THE
THEORY OF SOUND

Rayleigh

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THEORY OF SOUND

BY

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HONORARY FELLOW OF TRINITY COLLEGE, CAMBRIDGE

WITH A HISTORICAL INTRODUCTION BY

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HISTORICAL INTRODUCTION

BY ROBERT BRUCE LINDSAY

THE current reprinting of Lord Rayleigh's "The Theory of Sound," first published in 1877, stimulates an inquiry into the reason why such a treatise still retains a position of importance in the literature of its field, when most scientific treatises of sixty-five years ago now possess for the most part historical interest only, and have long since been superseded by twentieth century standard works. It has seemed appropriate on this occasion to review briefly the historical development of the subject of acoustics in which this situation has occurred, and to pay some tribute to the character and contributions of the author of a book which continues to show such vitality. It is hoped that the following introductory comments will enhance the pleasure of those who continue to turn to Rayleigh's treatise for enlightenment and guidance in acoustics.

I. BIOGRAPHICAL SKETCH OF JOHN WILLIAM STRUTT, THIRD BARON RAYLEIGH (1842-1919)

The author of "The Theory of Sound" occupies an unusual position in the history of British physics if only because, while there are numerous examples of men raised to the peerage as a reward for outstanding scientific work, it is rare to find a peer by inheritance devoting himself to science. Lord Rayleigh was born John William Strutt, the eldest son of the second Baron Rayleigh of Terling Place, Witham in the county of Essex. His immediate ancestors were country gentlefolk with little or no interest in scientific pursuits, though one of his grandmothers was descended from a brother of Robert Boyle. In his boyhood Rayleigh exhibited no unusual precocity but apparently displayed the average boy's interest in the world about him. His schooling was rather scattered, short stays at Eton and Harrow being terminated by ill-health. He finally spent the four years preceding college at a small boarding school kept by a Rev. Mr. Warner in Highstead, Torquay, where he showed no interest in classics but began to develop decided competence in mathematics.

In 1861 at the age of nearly 20, young Rayleigh went up to Cambridge and entered Trinity College. Here he became a pupil of

E. J. Routh, the famous "coach" in applied mathematics. It was under the guidance of Routh that he acquired the grasp of mathematics which stood him in such good stead in his later research. The system has often been criticized, but it ground the methods of advanced mathematical analysis essential for the physical scientist so thoroughly into the candidate that they became a natural part of his very being. It was not rigorous mathematics in the pure sense, but it was vigorous mathematics, which served to cultivate a keen appreciation of the particular method best suited for the solution of any particular problem. Rayleigh also stated in after life that he had profited greatly from the Cambridge lectures of Sir George G. Stokes, who though Lucasian professor of mathematics was greatly interested in experimental physics and performed many experimental demonstrations for his classes. In the Mathematical Tripos of 1865, Rayleigh came out as Senior Wrangler and also became first Smith's Prizeman. By this time he had clearly decided on a scientific career, though the propriety of this was considered by some members of his family rather doubtful in view of the social obligations inherent in his ultimate succession to his father's title and position. Rayleigh seems to have felt that such obligations should not be allowed to interfere with his scientific work. In 1866 he was elected Fellow of Trinity College, thus further emphasizing his scholarly leanings. Curiously enough he replaced the usual grand tour of the continent with a trip to the United States, then in the throes of reconstruction after the Civil War.

In 1868 immediately after his return from America Rayleigh purchased an outfit of experimental equipment. There was at that time no university physical laboratory, though certain professors possessed apparatus for their own experimental purposes and for demonstrations. Students received little or no direct encouragement to embark on experimental investigations for themselves. This may seem strange when one recalls that Cambridge had been for long the home of Newton. Moreover, long before Rayleigh's undergraduate days the immortal experiments of Young, Davy and Faraday, to mention only a few, had already shed undying lustre on British science. But this research had been carried on, by and large, outside the universities, which thus remained quite out of the current of real scientific progress in physics well past the first half of the nineteenth century. It was not until 1871 that Cambridge University established a professorship of experimental physics; in 1873 the Cavendish Laboratory was erected through the munificence of the Chancellor of the University, the eighth Duke of Devonshire. James Clerk Maxwell was elected the first Cavendish professor and served from 1871 to his untimely death in 1879. For the first time practical instruction from a distinguished physicist was provided at Cambridge.

To return to Rayleigh: it is interesting to observe that his first experimental investigations were on electricity and concerned the action of alternating currents on a galvanometer. The results were presented in a paper (his first) to the Norwich meeting of the British Association for the Advancement of Science in 1868. But he was soon thereafter deeply immersed in other things, including color vision and the pitch of resonators. The latter was his first work in acoustics and was apparently stimulated by his reading Helmholtz's famous work "On the Sensations of Tone" (1863). There was much correspondence about this and kindred matters with Maxwell, who was always eager to help along a youthful colleague. Rayleigh's experimental work was carried out at Terling in a rather crudely improvised laboratory. Later when the estate became his home by inheritance, more elaborate arrangements were made.

In 1871 Rayleigh married Evelyn Balfour, the sister of Arthur James Balfour, who was destined to gain much celebrity as a scholar, philosopher and statesman. He had become acquainted with Balfour as a fellow student at Cambridge. Shortly after his marriage a serious attack of rheumatic fever threatened for a time to cut short his career and left him much weakened in health. An excursion to Egypt was undertaken as a recuperative measure, and it was on a house boat trip up the Nile late in 1872 that the "Theory of Sound" had its genesis, the first part having been written with no access to a large library. The preparation of the treatise eventually extended over many years, and the two well-known volumes did not make their appearance from the press until 1877. In the meantime Rayleigh had succeeded to his father's title and had settled down at Terling. Changes were made to enable him to embark on more elaborate laboratory work, including experiments in acoustics and optics. It was during the period from 1871 to 1879 that he gave much attention to the diffraction of light and made copies of diffraction gratings. These investigations led to the introduction of the present standard definition of resolving power, a quantity of the utmost importance in specifying the performance of any optical instrument.

The premature death of Clerk Maxwell in 1879 left the Cavendish professorship vacant. Pressed by many scientific friends to stand for the post, Rayleigh finally consented, being partly influenced in his decision by the loss of income from his estate due to the agricultural depression of the late 70's. It does not appear that he ever contemplated retaining the professorship for an indefinite period, and indeed he ultimately limited his tenure to five years. The pedagogical duties of the Cavendish professor were not onerous: he was required to be in residence for eighteen weeks during the academic year and to deliver at least forty lectures in the course of this period. Rayleigh, however, had no desire to interpret the job as a sinecure. He embarked vigorously on a program of developing elementary laboratory instruction in a really elaborate way. It is difficult to appreciate today what a task such a program involved sixty years ago. Collegiate instruction in practical physics was almost a new thing, and there was little to go on save the teacher's imagination. Under Rayleigh's direction his demonstrators Glazebrook and Shaw, both of whom later became men of note, the former in applied physics and the latter in meteorology, developed laboratory courses for large classes in heat, electricity and magnetism, properties of matter, optics and acoustics. This was pioneer work of high order and had a beneficial influence on the teaching of physics throughout England and ultimately elsewhere.

Rayleigh was impressed at this time with the desirability of co-operative research on a problem of importance and selected for this purpose the redetermination of the standard electrical units. In particular he wished to undertake a new evaluation of the relation between the ohm, the practical unit of electrical resistance, and the electromagnetic unit of resistance. The first precision work on this problem had been carried out in 1863-64 under the auspices of the British Association with Maxwell in charge. Later work by others had disclosed considerable discrepancies. Rayleigh and his collaborators devoted three years of labor to a repetition of the original experiments with greater attention to sources of error. It is a tribute to Rayleigh's great experimental care that his final results have not been appreciably altered by more modern work. He appeared to possess the uncanny power to make the simplest of equipment produce the utmost in precision.

In December 1884 Rayleigh returned to Terling, which he made his scientific headquarters for the remainder of his life. It was close enough to London to permit frequent visits to the metropolis for the performance of official duties in connection with government or the various professional societies in which he played a prominent role. But he clearly enjoyed having his laboratory in his own home. Probably many contemporaries in the peerage, as well as the tenants on the estate, thought him a trifle queer, but he went his way with typical British imperturbability. The laboratory could hardly be considered elaborate even when judged by contemporary standards. Rayleigh had a hatred of superfluous elegance and always stressed the desirability of simplicity in all research apparatus. Some of this feeling was undoubtedly inspired by his constitutional aversion to unnecessary expenditure; there was also a profound philosophical implication in the method which may be of value to the present day investigator, even when surrounded by highly intricate and sophisticated apparatus.

The life of a scientist working at his desk or in his laboratory has little to offer in the way of the dramatic, at least to the man in the street. It is inevitable that mankind in the large should find more emotional satisfaction in the contemplation of man's relations with his fellow creatures than in his relations with the physical environment. For the most part, too, scientific investigations involve a train of reasoning unfamiliar and intricate to the general run of people. Occasionally, however, a scientific discovery will be made which involves a relatively simple and clear cut situation, while at the same time it solves a puzzle originally as baffling as any detective story mystery. This was the case with the most dramatic popular episode in Rayleigh's career, namely the discovery of the rare gas argon in the atmosphere.

Already in his address to the Mathematics and Physics Section of the British Association at the Southampton meeting in 1882, Rayleigh had called attention to the desirability of a more precise determination of the densities of the so-called permanent gases, oxygen, hydrogen and nitrogen. The importance of this lay in its bearing on the problem of the atomic weights of the elements and hence the whole foundation of chemistry. This job Rayleigh now set for himself and devoted to it a good part of his own time and that of a skilled assistant for the better part of ten years, culminating in the famous joint announcement with Sir William Ramsay of the isolation of argon in 1895. The story is too well known for detailed repetition here. It furnishes a classic example of the importance of following up a small experimental discrepancy lying outside the limit of reasonable experimental error, in this case the difference between the density of nitrogen prepared from nitrogen compounds and nitrogen obtained by removing the oxygen of the air. It seems easy to say now that the larger value of the latter points directly to the existence in the air of a small amount of a gas heavier than nitrogen. But in 1895 this was not so simple and neither was the task of isolating the new gas. It is not too much to say that the subsequent discovery of all the other rare gases of the atmosphere was directly due to Rayleigh's patient, ingenious and methodical investigation.

From 1887 to 1905, Lord Rayleigh served as Professor of Natural Philosophy at the Royal Institution of Great Britain as successor to John Tyndall, who in turn had succeeded Faraday. Unlike his predecessors Rayleigh spent comparatively little time in the laboratory of the Institution, confining his activity to the annual course of public lectures. These continued the tradition established by Faraday and Tyndall in covering the whole gamut of topics of physical interest with a profusion of experimental demonstrations. Sir Arthur Schuster says of Rayleigh in this connection: "Though not by nature a ready speaker, his lectures were effective." At any rate the auditor could always be confident that the speaker thoroughly understood what he was talking about.

In 1896 Rayleigh was appointed Scientific Adviser to Trinity House, a very ancient organization, dating back to Henry VIII, and having as its duties the erection and maintenance of such coastal installations as lighthouses, buoys and the like. For the next fifteen years he served faithfully and made numerous inspection trips. Some of his later work in optics and acoustics was suggested by problems arising in connection with the tests of lights and fog-signals. In spite of his devotion to his laboratory research, Rayleigh gave willingly of his time and energy to the deliberations of scientific committees of government and the various societies to which he belonged. Thus he was one of the leaders of the movement which led to the establishment of the National Physical Laboratory (the British counterpart of the National Bureau of Standards in Washington), and presided over the Executive Committee of the Laboratory until shortly before his death. He also served as President of the Advisory committee on Aeronautics from its inception in 1909 (at the instance of Prime Minister Asquith) until the time of his death. The activities of this committee were particularly important during the first world war from 1914-1918.

Among Lord Rayleigh's other public positions there is space only to mention his presidency of the Royal Society from 1905-1908 and his service as Chancellor of Cambridge University from 1908 until his death. Honors came to him in heaping measure, notable among them the Order of Merit, of which he was one of the first recipients in 1902, and the Nobel Prize in Physics in 1904.

Unlike most scientific men, Rayleigh was able to continue his work until his death, though he survived to the ripe old age of 76. He died on June 30, 1919, with three papers still unpublished. It is interesting that the last of these was one on acoustics: he never got over his interest in sound.

The opinion of his contemporaries and successors places Rayleigh in that great group of nineteenth century physicists that have made British science famous all over the world, the group whose other members were Kelvin, Maxwell and Stokes. His position in the history of science is a great one. It is good to recall that he was above all a modest man and it is impossible to accept as otherwise than sincere the remarks he made when he received the Order of Merit: "the only merit of which he personally was conscious was that of having pleased himself by his studies, and any results that may have been due to his researches were owing to the fact that it had been a pleasure to him to become a physicist."

II. HISTORICAL DEVELOPMENT OF ACOUSTICS TO THE TIME OF RAYLEIGH

Introduction. Sound plays in our daily lives a part scarcely less important than motion and light, and the sense of hearing though by no means esteemed so precious as the sense of sight and the ability to locomote is yet so prized that the production of efficient hearing aids for the deaf is fast becoming a major industry. Life is full of sounds and we want to hear the pleasant and vital ones, while shunning the unpleasant and dangerous variety. All told we are becoming steadily more sound conscious, as the relatively enormous growth of the telephone, radio, phonographic recording and talking motion picture industries sufficiently attests.

In view of its importance, it might be supposed that the science of sound, technically known as acoustics, would loom as a substantial item in the history of the development of physical ideas. Strangely enough, in the standard histories this is by no means the case: the history of acoustics has been largely a neglected subject. A possible reason for this has been advanced by Whewell in his "History of the Inductive Sciences." The basic theory of the origin, propagation and reception of sound was proposed at a very early stage in the development of human thought in substantially the form which we accept today: the ancient Greeks, according to the most reasonable interpretation of the records, evidently were aware that sound somehow arises from the motion of the parts of bodies, that it is transmitted by the air through some undefined motion of the latter and in this way ultimately striking the ear produces the sensation of hearing. Vague as these ideas were they were yet clarity itself compared with the ancient views on the motion of solid bodies as well as on light and heat. The latter branches of physics had to go through a long course of development in which theory succeeded theory until the present stage was reached. As Whewell emphasizes, in acoustics the basic theory was laid down early and all that was needed was its implementation by the necessary analysis and its application to new problems as they arose. On the theoretical side the history of acoustics thus tends to be merged in the larger development of mechanics as a whole.

It has seemed eminently worth while, however, in connection with a re-issue of the greatest single work ever published in acoustics to take advantage of the occasion to review the history of those parts of mechanics and other branches of physics which have had a definite bearing on acoustical theory. In a small measure this may serve to supplement D. C. Miller's interesting "Anecdotal History of the Science of Sound" (1935), which is devoted mainly to a resumé of the experimental phenomena.

The problems of acoustics as already indicated are most conveniently divided into three main groups, viz: 1) the production of sound, 2) the propagation of sound, and 3) the reception of sound. We shall find it advantageous to organize the following historical outline accordingly.

The Production of Sound. The fact that when a solid body is struck a sound is produced must have been observed from the very earliest times. The additional fact that under certain circumstances the sounds produced are particularly agreeable to the ear furnished the basis for the creation of music, which also originated long before the beginning of recorded history. But music was an art for centuries before its nature began to be examined in a scientific manner. It is usually assumed that the first Greek philosopher to study the origin of musical sounds was Pythagoras in the 6th century B.C. He is supposed to have discovered that of two stretched strings fastened at the ends the higher note is emitted by the shorter one, and that indeed if one has twice the length of the other, the shorter will emit a note an octave above the other. By this time the notion of pitch had, of course, been developed, but its association with the frequency of the vibrations of the sounding body was probably not understood, and it does not appear that this concept emerged until the time of Galileo Galilei (1564-1642), the founder of modern physics. At the very end of the "First Day" of Galileo's "Discourses Concerning Two New Sciences," first published in 1638, the reader will find a remarkable discussion of the vibrations of bodies. Beginning with the well known observations on the isochronism of the simple pendulum and the dependence of the frequency of vibration on the length of the suspension, Galileo goes on to describe the phenomenon of sympathetic vibrations or resonance by which the vibrations of one body can produce similar vibrations in another distant body. He reviews the common notions about the relation of the pitch of a vibrating string to its length and then expresses the opinion that the physical meaning of the relation is to be found in the number of vibrations per unit time. He says he was led to this point of view by an experiment in which he scraped a brass plate with an iron chisel and found that when a pure note of definite pitch was emitted the chisel cut the plate in a number of fine lines. When the pitch was high the lines were close together, while when the pitch was lower they were farther apart. Galileo was actually able to tune two spinet strings with two of these scraping tones; when the musical interval between the string notes was judged by the ear to be a fifth, the number of lines produced in the corresponding scrapings in the same total time interval bore precisely the ratio 3:2. The presumption is that if the octave had been tuned the ratio would have been 2:1, etc. It seems plain from a careful reading of Galileo's writings that he had a clear understanding of the dependence of the frequency of a stretched string on the length, tension and density. There was, of course, no question then of a dynamical discussion of the actual motion of the string: the theory of mechanics had not advanced far enough for that. But Galileo did make an interesting comparison between the vibrations of strings and pendulums in the endeavor to understand the reason why sounds of certain frequencies, i.e., those whose frequencies are in the ratio of two small integers, appear to the ear to combine pleasantly whereas others not possessing this property sound discordant. He observed that a set of pendulums of different lengths, set oscillating about a common axis and viewed in the original plane of their equilibrium positions present to the eye a pleasing pattern if the frequencies are simply commensurable, whereas they form a complicated jumble otherwise. This is a kinematic observation of great ingenuity and illustrates the fondness of the great Italian genius for analogy in physical description.

Credit is usually given to the Franciscan friar, Marin Mersenne (1588-1648) for the first correct published account of the vibrations of strings. This occurred in his "Harmonicorum Liber" published in Paris in 1636, two years before the appearance of Galileo's famous treatise on mechanics. However, it is now clear that Galileo's actual discovery antedated that of Mersenne. The latter did add one very important point: he actually measured the frequency of vibration of a long string and from this inferred the frequency of a shorter one of the same density and tension which gave a musical note. This was apparently the first direct determination of the frequency of a musical sound.

Though later experimenters like Robert Hooke (1635-1703) tried to connect frequency of vibration with pitch by allowing a cog wheel to run against a piece of cardboard, the most thorough-going pioneer studies of this matter were made by Joseph Sauveur (1653-1716), who incidentally first suggested the name *acoustics* for the science of sound. He employed an ingenious use of the beats between the sounds from two organ pipes which were adjudged by the ear to be a semi tone apart, i.e., having frequencies in the ratio 15/16. By experiment he found that when sounded together the pipes gave 6 beats a second. By treating this number as the difference between the frequencies of the pipes the conclusion was that these latter numbers were 90 and 96 respectively. Sauveur also worked with strings and calculated (1700) by a somewhat dubious method the frequency of a given stretched string from the measured sag of the central point. It was reserved to the English methematician Brook Taylor (1685-1731), the celebrated author of Taylor's Theorem on infinite series, to be the first to work out a strictly dynamical solution of the problem of the vibrating string. This was published in 1713 and was based on an assumed curve for the shape of the string of such a character that every point would reach the rectilinear position in the same time. From the equation of this curve and the Newtonian equation of motion he was able to derive a formula for the frequency of vibration agreeing with the experimental law of Galileo and Mersenne. Though only a special case, Taylor's treatment paved the way for the more elaborate mathematical techniques of Daniel Bernoulli (1700-1782), D'Alembert (1717-1783) and Euler (1707-1783), involving the introduction of partial derivatives and the representation of the equation of motion in the modern fashion.

In the meantime it had already been observed, notably by Wallis (1616-1703) in England as well as by Sauveur in France, that a stretched string can vibrate in parts with certain points, which Sauveur called *nodes*, at which no motion ever takes place, whereas very violent motion takes place at intermediate points called *loops*. It was soon recognized that such vibrations correspond to higher frequencies than that associated with the simple vibration of the string as a whole without nodes, and indeed that the frequencies are integral multiples of the frequency of the simple vibration. The corresponding emitted sounds were called by Sauveur the *harmonic* tones, while the sound associated with the simple vibration was named the *fundamental*. The notation thus introduced (about 1700) has survived to the present day. Sauveur noted the additional important fact that a vibrating string could produce the sounds corresponding to several of its harmonics at the same time. The dynamical explanation of this vibration was provided by Daniel Bernoulli in a celebrated memoir published by the Berlin Academy in 1755. Here he showed that it is possible for a string to vibrate in such a way that a multitude of simple harmonic oscillations are present at the same time and that each contributes independently to the resultant vibration, the displacement at any point of the string at any instant being the algebraic sum of the displacements for each simple harmonic node. This is the famous principle of the *coexistence of small oscillations*, also referred to as the *superposition principle*. It has proved of the utmost importance in the development of the theory of oscillations, though curiously enough its validity was at first strenuously doubted by D'Alembert and Euler, who saw at once that it led to the possibility of expressing any arbitrary function, e.g., the intial shape of a vibrating string, in terms of an infinite series of sines and cosines. The state of mathematics in the middle of the 18th century hardly permitted so bold a result. However, in 1822 Fourier (1768-1830) in his "Analytical Theory of Heat" did not hesitate to develop his celebrated theorem on this type of expansion with consequences of the greatest value for the advancement of acoustics.

The problem of the vibrating string was fully solved in elegant analytical fashion by J. L. Lagrange (1736-1813) in an extensive memoir of the Turin Academy in 1759. Here he supposed the string made up of a finite number of equally spaced identical mass particles and studied the motion of this system, establishing the existence of a number of independent frequencies equal to the number of particles. When he passed to the limit and allowed the number of particles to become infinitely great and the mass of each correspondingly small, these frequencies were found to be precisely the harmonic frequencies of the stretched string. The method of Lagrange was adopted by Rayleigh in his "Theory of Sound" and is indeed standard practise to-day, though most elementary books now develop the differential equation of motion of the string treated as a continuous medium by the method first set forth by D'Alembert in a memoir of the Berlin Academy of 1750. This differential equation we now call the wave equation, though the savants of the middle 18th century did not stress this interpretation.

In the memoir of Lagrange just referred to there is also a treatment of the sounds produced by organ pipes and musical wind instruments in general. The basic experimental facts were already known and

Lagrange was able to predict theoretically the approximate harmonic frequencies of closed and open pipes. The boundary conditions gave some trouble, as indeed they do to this day; in any case the problem impinges rather closely on the propagation of sound and as such is better treated in the next section.

The extension of the methods described in the preceding paragraphs to the vibrations of extended solid bodies like bars and plates naturally demanded a knowledge of the relation between the deformability of a solid body and the deforming force. Fortunately this problem had already been solved by Hooke, who in 1660 discovered and in 1676 announced in the form of the anagram *C E I I I N O S S S T T U V* the law "ut tensio sic vis" connecting the stress and strain for bodies undergoing *elastic* deformation. This law of course forms the basis for the whole mathematical theory of elasticity including elastic vibrations giving rise to sound. Its application to the vibrations of bars supported and clamped in various ways appears to have been made first by Euler in 1744 and Daniel Bernoulli in 1751, though it must be emphasized that dates of publication of memoirs do not always reflect accurately the time of discovery. The method used involved the variation of the expression for the work done in bending the bar. It is essentially that employed by Rayleigh in his treatise and leads of course to the well known equation of the fourth order in the space derivatives.

The corresponding analytical solution of the vibrations of a solid elastic plate came much later, though much experimental information was obtained in the latter part of the 18th century by the German E. F. F. Chladni (1756-1824), one of the greatest experimental acousticians. In 1787 he published his celebrated treatise "Entdeckungen über die Theorie des Klanges" in which he described his method of using sand sprinkled on vibrating plates to show the nodal lines. His figures were very beautiful and in a general way could be accounted for by considerations similar to those relating to vibrating strings. The exact forms, however, defied analysis for many years, even after the publication of Chladni's classic work "Die Akustik" in 1802. Napoleon provided for the Institute of France a prize of 3000 francs to be awarded for a satisfactory mathematical theory of the vibrations of plates. The prize was awarded in 1815 to Mlle. Sophie Germain, who gave the correct fourth order differential equation. Her choice of boundary conditions proved, however, to be incorrect. It was not until 1850 that Kirchhoff (1824-1887) gave a more accurate theory. The problem still provides considerable interest for workers even at the present time, both along theoretical and experimental lines.

In the meantime the analogous problem of the vibrations of a flexible membrane, important for the understanding of the sounds emitted by drum heads, was solved first by S. D. Poisson (1781-1840), though he did not complete the case of the circular membrane. This was done by Clebsch (1833-1872) in 1862. It is significant that most of the theoretical work on vibration problems during the 19th century was done by persons who called themselves mathematicians. This was natural though perhaps somewhat unfortunate, since the choice of conditions did not always reflect actual experimentally attainable situations. Rayleigh's own work did much to rectify this condition, and nowadays the experimental and theoretical acousticians work hand in hand. The importance of this is evident in the design of such modern instruments as loud speakers and quartz crystal vibrators.

A more complete description of the historical development of sound producers would, of course, necessarily pay much attention to musical instruments. Unfortunately this development lay rather aside from the scientific progress in acoustics, a situation which has persisted in large measure even to recent times. There are signs, however, that the designers of new musical instruments are paying more attention to acoustical principles than previously was the case, and that the theory of acoustics will have a greater influence on music in the future than it has had in the past.

We have now brought our brief sketch of the production of sound up to the time of Rayleigh. We shall therefore proceed with the equally important problem of the propagation of sound.

The Propagation of Sound. From the earliest recorded observations there has been rather general agreement that sound is conveyed from one point in space to another through some activity of the air. Aristotle, indeed, emphasizes that there is actual motion of the air involved, but as was often the case with his notions on physics his expressions are rather vague. Since in the transmission of sound the air certainly does not appear to move, it is not surprising that other philosophers denied Aristotle's view. Thus even during the Galilean period the French philosopher Gassendi (1592-1655), in his revival of the atomic theory, attributed the propagation of sound to the emission of a stream of fine, invisible particles from the sounding body which, after moving through the air, are able to affect the ear. Otto von Guericke (1602-1686) expressed great doubt that sound is conveyed by a motion of the air, observing that sound is transmitted better when the air is still than when there is a wind. Moreover he had tried around the middle of the 17th century the experiment of ringing a bell in a jar which was evacuated by means of his air pump, and claimed that he could still hear the sound. As a matter of fact, the first to try the bell-in-vacuum experiment was apparently the Jesuit Athanasius Kircher (1602-1680). He described it in his book "Musurgia Universalis", published in 1650, and concluded that air is not necessary for the transmission of sound. Undoubtedly there was not sufficient care to avoid transmission through the walls of the vessel. In 1660 Robert Boyle (1627-1691) in England repeated the experiment with a much improved air pump and more careful arrangements, and finally observed the now well known decrease in the intensity of the sound as the air is pumped out. He definitely concluded that the air is a medium for acoustic transmission, though presumably not the only one.

If air is the principal medium for the transmission of sound, the next question is: how rapidly does the propagation take place? As early as 1635 Gassendi while in Paris made measurements of the velocity of sound in air, using fire arms and assuming the passage of light as effectively infinite. His value was 1473 Paris feet per second. (The Paris foot is equivalent approximately to 32.48 cm.) Later by more careful measurements Mersenne showed this figure to be too high, obtaining 1380 Paris feet per second or about 450 meters/sec. Gassendi did note one matter of importance, namely that the velocity is independent of the pitch of the sound, thus discrediting the view of Aristotle, who had taught that high notes are transmitted faster than low notes. On the other hand Gassendi made the mistake of believing that the wind has no effect on the measured velocity of sound. In 1656 the Italian Borelli (1608-1679) and his colleague Viviani (1622-1703) made a more careful measurement and obtained 1077 Paris feet per second or 350 meters/sec. It is clear that all these values suffer from the lack of reference to the temperature, humidity and wind velocity conditions. It was not until 1740 that the Italian Branconi showed definitely by some experiments performed at Bologna that the velocity of sound in air increases with the temperature. Probably the first open air measurement of the velocity of sound that can be reckoned at all precise in the modern sense was carried out under the direction of the Academy of Sciences of Paris in 1738, using cannon fire. When reduced to 0°C the result was 332 meters/sec. Careful repetitions during the rest of the 18th century and the first half of the 19th century gave results differing by only a few meters per second from this value. The best modern value is $331.36 + 0.08$ meters per second in still air under standard conditions of temperature and pressure (0°C and 76 cm of Hg. pressure).

This value is taken from D. C. Miller's "Sound Waves: Their Shape and Speed" (1937).

In 1808 the French physicist J. B. Biot (1774-1862) made the first experiments on the velocity of sound in solid media, using for this purpose an iron water pipe in Paris nearly 1000 meters long. By comparing the times of arrival of sound through the metal and the air respectively it was established that the velocity of the compressional wave in the solid metal is many times greater than that through the air. As a matter of fact Chladni, whose work on the vibrations of solids has been mentioned earlier in this sketch, had already measured the velocity of elastic waves in rods in connection with his study of the vibration of solids, with results in general agreement with those of Biot.

J. D. Colladon and the mathematician J. C. F. Sturm (1803-1855) in the year 1826 investigated the transmission of sound through water in Lake Geneva, in Switzerland, using a sound and flash arrangement. The velocity was found to be 1435 meters/sec. at 8°C.

To return to the propagation of sound through air, though it had very early been compared with the motion of ripples on the surface of water, the first theoretical attempt to theorize seriously about a *wave* theory of sound was made by Isaac Newton (1642-1727), who in the second book of his Principia (1687) (Propositions 47, 48 and 49) compares the propagation of sound to pulses produced when a vibrating body moves the adjacent portions of the surrounding medium and these in turn move those next adjacent to themselves and so on. Newton here made some rather specific and arbitrary assumptions, among them the hypothesis that when a pulse is propagated through a fluid the particles of the fluid always move in simple harmonic motion, or, as he puts it "are always accelerated or retarded according to the law of the oscillating pendulum". He indeed affects to prove this as a theorem, but inspection fails to reveal any demonstration save that if it is true for one particle it will be true for all. He then assumes that the elastic medium under consideration is subject to the pressure produced by a homogeneous medium of height h and density equal to the density of the medium under consideration. Newton further imagines a pendulum whose length between the point of suspension and center of oscillation is h. It is then proved that in one period of the pendulum the pulse will travel a distance of $2\pi h$. But since the period of the pendulum is $2\pi\sqrt{h/g}$, it follows that the velocity of the pulse is $\sqrt{gh}$, and since for a homogeneous fluid of density ρ the pressure p produced at the bottom of a column of height h is $p = \rho h g$, it follows that the pulse velocity is $\sqrt{p/\rho}$.

