$\left(\frac{\partial W}{\partial x_1}\right)^2 + \left(\frac{\partial W}{\partial x_2}\right)^2 + \cdots + \left(\frac{\partial W}{\partial x_n}\right)^2.$$

This arises from T if one substitutes there

$$\frac{\partial T}{\partial x_1'} = \frac{\partial W}{\partial x_1}, \frac{\partial T}{\partial x_2'}, = \frac{\partial W}{\partial x_2}, \ldots, \frac{\partial T}{\partial x_n'} = \frac{\partial W}{\partial x_n}.$$

One finds the expression to which the above one goes over on changing the variables x to the variables λ, according to Lecture 19, if one uses in the transformed expression the $2T$ the equations:

$$\frac{\partial T}{\partial \lambda_1'} = \frac{\partial W}{\partial \lambda_1}, \frac{\partial T}{\partial \lambda_2'} = \frac{\partial W}{\partial \lambda_2}, \ldots \frac{\partial T}{\partial \lambda_n'} = \frac{\partial T}{\partial \lambda_n}.$$

In the present case, according to equation (26.10),

$$4\frac{\partial T}{\partial \lambda_i'} = M_i \lambda_i' = 4\frac{\partial W}{\partial \lambda_i},$$

so one has to set $\lambda_i' = \frac{4}{M_i}\frac{\partial W}{\partial \lambda_i}$ in

$$2T = \frac{1}{4}\left(M_1 \lambda_1'^2 + \cdots + M_n \lambda_n'^2\right),$$

and obtain in this way

$$\left(\frac{\partial W}{\partial x_1}\right)^2 + \left(\frac{\partial W}{\partial x_2}\right)^2 + \cdots + \left(\frac{\partial W}{\partial x_n}\right)^2$$
$$= \frac{1}{4}\left\{\frac{1}{M_1}\left(\frac{\partial W}{\partial \lambda_1}\right)^2 + \frac{1}{M_2}\left(\frac{\partial W}{\partial \lambda_2}\right)^2 \cdots + \frac{1}{M_n}\left(\frac{\partial W}{\partial \lambda_1}\right)^2\right\}, \quad (26.11)$$

where M_i is to be determined in accordance with (26.6), or what is the same

$$\sum\left(\frac{\partial W}{\partial x_i}\right)^2$$
$$= 4\sum \frac{(a_1+\lambda_i)(a_2+\lambda_i)\ldots(a_n+\lambda_i)}{(\lambda_i-\lambda_1)\ldots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\ldots(\lambda_i-\lambda_n)}\left(\frac{\partial W}{\partial \lambda_i}\right)^2. \quad (26.12)$$

Lecture 27

Geometric significance of elliptic coordinates on the plane and in space. Quadrature of the surface of an ellipsoid. Rectification of its lines of curvature

Let us now examine more closely the geometric significance that the substitution, introduced in the preceding lecture, has for $n = 2$ and $n = 3$. For the case of two variables one has the equation

$$\frac{x_1^2}{a_1 + \lambda} + \frac{x_2^2}{a_2 + \lambda} = 1.$$

If one looks upon x_1 and x_2 as orthogonal coordinates, this is then the equation of a conic section and in fact of an ellipse when λ lies between the limits $-a_1$, and $+\infty$, so that both the denominators are positive. It is an equation of a hyperbola if λ lies between $-a_1$ and $-a_2$, that is, the first denominator is negative and the second positive. If λ varies while a_1 and a_2 remain constant, then this equation represents a system of confocal conic sections. If x_1 and x_2 are given, then there are always two values of λ which satisfy the equations, one of which lies between $-a_1$ and ∞ and the other between $-a_1$ and $-a_2$ i.e. of the system of confocal conic sections, two always pass through a given point, and in fact one is an ellipse and the other a hyperbola. The variables λ_1 and λ_2 introduced for x_1 and x_2 therefore, speaking geometrically, represent the points in the plane determined through the ellipse and the hyperbola which go through them and have two given points for foci. If one sets λ_1 = constant, one obtains all points on an ellipse of the system of confocal conic sections. If one sets λ_2 = constant, then this we get all points on a hyperbola. The two systems of confocal ellipses and hyperbolas have this in common with the ordinary coordinate system, that two curves of *one* system do not intersect and that any curve of one system intersects with all curves of the other system orthogonally. In fact if one of the ellipses and one of the hyperbolas intersect at the point (x_1, x_2) then,

$$E = \frac{x_1^2}{a_1+\lambda_1} + \frac{x_2^2}{a_2+\lambda_1} = 1, \quad H = \frac{x_1^2}{a_1+\lambda_2} + \frac{x_2^2}{a_2+\lambda_2} = 1,$$

then the normals at the points (x_1, x_2) to the ellipse and the hyperbola form with the coordinate axes angles whose cosines behave as $\frac{\partial E}{\partial x_1} : \frac{\partial E}{\partial x_2}$ and similarly $\frac{\partial H}{\partial x_1} : \frac{\partial H}{\partial x_2}$. If these normals are to be mutually orthogonal, then the relation

$$\frac{\partial E}{\partial x_1} \cdot \frac{\partial H}{\partial x_1} + \frac{\partial E}{\partial x_2} \cdot \frac{\partial H}{\partial x_2} = 0,$$

or

$$\frac{x_1^2}{(a_1+\lambda_1)(a_1+\lambda_2)} + \frac{x_2^2}{(a_2+\lambda_1)(a_2+\lambda_2)} = 0$$

must hold. Since this is identical with the equation (26.4) of the preceding lecture, the orthogonality of the ellipse and the hyperbola is hereby proved. From this arises a simplification for the determination of the surface element. In general this is equal to

$$\left(\frac{\partial x_1}{\partial \lambda_1} \frac{\partial x_2}{\partial \lambda_2} - \frac{\partial x_1}{\partial \lambda_2} \frac{\partial x_2}{\partial \lambda_1} \right) d\lambda_1 d\lambda_2,$$

in the present case one needs only to multiply the arc lengths of the ellipse and the hyperbola with each other. According to formula (26.9) of the preceding lecture, the square of the element of arc of an arbitrary curve is

$$4(dx_1^2 + dx_2^2) = \frac{\lambda_1 - \lambda_2}{(a_1+\lambda_1)(a_2+\lambda_1)} d\lambda_1^2 + \frac{\lambda_2 - \lambda_1}{(a_1+\lambda_2)(a_2+\lambda_2)} d\lambda_2^2. \tag{27.1}$$

This gives the element of an arc of an ellipse if one sets $\lambda_1 =$ constant, so that $d\lambda_1 = 0$, and that of a hyperbola if one sets $\lambda_2 =$ constant, so that $d\lambda_2 = 0$. The elements of arc are therefore

$$\frac{1}{2} d\lambda_2 \sqrt{\frac{\lambda_2 - \lambda_1}{(a_1+\lambda_2)(a_2+\lambda_2)}} \quad \text{and} \quad \frac{1}{2} d\lambda_1 \sqrt{\frac{\lambda_1 - \lambda_2}{(a_1+\lambda_1)(a_2+\lambda_1)}},$$

and the surface element is the product of these, i.e.

$$\frac{\frac{1}{4}(\lambda_1 - \lambda_2)d\lambda_1 d\lambda_2}{\sqrt{-(a_1 + \lambda_1)(a_2 + \lambda_1)(a_1 + \lambda_2)(a_2 + \lambda_2)}}.$$

Entirely analogous considerations hold for three variables, i.e. for three-dimensional space. Let x_1, x_2, x_3 be rectangular coordinates; then the equation

$$\frac{x_1^2}{a_1 + \lambda} + \frac{x_2^2}{a_2 + \lambda} + \frac{x_3^2}{a_3 + \lambda} = 1$$

represents when λ varies, a system of confocal surfaces of the second degree. The theorems on confocal surfaces of the second degree (i.e. those in which the principal sections have the same foci) belong to the most remarkable ones in analytic geometry; I have made known some of the most important ones for the first time in Vol.XII of *Crelle's* Journal[1]. Since *Chasles*, in his "Aperçu historique"[2], designates these as new, without mentioning the priority of my work, we must remind ourselves that, in this work, all articles written in German in *Crelles* Journal have been ignored.[3]

The confocal surfaces can be divided in three systems; a system of ellipsoids for which λ lies between $-a_1$ and $+\infty$, a system of hyperboloids of one sheet for which λ lies between $-a_1$ and $-a_2$, and a system of hyperboloids of two sheets for which λ lies between $-a_2$ and $-a_3$. In the first case the denominators $a_1 + \lambda_1$, $a_2 + \lambda$ $a_3 + \lambda$ are all positive; in the second case $a_1 + \lambda$ and is negative, while $a_2 + \lambda$ and $a_3 + \lambda$ are positive; the third case $a_1 + \lambda$ and $a_2 + \lambda$ are negative and $a_3 + \lambda$ positive. For every point (x_1, x_2, x_3) three values of λ, $\lambda_1, \lambda_2, \lambda_3$ exist, which satisfy the above equation, and in fact, λ_1 corresponds to an ellipsoid, λ_2 to an one-sheeted hyperboloid and λ_3 to a two-sheeted hyperboloid. Of a given system of confocal surfaces of the second degree, through a given point pass *one* ellipsoid, *one* one-sheeted hyperboloid and *one* two-sheeted hyperboloid. Each of these three systems intersects the other two orthogonally. *Binet* has proved for the first time that the curves of intersection are, at the same time, the lines of curvature of these surfaces. *Charles Dupin* has shown in his "Développments de géometrie"

[1]Letter to Steiner, p.137
[2]Note XXXI, p 384
[3]Aperçu historique, p. 215, Remark

that this theorem always holds when three systems of surfaces intersect mutually orthogonally. *Lamé* has in more recent times made interesting applications of the theory of confocal surfaces to mathematical physics.

That these confocal surfaces passing through a given point in space intersect at right angles follows from the geometric significance of equation (26.4) of the previous lecture. It is self-evident also that the three curves of intersection of every two of these confocal surfaces are at right angles to one another. Hence it follows that any two of the elements of arcs of these curves of intersection multiplied by each other lead to the surface element of the confocal surface containing the two elements of arc, and that the product of all three elements of arcs of the curves of intersections represents the volume element in the coordinate system $(\lambda_1, \lambda_2, \lambda_3)$.

The expression for the square of the element of arc of any space-curve is, according to formula (26.9) of the previous lecture

$$\begin{aligned} dx_1^2 + dx_2^2 + dx_3^2 &= \frac{1}{4}\left\{\frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{(a_1+\lambda_1)(a_2+\lambda_1)(a_3+\lambda_1)} d\lambda_1^2\right. \\ &\quad + \frac{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)} d\lambda_2^2 \\ &\quad \left.\frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)} d\lambda_3^2\right\}. \qquad (27.2) \end{aligned}$$

If one sets one of the quantities $\lambda_1, \lambda_2, \lambda_3$ constant in this expression, then it refers to a curve which lies on one of the confocal surfaces, for example, on an ellipsoid for a constantλ_1. If one further sets in the expression two of the quantities $\lambda_1, \lambda_2, \lambda_3$ constant, then it refers to the intersection curve mentioned above, and in fact to those which lie on a confocal ellipsoid if one sets λ_1 and λ_2 or λ_1 and λ_3 constant. On the other hand we get the curves of intersection of two confocal hyperboloids if we set λ_2 and λ_3 constant. Accordingly, one obtains the elements of arcs of the curves of intersection on an ellipsoid

$$\frac{1}{2} d\lambda_3 \sqrt{\frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{(c_1+\lambda_3)(c_2+\lambda_3)(c_3+\lambda_3)}} \quad \text{and}$$

$$\frac{1}{2} d\lambda_2 \sqrt{\frac{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}} \qquad (27.3)$$

and for the surfaces element of the ellipsoid

$$\frac{\lambda_2-\lambda_3}{4}d\lambda_2 d\lambda_3$$
$$\sqrt{\frac{-(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}}.$$

If one integrates this differential and extends it over all possible values of λ_2 and λ_3, i.e. from $\lambda_2 = -a_2$ to $\lambda_2 = -a_1$, and from $\lambda_3 = -a_3$ to $\lambda_3 = -a_2$, then one obtains an octant of the whole surface of the ellipsoid. This double integral, however, breaks up into the sum of two products of simple integrals and gives for the surface of the ellipsoid the expression

$$2\int_{-a_2}^{-a_1} d\lambda_2\lambda_2\sqrt{\frac{(\lambda_2-\lambda_1)}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}}.$$
$$\cdot\int_{-a_3}^{-a_2} d\lambda_3\sqrt{\frac{-(\lambda_3-\lambda_1)}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}}$$
$$-2\int_{-a_2}^{-a_1} d\lambda_2\sqrt{\frac{(\lambda_2-\lambda_1)}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}}.$$
$$\int_{-a_3}^{-a_2} d\lambda_3\lambda_3\sqrt{\frac{-(\lambda_3-\lambda_2)}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}} \tag{27.4}$$

which is made up of elliptic integrals. This is the way in which *Legendre*[4] has found the quadrature of the surface of the ellipsoid. His work is of the greatest importance because thereby, for the first time, the lines of curvatures were applied as the analytical tool for the transformation of coordinates. If in the expression above, one takes the integrals over arbitrarily narrow limits, one obtains then, not the surface of the whole ellipsoid, but a piece of it which is enclosed between two lines of curvature of one kind and two of the other kind.

In order to obtain the volume element, one must multiply the surface element of the ellipsoid with the element of arc of the curve of intersection formed by the two hyperboloids. For this element of arc one obtains, when one sets λ_2 and λ_3 constant, the expression

$$\frac{1}{2}d\lambda_1\sqrt{\frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{(a_1+\lambda_1)(a_2+\lambda_1)(a_3+\lambda_1)}},$$

[4]Exercises de calcul intégral, I, p.185, or Traité de fonictiones elliptiques, I, p.352.

consequently, the volume element is

$$\frac{\frac{1}{8}(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)d\lambda_1 d\lambda_2 d\lambda_3}{\sqrt{-(a_1+\lambda_1)(a_2+\lambda_1)(a_3+\lambda_1)(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}}.$$

When one integrates this differential thrice, and in fact over such limits as do not exceed the possible values of λ_1, λ_2, and λ_3, one arrives at a space which is bounded by two confocal ellipsoids, two confocal one-sheeted hyperboloids and two confocal two-sheeted hyperboloids. The triple integral breaks up completely into six terms each of which is a product of these simple integrals.

The two elements of arc

$$\frac{1}{2}d\lambda_3\sqrt{\frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}} \quad \text{and}$$

$$\frac{1}{2}d\lambda_2\sqrt{\frac{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}}$$

which we multiplied together for the quadrature of the ellipsoid, are, according to *Binet*'s theorem, the elements of the lines of curvature on the ellipsoid. The integration of the elements gives the length of the lines of curvature, and we obtain for the length of their arcs the integrals

$$\frac{1}{2}\int d\lambda_3\sqrt{\frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}} \quad \text{and}$$

$$\frac{1}{2}\int d\lambda_2\sqrt{\frac{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}} \tag{27.5}$$

which belong to Abelian integrals and indeed of the genus which follows next to the elliptic integrals.

Lecture 28

The shortest line on the tri-axial ellipsoid. The problem of map projection

The formulae of the last two lectures lead to a very simple way of determining the shortest line on a tri-axial ellipsoid, already mentioned in lecture 22 (p.194), but not carried out so far. This will be described by a mass point constrained to remain on the surface of the ellipsoid, without any force acting on it, only driven by an initial push. So the force function U vanishes in this case.

If x_1, x_2, x_3 denote the rectangular coordinates, referred to the axes of the ellipsoid, of the moving point, then the constraint which holds in order that it remain on the ellipsoid, is expressed by the equation of constraint

$$\frac{x_1^2}{a_1 + \lambda_1} + \frac{x_2^2}{a_2 + \lambda_1} + \frac{x_3^2}{a_3 + \lambda_1} = 1.$$

It now depends on representing x_1, x_2, x_3 as functions of two new variables so that these inserted in the equation of constraint, satisfy it identically. Such values are those which we have found for x_1^2, x_2^2, x_3^2 in $\lambda_1, \lambda_2, \lambda_3$ if we look upon λ_1 as constant, λ_2, λ_3 as variable. We have to express the *vis viva* through the quantities λ_2, λ_3, which take the place of the variables denoted earlier by q, and their differential coefficients $\lambda_2' = \frac{d\lambda_2}{dt}$, $\lambda_3' = \frac{d\lambda_3}{dt}$, then to introduce for λ_2', λ_3' the new variables $\mu_2 = \frac{\partial T}{\partial \lambda_2'}$, $\mu_3 = \frac{\partial T}{\partial \lambda_3'}$, which correspond to the quantities denoted earlier by p, and to set $\mu_2 = \frac{\partial T}{\partial \lambda_2'} = \frac{\partial W}{\partial \lambda_2}$, $\mu_3 = \frac{\partial T}{\partial \lambda_3'} = \frac{\partial W}{\partial \lambda_3}$. In this way T expressed through $\lambda_2, \lambda_3, \frac{\partial W}{\partial \lambda_2}, \frac{\partial W}{\partial \lambda_3}$ and the equation $T + \alpha = 0$, which we can write in the form $T = h$ if one sets $\alpha = -h$, is the partial differential equation of the problem, through which W is defined as a function of λ_2 and λ_3. If one restricts the number of variables in equation (26.10) of lecture 26 to three, then one obtains for the 'vis viva' $2T$ the transformation formula

$$2T = x_1'^2 + x_2'^2 + x_3'^2 = \frac{1}{4}M_1\lambda_1'^2 + \frac{1}{4}M_2\lambda_2'^2 + \frac{1}{4}M_3\lambda_3'^2,$$

where

$$\begin{aligned} M_1 &= \frac{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{(a_1 + \lambda_1)(a_2 + \lambda_1)(a_3 + \lambda_1)}, \\ M_2 &= \frac{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}{(a_1 + \lambda_2)(a_2 + \lambda_2)(a_3 + \lambda_2)}, \\ M_3 &= \frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{(a_1 + \lambda_3)(a_2 + \lambda_3)(a_3 + \lambda_3)}. \end{aligned}$$

Since, however, the motion takes place on an ellipsoid, so λ_1 is constant, $\lambda_1' = 0$ and

$$2T = \frac{1}{4}M_2\lambda_2'^2 + \frac{1}{4}M_3\lambda_3'^2.$$

Hence we have

$$\begin{aligned} \frac{\partial T}{\partial \lambda_2'} &= \frac{1}{4}M_2\lambda_2' = \frac{\partial W}{\partial \lambda_2}, \frac{\partial T}{\partial \lambda_3'} = \frac{1}{4}M_3\lambda_3' = \frac{\partial W}{\partial \lambda_3}, \\ \lambda_2' &= \frac{4}{M_2}\frac{\partial W}{\partial \lambda_2}, \lambda_3' = \frac{4}{M_3}\frac{\partial W}{\partial \lambda_3}, \end{aligned}$$

and one obtains for $2T$ the expression

$$2T = \frac{4}{M_2}\left(\frac{\partial W}{\partial \lambda_2}\right)^2 + \frac{4}{M_3}\left(\frac{\partial W}{\partial \lambda_3}\right)^2.$$

The partial differential equation sought for is accordingly

$$\begin{aligned} T &= 2\frac{(a_1 + \lambda_3)(a_2 + \lambda_2)(a_3 + \lambda_2)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}\left(\frac{\partial W}{\partial \lambda_2}\right)^2 \\ &\quad + 2\frac{(a_1 + \lambda_3)(a_2 + \lambda_3)(a_3 + \lambda_3)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\left(\frac{\partial W}{\partial \lambda_2}\right)^2 = h, \end{aligned}$$

or,

$$\begin{aligned} &\frac{(a_1 + \lambda_2)(a_2 + \lambda_2)(a_3 + \lambda_2)}{\lambda_2 - \lambda_1}\left(\frac{\partial W}{\partial \lambda_2}\right)^2 \\ &\quad - \frac{(a_1 + \lambda_3)(a_2 + \lambda_3)(a_3 + \lambda_3)}{\lambda_3 - \lambda_1}\left(\frac{\partial W}{\partial \lambda_3}\right)^2 \\ &\quad = \frac{1}{2}h(\lambda_2 - \lambda_3). \end{aligned} \tag{28.1}$$

This partial differential equation again splits up into two ordinary differential equations, each of which contains only one independent variable. Again, if one adds and subtracts an arbitrary constant on the right hand side, one obtains the two ordinary differential equations

$$\begin{aligned}\frac{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}{\lambda_2-\lambda_1}\left(\frac{\partial W}{\partial \lambda_2}\right)^2 &= \frac{1}{2}h(\lambda_2+\beta)\\ \frac{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}{\lambda_3-\lambda_1}\left(\frac{\partial W}{\partial \lambda_3}\right)^2 &= \frac{1}{2}h(\lambda_3+\beta).\end{aligned}$$

The coefficient of $\left(\frac{\partial W}{\partial \lambda_2}\right)^2$ is positive, since of the three factors in the numerator only the first is negative and $\lambda_2-\lambda_1$ is in any case negative, therefore $\frac{1}{2}h(\lambda_2+\beta)$ must be positive; the coefficient of $\left(\frac{\partial W}{\partial \lambda_3}\right)^2$ on the contrary is negative, since the first two factors of the numerator are negative and the denominator $\lambda_3-\lambda_1$ also negative, consequently $\frac{1}{2}h(\lambda_3+\beta)$ must be negative. The constant h is however positive, because it is equal to half the *vis viva*, a positive quantity by its nature. Since, $\lambda_2+\beta$ must be positive, $\lambda_3+\beta$ negative, one has the inequalities

$$\beta+\lambda_2>0,\ \beta+\lambda_3<0,\ -\lambda_2<\beta<-\lambda_3,$$

two conditions that are mutually consistent since $\lambda_2>\lambda_3$.

We obtain the following complete solution of the partial differential equation (28.1) from the above ordinary differential equations,

$$\begin{aligned}W &= \sqrt{\frac{h}{2}}\left\{\int d\lambda_2\sqrt{\frac{(\lambda_2-\lambda_1)(\lambda_2+\beta)}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}}\right.\\ &\qquad \left.+\int d\lambda_3\sqrt{\frac{(\lambda_3-\lambda_1)(\lambda_3+\beta)}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}}\right\}. \qquad (28.2)\end{aligned}$$

One obtains from this the equation for the shortest line on the triaxial ellipsoid, $\frac{\partial W}{\partial \beta}=$ constants, or

$$\begin{aligned}&\int d\lambda_2\sqrt{\frac{\lambda_2-\lambda_1}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)(\beta+\lambda_2)}}\\ &\quad+\int d\lambda_3\sqrt{\frac{\lambda_3-\lambda_1}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)(\beta+\lambda_3)}} = \text{constant}\end{aligned} \qquad (28.3)$$

The equation for the time is $\tau - t = \frac{\partial W}{\partial \alpha} = -\frac{\partial W}{\partial h}$, or since W depends on h through the factor $\sqrt{h}$, and accordingly $\frac{\partial W}{\partial h} = \frac{1}{2h} W$,

$$t - \tau = \frac{1}{2h} W. \tag{28.4}$$

If s denotes the arc of the shortest line, reckoned from the point on which the moving point finds itself at time τ, then the theorem of *vis viva* gives $T = \frac{1}{2}\left(\frac{ds}{dt}\right)^2 = h$, $ds = \sqrt{2h}dt$,

$$s = \sqrt{2h}(t - \tau).$$

Hence one obtains, by comparing with (28.4), the equation for the arc $s = \frac{1}{\sqrt{2h}} W$, or

$$\begin{aligned} s \;\; = \;\; & \frac{1}{2}\left\{ \int d\lambda_2 \sqrt{\frac{(\lambda_2 - \lambda_1)(\lambda_2 + \beta)}{(a_1 + \lambda_2)(a_2 + \lambda_2)(a_3 + \lambda_2)}}, \right. \\ & \left. + \int d\lambda_3 \sqrt{\frac{(\lambda_3 - \lambda_1)(\lambda_3 + \beta)}{(a_1 + \lambda_3)(a_2 + \lambda_3)(a_3 + \lambda_3)}} \right\}, \end{aligned}$$

whereby also the length of the shortest line is calculated.

We have thus solved a problem hitherto considered unsolvable, by merely using the partial differential equation. Though the substitution applied is an essential requirement for this solution, the method of reduction to a partial differential equation simplifies the procedure significantly. In fact, *Minding* found that, when he wished to apply the substitution published by me,he would not have been able to overcome the difficulties in the usual way of integrating an ordinary differential equation if he did not already know my result.

Through the same substitution, which has already given us the solution of many difficult problems, we can also settle the problem of map-projection for the tri-axial ellipsoid. Among the different ways of representing a curved surface on a plane, as is necessary for a map, one prefers, above all, the method of projection in which infinitely small elements remain similar. In the preceding century *Lambert* had been concerned with various aspects of this projection, of which one can learn in detail from his contributions to mathematics. Because, of these *Lambert*'s colleague at that time, *Lagrange*, was induced to undertake an investigation from the same standpoint and gave the solution completely for all surfaces of revolution. The Copenhagan Academy which later announced a prize for the solution of this problem for all curved surfaces awarded it to the treatise sent in by *Gauss*. In this, *Lagarange*'s work, to which only little had to be added, finds no mention.

The leading idea for the solution of the problem of map projection is the following. Suppose one connects a point on the surface with points infinitesimaly close, and the same is done for corresponding points on the plane. Then in order that the infinitesimally small elements be similar, the corresponding lengths must be proportional, and conversely if the corresponding lengths are proportional, then the infinitely small elements are similar. This proportionality is to be expressed analytically.

Let the coordinates x, y, z of a point on the surface be given as functions of two quantities p, q; then the square of the element of arc of any curve on the surface is represented by the expression

$$ds^2 = dx^2 + dy^2 + dx^2 = Adp^2 + 2Bdpdq + Cdq^2.$$

The square of the corresponding element of arc in the plane is

$$d\sigma^2 = du^2 + dv^2,$$

where u and v denote rectangular coordinates in the plane. In order that the infinitely small elements of length be mutually proportional, $d\sigma^2$ must be equal to mds^2, where m can be any function of p and q. The correlation system between the quantities μ, ν and p, q must be such that the equation

$$du^2 + dv^2 = m(Adp^2 + 2Bdpdq + Cdq^2)$$

hold, where $\sqrt{m}$ represents the similarity-ratio.

This differential equation can be satisfied in the following way. One resolves $Adp^2 + 2Bdpdq + Cdq^2$ into two linear factors

$$\sqrt{A}dp + \left(\frac{B}{\sqrt{A}} + \sqrt{C - \frac{B^2}{A}}\sqrt{-1}\right) dq,$$
$$\sqrt{A}dp + \left(\frac{B}{\sqrt{A}} - \sqrt{C - \frac{B^2}{A}}\sqrt{-1}\right) dq,$$

and considers m itself resolved into the factors $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$;

then the differential equation above breaks up into the following two:

$$\begin{aligned} du + dv\sqrt{-1} &= (a + b\sqrt{-1}) \\ &\left\{\sqrt{A}dp + \left(\frac{B}{\sqrt{A}} + \sqrt{C - \frac{B^2}{A}}\sqrt{-1}\right) dq\right\}, \\ du - dv\sqrt{-1} &= (a - b\sqrt{-1}) \\ &\left\{\sqrt{A}dp + \left(\frac{B}{\sqrt{A}} - \sqrt{C - \frac{B^2}{A}}\sqrt{-1}\right) dq\right\}. \end{aligned}$$

If one can now determine a and b so that the right hand sides of these equations became perfect differentials, then one obtains u and v by integration as functions of p and q. Determining the integrating factor $a \pm b\sqrt{-1}$ is nothing else than integrating the differential equations

$$\begin{aligned} 0 &= \sqrt{A}dp + \left(\frac{B}{\sqrt{A}} + \sqrt{C - \frac{B^2}{A}}\sqrt{-1}\right) dq, \\ 0 &= \sqrt{A}dp + \left(\frac{B}{\sqrt{A}} - \sqrt{C - \frac{B^2}{A}}\sqrt{-1}\right) dq \end{aligned}$$

and this integration is the problem to be solved finally. If $B = 0$, then factors $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ must be found which make

$$\sqrt{A}dp + \sqrt{C}\sqrt{-1}dq \text{ and } \sqrt{A}dp - \sqrt{C}\sqrt{-1}dq$$

integrable, and then $\sqrt{a^2 + b^2}$ is the similarity-ratio.

If the surface is a tri-axial ellipsoid, on introducing the quantities $\lambda_1, \lambda_2, \lambda_3$ of which λ_1 is set constant, one obtains, as a consequence of equation (27.2) of Lecture 27, for the element of arc of any curve on the same, the expression

$$\begin{aligned} ds^2 &= Ad\lambda_2^2 + Cd\lambda_3^2 \\ &= \frac{1}{4}\frac{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}{(a_1 + \lambda_2)(a_2 + \lambda_2)(a_3 + \lambda_2)}d\lambda_2^2 \\ &\quad + \frac{1}{4}\frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{(a_1 + \lambda_3)(a_2 + \lambda_3)(a_3 + \lambda_3)}d\lambda_3^2, \end{aligned}$$

and one has to find the factors which make the expressions

$$\frac{1}{2}\sqrt{\frac{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}}\,d\lambda_2$$
$$+\frac{1}{2}\sqrt{\frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}}\,d\lambda_2\sqrt{-1}$$
$$\frac{1}{2}\sqrt{\frac{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}}\,d\lambda_2$$
$$-\frac{1}{2}\sqrt{\frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}}\,d\lambda_2\sqrt{-1},$$

integrable. These factors are $\frac{2}{\sqrt{\lambda_2-\lambda_3}}$ for both expression, therefore $a = \frac{2}{\sqrt{\lambda_2-\lambda_3}}$, $b = 0$, and the differential equations which give the correlation between u, v and p, q are

$$\begin{aligned} du + dv\sqrt{-1} &= \sqrt{\frac{\lambda_2-\lambda_1}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}}\,d\lambda_2 \\ &\quad +\sqrt{\frac{\lambda_1-\lambda_3}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}}\sqrt{-1}\,d\lambda_3, \\ du - dv\sqrt{-1} &= \sqrt{\frac{\lambda_2-\lambda_1}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}}d\lambda_2 \\ &\quad -\sqrt{\frac{\lambda_1-\lambda_3}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}}\sqrt{-1}d\lambda_3. \end{aligned}$$

Hence it follows that

$$\begin{aligned} u &= \int d\lambda_2\sqrt{\frac{\lambda_2-\lambda_1}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}}, \\ v &= \int d\lambda_3\sqrt{\frac{\lambda_1-\lambda_3}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}}, \end{aligned}$$

and the similarity ratio is

$$\sqrt{m} = \sqrt{a^2+b^2} = \frac{2}{\sqrt{\lambda_2-\lambda_3}},$$

and the quantity $\sqrt{m}$ so determined must then be multiplied by the length on the ellipsoid in order to give the corresponding length in the plane.

The formulae we have found for the shortest line on the tri-axial ellipsoid undergoes an important change for the case of an ellipsoid of revolution. There are two cases to be distinguished; the first is that of the oblate spheroid in which the two bigger axes are equal, where then $a_2 = a_3$; the second is that of the prolate spheroid in which the two smaller axes are equal, so $a_2 = a_1$. We shall consider only the first of these two cases, the second is to be handled entirely analogously. Here one proceeds in the well-known way, namely one supposes the difference between a_2 and a_3 to be infinitesimal and finally lets them coincide. So at first let

$$a_3 = a_2 + \omega,$$

where ω denotes an infinitesimal quantity. According to the general considerations, λ_3 lies between $-a_2$ and $-a_3$, so in the present case between $-a_2$ and $-(a_2 + \omega)$; one can therefore set

$$\lambda_3 = -(a_2 + \omega \sin^2 \varphi),$$

i.e.,

$$\begin{aligned} a_2 + \lambda_3 &= -\omega \sin^2 \varphi, \\ a_3 + \lambda_3 &= \omega - \omega \sin^2 \varphi = \omega \cos^2 \varphi \\ d\lambda_3 &= -\omega.2 \sin \varphi \cos \varphi d\varphi. \end{aligned}$$

Hence it follows that

$$\frac{d\lambda_3}{\sqrt{-(a_2 + \lambda_3)(a_3 + \lambda_3)}} = -2d\varphi.$$

We have to substitute this in the equation of the shortest line, i.e. in the equation

$$\begin{aligned} &\int d\lambda_2 \sqrt{\frac{\lambda_2 - \lambda_1}{(a_1 + \lambda_2)(a_2 + \lambda_2)(a_3 + \lambda_2)(\beta + \lambda_2)}} \\ &\quad + \int d\lambda_3 \sqrt{\frac{\lambda_3 - \lambda_1}{(a_1 + \lambda_3)(a_2 + \lambda_3)(a_3 + \lambda_3)(\beta + \lambda_3)}} \\ &\quad = \text{constant}. \end{aligned} \tag{28.5}$$

Of the factors under the radical in the first integral, $a_2+\lambda_2$ and $a_3+\lambda_2$ become equal, the integral is transformed into an elliptic integral. The second, however, goes over to

$$-2\sqrt{\frac{a_2+\lambda_1}{(a_1-a_2)(\beta-a_2)}}\int d\varphi=-2\sqrt{\frac{a_2+\lambda_1}{(a_1-a_2)(\beta-a_2)}}\varphi,$$

and equation (28.5) takes the form

$$\int\frac{d\lambda_2}{a_2+\lambda_2}\sqrt{\frac{\lambda_2-\lambda_1}{(a_1+\lambda_2)(\beta+\lambda_2)}}-2\sqrt{\frac{a_1+\lambda_1}{(a_1-a_2)(\beta-a_2)}}\varphi=\text{constant.}$$

The expressions for the coordinates of a point on the tri-axial ellipsoid were

$$\begin{aligned}
x_1 &= \sqrt{\frac{(a_1+\lambda_1)(a_1+\lambda_2)(a_1+\lambda_3)}{(a_1-a_2)(a_1-a_3)}},\\
x_2 &= \sqrt{\frac{(a_2+\lambda_1)(a_2+\lambda_2)(a_2+\lambda_3)}{(a_2-a_1)(a_2-a_3)}},\\
x_3 &= \sqrt{\frac{(a_3+\lambda_1)(a_3+\lambda_2)(a_3+\lambda_3)}{(a_3-a_1)(a_3-a_2)}}.
\end{aligned}$$

In the case of the oblate spheroid these become,

$$\begin{aligned}
x_1 &= \sqrt{\frac{a_1+\lambda_1}{a_1-a_2}}\sqrt{a_1+\lambda_2},\\
x_2 &= \sqrt{\frac{a_2+\lambda_1}{a_2-a_1}}\sqrt{a_2+\lambda_2}\sin\varphi,\\
x_3 &= \sqrt{\frac{a_1+\lambda_1}{a_2-a_1}}\sqrt{a_2+\lambda_2}\cos\varphi.
\end{aligned}$$

Since the general formulae for x_2 and x_3 are interchanged if a_2 and a_3 are interchanged, a superficial consideration would make us believe that when $a_2=a_3$, also x_2 must be equal to x_3; this, however, as we see, is by no means the case. The formulae holding then are the same which one obtains when one expresses through λ_2 and λ_3 the coordinates x_1 and $\sqrt{x_2^2+x_3^2}$ of the meridians of the spheroid after the substitution holding for the plane, and introduces the angle φ for the longitude on the spheroid.