This demonstration was severely criticized by Lagrange in his Turin memoir of 1759 (already mentioned) as well as in the later one of 1760, and indeed one must admit the conditions laid down by Newton are highly specialized: an elastic wave need not be harmonic, nor should the velocity depend on this assumption. Lagrange gave a more rigorous general derivation, the outcome of which, however, must have surprised him, for it led to precisely Newton's result. When the relevant data for air at 60°F are substituted into Newton's formula, the velocity proves to be about 945 feet/second. At the time of his deduction this was not in bad order of magnitude agreement with the observed velocity of sound in air under the conditions cited. However, the more accurate measurements consistently turned out higher, and Newton was himself dissatisfied; hence, in the second edition of the Principia (1713) he revised his theory to try to bring it into better agreement with the best experimental value of the time, viz., 1142 feet/second. His explanation was so obviously *ad hoc* that it should have failed to carry conviction. However, no further serious question about the matter appears to have been raised until 1816 when Laplace suggested that in the previous determinations an error had been made in using the isothermal volume elasticity of the air, i.e. the pressure itself, thereby assuming that the elastic motions of the air particles take place at constant temperature. In view of the rapidity of the motions, however, it seemed to him more reasonable to suppose that the compressions and rarefactions follow the adiabatic law in which the changes in temperature lead to a higher value of the elasticity, namely, the product of the pressure by the ratio γ of the two specific heats of the air. At the time of Laplace's first investigation rather crude experiments had indeed indicated the existence of two specific heats of a gas, but the values were not known with much precision. Laplace used some data of the experimentalists, LaRoche and Berard, giving $\gamma = 1.5$ and leading to a value of the velocity of sound at 6°C equal to 345.9 meters/sec. The best experimental value obtained up to that time by members of the Academy was 337.18 meters/sec. for this temperature. Laplace did not consider this discrepancy serious. He returned to the problem later and included a chapter on the velocity of sound in air in his famous "Méchanique Céleste" (1825). By that time Clément and Désormes had performed their well-known experiment on the determination of γ (1819) and had obtained the value of 1.35 leading to 332.9 meters/sec. for the velocity. Some years later the more accurate value of $\gamma = 1.41$ led to complete agreement with the measured velocity. The theory of

Laplace is so well established that it is now common practice to determine γ for various gases by precision measurements of the velocity of sound.

As has already been remarked the first treatment of the partial differential equation of wave motion came with D'Alembert in 1750 in connection with the vibration of strings. The rest of the 18th century saw numerous attempts to theorize about waves in continuous media, such as waves on the surface of water and the like. These had value in connection with acoustics only to the extent that they rendered the use of the wave equation familiar to workers in sound. By the end of the 18th century the general treatment of the solution of the wave equation for sound in tubes, for example, subject to the boundary conditions at the ends, had been pretty well established, and the predicted harmonic frequencies checked with experiment with reasonable accuracy. Of course there were discrepancies leading to end corrections and so forth, which were never fully cleared up until the time of Rayleigh. It was not until 1868 that A. Kundt (1839-1894) developed his simple but effective method of dust figures for studying experimentally the propagation of sound in tubes and in particular measuring sound velocity from standing wave patterns.

In the meantime the more difficult problem of the propagation of a compressional wave in a three dimensional fluid medium had been attacked by Poisson in a celebrated memoir of 1820. The method was essentially that adopted by Rayleigh in Chap. XIV of "Theory of Sound". Three years before in a similar memoir, 100 pages long, Poisson had given the most elaborate theory up to that time of the propagation of sound in tubes, including the theory of stationary air waves for tubes of finite length, both open and closed. He even considered the possibility of an end correction in the case of an open tube to take care of the fact that the condensation cannot be considered precisely zero at the open end. It remained, however, for Hermann von Helmholtz (1821-1894) in 1860 to give a more thorough treatment of this question. The special case of an abrupt change in cross-section was also studied by Poisson along with the reflection and transmission of sound at normal incidence on the boundary of two different fluids. Much modern work of practical significance was anticipated in this great study of Poisson.

The more difficult problem of the reflection and refraction of a plane sound wave incident *obliquely* on the boundary of two different fluids was solved by the self-taught Nottingham genius George Green (1793-1841) in 1838. This served to emphasize both the similarities and differences between the reflection and refraction of sound and light. It should be recalled that sound waves in fluids, being strictly compressional, are longitudinal, whereas light waves are transverse. Hence light waves can be polarized, and sound waves in fluids can not. Of course elastic waves in an extended solid can be both longitudinal and transverse, more accurately irrotational and solenoidal. This was realized by Poisson in his study of isotropic elastic media of 1829. The direct significance of this for acoustics is, of course, not great, but it had an important bearing on the elastic solid theory of light, which was actively pursued during the middle decades of the 19th century. The connection with modern geophysics (seismological waves) is obvious.

The Reception of Sound. In the historical development of acoustics up to very recent times the only sound receiver of interest has been the human ear and the reception of sound has been largely the study of the acoustical behavior of this organ. In this connection it is interesting to observe that no completely acceptable theory of audition has ever been proposed, and how we hear still remains a puzzling problem in modern psychophysics.

After the relation between pitch and frequency had been established it became an interesting task to determine the frequency limits of audibility. F. Savart (1791-1841) using fans and toothed wheels (1830) placed the minimum audible frequency at 8 vibrations per second and the upper limit at 24,000 vibrations per second. The later investigators Seebeck (1770-1831), Biot (1774-1862), K. R. Koenig (1832-1901) and Hermann von Helmholtz obtained values for the lower limit ranging from 16 to 32 vibrations per second. In such matters there are bound to be individual differences. These play an even greater role in the upper limit of audibility, which not only can vary many thousand vibrations per second from person to person, but for each individual usually decreases with age. The most elaborate studies on audibility during the 19th century were made by Koenig, who devoted a lifetime to the production of precision sources of sound of controlled frequency, such as tuning forks, rods, strings and pipes. The electrically driven fork also originated with him.

The closely related problem of the minimum sound amplitude or intensity necessary for audibility was apparently first studied by Toepler (1836-1912) and Boltzmann (1844-1906) in 1870. The more recent work dates from Rayleigh.

In 1843, Georg S. Ohm, the author of the famous law of electric currents, put forward a law of audition according to which all musical tones arise from simple harmonic vibrations of definite frequency, and the particular quality of actual musical sounds is due to combinations of simple tones of commensurable frequencies. He held, moreover, that the ear is able to analyze any complex note into the set of simple tones in terms of which it may be expanded mathematically by means of Fourier's theorem. This law has stimulated a host of researches in physiological acoustics. The greatest of these in the pre-Rayleigh period were undoubtedly those of Helmholtz, whose treatise "Die Lehre von den Tonempfindungen als Physiologische Grundlage für die Theorie der Musik", published in 1862, ranks as one of the great masterpieces of acoustics. Here he gave the first elaborate theory of the mechanism of the ear, the so-called resonance theory, and was able to justify theoretically the law of Ohm. In the course of his investigations he invented the resonator, now so well known by his name and employed in modern acoustics for many applications. Helmholtz developed the theory of summation and difference tones and in general laid the ground work for all subsequent research in the field of audition. One of the greatest physicists of the 19th century, he touched no field that he did not enrich with his experimental and theoretical genius.

Since the reception of sound by the ear in enclosed spaces like rooms and auditoriums is a common experience, it is proper that some attention should be paid here to what has come to be called architectural acoustics. The first discussion of the problem of improving hearing in rooms was limited to purely geometrical considerations, such as the installation of sounding boards and other reflectors. A Boston physician, J. B. Upham, in 1853 wrote several papers indicating a much clearer grasp of the more important matter involved, namely the reverberation or multiple reflection of the sound from all the surfaces of the room. He also showed how the reverberation time could be reduced by the installation of fabric curtains and upholstered furnishings. In 1856 Joseph Henry, the celebrated American physicist, who became the first secretary of the Smithsonian Institution, made a study of auditorium acoustics which reflects a thorough understanding of all the factors involved, though his suggestions were all of a qualitative character. In spite of this the subject was completely neglected by architects, and attempts were often made to correct gross acoustical defects by such inadequate, if not absurd, devices as stringing wires, etc. The real quantitative foundation of architectural acoustics dates from W. C. Sabine (1868-1919) in 1900.

Special devices for the amplification of sound received by the ear go back a long way. Horns for the production of sounds are of great antiquity. It is uncertain just when the suggestion arose that they might be used to improve the *reception* of sound. At all events the Jesuit Athanasius Kircher, already mentioned in this sketch, in 1650 designed a parabolic horn as a hearing aid as well as a speaking trumpet, and evidently realized the importance of the flare in the amplification. Robert Hooke suggested the possibility of a device to magnify the sounds of the body, but it seems to have been reserved for the French physician René Laënnec (1781-1826) actually to invent and employ the stethoscope for clinical purposes (1819). Sir Charles Wheatstone (1802-1875) in 1827 developed a similar instrument which he termed a microphone, a name now applied to an exclusively electrical device for the reception of sound. Koenig also invented a new type of stethoscope. The theoretical and experimental improvement of instruments like horns and other sound receivers of similar type has been and still is an important feature of modern acoustics.

All through the historical development of physics there has been a tendency to reduce the observation of physical phenomena and particularly experimental measurements to something which can be *seen*. Practically all physical instruments involve this principle and employ a pointer or a spot of light moving on a scale. It was therefore inevitable that attempts would be made to study sound phenomena visually, and this of course was especially necessary for the investigation of sounds whose frequencies lie outside the range of audibility of the ear. One of the first moves in this direction was the observation by John LeConte (1818-1891) that musical sounds can produce jumping in a gas flame if the pressure is properly adjusted (1858). The sensitive flame, as it later came to be called, was developed to a high pitch of excellence by John Tyndall (1820-1893), who used it for the detection of high frequency sounds and the study of the reflection, refraction and diffraction of sound waves. It still provides a very effective lecture demonstration but for practical purposes has been superseded in recent times by various types of electrical microphones.

In the endeavor to make visible the form of a sound wave Koenig about 1860 invented the manometric flame device which consists of a box through which gas flows to a burner. One side of the box is a flexible membrane. When sound waves impinge on the membrane the changes in pressure produce corresponding fluctuations in the flame which can be made visible by reflecting the light of the flame from a rapidly rotating mirror. Another attempt to visualize sound waves was made by Leon Scott in 1857 in his "phonautograph" in which a flexible diaphragm at the throat of a receiving horn was attached to a stylus which in turn touched a smoked rotating drum surface and traced out a curve corresponding to the incident sound. This was the precursor of the phonograph. An equally ambitious attempt to record sound was made by Eli Whitney Blake (1836-1895), the first Hazard Professor of Physics in Brown University, who in 1878 made a microphone by attaching a small metallic mirror to a vibrating disc at the back of a telephone mouthpiece. By reflecting a beam of light from the mirror, Blake succeeded in photographing the sounds of human speech. Such studies were much advanced by D. C. Miller, (1866-1941) who invented a similar instrument in the "phonodeik" and made very elaborate photographs of sound wave forms.

III. RAYLEIGH'S CONTRIBUTIONS TO ACOUSTICS AND THEIR SIGNIFICANCE FOR MODERN DEVELOPMENTS

The results of Rayleigh's work in acoustics are embodied in his treatise "The Theory of Sound" and in 128 published articles, the first of which appeared in 1870 (his fourth paper) and the last in 1919—this was his last published paper and appeared in print after his death. Except for the years 1895, 1896 and 1906, there was not a year from 1870 to 1919 in which an article having a definite connection with acoustics did not appear. This record of devotion to a single department of thought is undoubtedly unique in the annals of science and becomes all the more remarkable when we recall that this activity was accompanied by unchecked attention to a host of other problems extending over the whole field of physics, leading to a total of nearly 450 publications.

Lord Rayleigh appeared on the acoustical scene when the time was precisely ripe for a synthesis of experimental phenomena and rather highly developed theory, much of which was, however, too idealized for practical application. On the other hand much of the experimental work had been discussed in rather empirical fashion with little attempt at a dynamical explanation. Rayleigh's interest in acoustics appears to have been started through the advice of Professor W. F. Donkin, Savilian Professor of Astronomy at Oxford, that he ought to learn to read German. Rayleigh followed the suggestion and the first scientific work he read was Helmholtz's treatise "Lehre von der Tonempfindung". Certain references here to the properties of acoustic resonators attracted his attention and led to his first elaborate research, reported on in a long paper on the theory of resonance in the *Philosophical Transactions of the Royal Society* in 1870. This article furnishes a clear indication of the method of thinking about problems that remained characteristic of all Rayleigh's later work. He endeavored to develop the mathematical theory of the subject in a form related as closely as possible to experimentally realizable situations, and then followed up the results by the attempt at direct experimental verification. There was no pretense of an over-elaborate method of measurement, but the precision was fully sufficient in view of the inevitable limitations of the theory of aerial vibrations. In this paper Rayleigh first introduced the useful concept of the *acoustic conductivity* of an orifice. It has remained a standard acoustical quantity ever since, even if rather difficult to to estimate theoretically for all sorts of openings.

It was evidently not long after the publication of his researches on resonance that Rayleigh conceived the desirability of writing a treatise on acoustics. His reasons for the step are amply set forth in the preface to the first edition of "The Theory of Sound" and need no repetition here. In preparation for his task he studied in detail the general theory of vibrations of a dynamical system about a state of equilibrium and uncovered a number of general results of great interest. These were presented in the *Proceedings of the London Mathematical Society* in 1873 and include such theorems as that the increase in the mass of any part of a vibrating system can never lead to a decrease in any period of the motion. Here he also introduced his famous dissipation function for a system subject to damping forces proportional to the component velocities and finally proved a very general reciprocity theorem of which the one generally known by the name of Helmholtz is a special case. This theorem has been of the greatest importance in comparing the efficiency of acoustical devices as emitters and receivers of radiation energy. As before, a characteristic feature of these articles is the skillful combination of theory and observation or experiment. Rarely does one find a mass of analysis without illustrations from experience, and Rayleigh was always very keen to follow up supposed experimental exceptions to theoretically deduced laws. Usually his uncanny insight into the important things led him to the correct explanation of apparent difficulties.

In 1877, the year of the publication of "The Theory of Sound", Rayleigh inaugurated the custom of publishing collections of miscellaneous acoustical phenomena which he had himself observed. These were continued at intervals for the rest of his life, being published for the most part in the *Philosophical Magazine*. Among the earlier subjects investigated were the perception of the direction of a sound source, the diffracting effect of the head on spoken and received sound, the end correction of an organ pipe, sensitive flames, Aeolian tones, acoustical shadows, etc.

"The Theory of Sound" was published in June, 1877. Though, as his son remarks "the sale was not wholly unprofitable", it was hardly a best seller. Those interested in the general field realized its importance, but the possible fundamental significance of the work for future applications of sound to a host of practical problems could scarcely be properly estimated at that time. Helmholtz, it is true, reviewed the volumes in *Nature* and compared the treatment to the famous unfinished "Treatise on Natural Philosophy" of Thomson and Tait. He pressed, indeed, for a third volume on physiological acoustics and the maintenance of acoustic vibrations. Rather wisely, it seems, Rayleigh refrained from this and contented himself with enriching the literature of acoustics for the following forty years with a succession of attractive papers on a wide variety of topics, many if not most of which were a direct outgrowth of the treatment inaugurated in his treatise.

While it would be gratuitous in the extreme to present a detailed analysis of the contents of "The Theory of Sound" to the reader who has the book before him, it is difficult to refrain from emphasizing briefly some of the features which have made the treatise such a mine of information for all workers in acoustics from Rayleigh's day to the present time.

Though written in the rather informal style which characterized practically all of Rayleigh's published work, the book reflects clearly a great deal of careful planning with respect to its logical structure. The author was evidently impressed by the importance of the subdivision of the subject into the two principal sections: the production and propagation of sound. Hence the whole of the first half (the first volume in the original edition), with the exception of an introductory chapter, is devoted to the vibrations of dynamical systems, naturally with special emphasis on those giving rise to acoustically interesting radiation. In contrast to the usual continental European method of writing a treatise, Rayleigh's treatment opens with the simplest possible case, namely the oscillations of a system of one degree of freedom, and each element of the theory is accompanied by a definite experimental illustration.

The simple case is followed by two chapters on the general theory of vibrations of a system of n degrees of freedom, largely a development of his 1873 paper mentioned just above. It was here he emphasized the value of the method of obtaining an approximation to the lowest frequency of vibration of a complicated system in which the direct solution of the differential equations is impracticable. This procedure, which makes use of the expressions for the maximum potential and kinetic energies, was later generalized by Ritz and is now usually known as the Rayleigh-Ritz method. It has proved of value in handling not only all sorts of involved vibration problems but also problems in quantum mechanics. Applications to acoustics occur frequently throughout "The Theory of Sound", particularly with reference to non-uniform strings, bars, membranes and plates. Throughout his treatise Rayleigh displays great fondness for the use of energy considerations and uses the energy method (virtual work) freely for setting up the differential equations of motion of different types of vibrating systems. It is scarcely an exaggeration to say that there is no vibrating system likely to be encountered in practice which cannot be tackled successfully by the methods set forth in the first ten chapters of Rayleigh's treatise. Even the worker in the field of non-linear systems, a department of increasing practical importance in modern vibration theory, will find useful basic hints in Rayleigh. The reader should, indeed, be cautioned not to consider "The Theory of Sound" as a mere reference book. One who goes to it in this frame of mind is apt to be disappointed. It is a rather closely knit work in which the author, having developed certain methods, feels free to refer the reader to them again and again. Hence reading Rayleigh is a real process of discovery, not always easy but constantly challenging and illuminating. One rather trivial mathematical detail may properly be mentioned at this point. Rayleigh's mathematical notation is standard in nearly all respects from the standpoint of present-day fashions, but he never uses the round ∂ to denote the distinction between partial and ordinary derivatives. Presumably he felt that the reader with a suitable grasp of the physical meaning of the mathematical processes would have no difficulty in distinguishing the one type of derivative from the other.

The last thirteen chapters of "The Theory of Sound" are devoted primarily to acoustic radiation through fluid media. This is by far the more difficult part of the subject matter of acoustics and has remained so to the present time. Since there is no such thing as a perfect fluid the exact hydrodynamical equations describing with precision the motion of a compressional disturbance in a fluid medium like air or water must necessarily be extremely complicated. It has therefore proved desirable to approximate, and it is just here that the judgment of the physicist plays a significant role. Rayleigh possessed the power of assessing a problem from the point of view of the best possible approximation to lead to a physically useful result. This is particularly well illustrated by his studies of the diffraction and scattering of sound by obstacles, which is by no means so easy to study theoretically as in the analogous case of optics, largely because of the approximate character of the equations and the relatively large wave length of audible sound. Another illustration is the acoustic radiation into a surrounding fluid medium from a vibrating sphere or plate. The whole modern study of high frequency acoustic beams is based on this work. Still another example is provided by the effect of viscosity and heat conduction on the propagation of sound. Here the fundamental theory had already been worked out by other men like Stokes and Kirchhoff, but Rayleigh seemed able to seize on the useful applications to the transmission of sound through narrow tubes and the interstices of fabrics. He was aware that these effects are inadequate to account for the actually observed absorption of sound in three dimensional fluid media like the atmosphere or the sea. Progress in the solution of this problem at the present time is actually being made along the lines of a hint thrown out by Rayleigh in a paper on the cooling of air by radiation and conduction published in 1899.

A second revised and enlarged edition of "The Theory of Sound" was brought out by the author between the years 1894 and 1896, embodying the results of his investigations in the seventeen years which had elapsed since the first appearance of the book. No further revisions or reprintings were made until after Rayleigh's death. This would seem to reflect a rather stagnant state of acoustics during the first two decades of the twentieth century. Compared with the activity of university physical laboratories in other fields this must be considered the truth: academically, acoustics became, by and large, an uninteresting subject. In the meantime, however, the development of certain technological fields such as telephony, both with and without wires, as well as acoustic signalling under water and architectural acoustics, made it imperative for engineers to gain a better understanding of the theory of acoustics. The large industrial concerns began to make use of the subject in their research and development laboratories, and the whole field received a stimulus such as it probably never could have gained from the side of academic workers. We may say that acoustics was rediscovered and along with it Rayleigh's book. Reprintings were called for in rapid succession in 1926 and 1929 at about the time of the founding of the Acoustical Society of America (1928). At the same time numerous books began to appear whose main purpose was largely to interpret Rayleigh's work to the new workers in the subject, and to apply the methods of his treatise to a multitude of new and practical problems.

It would be absurd to maintain that the whole of acoustics is to be found within the covers of "The Theory of Sound". Rayleigh himself in some 60 papers published between 1900 and his death advanced the subject mightily and called attention to many problems which have turned out to be of great significance in recent applications. Among these foreshadowings of the future must be reckoned his use of electric circuit analogies in connection with the forced vibrations of acoustical resonators and other systems. This procedure has developed to such an extent that the modern acoustical engineer, using electrical equipment for most of his practical work, invariably insists on expressing all acoustical systems in terms of their electrical analogues. Other striking anticipations by Rayleigh of modern acoustical considerations concern the use of conical horns for the production and reception of sound in signalling, the acoustic shadow of a sphere (of particular significance in the diffraction effect of a microphone), the pressure of acoustic radiation (used in the measurement of sound intensity, especially in supersonics), the binaural effect in sound perception, the possible regime of sound waves of finite amplitude (explosion waves and those associated with gun fire) and the selective transmission of waves through stratified media (acoustical filtration). The list could easily be extended, but this will suffice to suggest to the contemporary worker in acoustics his debt to Rayleigh's foresight.

No one can foresee the future of the science of acoustics as, on the one hand, it reaches out into new realms of application in the engineering fields of the recording and reproduction of sound, the creation of more comfortable environments for the hearing of sound and the development of adequate hearing aids for the deaf, and, on the other hand, joins forces with pure physics and chemistry in the endeavor to learn more about the solid, liquid and gaseous states of matter, particularly through the agency of supersonics. It is safe to predict, however, that for a long time to come Lord Rayleigh's "The Theory of Sound" will be a *vade mecum* for both the pure and applied acoustician.

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PREFACE.

IN the work, of which the present volume is an instalment, my endeavour has been to lay before the reader a connected exposition of the theory of sound, which should include the more important of the advances made in modern times by Mathematicians and Physicists. The importance of the object which I have had in view will not, I think, be disputed by those competent to judge. At the present time many of the most valuable contributions to science are to be found only in scattered periodicals and transactions of societies, published in various parts of the world and in several languages, and are often practically inaccessible to those who do not happen to live in the neighbourhood of large public libraries. In such a state of things the mechanical impediments to study entail an amount of unremunerative labour and consequent hindrance to the advancement of science which it would be difficult to overestimate.

Since the well-known Article on Sound in the *Encyclopædia Metropolitana*, by Sir John Herschel (1845), no complete work has been published in which the subject is treated mathematically. By the premature death of Prof. Donkin the scientific world was deprived of one whose mathematical attainments in combination with a practical knowledge of music qualified him in a special manner to write on Sound. The first part of his *Acoustics* (1870), though little more than a fragment, is sufficient to shew that my labours would have been unnecessary had Prof. Donkin lived to complete his work.

In the choice of topics to be dealt with in a work on Sound, I have for the most part followed the example of my predecessors. To a great extent the theory of Sound, as commonly understood, covers the same ground as the theory of Vibrations in general; but, unless some limitation were admitted, the consideration of such subjects as the Tides, not to speak of Optics, would have to be included. As a general rule we shall confine ourselves to those classes of vibrations for which our ears afford a ready made and wonderfully sensitive instrument of investigation. Without ears we should hardly care much more about vibrations than without eyes we should care about light.

The present volume includes chapters on the vibrations of systems in general, in which, I hope, will be recognised some novelty of treatment and results, followed by a more detailed consideration of special systems, such as stretched strings, bars, membranes, and plates. The second volume, of which a considerable portion is already written, will commence with aërial vibrations.

My best thanks are due to Mr H. M. Taylor of Trinity College, Cambridge, who has been good enough to read the proofs. By his kind assistance several errors and obscurities have been eliminated, and the volume generally has been rendered less imperfect than it would otherwise have been.

Any corrections, or suggestions for improvements, with which my readers may favour me will be highly appreciated.

TERLING PLACE, WITHAM,
April, 1877.

IN this second edition all corrections of importance are noted, and new matter appears either as fresh sections, e.g. § 32 *a*, or enclosed in square brackets [ ]. Two new chapters X A, X B are interpolated, devoted to *Curved Plates or Shells*, and to *Electrical Vibrations*. Much of the additional matter relates to the more difficult parts of the subject and will be passed over by the reader on a first perusal.

In the mathematical investigations I have usually employed such methods as present themselves naturally to a physicist. The pure mathematician will complain, and (it must be confessed) sometimes with justice, of deficient rigour. But to this question there are two sides. For, however important it may be to maintain a uniformly high standard in pure mathematics, the physicist may occasionally do well to rest content with arguments which are fairly satisfactory and conclusive from his point of view. To his mind, exercised in a different order of ideas, the more severe procedure of the pure mathematician may appear not more but less demonstrative. And further, in many cases of difficulty to insist upon the highest standard would mean the exclusion of the subject altogether in view of the space that would be required.

In the first edition much stress was laid upon the establishment of general theorems by means of Lagrange's method, and I am more than ever impressed with the advantages of this procedure. It not unfrequently happens that a theorem can be thus demonstrated in all its generality with less mathematical apparatus than is required for dealing with particular cases by special methods.

During the revision of the proof-sheets I have again had the very great advantage of the cooperation of Mr H. M. Taylor, until he was unfortunately compelled to desist. To him and to several other friends my thanks are due for valuable suggestions.

July, 1894.

EDITORIAL NOTE FOR THE 1929 RE-ISSUE

In this re-issue, a few pencilled corrections and references in the Author's own copy have been made use of. Otherwise no change has been made.

EDITORIAL NOTE FOR THE PRESENT 1945 RE-ISSUE

In this re-issue, a Historical Introduction by Robert Bruce Lindsay had been added, and both volumes are bound as one. The text remains the same as the 1929 re-issue.

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CHAPTER I.

INTRODUCTION.

1. THE sensation of sound is a thing *sui generis*, not comparable with any of our other sensations. No one can express the relation between a sound and a colour or a smell. Directly or indirectly, all questions connected with this subject must come for decision to the ear, as the organ of hearing; and from it there can be no appeal. But we are not therefore to infer that all acoustical investigations are conducted with the unassisted ear. When once we have discovered the physical phenomena which constitute the foundation of sound, our explorations are in great measure transferred to another field lying within the dominion of the principles of Mechanics. Important laws are in this way arrived at, to which the sensations of the ear cannot but conform.

2. Very cursory observation often suffices to shew that sounding bodies are in a state of vibration, and that the phenomena of sound and vibration are closely connected. When a vibrating bell or string is touched by the finger, the sound ceases at the same moment that the vibration is damped. But, in order to affect the sense of hearing, it is not enough to have a vibrating instrument; there must also be an uninterrupted communication between the instrument and the ear. A bell rung *in vacuo*, with proper precautions to prevent the communication of motion, remains inaudible. In the air of the atmosphere, however, sounds have a universal vehicle, capable of conveying them without break from the most variously constituted sources to the recesses of the ear.

3. The passage of sound is not instantaneous. When a gun is fired at a distance, a very perceptible interval separates the report from the flash. This represents the time occupied by sound in travelling from the gun to the observer, the retardation of the flash due to the finite velocity of light being altogether negligible. The first accurate experiments were made by some members of the French Academy, in 1738. Cannons were fired, and the retardation of the reports at different distances observed. The principal precaution necessary is to reverse alternately the direction along which the sound travels, in order to eliminate the influence of the motion of the air in mass. Down the wind, for instance, sound travels relatively to the earth faster than its proper rate, for the velocity of the wind is added to that proper to the propagation of sound in still air. For still dry air at a temperature of 0°C., the French observers found a velocity of 337 metres per second. Observations of the same character were made by Arago and others in 1822; by the Dutch physicists Moll, van Beek and Kuytenbrouwer at Amsterdam; by Bravais and Martins between the top of the Faulhorn and a station below; and by others. The general result has been to give a somewhat lower value for the velocity of sound—about 332 metres per second. The effect of alteration of temperature and pressure on the propagation of sound will be best considered in connection with the mechanical theory.

4. It is a direct consequence of observation, that within wide limits, the velocity of sound is independent, or at least very nearly independent, of its intensity, and also of its pitch. Were this otherwise, a quick piece of music would be heard at a little distance hopelessly confused and discordant. But when the disturbances are very violent and abrupt, so that the alterations of density concerned are comparable with the whole density of the air, the simplicity of this law may be departed from.

5. An elaborate series of experiments on the propagation of sound in long tubes (water-pipes) has been made by Regnault[1]. He adopted an automatic arrangement similar in principle to that used for measuring the speed of projectiles. At the moment when a pistol is fired at one end of the tube a wire conveying an electric current is ruptured by the shock. This causes the withdrawal of a tracing point which was previously marking a line on a revolving drum. At the further end of the pipe is a stretched membrane so arranged that when on the arrival of the sound it yields to the

[1] *Mémoires de l'Académie de France*, t. XXXVII.

impulse, the circuit, which was ruptured during the passage of the sound, is recompleted. At the same moment the tracing point falls back on the drum. The blank space left unmarked corresponds to the time occupied by the sound in making the journey, and, when the motion of the drum is known, gives the means of determining it. The length of the journey between the first wire and the membrane is found by direct measurement. In these experiments the velocity of sound appeared to be not quite independent of the diameter of the pipe, which varied from $0^{m}\cdot108$ to $1^{m}\cdot100$. The discrepancy is perhaps due to friction, whose influence would be greater in smaller pipes.

6. Although, in practice, air is usually the vehicle of sound, other gases, liquids and solids are equally capable of conveying it. In most cases, however, the means of making a direct measurement of the velocity of sound are wanting, and we are not yet in a position to consider the indirect methods. But in the case of water the same difficulty does not occur. In the year 1826, Colladon and Sturm investigated the propagation of sound in the Lake of Geneva. The striking of a bell at one station was simultaneous with a flash of gunpowder. The observer at a second station measured the interval between the flash and the arrival of the sound, applying his ear to a tube carried beneath the surface. At a temperature of 8°C., the velocity of sound in water was thus found to be 1435 metres per second.

7. The conveyance of sound by solids may be illustrated by a pretty experiment due to Wheatstone. One end of a metallic wire is connected with the sound-board of a pianoforte, and the other taken through the partitions or floors into another part of the building, where naturally nothing would be audible. If a resonance-board (such as a violin) be now placed in contact with the wire, a tune played on the piano is easily heard, and the sound seems to emanate from the resonance-board. [Mechanical telephones upon this principle have been introduced into practical use for the conveyance of speech.]

8. In an open space the intensity of sound falls off with great rapidity as the distance from the source increases. The same amount of motion has to do duty over surfaces ever increasing as the squares of the distance. Anything that confines the sound will tend to diminish the falling off of intensity. Thus over the flat surface of still water, a sound carries further than over broken ground; the corner between a smooth pavement and a vertical wall is still better; but the most effective of all is a tube-like enclosure, which prevents spreading altogether. The use of speaking tubes to facilitate communication between the different parts of a building is well known. If it were not for certain effects (frictional and other) due to the sides of the tube, sound might be thus conveyed with little loss to very great distances.

9. Before proceeding further we must consider a distinction, which is of great importance, though not free from difficulty. Sounds may be classed as musical and unmusical; the former for convenience may be called *notes* and the latter *noises*. The extreme cases will raise no dispute; every one recognises the difference between the note of a pianoforte and the creaking of a shoe. But it is not so easy to draw the line of separation. In the first place few notes are free from all unmusical accompaniment. With organ pipes especially, the hissing of the wind as it escapes at the mouth may be heard beside the proper note of the pipe. And, secondly, many noises so far partake of a musical character as to have a definite pitch. This is more easily recognised in a sequence, giving, for example, the common chord, than by continued attention to an individual instance. The experiment may be made by drawing corks from bottles, previously tuned by pouring water into them, or by throwing down on a table sticks of wood of suitable dimensions. But, although noises are sometimes not entirely unmusical, and notes are usually not quite free from noise, there is no difficulty in recognising which of the two is the simpler phenomenon. There is a certain smoothness and continuity about the musical note. Moreover by sounding together a variety of notes—for example, by striking simultaneously a number of consecutive keys on a pianoforte—we obtain an approximation to a noise; while no combination of noises could ever blend into a musical note.

10. We are thus led to give our attention, in the first instance, mainly to musical sounds. These arrange themselves naturally in a certain order according to *pitch*—a quality which all can appreciate to some extent. Trained ears can recognise an enormous number of gradations—more than a thousand, probably, within the compass of the human voice. These gradations of pitch are not, like the degrees of a thermometric scale, without special mutual relations. Taking any given note as a starting point, musicians can single out certain others, which bear a definite relation to the first, and are known as its octave, fifth, &c. The corresponding differences of pitch are called *intervals*, and are spoken of as always the same for the same relationship. Thus, wherever they may occur in the scale, a note and its octave are separated by *the interval of the octave*. It will be our object later to explain, so far as it can be done, the origin and nature of the consonant intervals, but we must now turn to consider the physical aspect of the question.

Since sounds are produced by vibrations, it is natural to suppose that the simpler sounds, viz. musical notes, correspond to *periodic* vibrations, that is to say, vibrations which after a certain interval of time, called the *period*, repeat themselves with perfect regularity. And this, with a limitation presently to be noticed, is true.

11. Many contrivances may be proposed to illustrate the generation of a musical note. One of the simplest is a revolving wheel whose milled edge is pressed against a card. Each projection as it strikes the card gives a slight tap, whose regular recurrence, as the wheel turns, produces a note of definite pitch, *rising in the scale, as the velocity of rotation increases*. But the most appropriate instrument for the fundamental experiments on notes is undoubtedly the Siren, invented by Cagniard de la Tour. It consists essentially of a stiff disc, capable of revolving about its centre, and pierced with one or more sets of holes, arranged at equal intervals round the circumference of circles concentric with the disc. A windpipe in connection with bellows is presented perpendicularly to the disc, its open end being opposite to one of the circles, which contains a set of holes. When the bellows are worked, the stream of air escapes freely, if a hole is opposite to the end of the pipe; but otherwise it is obstructed. As the disc turns, a succession of puffs of air escape through it, until, when the velocity is sufficient, they blend into a note, whose pitch rises continually with the rapidity of the puffs. We shall have occasion later to describe more elaborate forms of the Siren, but for our immediate purpose the present simple arrangement will suffice.

12. One of the most important facts in the whole science is exemplified by the Siren—namely, that the pitch of a note depends upon the period of its vibration. The size and shape of the holes, the force of the wind, and other elements of the problem may be varied; but if the number of puffs in a given time, such as one second, remains unchanged, so also does the pitch. We may even dispense with wind altogether, and produce a note by allowing the corner of a card to tap against the edges of the holes, as they revolve; the pitch will still be the same. Observation of other sources of sound, such as vibrating solids, leads to the same conclusion, though the difficulties are often such as to render necessary rather refined experimental methods.

But in saying that pitch depends upon period, there lurks an ambiguity, which deserves attentive consideration, as it will lead us to a point of great importance. If a variable quantity be periodic in any time τ, it is also periodic in the times 2τ, 3τ, &c. Conversely, a recurrence within a given period τ, does not exclude a more rapid recurrence within periods which are the aliquot parts of τ. It would appear accordingly that a vibration really recurring in the time $\frac{1}{2}\tau$ (for example) may be regarded as having the period τ, and therefore by the law just laid down as producing a note of the pitch defined by τ. The force of this consideration cannot be entirely evaded by defining as the period the *least* time required to bring about a repetition. In the first place, the necessity of such a restriction is in itself almost sufficient to shew that we have not got to the root of the matter; for although a right to the period τ may be denied to a vibration repeating itself rigorously within a time $\frac{1}{2}\tau$, yet it must be allowed to a vibration that may differ indefinitely little therefrom. In the Siren experiment, suppose that in one of the circles of holes containing an even number, every alternate hole is displaced along the arc of the circle by the same amount. The displacement may be made so small that no change can be detected in the resulting note; but the periodic time on which the pitch depends has been doubled. And secondly it is evident from the nature of periodicity, that the superposition on a vibration of period τ, of others having periods $\frac{1}{2}\tau$, $\frac{1}{3}\tau$...&c., does not disturb the period τ, while yet it cannot be supposed that the addition of the new elements has left the quality of the sound unchanged. Moreover, since the pitch is not affected by their presence, how do we know that elements of the shorter periods were not there from the beginning?

13. These considerations lead us to expect remarkable relations between the notes whose periods are as the reciprocals of the natural numbers. Nothing can be easier than to investigate the question by means of the Siren. Imagine two circles of holes, the inner containing any convenient number, and the outer twice as many. Then at whatever speed the disc may turn, the period of the vibration engendered by blowing the first set will necessarily be the double of that belonging to the second. On making the experiment the two notes are found to stand to each other in the relation of octaves; and we conclude that *in passing from any note to its octave, the frequency of vibration is doubled.* A similar method of experimenting shews, that to the ratio of periods 3 : 1 corresponds the interval known to musicians as the *twelfth,* made up of an octave and a fifth; to the ratio of 4 : 1, the double octave; and to the ratio 5 : 1, the interval made up of two octaves and a *major third.* In order to obtain the intervals of the fifth and third themselves, the ratios must be made 3 : 2 and 5 : 4 respectively.

14. From these experiments it appears that if two notes stand to one another in a fixed relation, then, no matter at what part of the scale they may be situated, their periods are in a certain constant ratio characteristic of the relation. The same may be said of their *frequencies*[1], or the number of vibrations which they execute in a given time. The ratio 2 : 1 is thus characteristic of the octave interval. If we wish to combine two intervals,—for instance, starting from a given note, to take a step of an octave and then another of a fifth in the same direction, the corresponding ratios must be compounded:

$$\frac{2}{1} \times \frac{3}{2} = \frac{3}{1}.$$

The twelfth part of an octave is represented by the ratio $\sqrt[12]{2} : 1$, for this is the step which repeated twelve times leads to an octave above the starting point. If we wish to have a measure of intervals in the proper sense, we must take not the characteristic ratio itself, but the logarithm of that ratio. Then, and then only, will the measure of a compound interval be the *sum* of the measures of the components.

[1] A single word to denote the number of vibrations executed in the unit of time is indispensable: I know no better than 'frequency,' which was used in this sense by Young. The same word is employed by Prof. Everett in his excellent edition of Deschanel's *Natural Philosophy.*

15. From the intervals of the octave, fifth, and third considered above, others known to musicians may be derived. The difference of an octave and a fifth is called a *fourth*, and has the ratio $2 \div \frac{3}{2} = \frac{4}{3}$. This process of subtracting an interval from the octave is called *inverting* it. By inverting the major third we obtain the minor sixth. Again, by subtraction of a major third from a fifth we obtain the minor third; and from this by inversion the major sixth. The following table exhibits side by side the names of the intervals and the corresponding ratios of frequencies:

Octave	2 : 1
Fifth	3 : 2
Fourth	4 : 3
Major Third	5 : 4
Minor Sixth	8 : 5
Minor Third	6 : 5
Major Sixth	5 : 3

These are all the consonant intervals comprised within the limits of the octave. It will be remarked that the corresponding ratios are all expressed by means of *small* whole numbers, and that this is more particularly the case for the more consonant intervals.

The notes whose frequencies are multiples of that of a given one, are called its *harmonics*, and the whole series constitutes a *harmonic scale*. As is well known to violinists, they may all be obtained from the same string by touching it lightly with the finger at certain points, while the bow is drawn.

The establishment of the connection between musical intervals and definite ratios of frequency—a fundamental point in Acoustics—is due to Mersenne (1636). It was indeed known to the Greeks in what ratios the lengths of strings must be changed in order to obtain the octave and fifth; but Mersenne demonstrated the law connecting the length of a string with the period of its vibration, and made the first determination of the actual rate of vibration of a known musical note.

16. On any note taken as a key-note, or *tonic*, a *diatonic* scale may be founded, whose derivation we now proceed to explain. If the key-note, whatever may be its absolute pitch, be called Do, the fifth above or dominant is Sol, and the fifth below or subdominant is Fa. The common chord on any note is produced by combining it with its major third, and fifth, giving the ratios of frequency $1 : \frac{5}{4} : \frac{3}{2}$ or $4 : 5 : 6$. Now if we take the common chord on the tonic, on the dominant, and on the subdominant, and transpose them when necessary into the octave lying immediately above the tonic, we obtain notes whose frequencies arranged in order of magnitude are:

Do	Re	Mi	Fa	Sol	La	Si	Do
$1,$	$\frac{9}{8},$	$\frac{5}{4},$	$\frac{4}{3},$	$\frac{3}{2},$	$\frac{5}{3},$	$\frac{15}{8},$	$2.$

Here the common chord on Do is Do—Mi—Sol, with the ratios $1 : \frac{5}{4} : \frac{3}{2}$; the chord on Sol is Sol—Si—Re, with the ratios $\frac{3}{2} : \frac{15}{8} : 2 \times \frac{9}{8} = 1 : \frac{5}{4} : \frac{3}{2}$; and the chord on Fa is Fa—La—Do, still with the same ratios. The scale is completed by repeating these notes above and below at intervals of octaves.

If we take as our Do, or key-note, the lower c of a tenor voice, the diatonic scale will be

c d e f g a b c′.

Usage differs slightly as to the mode of distinguishing the different octaves; in what follows I adopt the notation of Helmholtz. The octave below the one just referred to is written with capital letters—C, D, &c.; the next below that with a suffix—$C_{,}$, $D_{,}$, &c.; and the one beyond that with a double suffix—$C_{,,}$, &c. On the other side accents denote elevation by an octave—c′, c″, &c. The notes of the four strings of a violin are written in this notation, g—d′—a′—e″. The middle c of the pianoforte is c′. [In French notation c′ is denoted by ut_3.]

17. With respect to an absolute standard of pitch there has been no uniform practice. At the Stuttgard conference in 1834, $c' = 264$ complete vibrations per second was recommended. This corresponds to $a' = 440$. The French pitch makes $a' = 435$. In Handel's time the pitch was much lower. If c′ were taken at 256 or 2^8, all the c's would have frequencies represented by powers of 2. This pitch is usually adopted by physicists and acoustical instrument makers, and has the advantage of simplicity.

The determination *ab initio* of the frequency of a given note is an operation requiring some care. The simplest method in principle is by means of the Siren, which is driven at such a rate as to give a note in unison with the given one. The number of turns effected by the disc in one second is given by a counting apparatus, which can be thrown in and out of gear at the beginning and end of a measured interval of time. This multiplied by the number of effective holes gives the required frequency. The consideration of other methods admitting of greater accuracy must be deferred.

18. So long as we keep to the diatonic scale of c, the notes above written are all that are required in a musical composition. But it is frequently desired to change the key-note. Under these circumstances a singer with a good natural ear, accustomed to perform without accompaniment, takes an entirely fresh departure, constructing a new diatonic scale on the new key-note. In this way, after a few changes of key, the original scale will be quite departed from, and an immense variety of notes be used. On an instrument with fixed notes like the piano and organ such a multiplication is impracticable, and some compromise is necessary in order to allow the same note to perform different functions. This is not the place to discuss the question at any length; we will therefore take as an illustration the simplest, as well as the commonest case—modulation into the key of the dominant.

By definition, the diatonic scale of c consists of the common chords founded on c, g and f. In like manner the scale of g consists of the chords founded on g, d and c. The chords of c and g are then common to the two scales; but the third and fifth of d introduce new notes. The third of d written f♯ has a frequency $\frac{9}{8} \times \frac{5}{4} = \frac{45}{32}$, and is far removed from any note in the scale of c. But the fifth of d, with a frequency $\frac{9}{8} \times \frac{3}{2} = \frac{27}{16}$, differs but little from a, whose frequency is $\frac{5}{3}$. In ordinary keyed instruments the interval between the two, represented by $\frac{81}{80}$, and called a *comma*, is neglected, and the two notes by a suitable compromise or *temperament* are identified.

19. Various systems of temperament have been used; the simplest and that now most generally used, or at least aimed at, is the *equal* temperament. On referring to the table of frequencies for the diatonic scale, it will be seen that the intervals from Do to Re, from Re to Mi, from Fa to Sol, from Sol to La, and from La to Si, are nearly the same, being represented by $\frac{9}{8}$ or $\frac{10}{9}$; while the intervals from Mi to Fa and from Si to Do, represented by $\frac{16}{15}$, are about half as much. The equal temperament treats these approximate relations as exact, dividing the octave into twelve equal parts called mean semitones. From these twelve notes the diatonic scale belonging to any key may be selected according to the following rule. Taking the key-note as the first, fill up the series with the third, fifth, sixth, eighth, tenth, twelfth and thirteenth notes, counting upwards. In this way all difficulties of modulation are avoided, as the twelve notes serve as well for one key as for another. But this advantage is obtained at a sacrifice of true intonation. The equal temperament third, being the third part of an octave, is represented by the ratio $\sqrt[3]{2} : 1$, or approximately 1·2599, while the true third is 1·25. The tempered third is thus higher than the true by the interval 126 : 125. The ratio of the tempered fifth may be obtained from the consideration that seven semitones make a fifth, while twelve go to an octave. The ratio is therefore $2^{\frac{7}{12}} : 1$, which $= 1{\cdot}4983$. The tempered fifth is thus too low in the ratio 1·4983 : 1·5, or approximately 881 : 882. This error is insignificant; and even the error of the third is not of much consequence in quick music on instruments like the pianoforte. But when the notes are *held*, as in the harmonium and organ, the consonance of chords is materially impaired.

20. The following Table, giving the twelve notes of the chromatic scale according to the system of equal temperament, will be convenient for reference[1]. The standard employed is a′ = 440; in

	C,,	C,	C	c	c′	c″	c‴	c⁗
C	16·35	32·70	65·41	130·8	261·7	523·3	1046·6	2093·2
C♯	17·32	34·65	69·30	138·6	277·2	544·4	1108·8	2217·7
D	18·35	36·71	73·42	146·8	293·7	587·4	1174·8	2349·6
D♯	19·44	38·89	77·79	155·6	311·2	622·3	1244·6	2489·3
E	20·60	41·20	82·41	164·8	329·7	659·3	1318·6	2637·3
F	21·82	43·65	87·31	174·6	349·2	698·5	1397·0	2794·0
F♯	23·12	46·25	92·50	185·0	370·0	740·0	1480·0	2960·1
G	24·50	49·00	98·00	196·0	392·0	784·0	1568·0	3136·0
G♯	25·95	51·91	103·8	207·6	415·3	830·6	1661·2	3322·5
A	27·50	55·00	110·0	220·0	440·0	880·0	1760·0	3520·0
A♯	29·13	58·27	116·5	233·1	466·2	932·3	1864·6	3729·2
B	30·86	61·73	123·5	246·9	493·9	987·7	1975·5	3951·0

[1] Zamminer, *Die Musik und die musikalischen Instrumente.* Giessen, 1855.

order to adapt the Table to any other absolute pitch, it is only necessary to multiply throughout by the proper constant.

The ratios of the intervals of the equal temperament scale are given below (Zamminer):—

Note.	Frequency.	Note.	Frequency.
c	$= 1{\cdot}00000$	f♯	$2^{\frac{6}{12}} = 1{\cdot}41421$
c♯	$2^{\frac{1}{12}} = 1{\cdot}05946$	g	$2^{\frac{7}{12}} = 1{\cdot}49831$
d	$2^{\frac{2}{12}} = 1{\cdot}12246$	g♯	$2^{\frac{8}{12}} = 1{\cdot}58740$
d♯	$2^{\frac{3}{12}} = 1{\cdot}18921$	a	$2^{\frac{9}{12}} = 1{\cdot}68179$
e	$2^{\frac{4}{12}} = 1{\cdot}25992$	a♯	$2^{\frac{10}{12}} = 1{\cdot}78180$
f	$2^{\frac{5}{12}} = 1{\cdot}33484$	b	$2^{\frac{11}{12}} = 1{\cdot}88775$

$c' = 2{\cdot}000$

21. Returning now for a moment to the physical aspect of the question, we will assume, what we shall afterwards prove to be true within wide limits,—that, when two or more sources of sound agitate the air simultaneously, the resulting disturbance at any point in the external air, or in the ear-passage, is the simple sum (in the extended geometrical sense) of what would be caused by each source acting separately. Let us consider the disturbance due to a simultaneous sounding of a note and any or all of its harmonics. By definition, the complex whole forms a note having the same period (and therefore pitch) as its gravest element. We have at present no criterion by which the two can be distinguished, or the presence of the higher harmonics recognised. And yet—in the case, at any rate, where the component sounds have an independent origin—it is usually not difficult to detect them by the ear, so as to effect an analysis of the mixture. This is as much as to say that a strictly periodic vibration may give rise to a sensation which is not simple, but susceptible of further analysis. In point of fact, it has long been known to musicians that under certain circumstances the harmonics of a note may be heard along with it, even when the note is due to a single source, such as a vibrating string; but the significance of the fact was not understood. Since attention has been drawn to the subject, it has been proved (mainly by the labours of Ohm and Helmholtz) that almost all musical notes are highly compound, consisting in fact of the notes of a harmonic scale, from which in particular cases one or more members may be missing. The reason of the uncertainty and difficulty of the analysis will be touched upon presently.

22. That kind of note which the ear cannot further resolve is called by Helmholtz in German a '*ton.*' Tyndall and other recent writers on Acoustics have adopted 'tone' as an English equivalent, —a practice which will be followed in the present work. The thing is so important, that a convenient word is almost a matter of necessity. *Notes* then are in general made up of *tones,* the pitch of the note being that of the gravest tone which it contains.

23. In strictness the quality of pitch must be attached in the first instance to simple tones only; otherwise the difficulty of discontinuity before referred to presents itself. The slightest change in the nature of a note may lower its pitch by a whole octave, as was exemplified in the case of the Siren. We should now rather say that the effect of the slight displacement of the alternate holes in that experiment was to introduce a new feeble tone an octave lower than any previously present. This is sufficient to alter the period of the whole, but the great mass of the sound remains very nearly as before.

In most musical notes, however, the fundamental or gravest tone is present in sufficient intensity to impress its character on the whole. The effect of the harmonic overtones is then to modify the *quality* or *character*[1] of the note, independently of pitch. That such a distinction exists is well known The notes of a violin, tuning fork, or of the human voice with its different vowel sounds, &c., may all have the same pitch and yet differ independently of loudness; and though a part of this difference is due to accompanying noises, which are extraneous to their nature as notes, still there is a part which is not thus to be accounted for. Musical notes may thus be classified as variable in three ways: First, *pitch.* This we have already sufficiently considered. Secondly, *character,* depending on the proportions in which the harmonic overtones are combined with the fundamental: and thirdly, *loudness.* This has to be taken last, because the ear is not capable of comparing (with any precision) the loudness of two notes which differ much in pitch or character. We shall indeed in a future chapter give a mechanical measure of the intensity of sound, including in one system all gradations of pitch; but this is nothing to the point.

[1] German, 'Klangfarbe'—French, 'timbre.' The word 'character' is used in this sense by Everett.

We are here concerned with the intensity of the sensation of sound, not with a measure of its physical cause. The difference of loudness is, however, at once recognised as one of more or less; so that we have hardly any choice but to regard it as dependent *cæteris paribus* on the magnitude of the vibrations concerned.

24. We have seen that a musical note, as such, is due to a vibration which is necessarily periodic; but the converse, it is evident, cannot be true without limitation. A periodic repetition of a noise at intervals of a second—for instance, the ticking of a clock—would not result in a musical note, be the repetition ever so perfect. In such a case we may say that the fundamental tone lies outside the limits of hearing, and although some of the harmonic overtones would fall within them, these would not give rise to a musical note or even to a chord, but to a noisy mass of sound like that produced by striking simultaneously the twelve notes of the chromatic scale. The experiment may be made with the Siren by distributing the holes quite irregularly round the circumference of a circle, and turning the disc with a moderate velocity. By the construction of the instrument, everything recurs after each complete revolution.

25. The principal remaining difficulty in the theory of notes and tones, is to explain why notes are sometimes analysed by the ear into tones, and sometimes not. If a note is really complex, why is not the fact immediately and certainly perceived, and the components disentangled? The feebleness of the harmonic overtones is not the reason, for, as we shall see at a later stage of our inquiry, they are often of surprising loudness, and play a prominent part in music. On the other hand, if a note is sometimes perceived as a whole, why does not this happen always? These questions have been carefully considered by Helmholtz[1], with a tolerably satisfactory result. The difficulty, such as it is, is not peculiar to Acoustics, but may be paralleled in the cognate science of Physiological Optics.

The knowledge of external things which we derive from the indications of our senses, is for the most part the result of inference. When an object is before us, certain nerves in our retinæ are excited, and certain sensations are produced, which we are accustomed to associate with the object, and we forthwith infer its presence. In the case of an unknown object the process is much

[1] *Tonempfindungen*, 3rd edition, p. 98.

the same. We interpret the sensations to which we are subject so as to form a pretty good idea of their exciting cause. From the slightly different perspective views received by the two eyes we infer, often by a highly elaborate process, the actual relief and distance of the object, to which we might otherwise have had no clue. These inferences are made with extreme rapidity and quite unconsciously. The whole life of each one of us is a continued lesson in interpreting the signs presented to us, and in drawing conclusions as to the actualities outside. Only so far as we succeed in doing this, are our sensations of any use to us in the ordinary affairs of life. This being so, it is no wonder that the study of our sensations themselves falls into the background, and that subjective phenomena, as they are called, become exceedingly difficult of observation. As an instance of this, it is sufficient to mention the 'blind spot' on the retina, which might *a priori* have been expected to manifest itself as a conspicuous phenomenon, though as a fact probably not one person in a hundred million would find it out for themselves. The application of these remarks to the question in hand is tolerably obvious. In the daily use of our ears our object is to disentangle from the whole mass of sound that may reach us, the parts coming from sources which may interest us at the moment. When we listen to the conversation of a friend, we fix our attention on the sound proceeding from him and endeavour to grasp that as a whole, while we ignore, as far as possible, any other sounds, regarding them as an interruption. There are usually sufficient indications to assist us in making this partial analysis. When a man speaks, the whole sound of his voice rises and falls together, and we have no difficulty in recognising its unity. It would be no advantage, but on the contrary a great source of confusion, if we were to carry the analysis further, and resolve the whole mass of sound present into its component tones. Although, as regards sensation, a resolution into tones might be expected, the necessities of our position and the practice of our lives lead us to stop the analysis at the point, beyond which it would cease to be of service in deciphering our sensations, considered as signs of external objects[1].

But it may sometimes happen that however much we may wish to form a judgment, the materials for doing so are absolutely

[1] Most probably the power of attending to the important and ignoring the unimportant part of our sensations is to a great extent inherited—to how great an extent we shall perhaps never know.

wanting. When a note and its octave are sounding close together and with perfect uniformity, there is nothing in our sensations to enable us to distinguish, whether the notes have a double or a single origin. In the mixture stop of the organ, the pressing down of each key admits the wind to a group of pipes, giving a note and its first three or four harmonics. The pipes of each group always sound together, and the result is usually perceived as a single note, although it does not proceed from a single source.

26. The resolution of a note into its component tones is a matter of very different difficulty with different individuals. A considerable effort of attention is required, particularly at first; and, until a habit has been formed, some external aid in the shape of a suggestion of what is to be listened for, is very desirable.

The difficulty is altogether very similar to that of learning to draw. From the machinery of vision it might have been expected that nothing would be easier than to make, on a plane surface, a representation of surrounding solid objects; but experience shews that much practice is generally required.

We shall return to the question of the analysis of notes at a later stage, after we have treated of the vibrations of strings, with the aid of which it is best elucidated; but a very instructive experiment, due originally to Ohm and improved by Helmholtz, may be given here. Helmholtz[1] took two bottles of the shape represented in the figure, one about twice as large as the other. These were blown by streams of air directed across the mouth and issuing from gutta-percha tubes, whose ends had been softened and pressed flat, so as to reduce the bore to the form of a narrow slit, the tubes being in connection with the same bellows. By pouring in water when the note is too low and by partially obstructing the mouth when the note is too high, the bottles may be made to give notes with the exact interval of an octave, such as b and b′. The larger bottle, blown alone, gives a somewhat muffled sound similar in character to the vowel U; but, when both bottles are blown, the character of the resulting sound is sharper, resembling rather the vowel O. For a short time after the notes had been heard separately Helmholtz was able to distinguish them in the mixture;

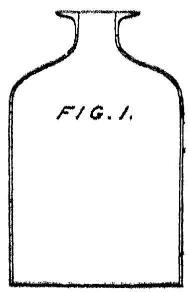


FIG. 1.

[1] *Tonempfindungen*, p. 109.

but as the memory of their separate impressions faded, the higher note seemed by degrees to amalgamate with the lower, which at the same time became louder and acquired a sharper character. This blending of the two notes may take place even when the high note is the louder.

27. Seeing now that notes are usually compound, and that only a particular sort called tones are incapable of further analysis, we are led to inquire what is the physical characteristic of tones, to which they owe their peculiarity? What sort of periodic vibration is it, which produces a simple tone? According to what mathematical function of the time does the pressure vary in the passage of the ear? No question in Acoustics can be more important.

The simplest periodic functions with which mathematicians are acquainted are the circular functions, expressed by a sine or cosine; indeed there are no others at all approaching them in simplicity. They may be of any period, and admitting of no other variation (except magnitude), seem well adapted to produce simple tones. Moreover it has been proved by Fourier, that the most general single-valued periodic function can be resolved into a series of circular functions, having periods which are submultiples of that of the given function. Again, it is a consequence of the general theory of vibration that the particular type, now suggested as corresponding to a simple tone, is the only one capable of preserving its integrity among the vicissitudes which it may have to undergo. Any other kind is liable to a sort of physical analysis, one part being differently affected from another. If the analysis within the ear proceeded on a different principle from that effected according to the laws of dead matter outside the ear, the consequence would be that a sound originally simple might become compound on its way to the observer. There is no reason to suppose that anything of this sort actually happens. When it is added that according to all the ideas we can form on the subject, the analysis within the ear must take place by means of a physical machinery, subject to the same laws as prevail outside, it will be seen that a strong case has been made out for regarding tones as due to vibrations expressed by circular functions. We are not however left entirely to the guidance of general considerations like these. In the chapter on the vibration of strings, we shall see that in many cases theory informs us beforehand of the nature of the vibration executed by a string, and in particular whether any specified simple vibration is a component or not. Here we have a decisive test. It is found by experiment that, whenever according to theory any simple vibration is present, the corresponding tone can be heard, but, whenever the simple vibration is absent, then the tone cannot be heard. We are therefore justified in asserting that simple tones and vibrations of a circular type are indissolubly connected. This law was discovered by Ohm.

CHAPTER II.

HARMONIC MOTIONS.

28. THE vibrations expressed by a circular function of the time and variously designated as *simple*, *pendulous*, or *harmonic*, are so important in Acoustics that we cannot do better than devote a chapter to their consideration, before entering on the dynamical part of our subject. The quantity, whose variation constitutes the 'vibration,' may be the displacement of a particle measured in a given direction, the pressure at a fixed point in a fluid medium, and so on. In any case denoting it by u, we have

$$u = a \cos\left(\frac{2\pi t}{\tau} - \epsilon\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

in which a denotes the *amplitude*, or extreme value of u; τ is the *periodic time*, or *period*, after the lapse of which the values of u recur; and ϵ determines the *phase* of the vibration at the moment from which t is measured.

Any number of harmonic vibrations *of the same period* affecting a variable quantity, compound into another of the same type, whose elements are determined as follows:

$$u = \Sigma a \cos\left(\frac{2\pi t}{\tau} - \epsilon\right)$$

$$= \cos\frac{2\pi t}{\tau}\, \Sigma a \cos\epsilon + \sin\frac{2\pi t}{\tau}\, \Sigma a \sin\epsilon$$

$$= r\cos\left(\frac{2\pi t}{\tau} - \theta\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

if $$r = \{(\Sigma a \cos\epsilon)^2 + (\Sigma a \sin\epsilon)^2\}^{\frac{1}{2}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and $$\tan\theta = \Sigma a \sin\epsilon \div \Sigma a \cos\epsilon \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

For example, let there be two components,

$$u = a\cos\left(\frac{2\pi t}{\tau} - \epsilon\right) + a'\cos\left(\frac{2\pi t}{\tau} - \epsilon'\right);$$

then
$$r = \{a^2 + a'^2 + 2aa'\cos(\epsilon - \epsilon')\}^{\frac{1}{2}} \quad\text{.....................}(5),$$

$$\tan\theta = \frac{a\sin\epsilon + a'\sin\epsilon'}{a\cos\epsilon + a'\cos\epsilon'} \quad\text{...............................}(6).$$

Particular cases may be noted. If the phases of the two components agree,

$$u = (a + a')\cos\left(\frac{2\pi t}{\tau} - \epsilon\right).$$

If the phases differ by half a period,

$$u = (a - a')\cos\left(\frac{2\pi t}{\tau} - \epsilon\right),$$

so that if $a' = a$, u vanishes. In this case the vibrations are often said to *interfere*, but the expression is rather misleading. Two sounds may very properly be said to interfere, when they together cause silence; but the mere superposition of two vibrations (whether rest is the consequence, or not) cannot properly be so called. At least if this be interference, it is difficult to say what non-interference can be. It will appear in the course of this work that when vibrations exceed a certain intensity they no longer compound by mere addition; *this* mutual action might more properly be called interference, but it is a phenomenon of a totally different nature from that with which we are now dealing.

Again, if the phases differ by a quarter or by three-quarters of a period, $\cos(\epsilon - \epsilon') = 0$, and

$$r = \{a^2 + a'^2\}^{\frac{1}{2}}.$$

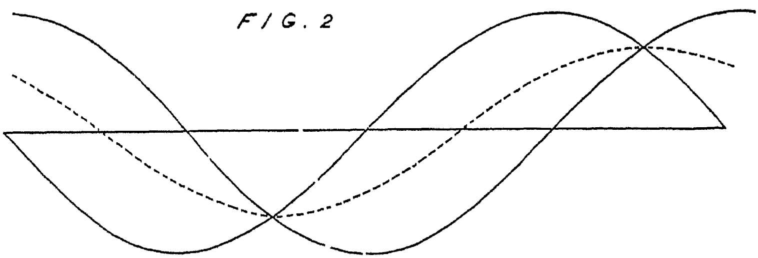
Harmonic vibrations of given period may be represented by lines drawn from a pole, the lengths of the lines being proportional to the amplitudes, and the inclinations to the phases of the vibrations. The resultant of any number of harmonic vibrations is then represented by the geometrical resultant of the corresponding lines. For example, if they are disposed symmetrically round the pole, the resultant of the lines, or vibrations, is zero.