One obtains, also for the map-projection, considered previously, special formulae for application to the spheroid. This special case of projection is called stereographic. Its characteristic properly is that homologous curves on the surface and on the plane intersect at the same angle. This is another expression for the similarly of the infinitesimal elements.

The partial differential equations whose integration gave us the equation of the shortest line on the ellipsoid was of the form

$$\frac{f(\lambda_2)\left(\frac{\partial W}{\partial \lambda_2}\right)^2 - f(\lambda_3)\left(\frac{\partial W}{\partial \lambda_3}\right)^2}{\lambda_2 - \lambda_3} = \text{constant},$$

where

$$f(\lambda) = \frac{(a_1 + \lambda)(a_2 + \lambda)(a_3 + \lambda)}{\lambda - \lambda_1}.$$

There is a constant on the right hand side of this equation, because we have assumed that the moving point is not subject to any force other than an initial push. One can now pose the question: of what nature should be the force acting on the point in order that the differential equation emerging can be integrated by the same method as has been used so far. For this purpose it must be possible to bring the force function, as one sees easily,to the form,

$$\frac{\chi(\lambda_2) + \psi(\lambda_3)}{\lambda_2 - \lambda_3},$$

because then the separation into two ordinary differential equations succeeds. But one cannot in general attach any mechanical significance to this analytical form; we shall consider only one case where one such is possible, namely the case in which the force function has the form $\lambda_2 + \lambda_3$, the expression that can be brought to the form $\frac{\lambda_2^2 - \lambda_3^2}{\lambda_2 - \lambda_3}$, and so belongs to the category under discussion. This case corresponds to the mechanical problem in which the point moving on the surface of an ellipsoid is subject to a force which attracts it towards the centre with a magnitude proportional to its distance from the same. In fact, in this case the force which acts on the point in the direction of the radius-vector extending from the centre is kr, consequently the force function is $\frac{1}{2}kr^2 = \frac{1}{2}k(x_1^2 + x_2^2 + x_3^2)$. Let us recall the general expression for $x_1^2, x_2^2, \ldots, x_n^2$ expressed in terms of $\lambda_1, \lambda_2, \ldots, \lambda_n$ given in lecture 26, equation (26.2); this expression is

$$\begin{aligned} x_m^2 &= \frac{(a_m+\lambda_1)(a_m+\lambda_2)\dots(a_m+\lambda_n)}{(a_m-a_1)(a_m-a_2)\dots(a_m-a_{m-1})(a_m-a_{m+1})\dots(a_m-a_n)} \\ &= \frac{a_m^n+(\lambda_1+\lambda_2+\cdots+\lambda_n)a_m^{n-1}+\cdots+\lambda_1\lambda_2\dots\lambda_n}{(a_m-a_1)(a_m-a_2)\dots(a_m-a_{m-1})(a_m-a_{m+1})\dots(a_m-a_n)}; \end{aligned}$$

According to the well- known theorem on partial fractions the remarkable formula

$$x_1^2+x_2^2+\cdots+x_n^2 = a_1+a_2+\cdots+a_n+\lambda_1+\lambda_2+\cdots+\lambda_n.$$

For $n=3$,

$$x_1^2+x_2^2+x_3^2 = a_1+a_2+a_3+\lambda_1+\lambda_2+\lambda_3.$$

In the case considered by us λ_1 is constant, so we obtain for the force-function

$$\frac{1}{2}k(x_1^2+x_2^2+x_3^2) = \frac{1}{2}k(\lambda_2+\lambda_3)+\text{ constant},$$

so that in this case the partial differential equation can be integrated with the same ease as earlier.

One can extend these considerations still further and take it that the force is not directed to the centre of the ellipsoid. In the case just considered, the force in the direction of the radius-vector was kr, therefore the components of the force in the direction of the coordinate axes are kx_1, kx_2, kx_3. If we now give the coordinates different coefficients m_1, m_2, m_3, then the integration would still be possible if we impose an equation of constraint on these constants. In fact, if the components in the directions of the coordinates axes are m_1x_1, m_2x_2, m_3x_3, then the force-function has the expression

$$\begin{aligned} \frac{1}{2}(m_1x_1^2+m_2x_2^2+m_3x_3^2) &= \frac{1}{2}m_1\frac{(a_1+\lambda_1)(a_1+\lambda_2)(a_1+\lambda_3)}{(a_1-a_2)(a_1-a_3)} \\ &\quad +\frac{1}{2}m_2\frac{(a_2+\lambda_1)(a_2+\lambda_2)(a_2+\lambda_3)}{(a_2-a_1)(a_2-a_3)} \\ &\quad \frac{1}{2}m_3\frac{(a_3+\lambda_1)(a_3+\lambda_2)(a_3+\lambda_3)}{(a_3-a_1)(a_3-a_2)}, \end{aligned}$$

which can be represented in the form

$$A + B(\lambda_2 + \lambda_3) + C\lambda_2\lambda_3,$$

and is therefore of the right form if C vanishes, i.e., if

$$\frac{m_1(a_1+\lambda_1)}{(a_1-a_2)(a_1-a_3)} + \frac{m_2(a_2+\lambda_1)}{(a_2-a_1)(a_2-a_3)} + \frac{m_3(a_3+\lambda_1)}{(a_3-a_1)(a_3-a_2)} = 0.$$

If this equation of constraint is satisfied by the values m_1, m_2, m_3, then the earlier integration methods can be used.

Lecture 29

Attraction of a point by two fixed centres

We now consider the motion of a point attracted by two fixed centres. Let us restrict ourselves first to the case in which the motion takes place in a plane, which is always the case when the direction of the initial velocity lies in the same plane as the line joining the fixed centres. Let this connecting line be the axis x_2, and the axis x_1 is at right angles to it at the mid point between the centres at a distance $2f$ from each other. If we now express x_1 and x_2 in terms of λ_1 and λ_2 and choose the constants a_1 and a_2 of the substitution in such a way that the two centres fall at the foci of the confocal system, then the differential equation to be integrated is

$$\frac{(a_1+\lambda_1)(a_2+\lambda_1)}{\lambda_1-\lambda_2}\left(\frac{\partial W}{\partial \lambda_1}\right)^2 + \frac{(a_1+\lambda_2)(a_2+\lambda_2)}{\lambda_2-\lambda_1}\left(\frac{\partial W}{\partial \lambda_2}\right)^2 = \frac{1}{2}U + \frac{1}{2}h, \tag{29.1}$$

when U likewise must be expressed in terms ofλ_1 and λ_2.

If the distances of the attracted point from the two centres are r and r_1, then one has

$$r^2 = (x_2+f)^2 + x_1^2, r_1^2 = (x_2-f)^2 + x_1^2,$$

or

$$r^2 = x_1^2 + x_2^2 + f^2 + 2fx_2, r_1^2 = x_1^2 + x_2^2 + f^2 - 2fx_2.$$

According to the fundamental property of the ellipse,

$$f^2 = (a_2+\lambda_1) - (a_1+\lambda_1) = a_2 - a_1;$$

the substitution

$$x_1 = \sqrt{\frac{(a_1+\lambda_1)(a_1+\lambda_2)}{a_1-a_2}}, x_2 = \sqrt{\frac{(a_2+\lambda_1)(a_2+\lambda_2)}{a_2-a_1}} \tag{29.2}$$

leads further, as we know, to the equation

$$x_1^2 + x_2^2 = a_1 + a_2 + \lambda_1 + \lambda_2;$$

therefore

$$\begin{aligned} r^2 &= x_1^2 + x_2^2 + f^2 + 2fx_2 \\ &= 2a_2 + \lambda_1 + \lambda_2 + 2\sqrt{(a_1+\lambda_1)(a_2+\lambda_2)} \\ &= \left\{\sqrt{a_2+\lambda_1} + \sqrt{a_2+\lambda_2}\right\}^2, \\ r_1^2 &= x_1^2 + x_2^2 + f^2 - 2fx_2 \\ &= 2a_2 + \lambda_1 + \lambda_2 - 2\sqrt{(a_2+\lambda_1)(a_2+\lambda_2)} \\ &= \left\{\sqrt{a_2+\lambda_1} - \sqrt{a_2+\lambda_2}\right\}^2. \end{aligned}$$

So

$$r = \sqrt{a_2+\lambda_1} + \sqrt{a_2+\lambda_2}, r_1 = \sqrt{a_2+\lambda_1} - \sqrt{a_2+\lambda_2}.$$

If one substitutes these expressions in the force-function

$$U = \frac{m}{r} + \frac{m_1}{r_1} = \frac{mr_1 + m_1 r}{rr_1},$$

it gives

$$U = \frac{(m+m_1)\sqrt{a_2+\lambda_1} - (m-m_1)\sqrt{a_2+\lambda_2}}{\lambda_1-\lambda_2}.$$

If one inserts this value of U in the partial differential equation (29.1) and multiplies by $\lambda_1 - \lambda_2$, then one obtains

$$\begin{aligned} &(a_1+\lambda_1)(a_2+\lambda_1)\left(\frac{\partial W}{\partial \lambda_1}\right)^2 - (a_1+\lambda_2)(a_2+\lambda_2)\left(\frac{\partial W}{\partial \lambda_2}\right)^2 \\ &= \frac{1}{2}h\lambda_1 + \frac{1}{2}(m+m_1)\sqrt{a_2+\lambda_1} \\ &\quad - \left\{\frac{1}{2}h\lambda_2 + \frac{1}{2}(m-m_1)\sqrt{a_2+\lambda_2}\right\}, \end{aligned} \tag{29.3}$$

and by introducing an arbitrary constant β, this equation can be resolved into two ordinary differential equations

$$\left(\frac{\partial W}{\partial \lambda_1}\right)^2 = \frac{\frac{1}{2}h\lambda_1 + \frac{1}{2}(m+m_1)\sqrt{a_2+\lambda_1} + \beta}{(a_1+\lambda_1)(a_2+\lambda_1)},$$
$$\left(\frac{\partial W}{\partial \lambda_2}\right)^2 = \frac{\frac{1}{2}h\lambda_2 + \frac{1}{2}(m-m_1)\sqrt{a_2+\lambda_2} + \beta}{(a_1+\lambda_2)(a_2+\lambda_2)},$$

so

$$W = \int d\lambda_1 \sqrt{\frac{\frac{1}{2}h\lambda_1 + \frac{1}{2}(m+m_1)\sqrt{a_2+\lambda_1} + \beta}{(a_1+\lambda_1)(a_2+\lambda_1)}} + \int d\lambda_2 \sqrt{\frac{\frac{1}{2}h\lambda_2 + \frac{1}{2}(m-m_1)\sqrt{a_2+\lambda_2} + \beta}{(a_1+\lambda_2)(a_2+\lambda_2)}}. \quad (29.4)$$

If we want to eliminate the irrational quantity under the square root sign, we set

$$\sqrt{a_2+\lambda_1} = p, \sqrt{a_2+\lambda_2} = q.$$

and obtain

$$W = \int dp \sqrt{\frac{2(hp^2 + (m+m_1)p + 2\beta - ha_2)}{p^2 - f^2}} + \int dq \sqrt{\frac{2(hq^2 + (m-m_1)q + 2\beta - ha_2)}{q^2 - f^2}}.$$

From (29.4), the integral equations are obtained in the form

$$\beta' = \frac{\partial W}{\partial \beta}, t - \tau = \frac{\partial W}{\partial h}.$$

In the first volume of the Turin Memoirs *Lagrange* has attempted to find the force which one can add to the attraction towards two fixed centres to carry out the integration, without *Euler*'s solution of this problem ceasing to hold. Although this investigation did not lead to any new results, it is still of great interest, and in fact not only for the state of science of that time, but also of the present. The force which, according to *Lagrange*, one can add is an attraction proportional to the distance, directed to a point lying at the middle of the two fixed centres.

This agrees fully with what we find, in retrospect, for the shortest line on the ellipsoid. Since through this force the term $\frac{1}{2}k(x_1^2 + x_2^2) = \frac{1}{2}k(\lambda_1 + \lambda_2 + a_1 + a_2)$ comes in addition in the force function, so does on the right hand side of the partial differential equation i.e. in $\frac{1}{2}U(\lambda_1 - \lambda_2)$, the expression $\psi(\lambda_1) - \psi(\lambda_2)$, if one sets $\psi(\lambda) = \frac{k}{4}\left\{\lambda^2 + (a_1 + a_2)\lambda\right\}$. At the same time $\psi(\lambda_1)$ and $\psi(\lambda_2)$ are respectively the terms by which the numerators under the square root sign in the integrals in λ_1 and λ_2 of the expression for W in (29.4) are to be increased.

Through the above formula we have completely solved the problem of attraction of a point by two fixed centres when the motion takes place in a plane; it remains now yet to reduce the general case to this. This happends through the principle of surface area.

In order to treat the problem in its greatest generality, we shall assume that a point is attracted not by two, but by an arbitrary number of fixed centres which lie in a straight line. Then, and even when a constant force acts parallel to this line, the principle of surface area holds with respect to the plane which is at right angles to this line. If now the initial velocity of the moving point lies on the same plane as the straight line, then the whole motion takes place on that plane, and one has no need to apply the surface area theorem. If, on the other hand, the initial velocity does not lie on the same plane as that line, then the point describes a curve of double curvature. Here it is of the great advantage to resolve the motion into two components. If one thinks of a plane through the point and the line which contains the centres, one can take that this rotates about the line and further, that the point itself moves on the rotating plane. In general, this resolution, which is possible under all circumstances, will not provide any simplification, but in the case considered it is possible, through the principle of surface area, to separate the motion of the point in the plane from the rotatory motion, so that one first seeks the motion of the point on the plane and after this has been found, obtains the angle of rotation of this plane (reckoned from a definite position of the same) through a simple quadrature. As we shall see, the differential equations of motion of this point in the rotating plane differs from the differential equations one obtains if the motion remains entirely on a plane. The only difference is that a term is added, which is proportional to $\frac{1}{r^3}$, where r represents the distance of the point from the line containing the centres. Let this line on which the fixed centres lie be the x-axis; if we further represent the differential equations of motion of the point in the usual way, without actually writing down the expression for the force, through the formula

$$\frac{d^2x}{dt^2} = X, \frac{d^2y}{dt^2} = Y, \frac{d^2z}{dt^2} = Z,$$

then the equation of constraint

$$yZ - zY = 0$$

holds. This equation which says that the forces Y, Z behave as the coordinates y, z, i.e. the directions of their components go through the x-axis is equivalent to the principle of surface area; for, if one sets $\frac{d^2y}{dt^2}$ and $\frac{d^2z}{dt^2}$ for Y and Z, then one obtains

$$y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2} = 0.$$

and hence, by integrating

$$y\frac{dz}{dt} - z\frac{dy}{dt} = \alpha.$$

In order to separate the motion of the point on the plane passing through the x-axis from the rotational motion of this plane, we must set

$$y = r\cos\varphi, z = r\sin\varphi,$$

so that x, r are the coordinates of the point in the rotating plane and φ the angle of rotation, reckoned from the $x - y$-plane.

Then one has

$$\begin{aligned} r &= \sqrt{y^2+z^2}, \\ \frac{dr}{dt} &= \frac{y\frac{dy}{dt} + z\frac{dz}{dt}}{\sqrt{y^2+z^2}}, \\ \frac{d^2r}{dt^2} &= \frac{y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2}}{\sqrt{y^2+z^2}} + \frac{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}{\sqrt{y^2+z^2}} - \frac{\left(y\frac{dy}{dt} + z\frac{dz}{dt}\right)^2}{(y^2+z^2)^{\frac{3}{2}}}. \end{aligned}$$

The last two terms combined into a single one give

$$\frac{(y^2+z^2)\left\{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} - \left(y\frac{dy}{dt} + z\frac{dz}{dt}\right)^2}{(y^2+z^2)^{\frac{3}{2}}},$$

or, according to a well-known formula

$$(y^2+z^2)\left\{\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}=\left(y\frac{dy}{dt}+z\frac{dz}{dt}\right)^2+\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)^2,$$

this is

$$\frac{\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)^2}{(y^2+z^2)^{\frac{3}{2}}},$$

or, finally, on using the surface area theorem,

$$\frac{\alpha^2}{r^3}.$$

One has then the equation

$$\frac{d^2r}{dt^2}=\frac{y\frac{d^2y}{dt^2}+z\frac{d^2z}{dt^2}}{\sqrt{y^2+z^2}}+\frac{\alpha^2}{r^3}=\frac{yY+zZ}{r}+\frac{\alpha^2}{r^3}.$$

Now let R be the force which acts on the point in the direction at right angles to the x-axis thus it is the resultant of the forces Y, Z. Then one has

$$\begin{aligned} Y &= \frac{y}{r}R, Z=\frac{z}{r}R, \\ yY+zZ &= \frac{y^2+z^2}{r}R=rR, \end{aligned}$$

and therefore

$$\frac{d^2r}{dt}=R+\frac{\alpha^2}{r^3}.$$

So we have the two equations of motion of the point in the rotating plane in the form

$$\frac{d^2x}{dt^2}=X, \frac{d^2r}{dt^2}=R+\frac{\alpha^2}{r^3}. \tag{29.5}$$

In the case we are considering the force is entirely independent of the rotation angle φ, so X and R depend only on x and r. One can therefore integrate both the equations and obtain, after determining x and r through the integral equations as functions of t, the rotation angle φ from the surface-area theorem. This changes, on introduction of r and φ, into

$$r^2\frac{d\varphi}{dt}=\alpha, \tag{29.6}$$

so that φ is determined through the formula

$$\varphi = \alpha \int \frac{dt}{r^2}.$$

We have accordingly reduced the original system of differential equations of the sixth order in x, y, z, t into a system of the fourth order in x, r, t; and since t does not enter in this explicitly, one can reduce it to one of the third order, by which one brings it to the form

$$dx : dr : dx' : dr' = x' : r' : X : \left(R + \frac{\alpha^2}{r^3}\right). \tag{29.7}$$

If one knows two integrals of this system, then one obtains a third through the principle of the last multiplier, and hence the time, through quadrature. If, for example, all the variables x, x' and r' are expressed through r, then

$$t = \int \frac{dr}{r'}.$$

With the help of this equation one can express φ, before r is expressed through t, as an integral over r:

$$\varphi = \alpha \int \frac{dr}{r^2 r'}.$$

It now turns out that for solving the problem completely, one has only to know two integrals of the system (29.7) of the third order. But then the theorem of *vis viva* which, as is well-known,always holds for attraction by fixed centres and for mutual attraction, gives one of these integrals. In fact, if in the equation

$$\frac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} = \int (Xdx + Ydy + Zdz),$$

we set,

$$\begin{aligned} Y &= \frac{y}{r}R, z = \frac{z}{r}R, \\ Ydy + Zdz &= R\frac{ydy + zdz}{r} = Rdr, \end{aligned}$$

further,

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2$$

or, since $\frac{d\varphi}{dt} = \frac{\alpha^\alpha}{r^2}$ according to the surface area theorem,

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + \frac{\alpha^2}{r^2},$$

then one obtains

$$\frac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2\right\} = \int (Xdx + Rdr) - \frac{1}{2}\frac{\alpha^2}{r^2},$$

which is an integral equation of the system (29.7). It now comes to finding a single integral. The problem of attraction of a point by an arbitrary number of fixed centres which lie on a line, and on which a constant force parallel to this line can act, is accordingly reduced to finding a single integral equation of a system of the second order.

If there are only two fixed centres, one finds this integral equation by the method explained at the beginning of this lecture. The coordinates x and r are the same as those which were denoted above by x_2 and x_1; but the force function is no longer the same. If the entire motion takes place in a plane, its value is $\int(Xdx + Rdr)$, but now, on the other hand, there comes in addition the term $-\frac{1}{2}\frac{\alpha^2}{r^2}$, or in the earlier rotation,

$$-\frac{1}{2}\frac{\alpha^2}{x_1^2}.$$

In order that after the addition of this term to the force function the partial differential equation (29.1) be integrable by the same method, one must be able to bring it to the form $\frac{1}{\lambda_1-\lambda_2}(\chi(\lambda_1) + \psi(\lambda_2))$, and this is actually the case. Since from (29.2)

$$x_1^2 = \frac{(a_1+\lambda_1)(a_1+\lambda_2)}{a_1 - a_2},$$

and so by decomposition into partial fractions,

$$\begin{aligned} -\frac{1}{2}\frac{\alpha^2}{x_1^2} &= \frac{1}{2}\alpha^2\frac{a_2-a_1}{(a_1+\lambda_1)(a_1+\lambda_2)} \\ &= -\frac{1}{2}\alpha^2\frac{a_2-a_1}{\lambda_1-\lambda_2}\left\{\frac{1}{a_1+\lambda_1} - \frac{1}{a_1+\lambda_2}\right\}. \end{aligned}$$

On the right hand side of equation (29.3), or what is the same, to $\frac{1}{2}U(\lambda_1 - \lambda_2)$ there appears the expression

$$\begin{aligned}&\frac{1}{4}\alpha^2(a_2-a_1)\left\{\frac{1}{a_1+\lambda_1}-\frac{1}{a_1+\lambda_2}\right\}\\&\quad=-\frac{1}{4}\alpha^2 f^2\frac{1}{a_1+\lambda_1}+\frac{1}{4}\alpha^2 f^2\frac{1}{a_1+\lambda_2},\end{aligned}$$

and consequently we obtain the partial differential equation

$$\begin{aligned}&(a_1+\lambda_1)(a_2+\lambda_1)\left(\frac{\partial W}{\partial\lambda_1}\right)^2-(a_1+\lambda_2)(a_2+\lambda_2)\left(\frac{\partial W}{\partial\lambda_2}\right)^2\\&=\ \frac{1}{2}h\lambda_1+\frac{1}{2}(m+m_1)\sqrt{a_2+\lambda_1}\\&\quad-\frac{1}{4}\alpha^2f^2\frac{1}{a_1+\lambda_1}-\left\{\frac{1}{2}h\lambda_2+\frac{1}{2}(m-m_1)\sqrt{a_2+\lambda_2}\right.\\&\quad\left.-\frac{1}{4}\alpha^2f^2\frac{1}{a_1+\lambda_2}\right\}.\end{aligned}$$

From this one gets

$$\begin{aligned}W\ &=\ \int d\lambda_1\sqrt{\frac{\frac{1}{2}h\lambda_1+\frac{1}{2}(m-m_1)\sqrt{a_2+\lambda_1}-\frac{1}{4}\alpha^2f^2\frac{1}{a_1+\lambda_1}+\beta}{(a_1+\lambda_1)(a_2+\lambda_1)}}\\&\quad+\int d\lambda_2,\sqrt{\frac{\frac{1}{2}h\lambda_2+\frac{1}{2}(m-m_1)\sqrt{a_2+\lambda_2}-\frac{1}{4}\alpha^2f^2\frac{1}{a_1+\lambda_2}+\beta}{(a_1+\lambda_2)(a_2+\lambda_2)}}\end{aligned} \tag{29.8}$$

and hence, by differentiation with respect to the constant β, the sought for second integral equation of the system (29.7):

$$\beta' = \frac{\partial W}{\partial \beta}. \tag{29.9}$$

This is the equation of the curve which the moving point describes in the rotating plane. There is now only the determination of the rotation angle φ to be carried out, for which, however, a difficulty remains. Namely, if one expresses the differential of φ, which by equation (29.6) is given in the present notation by

$$d\varphi = \alpha\frac{dt}{x_1^2},$$

in terms of λ_1 and λ_2, then one does not obtain a complete differential. For, the differential of t is given, when one substitutes its value (29.8) for W in the equation serving for the determination of the time

$$t - \tau = \frac{\partial W}{\partial h},$$

in the form

$$dt = F_1(\lambda_1)d\lambda_1 + F_2(\lambda_2)d\lambda_2,$$

and this expression multiplied by

$$\frac{\alpha}{x_1^2} = \frac{\alpha(a_1 - a_2)}{(a_1 + \lambda_1)(a_1 + \lambda_2)}$$

does not directly give a total differential, but it can be trasformed into one with the help of equation (29.9) connecting the variables λ_1 and λ_2.

One can avoid this difficulty if one reduces the problem of attraction by two fixed centres also in three dimensions wholly to a partial differential equation, without regard to any special considerations. The general partial differential equation for a free motion for which the theorem of *vis viva* holds is

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2 = 2U + 2h.$$

If we introduce polar coordinates for y and z and set

$$y = r\cos\varphi, z = r\sin\varphi,$$

then we obtain

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \varphi}\right)^2 = 2U + 2h.$$

Since the variable φ does not occur in U one can, in accordance with the general method used often, set

$$W = W_1 + \alpha\varphi$$

where W_1 is a function only of x, r and not φ. Therefore

$$\frac{\partial W}{\partial x} = \frac{\partial W_1}{\partial x}, \frac{\partial W}{\partial r} = \frac{\partial W_1}{\partial r}, \frac{\partial W}{\partial \varphi} = \alpha,$$

and the partial differential equation for W changes into

$$\left(\frac{\partial W_1}{\partial x}\right)^2 + \left(\frac{\partial W_1}{\partial r}\right)^2 = 2U - \frac{\alpha^2}{r^2} + 2h.. \tag{29.10}$$

This differential equation agrees exactly with what we have obtained above through the reduction of the motion in three dimensions to the motion on the rotating plane. The earlier consideration also showed that if the term $\frac{\alpha^2}{2r^2}$ is removed from U, the constant α introduced now is exactly the same as the one so designated earlier. The expression (29.8) given above for W therefore satisfies the differential equation (29.10) for W_1 and one finds W from the same equation, through the relation

$$W = W_1 + \alpha\varphi.$$

Hence follow the two integral equations

$$\beta' = \frac{\partial W}{\partial \beta} = \frac{\partial W_1}{\partial \beta}, \alpha' = \frac{\partial W}{\partial \alpha} = \frac{\partial W_1}{\partial \alpha} + \varphi,$$

of which the first is the one we already found above, while the second leads to the value of φ through the equation $\alpha' - \varphi = \frac{\partial W_1}{\partial \alpha}$. Here the expression (29.8) for W is to be substituted in place of W_1. The two integral equations together define the curve of double curvature on which the point moves, and therefore

$$\beta' = \frac{\partial W}{\partial \beta} \text{ and } \alpha' - \varphi = \frac{\partial W}{\partial \alpha},$$

where

$$\begin{aligned} W \;=\; & \int d\lambda_1 \sqrt{\frac{\frac{1}{2}h\lambda_1 + \frac{1}{2}(m+m_1)\sqrt{a_2+\lambda_1} - \frac{1}{4}\alpha f^2 \frac{1}{\alpha_1+\lambda_1} + \beta}{(a_1+\lambda_1)(a_2+\lambda_1)}} \\ & + \int d\lambda_2 \sqrt{\frac{\frac{1}{2}h\lambda_1 + \frac{1}{2}(m-m_1)\sqrt{a_2-\lambda_2} - \frac{1}{4}\alpha f^2 \frac{1}{\alpha_1+\lambda_2} + \beta}{(a_1+\lambda_2)(a_2+\lambda_2)}} \end{aligned}$$

and time is expressed through the equation

$$t - \tau = \frac{\partial W}{\partial h}.$$

On carrying out the differentiation, one obtains the ready formulae

$$
\begin{aligned}
\beta' &= \int \frac{\frac{1}{2}d\lambda_1}{\sqrt{a_2+\lambda_1}\sqrt{\left[\frac{1}{2}h\lambda_1+\frac{1}{2}(m+m_1)\sqrt{a_2+\lambda_1}+\beta\right](a_1+\lambda_1)-\frac{1}{4}\alpha^2 f^2}} + \\
&+ \int \frac{\frac{1}{2}d\lambda_2}{\sqrt{a_2+\lambda_2}\sqrt{\left[\frac{1}{2}h\lambda_2+\frac{1}{2}(m-m_1)\sqrt{a_2+\lambda_2}+\beta\right](a_1+\lambda_2)-\frac{1}{4}\alpha^2 f^2}}, \\
\varphi - \alpha' &= \int \frac{\frac{1}{4}\alpha f^2 d\lambda_1}{(a_1+\lambda_1)\sqrt{a_2+\lambda_1}\sqrt{\left[\frac{1}{2}h\lambda_1+\frac{1}{2}(m+m_1)\sqrt{a_2+\lambda_1}+\beta\right](a_1+\lambda_1)-\frac{1}{4}\alpha^2 f^2}} + \\
&+ \int \frac{\frac{1}{4}\alpha f^2 d\lambda_2}{(a_1+\lambda_2)\sqrt{a_2+\lambda_2}\sqrt{\left[\frac{1}{2}h\lambda_2+\frac{1}{2}(m-m_1)\sqrt{a_2+\lambda_2}+\beta\right](a_1+\lambda_1)-\frac{1}{4}\alpha^2 f^2}}, \\
t - \tau &= \int \frac{\frac{1}{4}\lambda_1 d\lambda_1}{\sqrt{a_2+\lambda_1}\sqrt{\left[\frac{1}{2}h\lambda_1+\frac{1}{2}(m+m_1)\sqrt{a_2+\lambda_1}+\beta\right](a_1+\lambda_1)-\frac{1}{4}\alpha^2 f^2}} + \\
&+ \int \frac{\frac{1}{4}\lambda_2 d\lambda_2}{\sqrt{a_2+\lambda_2}\sqrt{\left[\frac{1}{2}h\lambda_2+\frac{1}{2}(m-m_1)\sqrt{a_2+\lambda_2}+\beta\right](a_1+\lambda_2)-\frac{1}{4}\alpha^2 f^2}}.
\end{aligned}
$$

Here also one can, as above, remove the irrational quantities under the square root sign if one introduces in place of λ_1, λ_2 the quantities

$$\sqrt{a_2+\lambda_1} = p, \sqrt{a_1+\lambda_2} = q$$

as variables.

Lecture 30

Abel's Theorem

Finally, in order to exhibit a specially remarkable example of the importance of the substitution introduced in Lecture 26, which has already given us the solutions of a series of problems of mechanics, we shall apply it to Abel's theorem. This theorem likewise concerns a certain system of ordinary differential equations and gives two different systems of integral equations of the same, one of which is expressed through transcendental functions and other though purely algebraic ones. These systems of integral equations, so different in form, are nevertheless completely identical.

According to our method, the system of ordinary differential equations is reduced to a first order partial differential equation. A complete solution of this is sought, and the differential coefficients of the same with respect to arbitrary constants, set equal to new constants, lead to the system of integral equations. The solution of the partial differential equation, however, can take forms very different from one another. Looking for these different forms, one obtains the forms of different systems of integral equations which, however, must agree with one another in their meaning. This is the way in which we shall prove *Abel*'s theorem. We proceed from the partial differential equation

$$\left(\frac{\partial V}{\partial x_1}\right)^2 + \left(\frac{\partial V}{\partial x_2}\right)^2 + \cdots + \left(\frac{\partial V}{\partial x_n}\right)^2 = 2h, \tag{30.1}$$

which, for $n = 3$, corresponds to the simplest mechanical problem, of rectilinear uniform motion in three dimensions. This replaces the ordinary differential equations

$$\frac{d^2x_1}{dt^2} = 0, \frac{d^2x_2}{dt^2} = 0, \ldots, \frac{d^2x_n}{dt^2} = 0.$$

By the use of substitution introduced in Lecture 26, one obtains *Abel's* theorem, and indeed in a much more explicit form, than was given by *Abel.*

Since the variables $x_1, x_2, \ldots, x_n$ themselves do not occur in the equation (30.1), one obtains a complete solution V if one sets

$$V = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n. \tag{30.2}$$

Because the constants, $\alpha_1, \alpha_2, \ldots, \alpha_n$ have to satisfy only the condition

$$\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_{n-1}^2 + \alpha_n^2 = 2h,$$

so that

$$\alpha_n = \sqrt{2h - \alpha_1^2 - \alpha_2^2 - \cdots - \alpha_{n-1}^2},$$

and V contains therefore $n-1$ constants, except for one constant which one can still add, it is a complete solution. As integral equations one obtains

$$\frac{\partial V}{\partial \alpha_1} = \alpha_1', \frac{\partial V}{\partial \alpha_2} = \alpha_2', \ldots, \frac{\partial V}{\partial \alpha_{n-1}} = \alpha_{n-1}', \frac{\partial V}{\partial h} = t - \tau,$$

or,

$$\begin{aligned} x_1 - \frac{\alpha_1}{\alpha_n} x_n &= \alpha_1' \\ x_2 - \frac{\alpha_2}{\alpha_n} x_n &= \alpha_2' \\ &\cdots \\ x_{n-1} - \frac{\alpha_{n-1}}{\alpha_n} x_n &= \alpha_{n-1}' \\ \frac{1}{\alpha_n} x_n &= t - \tau, \end{aligned}$$

and finally, if one substitutes the last equation in the others,

$$\begin{aligned} x_1 &= \alpha_1(t-\tau) + \alpha_1', \\ x_2 &= \alpha_2(t-\tau) + \alpha_2', \\ &\ldots, \\ x_{n-1} &= \alpha_{n-1}(t-\tau) + \alpha_{n-1}', \\ x_n &= \alpha_n(t-\tau) \end{aligned} \tag{30.3}$$

which in fact are the equations of rectilinear motion for $n = 3$.