29. If we measure off along an axis of x distances proportional to the time, and take u for an ordinate, we obtain the harmonic curve, or curve of sines,

$$u = a \cos\left(\frac{2\pi x}{\lambda} - \epsilon\right),$$

where λ, called the wave-length, is written in place of τ, both quantities denoting the range of the independent variable corresponding to a complete recurrence of the function. The harmonic curve is thus the locus of a point subject at once to a uniform motion, and to a harmonic vibration in a perpendicular direction. In the next chapter we shall see that the vibration of a tuning fork is simple harmonic; so that if an excited tuning fork be moved with uniform velocity parallel to the line of its handle, a tracing point attached to the end of one of its prongs describes a harmonic curve, which may be obtained in a permanent form by allowing the tracing point to bear gently on a piece of smoked paper. In Fig. 2 the continuous lines are two harmonic curves of the same wave-length and amplitude, but of different

FIG. 2



phases; the dotted curve represents half their resultant, being the locus of points midway between those in which the two curves are met by any ordinate.

30. If two harmonic vibrations of different periods coexist,

$$u = a \cos\left(\frac{2\pi t}{\tau} - \epsilon\right) + a' \cos\left(\frac{2\pi t}{\tau'} - \epsilon'\right).$$

The resultant cannot here be represented as a simple harmonic motion with other elements. If τ and τ' be incommensurable, the value of u never recurs; but, if τ and τ' be in the ratio of two whole numbers, u recurs after the lapse of a time equal to the least common multiple of τ and τ'; but the vibration is not simple harmonic. For example, when a note and its fifth are sounding together, the vibration recurs after a time equal to twice the period of the graver.

One case of the composition of harmonic vibrations of different periods is worth special discussion, namely, when the difference of the periods is small. If we fix our attention on the course of things during an interval of time including merely a few periods, we see that the two vibrations are nearly the same as if their periods were absolutely equal, in which case they would, as we know, be equivalent to another simple harmonic vibration of the same period. For a few periods then the resultant motion is approximately simple harmonic, but the same harmonic will not continue to represent it for long. The vibration having the shorter period continually gains on its fellow, thereby altering the difference of phase on which the elements of the resultant depend. For simplicity of statement let us suppose that the two components have equal amplitudes, frequencies represented by m and n, where $m-n$ is small, and that when first observed their phases agree. At this moment their effects conspire, and the resultant has an amplitude double of that of the components. But after a time $1 \div 2(m-n)$ the vibration m will have gained half a period relatively to the other; and the two, being now in complete disagreement, neutralize each other. After a further interval of time equal to that above named, m will have gained altogether a whole vibration, and complete accordance is once more re-established. The resultant motion is therefore approximately simple harmonic, with an amplitude not constant, but varying from zero to twice that of the components, the frequency of these alterations being $m-n$. If two tuning forks with frequencies 500 and 501 be equally excited, there is every second a rise and fall of sound corresponding to the coincidence or opposition of their vibrations. This phenomenon is called beats. We do not here fully discuss the question how the ear behaves in the presence of vibrations having nearly equal frequencies, but it is obvious that if the motion in the neighbourhood of the ear almost cease for a considerable fraction of a second, the sound must appear to fall. For reasons that will afterwards appear, beats are best heard when the interfering sounds are simple tones. Consecutive notes of the stopped diapason of the organ shew the phenomenon very well, at least in the lower parts of the scale. A permanent interference of two notes may be obtained by mounting two stopped organ pipes of similar construction and identical pitch side by side on the same wind chest. The vibrations of the two pipes adjust themselves to complete opposition, so that at a little distance nothing can be heard, except the hissing of the wind If by a rigid wall between the two pipes one sound could be cut off, the other would be instantly restored. Or the balance, on which silence depends, may be upset by connecting the ear with a tube, whose other end lies close to the mouth of one of the pipes.

By means of beats two notes may be tuned to unison with great exactness. The object is to make the beats as slow as possible, since the number of beats in a second is equal to the difference of the frequencies of the notes. Under favourable circumstances beats so slow as one in 30 seconds may be recognised, and would indicate that the higher note gains only two vibrations a *minute* on the lower. Or it might be desired merely to ascertain the difference of the frequencies of two notes nearly in unison, in which case nothing more is necessary than to count the number of beats. It will be remembered that the difference of frequencies does not determine the *interval* between the two notes; that depends on the *ratio* of frequencies. Thus the rapidity of the beats given by two notes nearly in unison is doubled, when both are taken an exact octave higher.

Analytically

$$u = a\cos(2\pi mt - \epsilon) + a'\cos(2\pi nt - \epsilon'),$$

where $m - n$ is small.

Now $\cos(2\pi nt - \epsilon')$ may be written

$$\cos\{2\pi mt - 2\pi(m-n)t - \epsilon'\},$$

and we have

$$u = r\cos(2\pi mt - \theta) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where

$$r^2 = a^2 + a'^2 + 2aa'\cos\{2\pi(m-n)t + \epsilon' - \epsilon\}\ldots\ldots(2),$$

$$\tan\theta = \frac{a\sin\epsilon + a'\sin\{2\pi(m-n)t + \epsilon'\}}{a\cos\epsilon + a'\cos\{2\pi(m-n)t + \epsilon'\}}\ldots\ldots\ldots(3).$$

The resultant vibration may thus be considered as harmonic with elements r and θ, which are not constant but slowly varying functions of the time, having the frequency $m - n$. The amplitude r is at its maximum when

$$\cos\{2\pi(m-n)t + \epsilon' - \epsilon\} = +1,$$

and at its minimum when

$$\cos\{2\pi(m-n)t + \epsilon' - \epsilon\} = -1,$$

the corresponding values being $a + a'$ and $a - a'$ respectively.

31. Another case of great importance is the composition of vibrations corresponding to a tone and its harmonics. It is known that the most general single-valued finite periodic function can be expressed by a series of simple harmonics—

$$u = a_0 + \Sigma_{n=1}^{n=\infty} a_n \cos\left(\frac{2\pi nt}{\tau} - \epsilon_n\right) \dots\dots\dots\dots\dots (1),$$

a theorem usually quoted as Fourier's. Analytical proofs will be found in Todhunter's *Integral Calculus* and Thomson and Tait's *Natural Philosophy;* and a line of argument almost if not quite amounting to a demonstration will be given later in this work. A few remarks are all that will be required here.

Fourier's theorem is not obvious. A vague notion is not uncommon that the infinitude of arbitrary constants in the series of necessity endows it with the capacity of representing an arbitrary periodic function. That this is an error will be apparent, when it is observed that the same argument would apply equally, if one term of the series were omitted; in which case the expansion would not in general be possible.

Another point worth notice is that simple harmonics are not the only functions, in a series of which it is possible to expand one arbitrarily given. Instead of the simple elementary term

$$\cos\left(\frac{2\pi nt}{\tau} - \epsilon_n\right),$$

we might take

$$\cos\left(\frac{2\pi nt}{\tau} - \epsilon_n\right) + \frac{1}{2}\cos\left(\frac{4\pi nt}{\tau} - \epsilon_n\right),$$

formed by adding a similar one in the same phase of half the amplitude and period. It is evident that these terms would serve as well as the others; for

$$\begin{aligned}\cos\left(\frac{2\pi nt}{\tau} - \epsilon_n\right) = & \left\{\cos\left(\frac{2\pi nt}{\tau} - \epsilon_n\right) + \frac{1}{2}\cos\left(\frac{4\pi nt}{\tau} - \epsilon_n\right)\right\} \\ & - \frac{1}{2}\left\{\cos\left(\frac{4\pi nt}{\tau} - \epsilon_n\right) + \frac{1}{2}\cos\left(\frac{8\pi nt}{\tau} - \epsilon_n\right)\right\} \\ & + \frac{1}{4}\left\{\cos\left(\frac{8\pi nt}{\tau} - \epsilon_n\right) + \frac{1}{2}\cos\left(\frac{16\pi nt}{\tau} - \epsilon_n\right)\right\} \\ & - \dots\dots \textit{ad infin.},\end{aligned}$$

so that each term in Fourier's series, and therefore the sum of the series, can be expressed by means of the double elementary terms now suggested. This is mentioned here, because students, not being acquainted with other expansions, may imagine that simple harmonic functions are by nature the only ones qualified to be the elements in the development of a periodic function. The reason of the preeminent importance of Fourier's series in Acoustics is the mechanical one referred to in the preceding chapter, and to be explained more fully hereafter, namely, that, in general, simple harmonic vibrations are the only kind that are propagated through a vibrating system without suffering decomposition.

32. As in other cases of a similar character, e.g. Taylor's theorem, if the possibility of the expansion be known, the coefficients may be determined by a comparatively simple process. We may write (1) of § 31

$$u = A_0 + \Sigma_{n=1}^{n=\infty} A_n \cos\frac{2n\pi t}{\tau} + \Sigma_{n=1}^{n=\infty} B_n \sin\frac{2n\pi t}{\tau} \;\;\ldots\ldots(1).$$

Multiplying by $\cos(2n\pi t/\tau)$ or $\sin(2n\pi t/\tau)$, and integrating over a complete period from $t=0$ to $t=\tau$, we find

$$\left.\begin{aligned} A_n &= \frac{2}{\tau}\int_0^\tau u\cos\frac{2n\pi t}{\tau}\,dt \\ B_n &= \frac{2}{\tau}\int_0^\tau u\sin\frac{2n\pi t}{\tau}\,dt \end{aligned}\right\} \;\;\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

An immediate integration gives

$$A_0 = \frac{1}{\tau}\int_0^\tau u\,dt \;\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

indicating that A_0 is the *mean* value of u throughout the period.

The degree of convergency in the expansion of u depends in general on the continuity of the function and its derivatives. The series formed by successive differentiations of (1) converge less and less rapidly, but still remain convergent, and arithmetical representatives of the differential coefficients of u, so long as these latter are everywhere finite. Thus (Thomson and Tait, § 77), if all the derivatives up to the m^{th} inclusive be free from infinite values, the series for u is more convergent than one with

$$1,\; \frac{1}{2^m},\; \frac{1}{3^m},\; \frac{1}{4^m},\ldots\ldots\&c.,$$

for coefficients.

32 a. The general explanation of the beats heard when two pure tones nearly in unison are sounded simultaneously has been discussed in § 30. But the occurrence of beats is not confined to the case of approximate unison, at least when we have to deal with compound notes. Suppose for example that the interval is an octave. The gravêr note then usually includes a tone coincident in pitch with the fundamental tone of the higher note. If the interval be disturbed, the previously coincident tones separate from one another, and give rise to beats of the same frequency as if they existed alone. There is usually no difficulty in observing these beats; but if one or both of the component tones concerned be very faint, the aid of a resonator may be invoked.

In general we may consider that each consonant interval is characterized by the coincidence of certain component tones, and if the interval be disturbed the previously coincident tones give rise to beats. Of course it may happen in any particular case that the tones which would coincide in pitch are absent from one or other of the notes. The disturbance of the interval would then, according to the above theory, not be attended by beats. In practice faint beats are usually heard; but the discussion of this phenomenon, as to which authorities are not entirely agreed, must be postponed.

33. Another class of compounded vibrations, interesting from the facility with which they lend themselves to optical observation, occur when two harmonic vibrations affecting the same particle are executed *in perpendicular directions*, more especially when the periods are not only commensurable, but in the ratio of two *small* whole numbers. The motion is then completely periodic, with a period not many times greater than those of the components, and the curve described is re-entrant. If u and v be the co-ordinates, we may take

$$u = a\cos(2\pi nt - \epsilon), \quad v = b\cos 2\pi n't \qquad\qquad (1).$$

First let us suppose that the periods are equal, so that $n' = n$; the elimination of t gives for the equation of the curve described,

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} - \frac{2uv}{ab}\cos\epsilon - \sin^2\epsilon = 0 \qquad\qquad (2),$$

representing in general an ellipse, whose position and dimensions depend upon the amplitudes of the original vibrations and upon the difference of their phases. If the phases differ by a quarter period, $\cos\epsilon = 0$, and the equation becomes

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1.$$

In this case the axes of the ellipse coincide with those of co-ordinates. If further the two components have equal amplitudes, the locus degenerates into the circle

$$u^2 + v^2 = a^2,$$

which is described with uniform velocity. This shews how a uniform circular motion may be analysed into two rectilinear harmonic motions, whose directions are perpendicular.

If the phases of the components agree, $\epsilon = 0$, and the ellipse degenerates into the coincident straight lines

$$\left(\frac{u}{a} - \frac{v}{b}\right)^2 = 0;$$

or if the difference of phase amount to half a period, into

$$\left(\frac{u}{a} + \frac{v}{b}\right)^2 = 0.$$

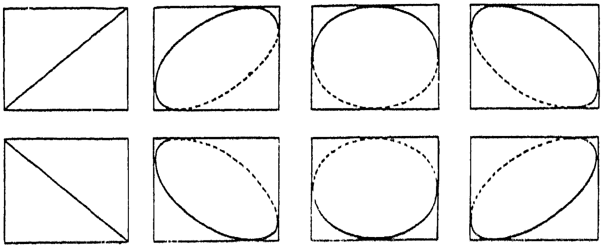
When the unison of the two vibrations is exact, the elliptic path remains perfectly steady, but in practice it will almost always happen that there is a slight difference between the periods. The consequence is that though a fixed ellipse represents the curve described with sufficient accuracy for a few periods, the ellipse itself gradually changes in correspondence with the alteration in the magnitude of ϵ. It becomes therefore a matter of interest to consider the system of ellipses represented by (2), supposing a and b constants, but ϵ variable.

Since the extreme values of u and v are $\pm a$, $\pm b$ respectively, the ellipse is in all cases inscribed in the rectangle whose sides are $2a$, $2b$. Starting with the phases in agreement, or $\epsilon = 0$, we have the ellipse coincident with the diagonal $\frac{u}{a} - \frac{v}{b} = 0$. As ϵ increases from 0 to $\frac{1}{2}\pi$, the ellipse opens out until its equation becomes

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1.$$

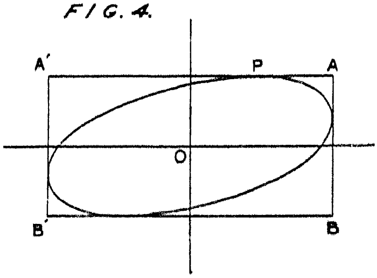
From this point it closes up again, ultimately coinciding with the other diagonal $\frac{u}{a} + \frac{v}{b} = 0$, corresponding to the increase of ϵ from $\frac{1}{2}\pi$ to π. After this, as ϵ ranges from π to 2π, the ellipse retraces its course until it again coincides with the first diagonal. The sequence of changes is exhibited in Fig. 3.

FIG. 3.



The ellipse, having already four given tangents, is completely determined by its point of contact P (Fig. 4) with the line $v = b$.

FIG. 4.



In order to connect this with ϵ, it is sufficient to observe that when $v = b$, $\cos 2\pi nt = 1$; and therefore $u = a \cos \epsilon$. Now if the elliptic paths be the result of the superposition of two harmonic vibrations of nearly coincident pitch, ϵ varies uniformly with the time, so that P itself executes a harmonic vibration along AA' with a frequency equal to the difference of the two given frequencies.

34. Lissajous[1] has shewn that this system of ellipses may be regarded as the different aspects of one and the same ellipse described on the surface of a transparent cylinder. In Fig. 5

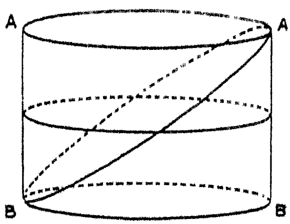


FIG. 5

[1] *Annales de Chimie* (3) LI. 147, 1857.

$AA'B'B$ represents the cylinder, of which AB' is a plane section. Seen from an infinite distance in the direction of the common tangent at A to the plane sections, the cylinder is projected into a rectangle, and the ellipse into its diagonal. Suppose now that the cylinder turns upon its axis, carrying the plane section with it. Its own projection remains a constant rectangle in which the pro-

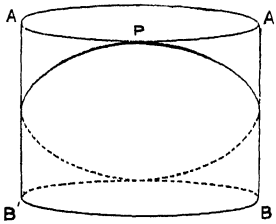


FIG. 6

jection of the ellipse is inscribed. Fig. 6 represents the position of the cylinder after a rotation through a right angle. It appears therefore that by turning the cylinder round we obtain in succession all the ellipses corresponding to the paths described by a point subject to two harmonic vibrations of equal period and fixed amplitudes. Moreover if the cylinder be turned continuously with uniform velocity, which insures a harmonic motion for P, we obtain a complete representation of the varying orbit described by the point when the periods of the two components differ slightly, each complete revolution answering to a gain or loss of a single vibration[1]. The revolutions of the cylinder are thus synchronous with the beats which would result from the composition of the two vibrations, if they were to act in the same direction.

35. Vibrations of the kind here considered are very easily realized experimentally. A heavy pendulum-bob, hung from a fixed point by a long wire or string, describes ellipses under the action of gravity, which may in particular cases, according to the circumstances of projection, pass into straight lines or circles. But in order to see the orbits to the best advantage, it is necessary that they should be described so quickly that the impression on the retina made by the moving point at any part of its course has not time to fade materially, before the point comes round again to renew its action. This condition is fulfilled by the vibration of a silvered bead (giving by reflection a luminous point), which is

[1] By a vibration will always be meant in this work a *complete* cycle of changes.

attached to a straight metallic wire (such as a knitting-needle), firmly clamped in a vice at the lower end. When the system is set into vibration, the luminous point describes ellipses, which appear as fine lines of light. These ellipses would gradually contract in dimensions under the influence of friction until they subsided into a stationary bright point, without undergoing any other change, were it not that in all probability, owing to some want of symmetry, the wire has slightly differing periods according to the plane in which the vibration is executed. Under these circumstances the orbit is seen to undergo the cycle of changes already explained.

36. So far we have supposed the periods of the component vibrations to be equal, or nearly equal; the next case in order of simplicity is when one is the double of the other. We have

$$u = a\cos(4n\pi t - \epsilon), \quad v = b\cos 2n\pi t.$$

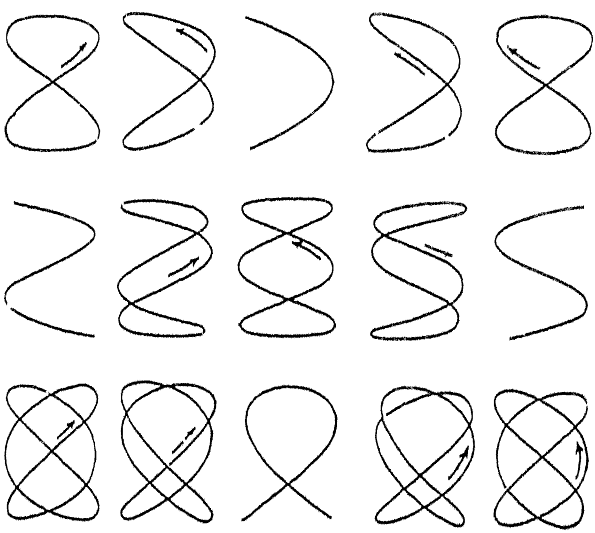
The locus resulting from the elimination of t may be written

$$\frac{u}{a} = \cos\epsilon\left(2\frac{v^2}{b^2} - 1\right) + 2\sin\epsilon\frac{v}{b}\sqrt{1 - \frac{v^2}{b^2}} \quad \text{.........(1)},$$

which for all values of ϵ represents a curve inscribed in the rectangle $2a$, $2b$. If $\epsilon = 0$, or π, we have

$$v^2 = \frac{b^2}{2}\left(1 \pm \frac{u}{a}\right),$$

FIG. 7.

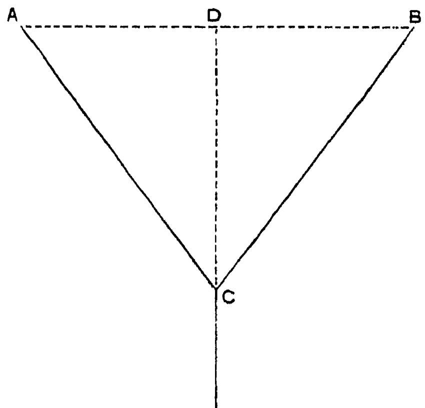


representing parabolas. Fig. 7 shews the various curves for the intervals of the octave, twelfth, and fifth.

To all these systems Lissajous' method of representation by the transparent cylinder is applicable, and when the relative phase is altered, whether from the different circumstances of projection in different cases, or continuously owing to a slight deviation from exactness in the ratio of the periods, the cylinder will appear to turn, so as to present to the eye different aspects of the same line traced on its surface.

37. There is no difficulty in arranging a vibrating system so that the motion of a point shall consist of two harmonic vibrations in perpendicular planes, with their periods in any assigned ratio. The simplest is that known as Blackburn's pendulum. A wire ACB is fastened at A and B, two fixed points at the same level. The bob P is attached to its middle point by another wire CP. For vibrations in the plane of the diagram, the point of suspension is practically C, provided that the wires are sufficiently stretched; but for a motion perpendicular to this plane, the bob turns about D, carrying the wire ACB with it. The periods of vibration in

FIG. 8.



the principal planes are in the ratio of the square roots of CP and DP. Thus if $DC = 3CP$, the bob describes the figures of the octave. To obtain the sequence of curves corresponding to approximate unison, ACB must be so nearly tight, that CD is relatively small.

38. Another contrivance called the kaleidophone was originally invented by Wheatstone. A straight thin bar of steel carrying a bead at its upper end is fastened in a vice, as explained in a previous paragraph. If the section of the bar is square, or circular, the period of vibration is independent of the plane in which it is performed. But let us suppose that the section is a rectangle with unequal sides. The stiffness of the bar—the force with which it resists bending—is then greater in the plane of greater thickness, and the vibrations in this plane have the shorter period. By a suitable adjustment of the thicknesses, the two periods of vibration may be brought into any required ratio, and the corresponding curve exhibited.

The defect in this arrangement is that the same bar will give only one set of figures. In order to overcome this objection the following modification has been devised. A slip of steel is taken whose rectangular section is very elongated, so that as regards bending in one plane the stiffness is so great as to amount practically to rigidity. The bar is divided into two parts, and the broken ends reunited, the two pieces being turned on one another through a right angle, so that the plane, which contains the small thickness of one, contains the great thickness of the other. When the compound rod is clamped in a vice at a point below the junction, the period of the vibration in one direction, depending almost entirely on the length of the upper piece, is nearly constant; but that in the second direction may be controlled by varying the point at which the lower piece is clamped.

39. In this arrangement the luminous point itself executes the vibrations which are to be observed; but in Lissajous' form of the experiment, the point of light remains really fixed, while its *image* is thrown into apparent motion by means of successive reflection from two vibrating mirrors. A small hole in an opaque screen placed close to the flame of a lamp gives a point of light, which is observed after reflection in the mirrors by means of a small telescope. The mirrors, usually of polished steel, are attached to the prongs of stout tuning forks, and the whole is so disposed that when the forks are thrown into vibration the luminous point appears to describe harmonic motions in perpendicular directions, owing to the angular motions of the reflecting surfaces. The amplitudes and periods of these harmonic motions depend upon those of the corresponding forks, and may be made such as to give with enhanced brilliancy any of the figures possible with the kaleidophone. By a similar arrangement it is possible to project the figures on a screen. In either case they gradually contract as the vibrations of the forks die away.

40. The principles of this chapter have received an important application in the investigation of rectilinear periodic motions. When a point, for instance a particle of a sounding string, is vibrating with such a period as to give a note within the limits of hearing, its motion is much too rapid to be followed by the eye; so that, if it be required to know the character of the vibration, some indirect method must be adopted. The simplest, theoretically, is to compound the vibration under examination with a uniform motion of translation in a perpendicular direction, as when a tuning-fork draws a harmonic curve on smoked paper. Instead of moving the vibrating body itself, we may make use of a revolving mirror, which provides us with an *image* in motion. In this way we obtain a representation of the function characteristic of the vibration, with the abscissa proportional to time.

But it often happens that the application of this method would be difficult or inconvenient. In such cases we may substitute for the uniform motion a harmonic vibration of suitable period in the same direction. To fix our ideas, let us suppose that the point, whose motion we wish to investigate, vibrates vertically with a period τ, and let us examine the result of combining with this a horizontal harmonic motion, whose period is some multiple of τ, say, $n\tau$. Take a rectangular piece of paper, and with axes parallel to its edges draw the curve representing the vertical motion (by setting off abscissæ proportional to the time) on such a scale that the paper just contains n repetitions or waves, and then bend the paper round so as to form a cylinder, with a re-entrant curve running round it. A point describing this curve in such a manner that it revolves uniformly about the axis of the cylinder will appear from a distance to combine the given vertical motion of period τ, with a horizontal harmonic motion of period $n\tau$. Conversely therefore, in order to obtain the representative curve of the vertical vibrations, the cylinder containing the apparent path must be imagined to be divided along a generating line, and developed into a plane. There is less difficulty in conceiving the cylinder and the situation of the curve upon it, when the adjustment of the periods is not quite exact, for then the cylinder appears to turn, and the contrary motions serve to distinguish those parts of the curve which lie on its nearer and further face.

41. The auxiliary harmonic motion is generally obtained optically, by means of an instrument called a vibration-microscope invented by Lissajous. One prong of a large tuning-fork carries a lens, whose axis is perpendicular to the direction of vibration; and which may be used either by itself, or as the object-glass of a compound microscope formed by the addition of an eye-piece independently supported. In either case a stationary point is thrown into apparent harmonic motion along a line parallel to that of the fork's vibration.

The vibration-microscope may be applied to test the rigour and universality of the law connecting *pitch* and *period*. Thus it will be found that any point of a vibrating body which gives a pure musical note will appear to describe a re-entrant curve, when examined with a vibration-microscope whose note is in strict unison with its own. By the same means the ratios of frequencies characteristic of the consonant intervals may be verified; though for this latter purpose a more thoroughly acoustical method, to be described in a future chapter, may be preferred.

42. Another method of examining the motion of a vibrating body depends upon the use of intermittent illumination[1]. Suppose, for example, that by means of suitable apparatus a series of electric sparks are obtained at regular intervals τ. A vibrating body, whose period is also τ, examined by the light of the sparks must appear at rest, because it can be seen only in one position. If, however, the period of the vibration differ from τ ever so little, the illuminated position varies, and the body will appear to vibrate slowly with a frequency which is the difference of that of the spark and that of the body. The type of vibration can then be observed with facility.

The series of sparks can be obtained from an induction-coil, whose primary circuit is periodically broken by a vibrating fork, or by some other interrupter of sufficient regularity. But a better result is afforded by sunlight rendered intermittent with the aid of a fork, whose prongs carry two small plates of metal, parallel to the plane of vibration and close together. In each plate is a slit

[1] Plateau, *Bull. de l'Acad. roy. de Belgique*, t. III, p. 364, 1836.

parallel to the prongs of the fork, and so placed as to afford a free passage through the plates when the fork is at rest, or passing through the middle point of its vibrations. On the opening so formed, a beam of sunlight is concentrated by means of a burning-glass, and the object under examination is placed in the cone of rays diverging on the further side[1]. When the fork is made to vibrate by an electro-magnetic arrangement, the illumination is cut off except when the fork is passing through its position of equilibrium, or nearly so. The flashes of light obtained by this method are not so instantaneous as electric sparks (especially when a jar is connected with the secondary wire of the coil), but in my experience the regularity is more perfect. Care should be taken to cut off extraneous light as far as possible, and the effect is then very striking.

A similar result may be arrived at by looking at the vibrating body through a series of holes arranged in a circle on a revolving disc. Several series of holes may be provided on the same disc, but the observation is not satisfactory without some provision for securing uniform rotation.

Except with respect to the sharpness of definition, the result is the same when the period of the light is any multiple of that of the vibrating body. This point must be attended to when the revolving wheel is used to determine an unknown frequency.

When the frequency of intermittence is an exact multiple of that of the vibration, the object is seen without apparent motion, but generally in more than one position. This condition of things is sometimes advantageous.

Similar effects arise when the frequencies of the vibrations and of the flashes are in the ratio of two small whole numbers. If, for example, the number of vibrations in a given time be half as great again as the number of flashes, the body will appear stationary, and in general double.

42 *a.* We have seen (§ 28) that the resultant of two isoperiodic vibrations of equal amplitude is wholly dependent upon their phase relation, and it is of interest to inquire what we are to expect from the composition of a large number (n) of equal vibrations of amplitude unity, of the same period, and of phases accidentally determined. The intensity of the resultant, represented by the square of the amplitude § 245, will of course depend upon the

[1] Töpler, *Phil. Mag.* Jan. 1867.

precise manner in which the phases are distributed, and may vary from n^2 to zero. But is there a definite intensity which becomes more and more probable when n is increased without limit?

The nature of the question here raised is well illustrated by the special case in which the possible phases are restricted to two *opposite* phases. We may then conveniently discard the idea of phase, and regard the amplitudes as at random *positive* or *negative*. If all the signs be the same, the intensity is n^2; if, on the other hand, there be as many positive as negative, the result is zero. But although the intensity may range from 0 to n^2, the smaller values are more probable than the greater.

The simplest part of the problem relates to what is called in the theory of probabilities the "expectation" of intensity, that is, the mean intensity to be expected after a great number of trials, in each of which the phases are taken at random. The chance that all the vibrations are positive is $(\frac{1}{2})^n$, and thus the expectation of intensity corresponding to this contingency is $(\frac{1}{2})^n . n^2$. In like manner the expectation corresponding to the number of positive vibrations being $(n-1)$ is

$$(\tfrac{1}{2})^n n\,(n-2)^2,$$

and so on. The whole expectation of intensity is thus

$$\frac{1}{2^n}\left\{1 . n^2 + n\,(n-2)^2 + \frac{n\,(n-1)}{1 . 2}\,(n-4)^2 + \frac{n\,(n-1)\,(n-2)}{1 . 2 . 3}\,(n-6)^2 + \ldots\right\} \quad\ldots\ldots(1).$$

Now the sum of the $(n+1)$ terms of this series is simply n, as may be proved by comparison of coefficients of x^2 in the equivalent forms

$$(e^x + e^{-x})^n = 2^n\,(1 + \tfrac{1}{2}x^2 + \ldots)^n$$
$$= e^{nx} + n\,e^{(n-2)x} + \frac{n\,(n-1)}{1 . 2}\,e^{(n-4)x} + \ldots.$$

The expectation of intensity is therefore n, and this whether n be great or small.

The same conclusion holds good when the phases are unrestricted. From (3) § 28, if $a_1 = a_2 = \ldots = 1$,

$$r^2 = (\cos\epsilon_1 + \cos\epsilon_2 + \ldots)^2 + (\sin\epsilon_1 + \sin\epsilon_2 + \ldots)^2$$
$$= n + 2\Sigma\cos(\epsilon_2 - \epsilon_1) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where under the sign of summation are to be included the cosines of the $\frac{1}{2}n\,(n-1)$ differences of phase. When the phases are accidental, the sum is as likely to be positive as negative, and thus the mean value of r^2 is n.

The reader must be on his guard here against a fallacy which has misled some eminent authors. We have not proved that when n is large there is any tendency for a single combination to give an intensity equal to n, but the quite different proposition that in a large number of trials, in each of which the phases are distributed at random, the *mean* intensity will tend more and more to the value n. It is true that even in a single combination there is no reason why any of the cosines in (2) should be positive rather than negative. From this we may infer that when n is increased the sum of the terms tends to vanish in comparison with the number of terms; but, the number of the terms being of the order n^2, we can infer nothing as to the value of the sum of the series in comparison with n.

So far there is no difficulty; but a complete investigation of this subject involves an estimate of the relative probabilities of resultants lying within assigned limits of magnitude. For example, we ought to be able to say what is the probability that the intensity due to a large number (n) of equal components is less than $\frac{1}{2}n$. This problem may conveniently be considered here, though it is naturally beyond the reach of elementary methods. We will commence by taking it under the restriction that the phases are of two opposite kinds only.

Adopting the statistical method of statement, let us suppose that there are an immense number N of independent combinations, each consisting of n unit vibrations, positive or negative, and combined accidentally. When N is sufficiently large, the statistics become regular; and the number of combinations in which the resultant amplitude is found equal to x may be denoted by $N \,.\, f(n, x)$, where f is a definite function of n and x. Now suppose that each of the N combinations receives another random contribution of ± 1, and inquire how many of them will subsequently possess a resultant x. It is clear that those only can do so which originally had amplitudes $x - 1$, or $x + 1$. *Half* of the former, and *half* of the latter number will acquire the amplitude x, so that the number required is

$$\tfrac{1}{2}Nf(n, x-1) + \tfrac{1}{2}Nf(n, x+1).$$

But this must be identical with the number corresponding to $n+1$ and x, so that

$$f(n+1, x) = \tfrac{1}{2}f(n, x-1) + \tfrac{1}{2}f(n, x+1) \quad \text{.........(3).}$$

This equation of differences holds good for all integral values of x and for all positive integral values of n. If $f(n, x)$ be given for one value of n, the equation suffices to determine $f(n, x)$ for all higher integral values of n. For the present purpose the initial value of n is zero. In that case we know that $f(x) = 0$ for all values of x other than zero, and that when $x = 0$, $f(0, 0) = 1$.

The problem proposed in the above form is perfectly definite; but for our immediate object it suffices to limit ourselves to the supposition that n is great, regarding $f(n, x)$ as a continuous function of continuous variables n and x, much as in the analogous problem of §§ 120, 121, 122.

Writing (3) in the form

$$f(n+1, x) - f(n, x) = \tfrac{1}{2}f(n, x-1) + \tfrac{1}{2}f(n, x+1) - f(n, x) \ldots (4),$$

we see that the left-hand member may then be identified with df/dn, and the right-hand member with $\frac{1}{2}d^2f/dx^2$, so that under these circumstances the differential equation to which (3) reduces is of the well-known form

$$\frac{df}{dn} = \frac{1}{2}\frac{d^2f}{dx^2} \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The analogy with the conduction of heat is indeed very close; and the methods developed by Fourier for the solution of problems in the latter subject are at once applicable. The special condition here is that initially, that is when $n = 0$, f must vanish for all values of x other than zero. As may be verified by differentiation, the special solution of (5) is then

$$f(n, x) = \frac{A}{\sqrt{n}} e^{-x^2/2n} \ldots\ldots\ldots\ldots\ldots (6),$$

in which A is an arbitrary constant to be determined from the consideration that the whole number of combinations is N. Thus, if dx be large in comparison with unity, the number of combinations which have amplitudes between x and $x + dx$ is

$$\frac{AN}{\sqrt{n}} e^{-x^2/2n}\, dx\,;$$

while
$$\frac{AN}{\sqrt{n}} \int_{-\infty}^{+\infty} e^{-x^2/2n}\, dx = N,$$

so that in virtue of the known equality

$$\int_{-\infty}^{+\infty} e^{-z^2}\, dz = \sqrt{\pi},$$

$$A \,.\, \sqrt{2\pi} = 1.$$

The final result for the number of combinations which have amplitudes between x and $x+dx$ is accordingly

$$\frac{N}{\sqrt{(2\pi n)}} e^{-x^2/2n}\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$

The *mean* intensity is expressed by

$$\frac{1}{\sqrt{(2\pi n)}} \int_{-\infty}^{+\infty} x^2 e^{-x^2/2n}\, dx = n,$$

as before.

We will now pass on to the more important problem in which the phases of the n unit vibrations are distributed at random over the entire period. In each combination the resultant amplitude is denoted by r and the phase (referred to a given epoch) by θ; and rectangular coordinates are taken so that

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Thus any point (x, y) in the plane of reference represents a vibration of amplitude r and phase θ, and the whole system of N vibrations is represented by a distribution of points, whose density it is our object to determine. Since no particular phase can be singled out for distinction, we know beforehand that the density of distribution will be independent of θ.

Of the infinite number N of points we suppose that

$$Nf(n, x, y)\, dx\, dy$$

are to be found within the infinitesimal area $dx\,dy$, and we will inquire as before how this number would be changed by the addition to the n component vibrations of one more unit vibration of accidental phase. Any vibration which after the addition is represented by the point x, y must before have corresponded to the point

$$x' = x - \cos\phi, \quad y' = y - \sin\phi,$$

where ϕ represents the phase of the additional unit vibration. And, if for the moment ϕ be regarded as given, to the area $dx\,dy$ corresponds an equal area $dx'dy'$. Again, all values of ϕ being equally probable, the factor necessary under this head is $d\phi/2\pi$. Accordingly the whole number to be found in $dx\,dy$ after the superposition of the additional unit is

$$N dx\,dy \int_0^{2\pi} f(n, x', y')\, d\phi/2\pi\,;$$

and this is to be equated to

$$N dx\,dy\, f(n+1, x, y)\,;$$

so that

$$f(n+1, x, y) = \int_0^{2\pi} f(n, x', y')\, d\phi/2\pi \quad \ldots\ldots\ldots \quad (8).$$

The value of $f(n, x', y')$ is obtained by introduction of the values of x', y' and expansion:

$$f(x', y') = f(x, y) - \frac{df}{dx}\cos\theta - \frac{df}{dy}\sin\theta + \frac{1}{2}\frac{d^2f}{dx^2}\cos^2\theta$$

$$+ \frac{d^2f}{dxdy}\cos\theta\sin\theta + \frac{1}{2}\frac{d^2f}{dy^2}\sin^2\theta + \ldots,$$

so that

$$\int_0^{2\pi} f(n, x', y')\, d\phi/2\pi = f(n, x, y) + \frac{1}{4}\frac{d^2f}{dx^2} + \frac{1}{4}\frac{d^2f}{dy^2} + \ldots\ldots$$

Also, n being very great,

$$f(n+1, x, y) - f(n, x, y) = df/dn\,;$$

and (8) reduces to

$$\frac{df}{dn} = \frac{1}{4}\left(\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

the usual equation for the conduction of heat in two dimensions.

In addition to (9), f has to satisfy the special condition of evanescence when $n = 0$ for all points other than the origin. The appropriate solution is necessarily symmetrical round the origin, and takes the form

$$f(n, x, y) = An^{-1}e^{-(x^2+y^2)/n} \ldots\ldots\ldots\ldots\ldots (10),$$

as may be verified by differentiation. The constant A is to be determined by the condition that the whole number is N. Thus

$$N = NAn^{-1}\iint e^{-(x^2+y^2)/n}\,dxdy = NA2\pi n^{-1}\int_0^\infty e^{-r^2/n}r\,dr = \pi AN\,;$$

and the number of vibrations within the area $dxdy$ becomes

$$\frac{N}{\pi n}e^{-(x^2+y^2)/n}\,dx\,dy \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11).$$

If we wish to find the number of vibrations which have amplitudes between r and $r + dr$, we must introduce polar coordinates and integrate with respect to θ. The required number is thus

$$2Nn^{-1}e^{-r^2/n}r\,dr \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)^1.$$

The result may also be expressed by saying that the *probability* of a resultant amplitude between r and $r + dr$ when a large number n of unit vibrations are compounded at random is

$$2n^{-1}e^{-r^2/n}r\,dr \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

[1] *Phil. Mag.* Aug. 1880.

The mean intensity is given by

$$2n^{-1}\int_0^\infty e^{-r^2/n}r^3dr = n,$$

as was to be expected.

The probability of a resultant amplitude less than r is

$$2n^{-1}\int_0^r e^{-r^2/n}r\,dr = 1 - e^{-r^2/n} \quad\text{..............}\quad (14),$$

or, which is the same thing, the probability of a resultant amplitude *greater* than r is

$$e^{-r^2/n} \quad\text{..........................}\quad (15).$$

The following table gives the probabilities of *intensities* less than the fractions of n named in the first column. For example, the probability of intensity less than n is ·6321.

·05	·0488	·80	·5506
·10	·0952	1·00	·6321
·20	·1813	1·50	·7768
·40	·3296	2·00	·8647
·60	·4512	3·00	·9502

It will be seen that, however great n may be, there is a reasonable chance of considerable relative fluctuations of intensity in different combinations.

If the amplitude of each component be α, instead of unity, as we have hitherto supposed for brevity, the probability of a resultant amplitude between r and $r + dr$ is

$$\frac{2}{n\alpha^2}e^{-r^2/n\alpha^2}r\,dr \quad\text{......................}\quad (16).$$

The result is thus a function of n and α only through $n\alpha^2$, and would be unchanged if for example the amplitude became $\frac{1}{2}\alpha$ and the number $4n$. From this it follows that the law is not altered, even if the components have different amplitudes, provided always that the whole number of each kind is very great; so that if there be n components of amplitude α, n' of amplitude β, and so on, the probability of a resultant between r and $r + dr$ is

$$\frac{2}{n\alpha^2 + n'\beta^2 + \dots}e^{-\frac{r^2}{n\alpha^2+n'\beta^2+\dots}}r\,dr \quad\text{........}\quad (17).$$

That this is the case may perhaps be made more clear by the consideration of a particular case. Let us suppose in the first place that $n + 4n'$ unit vibrations are compounded at random.

The appropriate law is given at once by (13) on substitution of $n+4n'$ for n, that is

$$2\,(n+4n')^{-1}\,e^{-r^2/(n+4n')}\,r\,dr \quad\ldots\ldots\ldots\ldots\ldots\ldots(18).$$

Now the combination of $n+4n'$ unit vibrations may be regarded as arrived at by combining a random combination of n unit vibrations with a second random combination of $4n'$ units, and the second random combination is the same as if due to a random combination of n' vibrations each of amplitude 2. Thus (18) applies equally well to a random combination of $(n+n')$ vibrations, n of which are of amplitude unity and n' of amplitude 2.

Although the result has no application to the theory of vibrations, it may be worth notice that a similar method applies to the composition *in three dimensions* of unit vectors, whose directions are accidental. The equation analogous to (8) gives in place of (9)

$$\frac{df}{dn}=\frac{1}{6}\left(\frac{d^2f}{dx^2}+\frac{d^2f}{dy^2}+\frac{d^2f}{dz^2}\right).$$

The appropriate solution, analogous to (13), is

$$3\sqrt{\left(\frac{6}{\pi n^3}\right)}\,e^{-r^2/\frac{2}{3}n}\,r^2dr \quad\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

expressing the probability of a resultant amplitude lying between r and $r+dr$.

Here again the mean value of r^2, to be expected in a great number of independent combinations, is n.

CHAPTER III.

SYSTEMS HAVING ONE DEGREE OF FREEDOM.

43. THE material systems, with whose vibrations Acoustics is concerned, are usually of considerable complication, and are susceptible of very various modes of vibration, any or all of which may coexist at any particular moment. Indeed in some of the most important musical instruments, as strings and organ-pipes, the number of independent modes is theoretically infinite, and the consideration of several of them is essential to the most practical questions relating to the nature of the consonant chords. Cases, however, often present themselves, in which one mode is of paramount importance; and even if this were not so, it would still be proper to commence the consideration of the general problem with the simplest case—that of one degree of freedom. It need not be supposed that the mode treated of is the only one possible, because so long as vibrations of other modes do not occur their possibility under other circumstances is of no moment.

44. The condition of a system possessing one degree of freedom is defined by the value of a single co-ordinate u, whose origin may be taken to correspond to the position of equilibrium. The kinetic and potential energies of the system for any given position are proportional respectively to $\dot{u}^2$ and u^2:—

$$T = \tfrac{1}{2} m\dot{u}^2, \quad V = \tfrac{1}{2} \mu u^2 \qquad (1),$$

where m and μ are in general functions of u. But if we limit ourselves to the consideration of positions *in the immediate neighbourhood of that corresponding to equilibrium*, u is a small quantity, and m and μ are sensibly constant. On this understanding we now proceed. If there be no forces, either resulting from internal friction or viscosity, or impressed on the system from without, the whole energy remains constant. Thus

$$T + V = \text{constant}.$$

Substituting for T and V their values, and differentiating with respect to the time, we obtain the equation of motion

$$m\ddot{u} + \mu u = 0 \quad\text{.........................(2)}$$

of which the complete integral is

$$u = a \cos(nt - \alpha) \quad\text{......................(3),}$$

where $n^2 = \mu \div m$, representing a *harmonic* vibration. It will be seen that the period alone is determined by the nature of the system itself; the amplitude and phase depend on collateral circumstances. If the differential equation were exact, that is to say, if T were strictly proportional to $\dot{u}^2$, and V to u^2, then, without any restriction, the vibrations of the system about its configuration of equilibrium would be accurately harmonic. But in the majority of cases the proportionality is only approximate, depending on an assumption that the displacement u is always small—how small depends on the nature of the particular system and the degree of approximation required; and then of course we must be careful not to push the application of the integral beyond its proper limits.

But, although not to be stated without a limitation, the principle that the vibrations of a system about a configuration of equilibrium have a period depending on the structure of the system and not on the particular circumstances of the vibration, is of supreme importance, whether regarded from the theoretical or the practical side. If the pitch and the loudness of the note given by a musical instrument were not within wide limits independent, the art of the performer on many instruments, such as the violin and pianoforte, would be revolutionized.

The periodic time

$$\tau = \frac{2\pi}{n} = 2\pi\sqrt{\frac{m}{\mu}} \quad\text{.................... (4),}$$

so that an increase in m, or a decrease in μ, protracts the duration of a vibration. By a generalization of the language employed in the case of a material particle urged towards a position of equilibrium by a spring, m may be called the inertia of the system, and

μ the force of the equivalent spring. Thus an augmentation of mass, or a relaxation of spring, increases the periodic time. By means of this principle we may sometimes obtain limits for the value of a period, which cannot, or cannot easily, be calculated exactly.

45. The absence of all forces of a frictional character is an ideal case, never realized but only approximated to in practice. The original energy of a vibration is always dissipated sooner or later by conversion into heat. But there is another source of loss, which though not, properly speaking, dissipative, yet produces results of much the same nature. Consider the case of a tuning-fork vibrating *in vacuo* The internal friction will in time stop the motion, and the original energy will be transformed into heat. But now suppose that the fork is transferred to an open space. In strictness the fork and the air surrounding it constitute a single system, whose parts cannot be treated separately. In attempting, however, the exact solution of so complicated a problem, we should generally be stopped by mathematical difficulties, and in any case an approximate solution would be desirable. The effect of the air during a few periods is quite insignificant, and becomes important only by accumulation. We are thus led to consider its effect as a *disturbance* of the motion which would take place *in vacuo*. The disturbing force is periodic (to the same approximation that the vibrations are so), and may be divided into two parts, one proportional to the acceleration, and the other to the velocity. The former produces the same effect as an alteration in the mass of the fork, and we have nothing more to do with it at present. The latter is a force arithmetically proportional to the velocity, and always acts in opposition to the motion, and therefore produces effects of the same character as those due to friction. In many similar cases the loss of motion by communication may be treated under the same head as that due to dissipation proper, and is represented in the differential equation with a degree of approximation sufficient for acoustical purposes by a term proportional to the velocity. Thus

$$\ddot{u} + \kappa\dot{u} + n^2u = 0 \quad\text{.........................}\quad (1)$$

is the equation of vibration for a system with one degree of freedom subject to frictional forces. The solution is

$$u = Ae^{-\frac{1}{2}\kappa t}\cos\{\sqrt{n^2 - \tfrac{1}{4}\kappa^2}\,.\,t - \alpha\} \quad\text{.................}\quad (2).$$

If the friction be so great that $\frac{1}{4}\kappa^2 > n^2$, the solution changes its form, and no longer corresponds to an oscillatory motion; but in all acoustical applications κ is a small quantity. Under these circumstances (2) may be regarded as expressing a harmonic vibration, whose amplitude is not constant, but diminishes in geometrical progression, when considered after equal intervals of time. The difference of the logarithms of successive extreme excursions is nearly constant, and is called the Logarithmic Decrement. It is expressed by $\frac{1}{4}\kappa\tau$, if τ be the periodic time.

The frequency, depending on $n^2 - \frac{1}{4}\kappa^2$, involves only the second power of κ; so that to the first order of approximation *the friction has no effect on the period*,—a principle of very general application.

The vibration here considered is called the *free* vibration. It is that executed by the system, when disturbed from equilibrium, and then *left to itself*.

46. We must now turn our attention to another problem, not less important,—the behaviour of the system, when subjected to an external force varying as a harmonic function of the time. In order to save repetition, we may take at once the more general case including friction. If there be no friction, we have only to put in our results $\kappa = 0$. The differential equation is

$$\ddot{u} + \kappa\dot{u} + n^2 u = E\cos pt \quad\quad (1).$$

Assume

$$u = a\cos(pt - \epsilon) \quad\quad (2),$$

and substitute:

$$a(n^2 - p^2)\cos(pt - \epsilon) - \kappa pa\sin(pt - \epsilon)$$
$$= E\cos\epsilon\cos(pt - \epsilon) - E\sin\epsilon\sin(pt - \epsilon);$$

whence, on equating coefficients of $\cos(pt - \epsilon)$, $\sin(pt - \epsilon)$,

$$\left.\begin{aligned} a(n^2 - p^2) &= E\cos\epsilon \\ a\,.\,p\kappa &= E\sin\epsilon \end{aligned}\right\} \quad\quad (3),$$

so that the solution may be written

$$u = \frac{E\sin\epsilon}{p\kappa}\cos(pt - \epsilon) \quad\quad (4),$$

where

$$\tan\epsilon = \frac{p\kappa}{n^2 - p^2} \quad\quad (5).$$

This is called a *forced* vibration; it is the response of the system to a force imposed upon it from without, and is maintained by the continued operation of that force. The amplitude is proportional to E—the magnitude of the force, and the period is the same as that of the force.

Let us now suppose E given, and trace the effect on a given system of a variation in the period of the force. The effects produced in different cases are not strictly similar; because the frequency of the vibrations produced is always the same as that of the force, and therefore variable in the comparison which we are about to institute. We may, however, compare the energy of the system in different cases at the moment of passing through the position of equilibrium. It is necessary thus to specify the moment at which the energy is to be computed in each case, because the total energy is not invariable throughout the vibration. During one part of the period the system receives energy from the impressed force, and during the remainder of the period yields it back again.

From (4), if $u = 0$,

$$\text{energy} \propto \dot{u}^2 \propto \sin^2 \epsilon,$$

and is therefore a maximum, when $\sin \epsilon = 1$, or, from (5), $p = n$. If the maximum kinetic energy be denoted by T_0, we have

$$T = T_0 \sin^2 \epsilon \qquad\qquad (6).$$

The kinetic energy of the motion is therefore the greatest possible, when the period of the force is that in which the system would vibrate freely under the influence of its own elasticity (or other internal forces), *without friction*. The vibration is then by (4) and (5),

$$u = \frac{E}{n\kappa} \sin nt;$$

and, if κ be small, its amplitude is very great. Its phase is a quarter of a period behind that of the force.

The case, where $p = n$, may also be treated independently. Since the period of the actual vibration is the same as that natural to the system.

$$\ddot{u} + n^2 u = 0,$$

so that the differential equation (1) reduces to

$$\kappa \dot{u} = E \cos pt,$$

whence by integration

$$u = \frac{E}{\kappa} \int \cos pt \, dt = \frac{E}{p\kappa} \sin pt,$$

as before.

If p be less than n, the retardation of phase relatively to the force lies between zero and a quarter period, and when p is greater than n, between a quarter period and a half period.

In the case of a system devoid of friction, the solution is

$$u = \frac{E}{n^2 - p^2} \cos pt \qquad (7).$$

When p is smaller than n, the phase of the vibration agrees with that of the force, but when p is the greater, the sign of the vibration is changed. The change of phase from complete agreement to complete disagreement, which is gradual when friction acts, here takes place abruptly as p passes through the value n. At the same time the expression for the amplitude becomes infinite. Of course this only means that, in the case of equal periods, friction *must* be taken into account, however small it may be, and however insignificant its result when p and n are not approximately equal. The limitation as to the magnitude of the vibration, to which we are all along subject, must also be borne in mind.

That the excursion should be at its maximum in one direction while the generating force is at its maximum in the opposite direction, as happens, for example, in the canal theory of the tides, is sometimes considered a paradox. Any difficulty that may be felt will be removed by considering the extreme case, in which the "spring" vanishes, so that the natural period is infinitely long. In fact we need only consider the force acting on the bob of a common pendulum swinging freely, in which case the excursion on one side is greatest when the action of gravity is at its maximum in the opposite direction. When on the other hand the inertia of the system is very small, we have the other extreme case in which the so-called equilibrium theory becomes applicable, the force and excursion being in the same phase.

When the period of the force is longer than the natural period, the effect of an increasing friction is to introduce a retardation in the phase of the displacement varying from zero up to a quarter period. If, however, the period of the natural vibration be the longer, the original retardation of half a period is diminished by something short of a quarter period; or the effect of friction is to *accelerate* the phase of the displacement estimated from that corresponding to the absence of friction. In either case the influence of friction is to cause an approximation to the state of things that would prevail if friction were paramount.

If a force of nearly equal period with the free vibrations vary slowly to a maximum and then slowly decrease, the displacement does not reach its maximum until after the force has begun to diminish. Under the operation of the force at its maximum, the vibration continues to increase until a certain limit is approached, and this increase continues for a time even although the force, having passed its maximum, begins to diminish. On this principle the retardation of spring tides behind the days of new and full moon has been explained[1].

47. From the linearity of the equations it follows that the motion resulting from the simultaneous action of any number of forces is the simple sum of the motions due to the forces taken separately. Each force causes the vibration proper to itself, without regard to the presence or absence of any others. The peculiarities of a force are thus in a manner transmitted into the motion of the system. For example, if the force be periodic in time τ, so will be the resulting vibration. Each harmonic element of the force will call forth a corresponding harmonic vibration in the system. But since the retardation of phase ϵ, and the ratio of amplitudes $a : E$, is not the same for the different components, the resulting vibration, though periodic in the same time, is different in *character* from the force. It may happen, for instance, that one of the components is isochronous, or nearly so, with the free vibration, in which case it will manifest itself in the motion out of all proportion to its original importance. As another example we may consider the case of a system acted on by two forces of nearly equal period. The resulting vibration, being compounded of two nearly in unison, is intermittent, according to the principles explained in the last chapter.

To the motions, which are the immediate effects of the impressed forces, must always be added the term expressing free vibrations, if it be desired to obtain the most general solution. Thus in the case of one impressed force,

$$u = \frac{E \sin \epsilon}{p\kappa} \cos (pt - \epsilon) + Ae^{-\frac{1}{2}\kappa t} \cos \{\sqrt{n^2 - \tfrac{1}{4}\kappa^2} \,.\, t - \alpha\} \,\ldots\ldots\ldots\, (1),$$

where A and α are arbitrary.

48. The distinction between *forced* and *free* vibrations is very important, and must be clearly understood. The period of the

[1] Airy's *Tides and Waves*, Art. 328.

former is determined solely by the force which is supposed to act on the system from without; while that of the latter depends only on the constitution of the system itself. Another point of difference is that so long as the external influence continues to operate, a forced vibration is permanent, being represented strictly by a harmonic function; but a free vibration gradually dies away, becoming negligible after a time. Suppose, for example, that the system is at rest when the force $E \cos pt$ begins to operate. Such finite values must be given to the constants A and α in (1) of § 47, that both u and $\dot{u}$ are initially zero. At first then there is a free vibration not less important than its rival, but after a time friction reduces it to insignificance, and the forced vibration is left in complete possession of the field. This condition of things will continue so long as the force operates. When the force is removed, there is, of course, no discontinuity in the values of u or $\dot{u}$, but the forced vibration is at once converted into a free vibration, and the period of the force is exchanged for that natural to the system.

During the coexistence of the two vibrations in the earlier part of the motion, the curious phenomenon of beats may occur, in case the two periods differ but slightly. For, n and p being nearly equal, and κ small, the initial conditions are approximately satisfied by

$$u = a\cos(pt - \epsilon) - ae^{-\frac{1}{2}\kappa t}\cos\{\sqrt{n^2 - \tfrac{1}{4}\kappa^2}\,.\,t - \epsilon\}.$$

There is thus a rise and fall in the motion, so long as $e^{-\frac{1}{2}\kappa t}$ remains sensible. This intermittence is very conspicuous in the earlier stages of the motion of forks driven by electro-magnetism (§ 63), [and may be utilized when it is desired to adjust n and p to equality. The initial beats are to be made slower and slower, until they cease to be perceptible. The vibration then swells continuously to a maximum.]

49. Vibrating systems of one degree of freedom may vary in two ways according to the values of the constants n and κ. The distinction of pitch is sufficiently intelligible; but it is worth while to examine more closely the consequences of a greater or less degree of damping. The most obvious is the more or less rapid extinction of a free vibration. The effect in this direction may be measured by the number of vibrations which must elapse before the amplitude is reduced in a given ratio. Initially the amplitude may be taken as unity; after a time t, let it be θ. Thus $\theta = e^{-\frac{1}{2}\kappa t}$.

If $t = x\tau$, we have $x = -\frac{2}{\kappa\tau}\log\theta$. In a system subject to only a moderate degree of damping, we may take approximately,

$$\tau = 2\pi \div n;$$

so that

$$x = -\frac{n}{\kappa\pi}\log\theta \quad\text{.......................(1).}$$

This gives the number of vibrations which are performed, before the amplitude falls to θ.

The influence of damping is also powerfully felt in a forced vibration, when there is a near approach to isochronism. In the case of an exact equality between p and n, it is the damping alone which prevents the motion becoming infinite. We might easily anticipate that when the damping is small, a comparatively slight deviation from perfect isochronism would cause a large falling off in the magnitude of the vibration, but that with a larger damping the same precision of adjustment would not be required. From the equations

$$T = T_0 \sin^2\epsilon, \quad \tan\epsilon = \frac{\kappa p}{n^2 - p^2},$$

we get

$$\frac{n^2 - p^2}{\kappa p} = \sqrt{\frac{T_0 - T}{T}} \quad\text{.....................(2);}$$

so that if κ be small, p must be very nearly equal to n, in order to produce a motion not greatly less than the maximum.

The two principal effects of damping may be compared by eliminating κ between (1) and (2). The result is

$$\frac{\log\theta}{x} = \pi\left(\frac{p}{n} - \frac{n}{p}\right)\sqrt{\frac{T}{T_0 - T}} \quad\text{.................(3),}$$

where the sign of the square root must be so chosen as to make the right-hand side negative.

If, when a system vibrates freely, the amplitude be reduced in the ratio θ after x vibrations; then, when it is acted on by a force (p), the energy of the resulting motion will be less than in the case of perfect isochronism in the ratio $T : T_0$. It is a matter of indifference whether the forced or the free vibration be the higher; all depends on the *interval*.

In most cases of interest the interval is small; and then, putting $p = n + \delta n$, the formula may be written,

$$\frac{\log\theta}{x} = \frac{2\pi\delta n}{n}\sqrt{\frac{T}{T_0 - T}} \quad\text{.....................(4).}$$

The following table calculated from these formulæ has been given by Helmholtz[1]:

Interval corresponding to a reduction of the resonance to one-tenth. $T : T_0 = 1 : 10$.	Number of vibrations after which the intensity of a free vibration is reduced to one-tenth. $\theta^2 = \frac{1}{10}$.
$\frac{1}{8}$ tone.	38·00
$\frac{1}{4}$ tone.	19·00
$\frac{1}{2}$ tone.	9·50
$\frac{3}{4}$ tone.	6·33
Whole tone.	4·75
$\frac{5}{4}$ tone.	3·80
$\frac{6}{4}$ tone = minor third.	3·17
$\frac{7}{4}$ tone.	2·71
Two whole tones = major third.	2·37

Formula (4) shews that, when δn is small, it varies *cœteris paribus* as $\frac{1}{x}$.

50. From observations of forced vibrations due to known forces, the natural period and damping of a system may be determined. The formulæ are

$$u = \frac{E \sin \epsilon}{p\kappa} \cos (pt - \epsilon),$$

where
$$\tan \epsilon = \frac{p\kappa}{n^2 - p^2}.$$

On the equilibrium theory we should have

$$u = \frac{E}{n^2} \cos pt.$$

The ratio of the actual amplitude to this is

$$\frac{E \sin \epsilon}{p\kappa} : \frac{E}{n^2} = \frac{n^2 \sin \epsilon}{p\kappa}.$$

If the equilibrium theory be known, the comparison of amplitudes tells us the value of $\frac{n^2 \sin \epsilon}{p\kappa}$, say

$$\frac{n^2 \sin \epsilon}{p\kappa} = a,$$

[1] *Tonempfindungen*, 3rd edition, p. 221.

and ϵ is also known, whence

$$n^2 = p^2 \div \left(1 - \frac{\cos\epsilon}{a}\right), \text{ and } \kappa = \frac{p\sin\epsilon}{a - \cos\epsilon}\text{...........(1).}$$

51. As has been already stated, the distinction of forced and free vibrations is important; but it may be remarked that most of the forced vibrations which we shall have to consider as affecting a system, take their origin ultimately in the motion of a second system, which influences the first, and is influenced by it. A vibration may thus have to be reckoned as forced in its relation to a system whose limits are fixed arbitrarily, even when that system has a share in determining the period of the force which acts upon it. On a wider view of the matter embracing both the systems, the vibration in question will be recognized as free. An example may make this clearer. A tuning-fork vibrating in air is part of a compound system including the air and itself, and in respect of this compound system the vibration is free. But although the fork is influenced by the reaction of the air, yet the amount of such influence is small. For practical purposes it is convenient to consider the motion of the fork as given, and that of the air as forced. No error will be committed if the *actual* motion of the fork (as influenced by its surroundings) be taken as the basis of calculation. But the peculiar advantage of this mode of conception is manifested in the case of an approximate solution being required. It may then suffice to substitute for the actual motion, what would be the motion of the fork in the absence of air, and afterwards introduce a correction, if necessary.