If we now introduce the variables λ in place of the variables x in equation (30.1), we then obtain, according to formula (26.12) of Lecture 26,

$$\sum_{i=1}^{i=n} \frac{(a_1+\lambda_i)(a_2+\lambda_i)\ldots(a_n+\lambda_i)}{(\lambda_i-\lambda_1)(\lambda_i-\lambda_2)\ldots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\ldots(\lambda_i-\lambda_n)} \cdot \left(\frac{\partial V}{\partial \lambda_i}\right)^2 = \frac{1}{2}h. \tag{30.4}$$

One does not know immediately in what way the variables in this equation can be separated from one another. But it is only necessary to remind oneself of the lemma from the theory of partial fractions given in Lecture 26 (p.226), from which the following formula:

$$\frac{1}{2}h = \sum_{i=1}^{i=n} \frac{c+c_1\lambda_i+c_2\lambda_i^2+\ldots+c_{n-2}\lambda_i^{n-2}+\frac{1}{2}h\lambda_i^{n-1}}{(\lambda_i-\lambda_1)(\lambda_i-\lambda_2)\ldots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\ldots(\lambda_i-\lambda_n)} \tag{30.5}$$

where $c, c_1, \cdots, c_{n-2}$ are arbitrary constants,is obtained, and is substituted for $\frac{1}{2}h$ in (30.4). If the following equation derived from the preceding one,

$$\sum_{i=1}^{i=n} \frac{(a_1+\lambda_i)(a_2+\lambda_i)\ldots(a_n+\lambda_i)}{(\lambda_i-\lambda_1)(\lambda_i-\lambda_2)\ldots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\ldots(\lambda_i-\lambda_n)} \left(\frac{\partial V}{\partial \lambda_i}\right)^2 = \sum_{i=1}^{i=n} \frac{c+c_1\lambda_i+c_2\lambda_i^2+\ldots+c_{n-2}\lambda_i^{n-2}+\frac{1}{2}h\lambda_i^{n-1}}{(\lambda_i-\lambda_1)(\lambda_i-\lambda_2)\ldots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\ldots(\lambda_i-\lambda_n)} \tag{30.6}$$

is satisfied and one equates on both sides the corresponding terms and in this way breaks up the partial differential equation (30.6) into n ordinary differential equations

$$(a_1+\lambda_i)(a_2+\lambda_i)\ldots(a_n+\lambda_i)\left(\frac{\partial V}{\partial \lambda_i}\right)^2 = c+c_i\lambda_i+c_2\lambda_i^2+\cdots+c_{n-2}\lambda_i^{n-2}+\frac{1}{2}h\lambda_i^{n-1},$$

for $i = 1, 2, \ldots, n$; then one obtains for V the complete solution

$$V = \sum_{i=1}^{i=n} \int d\lambda_i \sqrt{\frac{c + c_1\lambda_i + c_2\lambda_i^2 + \ldots + c_{n-2}\lambda_i^{n-2} + \frac{1}{2}h\lambda_i^{n-1}}{(a_i + \lambda_i)(a_2 + \lambda_i)\ldots(a_n + \lambda_i)}}, \tag{30.7}$$

and hence follow the integral equations

$$\frac{\partial V}{\partial c} = c', \frac{\partial V}{\partial c_1} = c_1', \ldots, \frac{\partial V}{\partial c_{n-2}} = c_{n-2}', \frac{\partial V}{\partial h} = t - \tau,$$

which, on introducing the notation

$$\begin{aligned} f(\lambda) &= (c + c_1\lambda + c_2\lambda^2 + \cdots + c_{n-2'}\lambda^{n-2} + \frac{1}{2}h\lambda^{n-1}) \\ &\quad (a_1 + \lambda)(a_2 + \lambda)\ldots(a_n + \lambda), \end{aligned}$$

take the form

$$\begin{aligned} 2c' &= \sum \int \frac{d\lambda_i}{\sqrt{f(\lambda_i)}}, \\ 2c_1' &= \sum \int \frac{\lambda_i d\lambda_i}{\sqrt{f(\lambda_i)}}, \\ &\cdots \\ 2c_{n-2}' &= \sum \int \frac{\lambda_i^{n-2} d\lambda_i}{\sqrt{f(\lambda_i)}}, \\ 4(t - \tau) &= \sum \int \frac{\lambda_i^{n-1} d\lambda_i}{\sqrt{f(\lambda_i)}}. \end{aligned} \tag{30.8}$$

These are the transcendental integral equations of the system of ordinary differential equations

$$\begin{aligned} \sum \frac{d\lambda_i}{\sqrt{f(\lambda_i)}} &= 0, \sum \frac{\lambda_i d\lambda_i}{\sqrt{f(\lambda_i)}} = 0, \ldots, \\ \sum \frac{\lambda_i^{n-2} d\lambda_i}{\sqrt{f(\lambda_i)}} &= 0, \sum \frac{\lambda_i^{n-1} d\lambda_i}{\sqrt{f(\lambda_i)}} = 4dt., \end{aligned} \tag{30.9}$$

while the algebraic integral equations of the system are given in (30.3).

Abel's theorem consists in this algebraic integration of the differential equations (30.9), and indeed here it appears in a form which has the advantage over the form given originally by *Abel*, essentially to make investigations easier, otherwise associated with difficult investigations of the reality of the variables and the limits within which one has to take them. The above proof of *Abel*'s theorem is therefore something essentially new. Though *Richelot* has later[1] derived these results from *Abel*'s theorem itself, it is the method given here which has led to it first and in a natural way.

Since the constants $c, c_1, \ldots, c_{n-2}$ are entirely arbitrary, one must so determine them that the expressions standing under the radical sign, $f(\lambda_i)$, are positive, and with this all integrals are real.

Abel's theorem does not follow quite completely from whatever has been done so far. Because, for the function $f(\lambda)$ is of the $(2n-1)$th, that is of odd order and it is therefore necessary to consider specially the other case where $f(\lambda)$ is of order $2n$ and that appears here as more general. One obtains that by adding other terms to the constant $2h$ on the right hand side of the partial differential equation (30.1). The integration method applied remains valid if one adds to h the sum of the squares $x_1^2 + x_2^2 + \cdots + x_n^2$ multiplied by a constant k. In terms of the variables λ this expression takes the form

$$k(x_1^2 + x_2^2 + \cdots + x_n^2) = k(a_1 + a_2 + \cdots + a_n + \lambda_1 + \lambda_2 + \cdots + \lambda_n).$$

If we introduce for h a new constant

$$h' = h + k(a_1 + a_2 + \cdots + a_n).$$

we have on the right hand side of (30.4) the expression $\frac{1}{2}h' + \frac{1}{2}k(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$ in place of $\frac{1}{2}h$. If by use of the lemma mentioned above we transform the same in a way analogous to equation (30.5), then we find that the right sides of equation (30.5) and (30.6) do not change any further than that under the summation sign in the numerator, the term

$$\frac{1}{2}k\lambda_i^n$$

occurs, and h changes into h'. In the transcendental equation (30.8) of *Abel*'s theorem now appears correspondingly, in place of the earlier

[1] *Crelles* Journal XXIII, p.354

function $f(\lambda)$ of order $2n-1$, the function of order $2n$;

$$f(\lambda) = \left\{c + c_i\lambda + c_2\lambda^2 + \cdots + c_{n-2}\lambda^{n-2} + \frac{1}{2}h'\lambda^{n-1} + \frac{1}{2}h\lambda^n\right\} (a_1+\lambda)(a_2+\lambda)\dots(a_n+\lambda). \quad (30.10)$$

The algebraic integral equations are a little more complicated in this case. The partial differential equation expressed in terms of x_1, $x_2, \dots, x_n$ becomes

$$\left(\frac{\partial V}{\partial x_1}\right)^2 + \left(\frac{\partial V}{\partial x_2}\right)^2 + \cdots + \left(\frac{\partial V}{\partial x_n}\right)^2 = 2h + 2k(x_1^2 + x_2^2 + \cdots + x_n^2), \quad (30.11)$$

and therefore can be separated as follows,

$$\left(\frac{\partial V}{\partial x_1}\right)^2 = 2kx_1^2 + \beta_1, \left(\frac{\partial V}{\partial x_2}\right)^2 = 2kx_2^2 + \beta_2 \dots, \left(\frac{\partial V}{\partial x_n}\right)^2 = 2kx_n^2 + \beta_n,$$

where

$$\beta_1 + \beta_2 + \cdots + \beta_n = 2h.$$

From this one finds

$$V = \int \sqrt{2kx_1^2 + \beta_1}dx_1 + \int \sqrt{2kx_2^2 + \beta_2}dx_2 + \cdots + \int \sqrt{2kx_n^2 + \beta_n}dx_n.$$

If one now thinks of β_n as expressed with the help of the above relation through h and the remaining β's, and denotes the differential coefficients constructed on this hypothesis by brackets, then to the ordinary differential equations corresponding to the partial differential equation (30.11) belong the integrals

$$\left(\frac{\partial V}{\partial \beta_1}\right) = \beta_1', \left(\frac{\partial V}{\partial \beta_2}\right) = \beta_2', \dots, \left(\frac{\partial V}{\partial \beta_{n-1}}\right) = \beta_{n-1}', \left(\frac{\partial V}{\partial h}\right) = t - \tau.$$

If, on the contrary, one denotes without brackets the differential coefficients of V formed without considering the relation between the quantities $\beta_1, \beta_2, \dots, \beta_n$, then

$$\left(\frac{\partial V}{\partial \beta_1}\right) = \frac{\partial V}{\partial \beta_1} - \frac{\partial V}{\partial \beta_n}, \left(\frac{\partial V}{\partial \beta_2}\right) = \frac{\partial V}{\partial \beta_2} - \frac{\partial V}{\partial \beta_n}, \dots, \left(\frac{\partial V}{\partial h}\right) = 2\left(\frac{\partial V}{\partial \beta_n}\right).$$

One can therefore give a symmetrical form to the integral equations by introducing the notation $\tau_1, \tau_2, \dots \tau_n$ for the constants $2\beta_1' - \tau$, $2\beta_2' - \tau, \dots, -\tau$,

$$2\frac{\partial V}{\partial \beta_1} = \int \frac{dx_1}{\sqrt{2kx_1^2 + \beta_1}} = t + \tau_1,$$
$$2\frac{\partial V}{\partial \beta_2} = \int \frac{dx_2}{\sqrt{2kx_2^2 + \beta_2}} = t + \tau_2,$$
$$2\frac{\partial V}{\partial \beta_n} = \int \frac{dx_n}{\sqrt{2kx_n^2 + \beta_n}} = t + \tau_n.$$

These equations, to be sure, do not immediately express an algebraic relation between the variables x. But the relation appears immediately if one determines the values of all integrals leading to circular arcs or all those leading to logarithms, and notices that the values of the variables x are expressed, either in terms of sines and cosines, or in terms exponentials, whose argument multiplied by t gives one and the the same constant. Therefore one obtains algebraic relations if one eliminates t between the above equations. The values of the variables x can be given in the form

$$x_1 = \sqrt{\frac{-\beta_1}{2k}} \sin\left[\sqrt{-2k}(t + \tau_1)\right],$$
$$x_2 = \sqrt{\frac{-\beta_2}{2k}} \sin\left[\sqrt{-2k}(t + \tau_2)\right],$$
$$x_n = \sqrt{\frac{-\beta_n}{2k}} \sin\left[\sqrt{-2k}(t + \tau_n)\right].$$

The relations resulting from the elimination of t between these equations can be so represented that only one is of second degree and the rest linear in $x_1, x_2, \dots x_n,$.

The system of ordinary differential equations which corresponds to the partial differential equation (30.11) is

$$\frac{d^2x_1}{dt^2} = 2kx_1, \frac{d^2x_2}{dt^2} = 2kx_2, \dots, \frac{d^2x_n}{dt^2} = 2kx_n. \tag{30.12}$$

One sees from the preciding that if one starts from the differential equation (30.9) in $\lambda_1, \lambda_2, \dots, \lambda_n$, under the assumption that $f(\lambda)$ is an integral function (30.10) of degree $2n$, and carries out the substitution of the variables $x_1, x_2, \dots, x_n$ for $\lambda_1, \lambda_2, \dots, \lambda_n$, one must arrive at the simple differential equations for $x_1, x_2, \dots, x_n$ (30.12). I have given this method of investigation in my article on *Abel*'s theorem in Vol.24 of *Crelles* Journal, without going to the source uncovered here.

In the first volume of the Turin Memoirs, in the article on the attraction by two fixed centres, *Lagrange* has proved in an analogous way the fundamental theorem of elliptic transcendents, which is a special case ($n = 2$) of this investigation.

Lecture 31

General investigations of the partial differential equations of the first order. Different forms of the integrability conditions

We shall now concern ourselves with the general investigations of first order partial differential equations. We assume that the function to be found does not itself appear in the differential equation. This assumption is not an essential restriction since the general case can always be reduced to this. In fact, if the given differential equation contains the function V to be found has the form

$$0 = \Phi(V, \frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \dots \frac{\partial V}{\partial q_n}, q_1, q_2, \dots, q_n),$$

then one introduces a new independent variable q and a new dependent variables W through the equation

$$W = qV;$$

then

$$\frac{\partial W}{\partial q} = V, \frac{\partial W}{\partial q_1} = q\frac{\partial V}{\partial q_1}, \dots, \frac{\partial W}{\partial q_n} = q\frac{\partial V}{\partial q_n},$$

so

$$V = \frac{\partial W}{\partial q}, \frac{\partial V}{\partial q_1} = \frac{1}{q}\frac{\partial W}{\partial q_1}, \dots, \frac{\partial V}{\partial q_n} = \frac{1}{q}\frac{\partial W}{\partial q_n}.$$

Therefore the given differential equation goes over into the following:

$$0 = \Phi\left(\frac{\partial W}{\partial q}, \frac{1}{q}\frac{\partial W}{\partial q_1}, \dots, \frac{1}{q}\frac{\partial W}{\partial q_n}, q_1, q_1, \dots q_n\right),$$

which indeed contains one more independent variable q, in which, however, W itself does not occur, but only its differential coefficients with respect to $q_1, q_2, \ldots, q_n, q$. We can therefore confine ourselves, without limiting the generality,to the case where

$$\varphi\left(\frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \ldots, \frac{\partial V}{\partial q_n}, q_1, q_2, \ldots, q_n\right) = 0$$

is the given differential equation and V itself does not occur in the equation. If for brevity we set

$$\frac{\partial V}{\partial q_i} = p_i,$$

we have accordingly the equation

$$\varphi(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots q_n) = 0. \tag{31.1}$$

If, for determining V, we wish to apply the same method that we used, following *Lagrange*, for the case $n = 2$, in Lecture 22, then we must find the quantities $p_1, p_2, \ldots, p_n$ as functions of $q_1, q_2, \ldots, q_n$ so that

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n \tag{31.2}$$

becomes a complete differential. But here we run into a peculiar difficulty. Since the equation (31.1) is already a relation between the quantities p and q, so we need yet another $n-1$ relations in order to express all the quantities $p_1, p_2, \ldots, p_n$ through $q_1, q_2, \ldots, q_n$. We have then $n-1$ functions of the variables $q_1, q_2, \ldots, q_n$ at our disposal and must so determine these that the expression (31.2) is a total differential. In order to satisfy this requirement, the $\frac{n(n-1)}{2}$ equations of constraint of the form

$$\frac{\partial p_i}{\partial q_k} = \frac{\partial p_k}{\partial q_i},$$

or, introducing the abbreviated notation

$$(i, k) = \frac{\partial p_i}{\partial q_k} - \frac{\partial p_k}{\partial q_i},$$

the $\frac{n(n-1)}{2}$ equations of constraint

$$(i, k) = 0$$

must be fulfilled, while we have only $n-1$ functions at our disposal. For $n=2$, indeed these numbers are equal, namely 1. In all other cases the first number is greater than the second.

This difficulty has prevented analysts up to now from extendng *Lagarange*'s method to a larger number of variables. We shall not allow this to deter us, since we know a priori that the problem, though it appears overdetermined, admits of a solution. We shall investigate how one can satisfy $\frac{n(n-1)}{2}$ equations of constraint with $n-1$ functions.

There is one circumstance to be noticed right away, which is of advantage in this investigation. Because of it the $\frac{n(n-1)}{2}$ equations of constraint will be brought into relation with one another. Namely, if i, i', i'' are three arbitrary indices, then one has the identity

$$\frac{\partial(i',i'')}{\partial q_i}+\frac{\partial(i'',i)}{\partial q_i'}+\frac{\partial(i,i')}{\partial q_i''}=0.$$

It does not indeed follow yet that if $(i'',i^i)=0$ and $(i,i')=0$, also (i'',i') vanishes, but only that the last expression is independent of q_i, so that if for any value of q_i it vanishes, it is always equal to zero.

In order to treat the above question exhaustively, we must first transform the constraint equations. In the present form of these equations, $\frac{\partial p_i}{\partial q_k}=\frac{\partial p_k}{\partial q_i}$, p's are looked upon only as functions of q i.e. we assume that the n relations between the quantitites p and q, of which one is given by the equation (31.1), while we have at our disposal the remaining $n-1$, have been solved for the n quantities $p_1, p_2, \ldots, p_n$. For the investigtion in question this form is too explicit. We shall make another hypothesis on the expression of $p_1, p_2, \ldots, p_n$; and assume that we have

p_n expressed as a function of $q_1, q_2, \ldots q_n$

p_{n-1} expressed as a function of $p_n, q_1, q_2, \ldots q_n$,

p_{n-2} expressed as a function of $p_{n-1}, p_n, q_1, q_2, \ldots, q_n$,

...

p_i expressed as a function of $p_{i+1}, \ldots, p_{n-1}, p_n, q_1, q_2, \ldots, q_n$,

...

p_1 expressed as afunction of $p_2, p_3, \ldots, p_{n-1}, p_n, q_1, q_2, \ldots, q_n$.

We shall write without brackets the differential coefficients of p_i with respect to $p_{i+1}, p_{i+2}, \ldots, p_n$, $q_1, q_2, \ldots q_n$ obtained on this hypothesis, while we shall enclose in brackets the differential coefficients formed under the original hypothesis, according to which all p are functions only of $q_1, q_2, \ldots, q_n$. This change in the way of representation requires that we should convert the differential coefficients appearing in the $\frac{n(n-1)}{2}$ constraint equations and now bracketed into others, which will be done now.

We can arrange the $\frac{n(n-1)}{2}$ equations of constraint in the following way:

$$\begin{aligned}
\left(\frac{\partial p_1}{\partial q_2}\right) &= \left(\frac{\partial p_2}{\partial q_1}\right), \left(\frac{\partial p_1}{\partial q_3}\right) = \left(\frac{\partial p_3}{\partial q_1}\right), \ldots, \left(\frac{\partial p_1}{\partial q_{m+1}}\right) \\
&= \left(\frac{\partial p_{m+1}}{\partial q_1}\right), \ldots, \left(\frac{\partial p_1}{\partial q_n}\right) = \left(\frac{\partial p_n}{\partial q_1}\right), \\
\left(\frac{\partial p_2}{\partial q_3}\right) &= \left(\frac{\partial p_3}{\partial q_2}\right), \ldots, \left(\frac{\partial p_2}{\partial q_{m+1}}\right) = \left(\frac{\partial p_{m+1}}{\partial q_2}\right), \ldots, \left(\frac{\partial p_2}{\partial q_n}\right) \\
&= \left(\frac{\partial p_n}{\partial q_2}\right) \\
&\ldots\ldots\ldots \\
\left(\frac{\partial p_m}{\partial q_{m+1}}\right) &= \left(\frac{\partial p_{m+1}}{\partial q_m}\right), \ldots, \left(\frac{\partial p_m}{\partial q_n}\right) = \left(\frac{\partial p_n}{\partial q_m}\right), \ldots, \\
&\quad \left(\frac{\partial p_{n-1}}{\partial q_n}\right) = \left(\frac{\partial p_n}{\partial q_{n-1}}\right) \qquad (31.3)
\end{aligned}$$

Any one these equations, say $\left(\frac{\partial p_i}{\partial q_k}\right) = \left(\frac{\partial p_k}{\partial q_i}\right)$, after the term on the right is brought to the left, is denoted by $(i, k) = 0$, so that the equations in the mth row, for example

$$\begin{aligned}
\left(\frac{\partial p_m}{\partial q_{m+1}}\right) = \left(\frac{\partial p_{m+1}}{\partial q_m}\right), \left(\frac{\partial p_m}{\partial q_{m+2}}\right) &= \left(\frac{\partial p_{m+2}}{\partial q_m}\right), \ldots, \left(\frac{\partial p_m}{\partial q_n}\right) \\
&= \left(\frac{\partial p_n}{\partial q_m}\right)
\end{aligned}$$

are represented in abbreviated form by

$$(m, m+1) = 0, (m, m+2) = 0, \ldots, (m, n) = 0.$$

Now if i is any one of the indices $m+1, m+2, \ldots, n$, then one has

$$\begin{aligned}
\left(\frac{\partial p_m}{\partial q_i}\right) &= \frac{\partial p_m}{\partial p_{m+1}} \left(\frac{\partial p_{m+1}}{\partial q_i}\right) + \frac{\partial p_m}{\partial p_{m+2}} \left(\frac{\partial p_{m+2}}{\partial q_i}\right) \\
&\quad + \cdots + \frac{\partial p_m}{\partial p_n} \left(\frac{\partial p_n}{\partial q_i}\right) + \frac{\partial p_m}{\partial q_i},
\end{aligned}$$

or, if we replace $\left(\frac{\partial p_{m+1}}{\partial q_i}\right), \left(\frac{\partial p_{m+2}}{\partial q_i}\right), \ldots, \left(\frac{\partial p_n}{\partial q_i}\right)$ by the differential coefficients of p_i with the help of the equations of condition (31.3),

$$\begin{aligned}\left(\frac{\partial p_m}{\partial q_i}\right) &= \frac{\partial p_m}{\partial p_{m+1}}\left(\frac{\partial p_i}{\partial q_{m+1}}\right) + \frac{\partial p_m}{\partial p_{m+2}}\left(\frac{\partial p_i}{\partial q_{m+2}}\right) \\ &\quad + \cdots + \frac{\partial p_m}{\partial p_n}\left(\frac{\partial p_i}{\partial q_n}\right) + \left(\frac{\partial p_m}{\partial q_i}\right).\end{aligned}$$

The equations of constraint of the mth row, if we write them in the reverse order starting with $(m,n) = 0$, will therefore be

$$\begin{aligned}
\frac{\partial p_m}{\partial p_{m+1}}\left(\frac{\partial p_n}{\partial q_{m+1}}\right) &+ \frac{\partial p_m}{\partial p_{m+2}}\left(\frac{\partial p_n}{\partial q_{m+2}}\right) + \cdots + \frac{\partial p_m}{\partial p_n}\left(\frac{\partial p_n}{\partial q_n}\right) \\
&+ \frac{\partial p_m}{\partial q_n} = \left(\frac{\partial p_n}{\partial q_m}\right), \\
\frac{\partial p_m}{\partial p_{m+1}}\left(\frac{\partial p_{n-1}}{\partial q_{m+1}}\right) &+ \frac{\partial p_m}{\partial p_{m+2}}\left(\frac{\partial p_{n-1}}{\partial q_{m+2}}\right) + \cdots + \frac{\partial p_m}{\partial p_n}\left(\frac{\partial p_{n-1}}{\partial q_n}\right) \\
&+ \frac{\partial p_m}{\partial q_{n-1}} = \left(\frac{\partial p_{n-1}}{\partial q_m}\right), \\
\cdots\cdots\cdots\cdots &\cdots \cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial p_m}{\partial p_{m+1}}\left(\frac{\partial p_i}{\partial q_{m+1}}\right) &+ \frac{\partial p_m}{\partial p_{m+2}}\left(\frac{\partial p_i}{\partial q_{m+2}}\right) + \cdots + \frac{\partial p_m}{\partial p_n}\left(\frac{\partial p_i}{\partial q_n}\right) \\
&+ \frac{\partial p_m}{\partial q_i} = \left(\frac{\partial p_i}{\partial q_m}\right), \\
&\cdots\cdots\cdots \\
\frac{\partial p_m}{\partial p_{m+1}}\left(\frac{\partial p_{m+1}}{\partial q_{m+1}}\right) &+ \frac{\partial p_m}{\partial p_{m+2}}\left(\frac{\partial p_{m+1}}{\partial q_{m+2}}\right) + \cdots + \frac{\partial p_m}{\partial p_n}\left(\frac{\partial p_{m+1}}{\partial q_n}\right) \\
&+ \frac{\partial p_m}{\partial q_{m+1}} = \left(\frac{\partial p_{m+1}}{\partial q_m}\right), \qquad (31.4)
\end{aligned}$$

a system of equations which, after shifting the term on the right hand side to the left hand side, we can represent in the abbreviated notation

$$((m,n)) = 0, ((m,n-1)) = 0, \ldots, ((m,i)) = 0, \ldots, ((m,m+1)) = 0.$$

These equations are no longer identical with those in the mth row of the system (31.3), because in their construction we have taken the help of the equations of the following row of this system. The relation of the equations of the two systems are expressed by the connection

$$\begin{aligned}((m,i)) &= (m,i) - \frac{\partial p_m}{\partial p_{m+1}}(m+1,i) - \ldots, -\frac{\partial p_m}{\partial p_{i-1}}(i-1,i) \\ &\quad +\frac{\partial p_m}{\partial p_{i+1}}(i,i+1) + \cdots + \frac{\partial p_m}{\partial p_n}(i,n).\end{aligned}$$

If however one applies to *all* the horizontal rows of the system (31.3) the same transformation by means of which the equations (31.4) were derived from the mth horizontal row, then *the transformed system is identical with the original system (31.3).* In order to see this, we write the transformed system in the reverse order, so the following:

$$\begin{aligned}((n-1,n)) &= \; = 0, \\ ((n-2,n)) &= 0, ((n-2,n-1)) = 0, \\ ((n-3,n)) &= 0, ((n-3,n-1)) = 0, ((n-3,n-2)) = 0.\end{aligned}$$

Then

$$\begin{aligned}((n-1,n)) &= (n-1,n), \\ ((n-2,n)) &= (n-2,n) - \frac{\partial p_{n-2}}{\partial p_{n-1}}(n-1,n), \\ ((n-3,n)) &= (n-3,n) - \frac{\partial p_{n-3}}{\partial p_{n-2}}(n-2,n) \\ &\quad -\frac{\partial p_{n-3}}{\partial p_{n-1}}(n-1,n), \\ &\quad \cdots\cdots\cdots \\ ((n-2,n-1)) &= (n-2,n-1) + \frac{\partial p_{n-2}}{\partial p_n}(n-2,n), \\ ((n-3,n-1)) &= (n-3,n-1) - \frac{\partial p_{n-3}}{\partial p_{n-2}}(n-2,n-1) \\ &\quad +\frac{\partial p_{n-3}}{\partial p_n}(n-1,n), \\ &\quad \cdots\cdots\cdots \\ ((n-3,n-2)) &= (n-3,n-2) + \frac{\partial p_{n-3}}{\partial p_{n-1}}(n-2,n-1) \\ &\quad +\frac{\partial p_{n-3}}{\partial p_n}(n-2,n),\end{aligned}$$

Thus one sees that the original equations follow from the new equations, so that the two systems are equivalent.

In order to remove from the system of equations (31.4) the bracketed differential coefficients entirely, one builds from the same the new system

$$\begin{aligned}
((m,n)) &= 0,\\
((m,n-1)) &- \frac{\partial p_{n-1}}{\partial p_n}((m,n)) = 0,\\
&\cdots\cdots\cdots\\
((m,i)) &- \frac{\partial p_i}{\partial p_{i+1}}((m,i+1)) - \cdots - \frac{\partial p_i}{\partial p_n}(m,n)) = 0,\\
&\cdots\cdots\cdots\\
((m,m+1)) &- \frac{\partial p_{m+1}}{\partial p_{m+2}}((m,m+2)) - \cdots - \frac{\partial p_{m+1}}{\partial p_n}((m,n)) = 0;
\end{aligned}$$

then by virtue of the equations

$$\begin{aligned}
\left(\frac{\partial p_n}{\partial q_k}\right) &= \frac{\partial p_n}{\partial q_k},\\
\left(\frac{\partial p_{n-1}}{\partial q_k}\right) &= \frac{\partial p_{n-1}}{\partial p_n}\left(\frac{\partial p_n}{\partial q_k}\right) + \frac{\partial p_{n-1}}{\partial q_k},\\
&\cdots\cdots\cdots\\
\left(\frac{\partial p_i}{\partial q_k}\right) &= \frac{\partial p_i}{\partial p_{i+1}}\left(\frac{\partial p_{i+1}}{\partial q_k}\right) + \cdots + \frac{\partial p_i}{\partial p_n}\left(\frac{\partial p_n}{\partial q_k}\right) + \frac{\partial p_i}{\partial q_k},\\
&\cdots\cdots\cdots
\end{aligned}$$

the bracketed differential coefficients disappear entirely from the new system and one has

$$\begin{aligned}
\frac{\partial p_m}{\partial p_{m+1}}\frac{\partial p_n}{\partial q_{m+1}} &+ \frac{\partial p_m}{\partial p_{m+2}}\frac{\partial p_n}{\partial q_{m+2}} + \cdots + \frac{\partial p_m}{\partial p_n}\frac{\partial p_n}{\partial q_n} + \frac{\partial p_m}{\partial q_n} = \frac{\partial p_n}{\partial q_m}\\
\frac{\partial p_m}{\partial p_{m+1}}\frac{\partial p_{n-1}}{\partial q_{m+1}} &+ \frac{\partial p_m}{\partial p_{m+2}}\frac{\partial p_{n-1}}{\partial q_{m+2}} + \cdots + \frac{\partial p_m}{\partial p_n}\frac{\partial p_{n-1}}{\partial q_n} + \frac{\partial p_m}{\partial q_{n-1}} - \frac{\partial p_{n-1}}{\partial p_n}\\
&\frac{\partial p_m}{\partial q_n} = \frac{\partial p_{n-1}}{\partial q_m}\\
&\cdots\cdots\cdots\\
\frac{\partial p_m}{\partial p_{m+1}}\frac{\partial p_i}{\partial q_{m+1}} &+ \frac{\partial p_m}{\partial p_{m+2}}\frac{\partial p_i}{\partial q_{m+2}} + \cdots + \frac{\partial p_m}{\partial p_n}\frac{\partial p_i}{\partial q_n} + \frac{\partial p_m}{\partial q_i} - \frac{\partial p_i}{\partial p_{i+1}}\\
&\frac{\partial p_m}{\partial q_{i+1}} - \frac{\partial p_i}{\partial p_{i+2}}\frac{\partial p_m}{\partial q_{i+2}} - \frac{\partial p_i}{\partial p_n}\frac{\partial p_m}{\partial q_n} = \frac{\partial p_i}{\partial q_m}\\
&\cdots\cdots\cdots\\
\frac{\partial p_m}{\partial p_{m+1}}\frac{\partial p_{m+1}}{\partial q_{m+1}} &+ \frac{\partial p_m}{\partial p_{m+2}}\frac{\partial p_{m+1}}{\partial q_{m+2}} + \cdots + \frac{\partial p_m}{\partial p_n}\frac{\partial p_{m+1}}{\partial q_n} + \frac{\partial p_{m+1}}{\partial p_{m+2}}\frac{\partial p_m}{\partial q_{m+2}}\\
&-\frac{\partial p_{m+1}}{\partial p_{m+3}}\frac{\partial p_m}{\partial q_{m+3}} - \frac{\partial p_{m+1}}{\partial p_m}\frac{\partial p_m}{\partial q_n} = \frac{\partial p_{m+1}}{\partial q_m} \qquad (31.5)
\end{aligned}$$

This system is equivalent to the system (31.4), so that not only the equations (31.4) can be derived from the equation (31.5) but also the latter from the former, as is evident from the construction of the equation (31.5).

All equations of system (31.5) are contained in the following general scheme:

$$\frac{\partial p_m}{\partial p_{m+1}}\frac{\partial p_i}{\partial q_{m+1}} + \frac{\partial p_m}{\partial p_{m+2}}\frac{\partial p_i}{\partial q_{m+2}} + \cdots + \frac{\partial p_m}{\partial p_n}\frac{\partial p_i}{\partial q_n} - \frac{\partial p_i}{\partial p_{i+1}}\frac{\partial p_m}{\partial q_{i+1}} - \frac{\partial p_i}{\partial p_{i+2}}\frac{\partial p_m}{\partial q_{i+2}} - \frac{\partial p_i}{\partial p_n}\frac{\partial p_m}{\partial p_n} = \frac{\partial p_i}{\partial q_m}$$

or

$$\sum_{k=m+1}^{k=n}\frac{\partial p_m}{\partial p_k}\frac{\partial p_i}{\partial q_k} - \sum_{k=i+1}^{k=n}\frac{\partial p_i}{\partial p_k}\frac{\partial p_m}{\partial q_k} + \frac{\partial p_m}{\partial q_i} - \frac{\partial p_i}{\partial q_m} = 0.$$

This equation is, with the exception of the last two terms, entirely symmetrical; for though the second sum is extended only over the values $i+1$ to n while the first only extends over the values $m+1$ to i, this rests on the fact that in our hypothesis on p_i, only the variables $p_{i+1}, p_{i+2}, \ldots p_n$ occur, but not the variables $p_1, p_2, \ldots, p_{i-1}$, so that the quantities $\frac{\partial p_i}{\partial p_k}$ are different from zero only for $k > i$.