52. Illustrations of the principles of this chapter may be drawn from all parts of Acoustics. We will give here a few applications which deserve an early place on account of their simplicity or importance.

A string or wire ACB is stretched between two fixed points A and B, and at its centre carries a mass M, which is supposed to be so considerable as to render the mass of the string itself negligible. When M is pulled aside from its position of equilibrium, and then let go, it executes along the line CM vibrations, which are the subject of inquiry. $AC = CB = a$. $CM = x$. The tension of the string in the position of equilibrium depends on the amount of the stretching to which it has been subjected. In any other position the tension is greater; but we limit ourselves to the case of vibrations so small that the additional stretching is a negligible fraction of the whole. On this condition the tension may be treated as constant. We denote it by T.

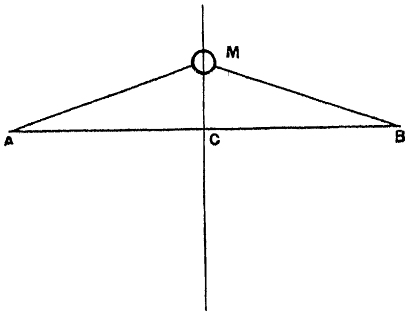


FIG. 9.

Thus, kinetic energy $= \frac{1}{2}M\dot{x}^2$,

and

potential energy $= 2T\{\sqrt{a^2+x^2} - a\} = T\dfrac{x^2}{a}$ approximately.

The equation of motion (which may be derived also independently) is therefore

$$M\ddot{x} + 2T\frac{x}{a} = 0 \qquad\qquad (1),$$

from which we infer that the mass M executes harmonic vibrations, whose period

$$\tau = 2\pi \div \sqrt{\frac{2T}{aM}} \qquad\qquad (2).$$

The amplitude and phase depend of course on the initial circumstances, being arbitrary so far as the differential equation is concerned.

Equation (2) expresses the manner in which τ varies with each of the independent quantities T, M, a: results which may all be obtained by consideration of the *dimensions* (in the technical sense) of the quantities involved. The argument from dimensions is so often of importance in Acoustics that it may be well to consider this first instance at length.

In the first place we must assure ourselves that of all the quantities on which τ may depend, the only ones involving a reference to the three fundamental units—of length, time, and mass—are a, M, and T. Let the solution of the problem be written

$$\tau = f(a, M, T) \qquad (3).$$

This equation must retain its form unchanged, whatever may be the fundamental units by means of which the four quantities are numerically expressed, as is evident, when it is considered that in deriving it no assumptions would be made as to the magnitudes of those units. Now of all the quantities on which f depends, T is the only one involving time; and since its dimensions are (Mass) (Length) $(\text{Time})^{-2}$, it follows that when a and M are constant, $\tau \propto T^{-\frac{1}{2}}$; otherwise a change in the unit of time would necessarily disturb the equation (3). This being admitted, it is easy to see that in order that (3) may be independent of the unit of length, we must have $\tau \propto T^{-\frac{1}{2}} . a^{\frac{1}{2}}$, when M is constant; and finally, in order to secure independence of the unit of mass,

$$\tau \propto T^{-\frac{1}{2}} . M^{\frac{1}{2}} . a^{\frac{1}{2}}.$$

To determine these indices we might proceed thus:—assume

$$\tau \propto T^x . M^y . a^z;$$

then by considering the dimensions in time, space, and mass, we obtain respectively

$$1 = -2x, \quad 0 = x + z, \quad 0 = x + y,$$

whence as above $\quad x = -\frac{1}{2}, \quad y = \frac{1}{2}, \quad z = \frac{1}{2}.$

There must be no mistake as to what this argument does and does not prove. We have *assumed* that there is a definite periodic time depending on no other quantities, having dimensions in space, time, and mass, than those above mentioned. For example, we have not proved that τ is independent of the amplitude of vibration. That, so far as it is true at all, is a consequence of the linearity of the approximate differential equation.

From the necessity of a complete enumeration of all the quantities on which the required result may depend, the method of dimensions is somewhat dangerous; but when used with proper care it is unquestionably of great power and value.

53. The solution of the present problem might be made the foundation of a method for the absolute measurement of pitch. The principal impediment to accuracy would probably be the difficulty of making M sufficiently large in relation to the mass of the wire, without at the same time lowering the note too much in the musical scale.

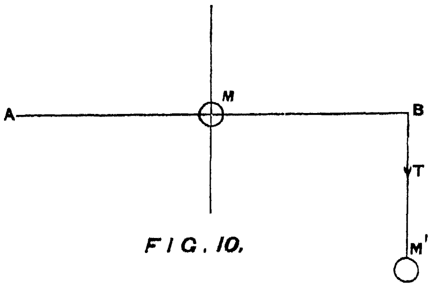


FIG. 10.

The wire may be stretched by a weight M' attached to its further end beyond a bridge or pulley at B. The periodic time would be calculated from

$$\tau = 2\pi \,.\, \sqrt{\frac{aM}{2gM'}} \quad\text{.......................(1).}$$

The ratio of $M' : M$ is given by the balance. If a be measured in feet, and $g = 32{\cdot}2$, the periodic time is expressed in seconds.

54. In an ordinary musical string the weight, instead of being concentrated in the centre, is uniformly distributed over its length. Nevertheless the present problem gives some idea of the nature of the gravest vibration of such a string. Let us compare the two cases more closely, supposing the amplitudes of vibration the same at the middle point.

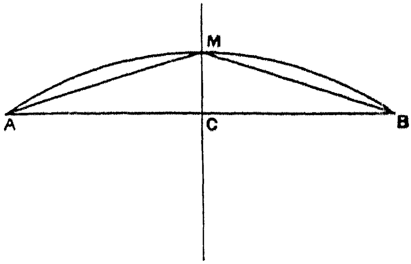


FIG. 11.

When the uniform string is straight, at the moment of passing through the position of equilibrium, its different parts are animated with a variable velocity, increasing from either end towards the centre. If we attribute to the whole mass the velocity of the centre, it is evident that the kinetic energy will be considerably over-estimated. Again, at the moment of maximum excursion, the uniform string is more stretched than its substitute, which follows the straight courses AM, MB, and accordingly the potential energy is diminished by the substitution. The concentration of the mass at the middle point at once increases the kinetic energy when $x=0$, and decreases the potential energy when $\dot{x}=0$, and therefore, according to the principle explained in § 44, prolongs the periodic time. For a string then the period is less than that calculated from the formula of the last section, on the supposition that M denotes the mass of the string. It will afterwards appear that in order to obtain a correct result we should have to take instead of M only $(4/\pi^2)M$. Of the factor $4/\pi^2$ by far the more important part, viz. $\frac{1}{2}$, is due to the difference of the kinetic energies.

55. As another example of a system possessing practically but one degree of freedom, let us consider the vibration of a spring, one end of which is clamped in a vice or otherwise held fast, while the other carries a heavy mass.

In strictness, this system like the last has an infinite number of independent modes of vibration; but, when the mass of the spring is relatively small, that vibration which is nearly independent of its inertia becomes so much the most important that the others may be ignored. Pushing this idea to its limit, we may regard the spring merely as the origin of a force urging the attached mass towards the position of equilibrium, and, if a certain point be not exceeded, in simple proportion to the displacement. The result is a harmonic vibration, with a period dependent on the stiffness of the spring and the mass of the load.

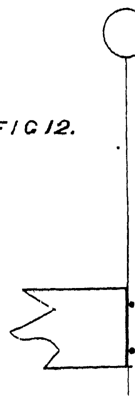


FIG 12.

56. In consequence of the oscillation of the centre of inertia, there is a constant tendency towards the communication of motion to the supports, to resist which adequately the latter must be very firm and massive. In order to obviate this inconvenience, two precisely similar springs and loads may be mounted on the same framework in a symmetrical manner. If the two loads perform vibrations of equal amplitude in such a manner that the motions are always opposite, or, as it may otherwise be expressed, with a phase-difference of half a period, the centre of inertia of the whole system remains at rest, and there is no tendency to set the framework into vibration. We shall see in a future chapter that this peculiar relation of phases will quickly establish itself, whatever may be the original disturbance. In fact, any part of the motion which does not conform to the condition of leaving the centre of inertia unmoved is soon extinguished by damping, unless indeed the supports of the system are more than usually firm.

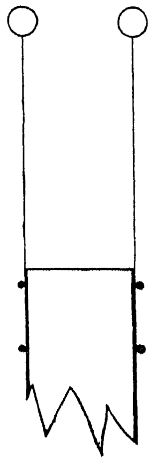


FIG 13

57. As in our first example we found a rough illustration of the fundamental vibration of a musical string, so here with the spring and attached load we may compare a uniform slip, or bar, of elastic material, one end of which is securely fastened, such for instance as the *tongue* of a *reed* instrument. It is true of course that the mass is not concentrated at one end, but distributed over the whole length; yet on account of the smallness of the motion near the point of support, the inertia of that part of the bar is of but little account. We infer that the fundamental vibration of a uniform rod cannot be very different in character from that which we have been considering. Of course for purposes requiring precise calculation, the two systems are sufficiently distinct; but where the object is to form clear ideas, precision may often be advantageously exchanged for simplicity.

In the same spirit we may regard the combination of two springs and loads shewn in Fig. 13 as a representation of a tuning-fork. The instrument, which has been much improved of late years, is indispensable to the acoustical investigator. On a large scale and for rough purposes it may be made by welding a cross piece on the middle of a bar of steel, so as to form a T, and then bending the bar into the shape of a horse-shoe. On the handle a screw should be cut. But for the better class of tuning-forks it is preferable to shape the whole out of one piece of steel.

A division running from one end down the middle of a bar is first made, the two parts opened out to form the prongs of the fork, and the whole worked by the hammer and file into the required shape. The two prongs must be exactly symmetrical with respect to a plane passing through the axis of the handle, in order that during the vibration the centre of inertia may remain unmoved, —unmoved, that is, in the direction in which the prongs vibrate.

The tuning is effected thus. To make the note higher, the equivalent inertia of the system must be reduced. This is done by filing away the ends of the prongs, either diminishing their thickness, or actually shortening them. On the other hand, to lower the pitch, the substance of the prongs near the bend may be reduced, the effect of which is to diminish the force of the spring, leaving the inertia practically unchanged; or the inertia may be increased (a method which would be preferable for temporary purposes) by loading the ends of the prongs with wax, or other material. Large forks are sometimes provided with moveable weights, which slide along the prongs, and can be fixed in any position by screws. As these approach the ends (where the velocity is greatest) the equivalent inertia of the system increases. In this way a considerable range of pitch may be obtained from one fork. The number of vibrations per second for any position of the weights may be marked on the prongs.

The relation between the pitch and the size of tuning-forks is remarkably simple. In a future chapter it will be proved that, provided the material remains the same and the shape constant, the period of vibration varies directly as the linear dimension. Thus, if the linear dimensions of a tuning-fork be doubled, its note falls an octave.

58. The note of a tuning-fork is a nearly pure tone. Immediately after a fork is struck, high tones may indeed be heard, corresponding to modes of vibration, whose nature will be subsequently considered; but these rapidly die away, and even while they exist, they do not blend with the proper tone of the fork, partly on account of their very high pitch, and partly because they do not belong to its harmonic scale. In the forks examined by Helmholtz the first of these overtones had a frequency from 5·8 to 6·6 times that of the proper tone.

Tuning-forks are now generally supplied with resonance cases, whose effect is greatly to augment the volume and purity of the sound, according to principles to be hereafter developed. In order to excite them, a violin or cello bow, well supplied with rosin, is drawn across the prongs in the direction of vibration. The sound so produced will last a minute or more.

59. As standards of pitch tuning-forks are invaluable. The pitch of organ-pipes rapidly varies with the temperature and with the pressure of the wind; that of strings with the tension, which can never be retained constant for long; but a tuning-fork kept clean and not subjected to violent changes of temperature or magnetization, preserves its pitch with great fidelity.

[But it must not be supposed that the vibrations of a fork are altogether independent of temperature. According to the observations of McLeod and Clarke[1] the frequency falls by ·00011 of its value for each degree Cent. of elevation.]

By means of beats a standard tuning-fork may be copied with very great precision. The number of beats heard in a second is the difference of the frequencies of the two tones which produce them; so that if the beats can be made so slow as to occupy half a minute each, the frequencies differ by only 1-30th of a vibration. Still greater precision might be obtained by Lissajous' optical method.

Very slow beats being difficult of observation, in consequence of the uncertainty whether a falling off in the sound is due to interference or to the gradual dying away of the vibrations, Scheibler adopted a somewhat modified plan. He took a fork slightly different in pitch from the standard—whether higher or lower is not material, but we will say, lower,—and counted the number of beats, when they were sounded together. About four beats a second is the most suitable, and these may be counted for perhaps a minute. The fork to be adjusted is then made slightly higher than the auxiliary fork, and tuned to give with it precisely the same number of beats, as did the standard. In this way a copy as exact as possible is secured. To facilitate the counting of the beats Scheibler employed pendulums, whose periods of vibration could be adjusted.

[The question between slow and quick beats depends upon the circumstances of the case. It seems to be sometimes supposed that quick beats have the advantage as admitting of greater

[1] *Phil. Trans.* 1880, p. 12.

relative accuracy of counting. But a little consideration shews that in a comparison of frequencies we are concerned not with the *relative*, but with the *absolute* accuracy of the counting. If we miscount the beats in a minute by one, it makes just the same error in the result, whether the whole number of beats be 60 or 240.

When the sounds are pure tones and are well maintained, it is advisable to use beats much slower than four per second. By choosing a suitable position it is often possible to make the intensities at the ear equal; and then the phase of silence, corresponding to antagonism of equal and opposite vibrations, is extremely well marked. Taking advantage of this we may determine slow beats with very great accuracy by observing the time which elapses between recurrences of silence. In favourable cases the whole number of beats in the period of observation may be fixed to within one-tenth or one-twentieth of a single beat, a degree of accuracy which is out of the question when the beats are quick. In this way beats of periods exceeding 30 seconds may be utilised with excellent effect[1].]

60. The method of beats was also employed by Scheibler to determine the absolute pitch of his standards. Two forks were tuned to an octave, and a number of others prepared to bridge over the interval by steps so small that each fork gave with its immediate neighbours in the series a number of beats that could be easily counted. The difference of frequency corresponding to each step was observed with all possible accuracy. Their sum, being the difference of frequencies for the interval of the octave, was equal to the frequency of that fork which formed the starting point at the bottom of the series. The pitch of the other forks could be deduced.

If consecutive forks give four beats per second, 65 in all will be required to bridge over the interval from c' (256) to c'' (512). On this account the method is laborious; but it is probably the most accurate for the original determination of pitch, as it is liable to no errors but such as care and repetition will eliminate. It may be observed that the essential thing is the measurement of the *difference* of frequencies for two notes, whose *ratio* of frequencies is independently known. If we could be sure of its accuracy, the interval of the fifth, fourth, or even major third, might

[1] Acoustical Observations, *Phil. Mag.* May, 1882, p. 342.

be substituted for the octave, with the advantage of reducing the number of the necessary interpolations. It is probable that with the aid of optical methods this course might be successfully adopted, as the corresponding Lissajous' figures are easily recognised, and their steadiness is a very severe test of the accuracy with which the ratio is attained.

[It is essential to the success of this method that the pitch of each of the numerous sounds employed should be definite, and in particular that the vibrations of any fork should take place at the same rate whether that fork be sounding in conjunction with its neighbour above or with its neighbour below. There is no reason to doubt that this condition is sufficiently satisfied in the case of independent tuning-forks; but an attempt to replace forks by a set of reeds, mounted side by side on a common wind-chest, has led to error, owing to a disturbance of pitch by mutual interaction[1].]

The frequency of large tuning-forks may be determined by allowing them to trace a harmonic curve on smoked paper, which may conveniently be mounted on the circumference of a revolving drum. The number of waves executed in a second of time gives the frequency.

In many cases the use of intermittent illumination described in § 42 gives a convenient method of determining an unknown frequency.

61. A series of forks ranging at small intervals over an octave is very useful for the determination of the frequency of any musical note, and is called Scheibler's Tonometer. It may also be used for tuning a note to any desired pitch. In either case the frequency of the note is determined by the number of beats which it gives with the forks, which lie nearest to it (on each side) in pitch.

For tuning pianofortes or organs, a set of twelve forks may be used giving the notes of the chromatic scale on the equal temperament, or any desired system. The corresponding notes are adjusted to unison, and the others tuned by octaves. It is better, however, to prepare the forks so as to give four vibrations per second less than is above proposed. Each note is then tuned a little higher than the corresponding fork, until they give when sounded together exactly four beats in the second. It will be

[1] *Nature*, XVII. pp. 12, 26; 1877.

observed that the addition (or subtraction) of a constant number to the frequencies is not the same thing as a mere displacement of the scale in absolute pitch.

In the ordinary practice of tuners a' is taken from a fork, and the other notes determined by estimation of fifths. It will be remembered that twelve true fifths are slightly in excess of seven octaves, so that on the equal temperament system each fifth is a little flat. The tuner proceeds upwards from a' by successive fifths, coming down an octave after about every alternate step, in order to remain in nearly the same part of the scale. Twelve fifths should bring him back to a. If this be not the case, the work must be readjusted, until all the twelve fifths are too flat by, as nearly as can be judged, the same small amount. The inevitable error is then impartially distributed, and rendered as little sensible as possible. The octaves, of course, are all tuned true. The following numbers indicate the order in which the notes may be taken:

$a\sharp$	b	c'	$c'\sharp$	d'	$d'\sharp$	e'	f'	$f'\sharp$	g'	$g'\sharp$	a'	$a'\sharp$	b'	c''	$c''\sharp$	d''	$d''\sharp$	e''
13	5	16	8	19	11	3	14	6	17	9	1	12	4	15	7	18	10	2

In practice the equal temperament is only approximately attained; but this is perhaps not of much consequence, considering that the system aimed at is itself by no means perfection.

Violins and other instruments of that class are tuned by true fifths from a'.

62. In illustration of *forced* vibration let us consider the case of a pendulum whose point of support is subject to a small horizontal harmonic motion. Q is the bob attached by a fine wire to a moveable point P. $OP = x_0$. $PQ = l$, and x is the horizontal co-ordinate of Q. Since the vibrations are supposed small, the vertical motion may be neglected, and the tension of the wire equated to the weight of Q. Hence for the horizontal motion $\ddot{x} + \kappa\dot{x} + \frac{g}{l}(x - x_0) = 0$.

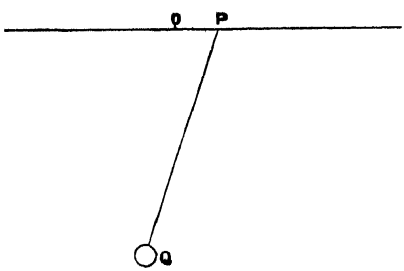


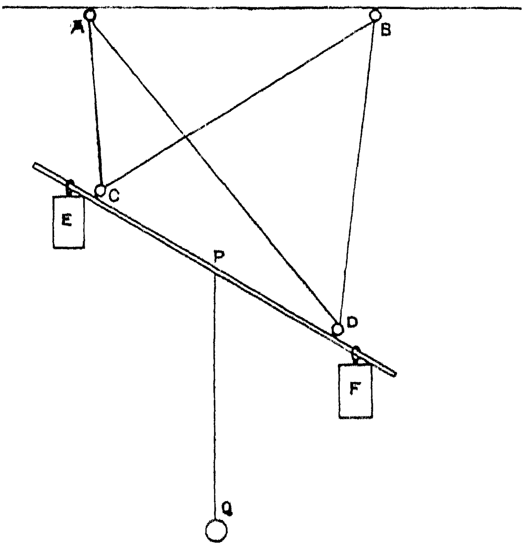
FIG. 14.

Now $x_0 \propto \cos pt$; so that putting $g \div l = n^2$, our equation takes the form already treated of, viz.

$$\ddot{x} + \kappa\dot{x} + n^2x = E \cos pt.$$

If p be equal to n, the motion is limited only by the friction. The assumed horizontal harmonic motion for P may be realized by means of a second pendulum of massive construction, which carries P with it in its motion. An efficient arrangement is shewn in the figure. A, B are iron rings screwed into a beam, or other firm

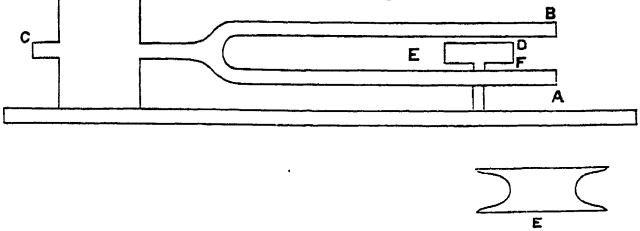
FIG 15.



support; C, D similar rings attached to a stout bar, which carries equal heavy weights E, F, attached near its ends, and is supported in a horizontal position at right angles to the beam by a wire passing through the four rings in the manner shewn. When the pendulum is made to vibrate, a point in the rod midway between C and D executes a harmonic motion in a direction parallel to CD, and may be made the point of attachment of another pendulum PQ. If the weights E and F be very great in relation to Q, the upper pendulum swings very nearly in its own proper period, and induces in Q a forced vibration of the same period. When the length PQ is so adjusted that the natural periods of the two pendulums are nearly the same, Q will be thrown into violent motion, even though the vibration of P be of but inconsiderable amplitude. In this case the difference of phase is about a quarter of a period, by which amount the upper pendulum is in advance. If the two periods be very different, the vibrations either agree or are completely opposed in phase, according to equations (4) and (5) of § 46.

63. A very good example of a forced vibration is afforded by a fork under the influence of an intermittent electric current,

FIG. 16.



whose period is nearly equal to its own. *ACB* is the fork; *E* a small electro-magnet, formed by winding insulated wire on an iron core of the shape shewn in *E* (similar to that known as 'Siemens' armature'), and supported between the prongs of the fork. When an intermittent current is sent through the wire, a periodic force acts upon the fork. This force is not expressible by a simple circular function; but may be expanded by Fourier's theorem in a series of such functions, having periods τ, $\frac{1}{2}\tau$, $\frac{1}{3}\tau$, &c. If any of these, of not too small amplitude, be nearly isochronous with the fork, the latter will be caused to vibrate; otherwise the effect is insignificant. In what follows we will suppose that it is the complete period τ which nearly agrees with that of the fork, and consequently regard the series expressing the periodic force as reduced to its first term.

In order to obtain the maximum vibration, the fork must be carefully tuned by a small sliding piece or by wax, until its natural period (without friction) is equal to that of the force. This is best done by actual trial. When the desired equality is approached, and the fork is allowed to start from rest, the force and complementary free vibration are of nearly equal amplitudes and frequencies, and therefore (§ 48) in the beginning of the motion produce *beats*, whose slowness is a measure of the accuracy of the adjustment. It is not until after the free vibration has had time to subside, that the motion assumes its permanent character. The vibrations of a tuning-fork properly constructed and mounted are subject to very little damping; consequently a very slight deviation from perfect isochronism occasions a marked falling off in the intensity of the resonance.

The amplitude of the forced vibration can be observed with sufficient accuracy by the ear or eye; but the experimental verification of the relations pointed out by theory between its phase and that of the force which causes it, requires a modified arrangement.

Two similar electro-magnets acting on similar forks, and included in the same circuit are excited by the same intermittent current. Under these circumstances it is clear that the systems will be thrown into similar vibrations, because they are acted on by equal forces. This similarity of vibrations refers both to phase and amplitude. Let us suppose now that the vibrations are effected in perpendicular directions, and by means of one of Lissajous' methods are optically compounded. The resulting figure is necessarily a straight line. Starting from the case in which the amplitudes are a maximum, viz. when the natural periods of both forks are the same as that of the force, let one of them be put a little out of tune. It must be remembered that whatever their natural periods may be, the two forks vibrate in perfect unison with the force, and therefore with one another. The principal effect of the difference of the natural periods is to destroy the synchronism of phase. The straight line, which previously represented the compound vibration, becomes an ellipse, and this remains perfectly steady, so long as the forks are not touched. Originally the forks are both a quarter period behind the force. When the pitch of one is slightly lowered, it falls still more behind the force, and at the same time its amplitude diminishes. Let the difference of phase between the two forks be ϵ', and the ratio of amplitudes of vibration $a : a_0$. Then by (6) of § 46

$$a = a_0 \cos \epsilon'.$$

The following table shews the simultaneous values of $a : a_0$ and ϵ'.

$a : a_0$	ϵ'
1·0	0
·9	25° 50′
·8	36° 52′
·7	45° 34′
·6	53° 7′
·5	60°
·4	66° 25′
·3	72° 32′
·2	78° 27′
·1	84° 15′[1]

[1] *Tonempfindungen*, 3rd edit[illegible] p. 190

It appears that a considerable alteration of phase in either direction may be obtained without very materially reducing the amplitude. When one fork is vibrating at its maximum, the other may be made to differ from it on either side by as much as 60° in phase, without losing more than half its amplitude, or by as much as 45°, without losing more than half its *energy*. By allowing one fork to vibrate 45° in advance, and the other 45° in arrear of the phase corresponding to the case of maximum resonance, we obtain a phase difference of 90° in conjunction with an equality of amplitudes. Lissajous' figure then becomes a circle.

[An intermittent electric current may also be applied to regulate the speed of a revolving body. The *phonic wheel*, invented independently by M. La Cour and by the author of this work[1], is of great service in acoustical investigations. It may take various forms; but the essential feature is the approximate closing of the magnetic circuit of an electro-magnet, fed with an intermittent current, by one or more soft iron armatures carried by the wheel and disposed symmetrically round the circumference. If in the revolution of the wheel the closest passage of the armature synchronises with the middle of the time of excitation, the electro-magnetic forces operating upon the armature during its advance and its retreat balance one another. If however the wheel be a little in arrear, the forces promoting advance gain an advantage over those hindering the retreat of the armature, and thus upon the whole the magnetic forces encourage the rotation. In like manner if the phase of the wheel be in advance of that first specified, forces are called into play which retard the motion. By a self-acting adjustment the rotation settles down into such a phase that the driving forces exactly balance the resistances. When the wheel runs lightly, and the electric appliances are moderately powerful, independent driving may not be needed. In this case of course the phase of closest passage must *follow* that which marks the middle of the time of magnetisation. If, as is sometimes advisable, there be an independent driving power, the phase of closest passage may either precede or follow that of magnetisation.

In some cases the oscillations of the motion about the phase into which it should settle down are very persistent and interfere with the applications of the instrument. A remedy may be found in a ring containing water or mercury, revolving concen-

[1] *Nature*, May 23, 1878.

trically. When the rotation is uniform, the fluid revolves like a solid body and then exercises no influence. But when from any cause the speed changes, the fluid persists for a time in the former motion, and thus brings into play forces tending to damp out oscillations.]

64. The intermittent current is best obtained by a fork-interrupter invented by Helmholtz. This may consist of a fork and electro-magnet mounted as before. The wires of the magnet are connected, one with one pole of the battery, and the other with a mercury cup. The other pole of the battery is connected with a second mercury cup. A U-shaped rider of insulated wire is carried by the lower prong just over the cups, at such a height that during the vibration the circuit is alternately made and broken by the passage of one end into and out of the mercury. The other end may be kept permanently immersed. By means of the periodic force thus obtained, the effect of friction is compensated, and the vibrations of the fork permanently maintained. In order to set another fork into forced vibration, its associated electro-magnet may be included, either in the same driving-circuit, or in a second, whose periodic interruption is effected by another rider dipping into mercury cups[1].

The *modus operandi* of this kind of self-acting instrument is often imperfectly apprehended. If the force acting on the fork depended only on its position—on whether the circuit were open or closed—the work done in passing through any position would be undone on the return, so that after a complete period there would be nothing outstanding by which the effect of the frictional forces could be compensated. Any explanation which does not take account of the retardation of the current is wholly beside the mark. The causes of retardation are two: irregular contact, and self-induction. When the point of the rider first touches the mercury, the electric contact is imperfect, probably on account of

[1] I have arranged several interrupters on the above plan, all the component parts being of home manufacture. The forks were made by the village blacksmith. The cups consisted of iron thimbles, soldered on one end of copper slips, the further end being screwed down on the base board of the instrument. Some means of adjusting the level of the mercury surface is necessary. In Helmholtz' interrupter a horse-shoe electro-magnet embracing the fork is adopted, but I am inclined to prefer the present arrangement, at any rate if the pitch be low. In some cases a greater motive power is obtained by a horse-shoe magnet acting on a soft iron armature carried horizontally by the upper prong and perpendicular to it. I have usually found a single Smee cell sufficient battery power.

adhering air. On the other hand, in leaving the mercury the contact is prolonged by the adhesion of the liquid in the cup to the amalgamated wire. On both accounts the current is retarded behind what would correspond to the mere position of the fork. But, even if the resistance of the circuit depended only on the position of the fork, the current would still be retarded by its self-induction. However perfect the contact may be, a finite current cannot be generated until after the lapse of a finite time, any more than in ordinary mechanics a finite velocity can be suddenly impressed on an inert body. From whatever causes arising[1], the effect of the retardation is that more work is gained by the fork during the retreat of the rider from the mercury, than is lost during its entrance, and thus a balance remains to be set off against friction.

If the magnetic force depended only on the position of the fork, the phase of its first harmonic component might be considered to be 180^0 in advance of that of the fork's own vibration. The retardation spoken of reduces this advance. If the phase-difference be reduced to 90^0, the force acts in the most favourable manner, and the greatest possible vibration is produced.

It is important to notice that (except in the case just referred to) the actual pitch of the interrupter differs to some extent from that natural to the fork according to the law expressed in (5) of § 46, ϵ being in the present case a prescribed phase-difference depending on the nature of the contacts and the magnitude of the self-induction. If the intermittent current be employed to drive a second fork, the maximum vibration is obtained, when the frequency of the fork coincides, not with the natural, but with the modified frequency of the interrupter.

The deviation of a tuning-fork interrupter from its natural pitch is practically very small; but the fact that such a deviation is possible, is at first sight rather surprising. The explanation (in the case of a small retardation of current) is, that during that half of the motion in which the prongs are the most separated, the electro-magnet acts in aid of the proper recovering power due to rigidity, and so naturally raises the pitch. Whatever the relation of phases may be, the force of the magnet may be divided into

[1] Any desired retardation might be obtained, in default of other means, by attaching the rider, not to the prong itself, but to the further end of a light straight spring carried by the prong and set into forced vibration by the motion of its point of attachment.

two parts respectively proportional to the velocity and displacement (or acceleration). To the first exclusively is due the sustaining power of the force, and to the second the alteration of pitch.