However, we can consider the problem of transformation of the constraint equations in a still more general way. Any one of them is

$$(i, i') = 0, \quad \text{or} \quad \left(\frac{\partial p_i}{\partial q_{i'}}\right) - \left(\frac{\partial p_{i'}}{\partial q_i}\right) = 0,$$

where $p_i, p_{i'}$ depend only on the quantities $q_1, q_2, \ldots, q_n$. If we now assume that p_i contains, besides the quantities $q_1, q_2, \ldots q_n$, also $p_k, p_\lambda, \ldots$, and likewise $p_{i'}$ contains besides the quantities $q_1, q_2, \ldots, q_n$ also $p_{k'}, p_{\lambda'}, \ldots$, and we write the differential coefficients with *this* hypothesis without brackets, then

$$\begin{aligned}\left(\frac{\partial p_i}{\partial q_{i'}}\right) &= \frac{\partial p_i}{\partial q_{i'}} + \frac{\partial p_i}{\partial p_k}\left(\frac{\partial p_k}{\partial q_{i'}}\right) + \frac{\partial p_i}{\partial p_\lambda}\left(\frac{\partial p_\lambda}{\partial q_{i'}}\right) + \cdots, \\ \left(\frac{\partial p_{i'}}{\partial q_i}\right) &= \frac{\partial p_{i'}}{\partial q_i} + \frac{\partial p_{i'}}{\partial p_{k'}}\left(\frac{\partial p_{k'}}{\partial q_i}\right) + \frac{\partial p_{i'}}{\partial p_{\lambda'}}\left(\frac{\partial p_{\lambda'}}{\partial q_i}\right) + \cdots,\end{aligned}$$

or, if we replace the differential coefficients $\left(\frac{\partial p_k}{\partial q_{i'}}\right), \left(\frac{\partial p\lambda}{\partial q_{i'}}\right), \ldots, \left(\frac{\partial p_{k'}}{\partial q_i}\right)$, $\left(\frac{\partial p_{\lambda'}}{\partial q_i}\right), \ldots$ by the differential coefficients of $p_{i'}$, and of p_i, which are equal according to the equations of condition (31.3),

$$\begin{aligned}\left(\frac{\partial p_i}{\partial q_{i'}}\right) &= \frac{\partial p_i}{\partial q_{i'}} + \frac{\partial p_i}{\partial p_\chi}\left(\frac{\partial p_{i'}}{\partial q_\chi}\right) + \frac{\partial p_i}{\partial p_\lambda}\left(\frac{\partial p_{i'}}{\partial q_\lambda}\right) + \cdots = \frac{\partial p_i}{\partial q_{i'}} \\ &\quad + \sum_\chi \frac{\partial p_i}{\partial p_\chi}\left(\frac{\partial p_{i'}}{\partial q_\chi}\right), \\ \left(\frac{\partial p_{i'}}{\partial q_i}\right) &= \frac{\partial p_{i'}}{\partial q_i} + \frac{\partial p_{i'}}{\partial p_{\chi'}}\left(\frac{\partial p_i}{\partial q_{\chi'}}\right) + \frac{\partial p_{i'}}{\partial p_{\lambda'}}\left(\frac{\partial p_i}{\partial q_{\lambda'}}\right) + \cdots = \frac{\partial p_{i'}}{\partial q_i} \\ &\quad + \sum_{\chi'} \frac{\partial p_{i'}}{\partial p_{\chi'}}\left(\frac{\partial p_i}{\partial q_{\chi'}}\right),\end{aligned}$$

where the summation over χ extends over the values $\chi, \lambda, \ldots$, and the summation over χ' over the values $\chi', \lambda', \ldots$. Through the introduction of these expressions, the equation of constraint $(i, i') = 0$ changes into

$$\frac{\partial p_i}{\partial q_{i'}} - \frac{\partial p_{i'}}{\partial q_i} + \sum_\chi \frac{\partial p_i}{\partial p_\chi}\left(\frac{\partial p_{i'}}{\partial q_\chi}\right) - \sum_{\chi'} \frac{\partial p_{i'}}{\partial p_{\chi'}}\left(\frac{\partial p_i}{\partial q_{\chi'}}\right) = 0. \qquad (31.6)$$

One can prove in general that the difference between the two sums which contain the bracketed differential coefficients does not change its value when the brackets are removed. In fact,

$$\left(\frac{\partial p_{i'}}{\partial q_\chi}\right) = \frac{\partial p_{i'}}{\partial q_\chi} + \sum_{\chi'} \frac{\partial p_{i'}}{\partial p_{\chi'}}\left(\frac{\partial p_{\chi'}}{\partial q_\chi}\right), \left(\frac{\partial p_{i'}}{\partial p_{\chi'}}\right) = \frac{\partial p_i}{\partial q_{\chi'}} + \sum_\chi \frac{\partial p_i}{\partial p_\chi}\left(\frac{\partial p_\chi}{\partial q_{\chi'}}\right),$$

therefore

$$\begin{aligned}&\sum_\chi \frac{\partial p_i}{\partial p_\chi}\left(\frac{\partial p_{i'}}{\partial q_\chi}\right) - \sum_{\chi'} \frac{\partial p_{i'}}{\partial p_{\chi'}}\left(\frac{\partial p_i}{\partial q_{\chi'}}\right) \\ &= \sum_\chi \frac{\partial p_i}{\partial p_\chi}\frac{\partial p_{i'}}{\partial q_\chi} - \sum_{k'} \frac{\partial p_{i'}}{\partial p_{k'}}\frac{\partial p_i}{\partial q_{k'}} + \sum_k \sum_{\chi'} \frac{\partial p_i}{\partial p_\chi}\frac{\partial p_{i'}}{\partial p_{\chi'}}\left(\frac{\partial p_{\chi'}}{\partial q_\chi}\right) \\ &\quad - \sum_{\chi'} \sum_\chi \frac{\partial p_{i'}}{\partial p_{\chi'}}\frac{\partial p_i}{\partial p_\chi}\left(\frac{\partial p_\chi}{\partial q_{\chi'}}\right);\end{aligned}$$

However, since the two double sums cancel each other because of the constraint equations $\left(\frac{\partial p_{\chi'}}{\partial q_\chi}\right) = \left(\frac{\partial p_\chi}{\partial q_{\chi'}}\right)$, we have

$$\sum_\chi \frac{\partial p_i}{\partial p_\chi}\left(\frac{\partial p_{i'}}{\partial q_\chi}\right) - \sum_{\chi'} \frac{\partial p_{i'}}{\partial p_{\chi'}}\left(\frac{\partial p_i}{\partial q_{\chi'}}\right) = \sum_\chi \frac{\partial p_i}{\partial p_\chi}\frac{\partial p_{i'}}{\partial q_\chi} - \sum_{\chi'} \frac{\partial p_i}{\partial p_{\chi'}}\frac{\partial p_i}{\partial q_{\chi'}},$$

and (31.6) changes into

$$\frac{\partial p_i}{\partial q_{i'}} - \frac{\partial p_{i'}}{\partial q_i} + \sum_{\chi} \frac{\partial p_i}{\partial p_\chi} \frac{\partial p_{i'}}{\partial q_\chi} - \sum_{\chi'} \frac{\partial p_{i'}}{\partial p_{\chi'}} \frac{\partial p_i}{\partial q_{\chi'}} = 0, \tag{31.7}$$

an equation which differs from the earlier only by the omission of the brackets.

Although we have derived (31.7) from $(i, i') = 0$, the two equations are not equivalent since we have used for the transformation only the following among the remaining equations of condition

$$\left(\frac{\partial p_\chi}{\partial q_{i'}}\right) = \left(\frac{\partial p_{i'}}{\partial q_\chi}\right), \left(\frac{\partial p_{\chi'}}{\partial q_i}\right) = \left(\frac{\partial p_i}{\partial q_{\chi'}}\right), \left(\frac{\partial p_{\chi'}}{\partial q_\chi}\right) = \left(\frac{\partial p_\chi}{\partial q_{\chi'}}\right),$$

and indeed for all values of χ and χ'.

Let us apply formula (31.7) to the case where the quantities p_1 and p_2 are expressed as functions of $p_3, p_4, \dots p_n$ $q_1, q_2, \dots, q_n$. Here we have to set $i = 1$, $i' = 2$ and χ as well as χ' take all values from 3 to n. We have therefore

$$0 = \frac{\partial p_1}{\partial q_2} - \frac{\partial p_2}{\partial q_1} + \left\{ \begin{array}{l} \frac{\partial p_1}{\partial p_3}\frac{\partial p_1}{\partial q_3} + \frac{\partial p_1}{\partial p_4}\frac{\partial p_2}{\partial q_4} + \cdots + \frac{\partial p_1}{\partial p_n}\frac{\partial p_2}{\partial q_n} - \\ -\frac{\partial p_2}{\partial p_3}\frac{\partial p_1}{\partial q_3} - \frac{\partial p_2}{\partial q_4}\frac{\partial p_1}{\partial q_4} - \cdots - \frac{\partial p_2}{\partial p_n}\frac{\partial p_1}{\partial q_n}. \end{array} \right. \tag{31.8}$$

In this equation, only the first two terms are asymmetrical and this is because of the preference we have given to the quantities p_1 and p_2, in that we have assumed that they are expressed explicitly through the remaining variables. The asymmetry disappears if we assume instead of this that there exist two equations which contain all these quantitites $p_1, p_2, \dots, p_n$ and $q_1, q_2, \dots, q_n$, and that one can solve these for arbitrary quantitites p_i and $p_{i'}$ even as for p_1 and p_2. Let the two equations be

$$\varphi = a, \quad \psi = b,$$

where φ and ψ are functions of $p_1, p_2, \dots, p_n$, $q_1, q_2, \dots, q_n$ and a, b denote constants. Then through this complete symmetry is restored. The partial differential coefficients of p_1 and p_2 which occur in equation (31.8) are replaced by the partial differential coefficients of φ and ψ. Since equation (31.8) has the form

$$0 = \frac{\partial p_1}{\partial q_2} - \frac{\partial p_2}{\partial q_1} + \sum_{k=1}^{k=n} \left(\frac{\partial p_1}{\partial p_k}\frac{\partial p_2}{\partial q_k} - \frac{\partial p_2}{\partial p_k}\frac{\partial p_1}{\partial q_k} \right), \tag{31.8*}$$

then for the transformation intended, it is required to express the quantities $\frac{\partial p_1}{\partial q_2} - \frac{\partial p_2}{\partial q_1}$ and $\frac{\partial p_1}{\partial p_k}\frac{\partial p_2}{\partial q_k} - \frac{\partial p_2}{\partial p_k}\frac{\partial p_1}{\partial q_k}$ through the partial differential coefficients of φ and ψ. Here we must consider, by virtue of the equations $\varphi = a$ and $\psi = b$, the quantities p_1, p_2, as funcctions of all the remaining $p_3, p_4, \ldots, p_n$, $q_1, q_2, \ldots, q_n$,. These, however, are considered independent of one another. By differentiation of the equation $\varphi = a$ and $\psi = b$ we obtain

$$\frac{\partial\varphi}{\partial p_1}\frac{\partial p_1}{\partial q_1} + \frac{\partial\varphi}{\partial p_2}\frac{\partial p_2}{\partial q_1} + \frac{\partial\varphi}{\partial q_1} = 0, \qquad \frac{\partial\varphi}{\partial p_1}\frac{\partial p_1}{\partial q_2} + \frac{\partial\varphi}{\partial p_2}\frac{\partial p_2}{\partial q_2} + \frac{\partial\varphi}{\partial q_2} = 0,$$
$$\frac{\partial\psi}{\partial p_1}\frac{\partial p_1}{\partial q_1} + \frac{\partial\psi}{\partial p_2}\frac{\partial p_2}{\partial q_1} + \frac{\partial\psi}{\partial q_1} = 0, \qquad \frac{\partial\psi}{\partial p_1}\frac{\partial p_1}{\partial q_2} + \frac{\partial\psi}{\partial p_2}\frac{\partial p_2}{\partial q_2} + \frac{\partial\psi}{\partial q_2} = 0.$$

From these are obtained, on introducing the notation

$$N = \frac{\partial\varphi}{\partial p_1}\frac{\partial\psi}{\partial p_2} - \frac{\partial\varphi}{\partial p_2}\frac{\partial\psi}{\partial p_1},$$

the values

$$\begin{aligned}
-N\frac{\partial p_2}{\partial q_1} &= \frac{\partial\varphi}{\partial p_1}\frac{\partial\psi}{\partial q_1} - \frac{\partial\psi}{\partial p_1}\frac{\partial\varphi}{\partial q_1}, \\
N\frac{\partial p_1}{\partial q_2} &= \frac{\partial\varphi}{\partial p_2}\frac{\partial\psi}{\partial q_2} - \frac{\partial\psi}{\partial p_2}\frac{\partial\varphi}{\partial q_2} \\
N\left\{\frac{\partial p_1}{\partial q_2} - \frac{\partial p_2}{\partial q_1}\right\} &= \frac{\partial\varphi}{\partial p_1}\frac{\partial\psi}{\partial q_1} + \frac{\partial\varphi}{\partial p_2}\frac{\partial\psi}{\partial q_2} - \frac{\partial\psi}{\partial p_1}\frac{\partial\varphi}{\partial q_1} - \frac{\partial\psi}{\partial p_2}\frac{\partial\varphi}{\partial q_2}.
\end{aligned} \tag{31.9}$$

On differentiation of the equation $\varphi = a$ and $\psi = b$ with respect to p_k and q_k, we obtain

$$\frac{\partial\varphi}{\partial p_1}\frac{\partial p_1}{\partial p_k} + \frac{\partial\varphi}{\partial p_2}\frac{\partial p_2}{\partial p_k} + \frac{\partial\varphi}{\partial p_k} = 0, \qquad \frac{\partial\varphi}{\partial p_1}\frac{\partial p_1}{\partial q_k} + \frac{\partial\varphi}{\partial p_2}\frac{\partial p_2}{\partial q_k} + \frac{\partial\varphi}{\partial q_k} = 0,$$
$$\frac{\partial\psi}{\partial p_1}\frac{\partial p_1}{\partial p_k} + \frac{\partial\psi}{\partial p_2}\frac{\partial p_2}{\partial p_k} + \frac{\partial\psi}{\partial p_k} = 0, \qquad \frac{\partial\psi}{\partial p_1}\frac{\partial p_1}{\partial q_k} + \frac{\partial\psi}{\partial p_2}\frac{\partial p_2}{\partial q_k} + \frac{\partial\psi}{\partial q_k} = 0. \tag{31.10}$$

Hence, on retaining the above meaning for N, for the partial differential coefficients of p_1 and p_2 with respect to p_k and q_k through solution of the linear equations standing one below the other, we obtain the values

$$\begin{aligned}
N\frac{\partial p_1}{\partial p_k} &= \frac{\partial\varphi}{\partial p_2}\frac{\partial\psi}{\partial p_k} - \frac{\partial\psi}{\partial p_2}\frac{\partial\varphi}{\partial p_k}, N\frac{\partial p_1}{\partial q_k} = \frac{\partial\varphi}{\partial p_2}\frac{\partial\psi}{\partial q_k} - \frac{\partial\psi}{\partial p_2}\frac{\partial\varphi}{\partial q_k} \\
-N\frac{\partial p_1}{\partial p_k} &= \frac{\partial\varphi}{\partial p_1}\frac{\partial\psi}{\partial p_k} - \frac{\partial\psi}{\partial p_1}\frac{\partial\varphi}{\partial p_k}, -N\frac{\partial p_2}{\partial q_k} = \frac{\partial\varphi}{\partial p_1}\frac{\partial\psi}{\partial p_k} - \frac{\partial\psi}{\partial p_1}\frac{\partial\varphi}{\partial q_k},
\end{aligned}$$

and if we now form the expression $\frac{\partial p_1}{\partial p_k}\frac{\partial p_2}{\partial q_k} - \frac{\partial p_2}{\partial p_k}\frac{\partial p_1}{\partial q_k}$, then we obtain an equation whose left hand side is divisible by the square of N, while the right hand side contains only N as a factor. After cancelling the common divisor N on both sides, we obtain the formula

$$N\left\{\frac{\partial p_1}{\partial p_k}\frac{\partial p_2}{\partial q_k} - \frac{\partial p_2}{\partial p_k}\frac{\partial p_1}{\partial q_k}\right\} = \frac{\partial \varphi}{\partial p_k}\frac{\partial \psi}{\partial q_k} - \frac{\partial \psi}{\partial p_k}\frac{\partial \psi}{\partial q_k}. \tag{31.11}$$

In the derivation of the above equation one can avoid removing the common divisor N if, for example, one solves for $\frac{\partial \varphi}{\partial p_1}$ and $\frac{\partial \varphi}{\partial p_2}$ the two equations standing in the first horizontal row in equations (31.10) and in the expression obtained for $\frac{\partial \varphi}{\partial p_1}$, substitutes in place of $\frac{\partial p_2}{\partial p_k}$ and $\frac{\partial p_2}{\partial q_k}$ their values obtained above. Using formulas (31.9) and (31.11) the equation (31.8*) is transformed into

$$0 = \frac{\partial \varphi}{\partial p_1}\frac{\partial \psi}{\partial q_1} + \frac{\partial \varphi}{\partial p_2}\frac{\partial \psi}{\partial q_2} - \frac{\partial \psi}{\partial p_1}\frac{\partial \varphi}{\partial q_1} - \frac{\partial \psi}{\partial p_2}\frac{\partial \varphi}{\partial q_2} + \sum_{k=3}^{k=n}\left\{\frac{\partial \varphi}{\partial p_k}\frac{\partial \psi}{\partial q_k} - \frac{\partial \psi}{\partial p_k}\frac{\partial \varphi}{\partial q_k}\right\}.$$

On combining all the terms we then have a sum extended from 1 to n:

$$0 = \sum_{k=1}^{k=n}\left\{\frac{\partial \varphi}{\partial p_k}\frac{\partial \psi}{\partial q_k} - \frac{\partial \psi}{\partial p_k}\frac{\partial \varphi}{\partial q_k}\right\} \tag{31.12}$$

and hence the theorem:

If $\varphi = a$ and $\psi = b$ are two arbitrary ones of the n equations which so determine $p_1, p_2, \ldots, p_n$ as functions of $q_1, q_2, \ldots, q_n$ that

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n$$

is a complete differential, then they must satisfy the conditions

$$0 = \left\{\begin{array}{l} \frac{\partial \varphi}{\partial p_1}\frac{\partial \psi}{\partial q_1} + \frac{\partial \varphi}{\partial p_2}\frac{\partial \psi}{\partial q_2} + \cdots + \frac{\partial \varphi}{\partial p_n}\frac{\partial \psi}{\partial q_n} \\ -\frac{\partial \psi}{\partial p_1}\frac{\partial \varphi}{\partial q_1} - \frac{\partial \psi}{\partial p_2}\frac{\partial \varphi}{\partial q_2} - \cdots - \frac{\partial \psi}{\partial p_n}\frac{\partial \varphi}{\partial q_n} \end{array}\right. \tag{31.13}$$

and indeed this equation is an identity since the arbitrary constants a and b do not occur in it.

The equation (31.12) includes the result given in (31.7) as a special case. For, if one takes the functions φ and ψ of the form

$$\begin{aligned} \varphi &= p_i - f(p_k, p_\lambda, \ldots, q_1, q_2, \ldots, q_n), \\ \psi &= p_{i'} - F(p_k, p_{\lambda'}, \ldots, q_1, q_2, \ldots, q_n), \end{aligned}$$

then equation (31.12) transformsr into the equation (31.7).

Lecture 32

Direct proof of the most general form of the integrability condition. Introduction of the function H, which set equal to an arbitrary constant determines the p as functions of the q

We shall prove *directly* the theorem we arrived at the end of the last lecture.

Let us suppose that we have solved the n equations which make $p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n$ a complete differential, and to which the equations $\varphi = a$, $\psi = b$ belong, and these have been solved for $p_1, p_2, \ldots p_n$, and that these values have been substituted in the equations $\varphi = a$ and $\psi = b$ so that these are satisfied identically. Consequently one obtains, by partial differentiation of $\varphi = a$ and $\psi = b$ with respect to any q, an identical equation if the p's are looked upon as functions of the q's. Thus, differentation of $\varphi = a$ with respect to q_i gives

$$\frac{\partial \varphi}{\partial p_1}\left(\frac{\partial p_1}{\partial q_i}\right) + \frac{\partial \varphi}{\partial p_2}\left(\frac{\partial p_2}{\partial q_i}\right) + \cdots + \frac{\partial \varphi}{\partial p_n}\left(\frac{\partial p_n}{\partial q_i}\right) + \frac{\partial \varphi}{\partial q_i} = 0,$$

or,

$$\sum_{k=1}^{k=n} \frac{\partial \varphi}{\partial p_k}\left(\frac{\partial p_k}{\partial q_i}\right) + \frac{\partial \varphi}{\partial q_i} = 0.$$

Similarly, differentiation of $\psi = b$ with respect to q_k gives

$$\sum_{i=1}^{i=n} \frac{\partial \psi}{\partial p_i}\left(\frac{\partial p_i}{\partial q_k}\right) + \frac{\partial \psi}{\partial q_k} = 0.$$

If one multiplies the first of these equations with $\frac{\partial \psi}{\partial p_i}$ and sums from 1 to n for i, and multiplies the second by $\frac{\partial \varphi}{\partial p_k}$ and sums from 1 to n for k, then one obtains the two results:

$$\sum_{i=1}^{i=n}\sum_{k=1}^{k=n}\frac{\partial\psi}{\partial p_i}\frac{\partial\varphi}{\partial p_k}\left(\frac{\partial p_k}{\partial q_i}\right)+\sum_{i=1}^{i=n}\frac{\partial\psi}{\partial p_i}\frac{\partial\varphi}{\partial q_i}=0,$$

$$\sum_{k=1}^{k=n}\sum_{i=1}^{i=n}\frac{\partial\varphi}{\partial p_k}\frac{\partial\psi}{\partial p_i}\left(\frac{\partial p_i}{\partial q_k}\right)+\sum_{k=1}^{k=n}\frac{\partial\varphi}{\partial p_k}\frac{\partial\psi}{\partial q_k}=0.$$

If one subtracts one equation from the other, then the double sums cancel; p's being determined from the n equations which make $p_1dq_1 + p_2dq_2+\cdots+p_ndq_n$ a complete differential, $\left(\frac{\partial p_i}{\partial q_k}\right)=\left(\frac{\partial p_k}{\partial q_i}\right)$, there remains

$$\sum_{k=1}^{k=n}\frac{\partial\varphi}{\partial p_k}\frac{\partial\psi}{\partial q_k}-\sum_{i=1}^{i=n}\frac{\partial\psi}{\partial p_i}\frac{\partial\varphi}{\partial q_i}=0,$$

or

$$\sum_{k=1}^{k=n}\left\{\frac{\partial\varphi}{\partial p_k}\frac{\partial\psi}{\partial q_k}-\frac{\partial\psi}{\partial p_k}\frac{\partial\varphi}{\partial q_k}\right\}=0, \tag{32.1}$$

a result which agrees with equation (31.12) of the previous lecture. One sees from this proof that for the derivation of equation (32.1) all the equations of constraint

$$\left(\frac{\partial p_i}{\partial q_k}\right)=\left(\frac{\partial p_k}{\partial q_i}\right)$$

are necessary, since only in virtue of this equality does the double sum extended over all values of i and k cancel.

As already remarked earlier the equation (32.1) assumes nothing more than that the equations $\varphi = a$ and $\psi = b$ are any two such equations as would make $p_1dq_1 + p_2dq_2 + \cdots + p_ndq_n$ a complete differential. In this generality a and b can be arbitrary constants as well as definite numerical values, e.g. zero. Also we need not fix anything as to the nature of the functions φ, ψ. The functions can themselves contain arbitrary constants but also can be without them.

According to these different circumstances one needs to check, whether the equation (32.1) is an identity or not. If a and b are not arbitrary constants, then it need not be an identity, but can be satisfied by the functions $\varphi = a$ and $\psi = b$ themselves. This case, however, seldom occurs. More frequently, if the equation (32.1) is not identically satisfied, it is a third among the n equations which makes $p_1dq_1+p_2dq_2+\cdots+p_ndq_n$

into a complete differential. Then from the equation (32.1) and one of the equations $\varphi = a$, $\psi = b$, a fourth equation is derived by simple differentiation. This again is either an identity, or a consequence of the three known up to now, or finally a fourth equation of the system, and so on. So it can come about that from $\varphi = a$ and $\psi = b$ one can through more differentiation derive n different equations which exhaust the system of n equations, but one cannot obtain more than n independent equations ($\varphi = a$ and $\psi = b$ included), since all must be satisfied by the n values of $p_1, p_2, \ldots, p_n$ which make $p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n$ a complete differential. We see that if we do not fix the character of the equations $\varphi = a$, $\psi = b$, then we cannot also say anything definite about the nature of equation (32.1).

A more definite determination can be made if we add to the requirement that $\varphi = a$, $\psi = b$ belong to the system of n equations which make $p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n$ a complete differential, also that

$$V = \int (p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n)$$

be a complete solution of the given partial differential equation which must contain also $n - 1$ arbitrary constants besides the constant which comes in V through addition. Let us assume that the given differential equation itself contains an undetermined constant h and is solved for it and has the form

$$\varphi(p_1, p_2, \ldots, p_n q_1, q_2, \ldots, q_n) = h,$$

and the complete solution V contains $n - 1$ other arbitrary constants $h_1, h_2, \ldots, h_{n-1}$; then

$$\frac{\partial V}{\partial q_1} = p_1, \frac{\partial V}{\partial q_2} = p_2, \ldots, \frac{\partial V}{\partial q_n} = p_n$$

are the right equations which make $p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n$ a complete differential and its integral a complete solution of the partial differential equation. We think of these n equations as solved for the n constants $h, h_1, h_2, \ldots, h_{n-1}$ contained therein and the result brought to the form

$$h = H, h_1 = H_1, h_2 = H_2, \ldots, h_{n-1} = H_{n-1},$$

where $H, H_1, H_1, \ldots, H_{n-1}$ are functions only of p_1, $p_2, \ldots, p_n$, q_1, $q_2, \ldots, q_n$, then the first equation $h = H$ is obviously none other than the given partial differential equation, since it is the only one which is free from the arbitrary constants $h_1, h_2, \ldots, h_{n-1}$. There exist then in any case, as we see, besides the given differential equation $h = H$, $n-1$ further such equations, linearly independent of one another, of the form

$$h_1 = H_1, h_2 = H_2, \ldots h_{n-1} = H_{n-1},$$

of such a nature that when the quantities $p_1, p_2, \ldots, p_n$ are determined from these equations, $\int(p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n)$ is a complete solution of the partial differential equation $h = H$. It is impossible to derive from these n equations

$$h = H, h_1 = H_1, \ldots, h_{n-1} = H_{n-1},$$

any other which would be entirely free of the constants h, h_1, $\ldots, h_{n-1}$; for, otherwise we could eliminate one of the quantities p from this equation and from $h = H$ and then arrive at a partial differential equation in which the number of variables for the differentiation would be one less than in the given one, and which neverthless satisfies the expression $V = \int(p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n)$. V cannot therefore be a *complete* solution of the equation $h = H$. It is therefore impossible to get rid of all the constants at one stroke. Hence, it follows that if we derive an equation from the n equations $h = H$, $h_1 = H_1, \ldots, h_{n-1} = H_{n-1}$ and obtain an equation free from all constants $h, h_1, \ldots, h_{n-1}$, then this must be an identity. This equation must similarly be satisfied by the values of the quantities p_1, $p_2, \ldots, p_n$ which we determined from the previous n equations. But these values of p_1, $p_2, \ldots, p_n$ contain further many independent quantities h, $h_1, \ldots, h_{n-1}$; therefore, any derived equation, if it is identically satisfied after the substitution of the values of p_1, $p_2, \ldots, p_n$ must be an identity also before the substitution. One such derived equation is the equation (32.1) if there the quantities H were substituted for φ and ψ; therefore

$$\frac{\partial H_i}{\partial p_1}\frac{\partial H_{i'}}{\partial q_1} + \frac{\partial H_i}{\partial p_2}\frac{\partial H_{i'}}{\partial q_2} + \cdots + \frac{\partial H_i}{\partial p_n}\frac{\partial H_{i'}}{\partial q_n} - $$
$$-\frac{\partial H_{i'}}{\partial p_1}\frac{\partial H_i}{\partial q_1} - \frac{\partial H_{i'}}{\partial p_2}\frac{\partial H_i}{\partial q_2} - \cdots - \frac{\partial H_{i'}}{\partial p_n}\frac{\partial H_i}{\partial q_n} = 0$$

is an *identity*. Thus in the case where $\varphi = a$ and $\psi = b$ belong to the system of equations $h_i = H_i$, there remains no doubt about the nature of the equation (32.1). Indeed we know that it must be an identity. Therefore the $\frac{n(n-1)}{2}$ equations which we obtain if we substitute for φ and ψ

all combinations in pairs of the quantities H_i, the equations of constraint must be satisfied by these quantities. We have in this way $\frac{n(n-1)}{2}$ further equations of constraint which must be fulfilled by the n functions of which one H, is known, while the remaining, $H_1, H_2, \ldots, H_{n-1}$ are to be found.

We now introduce the notation

$$\begin{aligned}(H_i, H_k) &= \frac{\partial H_i}{\partial p_1}\frac{\partial H_k}{\partial q_1} + \frac{\partial H_i}{\partial p_2}\frac{\partial H_k}{\partial q_2} + \cdots + \frac{\partial H_i}{\partial p_n}\frac{\partial H_k}{\partial q_n} - \\ &- \frac{\partial H_k}{\partial p_1}\frac{\partial H_i}{\partial q_1} - \frac{\partial H_k}{\partial p_2}\frac{\partial H_i}{\partial q_2} - \cdots - \frac{\partial H_k}{\partial p_n}\frac{\partial H_i}{\partial q_n}\end{aligned}$$

(which bears no relation to the notation (i, k) introduced in the previous lecture), so that for any arbitrary values of H_i, H_k,

$$(H_i, H_k) = -(H_k, H_i), (H_i, H_i) = 0.$$

If now $h = H$, $h_1 = H_1, \ldots, h_{n-1} = H_{n-1}$ are the equations which make V into a complete solution of the given partial differential equation $h = H$, then the quantities H must satisfy the $\frac{n(n-1)}{2}$ constraint equations which one obtains when one sets in

$$(H_i, H_k) = 0,$$

all possible combinations of two of the numbers $0, 1, \ldots, n-1$ for the indices i, k different from each other.

These $\frac{n(n-1)}{2}$ equations of constraint are necessary in order that the values of $p_1, p_2, \ldots, p_n$ arising from the equations $h_i = H_i$ make the expression

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n$$

a complete differential and its integral a complete solution of the given partial differential equation. It now remains to prove only that they are sufficient, i.e. when they are fulfilled, $p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n$ is actually a complete differential and with it the $\frac{n(n-1)}{2}$ equations

$$\left(\frac{\partial p_k}{\partial q_i}\right) = \left(\frac{\partial p_i}{\partial q_k}\right)$$

hold. (The second part of the assertion, that $\int(p_1 dq_1 + \cdots + p_n dq_n)$ is a complete solution follows automatically from this, since the constants

$h_1, h_2, \ldots, h_{n-1}$ are arbitrary and mutually independent). We have then to conclude that from the equations of constraint

$$(H_i, H_k) = 0$$

the equations of constraint

$$\left(\frac{\partial p_k}{\partial q_i}\right) = \left(\frac{\partial p_i}{\partial q_k}\right)$$

follow, just as above the former have been derived from the latter.

In order to lead to this proof, we must return to the equations which appeared at the beginning of this lecture for the direct proof of the equation (32.1). If we start only from the assumption that $\varphi = a$, $\psi = b$ belong to the system of n equations which serve for the determination of $p_1, p_2, \ldots, p_n$ as functions of $q_1, q_2, \ldots, q_n$, and that $\varphi = a$, $\psi = b$ are satisfied identically by these expressions of $p_1, p_2, \ldots, p_n$ in $q_1, q_2, \ldots, q_n$, then we obtain the equations

$$\sum_{i=1}^{i=n}\sum_{k=1}^{k=n} \frac{\partial \psi}{\partial p_i}\frac{\partial \varphi}{\partial p_k}\left(\frac{\partial p_k}{\partial q_i}\right) \quad + \quad \sum_{i=1}^{i=n} \frac{\partial \psi}{\partial p_i}\frac{\partial \varphi}{\partial q_i} = 0,$$

$$\sum_{i=1}^{i=n}\sum_{k=1}^{k=n} \frac{\partial \psi}{\partial p_i}\frac{\partial \varphi}{\partial p_k}\left(\frac{\partial p_i}{\partial q_k}\right) \quad + \quad \sum_{k=1}^{k=n} \frac{\partial \varphi}{\partial p_k}\frac{\partial \psi}{\partial q_k} = 0.$$

If we then assume the equations of constraint $\left(\frac{\partial p_k}{\partial q_i}\right) - \left(\frac{\partial p_i}{\partial q_k}\right) = 0$, the double sums cancel on subtraction and we obtain the new form of the equations of constraint. Now, where we cannot assume the constraint equations $\left(\frac{\partial p_k}{\partial q_k}\right) = \left(\frac{\partial p_i}{\partial q_k}\right)$, but want to prove them, we obtain on taking the difference of the two equations above and if we set H_α and H_β in place of the functions φ, ψ,

$$\begin{aligned} 0 \quad = \quad & \sum_{i=1}^{i=n}\sum_{k=1}^{k=n} \frac{\partial H_\alpha}{\partial p_k}\frac{\partial H_\beta}{\partial p_i}\left\{\left(\frac{\partial p_i}{\partial q_k}\right) - \left(\frac{\partial p_k}{\partial q_i}\right)\right\} \\ & + \sum_{i=1}^{i=n}\left\{\frac{\partial H_\alpha}{\partial p_i}\frac{\partial H_\beta}{\partial q_i} - \frac{\partial H_\beta}{\partial p_i}\frac{\partial H_\alpha}{\partial q_i}\right\}. \end{aligned} \tag{32.2}$$

This simple sum which forms the second member on the right hand side of this equation is none other than the expression denoted above by (H_α, H_β). The double sum which forms the first term can be reduced to

$\frac{n(n-1)}{2}$ terms, since the terms for which $i = k$ vanish. Of the remaining, any two of which go over into each other on interchanging i and k, can be combined into one. In this way equation (32.2) is transformed into

$$0 = \sum_{i,k} \left\{ \frac{\partial H_\alpha}{\partial p_k} \frac{\partial H_\beta}{\partial p_i} - \frac{\partial H_\beta}{\partial p_k} \frac{\partial H_\alpha}{\partial p_i} \right\} \left\{ \left(\frac{\partial p_i}{\partial q_k} \right) + \left(\frac{\partial p_k}{\partial q_i} \right) \right\} + (H_\alpha, H_\beta), \tag{32.2*}$$

where the summation is extended over all combinations of i, k different from each other. One obtains $\frac{n(n-1)}{2}$ such equations when one sets for H_α, H_β any two different ones of $H, H_1, \ldots, H_{n-1}$. So one obtains a system of $\frac{n(n-1)}{2}$ equation which are linear in the expressions $\left(\frac{\partial p_i}{\partial q_k}\right) - \left(\frac{\partial p_k}{\partial q_i}\right)$, and in which the (H_α, H_β) form the constant terms. It is to be proved that when the last quantities vanish, the first would all became equal to zero. Now in a system of linear equations, the vanishing of the unknown is always a necessary consequence of the vanishing of the constant terms if the determinant of the system is not equal to zero, in which case the values of the unknown remain undetermined. That this exceptional case does not occur here one can prove, without finding out the value of the determinant in question, through this that one can derive in the following simple way the solution formulae for (32.2*) from the form of the equations of the system given in (32.2). One puts for abbreviation

$$\frac{\partial H_\alpha}{\partial p_i} = a_i^{(\alpha)},$$

and denotes by R the determinant formed by the n^2 quantities $a_i^{(\alpha)}$, where α takes the values $0, 1, \ldots, n-1$ and i the values $1, 2, \ldots, n$ so that

$$R = \sum \pm a_1 a_2' a_3'' \ldots a_n^{(n-1)}.$$

Further one sets

$$A_i^{(\alpha)} = \frac{\partial R}{\partial a_i^{(\alpha)}}.$$

On introducing these notations and interchanging α and β, equations (32.2) can be written in the following way:

$$\sum_{i=1}^{i=n} \sum_{k=1}^{k=n} a_i^{(\alpha)} a_k^{(\beta)} \left\{ \left(\frac{\partial p_i}{\partial q_k} \right) - \left(\frac{\partial p_k}{\partial q_i} \right) \right\} = (H_\alpha, H_\beta). \tag{32.3}$$

This equation holds not only when distinct values from the series $0, 1, 2, \ldots n-1$ one sets for α and β, but also when both the indices are equal to one and the same value. In this last case the equation (32.3) is an identity, since in the only formally different equation (32.2*), all the terms then vanish individually.

If one multiplies equation (32.3) with $A_r^{(\alpha)}$, $A_s^{(\beta)}$, where r and s denote numbers from the series $1, 2. \dots, n$, then one may, according to the remark just made, sum from 0 to $n-1$ over each the indices α and β independently of each other. If one changes in the result the order of summation, which on the one hand is made over i and k and over α and β on the other, and denote by M_{ik} the double sum

$$M_{ik} = \sum_{\alpha=0}^{\alpha=n-1} \sum_{\beta=0}^{\beta=n-1} a_i^{(\alpha)} a_k^{(\beta)} A_r^{(\alpha)} A_s^{(\beta)} = \sum_{\alpha=0}^{\alpha=n-1} a_i^{(\alpha)} A_r^{(\alpha)} \sum_{\beta=0}^{\beta=n-1} a_k^{(\beta)} A_s^{(\beta)},$$

then this gives

$$\sum_{i=1}^{i=n} \sum_{k=1}^{k=n} M_{ik} \left\{ \left(\frac{\partial p_i}{\partial q_k} \right) - \left(\frac{\partial p_k}{\partial q_i} \right) \right\} = \sum_{\alpha=0}^{\alpha=n-1} \sum_{\beta=0}^{\beta=n-1} A_r^{(\alpha)} A_s^{(\beta)} (H_\alpha, H_\beta). \tag{32.4}$$

The simple sums whose product M_{ik} represents[1] are equal to to 0 or R according as i is different from r and k from s, or i coincides with r and k with s. Hence

$$M_{ik} = 0$$

except when $i = r$ and $k = s$, and in this case, $M_{r,s} = R^2$. Equation (32.4) then changes into

$$R^2 \left\{ \left(\frac{\partial p_r}{\partial q_s} \right) - \left(\frac{\partial p_s}{\partial q_r} \right) \right\} = \sum_{\alpha=0}^{\alpha=n-1} \sum_{\beta=0}^{\beta=n-1} A_r^{(\alpha)} A_s^{(\beta)} (H_\alpha, H_\beta).$$

Hence one sees that if (H_α, H_β) are all equal to 0, as we assume, then all the quantities $\left(\frac{\partial p_r}{\partial q_s}\right) - \left(\frac{\partial p_s}{\partial q_r}\right)$ also vanish, unless R is equal to zero. But the vanishing of the expression

$$R = \sum \pm a_1 a_2' a_3'' \dots a_n^{(n-1)} = \sum \pm \frac{\partial H}{\partial p_i} \frac{\partial H_1}{\partial p_2} \cdots \frac{\partial H_{n-1}}{\partial p_n}$$

signifies that the functions $H, H_1, \dots, H_{n-1}$ of the quantities p_1, $p_2, \dots, p_n$ are not independent of one another. Then the equations

[1]See Lecture 11, §3.

$H = h$, $H_1 = h_1, \ldots, H_{n-1} = h_{n-1}$ are not sufficient to determine the variables $p_1, p_2, \ldots, p_n$ as function of $q_1, q_2, \ldots, q_n$. Except for this single and obvious exceptional case, one can conversely derive the original equations of constraint

$$\left(\frac{\partial p_i}{\partial q_k}\right) = \left(\frac{\partial p_k}{\partial q_i}\right)$$

from the $\frac{n(n-1)}{2}$ equations of constraint

$$(H_\alpha, H_\beta) = 0.$$

Lecture 33

On the simultaneous solutions of two linear partial differential equations

The problem of integrating the given partial differential equation $H = h$ is now reduced to finding $n-1$ functions $H_1, H_2, \ldots, H_{n-1}$, independent of one another and also of H, of the variables $p_1, p_2, \ldots, p_n$, $q_1, q_2, \ldots, q_n$, which satisfy the $\frac{n(n-1)}{2}$ equations of constraint

$$(H_\alpha, H_\beta) = 0$$

(for the values $0, 1, \ldots, n-1$ of the indices α and β), and which one has to set equal to $n-1$ mutually independent arbitrary constants $h_1, h_2, \ldots, h_{n-1}$. Between any one of these $n-1$ functions, e.g. H_1, and the function H known to us, also the equation of constraint $(H, H_1) = 0$, holds i.e. H_1 satisfies the partial differential equation

$$\begin{aligned}\frac{\partial H}{\partial p_1}\frac{\partial H_1}{\partial q_1} &+ \frac{\partial H}{\partial p_2}\frac{\partial H_1}{\partial q_2} + \cdots + \frac{\partial H}{\partial p_n}\frac{\partial H_1}{\partial q_n} - \\ &- \frac{\partial H}{\partial q_1}\frac{\partial H_1}{\partial p_1} - \frac{\partial H}{\partial q_2}\frac{\partial H_1}{\partial p_2} - \cdots - \frac{\partial H}{\partial q_n}\frac{\partial H_1}{\partial p_n} = 0,\end{aligned}$$

or what is the same, $H_1 = h_1$ is an integral of the system of isoperimetric differential equations[1]

$$\begin{aligned}&dq_1 : dq_2 : \ldots : dq_n : dp_1 : dp_2 : \ldots : dq_n \\ &= \frac{\partial H}{\partial p_1} : \frac{\partial H}{\partial p_2} : \ldots : \frac{\partial H}{\partial p_n} : -\frac{\partial H}{\partial q_1} : -\frac{\partial H}{\partial q_2} : \ldots : -\frac{\partial H}{\partial q_n},\end{aligned}$$

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[1]See Lecture 19, p 172

which, for $H = T - U$, goes over to the system of differential equations of mechanics. A similar relation holds for the functions $H_2, \ldots, H_{n-1}$ which satisfy the analogous equations of constraint $(H, H_2) = 0, \ldots, (H, H_{n-1}) = 0$. All $n - 1$ equations

$$H_1 = h_1, H_2 = h_2, \ldots, H_{n-1} = h_{n-1}$$

are therefore integrals of the system of isoperimetric differential equations given above. But this determination of the functions $H_1, H_2, \ldots, H_{n-1}$ is not sufficient. Through this only the equations of constraint

$$(H, H_1) = 0, (H, H_2) = 0, \ldots, (H, H_{n-1}) = 0$$

are satisfied, and the remaining $\frac{n(n-1)}{2} - (n-1) = \frac{(n-1)(n-2)}{2}$ equations of constraint $(H_\alpha, H_\beta) = 0$ which, with the exception of H, hold between any two of the $n-1$ functions $H_1, H_2, \ldots, H_{n-1}$, will not be satisfied by the value of the functions so determined unless one has chosen the $n-1$ integrals just for that purpose. We cannot even know a priori whether for the first of the functions sought for, H_1, an entirely arbitrary integral may be taken, and whether the remaining $n - 2$ functions can be so determined that they, along with H and H_1, also satisfy all the previous conditions.

A more precise investigation shows that in fact H_1 can be chosen entirely arbitrarily among the integrals and it needs to satisfy only the condition

$$(H, H_1) = 0;$$

Whatever function H_1 one may take corresponding to this condition, there always exists a second function H_2 which simultaneously satisfies both the conditions

$$(H, H_2) = 0, (H_1, H_2) = 0;$$

that further, whatever function H_2 one may take corresponding to both these conditions, there always exists a third one, H_3, which simultaneously satisfies all the three conditions

$$(H, H_3) = 0, (H_1, H_3) = 0, (H_2, H_3) = 0.$$

One can continue in this way until all the functions $H_1, H_2, \ldots, H_{n-1}$ are determined.

We see that the present investigation forces on us the necessity of answering the question, whether and under what conditions is it possible to satisfy several partial differential equations simultaneously.

In order to handle this question in its greatest generality, let the linear partial differential equation be of the form

$$A_0\frac{\partial f}{\partial x_0}+A_1\frac{\partial f}{\partial x_1}+A_2\frac{\partial f}{\partial x_2}+\cdots+A_n\frac{\partial f}{\partial x_n}=0.$$

We shall denote by $A(f)$ the left side of this equation in which A_0, $A_1, \ldots A_n$ are given functions of $x_0, x_1, \ldots, x_n$, so that we can look upon the construction of such an expression as an operation done on the unknown function f. Let then

$$A(f)=A_0\frac{\partial f}{\partial x_0}+A_1\frac{\partial f}{\partial x_1}+\cdots+A_n\frac{\partial f}{\partial x_n}=\sum_{i=0}^{i=n}A_i\frac{\partial f}{\partial x_i},$$

and similarly,

$$B(f)=B_0\frac{\partial f}{\partial x_0}+B_1\frac{\partial f}{\partial x_1}+\cdots+B_n\frac{\partial f}{\partial x_n}=\sum_{k=0}^{k=n}B_k\frac{\partial f}{\partial x_k}.$$

$A(f)$ and $B(f)$ are two different operators of this sort with which one can operate on the function f. If we apply the two operators one after the other, we get, according as we begin with the operation A or the operation B, the two expression $B(A(f))$ and $A(B(f))$ which are defined through the equations

$$\begin{aligned}
B(A(f)) &= \sum_{k=0}^{k=n}B_k\frac{\partial}{\partial x_k}\left\{\sum_{i=0}^{i=n}A_i\frac{\partial f}{\partial x_i}\right\}=\sum_{k=0}^{k=n}\sum_{i=0}^{i=n}B_kA_i\frac{\partial^2 f}{\partial x_i\partial x_k}\\
&\quad+\sum_{k=0}^{k=n}\sum_{i=0}^{i=n}B_k\frac{\partial A_i}{\partial x_k}\frac{\partial f}{\partial x_i},\\
A(B(f)) &= \sum_{i=0}^{i=n}A_i\frac{\partial}{\partial x_i}\left\{\sum_{k=0}^{k=n}B_k\frac{\partial f}{\partial x_k}\right\}=\sum_{i=0}^{i=n}\sum_{k=0}^{k=n}A_iB_k\frac{\partial^2 f}{\partial x_i\partial x_k}\\
&\quad+\sum_{i=0}^{i=n}\sum_{k=0}^{k=n}A_i\frac{\partial B_k}{\partial x_i}\frac{\partial f}{\partial x_k}.
\end{aligned}$$

In both the expressions, only the terms multiplied by the second order differential coefficients of f are in general equal. In the difference of the two only those terms remain which contain the first order differential coefficients of f. For the difference, which we shall call $C(f)$, we obtain

$$\begin{aligned} C(f) &= B(A(f)) - A(B(f)) \\ &= \sum_{k=0}^{k=n}\sum_{i=0}^{i=n} B_k \frac{\partial A_i}{\partial x_k}\frac{\partial f}{\partial x_i} - \sum_{i=0}^{n}\sum_{k=0}^{n} A_i \frac{\partial B_k}{\partial x_i}\frac{\partial f}{\partial x_k} \\ &= \sum_{i=0}^{i=n}\left\{\sum_{k=0}^{k=n}\left(B_k\frac{\partial A_i}{\partial x_k} - A_k\frac{\partial B_i}{\partial x_k}\right)\right\}\frac{\partial f}{\partial x_i}, \end{aligned}$$

or, if we introduce the notation

$$\begin{aligned} C_i &= \sum_{k=0}^{k=n}\left(B_k\frac{\partial A_i}{\partial x_k} - A_k\frac{\partial B_i}{\partial x_k}\right) \\ &= \left\{ \begin{array}{l} B_0\frac{\partial A_i}{\partial x_0} + B_1\frac{\partial A_i}{\partial x_1} + \cdots + B_n\frac{\partial A_i}{\partial x_n} \\ -A_0\frac{\partial B_i}{\partial x_0} - A_1\frac{\partial B_i}{\partial x_1} - \cdots - A_n\frac{\partial B_i}{\partial x_n} \end{array} \right. \end{aligned}$$

then

$$C(f) = \sum_{i=0}^{i=n} C_i \frac{\partial f}{\partial x_i} = C_0\frac{\partial f}{\partial x_0} + C_1\frac{\partial f}{\partial x_1} + \cdots + C_n\frac{\partial f}{\partial x_n}.$$

If, as we shall assume in the following investigation, the $n+1$ equations hold:

$$C_0 = 0, C_1 = 0, \ldots, C_n = 0,$$

so that for the values $0, 1, \ldots, n$ of the index i the equations

$$C_i = \left\{ \begin{array}{l} B_0\frac{\partial A_i}{\partial x_0} + B_1\frac{\partial A_i}{\partial x_1} + \cdots + B_n\frac{\partial A_i}{\partial x_n} \\ -A_0\frac{\partial B_i}{\partial x_0} - A_1\frac{\partial B_i}{\partial x_1} - \cdots - A_n\frac{\partial B_i}{\partial x_n} \end{array} \right\} = 0$$

are satisfied, then one has

$$C(f) = B(A(f)) - A(B(f)) = 0$$

or,

$$B(A(f)) = A(B(f)),$$

i.e. it is equally valid whether one first applies the operation A and then the operation B, or first the operation B and then the operation A.

This independence of the order in which the operations A and B are applied is of great importance since it allows extension to an arbitrary number of repetitions of both the operations. If one denotes by $A^2, A^3, \ldots, A^m$ the operation A applied twice, thrice, $\ldots, m$ times one after the other, and likewise by $B^2, B^3, \ldots, B^n$ the operation B applied twice, thrice, $\ldots, m$ times, one after the other, then from the equation $B(A(f)) = A(B(f))$ follow the more general relation,

$$B^{m'}(A^m(f)) = A^m(B^{m'}(f)).$$

From this result one can derive the greatest use in the investigations of the two linear partial differential equations which satisfy the $n+1$ equations of constraint $C_i = 0$,

$$A(f) = 0, B(f) = 0,$$

partly to find the solutions of any single differential equation, partly their simultaneous solutions. If one assumes that a solution f_1 of the differential equation $A(f) = 0$ is known to us, one has identically

$$A(f_1) = 0,$$

from which follows

$$B(A(f_1)) = B(0) = 0.$$

But by our assumption, the $n+1$ equations of constraint $C_i = 0$ are satisfied, so one can reverse the sequence of the operations A and B, so form the equation

$$B(A(f_1)) = 0$$

one obtains the equation

$$A(B(f_1)) = 0,$$

i.e., $B(f_1)$ is likewise a solution of $A(f) = 0$. According to the nature of this solution three cases have to be distinguished, whereby one has to remember that the partial differential equation $A(f) = 0$ has, besides f_1 still $n-1$ solutions $f_2, f_3, \ldots, f_n$ independent of one another and of f_1, and moreover the obvious solution $f =$ constant. It may be that either $B(f_1)$ is first, a solution f_2 independent of f_1, or second, a function of f_1 which can also be a constant; third, it must be regarded as a special case when $B(f_1)$ is found equal to the constant value zero. We have then the three cases

$$B(f_1) = f_2, B(f_1) = F(f_1), B(f_1) = 0.$$

In the first case, we have found from the solution f_1 of the partial differential equation $A(f) = 0$ a second solution $f_2 = B(f_1)$; in the third case we have equally $A(f_1) = 0$ and $B(f_1) = 0$, i.e., f_1 is a simultaneous solution of $A(f) = 0$ and $B(f) = 0$; the second case will be discussed later.

In the first case, where $B(f_1)$ is a new solution f_2, one can proceed further in the same way, namely, since $A(f_2) = 0$, one so obtains $B(A(f_2)) = B(0) = 0$, by reversing the two operations

$$0 = A(B(f_2)) = A(B^2(f_1)),$$

i.e., $B^2(f_1)$ is a likewise a solution of $A(f) = 0$. Here again one has to distinguish three cases, namely

$$B^2(f_1) = f_3, B^2(f_1) = F(f_1, f_2), B^2(f_1) = B(f_2) = 0.$$

In the first case one has a third solution $f_3 = B^2(f_1)$ of $A(f) = 0$, independent of f_1 and f_2; in the third case, $f_2 = B(f_1)$ is a simultaneous solution of $A(f) = 0$ and $B(f) = 0$; we shall come back later to the second case in which $B^2(f_1)$ is a function of the earlier solutions f_1 and $f_2 = B(f_1)$ which can also go over into a non-vanishing constant. Through repeated applications of the operation B, from *one* solution f_1 arise the series of functions $f_1, B(f_1)$, $B^2(f_1)$, $B^3(f_1), \ldots$, all of which satisfy the partial differential equation $A(f) = 0$. Now, either the first n quantities of this series are mutually independent functions and form a complete system of solutions of the equation $A(f) = 0\ldots$ or one of the preceding quantities, say $B^m(f_1)$, is already a function of the foregoing $f_1, B(f_1), B^2(f_1), \ldots, B^{m-1}(f_1)$, which may also reduce to a nonvanishing constant, or to zero.

The case unfavourable to the finding of the solution of $A(f) = 0$ in which the entire cycle is not run through makes easy to find the simultaneous solutions of $A(f) = 0$ and $B(f) = 0$.

The most general solution of $A(f) = 0$ is an arbitrary function of n of its mutually independent solutions $f_1, f_2, \ldots, f_n$. In order to obtain a simultaneous solution of $A(f) = 0$ and $B(f) = 0$, this arbitrary function of $f_1, f_2, \ldots, f_n$ must be so determined that it also satisfies $B(f) = 0$. For this purpose, if we introduce in the expression $B(f)$, for n of the $n+1$ variables $x_0, x_1, \ldots, x_n$, e.g. for $x_1, x_2, \ldots, x_n$, the functions $f_1, f_2, \ldots, f_n$ as new variables and represent the differential coefficients of f constructed under this new hypothesis with $\left(\frac{\partial f}{\partial x_o}\right), \frac{\partial f}{\partial f_1}, \frac{\partial f}{\partial f_2}, \ldots, \frac{\partial f}{\partial f_n}$, where the new differential coefficient $\left(\frac{\partial f}{\partial x_0}\right)$ is completely different from the earlier $\frac{\partial f}{\partial x_0}$, then we have

$$\frac{\partial f}{\partial x_0} = \left(\frac{\partial f}{\partial x_0}\right) + \sum_{k=1}^{k=n} \frac{\partial f}{\partial f_k}\frac{\partial f_k}{\partial x_0},$$

and if i denotes the numbers from 1 to n,

$$\frac{\partial f}{\partial x_i} = \sum_{k=1}^{k=n} \frac{\partial f}{\partial f_k}\frac{\partial f_k}{\partial x_i};$$

therefore

$$\begin{aligned} B(f) &= B_0\left(\frac{\partial f}{\partial x_0}\right) + B_0\sum_{k=1}^{k=n} \frac{\partial f}{\partial f_k}\frac{\partial f_k}{\partial x_0} + \sum_{i=1}^{i=n} B_i \sum_{k=1}^{k=n} \frac{\partial f}{\partial f_k}\frac{\partial f_k}{\partial x_i} \\ &= B_0\left(\frac{\partial f}{\partial x_0}\right) + \sum_{k=1}^{k=n}\left\{\sum_{i=0}^{i=n} B_i \frac{\partial f_k}{\partial x_i}\right\}\frac{\partial f}{\partial f_k}, \end{aligned}$$

or, finally, since $\sum_{i=0}^{i=n} B_i \frac{\partial f_k}{\partial x_i}$ is none other than $B(f_k)$,

$$B(f) = B_0\left(\frac{\partial f}{\partial x_0}\right) + \sum_{k=1}^{k=n} B(f_k)\frac{\partial f}{\partial f_k}.$$

Now f, if it is a solution of $A(f) = 0$, can depend only on the functions f_k, but does not depend on x_0; so one has $\left(\frac{\partial f}{\partial x_0}\right) = 0$, and the equation $B(f) = 0$ reduces to

$$\sum_{k=1}^{k=n} B(f_k)\frac{\partial f}{\partial f_k} = 0,$$

i.e., to

$$B(f_1)\frac{\partial f}{\partial f_1} + B(f_2)\frac{\partial f}{\partial f_2} + \cdots + B(f_n)\frac{\partial f}{\partial f_n} = 0.$$

But as a consequence of the $n+1$ constraints assumed,

$$C_i = B(A_i) - A(B_i) = 0,$$

holding for $i = 0, 1, \ldots, n$, the solution f_i of $A(f) = 0$ and at the same time $B(f_i)$ a solution of $A(f) = 0$, the obvious solution $f =$ constant reckoned along with this; consequently, all the quantities $B(f_1)$,

$B(f_2), \ldots, B(f_n)$ are solutions of $A(f) = 0$; and since the most general solution of $A(f) = 0$ is an arbitrary function of $f_1, f_2, \ldots, f_n$, all the functions $B(f_1)$, $B(f_2), \ldots, B(f_n)$ are functions of the quantities $f_1, f_2, \ldots, f_n$; consequently, the equation

$$B(f_1)\frac{\partial f}{\partial f_1} + B(f_2)\frac{\partial f}{\partial f_2} + \cdots + B(f_n)\frac{\partial f}{\partial f_n} = 0$$

is a partial differential equation which defines f as a function of f_1, $f_2, \ldots, f_n$. It admits $n - 1$ mutually independent solutions $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ and its most general solution, which represents equally the most general simultaneous solution of $A(f) = 0$ and $B(f) = 0$, is therefore an arbitrary function $F(\varphi_1, \varphi_2, \ldots, \varphi_{n-1})$ of the preceding $n - 1$ mutually independent solutions. Accordingly, such simultaneous solutions exist if the $n + 1$ conditions $C_i = 0$ are satisfied.

We now show the use which the repeated application of the operation B on the solution f_1 of $A(f) = 0$ can be put to when it comes to the determination not of the most general solution but to a particular simultaneous solution of $A(f) = 0$ and $B(f) = 0$. I assume the quantities $B(f_1) = f_2$, $B^2(f_1) = f_3, \ldots, B^{m-1}(f_1) = f_m$, where m is less than or at most equal to n, to be solution of $A(f) = 0$, independent of one another and of f_1. On the other hand $B^m(f)$ is not a solution independent of $f_1, f_2, \ldots, f_m$; then there are two cases to be distinguished:

1. If $B^m(f_1)$ is equal to a function $F(f_1, f_2, \ldots, f_m)$ of f_1, f_2, $\ldots$, f_m, which can also go over into a constant nonvanishing value, then it is possible so to determine the simultaneous solution of $A(f) = 0$ and $B(f) = 0$ always that they depend only on $f_1, f_2, \ldots, f_m$, but do not contain the remaining f_{m+1}, $f_{m+2}, \ldots, f_n$. Then through this hypothesis the above partial differential equation which defines the simultaneous solution f as a function of $f_1, f_2, \ldots, f_n$ reduces to the following:

$$f_2\frac{\partial f}{\partial f_1} + f_3\frac{\partial f}{\partial f_2} + \cdots + f_m\frac{\partial f}{\partial f_{m-1}} + F(f_1, f_2, \ldots, f_m)\frac{\partial f}{\partial f_m} = 0,$$

which coincides with the system of ordinary differential equations

$$\begin{aligned} df_1 : df_2 : \ldots : df_{m-1} : df_m \\ = f_2 : f_3 : \ldots : f_m : F(f_1, f_2, \ldots, f_m). \end{aligned}$$

If we introduce in this system the further variable t so that one sets the m equal ratios equal to the ratio $dt : 1$, then one has

$$\frac{df_1}{dt} = f_2, \frac{df_2}{dt} = f_3, \dots, \frac{df_{m-1}}{dt} = f_m, \frac{df_m}{dt} = F(f_1, f_2, \dots, f_m),$$

or

$$f_2 = \frac{df_1}{dt}, f_3 = \frac{d^2 f_1}{dt^2}, \dots, f_m = \frac{d^{m-1} f_1}{dt^{m-1}}, \frac{df_m}{dt} = \frac{d^m f_1}{dt^m},$$

and consequently,

$$\frac{d^m f_1}{dt^m} = F\left(f_1, \frac{df_1}{dt}, \dots, \frac{d^{m-1} f_1}{dt^{m-1}}\right).$$

If now $\varphi_1 =$ constant is any integral whatsoever free of t of this mth differential equation, then $f = \varphi_1$ is a simultaneous solution of $A(f) = 0$ and $B(f) = 0$.

2. If $B^m(f_1) = 0$, then one has $0 = B(B^{m-1}(f_1)) = B(f_m)$ and $0 = A(f_m)$; so $f_m = B^{m-1}(f_1)$ is a simultaneous solution of $A(f) = 0$ and $B(f) = 0$.

There is an exception to the result obtained in 1. for $m = 1$, i.e., when already $B(f_1)$ is itself a function of f_1 or reduces to a constant different from zero. This one sees already from that the differential equation between f_1 and t is then of the first order, and so possesses no integral free of t. The partial differential equation which defines f as a function of $f_1, f_2, \dots, f_m$ then changes to

$$\frac{\partial f}{\partial f_1} = 0,$$

and gives the obvious solution $f =$ constant, which is not useful. In this case one cannot make any use of the solution f_1 alone, but it is necessary to know a second solution f_2 of the equation $A(f) = 0$. If one applies the operation B to f_2 as to f_1 earlier, and if $B(f_2)$ is not a function of f_2 alone, then one obtains, according to the previous procedure, a simultaneous solution of $A(f) = 0$ and $B(f) = 0$ from f_2. If, on the other hand, $B(f_2)$ is a function of f_2 alone, so that a simultaneous solution cannot be found from f_2 alone, then one finds one such through simultaneous use of f_1 and f_2. That is, if

$$B(f_1) = \Phi(f_1), B(f_2) = \Psi(f_2),$$

then one can assume that f is a function of f_1 and f_2 alone, and one obtains for the determination of this function the partial differential equation

$$\Phi(f_1)\frac{\partial f}{\partial f_1} + \Psi(f_2)\frac{\partial f}{\partial f_2} = 0,$$

which leads to the ordinary differential equation

$$df_1 : df_2 = \Phi(f_1) : \Psi(f_2),$$

and the expression

$$f = \int \frac{df_1}{\Phi(f_1)} - \int \frac{df_2}{\Psi(f_2)}$$

gives the simultaneous solution sought for.

Lecture 34

Application of the preceding investigation to the integration of partial differential equations of the first order, and in particular, to the case of mechanics. The theorem on the third integral derived from two given integrals of differential equations of dynamics

In order to apply the results obtained in the investigation of the previous lecture on the simultaneous solutions of linear partial differential equations to the case which led us to this investigation and from whcih we proceed to the integration of the partial differential equation $H = h$ (p.290), we shall first replace the $n+1$ independent variables $x_0, x_1, \ldots, x_n$ by an even number $2n$ of variables $x_1, x_2, \ldots, x_{2n}$, where indices we shall begin with 1 instead of 0, so that the expression $A(f)$ and $B(f)$ are now defined through the equations

$$\begin{aligned} A(f) &= A_1 \frac{\partial f}{\partial x_1} + A_2 \frac{\partial f}{\partial x_2} + \cdots + A_{2n} \frac{\partial f}{\partial x_{2n}}, \\ B(f) &= B_1 \frac{\partial f}{\partial x_1} + B_2 \frac{\partial f}{\partial x_2} + \cdots + B_{2n} \frac{\partial f}{\partial x_{2n}}, \end{aligned}$$

and the $2n$ equations of constraint

$$C = B(A_i) - A(B_i) = 0$$

hold for $i = 1, 2, \ldots, 2n$. Further, we may put p and q in place of the $2n$ independent variables so that

$$x_1 = q_1, x_2 = q_2, \ldots, x_n = q_n, x_{n+1} = p_1, x_{n+2} = p_2, \ldots, x_{2n} = p_n,$$

and finally, let the coefficients A_i, B_i be determined through the equations

$$\begin{aligned} A_1 &= \frac{\partial\varphi}{\partial p_1}, A_2 = \frac{\partial\varphi}{\partial p_2}, \dots, A_n = \frac{\partial\varphi}{\partial p_n}, \\ &\quad A_{n+1} = -\frac{\partial\varphi}{\partial q_1}, A_{n+2} = -\frac{\partial\varphi}{\partial q_2}, \dots, A_{2n} = -\frac{\partial\varphi}{\partial q_n}, \\ B_1 &= \frac{\partial\psi}{\partial p_1}, B_2 = \frac{\partial\psi}{\partial p_2}, \dots, B_n = \frac{\partial\psi}{\partial p_n}, \\ &\quad B_{n+1} = -\frac{\partial\psi}{\partial q_1}, B_{n+2} = -\frac{\partial\psi}{\partial q_2}, \dots, B_{2n} = -\frac{\partial\psi}{\partial q_n}. \end{aligned}$$

Then we obtain

$$\begin{aligned} A(f) &= \frac{\partial\varphi}{\partial p_1}\frac{\partial f}{\partial q_1} + \frac{\partial\varphi}{\partial p_2}\frac{\partial f}{\partial q_2} + \cdots + \frac{\partial\varphi}{\partial p_n}\frac{\partial f}{\partial q_n} \\ &\quad -\frac{\partial\varphi}{\partial q_1}\frac{\partial f}{\partial p_1} - \frac{\partial\varphi}{\partial q_2}\frac{\partial f}{\partial p_2} - \cdots - \frac{\partial\varphi}{\partial q_n}\frac{\partial f}{\partial p_n}, \\ B(f) &= \frac{\partial\psi}{\partial p_1}\frac{\partial f}{\partial q_1} + \frac{\partial\psi}{\partial p_2}\frac{\partial f}{\partial q_2} + \cdots + \frac{\partial\psi}{\partial p_n}\frac{\partial f}{\partial q_n} \\ &\quad -\frac{\partial\psi}{\partial q_1}\frac{\partial f}{\partial p_1} - \frac{\partial\psi}{\partial q_2}\frac{\partial f}{\partial p_2} - \cdots - \frac{\partial\psi}{\partial q_n}\frac{\partial f}{\partial p_n}, \end{aligned}$$

or, in the notation introduced in lecture 32 (p.285),

$$\begin{aligned} A(f) &= (\varphi, f), \\ B(f) &= (\psi, f). \end{aligned}$$

In order to obtain the values of the $2n$ quantities C_i for $i = 1, 2, \dots, 2n$, we divide them into two groups C_i and C_{n+i} for $i = 1, 2, \dots, n$; then one obtains

$$\begin{aligned} C_i &= B(A_i) - A(B_i) = \left(\psi, \frac{\partial\varphi}{\partial p_i}\right) - \left(\varphi, \frac{\partial\psi}{\partial q_i}\right), \\ C_{n+i} &= B(A_{n+i}) - A(B_{n+i}) = \left(\psi, -\frac{\partial\varphi}{\partial q_i}\right) - \left(\varphi, -\frac{\partial\psi}{\partial q_i}\right), \end{aligned}$$

or, when one takes into account the identity

$$\begin{aligned} (\psi, \varphi) &= -(\varphi, \psi) = (\varphi, -\psi), \\ -C_i &= \left(\frac{\partial\varphi}{\partial p_i}, \psi\right) + \left(\varphi, \frac{\partial\psi}{\partial p_i}\right), \\ C_{n+i} &= \left(\frac{\partial\varphi}{\partial q_i}, \psi\right) + \left(\varphi, \frac{\partial\psi}{\partial q_i}\right). \end{aligned}$$

Since the expression (φ, ψ) is a linear function of the differential coefficients of φ as well as the differential coefficients of ψ, the right sides of these equations are nothing but the derivatives of (φ, ψ) with respect to p_i and q_i, so,

$$\begin{aligned} C_i &= -\frac{\partial(\varphi, \psi)}{\partial p_i}, \\ C_{n+i} &= \frac{\partial(\varphi, \psi)}{\partial q_i}, \end{aligned}$$

and all the $2n$ equations of constraint $C_i = 0$, $C_{n+i} = 0$ are satisfied for $i = 1, 2, \ldots, n$, so long as

$$(\varphi, \psi) = 0$$

identically, i.e., so long as $f = \psi$ is a solution of the linear partial differential equation $A(f) = (\varphi, f) = 0$. If this one equation of condition

$$(\varphi, \psi) = 0$$

is satisfied, then there always exist simultaneous solutions of the equations

$$(\varphi, f) = 0, \quad (\psi, f) = 0,$$

and one can utilise the results of the preceding lecture for their determination.