65. The general phenomenon of resonance, though it cannot be exhaustively considered under the head of one degree of freedom, is in the main referable to the same general principles. When a forced vibration is excited in one part of a system, all the other parts are also influenced, a vibration of the same period being excited, whose amplitude depends on the constitution of the system considered as a whole. But it not unfrequently happens that interest centres on the vibration of an outlying part whose connection with the rest of the system is but loose. In such a case the part in question, provided a certain limit of amplitude be not exceeded, is very much in the position of a system possessing one degree of freedom and acted on by a force, which may be regarded as *given*, independently of the natural period. The vibration is accordingly governed by the laws we have already investigated. In the case of approximate equality of periods to which the name of resonance is generally restricted, the amplitude may be very considerable, even though in other cases it might be so small as to be of little account; and the precision required in the adjustment of the periods in order to bring out the effect, depends on the degree of damping to which the system is subjected.

Among bodies which resound without an extreme precision of tuning, may be mentioned stretched membranes, and strings associated with sounding-boards, as in the pianoforte and the violin. When the proper note is sounded in their neighbourhood, these bodies are caused to vibrate in a very perceptible manner. The experiment may be made by singing into a pianoforte the note given by any of its strings, having first raised the corresponding damper. Or if one of the strings belonging to any note be plucked (like a harp string) with the finger, its fellows will be set into vibration, as may immediately be proved by stopping the first.

The phenomenon of resonance is, however, most striking in cases where a very accurate equality of periods is necessary in order to elicit the full effect. Of this class tuning-forks, mounted on resonance boxes, are a conspicuous example. When the unison is perfect the vibration of one fork will be taken up by another across the width of a room, but the slightest deviation of pitch is sufficient to render the phenomenon almost insensible. Forks of 256 vibrations per second are commonly used for the purpose, and it is found that a deviation from unison giving only one beat in a second makes all the difference. When the forks are well tuned and close together, the vibration may be transferred backwards and forwards between them several times, by damping them alternately, with a touch of the finger.

Illustrations of the powerful effects of isochronism must be within the experience of every one. They are often of importance in very different fields from any with which acoustics is concerned. For example, few things are more dangerous to a ship than to lie in the trough of the sea under the influence of waves whose period is nearly that of its own natural rolling.

65 *a*. It has already (§ 30) been explained how the superposition of two vibrations of equal amplitude and of nearly equal frequency gives rise to a resultant in which the sound rises and falls in beats. If we represent the two components by $\cos 2\pi n_1 t$, $\cos 2\pi n_2 t$, the resultant is

$$2\cos\pi(n_1-n_2)\,t\,.\cos\pi(n_1+n_2)\,t \quad\ldots\ldots\ldots\ldots\ldots(1);$$

and it may be regarded as a vibration of frequency $\frac{1}{2}(n_1+n_2)$, and of amplitude $2\cos\pi(n_1-n_2)\,t$. In passing through zero the amplitude changes sign, which is equivalent to a change of phase of 180°, if the amplitude be regarded as always positive. This change of phase is readily detected by measurement in drawings traced by machines for compounding vibrations, and it is a feature of great importance. If a force of this character act upon a system whose natural frequency is $\frac{1}{2}(n_1+n_2)$, the effect produced is comparatively small. If the system start from rest, the successive impulses cooperate at first, but after a time the later impulses begin to destroy the effect of former ones. The greatest response would be given to forces of frequency n_1 and n_2, and not to a force of frequency $\frac{1}{2}(n_1+n_2)$.

If, as in some experiments of Prof. A. M. Mayer[1], an otherwise steady sound is rendered intermittent by the periodic interposition of an obstacle, a very different result is arrived at. In this case the phase is resumed after each silence without reversal. If a force of this character act upon an isochronous system, the effect is indeed less than if there were no intermittence; but as all the

[1] *Phil. Mag.* May, 1875.

impulses operate in the same sense without any antagonism, the response is powerful. One kind of intermittent vibration or force is represented by

$$2(1+\cos 2\pi mt)\cos 2\pi nt \quad\quad (2),$$

in which n is the frequency of the vibration, and m the frequency of intermittence[1]. The amplitude is here always positive, and varies between the values 0 and 4. By ordinary trigonometrical transformation (2) may be put in the form

$$2\cos 2\pi nt+\cos 2\pi(n+m)t+\cos 2\pi(n-m)t \quad\quad (3);$$

which shews that the intermittent vibration in question is equivalent to three simple vibrations of frequencies n, $n+m$, $n-m$. This is the explanation of the secondary sounds observed by Mayer.

The form (2) is of course only a particular case. Another in which the intensity of the intermittent sound rises more suddenly to its maximum is given by

$$4\cos^4 \pi mt\cos 2\pi nt \quad\quad (4),$$

which may be transformed into

$$\tfrac{3}{2}\cos 2\pi nt+\cos 2\pi(n+m)t+\cos 2\pi(n-m)t$$
$$+\tfrac{1}{4}\cos 2\pi(n+2m)t+\tfrac{1}{4}\cos 2\pi(n-2m)t \quad\quad (5).$$

There are here *four* secondary sounds, the frequencies of the two new ones differing twice as much as before from that of the primary sound.

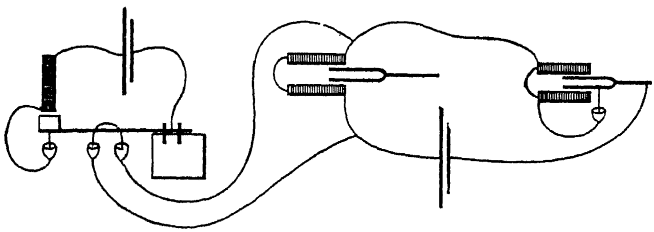
The theory of intermittent vibrations is well illustrated by electrically driven forks. A fork interrupter of frequency 128 gave a periodic current, by the passage of which through an electro-magnet a second fork of like pitch could be excited. The action of this current on the second fork could be rendered intermittent by short-circuiting the electro-magnet. This was effected by another interrupter of frequency 4, worked by an *independent* current from a Smee cell. To excite the main current a Grove cell was employed. When the contact of the second interrupter was permanently broken, so that the main current passed continuously through the electro-magnet, the fork was, of course, most powerfully affected when tuned to 128. Scarcely any response was observable when the pitch was changed to 124 or 132. But if the second interrupter were allowed to operate, so as

[1] Crum Brown and Tait. *Edin. Proc.* June, 1878. Acoustical Observations II. *Phil. Mag.* April, 1880.

to render the periodic current through the electro-magnet intermittent, then the fork would respond powerfully when tuned to 124 or 132 as well as when tuned to 128, but not when tuned to intermediate pitches, such as 126 or 130.

The operation of the intermittence in producing a sensitiveness which would not otherwise exist, is easily understood. When a fork of frequency 124 starts from rest under the influence of a force of frequency 128, the impulses cooperate at first, but after $\frac{1}{8}$ of a second the new impulses begin to oppose the earlier ones. After $\frac{1}{4}$ of a second, another series of impulses begins whose effect agrees with that of the first, and so on. Thus if all these impulses are allowed to act, the resultant effect is trifling; but if every alternate series is stopped off, a large vibration accumulates.

Fig. 16 *a.*



The most general expression for a vibration of frequency n, whose amplitude and phase are slowly variable with a frequency m, is

$$\left\{\begin{matrix} A_0 + A_1 \cos 2\pi mt + A_2 \cos 4\pi mt + A_3 \cos 6\pi mt + \ldots \\ + B_1 \sin 2\pi mt + B_2 \sin 4\pi mt + B_3 \sin 6\pi mt + \ldots \end{matrix}\right\} \cos 2\pi nt$$
$$+ \left\{\begin{matrix} C_0 + C_1 \cos 2\pi mt + C_2 \cos 4\pi mt + C_3 \cos 6\pi mt + \ldots \\ + D_1 \sin 2\pi mt + D_2 \sin 4\pi mt + D_3 \sin 6\pi mt + \ldots \end{matrix}\right\} \sin 2\pi nt \quad \ldots\ldots(6);$$

and this applies both to the case of beats (e.g. if A_1 only be finite) and to such intermittence as is produced by the interposition of an obstacle. The vibration in question is accordingly in all cases equivalent to a combination of simple vibrations of frequencies

$$n, \quad n+m, \quad n-m, \quad n+2m, \quad n-2m, \quad \&c.$$

It may be well here to emphasise that a simple vibration implies *infinite* continuance, and does not admit of variations of phase or amplitude. To suppose, as is sometimes done in optical speculations, that a train of simple waves may begin at a given epoch, continue for a certain time involving it may be a large number of periods, and ultimately cease, is a contradiction in terms.

66. The solution of the equation for free vibration, viz.

$$\ddot{u} + \kappa\dot{u} + n^2 u = 0 \qquad (1).$$

may be put into another form by expressing the arbitrary constants of integration A and α in terms of the initial values of u and $\dot{u}$, which we may denote by u_0 and $\dot{u}_0$. We obtain at once

$$u = e^{-\frac{1}{2}\kappa t}\left\{\dot{u}_0 \frac{\sin n't}{n'} + u_0\left(\cos n't + \frac{\kappa}{2n'}\sin n't\right)\right\} \qquad (2),$$

where $$n' = \sqrt{n^2 - \tfrac{1}{4}\kappa^2}.$$

If there be no friction, $\kappa = 0$, and then

$$u = \dot{u}_0 \frac{\sin nt}{n} + u_0 \cos nt \qquad (3).$$

These results may be employed to obtain the solution of the complete equation

$$\ddot{u} + \kappa\dot{u} + n^2 u = U \qquad (4),$$

where U is an explicit function of the time; for from (2) we see that the effect at time t of a velocity $\delta\dot{u}$ communicated at time t' is

$$u = \delta\dot{u}\, e^{-\frac{1}{2}\kappa(t-t')} \frac{\sin n'(t-t')}{n'}$$

The effect of U is to generate in time dt' a velocity Udt', whose result at time t will therefore be

$$u = \frac{1}{n'} U dt' e^{-\frac{1}{2}\kappa(t-t')} \sin n'(t-t'),$$

and thus the solution of (4) will be

$$u = \frac{1}{n'}\int^t e^{-\frac{1}{2}\kappa(t-t')} \sin n'(t-t')\, U\, dt' \qquad (5).$$

If there be no friction, we have simply

$$u = \frac{1}{n}\int^t \sin n(t-t')\, U\, dt' \qquad (6),$$

U being the force at time t'.

The lower limit of the integrals is so far arbitrary, but it will generally be convenient to make it zero.

On this supposition u and $\dot{u}$ as given by (6) vanish, when $t = 0$, and the complete solution is

$$u = e^{-\frac{1}{2}\kappa t}\left\{\dot{u}_0 \frac{\sin n't}{n'} + u_0\left(\cos n't + \frac{\kappa}{2n'}\sin n't\right)\right\}$$
$$+ \frac{1}{n'}\int_0^t e^{-\frac{1}{2}\kappa(t-t')} \sin n'(t-t')\, U\, dt' \qquad (7),$$

or if there be no friction

$$u = \dot{u}_0 \frac{\sin nt}{n} + u_0 \cos nt + \frac{1}{n}\int_0^t \sin n(t-t')\, U\, dt' \;\ldots\ldots\ldots (8).$$

When t is sufficiently great, the complementary terms tend to vanish on account of the factor $e^{-\frac{1}{2}\kappa t}$, and may then be omitted.

66 *a*. In § 66 we have limited the discussion to the case of greatest acoustical importance, that is, we have supposed that n' is real, as happens when n^2 is positive, and κ not too great. But a more general treatment of the problem of free vibrations is not without interest. Whatever may be the values of n^2 and κ, the solution of (1) § 66 may be expressed

$$u = Ae^{\mu_1 t} + Be^{\mu_2 t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where μ_1, μ_2 are the roots of

$$\mu^2 + \kappa\mu + n^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The case already discussed is that in which the values of μ are imaginary. The motion is then oscillatory, with amplitude which decreases if κ be positive, but increases if κ be negative.

But if n^2, though positive, be less than $\frac{1}{4}\kappa^2$, or if n^2 be negative, n' becomes imaginary, that is μ becomes real. The motion expressed by (1) is then non-oscillatory, and it depends upon the sign of μ whether it increases or diminishes with the time. From the solution of (2), viz.

$$\mu = -\tfrac{1}{2}\kappa \pm \tfrac{1}{2}\sqrt{(\kappa^2 - 4n^2)} \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

it is evident that if n^2 be positive (and less than $\frac{1}{4}\kappa^2$) the two values of μ are of the same sign, and that the sign is the opposite of that of κ. Hence if κ be positive, both terms in (1) diminish with the time, so that the system, however disturbed, subsides again into a state of rest. If, on the contrary, κ be negative, the motion increases without limit.

We have still to consider the case of n^2 negative. The real values of μ are then of *opposite* signs. It is possible so to start the system from a displaced position that it shall approach asymptotically the condition of rest in the configuration of equilibrium; but unless a special relation between displacement and velocity is satisfied, the motion tends to increase without limit. Under these circumstances the equilibrium must be regarded as *unstable*. In this sense stability requires that n^2 and κ be both positive.

A word may not be out of place as to the effect of an impressed force upon a statically unstable system. If in § 46 we suppose $\kappa = 0$, the solution (7) does not change its form merely because n^2 becomes negative. The fact that a system is susceptible of purely periodic motion under the operation of an external periodic force is therefore no evidence of stability.

67. For most acoustical purposes it is sufficient to consider the vibrations of the systems, with which we may have to deal, as infinitely small, or rather as similar to infinitely small vibrations. This restriction is the foundation of the important laws of isochronism for free vibrations, and of persistence of period for forced vibrations. There are, however, phenomena of a subordinate but not insignificant character, which depend essentially on the square and higher powers of the motion. We will therefore devote the remainder of this chapter to the discussion of the motion of a system of one degree of freedom, the motion not being so small that the squares and higher powers can be altogether neglected.

The approximate expressions for the kinetic and potential energies will be of the form

$$T = \tfrac{1}{2}(m_0 + m_1 u)\,\dot{u}^2, \quad V = \tfrac{1}{2}(\mu_0 + \mu_1 u)\,u^2.$$

If the sum of T and V be differentiated with respect to the time, we find as the equation of motion

$$m_0\ddot{u} + \mu_0 u + m_1 u\ddot{u} + \tfrac{1}{2}m_1\dot{u}^2 + \tfrac{3}{2}\mu_1 u^2 = \text{Impressed Force},$$

which may be treated by the method of successive approximation. For the sake of simplicity we will take the case where $m_1 = 0$, a supposition in no way affecting the essence of the question. The *inertia* of the system is thus constant, while the force of restitution is a composite function of the displacement, partly proportional to the displacement itself and partly proportional to its square—accordingly unsymmetrical with respect to the position of equilibrium. Thus for free vibrations our equation is of the form

$$\ddot{u} + n^2 u + \alpha u^2 = 0 \quad\quad (1),$$

with the approximate solution

$$u = A\cos nt \quad\quad (2),$$

where A—the amplitude—is to be treated as a small quantity.

Substituting the value of u expressed by (2) in the last term, we find

$$\ddot{u} + n^2 u = -\alpha\frac{A^2}{2}(1 + \cos 2nt),$$

whence for a second approximation to the value of u

$$u = A\cos nt - \frac{\alpha A^2}{2n^2} + \frac{\alpha A^2}{6n^2}\cos 2nt \ldots\ldots\ldots\ldots\ldots(3);$$

shewing that the proper tone (n) of the system is accompanied by its *octave* ($2n$), whose *relative* importance increases with the amplitude of vibration. A trained ear can generally perceive the octave in the sound of a tuning-fork caused to vibrate strongly by means of a bow, and with the aid of appliances, to be explained later, the existence of the octave may be made manifest to any one. By following the same method the approximation can be carried further; but we pass on now to the case of a system in which the recovering power is symmetrical with respect to the position of equilibrium. The equation of motion is then approximately

$$\ddot{u} + n^2u + \beta u^3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

which may be understood to refer to the vibrations of a heavy pendulum, or of a load carried at the end of a straight spring.

If we take as a first approximation $u = A\cos nt$, corresponding to $\beta = 0$, and substitute in the term multiplied by β, we get

$$\ddot{u} + n^2u = -\frac{\beta A^3}{4}\cos 3nt - \frac{3\beta A^3}{4}\cos nt.$$

Corresponding to the last term of this equation, we should obtain in the solution a term of the form $t\sin nt$, becoming greater without limit with t. This, as in a parallel case in the Lunar Theory, indicates that our assumed first approximation is not really an approximation at all, or at least does not *continue* to be such. If, however, we take as our starting point $u = A\cos mt$, with a suitable value for m, we shall find that the solution may be completed with the aid of periodic terms only. In fact it is evident beforehand that all we are entitled to assume is that the motion is approximately simple harmonic, with a period *approximately* the same, as if $\beta = 0$. A very slight examination is sufficient to shew that the term varying as u^3, not only may, but *must* affect the period. At the same time it is evident that a solution, in which the period is assumed wrongly, no matter by how little, must at length cease to represent the motion with any approach to accuracy.

We take then for the approximate equation

$$\ddot{u} + n^2u = -\frac{3\beta A^3}{4}\cos mt - \frac{\beta A^3}{4}\cos 3mt \ldots\ldots\ldots(5),$$

of which the solution will be

$$u = A \cos mt + \frac{\beta A^3}{4} \frac{\cos 3mt}{9m^2 - n^2} \quad\text{.................. (6),}$$

provided that m be taken so as to satisfy

$$A(-m^2 + n^2) = -\frac{3\beta A^3}{4},$$

or

$$m^2 = n^2 + \frac{3\beta A^2}{4} \quad\text{.......................... (7).}$$

The term in β thus produces two effects. It alters the pitch of the fundamental vibration, and it introduces the *twelfth* as a necessary accompaniment. The alteration of pitch is in most cases exceedingly small—depending on the square of the amplitude, but it is not altogether insensible. Tuning-forks generally rise a little, though very little, in pitch as the vibration dies away. It may be remarked that the same slight dependence of pitch on amplitude occurs when the force of restitution is of the form $n^2u + \alpha u^2$, as may be seen by continuing the approximation to the solution of (1) one step further than (3). The result in that case is

$$m^2 = n^2 - \frac{5\alpha^2 A^2}{6n^2} \quad\text{.................... (8)[1].}$$

The difference $m^2 - n^2$ is of the same order in A in both cases; but in one respect there is a distinction worth noting, namely, that in (8) m^2 is always less than n^2, while in (7) it depends on the sign of β whether its effect is to raise or lower the pitch. However, in most cases of the unsymmetrical class the change of pitch would depend partly on a term of the form αu^2 and partly on another of the form βu^3, and then

$$m^2 = n^2 - \frac{5\alpha^2 A^2}{6n^2} + \frac{3\beta A^2}{4} \quad\text{.............. (9)[1].}$$

[In all cases where the period depends upon amplitude, it is necessarily an *even* function thereof, a change of sign in the amplitude being merely equivalent to an alteration in phase of 180°.]

68. We now pass to the consideration of the vibrations forced on an unsymmetrical system by two harmonic forces

$$E \cos pt, \quad F \cos (qt - \epsilon).$$

[1] [A correction is here introduced, the necessity for which was pointed out to me by Dr Burton.]

The equation of motion is

$$\ddot{u} + n^2 u = -\alpha u^2 + E\cos pt + F\cos(qt-\epsilon) \;\ldots\ldots\; (1).$$

To find a first approximation we neglect the term containing α. Thus

$$u = e\cos pt + f\cos(qt-\epsilon) \;\ldots\ldots\ldots\ldots\; (2),$$

where

$$e = \frac{E}{n^2 - p^2}, \quad f = \frac{F}{n^2 - q^2} \;\ldots\ldots\ldots\ldots\; (3).$$

Substituting this in the term multiplied by α, we get

$$\ddot{u} + n^2 u = E\cos pt + F\cos(qt-\epsilon)$$

$$-\alpha\left[\frac{e^2+f^2}{2} + \frac{e^2}{2}\cos 2pt + \frac{f^2}{2}\cos 2(qt-\epsilon) + ef\cos\{(p-q)t+\epsilon\} + ef\cos\{(p+q)t-\epsilon\}\right]$$

whence as a second approximation for u

$$u = e\cos pt + f\cos(qt-\epsilon) - \frac{\alpha(e^2+f^2)}{2n^2} - \frac{\alpha e^2}{2(n^2-4p^2)}\cos 2pt$$

$$-\frac{\alpha f^2}{2(n^2-4q^2)}\cos 2(qt-\epsilon) - \frac{\alpha ef}{n^2-(p-q)^2}\cos\{(p-q)t+\epsilon\}$$

$$-\frac{\alpha ef}{n^2-(p+q)^2}\cos\{(p+q)t-\epsilon\} \;\ldots\ldots\ldots\ldots\; (4).$$

The additional terms represent vibrations having frequencies which are severally the doubles and the sum and difference of those of the primaries. Of the two latter the amplitudes are proportional to the product of the original amplitudes, shewing that the derived tones increase in relative importance with the intensity of their parent tones.

68a. If an isolated vibrating system be subject to internal dissipative influences, the vibrations cannot be permanent, since they are dependent upon an initial store of energy which suffers gradual exhaustion. In order that the motion may be maintained, the vibrating body must be in connection with a source of energy. We have already considered cases of this kind under the head of forced vibrations, where the system is subject to forces whose amplitude and phase are prescribed, independently of the behaviour of the system. Such forces may have their origin in revolving mechanism (such as electric alternators) governed so as to move at a uniform speed. But more frequently the forces under consideration depend upon the vibrations of other systems, and then the question as to how the vibrations are to be maintained represents itself. A good example is afforded by the case already discussed (§§ 63, 65) of a fork maintained in vibration electrically by means of currents governed by a fork interrupter. It has been pointed out that the performance of the latter depends upon the magnetic forces operative upon it differing in phase from the vibrations of the fork itself. With the interrupter may be classed for the present purpose almost all acoustical and musical instruments capable of providing a sustained sound. It may suffice to mention vibrations maintained by wind (organ-pipes, harmonium reeds, æolian harps, &c.), by heat (singing flames, Rijke's tubes, &c.), by friction (violin strings, finger-glasses), and the slower vibrations of clock pendulums and watch balance-wheels.

In considering whether proposed forces are of the right kind for the maintenance or encouragement of a vibration, it is often convenient to regard them as reduced to impulses. Suppose, to take a simple case, that a small horizontal positive impulse acts upon the bob of a vibrating pendulum. The effect depends, of course, upon the phase of the vibration at the instant of the impulse. If the bob be moving positively at the instant in question the vibration is encouraged, and this effect is a maximum when the positive motion is greatest, that is, when the impulse occurs at the moment of positive movement through the position of equilibrium. This is the condition of things aimed at in designing a clock escapement, for the effect of the force is then a maximum in encouraging the vibration, and a minimum (zero to the first order of approximation) in disturbing the period. Of course, if the impulse be half a period earlier or later than is above supposed, the effect is to discourage the vibration, again without altering the period. In like manner we see that if the impulse occur at a moment of maximum elongation the effect is concentrated upon the period, the vibration being neither encouraged nor discouraged.

In most cases the force acting upon a vibrating system in virtue of its connection with a source of energy may be regarded as harmonic. It may then be divided into two parts, one proportional to the displacement u (or to the acceleration $\ddot{u}$), the second proportional to the velocity $\dot{u}$. The inclusion of such forces does not alter the form of the equation of vibration

$$\ddot{u} + \kappa\dot{u} + n^2 u = 0 \text{ ....} (1).$$

By the first part (proportional to u) the pitch is modified, and by the second the coefficient of decay. If the altered κ be still positive, vibrations gradually die down; but if the effect of the included forces be to render κ negative, vibrations tend on the contrary to increase. The only case in which according to (1) a steady vibration is possible, is when the complete value of κ is zero. If this condition be satisfied, a vibration of any amplitude is permanently maintained.

When κ is negative, so that small vibrations tend to increase, a point is of course soon reached beyond which the approximate equations cease to be applicable. We may form an idea of the state of things which then arises by adding to equation (1) a term proportional to a higher power of the velocity. Let us take

$$\ddot{u} + \kappa\dot{u} + \kappa'\dot{u}^3 + n^2u = 0 \quad\text{.....................(2)},$$

in which κ and κ' are supposed to be small quantities. The approximate solution of (2) is

$$u = A\sin nt + \frac{\kappa' n A^3}{32}\cos 3nt \quad\text{.................(3)},$$

in which A is given by

$$\kappa + \tfrac{3}{4}\kappa' n^2 A^2 = 0 \quad\text{.......................(4)}.$$

From (4) we see that no steady vibration is possible unless κ and κ' have opposite signs. If κ and κ' be both positive, the vibration in all cases dies down; while if κ and κ' be both negative, the vibration (according to (2)) increases without limit. If κ be negative and κ' positive, the vibration becomes steady and assumes the amplitude determined by (4). A smaller vibration increases up to this point, and a larger vibration falls down to it. If on the other hand κ be positive, while κ' is negative, the steady vibration abstractedly possible is unstable, a departure in either direction from the amplitude given by (4) tending always to increase[1].

68 *b*. We will now consider briefly another and a very curious kind of maintenance, of which the peculiarity is that the maintaining influence operates with a frequency which is the double of that of the vibration maintained. Probably the best known example is that form of Melde's experiment, in which a fine string is maintained in transverse vibration by connecting one of its extremities with the vibrating prong of a massive tuning-fork,

[1] On Maintained Vibrations, *Phil. Mag.*, April, 1883

the direction of motion of the point of attachment being parallel to the length of the string. The effect of the motion is to render the tension of the string periodically variable; and at first sight there is nothing to cause the string to depart from its equilibrium condition of straightness. It is known, however, that under these circumstances the equilibrium may become unstable, and that the string may settle down into a state of permanent and vigorous vibration, whose period is the *double* of that of the fork.

As a simpler example, with but one degree of freedom, we may take a pendulum, formed of a bar of soft iron and vibrating upon knife-edges. Underneath is placed symmetrically a vertical bar electro-magnet, through which is caused to pass an electric current rendered intermittent by an interrupter whose frequency is twice that of the pendulum. The magnetic force does not tend to displace the pendulum from its equilibrium position, but produces the same sort of effect as if gravity were subject to a periodic variation of intensity.

A similar result is obtained by causing the point of support of the pendulum to vibrate in a *vertical* path. If we denote this motion by $\eta = \beta \sin 2pt$, the effect is as if gravity were variable by the term $4p^2\beta \sin 2pt$.

Of the same nature are the crispations observed by Faraday[1] and others upon the surface of water which oscillates vertically. Faraday arrived experimentally at the conclusion that there were two complete vibrations of the support for each complete vibration of the liquid.

In the following investigation[2], relative to the case of one degree of freedom, we shall start with the assumption that a steady vibration is in progress, and inquire under what conditions the assumed state of things is possible.

If the force of restitution, or "spring," of a body susceptible of vibration be subject to an imposed periodic variation, the differential equation takes the form

$$\ddot{u} + \kappa \dot{u} + (n^2 - 2\alpha \sin 2pt)\, u = 0 \quad \text{..............(1)},$$

in which κ and α are supposed to be small. A similar equation would apply approximately to the case of a periodic variation in the effective mass of the body. The motion expressed by the solution of (1) can be regular only when it keeps perfect time

[1] *Phil. Trans.* 1831, p. 299.
[2] *Phil. Mag.*, April, 1883.

with the imposed variations. It will appear that the necessary conditions cannot be satisfied rigorously by any simple harmonic vibration, but we may assume

$$u = A_1 \sin pt + B_1 \cos pt + A_3 \sin 3pt + B_3 \cos 3pt + A_5 \sin 5pt + \dots\dots\dots(2),$$

in which it is not necessary to provide for sines and cosines of even multiples of pt. If the assumption be justifiable, the solution in (2) must be convergent. Substituting in the differential equation, and equating to zero the coefficients of sin pt, cos pt, &c. we find

$$A_1(n^2 - p^2) - \kappa p B_1 - \alpha B_1 + \alpha B_3 = 0,$$
$$B_1(n^2 - p^2) + \kappa p A_1 - \alpha A_1 - \alpha A_3 = 0;$$
$$A_3(n^2 - 9p^2) - 3\kappa p B_3 - \alpha B_1 + \alpha B_5 = 0,$$
$$B_3(n^2 - 9p^2) + 3\kappa p A_3 + \alpha A_1 - \alpha A_5 = 0;$$
$$A_5(n^2 - 25p^2) - 5\kappa p B_5 - \alpha B_3 + \alpha B_7 = 0,$$
$$B_5(n^2 - 25p^2) + 5\kappa p A_5 + \alpha A_3 - \alpha A_7 = 0;$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

These equations shew that A_3, B_3 are of the order α relatively to A_1, B_1; that A_5, B_5 are of order α relatively to A_3, B_3, and so on. If we omit A_3, B_3 in the first pair of equations, we find as a first approximation,

$$A_1(n^2 - p^2) - (\kappa p + \alpha) B_1 = 0,$$
$$A_1(\kappa p - \alpha) + (n^2 - p^2) B_1 = 0;$$

whence $$\frac{B_1}{A_1} = \frac{n^2 - p^2}{\kappa p + \alpha} = \frac{\alpha - \kappa p}{n^2 - p^2} = \frac{\sqrt{(\alpha - \kappa p)}}{\sqrt{(\alpha + \kappa p)}} \dots\dots\dots\dots(3),$$

and $$(n^2 - p^2)^2 = \alpha^2 - \kappa^2 p^2 \dots\dots\dots\dots(4).$$

Thus, if α be given, the value of p necessary for a regular motion is definite; and p having this value, the regular motion is

$$u = P \sin(pt + \epsilon),$$

in which ϵ, being equal to $\tan^{-1}(B_1/A_1)$, is also definite. On the other hand, as is evident at once from the linearity of the original equation, there is nothing to limit the amplitude of vibration.

These characteristics are preserved however far it may be necessary to pursue the approximation. If A_{2m+1}, B_{2m+1} may be neglected, the first m pairs of equations determine the *ratios* of all the coefficients, leaving the absolute magnitude open; and they provide further an equation connecting p and α, by which the pitch is determined.

For the second approximation the second pair of equations give

$$A_3 = \frac{\alpha B_1}{n^2 - 9p^2}, \qquad B_3 = -\frac{\alpha A_1}{n^2 - 9p^2},$$

whence

$$u = P \sin(pt + \epsilon) + \frac{\alpha P}{9p^2 - n^2} \cos(3pt + \epsilon) \;\ldots\ldots\ldots(5),$$

and from the first pair

$$\tan \epsilon = \left\{n^2 - p^2 - \frac{\alpha^2}{n^2 - 9p^2}\right\} \div (\alpha + \kappa p) \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

while p is determined by

$$\left\{n^2 - p^2 - \frac{\alpha^2}{n^2 - 9p^2}\right\}^2 = \alpha^2 - \kappa^2 p^2 \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Returning to the first approximation, we see from (4) that the solution is possible only under the condition that α be not less than κp. If $\alpha = \kappa p$, then $p = n$; that is, the imposed variation in the "spring" must be exactly twice as quick as the natural vibration of the body would be in the absence of friction. From (3) it appears that in this case $\epsilon = 0$, which indicates that the spring is a minimum one-eighth of a period *after* the body has passed its position of equilibrium, and a maximum one-eighth of a period *before* such passage. Under these circumstances the greatest possible amount of energy is communicated to the system; and in the case contemplated it is just sufficient to balance the loss by dissipation, the adjustment being evidently independent of the amplitude.