The assertion made at the begining of the preceding lecture is hereby proved, according to which, if H_1 is any function whatsoever satisfying the condition $(H, H_1) = 0$, a second function H_2 can always be determined which satisfies simultaneously both the conditions $(H, H_2) = 0$ and $(H_1, H_2) = 0$. Indeed the investigation of the preceding lecture gives not only the proof of the existence, but also the means for the determination of H_2. The further continuation of the preceding investigation then gives, under the assumption of the function H_1 and H_2 so defined, the means of determining the new function H_3, which satisfies simultaneously the three conditions $(H_1, H_3) = 0$, $(H_2, H_3) = 0$, $(H, H_3) = 0$ and so on.

In the preceding lecture we have determined not only the simultaneous solutions of two linear partial differential equations $A(f) = 0$, $B(f) = 0$, which satisfy the conditions $C_i = B(A_i) - A(B_i) = 0$, but what is no less important, derived from *one* solution f_1 of $A(f) = 0$, through repeated application of the operation B, a series of new solutions

$B(f_1) = f_2$, $B(f_2) = f_3, \dots, B(f_{m-1}) = f_m$, until the repetition of that order leads to a solution $B(f_m) = f_{m+1}$ which is a function $F(f_1, f_2, \dots, f_m)$ of the previous ones, or is a constant which, in particular, can also be zero.

As we make an application of this to the present case we need a modification which rests on the following circumstance. In general $A(f) = 0$ possesses the obvious solution f =constant and besides this, and the solution $f = f_1$ which from the start is known to us. In the special case, however, where $A(f) = (\varphi, f)$, $B(f) = (\psi, f)$, while the equations of constraint $C_i = 0$ are satisfied through the identity $(\varphi, \psi) = 0$. If $f = f_1$ is a solution of $(\varphi, f) = 0$ we know already a second solution ψ besides f_1. Moreover, in addition to the obvious solution f = constant, there is the special solution $f = \varphi$. Hence f_{m+1} is not a new solution if it becomes equal to a function $F(\varphi, \psi, f_1, f_2, \dots, f_m)$, which contains φ and ψ besides $f_1, f_2, \dots, f_m$. With this in mind and without explicitly mentioning the case where the function F reduces to a constant including zero but including it in the notation $F(\varphi, \psi, f_1, f_2, \dots, f_m)$, we obtain the result:

If f_1 is a solution of the partial differential equation $(\varphi, f) = 0$ defining f, and if the constraint equation $(\varphi, \psi) = 0$ is satisfied, then $(\psi, f_1) = f_2$ is in turn a solution of $(\varphi, f) = 0$, and indeed in general a new solution. In particular cases, however, the solution can be a function $F(\varphi, \psi, f_1)$ of ψ, f_1 including the obvious one φ. If one continues this and sets $(\psi, f_2) = f_3$, $(\psi, f_3) = f_4, \dots$, $(\psi, f_{m-1}) = f_m$, $(\psi, f_m) = f_{m+1}$, one will in general obtain further new solutions $f_3, f_4, \dots, f_m$ of $(\varphi, f) = 0$, until f_{m+1} becomes a function $F(\varphi, \psi, f_1, f_2, \dots, f_m)$ of the already known $\psi, f_1, f_2, \dots, f_m$ and the obvious solution φ.

When one allows the function φ to coincide with the function H, which forms the left hand side of the partial differential equation $H = h$, then it is appropriate to change the remmaining notations also. We set $\varphi = H$, $\psi = H_1$, $f_1 = H_2$, $f_2 = H_3$ and so on, and the above result becomes:

If the equations $(H, H_1) = 0$ and $(H, H_2) = 0$ are satisfied, i.e., if H_1 and H_2 are solutions of the linear partial differential equation $(H, H_i) = 0$ defining H_i, then $(H_1, H_2) = H_3$ is likewise a solution of this differential equation and indeed in general a new solution. In special cases H_3 is a function of H, H_1, H_2. If one continues with this operation and sets $(H_1, H_3) = H_4$, $(H_1, H_4) = H_5, \dots, (H_1, H_{m-1}) = H_m$ $(H_1, H_m) = H_{m+1}$, one obtains in general new solutions $H_4, H_5, \dots, H_m$

of $(H, H_i) = 0$ until H_{m+1} is a function of the already known $H, H_1, \ldots, H_m$, including the obvious solution H.[1]

However, as we know, it is of equal significance whether we say that H_i are solutions of the linear partial differential equation $(H, H_i) = 0$ defining H_i, i.e., the equation

$$\begin{aligned}\frac{\partial H}{\partial p_1}\frac{\partial H_i}{\partial q_1} &+ \frac{\partial H}{\partial p_2}\frac{\partial H_i}{\partial q_2} + \cdots + \frac{\partial H}{\partial p_n}\frac{\partial H_i}{\partial q_n} - \\ -\frac{\partial H}{\partial q_1}\frac{\partial H_i}{\partial p_1} &- \frac{\partial H}{\partial q_2}\frac{\partial H_i}{\partial p_2} - \cdots - \frac{\partial H}{\partial q_n}\frac{\partial H_i}{\partial p_n} = 0,\end{aligned}$$

or whether we say that H_1 set equal to an arbitrary constant is an integral of the system of ordinary differential equations

$$\begin{aligned}&dq_1 : dq_2 : \ldots : dq_n : dp_1 : dp_2 : \ldots : dp_n \\ &= \frac{\partial H}{\partial p_1} : \frac{\partial H}{\partial p_2} : \ldots : \frac{\partial H}{\partial p_n} : -\frac{\partial H}{\partial q_1} : -\frac{\partial H}{\partial q_2} : \ldots : -\frac{\partial H}{\partial q_n},\end{aligned}$$

i.e., an integral free from t of the isoperimetric differential equations

$$\begin{aligned}\frac{dq_1}{dt} &= \frac{\partial H}{\partial p_1}, \frac{dq_2}{dt} = \frac{\partial H}{\partial p_2}, \ldots, \frac{dq_n}{dt} = \frac{\partial H}{\partial p_n}, \\ \frac{dp_1}{dt} &= -\frac{\partial H}{\partial q_1}, \frac{dp_2}{dt} = -\frac{\partial H}{\partial q_2}, \ldots, \frac{dp_n}{dt} = -\frac{\partial H}{\partial q_n},\end{aligned}$$

which, when one sets $H = T - U$, where T represents half the 'vis viva' and U the force function, transform into the system of differential equations of motion. We can therefore express the result obtained in the following form:

Let the system of isoperimetric differential equations

$$\begin{aligned}\frac{dq_1}{dt} &= \frac{\partial H}{\partial p_1}, \frac{dq_2}{dt} = \frac{\partial H}{\partial p_2}, \ldots, \frac{dq_n}{dt} = \frac{\partial H}{\partial p_n}, \\ \frac{dp_1}{dt} &= -\frac{\partial H}{\partial q_1}, \frac{dp_2}{dt} = -\frac{\partial H}{\partial q_2}, \ldots, \frac{dp_n}{dt} = -\frac{\partial H}{\partial q_n},\end{aligned}$$

in which H represents a function of the variables q_1, $q_2, \ldots, q_n$, p_1, $p_2, \ldots, p_n$, without t, and which for $H = T - U$ goes over to the system of differential equations of dynamics, be given. If one knows two

[1]It is not to be overlooked that the quantities $H_1, H_2, H_3, \ldots$ denote here any solutions whatsoever of the equation $(H, H_i) = 0$, and not the special system of these solutions which, set equal to constants, forms the equation leading to the complete solution of the partial differential equation $H = h$. (See Lecture 32, p.291)

integrals $H_1 = h_1$, $H_2 = h_2$ *of this system free from* t *and one forms the expression*

$$H_3 = (H_1, H_2) = \frac{\partial H_1}{\partial p_1}\frac{\partial H_2}{\partial q_1} + \frac{\partial H_1}{\partial p_2}\frac{\partial H_2}{\partial q_2} + \cdots + \frac{\partial H_1}{\partial p_n}\frac{\partial H_n}{\partial q_n} - \frac{\partial H_2}{\partial p_1}\frac{\partial H_1}{\partial q_1} - \frac{\partial H_2}{\partial p_2}\frac{\partial H_1}{\partial q_2} - \cdots - \frac{\partial H_2}{\partial p_2}\frac{\partial H_1}{\partial q_1},$$

then

$$H_3 = h_3,$$

where h_3 *denotes a third arbitrary constant, is in general a new integral of the system. In special cases* H_3 *may be a function of* H, H_1 *and* H_2, *or a constant numerical value, not excluding zero. In these cases* $H_3 = h_3$ *is not a new integral but an equation which is satisfied identically under the assumption of the earlier integrals* $H_1 = h_1$, $H_2 = h_2$ *and the obvious integral* $H = h$. *If one continues this operation and builds from* H_1 *and* H_3 *or* H_2 *and* H_3 *the expression* (H_1, H_3) *or* (H_2, H_3), *this set equal to a constant, gives a further new integral etc.*

This is one of the most remarkable theorems of the entire integral calculus, and for the special case in which one sets $H = T - U$, a fundamental theorem of analytical mechanics. Namely, it shows that when the theorem of *vis viva* holds, one can in general derive from two integrals of the differential equations of motion a third by a simple differentiation, from that a fourth, etc. So that one obtains either all integrals, or at least a number of them.

After I discovered this theorem I communicated it to the Academies of Berlin and Paris as an entirely new discovery. But I noticed soon after that this theorem had already been discovered and forgotton for 30 years, because one did not appreciate its real meaning, but had only used it as a lemma in an entirely different problem.

If one had integrated the above differential equations for a definite problem of mechanics and would, using the so-called perturbation theory developed by *Lagrange* and *Laplace*, determine the modifications of the motion by the addition of new smaller forces, then one would be led to a new expression put together from p_i and q_i, which is independent of time – a result which belongs to the greatest discoveries of the Geometers mentioned.*Poisson*, who carried out the investigation somewhat differently, found this expression independent of t to be precisely of the form (H_i, H_k). This theorem of *Poisson* was celebrated because of the difficulty of its proof. But one attached so little value to it that *Lagrange* did not mention it even once in the second edition of Mécanique

Analytique, but preferred his own formulas as simpler. But precisely this formula of *Poisson* agrees esstentially with the one given above. For, if those expressions (H_i, H_k) which for *Poisson* enter as coefficients in the perturbing function are independent of time, then they must be functions which in the original problem must be equal to constants. But this remark was missed by the Geometers earlier, and it needed in fact a fresh discovery to consider the true significane of the theorem.

A peculiar circumstance had contributed to the fact that nobody recognised the importance of this theorem discovered so long ago. Namely, the cases in which one applied the theorem were precisely those in which the newly constructed expression gave no new integral, but in which the resulting expression was identically equal to zero, or equal to a number different from zero, say $= 1$. These cases which in the general theory appear as exceptional are very frequent in practice. In order that an integral combined with a second should lead to all integrals one after another, must be one which is peculiar to the particular problem. But the first integrals which are found for a given problem are as a rule just those which follow from general principles (e.g. the conservation of surface area), and so are not peculiar to the specific problem. Therefore one cannot expect that all integrals could be derived from them.

We see that a certain polarity, i.e. a qualitative difference exists between the integrals. Earlier one did not know that any integral held for many equal values, and the only use one could make of them was to reduce the order of the system by one. But we see now that there exist certain integrals $H_1 = h_1$ and $H_2 = h_2$ from which one can derive all the othèr integrals. This case is indeed general. That is, if the equations $H_1 = h_1, H_2 = h_2, \ldots, H_m = h_m$ represent all the integrals and one constructs frm the left hand sides of these an arbitrary function

$$F(H_1, H_2, \ldots, H_m) = H_{m+1},$$

which can be given in advance, then one can, in an overwhelming majority of cases, derive from H_{m+1} and one of the given integrals, e.g. from H_{m+1} and H_1, all the remaining ones, and this is the general case, since H_{m+1} set equal to an arbitrary constant represents the most general form of an integral. The first integrals that one finds in the solution of a problem are not, as a rule, as H_{m+1} is, got together from those which belong specifically to the problem and the ones given by a general principle, but they are ordinary, only of a general nature, and therefore one does not obtain from them all the integrals of the problem.

The application of the general theorem to free motion gives the following theorem.

If one knows two integrals $\varphi = h_1$, $\psi = h_2$, independent of t, of the system

$$m\frac{d^2x_i}{dt^2} = \frac{\partial U}{dx_i}, m\frac{d^2y_i}{dt^2} = \frac{\partial U}{\partial y_i}, m\frac{d^2z_i}{dt^2} = \frac{\partial U}{\partial z_i},$$

and one constructs the expression

$$\begin{aligned}(\varphi,\psi) &= \sum\frac{1}{m_i}\left\{\frac{\partial\varphi}{\partial x_i'}\frac{\partial\psi}{\partial x_i}+\frac{\partial\varphi}{\partial y_i'}\frac{\partial\psi}{\partial y_i}+\frac{\partial\varphi}{\partial z_i'}\frac{\partial\psi}{\partial z_i}\right.\\ &\quad\left.-\frac{\partial\psi}{\partial x_i'}\frac{\partial\varphi}{\partial x_i}-\frac{\partial\psi}{\partial y_i'}\frac{\partial\varphi}{\partial y_i}-\frac{\partial\psi}{\partial z_i'}\frac{\partial\varphi}{\partial z_i}\right\}\end{aligned}$$

then, in general,

$$(\varphi,\psi) = h_3$$

is a new integral; in special cases, however, (φ,ψ) can also be a function of the constants h_1 h_2 and the constant h it occurring in the 'vis viva' $T - U = h$, is a pure numerical value, and indeed can also be equal to zero.

In this way one can derive from the two surface-area theorems the third one. For this purpose we have only to set

$$\varphi = \sum m_i(x_iy_i' - y_ix_i'), \psi = \sum m_i(x_iz_i' - z_ix_i'),$$

then

$$\begin{aligned}\frac{\partial\varphi}{\partial x_i} &= m_iy_i', \frac{\partial\varphi}{\partial y_i} = -m_ix_i', \frac{\partial\varphi}{\partial z_i} = 0,\\ \frac{\partial\varphi}{\partial x_i'} &= m_iy_i, \frac{\partial\varphi}{\partial y_i} = -m_ix_i, \frac{\partial\varphi}{\partial z_i'} = 0,\\ \frac{\partial\psi}{\partial x_i} &= m_iz_i', \frac{\partial\varphi}{\partial y_i} = 0, \frac{\partial\psi}{\partial z_i} = -m_ix_i',\\ \frac{\partial\varphi}{\partial x_i'} &= m_iz_i, \frac{\partial\psi}{\partial y_i'} = 0, \frac{\partial\psi}{\partial z_i'} = m_ix_i;\end{aligned}$$

therefore

$$(\varphi,\psi) = \sum m_i(y_i'z_i - y_iz_i'),$$

and so the third surface area theorem is

$$(\varphi,\psi) = h_3.$$

Poisson, in his famous treatise on the variation of constants in Vol.15 of the Journal of the Polytechnic School, makes an application of the perturbation theorems mentioned above to the perturbations of a rotatory motion about a fixed point. For this he is required to carry out the same computational operations as those which we have just made. That is why his computations contain the derivation of the third surface-area theorem, but he does not utter a single word on this remarkable result.

Similar observations can be made if one adds to the three surface-area theorems the three equations of conservation of the motion of the centre of gravity and investigate from how many of these six integrals the remaining arise.

Lecture 35

The two classes of integrals which one obtains according to *Hamilton*'s method for problems of mechanics. Determination of the value of (φ, ψ) for them

If from the system of differential equations

$$\begin{aligned} dt &: dq_1 : dq_2 : \ldots : dq_n : dp_1 : dp_2 : \ldots : dp_n \\ &= 1 : \frac{\partial H}{\partial p_1} : \frac{\partial H}{\partial p_2} : \ldots : \frac{\partial H}{\partial p_n} : -\frac{\partial H}{\partial q_1} : -\frac{\partial H}{\partial q_2} : \ldots : -\frac{\partial H}{\partial q_n}, \end{aligned} \tag{35.1}$$

which has obviously the integral $H = h$, two integrals $H_1 = h_1$, $H_2 = h_2$, independent of t, are given, one cannot, as we have seen, in general say a priori whether (H_1, H_2) set equal to an arbitrary constant is a new integral or whether it reduces to a constant dependent on h, h_1, h_2, or to a pure number, or even to zero. This question can be decided completely if $H_1 = h_1$ and $H_2 = h_2$ are integrals which belong to the system provided by *Hamilton*'s partial differential equation. Indeed, we shall see that if φ =constant and ψ =constant are two of Hamilton's integrals then (φ, ψ) is either $= 0$ or $= \pm 1$. Two integrals of this system then never give a new integral. In order to prove this theorem we need a lemma that shows what the expression (φ, ψ) becomes if, in φ and ψ, besides the quantities $q_1, q_2, \ldots, q_n$, $p_1, p_2, \ldots, p_n$, also other quantities $\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_k, \ldots, \tilde{\omega}_m$ occur, which are functions of $q_1, q_2, \ldots, q_n$ and $p_1, p_2, \ldots, p_n$. In this case one can form the differential coefficients of φ and ψ in the variables p and q, as well as of the expression (φ, ψ), in two different ways, according as one does take account of the variables p and q in $\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_k, \ldots, \tilde{\omega}_n$ or not. If we denote the differential coefficients formed in these two ways of φ and ψ with and without brackets, and the expression built from φ

and ψ with double brackets $((\varphi,\psi))$ or with simple brackets (φ,ψ), then

$$((\varphi,\psi)) = \sum_i \left\{ \left(\frac{\partial\varphi}{\partial p_i}\right)\left(\frac{\partial\psi}{\partial q_i}\right) - \left(\frac{\partial\psi}{\partial p_i}\right)\left(\frac{\partial\varphi}{\partial q_i}\right) \right\} \tag{35.2}$$

$$(\varphi,\psi) = \sum_i \left\{ \frac{\partial\varphi}{\partial p_i}\frac{\partial\psi}{\partial q_i} - \frac{\partial\psi}{\partial p_i}\frac{\partial\varphi}{\partial q_i} \right\}. \tag{35.3}$$

The summation over i extends over $1,2,\ldots,n$, and for the bracketed differential coefficients in (35.2) the following equations hold,

$$\left(\frac{\partial\varphi}{\partial p_i}\right) = \frac{\partial\varphi}{\partial p_i} + \sum_k \frac{\partial\varphi}{\partial\tilde{\omega}_k}\frac{\partial\tilde{\omega}_k}{\partial p_i}, \qquad \left(\frac{\partial\psi}{\partial p_i}\right) = \frac{\partial\psi}{\partial p_i} + \sum_{k'} \frac{\partial\psi}{\partial\tilde{\omega}_{k'}}\frac{\partial\tilde{\omega}_{k'}}{\partial p_i},$$

$$\left(\frac{\partial\varphi}{\partial q_i}\right) = \frac{\partial\varphi}{\partial q_i} + \sum_k \frac{\partial\varphi}{\partial\tilde{\omega}_k}\frac{\partial\tilde{\omega}_k}{\partial q_i}, \qquad \left(\frac{\partial\psi}{\partial q_i}\right) = \frac{\partial\psi}{\partial q_i} + \sum_{k'} \frac{\partial\psi}{\partial\tilde{\omega}_{k'}}\frac{\partial\tilde{\omega}_{k'}}{\partial q_i},$$

in which the summations over k and k' are to be taken from 1 to m. If one substitutes these in the expression (35.2), one obtains a simple sum in i, a double sum in i and k(or k'), and a triple sum in i,k,k'. Namely,

$$\begin{aligned} ((\varphi,\psi)) = & \sum_i \left(\frac{\partial\varphi}{\partial p_i}\frac{\partial\psi}{\partial q_i} - \frac{\partial\psi}{\partial p_i}\frac{\partial\varphi}{\partial q_i} \right) + \\ & + \sum_i\sum_{k'} \left(\frac{\partial\varphi}{\partial p_i}\frac{\partial\psi}{\partial\tilde{\omega}_{k'}}\frac{\partial\tilde{\omega}_{k'}}{\partial q_i} - \frac{\partial\varphi}{\partial q_i}\frac{\partial\psi}{\partial\tilde{\omega}_{k'}}\frac{\partial\tilde{\omega}_{k'}}{\partial p_i} \right) \\ & - \sum_i\sum_k \left(\frac{\partial\psi}{\partial p_i}\frac{\partial\varphi}{\partial\tilde{\omega}_k}\frac{\partial\tilde{\omega}_k}{\partial q_i} - \frac{\partial\psi}{\partial q_i}\frac{\partial\varphi}{\partial\tilde{\omega}_k}\frac{\partial\tilde{\omega}_k}{\partial p_i} \right) \\ & + \sum_i\sum_k\sum_{k'} \frac{\partial\varphi}{\partial\tilde{\omega}_k}\frac{\partial\psi}{\partial\tilde{\omega}_{k'}} \left(\frac{\partial\tilde{\omega}_k}{\partial p_i}\frac{\partial\tilde{\omega}_{k'}}{\partial q_i} - \frac{\partial\tilde{\omega}_{k'}}{\partial p_i}\frac{\partial\tilde{\omega}_k}{\partial q_i} \right), \end{aligned}$$

and if one reverses the order of summation in the double and triple sums and takes into account the definition given in (35.3) of the expression (φ,ψ) enclosed in simple brackets, one gets,

$$\begin{aligned} ((\varphi,\psi)) = & (\varphi,\psi) + \sum_{k'} \frac{\partial\psi}{\partial\tilde{\omega}_{k'}}(\varphi,\tilde{\omega}_{k'}) - \sum_k \frac{\partial\varphi}{\partial\tilde{\omega}_k}(\psi,\tilde{\omega}_k) \\ & + \sum_k\sum_{k'} \frac{\partial\varphi}{\partial\tilde{\omega}_k}\frac{\partial\psi}{\partial\tilde{\omega}_k}(\tilde{\omega}_k,\tilde{\omega}_{k'}). \end{aligned}$$

Since the summations in k and k' are extended over the same values from 1 to m, one can write k in place of k' in the second term. In the last term the numbers for which the values of k and k' coincide vanish because of the factor $(\tilde{\omega}_k, \tilde{\omega}_{k'})$. In the remaining numbers one can combine every two into one since $(\tilde{\omega}_{k'}, \tilde{\omega}_k) = -(\tilde{\omega}_k, \tilde{\omega}_{k'})$. Therefore one needs to take this sum only for combinations of any two values of k and k' different from each other and take $(\tilde{\omega}_k, \tilde{\omega}_{k'})$, multiplied by $\left(\frac{\partial\varphi}{\partial\tilde{\omega}_k}\frac{\partial\psi}{\partial\tilde{\omega}_{k'}} - \frac{\partial\psi}{\partial\tilde{\omega}_k}\frac{\partial\varphi}{\partial\tilde{\omega}_{k'}}\right)$; so one obtains finally

$$\begin{aligned}((\varphi,\psi)) &= (\varphi,\psi) + \sum_k \frac{\partial\psi}{\partial\tilde{\omega}_k}(\varphi,\tilde{\omega}_k) - \sum_k \frac{\partial\varphi}{\partial\tilde{\omega}_k}(\psi,\tilde{\omega}_k) \\ &\quad + \sum_{k,k'}\left(\frac{\partial\varphi}{\partial\tilde{\omega}_k}\frac{\partial\psi}{\partial\tilde{\omega}_{k'}} - \frac{\partial\varphi}{\partial\tilde{\omega}_{k'}}\frac{\partial\psi}{\partial\tilde{\omega}_k}\right)(\tilde{\omega}_k,\tilde{\omega}_{k'}). \qquad (35.4)\end{aligned}$$

For later use, we shall specialize the formula (35.4) by substituting for the quantities $\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n$, the n functions H, $H_1, \ldots, H_{n-1}$ already considered earlier,[1] free of arbitrary constants, and depending only on the values of the variables $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$, which, set equal to n mutually independent arbitrary constants $h, h_1, \ldots, h_{n-1}$, determine the variables p_1, p_2, $\ldots, p_n$ represented as functions of the variables q_1, $q_2, \ldots, q_n$, so that

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n$$

is a complete differential and its integral a complete solution V of the partial differential equation $H = h$. Then, as we have seen, we have evidently

$$(H_k, H_{k'}) = 0,$$

and consequently, in the general formula (35.4), the double sum in k and k' vanishes, and we obtain

$$((\varphi,\psi)) = (\varphi,\psi) + \sum_k \frac{\partial\psi}{\partial H_k}(\varphi, H_k) - \sum_k \frac{\partial\varphi}{\partial H_k}(\psi, H_k), \qquad (35.5)$$

where the summation is extended from $k = 0$ to $k = n - 1$.

We now specialize this formula still further. According to our assumption upto now, the functions φ and ψ contain the variables p and q first, explicitly, and secondly, implicitly, through the functions $H, H_1, \ldots, H_{n-1}$. At present we assume that the functions φ and ψ contain the variables p only in the second way, that is, *only implicitly,* a form

[1] See Lecture 32, p.283

which can always be arrived at through the introduction of the n quantities H as the new variables in place of the n quantities q. Consequently by this φ and ψ are expressed solely by $q_1, q_2, \ldots, q_n,\ H, H_1, \ldots, H_{n-1}$. So with this hypothesis there is an essential simplification for the expressions appearing in equation (35.5):

$$\begin{aligned}(\varphi,\psi) &= \sum_i \left(\frac{\partial\varphi}{\partial p_i}\frac{\partial\psi}{\partial q_i} - \frac{\partial\varphi}{\partial q_i}\frac{\partial\psi}{\partial p_i}\right),\\ (\varphi,H_k) &= \sum_i \left(\frac{\partial\varphi}{\partial p_i}\frac{\partial H_k}{\partial q_i} - \frac{\partial\varphi}{\partial q_i}\frac{\partial H_k}{\partial p_i}\right),\\ (\psi,H_k) &= \sum_i \left(\frac{\partial\psi}{\partial p_i}\frac{\partial H_k}{\partial q_i} - \frac{\partial\psi}{\partial q_i}\frac{\partial H_k}{\partial p_i}\right).\end{aligned}$$

Since the differential coefficients $\frac{\partial\varphi}{\partial p_i}, \frac{\partial\psi}{\partial p_i}$ vanish for all values of i,

$$(\varphi,\psi) = 0, (\varphi,H_k) = -\sum_i \frac{\partial\varphi}{\partial q_i}\frac{\partial H_k}{\partial p_i}, (\psi,H_k) = -\sum_i \frac{\partial\psi}{\partial q_i}\frac{\partial H_k}{\partial p_i},$$

and the general expression (35.5) for $((\varphi,\psi))$ now takes the simple form

$$((\varphi,\psi)) = -\sum_k \frac{\partial\psi}{\partial H_k}\sum_i \frac{\partial\varphi}{\partial q_i}\frac{\partial H_k}{\partial p_i} + \sum_k \frac{\partial\varphi}{\partial H_k}\sum_i \frac{\partial\psi}{\partial q_i}\frac{\partial H_k}{\partial p_i}. \qquad (35.6)$$

In this equation is contained the specialization of lemma (35.4) which we have to use for the consideration of Hamilton's form of the integrals.

In order to write down, under these assumptions, the integrals of the system of differential equation (35.1) in *Hamilton*'s form, let, with the notaion already used,

$$H = h, H_1 = h_1, \ldots, H_{n-1} = h_{n-1}$$

be the equations which so determine the values of $p_1, p_2, \ldots, p_n$, that

$$V = \int (p_1 dq_1 + p_2, dq_2 + \cdots + p_n dq_n)$$

is a complete solution of the partial differential equation $H = h$. Then, as we know,[2] the integral equations of the system (35.1) in *Hamilton*'s

[2]See lecture 20, p.184

form are

$$\begin{aligned}\frac{\partial V}{\partial q_1} &= p_1, \frac{\partial V}{\partial q_2} = p_2, \ldots, \frac{\partial V}{\partial q_n} = p_n, \\ \frac{\partial V}{\partial h} &= t + h', \frac{\partial V}{\partial h_1} = h'_1, \ldots, \frac{\partial V}{\partial h_{n-1}} = h'_{n-1},\end{aligned} \tag{35.7}$$

where $h', h'_1, \ldots, h'_{n-1}$ denote new arbitrary constants. But these integral equations are not all as yet solved for the arbitrary constants. In order to obtain them in this form, i.e. as *integrals*, in our terminology, let us substitute for the first half of the integral equations (35.7) the integrals equivalent to them

$$H = h, H_1 = h_1, \ldots, H_{n-1} = h_{n-1},$$

and in the second half of the same, which are already solved for the arbitrary constants $h', h'_1, \ldots, h'_{n-1}$, we substitute for h, h_1, $\ldots, h_{n-1}$ their values $H, H_1, \ldots, H_{n-1}$. Then, if H', H'_1, $\ldots, H'_{n-1}$ denote the functions of the variables $q_1, q_2, \ldots, q_n$, $p_1, p_2, \ldots, p_n$ into which the quantities $\frac{\partial V}{\partial h}, \frac{\partial V}{\partial h_1}, \ldots, \frac{\partial V}{\partial h_{n-1}}$ go over through this substitution, we obtain the integral equations of the second row of the system (35.7) in the form of the integrals

$$H' = t + h', H'_1 = h'_1, H'_2 = h'_2, \ldots, H'_{n-1} = h'_{n-1}.$$

The quantities $H', H'_1, \ldots, H'_{n-1}$ contain the variables $p_1, p_2, \ldots, p_n$ only implicitly by means of the quantities $H, H_1, \ldots, H_{n-1}$, since the function V and its differential coefficients $\frac{\partial V}{\partial h}, \frac{\partial V}{\partial h_1}, \ldots, \frac{\partial V}{\partial h_{n-1}}$ depend only on $q_1, q_2, \ldots, q_n$, $H, H_1, \ldots, H_{n-1}$. Then H', H'_1, $\ldots$, H'_{n-1} are precisely of the form in which the quantities φ and ψ in equation (35.6) have been represented under our assumption. The same holds, as is self-evident, for the quantities $H, H_1, \ldots, H_{n-1}$ if we consider them as functions of themselves, only then also the variables $q_1, q_2, \ldots, q_n$ do not come into them explicitly. The formula (35.6) for $((\varphi, \psi))$ then can be applied to expressions of the form $((H'_\alpha, H'_\beta))$ or $((H'_\alpha, H_\beta))$, where we shall omit double brackets from now on for simplifying our notation.

If in (35.6) one sets, first, $\varphi = H'_\alpha$ and $\psi = H'_\beta$, where α and β represent numbers of the series $0, 1, \ldots, n-1$, then we get,

$$(H'_\alpha, H'_\beta) = -\sum_k \frac{\partial H'_\beta}{\partial H_k} \sum_i \frac{\partial H'_\alpha}{\partial q_i} \frac{\partial H_k}{\partial p_i} + \sum_k \frac{\partial H'_\alpha}{\partial H_k} \sum_i \frac{\partial H'_\beta}{\partial q_i} \frac{\partial H_k}{\partial p_i}. \tag{35.8}$$

But, according, to our definition of H'_α,

$$H'_\alpha = \frac{\partial V}{\partial h_\alpha},$$

assuming that in $\frac{\partial V}{\partial h_\alpha}$ the H_k's are substituted for the h_k's Since from the equation

$$V = \int (p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n)$$

which determines V, the expression

$$\frac{\partial V}{\partial h_\alpha} = \int \left(\frac{\partial p_1}{\partial h_\alpha} dq_1 + \frac{\partial p_2}{\partial h_\alpha} dq_2 + \cdots + \frac{\partial p_n}{\partial h_\alpha} dq_n \right)$$

follows for the differential coefficients of V with respect to h_α. Hence partial differentiation with respect to q_i, gives

$$\frac{\partial \left(\frac{\partial V}{\partial h_\alpha} \right)}{\partial q_i} = \frac{\partial p_i}{\partial h_\alpha}.$$

After replacement of the quantities h_k by the corresponding quantities H'_k, one obtains,

$$\frac{\partial H'_\alpha}{\partial q_i} = \frac{\partial p_i}{\partial H_\alpha}. \qquad (35.9)$$

On using this equation, the sum over i occurring in formula (35.8) has the simple expression,

$$\begin{aligned} \sum_i \frac{\partial H'_\alpha}{\partial q_i} \frac{\partial H_k}{\partial p_i} &= \sum_i \frac{\partial H_k}{\partial p_i} \frac{\partial p_i}{\partial H_\alpha} = \frac{\partial H_k}{\partial H_\alpha}, \\ \sum_i \frac{\partial H'_\beta}{\partial q_i} \frac{\partial H_k}{\partial p_i} &= \sum_i \frac{\partial H_k}{\partial p_i} \frac{\partial p_i}{\partial H_\beta} = \frac{\partial H_k}{\partial H_\beta}, \end{aligned}$$

and (35.8) goes over to

$$(H'_\alpha, H'_\beta) = -\sum_k \frac{\partial H'_\beta}{\partial H_k} \frac{\partial H_k}{\partial H_\alpha} + \sum_k \frac{\partial H'_\alpha}{\partial H_k} \frac{\partial H_k}{\partial H_\beta},$$

or, since $\frac{\partial H_k}{\partial H_\alpha}$ vanishes for all values of k different from α, and equals 1 when $k = \alpha$,

$$(H'_\alpha, H'_\beta) = -\frac{\partial H'_\beta}{\partial H_\alpha} + \frac{\partial H'_\alpha}{\partial H_\beta}.$$

The right hand side of this equation is equal to zero; for, if V' denotes the function into which V changes when the h_k are replaced by the corresponding H_k, then

$$H' = \frac{\partial V'}{\partial H}, H_1' = \frac{\partial V'}{\partial H_1}, H_2' = \frac{\partial V'}{\partial H_2}, \dots, H_n' = \frac{\partial V'}{\partial H_n},$$

hence

$$\frac{\partial H_\alpha'}{\partial H_\beta} = \frac{\partial H_\beta'}{\partial H_\alpha},$$

and therefore it follows

$$(H_\alpha', H_\beta') = 0.$$

In order to transform expressions of the form (H_α', H_β) we set, in (35.6), $\varphi = H_\alpha', \psi = H_\beta$ for φ and ψ. Then we have

$$(H_\alpha', H_\beta) = -\sum_k \frac{\partial H_\beta}{\partial H_k} \sum_i \frac{\partial H_\alpha'}{\partial q_i}\frac{\partial H_k}{\partial p_i} + \sum_k \frac{\partial H_\alpha'}{\partial H_k} \sum_i \frac{\partial H_\beta}{\partial q_i}\frac{\partial H_k}{\partial p_i}. \qquad (35.10)$$

Using the equation (35.9), the first sum over i becomes,

$$\sum_i \frac{\partial H_\alpha'}{\partial q_i}\frac{\partial H_k}{\partial p_i} = \sum_i \frac{\partial H_k}{\partial p_i}\frac{\partial p_i}{\partial H_\alpha} = \frac{\partial H_k}{\partial H_\alpha}.$$

The second sum over i, on the contrary, vanishes, because we look upon $q_1, q_2, \dots, q_n$ and $H, H_1, \dots, H_{n-1}$ as independent variables. Thus H_β contains no q_i and the differential coefficients $\frac{\partial H_\beta}{\partial q_i}$ are all equal to zero. In this way, equation (35.10) changes into

$$(H_\alpha', H_\beta) = -\sum_k \frac{\partial H_\beta}{\partial H_k}\frac{\partial H_k}{\partial H_\alpha} = -\frac{\partial H_\beta}{\partial H_\alpha},$$

and since $\frac{\partial H_\beta}{\partial H_\alpha}$ is equal to either 0 or 1 according to whether β is different from or the same as α. Therefore, one has, for any two mutually distinct values of α and β,

$$(H_\alpha', H_\beta) = 0,$$

and on the other hand, when $\alpha = \beta$,

$$(H_\alpha', H_\alpha) = -1.$$

Finally, according to the equations of constraint through which the quantities H are defined,

$$(H_\alpha, H_\beta) = 0.$$

We have thus obtained the following identical equations for H_α and H'_α:

$$(H_\alpha, H_\beta) = 0, \quad (H'_\alpha, H'_\beta) = 0, \quad (H_\alpha, H'_\beta) = 0,$$

of which the first two hold for all values of α and β, the last however only for mutually distinct values of α and β, while for $\alpha = \beta$, the equation

$$(H_\alpha, H'_\alpha) = 1$$

holds. One can combine these results in the following theorem:

If the system of isoperimetric differential equations

$$\begin{aligned} \frac{dq_1}{dt} &= \frac{\partial H}{\partial p_1}, \frac{dq_2}{dt} = \frac{\partial H}{\partial p_2}, \ldots, \frac{dq_n}{dt} = \frac{\partial H}{\partial p_n}, \\ \frac{dp_1}{dt} &= -\frac{\partial H}{\partial q_1}, \frac{dp_2}{dt} = -\frac{\partial H}{\partial q_2}, \ldots, \frac{dp_n}{dt} = -\frac{\partial H}{\partial q_n}, \end{aligned} \qquad (35.1)$$

be given, in which H denotes a given function of the variables q_1, q_2, $\ldots, q_n$, p_1, $p_2, \ldots, p_n$, and which for $H = T - U$ go over into the system of differential equations of dynamics in case the principle of conservation of vis viva holds. One considers the partial differential equation

$$H = h,$$

in which we set $p_1 = \frac{\partial V}{\partial q_1}, p_2 = \frac{\partial V}{\partial q_2}, \ldots, p_n = \frac{\partial V}{\partial q_n}$, to which the system can be reduced. Let

$$H_1 = h_1, H_2 = h_2, \ldots, H_{n-1} = h_{n-1}$$

be the equations which, along with $H = h$, so determine p_1, p_2, $\ldots, p_n$ as functions of q_1, $q_2, \ldots, q_n$ that

$$p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n$$

is a complete differential, and its integral

$$V = \int (p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n)$$

is a complete solution of the partial differential equation $H = h$. If one now denotes by $H', H'_1, \ldots, H'_{n-1}$ the functions of the variables $q_1, q_2, \ldots, q_n$, $p_1, p_2, \ldots, p_n$ to which the differential coefficients

$\frac{\partial V}{\partial h}, \frac{\partial V}{\partial h_1}, \ldots, \frac{\partial V}{\partial h_{n-1}}$ *go over, when the constants* $h, h_1, \ldots, h_{n-1}$ *are replaced by the functions* $H, H_1, \ldots, H_{n-1}$*, and one puts the integrals belonging to the system of differential equations (35.1) in the Hamiltonian form, i.e. in the equations*

$$\begin{aligned} H = h, H_1 = h_1, \qquad & H_2 = h_2, \ldots, H_{n-1} = h_{n-1}, \\ H' = t + h', H_1' = h_1' \qquad & H_2' = h_2', \ldots, H_{n-1}' = h_{n-1}', \end{aligned}$$

then the $2n$ *functions* $H, H_1, \ldots, H_{n-1}$, $H', H_1', \ldots, H_{n-1}'$ *which form the left hand sides of these integrals have the property that when one substitutes in the expression*

$$\begin{aligned} (\varphi, \psi) \quad = \quad & \frac{\partial \varphi}{\partial p_1}\frac{\partial \psi}{\partial q_1} + \frac{\partial \varphi}{\partial p_2}\frac{\partial \psi}{\partial q_2} + \cdots + \frac{\partial \varphi}{\partial p_n}\frac{\partial \psi}{\partial q_n} \\ & -\frac{\partial \psi}{\partial p_1}\frac{\partial \varphi}{\partial q_1} - \frac{\partial \psi}{\partial p_2}\frac{\partial \varphi}{\partial q_2} - \cdots - \frac{\partial \psi}{\partial p_n}\frac{\partial \varphi}{\partial q_n} \end{aligned}$$

any two of the $2n$ *quantities* $H, H_1, \ldots, H_{n-1}$, $H', H_1', \ldots, h_{n-1}'$ *for* φ *and* ψ*, it vanishes, with certain exceptions of combinations of* H *and* H', H_1 *and* $H_1', \ldots, H_{n-1}$ *and* H_{n-1}'*, any of which when substituted for* φ *and* ψ*, makes the expression* (φ, ψ) *equal to unity.*

By means of this theorem one can obtain very simple formulas for the variations of constant, which will form the subject matter of the next lecture.

Lecture 36

Perturbation theory

When the theory of variations of constants is applied in Dynamics, one assumes that the system of differential equations of motion changes in that to the characteristic function H a characteristic function Ω is added, which may contain besides the variables $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$, also time explicitly, so that the differential equations change into the following:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} + \frac{\partial \Omega}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} - \frac{\partial \Omega}{\partial q_i}. \tag{36.1}$$

If Ω is small compared to H, one can use the values of the variables p_i and q_i of the unperturbed problem (with $\Omega = 0$) as approximate values in the perturbed problem, and so represent the new values p_i and q_i that they have the same analytic form, but that in place of the earlier arbitrary constants (or elements, in astronomical terminology) new functions of time appear. Unlike in the unperturbed problem where the variables p_i and q_i are to be looked upon as variables to be determined,in the perturbed problem one seeks rather those functions which occur in the place of the arbitrary constants or elements, i.e. the perturbed elements will be the variables of the new problem. This guarantees the advantage that one obtains as first approximations, not functions of time which have constants, but the constants themselves, the elements of the unperturbed problem.

We are now concerned with the setting up of the differential equations for the perturbed elements. We remind ourselves first of Hamilton's form of the integrals of the unperturbed problem, so the system considered in the preceding lecture:

$$\begin{aligned} H &= h, \qquad H_1 = h_1, \ldots, H_{n-1} = h_{n-1}, \\ H' &= h' + t, \qquad H_1' = h_1', \ldots, H_{n-1}' = h_{n-1}', \end{aligned} \tag{36.2}$$

and we denote any integral independent of t of the unperturbed problem by

$$\varphi = a,$$

where φ is a function of the variables $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$, and a represents an arbitrary constant, so that φ must represented as a function of $2n-1$ variables $H, H_1, \ldots, H_{n-1}, H'_1, \ldots, H'_{n-1}$ and a similar function of $2n-1$ constants $h, h_1, \ldots, h_{n-1}, h'_1, \ldots, h'_{n-1}$. In the perturbed problem a is no longer a constant, so that $\frac{da}{dt}$ is no more equal to zero, and one obtains more terms for $\frac{da}{dt}$, on using the differential equations (36.1), the expression

$$\begin{aligned} \frac{da}{dt} &= \sum_{i=1}^{i=n} \left(\frac{\partial \varphi}{\partial q_i} \frac{\partial q_i}{dt} + \frac{\partial \varphi}{\partial p_i} \frac{dp_i}{dt} \right) \\ &= \sum_{i=1}^{i=n} \left(\frac{\partial \varphi}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \varphi}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \\ &\quad + \sum_{i=1}^{i=n} \left(\frac{\partial \varphi}{\partial q_i} \frac{\partial \Omega}{\partial p_i} - \frac{\partial \varphi}{\partial p_i} \frac{\partial \Omega}{\partial q_i} \right), \end{aligned}$$

or, what is the same,

$$\frac{da}{dt} = (H, \varphi) + (\Omega, \varphi). \tag{36.3}$$

Since $\varphi = a$ is an integral independent of t of the unperturbed problem, φ satisfies the linear partial differential equation $(H, \varphi) = 0$, and the expression for $\frac{da}{dt}$ reduces to

$$\frac{da}{dt} = (\Omega, \varphi). \tag{36.3*}$$

The right hand side of this equation contains, besides t appearing explicitly in Ω, the $2n$ variables $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$, for which, however, we wish to introduce the $2n$ functions $H, H_1, \ldots, H_{n-1}$, $H'_1, H'_1, \ldots, H'_{n-1}$ as new variables. The introduction of these new variables in Ω transforms (Ω, φ) to

$$(\Omega, \varphi) = \sum_{k=0}^{k=n-1} \frac{\partial \Omega}{\partial H_k} (H_k, \varphi) + \sum_{k=0}^{k=n-1} \frac{\partial \Omega}{\partial H'_k} (H'_k, \varphi). \tag{36.4}$$

If we now introduce the new variables also in φ and take into account that φ is independent of one of them, H', so that $\frac{\partial\varphi}{\partial H'}$ vanishes, then we obtain for $(H_k,\varphi), (H'_k,\varphi)$ the transformation

$$\begin{aligned}(H_k,\varphi) &= \sum_{s=0}^{s=n-1}\frac{\partial\varphi}{\partial H_s}(H_k,H_s)+\sum_{s=1}^{s=n-1}\frac{\partial\varphi}{\partial H'_s}(H_k,H'_s),\\ (H'_k,\varphi) &= \sum_{s=0}^{s=n-1}\frac{\partial\varphi}{\partial H_s}(H'_k,H_s)+\sum_{s=1}^{s=n-1}\frac{\partial\varphi}{\partial H'_s}(H'_k,H'_s).\end{aligned}$$

But according to the theorem proved in the previous lecture, all the expressions (H_k,H_s), (H_k,H'_s), (H'_k,H_s), (H'_k,H'_s), vanish, with the exception of those (H_k,H'_s), (H'_k,H_s) in which k and s have the same value, and of these the first will be equal to $+1$ and the last to -1. Thereby the expression for (H_k,φ), (H'_k,φ), reduce to the simple values

$$(H_k,\varphi)=\frac{\partial\varphi}{\partial H'_k},\,(H'_k,\varphi)=-\frac{\partial\varphi}{\partial H_k}.$$

As a consequence, equation (36.4) goes over into

$$(\Omega,\varphi)=\sum_{k=1}^{k=n-1}\frac{\partial\Omega}{\partial H_k}\frac{\partial\varphi}{\partial H'_k}-\sum_{k=0}^{k=n-1}\frac{\partial\Omega}{\partial H'_k}\frac{\partial\varphi}{\partial H_k},$$

and equation (36.3)* finally gives for $\frac{da}{dt}$ the value

$$\frac{da}{dt}=\sum_{k=1}^{k=n-1}\frac{\partial\Omega}{\partial H_k}\frac{\partial\varphi}{\partial H'_k}-\sum_{k=0}^{k=n-1}\frac{\partial\Omega}{\partial H'_k}\frac{\partial\varphi}{\partial H_k}. \tag{36.5}$$

The partial differential coefficients of the perturbation function are here multiplied by the quantities $\frac{\partial\varphi}{\partial H'_k}$ and $-\frac{\partial\varphi}{\partial H_k}$, so by expressions which do not contain t explicitly, since t does not come in φ. This is the famous theorem of *Poisson*.

If we specialize the formula (36.5) by substituting for φ the individual functions $H, H_1,\ldots,H_{n-1}$, $H'_1,\ldots,H'_{n-1}$, and accordingly for a, the quantities $h,h_1,\ldots,h_{n-1}$, $h'_1,\ldots,h'_{n-1}$ at the same time, then we obtain, for $k=0,1,\ldots,n-1$,

$$\frac{dh_k}{dt}=-\frac{\partial\Omega}{\partial H'_k}, \tag{36.6}$$

and for $k = 1, \ldots, n-1$,

$$\frac{dh'_k}{dt} = \frac{\partial \Omega}{\partial H_k}. \tag{36.7}$$

It now remains to consider that integral of the unperturbed problem through which the time is introduced, i.e., the integral

$$H' = h' + t.$$

Since now $h' + t$ enters in place of a and H' in place of φ, so equation (36.3) changes into

$$\frac{dh'}{dt} + 1 = (H, H') + (\Omega, H'),$$

and since $(H, H') = 1$, one obtains

$$\frac{dh'}{dt} = (\Omega, H'),$$

an equation precisely of the form (36.3)*, only h' and H' appear in place of a and φ. Since in equation (36.4) H' can be introduced in place of φ, one obtains (Ω, H') is equal to the partial differential coefficient $\frac{\partial \Omega}{\partial H}$, and so finally

$$\frac{dh'}{dt} = \frac{\partial \Omega}{\partial H},$$

i.e. equation (36.7) holds also for $k = 0$.

The equations (36.2) which give the integrals of the unperturbed problem, are for the perturbed only the defining equations of the new variables $h, h_1, \ldots, h_{n-1}$, $h', h'_1, \ldots, h'_{n-1}$, and serve to express the old variables $q_1, q_2, \ldots, q_n$, $p_1, p_2, \ldots, p_n$ or functions of these, H, $H_1, \ldots, H_{n-1}$, $H', H'_1, \ldots, H'_{n-1}$, through the new variables. While one carries out these substitutions in the perturbation function, so replaces in it H, H_1, $\ldots, H_{n-1}$, H', $H'_1, \ldots, H'_{n-1}$ by $h, h_1, \ldots, h_{n-1}$, $h' + t, h'_1, \ldots, h'_{n-1}$ the differential coefficients $\frac{\partial \Omega}{\partial H_k}, \frac{\partial \Omega}{\partial H'_k}$ go over to $\frac{\partial \Omega}{\partial h_k}, \frac{\partial \Omega}{\partial h'_k}$, and one obtains for the variables, which in the perturbed problem take the place of the constants of the unperturbed, the differential equations

$$\begin{aligned} \frac{dh}{dt} &= -\frac{\partial \Omega}{\partial h'}, \frac{dh_1}{dt} = -\frac{\partial \Omega}{\partial h'_1}, \ldots, \frac{dh_{n-1}}{dt} = -\frac{\partial \Omega}{\partial h'_{n-1}}, \\ \frac{dh'}{dt} &= \frac{\partial \Omega}{\partial h}, \frac{dh'_1}{dt} = \frac{\partial \Omega}{\partial h_1}, \ldots, \frac{dh'_{n-1}}{dt} = \frac{\partial \Omega}{\partial h_{n-1}}. \end{aligned} \tag{36.8}$$

This system is of a form similar to the differential equations of the unperturbed problem, only in place of the variables q_1, q_2, $\ldots, q_n$, p_1, $p_2, \ldots, p_n$ and their function H, the variables h, h_1, $\ldots, h_{n-1}$, h', h'_1, $\ldots, h'_{n-1}$ and the function $-\Omega$ appear, of which moreover, the last contains the time t explicitly. The integration of this system is therefore, according to the earlier general consideration,[1] equivalent to a complete solution of the partial differential equation

$$\frac{\partial S}{\partial t} - \Omega = 0,$$

which defines S as a function of $t, .h, h_1, \ldots, h_{n-1}$, after the variables $h', h'_1, \ldots, h'_{n-1}$ have been replaced by the differential coefficients $\frac{\partial S}{\partial h}$, $\frac{\partial S}{\partial h_1}, \ldots, \frac{\partial S}{\partial h_{n-1}}$.

The differential equations of the perturbation problem formulated here agree with those given by *Lagrange* and *Laplace* in that the perturbed elements are the variables sought, and that the right hand sides of the differential equations are expressed through the differential coefficients of the perturbation function with respect to the perturbed elements. But for them, in general, all differential coefficients of the perturbation function occur in every differential equation and the coefficients of the same are expressions of the form (φ, ψ), the construction of which is very laborious. One finds something closer to what is given here in *Lagrange*'s *Mécanique Analytique*, in which the necessary lengthy computations are shortened with the greatest skill, as well as in *Encke*'s Astronomical Year book of 1837. In the simplest case of planetary perturbations, one has, according to the older formulae, to compute 15 expressions of the form (φ, ψ).

It was possible for us to simplify the differential equations because we assumed the elements of the unperturbed problem to be precisely in the form that they are given by *Hamilton*'s method, so that in each only *one* differential coefficient of the perturbing function occurs, and that the coefficient of the same is reduced to either plus or minus 1. This choice of the elements is of the greatest importance.It is for this reason that we discussed the geometrical significance of the arbitrary constants introduced in *Hamilton*'s method for the determination of planetary motion.

Instead of introducing the variables h_k, h'_k in place of the original variables p_i and q_i in the system of ordinary differential equations and

[1]See Lecture 20, p.173

then leading in an indirect way to the partial differential equation $\frac{\partial S}{\partial t} - \Omega = 0$, we shall, in the sequel, pose the problem of introducing these new variables directly in the partial differential equation

$$\frac{\partial V}{\partial t} + H + \Omega = 0, \tag{36.9}$$

which belongs to the perturbation problem expressed in its original variables. We assume a complete solution V_0 of the partial differential equation of the unperturbed problem

$$\frac{\partial V_0}{\partial t} + H = 0, \tag{36.10}$$

as known. This is required for the determination of the new variables h_k and h'_k, and we shall go directly from the partial differential equation (36.9) to the partial differential equation

$$\frac{\partial S}{\partial t} - \Omega = 0. \tag{36.11}$$

The partial differential equation (36.9), in which the quantities $p_1, p_2, \ldots, p_n$ are replaced by the partial differential coefficients $\frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \ldots, \frac{\partial V}{\partial q_n}$, is equivalent to the total differential equation

$$dV = -(H + \Omega)dt + p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n, \tag{36.12}$$

where in H and $\Omega, p_1, p_2, \ldots, p_n$ enter again in place of $\frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \ldots, \frac{\partial V}{\partial q_n}$.

While we introduce as new variables the functions which in the unperturbed problem are equal to arbitrary constants, we have to effect a substitution which is of the same nature as the one considered in Lecture 21, but more general than that. In the present case, as there, not only are the new variables to be introduced for the independent variables $q_1, q_2, \ldots, q_n, t$ and the function V found but the new variables are dependent on $p_1, p_2, \ldots, p_n$, i.e. the differential coefficients of V with respect to $q_1, q_2, \ldots, q_n$. The transformation under discussion takes place in the following way.

The partial differential equation of the unperturbed problem is

$$\frac{\partial V_0}{\partial t} + H = 0,$$

which, in Lecture 21, we have reduced to the equation

$$H = h$$

through the substitution

$$V_0 = W - ht.$$

The complete solution W of this partial differential equation is a function of $q_1, q_2, \ldots, q_n$ which contains, besides h, the $n-1$ arbitrary constants $h_1, h_2, \ldots, h_{n-1}$. If we have found them, then the system of integral equations of the unperturbed problem is

$$\frac{\partial W}{\partial q_1} = p_1, \frac{\partial W}{\partial q_2} = p_2, \ldots, \frac{\partial W}{\partial q_n} = p_n,$$
$$\frac{\partial W}{\partial h} = t + h', \frac{\partial W}{\partial h_1} = h'_1, \ldots, \frac{\partial W}{\partial h_{n-1}} = h'_{n-1}.$$

Since $h, h_1, \ldots, h_{n-1}$ are constants in the unperturbed problem, W satisfies the total differential equation

$$dW = p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n.$$

In the perturbation problem, on the contrary, there enter functions of time in place of the arbitrary constants, $h, h_1, \ldots, h_{n-1}$ are variables and there comes in the complete differential of W in addition the sum

$$\begin{aligned} &\frac{\partial W}{\partial h} dh + \frac{\partial W}{\partial h_1} dh_1 + \cdots + \frac{\partial W}{\partial h_n} dh_n \\ = \quad &(t + h')dh + h'_1 dh_1 + \cdots + h'_{n-1} dh_{n-1}. \end{aligned}$$

One has then in the perturbation problem

$$\begin{aligned} dW \quad = \quad & p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n + (t + h')dh \\ & + h'_1 dh_1 + \cdots + h'_{n-1} dh_{n-1} \end{aligned} \qquad (36.13)$$

This equation will be satisfied identically by the integral equations if one looks upon the earlier constants as variables, i.e. if the integral equations are no longer those of the unperturbed but of the perturbed problem. Therefore in these the equation is an *identity.* Therefore the total differential equation for dV is not altered if we subtract the equation (36.13) for dW from the former. Let us take the difference with opposite signs,so that it gives

$$\begin{aligned} d(W - V) \quad = \quad & (H + \Omega)dt + (t + h')dh + h'_1 \\ & dh_1 + \cdots + h'_{n-1} dh_{n-1}. \end{aligned}$$

By the integral equation of the perturbation problem, however, $H = h$ identically, consequently the terms $Hdt + tdh$ standing on the right hand side can be combined into $d(ht)$. If we bring this quantity to the left side, we obtain

$$d(W - ht - V) = \Omega dt + h'dh + h'_1 dh_1 + \cdots + h'_{n-1} dh_{n-1},$$

or, if we set

$$W - ht - V = V_0 - V = S,$$

then

$$dS = \Omega dt + h'dh + h'_1 dh_1 + \cdots + h'_{n-1} dh_{n-1},$$

and this total differential equation is equivalent to the partial differential equation obtained above,

$$\frac{\partial S}{\partial t} - \Omega = 0, \tag{36.11}$$

in which the quantities $h', h'_1, \ldots, h'_{n-1}$ are to be replaced by the differential coefficients $\frac{\partial S}{\partial h}, \frac{\partial S}{\partial h_1}, \ldots, \frac{\partial S}{\partial h_{n-1}}$. Finally, the partial differential equation (36.11) is the same as that to which the system of ordinary differential equations (36.8) can be reduced. Thus we have been led in the shortest way to the same system of differential equations

$$\begin{aligned} \frac{dh}{dt} &= -\frac{\partial \Omega}{\partial h'}, & \frac{dh_1}{dt} &= -\frac{\partial \Omega}{\partial h'_1}, \ldots, & \frac{dh_{n-1}}{dt} &= -\frac{\partial \Omega}{\partial h'_{n-1}} \\ \frac{dh'}{dt} &= \frac{\partial \Omega}{\partial h}, & \frac{dh'_1}{dt} &= \frac{\partial \Omega}{\partial h_1}, \ldots, & \frac{dh'_{n-1}}{dt} &= \frac{\partial \Omega}{\partial h_{n-1}} \end{aligned} \tag{36.8}$$

which we had found earlier in a different way.

This system of differential equations has the advantage that one finds the first corrections to the elements through mere quadratures. This follows if one looks upon the elements as constants in Ω and gives them the values they have in the unperturbed problem. Then Ω will be a function merely of time t,, and the corrected elements would be given through simple quadratures. The determination of the higher corrections is a difficult problem which cannot be gone into here.

Another remarkable system of formulae exists that is connected likewise to the introduction of the constants h, h_1, $\ldots$, h_{n-1}, h', h'_1, $\ldots$, h'_{n-1}, as elements. Namely, of the two principal forms under which we can represent the integral equations, we have so far considered these:

$$\begin{aligned} H &= h, & H_1 &= h_1, \ldots, H_{n-1} = h_{n-1}, \\ H' &= h' + t, & H'_1 &= h'_1, \ldots, H'_{n-1} = h'_{n-1}, \end{aligned}$$

in which the equations are solved for the abitrary constants h_k and h'_k, and H_k and H'_k are functions solely of the variables $q_1, q_2, \ldots, q_n$, $p_1, p_2, \ldots, p_n$. The second principal form is that in which the $2n$ variables $q_1, q_2, \ldots, q_n$, $p_1, p_2, \ldots, p_n$. are represented as functions of t and the constants $h, h_1, \ldots, h_{n-1}$, $h', h'_1, \ldots, h'_{n-1}$. According as one chooses one form or the other, in perturbation theory, one has to do with either the partial differential coefficients of the quantities H_k and H'_k in the variables q_i and p_i in the arbitrary constants h_k and h'_k, i.e., one must either, as *Poisson*, form the differential coefficients with respect to the variables of the functions which become equal to the elements, or, as *Lagrange*, the differential coefficients of the variables with respect to the elements. In either case one has to construct a system of $4n^2$ differential coefficients. The constants h_k and h'_k which one obtains through *Hamilton*'s form of the integral equations have the remarkable property, besides the one already mentioned, that the two systems of differential coefficients are either equal or opposite.

According to the theorem proved in the previous Lecture one has, namely

$$\begin{aligned}
(H_i, H) = 0, \quad & (H_i, H_1) = 0, \ldots, (H_i, H_{i-1}) = 0, \\
& (H_i, H_i) = 0, (H_i, H_{i+1}) = 0, \ldots, (H_i, H_{n-1}) = 0. \\
(H_i, H') = 0, \quad & (H_i, H'_1) = 0, \ldots, (H_i, H'_{i-1}) = 0, \\
& (H_i, H'_i) = 1, (H_i, H'_{i+1}) = 0, \ldots, (H_i, H'_{n-1}) = 0.
\end{aligned} \tag{36.14}$$

In these $2n$ equations, the $2n$ partial differential coefficients of H_i:

$$\frac{\partial H_i}{\partial q_1}, \frac{\partial H_i}{\partial q_2}, \ldots, \frac{\partial H_i}{\partial q_n}, \frac{\partial H_i}{\partial p_1}, \frac{\partial H_i}{\partial p_2}, \ldots, \frac{\partial H_i}{\partial p_n},$$

which we intend to consider as the unknowns of the system, enter linearly. As coefficients of these $2n$ unknowns we find in the equations (36.14) the $2n$ quantities

$$-\frac{\partial H}{\partial p_1}, -\frac{\partial H}{\partial p_2}, \ldots, -\frac{\partial H}{\partial p_n}, \frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2}, \ldots, \frac{\partial H}{\partial q_n},$$

and the corresponding quantities arising from the partial differentiation of $H_1, H_2, \ldots, H_{n-1}$, $H'_1, \ldots, H'_{i-1}$, $H'_i, H'_{i+1}, \ldots, H'_{n-1}$. On the right hand sides of all the equation (36.14) stands 0, with the single exception of the equation whose coefficients are differential coefficients of H'_i and in which the right hand side is equal to unity.

One obtains a similar system of linear equations, i.e. a system in which the coefficients and the right hand sides are entirely the same, for the differential coefficients of $-p_1, -p_2, \ldots, -p_n$, $q_1, q_2, \ldots, q_n$ with respect to h'_i In fact, the integrals

$$\begin{aligned} H &= h, \qquad H_1 = h_1, \ldots, H_i = h_i, \ldots, H_{n-1} = h_{n-1}, \\ H' &= t + h, \qquad H'_1 = h'_1, \ldots, H'_i = h'_i, \ldots, H'_{n-1} = h'_{n-1} \end{aligned}$$

become identical equations if one considers in them the variables q_1, $q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$ replaced by their values in t and the $2n$ arbitrary constants. Therefore, one can differentiate them partially with respect to any of the arbitrary constants, and obtain after differentiation in h'_i the system of equations

$$\begin{aligned} &\frac{\partial H}{\partial h'_i} = 0, \qquad \frac{\partial H_1}{\partial h'_i} = 0, \ldots, \frac{\partial H_{i-1}}{\partial h'_i} = 0, \\ &\frac{\partial H_i}{\partial h'_i} = 0, \qquad \frac{\partial H_{i+1}}{\partial h'_i} = 0, \ldots, \frac{\partial H_{n-1}}{\partial h'_i} = 0, \\ &\frac{\partial H'}{\partial h'_i} = 0, \qquad \frac{\partial H'_1}{\partial h'_i} = 0, \ldots, \frac{\partial H'_{i-1}}{\partial h'_i} = 0, \\ &\frac{\partial H'_i}{\partial h'_i} = 1, \qquad \frac{\partial H'_{i+1}}{\partial h'_i} = 0, \ldots, \frac{\partial H'_{n-1}}{\partial h'_i} = 0, \end{aligned} \tag{36.15}$$

of which the first, for example, can be given in expanded form in the following way:

$$\frac{\partial H}{\partial p_1}\frac{\partial p_1}{\partial h'_i} + \frac{\partial H}{\partial p_2}\frac{\partial p_2}{\partial h'_i} + \cdots + \frac{\partial H}{\partial p_n}\frac{\partial p_n}{\partial h'_i} + \frac{\partial H}{\partial q_1}\frac{\partial q_1}{\partial h'_i} + \cdots + \frac{\partial H}{\partial q_n}\frac{\partial q_n}{\partial h'_i} = 0.$$

This system differs from the system (36.14) only in that in place of the earlier unknowns

$$\frac{\partial H_i}{\partial q_1}, \frac{\partial H_i}{\partial q_2}, \ldots, \frac{\partial H_i}{\partial q_n}, \frac{\partial H_i}{\partial p_1}, \frac{\partial H_i}{\partial p_2}, \ldots, \frac{\partial H_i}{\partial p_n},$$

at present the quantities

$$\frac{-\partial p_1}{\partial h'_i}, \frac{-\partial p_2}{\partial h'_i}, \ldots, \frac{-\partial p_n}{\partial h'_i}, \frac{\partial q_1}{\partial h'_i}, \frac{\partial q_2}{\partial h'_i}, \ldots, \frac{\partial q_n}{\partial h'_i},$$

occur. But, if in two systems of linear equations the coefficients and the constant terms are equal, so are also the unknowns unless the common determinant of the system, i.e., in the present case, the expression

$$\sum \pm \frac{\partial H}{\partial q_1}\frac{\partial H_1}{\partial q_2} \cdots \frac{\partial H_{n-1}}{\partial q_n}\frac{\partial H'}{\partial p_1}\frac{\partial H'_1}{\partial p_2} \cdots \frac{\partial H'_{n-1}}{\partial p_n}$$

vanishes. This however is never the case, as otherwise the $2n$ quantities $H, H_1, \ldots, H_{n-1}, H', H'_1, \ldots, H'_{n-1}$ would not be mutually independent functions of the $2n$ variables $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$, and the system of integrals would be insufficient to determine these variables as functions of $h, h_1, \ldots, h_{n-1}, h' + t, h'_1, \ldots, h'_{n-1}$. Accordingly, the two systems of unknowns are mutually equal, i.e. one has

$$\begin{aligned} \frac{\partial q_1}{\partial h'_i} &= \frac{\partial H_i}{\partial p_1} & \frac{\partial q_2}{\partial h'_i} &= \frac{\partial H_i}{\partial p_2}, \ldots, \frac{\partial q_n}{\partial h'_i} = \frac{\partial H_i}{\partial p_n} \\ \frac{\partial p_1}{\partial h'_i} &= -\frac{\partial H_i}{\partial q_1}, & \frac{\partial p_2}{\partial h'_i} &= -\frac{\partial H_i}{\partial q_2}, \ldots, \frac{\partial p_n}{\partial h'_i} = -\frac{\partial H_i}{\partial q_n}. \end{aligned} \tag{36.16}$$

In addition to this system of formulae which one is led to by comparing the systems (36.14) and (36.15), there exists another one, which can be obtained from these by mere interchange. Namely, the systems (36.14) and (36.15) again give the right systems of equations if one writes, for all values of the index i, in place of the quantities without primes H_i, h_i, and with negative signs the corresponding quantities with primes, $-H'_i, -h'_i$,, on the other hand, in place of the quantities with the primes H'_i, h'_i, the corresponding quantities H_i, h_i without primes and with positive signs. This method of interchanging must therefore be applicable also to the system (36.16), and gives from them the new system of formulae

$$\begin{aligned} \frac{\partial q_1}{\partial h_i} &= -\frac{\partial H'_i}{\partial p_1}, & \frac{\partial q_2}{\partial h_i} &= -\frac{\partial H'_i}{\partial p_2}, \ldots, \frac{\partial q_n}{\partial h_i} = -\frac{\partial H'_i}{\partial p_n}, \\ \frac{\partial p_1}{\partial h_i} &= \frac{\partial H'_i}{\partial q_1} & \frac{\partial p_2}{\partial h_i} &= \frac{\partial H'_i}{\partial q_2}, \ldots, \frac{\partial p_n}{\partial h_i} = \frac{\partial H'_i}{\partial q_n}. \end{aligned} \tag{36.17}$$

We can combine the two systems of formula (36.16) and (36.17) into the four equations

$$\begin{aligned} \frac{\partial q_k}{\partial h'_i} &= \frac{\partial H_i}{\partial p_k}, & \frac{\partial q_k}{\partial h_i} &= -\frac{\partial H'_i}{\partial p_k}, \\ \frac{\partial p_k}{\partial h'_i} &= -\frac{\partial H_i}{\partial q_k}, & \frac{\partial p_k}{\partial h_i} &= \frac{\partial H'_i}{\partial q_k}, \end{aligned}$$

and express the result obtained in the following theorem:[2]

[2]This theorem was communicated to the Berlin Academy on 21 November, 1838, (See Monatsberichte a.d. J. 1838, p.178)

Consider the system of integrals expressed in Hamiltonian form:

$$H = h, \qquad H_1 = h_1, \ldots, H_{n-1} = h_{n-1},$$
$$H' = h' + t, \qquad H'_1 = h'_1, \ldots, H'_{n-1} = h'_{n-1}.$$

On the one hand, the constants h_i and h'_i are functions of the variables p_i, q_i and time t,. On the other hand, from the same equations, one expresses the same variables by means of the constants and t. Then the partial differential coefficients, constructed by the first method, of the constants with respect to the variables p_i and q_i, and the partial differential coefficients, constructed by the second method, of the variables p_i, q_i with respect to the constants, are, apart from sign, pairwise equal to one another.

Supplement

Jacobi was prevented by severe illness in the spring of 1843 from bringing his lectures on Dynamics to a conclusion. The plan of the same shows sufficiently that, at the end of the course, he had intended to carry out his method of integration of non-linear partial differential equatons of the first order, which has been found in a complete and worked out treatise written in 1835 among his papers left behind, and which has been published by me in Volume 60 of the Mathematical Journal on the basis of this treatise I seek to fill here the gap, in Jacobi's sense, which remained at the conclusion of his lectures on Dynamics. *C. Clebsch*

The integration of first order non-linear partial differential equations

In lecture 32 (pp.284, 285) the integration of the partial differential equation $f = h$ or $H = h$ was reduced to the system of $\frac{n(n-1)}{2}$ simultaneous equations

$$(H_i, H_k) = 0. \tag{S.1}$$

If the function H is determined in accordance with these equations, then the equations

$$H = h, H_1 = h_1, \ldots, H_{n-1} = h_{n-1} \tag{S.2}$$

lead to such an expression for p that

$$dV = p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n,$$

is a complete differential. However, instead of proceeding to the simultaneous integration of the system (S.1) with the help of the principle laid down in Lecture 34, one may now equally well pose the problem of finding the expressions which $p_1, p_2, \ldots, p_n$ take in consequence of the equation (S.2). Let us, as explained in lecture 31 (p.271), think of p_1

as expressed as a function of the quantities q and $p_2, p_3, \ldots, p_n$; then p_2 determined as a function of the quantities q and $p_3, p_4, \ldots, p_n$; etc. If $p_1, p_2, \ldots, p_i$ are found, one can, without going to search for p_{i+1}, expresses the earlier quantities through $p_{i+1}, p_{i+2}, \ldots, p_n$ and q. These i equations which the function p_{i+1} has then to satisfy simultaneously, one can find from equation (31.7) of Lecture 31 (p.278), if one replaces in these the i' th row by the numbers $1, 2, \ldots, i$ and put $i+1$ in place of i. Since then $p_{i'}$ depends only on $p_{i+1}, p_{i+2}, \ldots, p_n$, and p_{i+1} on the contrary only on $p_{i+2}, p_{i+3}, \ldots, p_n$, then the equations introduced give the following system:

$$\begin{aligned}
0 \ = \ & \frac{\partial p_{i+1}}{\partial q_1} - \frac{\partial p_1}{\partial q_{i+1}} + \frac{\partial p_{i+1}}{\partial p_{i+2}} \frac{\partial p_1}{\partial q_{i+2}} + \frac{\partial p_{i+1}}{\partial p_{i+3}} \frac{\partial p_1}{\partial q_{i+3}} \\
& + \cdots + \frac{\partial p_{i+1}}{\partial p_n} \frac{\partial p_1}{\partial q_n} - - \frac{\partial p_1}{\partial p_{i+1}} \frac{\partial p_{i+1}}{\partial p_{i+1}} - \frac{\partial p_1}{\partial p_{i+2}} \frac{\partial p_{i+1}}{\partial p_{i+2}} \\
& - \frac{\partial p_1}{\partial p_{i+3}} \frac{\partial p_{i+1}}{\partial q_{i+3}} - \cdots - \frac{\partial p_1}{\partial p_n} \frac{\partial p_{i+1}}{\partial q_n}, \\
0 \ = \ & \frac{\partial p_{i+1}}{\partial q_2} - \frac{\partial p_2}{\partial q_{i+1}} - \frac{\partial p_{i+1}}{\partial p_{i+2}} \frac{\partial p_2}{\partial q_{i+2}} + \frac{\partial p_{i+1}}{\partial p_{i+3}} \frac{\partial p_2}{\partial q_{i+3}} \\
& + \cdots + \frac{\partial p_{i+1}}{\partial p_n} \frac{\partial p_2}{\partial q_n} - - \frac{\partial p_2}{\partial p_{i+1}} \frac{\partial p_{i+1}}{\partial q_{i+1}} - \frac{\partial p_2}{\partial p_{i+2}} \frac{\partial p_{i+1}}{\partial q_{i+2}} \\
& - \frac{\partial p_2}{\partial p_{i+3}} \frac{\partial p_{i+1}}{\partial q_{i+3}} - \cdots - \frac{\partial p_2}{\partial p_n} \frac{\partial p_{i+1}}{\partial q_n}, \\
& \cdots\cdots\cdots\cdots \\
0 \ = \ & \frac{\partial p_{i+1}}{\partial q_i} - \frac{\partial p_i}{\partial q_{i+1}} + \frac{\partial p_{i+1}}{\partial p_{i+2}} \frac{\partial p_i}{\partial q_{i+2}} + \frac{\partial p_{i+1}}{\partial p_{i+3}} \frac{\partial p_i}{\partial q_{i+3}} \\
& + \cdots + \frac{\partial p_{i+1}}{\partial p_n} \frac{\partial p_i}{\partial q_n} - \frac{\partial p_i}{\partial p_{i+2}} \frac{\partial p_{i+1}}{\partial q_{i+2}} - \frac{\partial p_i}{\partial p_{i+1}} \frac{\partial p_{i+1}}{\partial q_{i+1}} \\
& - \frac{\partial p_i}{\partial p_{i+2}} \frac{\partial p_{i+1}}{\partial q_{i+2}} - \frac{\partial p_i}{\partial p_{i+3}} \frac{\partial p_{i+1}}{\partial q_{i+3}} - \cdots - \frac{\partial p_i}{\partial p_n} \frac{\partial p_{i+1}}{\partial q_n}.
\end{aligned} \tag{S.3}$$

We can so transform this system that we do not consider p_{i+1} as a function of $p_{i+2}, p_{i+3}, \ldots, p_n$, $q_1, q_2, \ldots, q_n$, but introduce an equation

$$f = \text{constant},$$

which holds between p_{i+1} and these quantities. Then, for $h > i+1$,

$$\frac{\partial f}{\partial p_{i+1}} \frac{\partial p_{i+1}}{\partial p_h} + \frac{\partial f}{\partial p_h} = 0,$$

and for any value of h,

$$\frac{\partial f}{\partial p_{i+1}}\frac{\partial p_{i+1}}{\partial q_h}+\frac{\partial f}{\partial q_h}=0.$$

If we multiply equation (S.3) by $\frac{\partial f}{\partial p_{i+1}}$, then they take the following form:

$$\begin{aligned}
0 &= \frac{\partial f}{\partial q_1}+\frac{\partial p_1}{\partial q_{i+1}}\frac{\partial f}{\partial p_{i+1}}+\frac{\partial p_1}{\partial q_{i+2}}\frac{\partial f}{\partial p_{i+2}}+\cdots+\frac{\partial p_1}{\partial q_n}\frac{\partial f}{\partial p_n}- \\
&\quad -\frac{\partial p_1}{\partial p_{i+1}}\frac{\partial f}{\partial q_{i+1}}-\frac{\partial p_1}{\partial p_{i+2}}\frac{\partial f}{\partial q_{i+2}}-\cdots-\frac{\partial p_1}{\partial p_n}\frac{\partial f}{\partial q_n}, \\
0 &= \frac{\partial f}{\partial q_2}+\frac{\partial p_2}{\partial_{q_{i+1}}}\frac{\partial f}{\partial p_{i+1}}+\frac{\partial p_2}{\partial q_{i+2}}\frac{\partial f}{\partial p_{i+2}}+\cdots+\frac{\partial p_2}{\partial q_n}\frac{\partial f}{\partial p_n}- \\
&\quad -\frac{\partial p_2}{\partial p_{i+1}}\frac{\partial f}{\partial q_{i+1}}-\frac{\partial p_2}{\partial p_{i+2}}\frac{\partial f}{\partial q_{i+2}}-\cdots-\frac{\partial p_2}{\partial p_n}\frac{\partial f}{\partial q_n} \\
&\cdots\cdots\cdots\cdots \\
0 &= \frac{\partial f}{\partial q_i}+\frac{\partial p_i}{\partial q_{i+1}}\frac{\partial f}{\partial p_{i+1}}+\frac{\partial p_i}{\partial q_{i+2}}\frac{\partial f}{\partial p_{i+2}}+\cdots+\frac{\partial p_i}{\partial q_n}\frac{\partial f}{\partial p_n}- \\
&\quad -\frac{\partial p_i}{\partial p_{i+1}}\frac{\partial f}{\partial q_{i+1}}-\frac{\partial p_i}{\partial p_{i+2}}\frac{\partial f}{\partial q_{i+2}}-\cdots-\frac{\partial p_i}{\partial p_n}\frac{\partial f}{\partial q_n}.
\end{aligned} \tag{S.4}$$

The simultaneous integration of this system rests on the theorems that were given at the end of Lecture 31 and in lecture 34. If p_χ is any one of the quantities $p_1, p_2, \ldots, p_i$ and

$$\varphi_\chi - p_\chi = 0$$

the equation by virtue of which p_χ is expressed in terms of $p_{i+1}, p_{i+2}, \ldots, p_n,\ q_1, q_2, \ldots, q_n$, then

$$\begin{aligned}
\frac{\partial(\varphi_\chi - p_\chi)}{\partial p_{i+h}} &= \frac{\partial \varphi_\chi}{\partial p_{i+h}} = \frac{\partial p_\chi}{\partial p_{i+h}}, \\
\frac{\partial(\varphi_\chi - p_\chi)}{\partial q_h} &= \frac{\partial \varphi_\chi}{\partial q_h} = \frac{\partial p_\chi}{\partial q_h};
\end{aligned}$$

however, if $h < i+1$, one has

$$\frac{\partial(\varphi_\chi - p_\chi)}{\partial p_h} = 0, \quad \frac{\partial(\varphi_\chi - p_\chi)}{\partial p_\chi} = -1.$$

The equation (S.4) can therefore, with the help of the notation (φ, ψ), be written thus:

$$(f, \varphi_1 - p_1) = 0, (f, \varphi_2 - p_2) = 0, \ldots, (f, \varphi_i - p_i) = 0. \tag{S.5}$$

If one now forms the expression $(\varphi_\chi - p_\chi, \varphi_\lambda - p_\lambda)$, where χ, λ denote any two of the numbers $1, 2, \dots, i$, then one finds

$$(\varphi_\chi - p_\chi, \varphi_\lambda - p_\lambda) = 0.$$

Thus $\varphi_\chi - p_\chi = 0$ as well as $\varphi_\lambda - p_\lambda = 0$ belong to the system of equations which determine p. According to the theorem given at the end of Lecture 31, the above expression must then vanish. Now it was given in Lecture 34 that if $(\varphi, \psi) = 0$, then from a solution F of the equation

$$(f, \varphi) = 0,$$

further solutions

$$F' = (F, \psi), F'' = (F', \psi) \text{ etc.}$$

can be derived. If we apply this to any two equations

$$(f, \varphi_\chi - p_\chi) = 0, (f, \varphi_\lambda - p_\lambda) = 0$$

of the system (S.5), then it follows that from any function F which satisfies the equation

$$(F, \varphi_\chi - p_\chi) = 0,$$

a series of other solutions of the same equation can be constructed, namely

$$F' = (F, \varphi_\lambda - p_\lambda), F'' = (F', \varphi_\lambda - p_\lambda) \text{ etc.}$$

Finally, then, follows the theorem:

If F is a simultaneous solution of the equations

$$(f, \varphi_1 - p_1) = 0, (f, \varphi_2 - p_2) = 0, \dots, (f, \varphi_{h-1} - p_{h-1}) = 0,$$

then also

$$F' = (F, \varphi_h - p_h), F'' = (F', \varphi_h - p_h), \dots,$$

are simultaneous solutions of the same equations.

Let us then assume that a common solution F of the first $h-1$ equation (S.5) has been found and that a solution is sought which satisfies also the h^{th} of these equations. Then the question arises whether there exists a function Φ which satisfies the last equation, which is a function of F, of the derived solutions $F', F'', \dots, F^{(\mu-1)}$ and of the quantities $q_h, q_{h+1}, , \dots, q_i$, which obviously satisfies the first $h-1$ equations (S.5) (or (S.4)). The number μ is restricted by the fact, that the $F^{(\mu)}$

can be expressed by the previous functions $F', F'', \ldots, F^{(\mu-1)}$ and by $q_h, q_{h+1}, \ldots, q_i$; and that

$$F^{(\mu)} = \Pi(F, F', \ldots, F^{(\mu-1)}, q_h, q_{h+1}, \ldots, q_i).$$

Now the total number of all simulaneous solutions that $h-1$ mutually independent linear partial differential equations in $2n-i$ variables q_1, $q_2, \ldots, q_n$, p_{i+1}, $p_{i+2}, \ldots, p_n$ can, in general, have is $2n-i-(h-1)$; therefore the number of arguments of the function Π can at most be equal to this number. Hence

$$\mu + i - (h-1) \leq 2n - i - (h-1),$$

or

$$\mu \leq 2(n-i).$$

If we now look upon a solution Φ of the equation

$$(\Phi, \varphi_h - p_h) = 0 \tag{S.6}$$

as a function of the arguments of Π alone, then we obtain

$$\begin{aligned} 0 &= (\Phi, \varphi_h, -p_h) \\ &= \frac{\partial \Phi}{\partial F}(F, \varphi_h - p_h) + \frac{\partial \Phi}{\partial F'}(F', \varphi_h - p_h) \\ &\quad + \cdots + \frac{\partial \Phi}{\partial F^{(\mu-1)}}(F^{(\mu-1)}, \varphi_h - p_h) + \\ &\quad + \frac{\partial \Phi}{\partial q_h}(q_h, \varphi_h - p_h) + \frac{\partial \Phi}{\partial q_{h+1}}(q_{h+1}, \varphi_h - p_h) \\ &\quad + \cdots + \frac{\partial \Phi}{\partial q_i}(q_i, \varphi_h - p_h). \end{aligned} \tag{S.7}$$

Since in the h^{th} equation of the system (S.4) or (S.5), differentiations are done only with respect to q_h and not $q_{h+1}, q_{h+2}, \ldots, q_i$, therefore the coefficients

$$(q_{h+1}, \varphi_h - p_h), (q_{h+2}, \varphi_h - p_h), \ldots, (q_i, \varphi_h - p_h)$$

vanish, and one finds besides that

$$(q_h, \varphi_h - p_h) = 1.$$

If one further takes into account the method of constructing the function F, then one sees that equation (S.6) or (S.7) transforms into the following:

$$\frac{\partial \Phi}{\partial q_h} + F' \frac{\partial \Phi}{\partial F} + F'' \frac{\partial \Phi}{\partial F'} + \cdots + \Pi \frac{\partial \Phi}{\partial F^{(\mu-1)}} = 0. \tag{S.8}$$

A look at this equation teaches us that it is actually possible to determine a function Φ in the given way, since the coefficients of this equation contain only the variables on which Φ was considered to be dependent.

In order to find a solution of equation (S.8), one needs only to look for an integral of the system

$$\frac{\partial F}{\partial q_h} = F', \frac{\partial F'}{\partial q_h} = F'', \ldots, \frac{\partial F^{(\mu-1)}}{\partial q_h} = \Pi,$$

or, what is the same, a first integral of the differential equations of the μth order

$$\frac{d^\mu F}{dq_h^\mu} = \Pi,$$

where in Π the quantities $F', F'', \ldots, F^{(\mu-1)}$ are to be replaced by $\frac{dF}{dq_h}$, $\frac{d^2F}{dq_h^2}, \ldots, \frac{d^{\mu-1}F}{dq_h^{\mu-1}}$.

One can state these results in the following theorem.

If one knows a simultaneous solution of the first $h-1$ equations of the system (S.4) or (S.5), then finding a solution which satisfies also the hth equation requires only the knowledge of a first integral of a differential equation whose order does not exceed $2(n-1)$.

To find a simultaneous solution of the system (S.5), one has only to carry out successively the procedure just gone through i times one after the other. One looks for a solution F of the first equation (S.5), or an integral of the system of $2(n-i)$ differential equations

$$\begin{aligned}
\frac{dp_{i+1}}{dq_1} &= \frac{\partial p_1}{\partial q_{i+1}}, & \frac{dp_{i+2}}{dq_1} &= \frac{\partial p_1}{\partial q_{i+2}}, \ldots, & \frac{dp_n}{dq_1} &= \frac{\partial p_1}{\partial q_n}, \\
\frac{dq_{i+1}}{dq_1} &= -\frac{\partial p_1}{\partial p_{i+1}}, & \frac{dq_{i+2}}{dq_1} &= -\frac{\partial p_1}{\partial p_{i+2}}, \ldots, & \frac{dq_n}{dq_1} &= -\frac{\partial p_1}{\partial p_n}.
\end{aligned}$$

One develops therefrom the other solutions of the same equation:

$$\begin{aligned}
F' &= (F, \varphi_2 - p_2), F'' = (F', \varphi_2 - p_2), \ldots, \\
&\quad F^{(\mu)} = \Pi(F, F', \ldots, F^{(\mu-1)}, q_2, q_3, \ldots, q_i).
\end{aligned}$$

Every first integral of the equation

$$\frac{d^\mu F}{dq_2^\mu} = \Pi\left(F, \frac{dF}{dq_2}, \ldots, \frac{d^{\mu-1}F}{dq_2^{\mu-1}}, q_2, q_3, \ldots, q_i\right)$$

which contains an arbitrary constant then leads to a solution which satisfies the first two of the equations (S.5). Let Φ be this solution. One then constructs

$$\begin{aligned}\Phi' &= (\Phi, \varphi_3 - p_3), \Phi'' = (\Phi', \varphi_3 - p_3), \ldots, \Phi^{(v)} = \Pi(\Phi, \Phi', \ldots, \\ &\quad \Phi^{(v-1)}, q_3, q_4, \ldots, q_i).\end{aligned}$$

Every first integral of the differential equation

$$\frac{d^v\Phi}{dq_3^v} = \Pi\left(\Phi, \frac{d\Phi}{dq_3}, \frac{d^2\Phi}{dq_3^2}, \ldots, \frac{d^{v-1}\Phi}{dq_3^{v-1}}, q_3, q_4, \ldots, q_i\right)$$

which contains an arbitrary constant then gives a function which satisfies the first three of the equation (S.5), etc.

Finding a simultaneous solution of the system (S.5), or (S.4) requires the knowledge of first integrals of i differential equations of which the first is of order $2(n-i)$, while the others can be of lower order.

The entire development of the integration procedure requires also the determination of p_1 from the given partial differential equation. If one has solved this, then one looks first for an integral of the system of $2(n-1)$ differential equations

$$\begin{aligned}&\frac{dp_2}{dq_1} = \frac{\partial p_1}{\partial q_2}, \quad \frac{dp_3}{dq_1} = \frac{\partial p_1}{\partial q_3}, \ldots, \frac{dp_n}{dq_1} = \frac{\partial p_1}{\partial q_n} \\ &\frac{dq_2}{dq_1} = -\frac{\partial p_1}{\partial p_2}, \quad \frac{dq_3}{dq_1} = -\frac{\partial p_1}{\partial p_3}, \ldots, \frac{dq_n}{dq_1} = -\frac{\partial p_1}{\partial p_n}.\end{aligned}$$

From the integral already found, one determines p_2 as a function of q and the following p's, and while one introduces these functions in the expression for p_1, one represents p_1 in the same way.

Secondly one looks for an integral of the system of $2(n-2)$ differential equations

$$\begin{aligned}&\frac{dp_3}{dq_1} = \frac{\partial p_1}{\partial q_3}, \quad \frac{dp_4}{dq_1} = \frac{\partial p_1}{\partial q_4}, \ldots, \frac{dp_n}{dq_1} = \frac{\partial p_1}{\partial q_n} \\ &\frac{dq_3}{dq_1} = -\frac{\partial p_1}{\partial p_3}, \quad \frac{dq_4}{dq_1} = -\frac{\partial p_1}{\partial p_4}, \ldots, \frac{dq_n}{dq_1} = -\frac{\partial p_1}{\partial p_n},\end{aligned}$$

where the differential coefficients of p_1 are to be taken in the new sense just stipulated. Let an integral of this system be F =constant. Let us construct

$$\begin{aligned} F' &= \frac{\partial F}{\partial q_2} + \frac{\partial p_2}{\partial q_3}\frac{\partial F}{\partial p_3} + \frac{\partial p_2}{\partial q_4}\frac{\partial F}{\partial p_4} + \cdots + \frac{\partial p_2}{\partial q_n}\frac{\partial F}{\partial p_n} - \\ &\quad -\frac{\partial p_2}{\partial p_3}\frac{\partial F}{\partial q_3} - \frac{\partial p_3}{\partial q_4}\frac{\partial F}{\partial q_4} - \cdots - \frac{\partial p_3}{\partial p_n}\frac{\partial F}{\partial q_n}, \\ F'' &= \frac{\partial F'}{\partial q_2} + \frac{\partial p_2}{\partial q_3}\frac{\partial F'}{\partial p_3} + \frac{\partial p_2}{\partial q_4}\frac{\partial F'}{\partial p_4} + \cdots + \frac{\partial p_2}{\partial q_n}\frac{\partial F'}{\partial p_n} - \\ &\quad -\frac{\partial p_2}{\partial p_3}\frac{\partial F'}{\partial q_3} - \frac{\partial p_3}{\partial q_4}\frac{\partial F'}{\partial q_4} - \cdots - \frac{\partial p_2}{\partial p_n}\frac{\partial F'}{\partial q_n}, \text{ etc.} \end{aligned}$$

till we are led to a function $F^{(\mu)}(\mu \leq 2(n-2))$, which itself can be represented as a function of $q_2, F, F', \ldots, F^{(\mu-1)}$. If this is

$$F^{(\mu)} = \Pi(F, F', \ldots, F^{(\mu-1)}, q_2).$$

we look for a first integral

$$\Phi\left(F, \frac{dF}{dq_2}, \frac{d^2F}{dq_2^2}, \ldots, \frac{d^{(\mu-1)}F}{dq_2^{\mu-1}}, q_2\right) = \text{constant}$$

of the differential equation of order μ :

$$\frac{d^\mu F}{dq_2^\mu} = \Pi\left(F, \frac{dF}{dq_2}, \frac{d^2F}{dq_2^2}, \ldots, \frac{d^{\mu-1}F}{dq_2^{\mu-1}}, q_2\right),$$

and we form the equation

$$\Phi\left(F, F', F'', \ldots, F^{(\mu-1)}, q_2\right) = \text{constant}.$$

This equation serves to determine p_3. If one has expressed this through $p_4, p_5, \ldots, p_n$ and q, and thereby represented p_1 and p_2 also as functions of these quantities, one then seeks *thirdly* an integral of the system of ordinary differential equations

$$\begin{aligned} \frac{dp_4}{dq_1} &= \frac{\partial p_1}{\partial q_4}, \quad \frac{dp_5}{dq_1} = \frac{\partial p_1}{\partial q_5}, \ldots, \frac{dp_n}{dq_1} = \frac{\partial p_1}{\partial q_n} \\ \frac{dq_4}{dq_1} &= -\frac{\partial p_1}{\partial p_4}, \quad \frac{dq_5}{dq_1} = -\frac{\partial p_1}{\partial p_5}, \ldots, \frac{dq_n}{dq_1} = -\frac{\partial p_1}{\partial p_n}. \end{aligned}$$

If this integral is $\Psi =$ constant, one constructs further

$$\begin{aligned}\Psi' = \frac{\partial \Psi}{\partial q_2} &+ \frac{\partial p_2}{\partial q_4}\frac{\partial \Psi}{\partial p_4} + \frac{\partial p_2}{\partial q_5}\frac{\partial \Psi}{\partial p_5} + \cdots + \frac{\partial p_2}{\partial q_n}\frac{\partial \Psi}{\partial p_n} - \\ &-\frac{\partial p_2}{\partial p_4}\frac{\partial \Psi}{\partial q_4} - \frac{\partial p_2}{\partial p_5}\frac{\partial \Psi}{\partial q_5} - \cdots - \frac{\partial p_2}{\partial p_n}\frac{\partial \Psi}{\partial q_n}, \\ \Psi'' = \frac{\partial \Psi'}{\partial q_2} &+ \frac{\partial p_2}{\partial q_4}\frac{\partial \Psi'}{\partial p_4} + \frac{\partial p_2}{\partial q_5}\frac{\partial \Psi'}{\partial p_5} + \cdots + \frac{\partial p_2}{\partial q_n}\frac{\partial \Psi'}{\partial p_n} - \\ &-\frac{\partial p_2}{\partial p_4}\frac{\partial \Psi'}{\partial q_4} - \frac{\partial p_2}{\partial p_5}\frac{\partial \Psi'}{\partial q_5} - \cdots - \frac{\partial p_2}{\partial p_n}\frac{\partial \Psi'}{\partial q_n},\end{aligned}$$

etc. till one is led to a function

$$\Psi^{(\nu)} = \Pi\left(\Psi, \Psi', \ldots, \Psi^{(\nu-1)}, q_2, q_3\right),$$

($\nu \leq 2(n-3)$), and seeks a first integral

$$X\left(\Psi, \frac{d\Psi}{dq_2}, \frac{d^2\Psi}{dq_2^2}, \ldots, \frac{d^{\nu-1}\Psi}{dq_2^{\nu-1}}, q_2, q_3\right) = \text{constant}$$

of the differential equation of the μth order:

$$\frac{d^\nu \Psi}{dq_2^\nu} = \Pi\left(\Psi, \frac{d\Psi}{dq_2}, \frac{d^2\Psi}{dq_2^2}, \ldots, \frac{d^{v-1}\Psi}{dq_2^{v-1}}, q_2, q_3\right).$$

From the function

$$X\left(\Psi, \Psi', \Psi'', \ldots, \Psi^{(v-1)}, q_2, q_3\right)$$

one now forms the further functions

$$\begin{aligned}X' = \frac{\partial X}{\partial q_3} &+ \frac{\partial p_3}{\partial q_4}\frac{\partial X}{\partial p_4} + \frac{\partial p_3}{\partial q_5}\frac{\partial X}{\partial p_5} + \cdots + \frac{\partial p_3}{\partial q_n}\frac{\partial X}{\partial p_n} - \\ &-\frac{\partial p_3}{\partial p_4}\frac{\partial X}{\partial q_4} - \frac{\partial p_3}{\partial p_5}\frac{\partial X}{\partial q_5} - \cdots - \frac{\partial p_3}{\partial p_n}\frac{\partial \chi}{\partial q_n}, \\ X'' = \frac{\partial X'}{\partial q_3} &+ \frac{\partial p_3}{\partial q_4}\frac{\partial X'}{\partial p_4} + \frac{\partial p_3}{\partial q_5}\frac{\partial \chi'}{\partial p_5} + \cdots + \frac{\partial p_3}{\partial q_n}\frac{\partial \chi'}{\partial p_n} - \\ &-\frac{\partial p_3}{\partial p_4}\frac{\partial \chi'}{\partial q_4} - \frac{\partial p_3}{\partial p_5}\frac{\partial \chi'}{\partial q_5} - \cdots - \frac{\partial p_3}{\partial p_5}\frac{\partial \chi'}{\partial q_n},\end{aligned}$$

etc. till one is led to the function $X^{(\rho)}$:

$$X^{(\rho)} = \Pi\left(X, X', \ldots, X^{(\rho-1)}, q_3\right),$$

where $\rho \leq 2(n-3)$. One seeks then a first integral

$$\Omega\left(X, \frac{dX}{dq_3}, \ldots, \frac{d^{\rho-1}X}{dq_3^{\rho-1}}, q_3\right) = \text{constant}$$

of the differential equation of order ρ :

$$\frac{d^\rho X}{dq_3^\rho} = \Pi\left(X, \frac{d\chi}{dq_3}, \ldots, \frac{d^{\rho-1}X}{dq_3^{\rho-1}}, q_3\right).$$

The equation

$$\Omega\left(X, X', \ldots, X^{(\rho-1)}, q_3\right) = \text{constant}$$

then serves to express p_4 through $p_5, p_6, \ldots, p_n$ and then q, and then also p_1, p_2, p_3through these quantities.

If one proceeds in this way, one is finally led to determine p_1, $p_2, \ldots, p_{n-1}$ as functions of p_n and the q's. One then seeks to express the last quantity p_n through the q alone. This requires that one first determines an integral Ξ of the system

$$\frac{dp_n}{dq_1} = \frac{\partial p_1}{\partial q_n}, \frac{dq_n}{dq_1} = -\frac{\partial p_1}{\partial p_n}.$$

One then forms

$$\begin{aligned} \Xi' &= \frac{\partial \Xi}{\partial q_2} + \frac{\partial p_2}{\partial q_n}\frac{\partial \Xi}{\partial p_n} - \frac{\partial p_2}{\partial p_n}\frac{\partial \Xi}{\partial q_n}, \\ \Xi'' &= \frac{\partial \Xi}{\partial q_2} + \frac{\partial p_2}{\partial q_n}\frac{\partial \Xi'}{\partial p_n} - \frac{\partial p_2}{\partial p_n}\frac{\partial \Xi'}{\partial q_n}, \end{aligned}$$

of which the latter, if not the farmer, can be expressed in terms of Ξ and Ξ, Ξ' respectively and the quantities $q_2, q_3, \ldots, q_{n-1}$. Then one integrates either, if

$$\Xi' = \Pi(\Xi, q_2, q_3, \ldots, q_{n-1}),$$

the equation

$$\frac{d\Xi}{dq_2} = \Pi(\Xi, q_2, q_3, \ldots, q_{n-1}),$$

or, if

$$\Xi'' = \Pi(\Xi, \Xi', q_2, q_3, \ldots, q_{n-1}),$$

the equation

$$\frac{d^2\Xi}{dq_2^2} = \Pi\left(\Xi, \frac{d\Xi}{dq_2}, q_2, q_3, \ldots, q_{n-1}\right).$$

Further, if one replaces the differential coefficient of Ξ by Ξ', one obtains in the first case a function $Y = Y(\Xi, q_2, q_3, \ldots, q_{n-1})$, in the second case a function $Y = Y(\Xi, \Xi', q_2, q_3, \ldots, q_{n-1})$. The following functions will then be derived from the function Y

$$\begin{aligned} Y' &= \frac{\partial Y}{\partial q_3} + \frac{\partial p_3}{\partial q_n}\frac{\partial Y}{\partial p_n} - \frac{\partial p_3}{\partial p_n}\frac{\partial Y}{\partial q_n}, \\ Y'' &= \frac{\partial Y'}{\partial q_3} + \frac{\partial p_3}{\partial q_n}\frac{\partial Y'}{\partial p_n} - \frac{\partial p_3}{\partial p_n}\frac{\partial Y'}{\partial q_n}, \end{aligned}$$

etc. If one continues this way, one arrives at a function Z from which one derives the functions

$$\begin{aligned} Z' &= \frac{\partial Z}{\partial q_{n-1}} + \frac{\partial p_{n-1}}{\partial q_n}\frac{\partial Z}{\partial p_n} - \frac{\partial p_{n-1}}{\partial p_n}\frac{\partial Z}{\partial q_n}, \\ Z'' &= \frac{\partial Z'}{\partial q_{n-1}} + \frac{\partial p_{n-1}}{\partial q_n}\frac{\partial Z'}{\partial p_n} - \frac{\partial p_{n-1}}{\partial p_n}\frac{\partial Z'}{\partial q_n}. \end{aligned}$$

If Z' is already a function Π of Z and q_{n-1}, one integrates the equation

$$\frac{dZ}{dq_{n-1}} = \Pi(Z, q_{n-1})$$

and its integral leads to the last equation by virtue of which p_n itself is expressed through q. If, however,

$$Z'' = \Pi(Z, Z', q_{n-1}),$$

one seeks a first integral of the differential equation of the second order

$$\frac{d^2Z}{dq_{n-1}^2} = \Pi\left(Z, \frac{dZ}{dq_{n-1}}, q_{n-1}\right).$$

If this integral is

$$\Theta\left(Z, \frac{dZ}{dq_{n-1}}, q_{n-1}\right) = \text{constant},$$

then

$$\Theta(Z, Z', q_{n-1}) = \text{ constant}$$

is the equation for determining p_n.

Through these operations the search for a complete solution of the given partial differential equation is carried so far that it only remains to carry out the integration

$$v = \int (p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n).$$

If one reduces all the systems occurring each to a differential equation of higher order, then one has in all to seek an integral each for

1	differential equation of order	$2(n-1)$,
2	differential equations of order	$2(n-2)$,
	...	
i	differential equations of order	$2(n-i)$,
	...	
$n-1$	differential equations of order	2.

But it is only in the most unfavourable cases that all differential equations actually reach the orders given here. In general in any class only *one* equation will reach this order. The orders of the others will be, more or less, lower.

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