If $\alpha < \kappa p$ sufficient energy cannot pass to maintain the motion, whatever may be the phase-relation; but if $\alpha > \kappa p$, the balance between energy supplied and energy dissipated may be attained by such an alteration of phase as shall diminish the former quantity to the required amount. The alteration of phase may for this purpose be indifferently in either direction; but if ϵ be positive, we must have

$$p^2 = n^2 - \sqrt{(\alpha^2 - \kappa^2 p^2)};$$

while if ϵ be negative

$$p^2 = n^2 + \sqrt{(\alpha^2 - \kappa^2 p^2)}.$$

If α be very much greater than κp, $\epsilon = \pm \frac{1}{4}\pi$. which indicates that when the system passes through its position of equilibrium the spring is at its maximum or at its minimum.

The inference from the equation that the adjustment of pitch must be absolutely rigorous for steady vibration will be subject to some modification in practice; otherwise the experiment could not succeed. In most cases n^2 is to a certain extent a function of amplitude; so that if n^2 have very nearly the required value, complete coincidence is attainable by the assumption of an amplitude of large and determinate amount without other alterations in the conditions of the system.

The reader who wishes to pursue this subject is referred to a paper by the Author "On the Maintenance of Vibrations by Forces of Double Frequency, and on the Propagation of Waves through a Medium endowed with a Periodic Structure,"[1] in which the analysis of Mr Hill[2] is applied to the present problem.

68 *c*. The determination of absolute pitch by means of the siren has already been alluded to (§ 17). In all probability first-rate results might be got by this method if proper provision, with the aid of a phonic wheel for example, were made for uniform speed. In recent years several experimenters have obtained excellent results by various methods; but a brief notice of these is all that our limits will allow.

One of the most direct determinations is that of Koenig[3], to whom the scientific world has long been indebted for the construction of much excellent apparatus. This depends upon a special instrument, consisting of a fork of 64 complete vibrations per second, the motion being maintained by a clock movement acting upon an escapement. A dial is provided marking ordinary time, and serves to record the number of vibrations executed. The performance of the fork is tested by a comparison between the instrument and any chronometer known to be keeping good time. The standard fork of 256 complete vibrations was compared with that of the instrument by observing the Lissajous's figure appropriate to the double octave.

M. Koenig has also investigated the influence of resonators upon the pitch of forks. Thus without a resonator a fork of 256 complete vibrations sounded in a satisfactory manner for about 90 seconds. A resonator of adjustable pitch was then brought into proximity, and the pitch, originally much graver than that of the

[1] *Phil. Mag.*, August, 1887.

[2] On the Part of the Motion of the Lunar Perigree which is a Function of the Mean Motions of the Sun and Moon, Acta Mathematica 8; 1, 1886. Mr Hill's work was first published in 1877.

[3] Wied. *Ann.* IX. p. 394, 1880.

fork, was gradually raised. Even when the resonator was still a minor third below the fork, there was observed a slight diminution in the duration of the vibratory movement, and at the same time an augmentation in the frequency of about ·005. As the natural note of the resonator approached nearer to that of the fork, this diminution in the time and this increase in frequency became more pronounced up to the immediate neighbourhood of unison; but at the moment when unison was established, the alteration of pitch suddenly disappeared, and the frequency became exactly the same as in the absence of the resonator. At the same time the sound was powerfully reinforced; but this exaggerated intensity fell off rapidly and the vibration died away after 8 or 10 seconds. The pitch of the resonator being again raised a little, the sound of the fork began to change in the opposite direction, being now as much too grave as before the unison was reached it had been too acute. The displacement then fell away by degrees, as the pitch of the resonator was further raised, and the duration of the vibrations gradually recovered its original value of about 90 seconds. The maximum disturbance in the frequency observed by Koenig was ·035 complete vibrations. For the explanation of these effects see § 117.

The temperature coefficient found by Koenig is ·000112, so that the pitch of a 256 fork falls ·0286 for each degree Cent. by which the temperature rises.

In determinations of absolute pitch[1] by the Author of this work an electrically maintained interrupter fork, whose frequency may for example be 32, was employed to drive a dependent fork of pitch 128. When the apparatus is in good order, there is a fixed relation between the two frequencies, the one being precisely four times the other. The higher is of course readily compared by beats, or by optical methods, with a standard of 128, whose accuracy is to be tested. It remains to determine the frequency of the interrupter fork itself.

For this purpose the interrupter is compared with the pendulum of a standard clock whose rate is known. The comparison may be direct, or the intervention of a phonic wheel (§ 63) may be invoked. In either case the pendulum of the clock is provided with a silvered bead upon which is concentrated the light from a lamp. Immediately in front of the pendulum is placed a screen perforated by a somewhat narrow vertical slit. The bright point of light

[1] *Nature*, XVII. p. 12, 1877; *Phil. Trans.* 1883, Part I. p. 316.

reflected by the bead is seen intermittently, either by looking over the prong of the interrupter or through a hole in the disc of the phonic wheel. In the first case there are 32 views per second, but in the latter this number is reduced by the intervention of the wheel. In the experiments referred to the wheel was so arranged that one revolution corresponded to four complete vibrations of the interrupter, and there were thus 8 views of the pendulum per second, instead of 32. Any deviation of the period of the pendulum from a precise multiple of the period of intermittence shews itself as a cycle of changes in the appearance of the flash of light, and an observation of the duration of this cycle gives the data for a precise comparison of frequencies.

The calculation of the results is very simple. Supposing in the first instance that the clock is correct, let a be the number of cycles per second (perhaps $\frac{1}{40}$) between the wheel and the clock. Since the period of a cycle is the time required for the wheel to gain, or lose, one revolution upon the clock, the frequency of revolution is $8 \pm a$. The frequency of the auxiliary fork is precisely 16 times as great, i.e. $128 \pm 16a$. If b be the number of beats per second between the auxiliary fork and the standard, the frequency of the latter is

$$128 \pm 16a \pm b.$$

An error in the mean rate of the clock is readily allowed for; but care is required to ascertain that the actual rate at the time of observation does not differ appreciably from the mean rate. To be quite safe it would be necessary to repeat the determinations at intervals over the whole time required to rate the clock by observation of the stars. In this case it would probably be convenient to attach a counting apparatus to the phonic wheel.

In the method of M'Leod and Clarke[1] time, given by a clock, is recorded automatically upon the revolving drum of a chronograph, which is maintained by a suitable governor in uniform rotation. The circumference of the drum is marked with a grating of equidistant lines parallel to the axis, and the comparison between the drum and the standard fork is effected by observation of the wavy pattern seen when the revolving grating is looked at past the edges of the vibrating prongs. These observers made a special investigation as to the effect of bowing a fork upon previously existing vibrations. Their conclusion is that in the case of unloaded forks no sensible change of phase occurs.

[1] *Phil. Trans.* 1880, Part I. p. 1.

In the chronographic method of Prof. A. M. Mayer[1] the fork under investigation is armed with a triangular fragment of thin sheet metal, one milligram in weight, and actually traces its vibrations as a curve of sines upon smoked paper. The time is recorded by small electric discharges from an induction apparatus, under the control of a clock, and delivered from the *same tracing point*. Although the disturbance due to the tracing point appears to be very small, it is doubtful whether this method could compete in respect of accuracy with those above described where the comparison with the standard is optical or acoustical. On the other hand, it has the advantage of not requiring a uniform rotation of the drum, and the apparatus lends itself with facility to the determination of small intervals of time after the manner originally proposed by T. Young[2].

68*d*. The methods hitherto described for the determination of absolute pitch, with the exception of that of Scheibler, may be regarded as rather mechanical in their character, and they depend for the most part upon somewhat special apparatus. It is possible, however, to determine pitch with fair accuracy with no other appliances than a common harmonium and a watch, and as the process is instructive in respect of the theory of overtones, a short account will here be given of it[3].

The fundamental principle is that the absolute frequencies of two musical notes can be deduced from the *interval* between them, i.e. the ratio of their frequencies, and the number of beats which they occasion in a given time when sounded together. For example, if x and y denote the frequencies of two notes whose interval is an equal temperament major third, we know that $y = 1{\cdot}25992\,x$. At the same time the number of beats heard in a second depending upon the deviation of the third from true intonation, is $4y - 5x$. In the case of the notes of a harmonium, which are rich in overtones, these beats are readily counted, and thus two equations are obtained from which the values of x and y are at once found.

Of course in practice the truth of an equal temperament third could not be taken for granted, but the difficulty thence arising would be easily met by including in the counting all the three

[1] National Academy of Sciences, Washington, *Memoirs*, Vol. III. p. 43, 1884.
[2] *Lectures*, Vol. I. p. 191.
[3] *Nature*, Jan. 23, 1879.

major thirds which together make up an octave. Suppose, for example, that the frequencies of c, e, $g\sharp$, c' are respectively x, y, z, $2x$, and that the beats per second between x and y are a, between y and z are b, and between z and $2x$ are c. Then

$$4y - 5x = a, \quad 4z - 5y = b, \quad 8x - 5z = c,$$

from which

$$x = \tfrac{1}{3}(25a + 20b + 16c),$$
$$y = \tfrac{1}{3}(32a + 25b + 20c),$$
$$z = \tfrac{1}{3}(40a + 32b + 25c).$$

In the above statements the octave c—c' is for simplicity supposed to be true. The actual error could readily be allowed for if required; but in practice it is not necessary to use c' at all, inasmuch as the third set of beats can be counted equally well between $g\sharp$ and c.

The principal objection to the method in the above form is that it presupposes the absolute constancy of the notes, for example, that y is the same whether it is being sounded in conjunction with x or in conjunction with z. This condition is very imperfectly satisfied by the notes of a harmonium.

In order to apply the fundamental principle with success, it is necessary to be able to check the accuracy of the interval which is supposed to be known, at the same time that the beats are being counted. If the interval be a major tone (9 : 8), its exactness is proved by the absence of beats between the ninth component of the lower and the eighth of the higher note, and a counting of the beats between the tenth component of the lower and the ninth of the higher note completes the necessary data for determining the absolute pitch.

The equal temperament whole tone (1·12246) is intermediate between the minor tone (1·11111) and the major tone (1·12500), but lies much nearer to the latter. Regarded as a disturbed major tone, it gives slow beats, and regarded as a disturbed minor tone it gives quick ones. Both sets of beats can be heard at the same time, and when counted (by two observers) give the means of calculating the absolute pitch of both notes. If x and y be the frequencies of the two notes, a and b the frequencies of the slow and quick beats respectively,

$$9x - 8y = a, \quad 9y - 10x = b,$$

whence

$$x = 9a + 8b, \quad y = 10a + 9b.$$

The application of this method in no way assumes the truth of the equal temperament whole tone, and in fact it is advantageous to flatten the interval somewhat, so as to make it lie more nearly midway between the major and the minor tone. In this way the rapidity of the quicker beats is diminished, which facilitates the counting.

The course of an experiment is then as follows. The notes C and D are sounded together, and at a given signal the observers begin counting the beats situated at about d'' and e'' on the scale. After the expiration of a measured interval of time a second signal is given, and the number of both sets of beats is recorded.

For further details of the method reference must be made to the original memoir, but one example of the results may be given here. The period being 10 minutes, the number of beats recorded were 2392 and 2341, giving $x = 67{\cdot}09$ as the pitch of C.

CHAPTER IV.

VIBRATING SYSTEMS IN GENERAL.

69. WE have now examined in some detail the oscillations of a system possessed of one degree of freedom, and the results, at which we have arrived, have a very wide application. But material systems enjoy in general more than one degree of freedom. In order to define their configuration at any moment several independent variable quantities must be specified, which, by a generalization of language originally employed for a point, are called the *co-ordinates* of the system, the number of independent co-ordinates being the *index of freedom*. Strictly speaking, the displacements possible to a natural system are infinitely various, and cannot be represented as made up of a finite number of displacements of specified type. To the elementary parts of a solid body any arbitrary displacements may be given, subject to conditions of continuity. It is only by a process of abstraction of the kind so constantly practised in Natural Philosophy, that solids are treated as rigid, fluids as incompressible, and other simplifications introduced so that the position of a system comes to depend on a finite number of co-ordinates. It is not, however, our intention to exclude the consideration of systems possessing infinitely various freedom; on the contrary, some of the most interesting applications of the results of this chapter will lie in that direction. But such systems are most conveniently conceived as limits of others, whose freedom is of a more restricted kind. We shall accordingly commence with a system, whose position is specified by a finite number of independent co-ordinates ψ_1, ψ_2, ψ_3, &c.

70. The main problem of Acoustics consists in the investigation of the vibrations of a system about a position of stable equilibrium, but it will be convenient to commence with the statical part of the subject. By the Principle of Virtual Velocities, if we reckon the co-ordinates ψ_1, ψ_2, &c. from the configuration of equilibrium, the potential energy of any other configuration will be a homogeneous quadratic function of the co-ordinates, provided that the displacement be sufficiently small. This quantity is called V, and represents the work that may be gained in passing from the actual to the equilibrium configuration. We may write

$$V = \tfrac{1}{2}c_{11}\psi_1^2 + \tfrac{1}{2}c_{22}\psi_2^2 + \ldots + c_{12}\psi_1\psi_2 + c_{23}\psi_2\psi_3 + \ldots\ldots(1).$$

Since by supposition the equilibrium is thoroughly stable, the quantities c_{11}, c_{22}, c_{12}, &c. must be such that V is positive for all real values of the co-ordinates.

71. If the system be displaced from the zero configuration by the action of given forces, the new configuration may be found from the Principle of Virtual Velocities. If the work done by the given forces on the hypothetical displacement $\delta\psi_1$, $\delta\psi_2$, &c. be

$$\Psi_1\delta\psi_1 + \Psi_2\delta\psi_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

this expression must be equivalent to δV, so that since $\delta\psi_1$, $\delta\psi_2$, &c. are independent, the new position of equilibrium is determined by

$$\frac{dV}{d\psi_1} = \Psi_1, \quad \frac{dV}{d\psi_2} = \Psi_2, \text{ \&c.}\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

or by (1) of § 70,

$$\left.\begin{array}{l} c_{11}\psi_1 + c_{12}\psi_2 + c_{13}\psi_3 + \ldots\ldots = \Psi_1 \\ c_{21}\psi_1 + c_{22}\psi_2 + c_{23}\psi_3 + \ldots\ldots = \Psi_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}\ldots\ldots\ldots\ldots(3),$$

where there is no distinction in value between c_{rs} and c_{sr}.

From these equations the co-ordinates may be determined in terms of the forces. If ∇ be the determinant

$$\nabla = \begin{vmatrix} c_{11}, & c_{12}, & c_{13}, & \ldots \\ c_{21}, & c_{22}, & c_{23}, & \ldots \\ c_{31}, & c_{32}, & c_{33}, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

the solution of (3) may be written

$$\left.\begin{array}{l} \nabla \cdot \psi_1 = \dfrac{d\nabla}{dc_{11}}\Psi_1 + \dfrac{d\nabla}{dc_{12}}\Psi_2 + \ldots\ldots \\ \nabla \cdot \psi_2 = \dfrac{d\nabla}{dc_{21}}\Psi_1 + \dfrac{d\nabla}{dc_{22}}\Psi_2 + \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}\ldots\ldots\ldots\ldots\ldots(5).$$

These equations determine ψ_1, ψ_2, &c. uniquely, since ∇ does not vanish, as appears from the consideration that the equations $dV/d\psi_1 = 0$, &c. could otherwise be satisfied by finite values of the co-ordinates, provided only that the *ratios* were suitable, which is contrary to the hypothesis that the system is thoroughly stable in the zero configuration.

72. If $\psi_1, \ldots \Psi_1, \ldots$ and $\psi_1', \ldots \Psi_1', \ldots$ be two sets of displacements and corresponding forces, we have the following reciprocal relation,

$$\Psi_1\psi_1' + \Psi_2\psi_2' + \ldots = \Psi_1'\psi_1 + \Psi_2'\psi_2 + \ldots\ldots\ldots\ldots(1),$$

as may be seen by substituting the values of the forces, when each side of (1) takes the form,

$$c_{11}\psi_1\psi_1' + c_{22}\psi_2\psi_2' + \ldots\ldots$$
$$\ldots + c_{12}(\psi_2\psi_1' + \psi_2'\psi_1) + c_{23}(\psi_3\psi_2' + \psi_3'\psi_2) + \ldots\ldots$$

Suppose in (1) that all the forces vanish except Ψ_2 and Ψ_1'; then

$$\Psi_2\psi_2' = \Psi_1'\psi_1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

If the forces Ψ_2 and Ψ_1' be of the same kind, we may suppose them equal, and we then recognise that a force of any type acting alone produces a displacement of a second type equal to the displacement of the first type due to the action of an equal force of the second type. For example, if A and B be two points of a rod supported horizontally in any manner, the vertical deflection at A, when a weight W is attached at B, is the same as the deflection at B, when W is applied at A[1].

73. Since V is a homogeneous quadratic function of the coordinates,

$$2V = \frac{dV}{d\psi_1}\psi_1 + \frac{dV}{d\psi_2}\psi_2 + \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or, if Ψ_1, Ψ_2, &c. be the forces necessary to maintain the displacement represented by ψ_1, ψ_2, &c.,

$$2V = \Psi_1\psi_1 + \Psi_2\psi_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If $\psi_1 + \Delta\psi_1$, $\psi_2 + \Delta\psi_2$, &c. represent another displacement for which the necessary forces are $\Psi_1 + \Delta\Psi_1$, $\Psi_2 + \Delta\Psi_2$, &c., the

[1] On this subject, see *Phil. Mag.*, Dec., 1874, and March, 1875.

corresponding potential energy is given by

$$2(V+\Delta V) = (\Psi_1+\Delta\Psi_1)(\psi_1+\Delta\psi_1) + \ldots$$
$$= 2V + \Psi_1\Delta\psi_1 + \Psi_2\Delta\psi_2 + \ldots$$
$$+ \Delta\Psi_1 . \psi_1 + \Delta\Psi_2 . \psi_2 + \ldots$$
$$+ \Delta\Psi_1 . \Delta\psi_1 + \Delta\Psi_2 . \Delta\psi_2 + \ldots,$$

so that we may write

$$2\Delta V = \Sigma\, \Psi . \Delta\psi + \Sigma\, \Delta\Psi . \psi + \Sigma\, \Delta\Psi . \Delta\psi \quad \ldots\ldots\ldots(3),$$

where ΔV is the difference of the potential energies in the two cases, and we must particularly notice that by the reciprocal relation, § 72 (1),

$$\Sigma\, \Psi . \Delta\psi = \Sigma\, \Delta\Psi . \psi \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

From (3) and (4) we may deduce two important theorems, relating to the value of V for a system subjected to given displacements, and to given forces respectively.

74. The first theorem is to the effect that, if given displacements (not sufficient by themselves to determine the configuration) be produced in a system by forces of corresponding types, the resulting value of V for the system so displaced, and in equilibrium, is as small as it can be under the given displacement conditions; and that the value of V for any other configuration exceeds this by the potential energy of the configuration which is the difference of the two. The only difficulty in the above statement consists in understanding what is meant by 'forces of corresponding types.' Suppose, for example, that the system is a stretched string, of which a given point P is to be subject to an obligatory displacement; the force of corresponding type is here a force applied at the point P itself. And generally, the forces, by which the proposed displacement is to be made, must be such as would do no work on the system, provided only that that displacement were *not* made.

By a suitable choice of co-ordinates, the given displacement conditions may be expressed by ascribing given values to the first r co-ordinates $\psi_1, \psi_2, \ldots \psi_r$, and the conditions as to the forces will then be represented by making the forces of the remaining types Ψ_{r+1}, Ψ_{r+2}, &c. vanish. If $\psi + \Delta\psi$ refer to any other configuration of the system, and $\Psi + \Delta\Psi$ be the corresponding forces, we are to suppose that $\Delta\psi_1, \Delta\psi_2$, &c. as far as $\Delta\psi_r$ all vanish. Thus for the first r suffixes $\Delta\psi$ vanishes, and for the remaining suffixes Ψ vanishes. Accordingly $\Sigma \Psi . \Delta\psi$ is zero, and therefore $\Sigma \Delta\Psi . \psi$ is also zero. Hence

$$2 \Delta V = \Sigma \Delta\Psi . \Delta\psi \quad \text{(I)},$$

which proves that if the given displacements be made in any other than the prescribed way, the potential energy is increased by the energy of the difference of the configurations.

By means of this theorem we may trace the effect on V of any relaxation in the stiffness of a system, subject to given displacement conditions. For, if after the alteration in stiffness the original equilibrium configuration be considered, the value of V corresponding thereto is by supposition less than before; and, as we have just seen, there will be a still further diminution in the value of V when the system passes to equilibrium under the altered conditions. Hence we conclude that a diminution in V as a function of the co-ordinates entails also a diminution in the actual value of V when a system is subjected to given displacements. It will be understood that in particular cases the diminution spoken of may vanish[1].

For example, if a point P of a bar clamped at both ends be displaced laterally to a given small amount by a force there applied, the potential energy of the deformation will be diminished by any relaxation (however local) in the stiffness of the bar.

75. The second theorem relates to a system displaced *by given forces*, and asserts that in this case the value of V in equilibrium is greater than it would be in any other configuration in which the system could be maintained at rest under the given forces, by the operation of mere constraints. We will shew that the *removal* of constraints increases the value of V.

The co-ordinates may be so chosen that the conditions of constraint are expressed by

$$\psi_1 = 0, \quad \psi_2 = 0, \quad \ldots \quad \psi_r = 0 \quad \text{(1)}.$$

We have then to prove that when Ψ_{r+1}, Ψ_{r+2}, &c. are given, the value of V is least when the conditions (1) hold. The second configuration being denoted as before by $\psi_1 + \Delta\psi_1$ &c., we see that for suffixes up to r inclusive ψ vanishes, and for higher suffixes $\Delta\Psi$ vanishes. Hence

$$\Sigma \psi . \Delta\Psi = \Sigma \Delta\psi . \Psi = 0,$$

[1] See a paper on General Theorems relating to Equilibrium and Initial and Steady Motions. *Phil. Mag.*, March. 1875.

and therefore

$$2\Delta V = \Sigma\, \Delta\Psi . \Delta\psi \text{......................} (2),$$

shewing that the increase in V due to the removal of the constraints is equal to the potential energy of the difference of the two configurations.

76. We now pass to the investigation of the initial motion of a system which starts from rest under the operation of given impulses. The motion thus acquired is independent of any potential energy which the system may possess when actually displaced, since by the nature of impulses we have to do only with the initial configuration itself. The initial motion is also independent of any forces of a finite kind, whether impressed on the system from without, or of the nature of viscosity.

If P, Q, R be the component impulses, parallel to the axes, on a particle m whose rectangular co-ordinates are x, y, z, we have by D'Alembert's Principle

$$\Sigma m\,(\dot{x}\,\delta x + \dot{y}\,\delta y + \dot{z}\,\delta z) = \Sigma\,(P\,\delta x + Q\,\delta y + R\,\delta z) \text{......} (1),$$

where $\dot{x}$, $\dot{y}$, $\dot{z}$ denote the velocities acquired by the particle in virtue of the impulses, and δx, δy, δz correspond to any arbitrary displacement of the system which does not violate the connection of its parts. It is required to transform (1) into an equation expressed by the independent generalized co-ordinates.

For the first side,

$$\Sigma m\,(\dot{x}\,\delta x + \dot{y}\,\delta y + \dot{z}\,\delta z) = \delta\psi_1\, \Sigma m \left(\dot{x}\frac{dx}{d\psi_1} + \dot{y}\frac{dy}{d\psi_1} + \dot{z}\frac{dz}{d\psi_1}\right)$$

$$+ d\psi_2\, \Sigma m \left(\dot{x}\frac{dx}{d\psi_2} + \dot{y}\frac{dy}{d\psi_2} + \dot{z}\frac{dz}{d\psi_2}\right) + \text{......}$$

$$= \delta\psi_1\, \Sigma m \left(\dot{x}\frac{d\dot{x}}{d\dot{\psi}_1} + \dot{y}\frac{d\dot{y}}{d\dot{\psi}_1} + \dot{z}\frac{d\dot{z}}{d\dot{\psi}_1}\right) + \text{......}$$

$$= \delta\psi_1 . \tfrac{1}{2}\Sigma m \frac{d}{d\dot{\psi}_1}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \text{.. ...}$$

$$= \delta\psi_1 \frac{dT}{d\dot{\psi}_1} + \delta\psi_2 \frac{dT}{d\dot{\psi}_2} + \text{..............................} (2),$$

where T, the kinetic energy of the system is supposed to be expressed as a function of $\dot{\psi}_1$, $\dot{\psi}_2$, &c.

On the second side,

$$\Sigma (P\delta x + Q\delta y + R\delta z) = \delta\psi_1 \Sigma m \left(P \frac{dx}{d\psi_1} + Q \frac{dy}{d\psi_1} + R \frac{dz}{d\psi_1}\right) + \ldots\ldots$$

$$= \xi_1 \delta\psi_1 + \xi_2 \delta\psi_2 + \ldots\ldots \quad (3),$$

if

$$\Sigma m \left(P \frac{dx}{d\psi_1} + Q \frac{dy}{d\psi_1} + R \frac{dz}{d\psi_1}\right) = \xi_1, \text{ \&c.}$$

The transformed equation is therefore

$$\left(\frac{dT}{d\dot{\psi}_1} - \xi_1\right) \delta\psi_1 + \left(\frac{dT}{d\dot{\psi}_2} - \xi_2\right) \delta\psi_2 + \ldots = 0 \quad \ldots\ldots (4),$$

where $\delta\psi_1$, $\delta\psi_2$, &c. are now completely independent. Hence to determine the motion we have

$$\frac{dT}{d\dot{\psi}_1} = \xi_1, \quad \frac{dT}{d\dot{\psi}_2} = \xi_2, \text{ \&c.} \quad \ldots\ldots (5),$$

where ξ_1, ξ_2, &c. may be considered as the generalized components of impulse.

77. Since T is a homogeneous quadratic function of the generalized co-ordinates, we may take

$$T = \tfrac{1}{2}a_{11}\dot{\psi}_1^2 + \tfrac{1}{2}a_{22}\dot{\psi}_2^2 + \ldots\ldots + a_{12}\dot{\psi}_1\dot{\psi}_2 + a_{23}\dot{\psi}_2\dot{\psi}_3 + \ldots\ldots (1),$$

whence

$$\left.\begin{array}{l} \xi_1 = \dfrac{dT}{d\dot{\psi}_1} = a_{11}\dot{\psi}_1 + a_{12}\dot{\psi}_2 + a_{13}\dot{\psi}_3 + \ldots\ldots \\ \xi_2 = \dfrac{dT}{d\dot{\psi}_2} = a_{21}\dot{\psi}_1 + a_{22}\dot{\psi}_2 + a_{23}\dot{\psi}_3 + \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots\ldots (2),$$

where there is no distinction in value between a_{rs} and a_{sr}.

Again, by the nature of T,

$$2T = \dot{\psi}_1 \frac{dT}{d\dot{\psi}_1} + \dot{\psi}_2 \frac{dT}{d\dot{\psi}_2} + \ldots\ldots = \xi_1\dot{\psi}_1 + \xi_2\dot{\psi}_2 + \ldots\ldots (3).$$

The theory of initial motion is closely analogous to that of the displacement of a system from a configuration of stable equilibrium by steadily applied forces. In the present theory the initial kinetic energy T bears to the velocities and impulses the same relations as in the former V bears to the displacements and forces respectively. In one respect the theory of initial motions is the more complete, inasmuch as T is exactly, while V is in general only approximately, a homogeneous quadratic function of the variables.

If $\dot{\psi}_1, \dot{\psi}_2, \ldots, \xi_1, \xi_2, \ldots$ denote one set of velocities and impulses for a system started from rest, and $\dot{\psi}_1', \dot{\psi}_2', \ldots, \xi_1', \xi_2', \ldots$ a second set, we may prove, as in § 72, the following reciprocal relation:

$$\xi_1'\dot{\psi}_1 + \xi_2'\dot{\psi}_2 + \ldots = \xi_1\dot{\psi}_1' + \xi_2\dot{\psi}_2' + \ldots\ldots\ldots\ldots(4)$$[1].

This theorem admits of interesting application to fluid motion. It is known, and will be proved later in the course of this work, that the motion of a frictionless incompressible liquid, which starts from rest, is of such a kind that its component velocities at any point are the corresponding differential coefficients of a certain function, called the velocity-potential. Let the fluid be set in motion by a prescribed arbitrary deformation of the surface S of a closed space described within it. The resulting motion is determined by the normal velocities of the elements of S, which, being shared by the fluid in contact with them, are denoted by du/dn, if u be the velocity-potential, which interpreted physically denotes the impulsive pressure, if the density be taken as unity. Hence by the theorem, if v be the velocity-potential of a second motion, corresponding to another set of arbitrary surface velocities dv/dn,

$$\iint u \frac{dv}{dn} dS = \iint v \frac{du}{dn} dS \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

—an equation immediately following from Green's theorem, if besides S there be only fixed solids immersed in the fluid. The present method enables us to attribute to it a much higher generality. For example, the immersed solids, instead of being fixed, may be free, altogether or in part, to take the motion imposed upon them by the fluid pressures.

78. A particular case of the general theorem is worthy of special notice. In the first motion let

$$\dot{\psi}_1 = A, \quad \dot{\psi}_2 = 0, \quad \xi_3 = \xi_4 = \xi_5 \ldots\ldots = 0;$$

and in the second,

$$\dot{\psi}_1' = 0, \quad \dot{\psi}_2' = A, \quad \xi_3' = \xi_4' = \xi_5' \ldots\ldots = 0.$$

Then

$$\xi_1 = \xi_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

In words, if, by means of a suitable impulse of the corresponding type, a given arbitrary velocity of one co-ordinate be impressed on a system, the impulse corresponding to a second co-ordinate necessary in order to prevent it from changing, is the same as would be required for the first co-ordinate, if the given velocity were impressed on the second.

[1] Thomson and Tait, § 313 (*f*).

As a simple example, take the case of two spheres A and B immersed in a liquid, whose centres are free to move along certain lines. If A be set in motion with a given velocity, B will naturally begin to move also. The theorem asserts that the impulse required to prevent the motion of B, is the same as if the functions of A and B were exchanged: and this even though there be other rigid bodies, C, D, &c., in the fluid, either fixed, or free in whole or in part.

The case of electric currents mutually influencing each other by induction is precisely similar. Let there be two circuits A and B, in the neighbourhood of which there may be any number of other wire circuits or solid conductors. If a unit current be suddenly developed in the circuit A, the electromotive impulse induced in B is the same as there would have been in A, had the current been forcibly developed in B.

79. The motion of a system, on which given arbitrary velocities are impressed by means of the necessary impulses of the corresponding types, possesses a remarkable property discovered by Thomson. The conditions are that $\dot\psi_1$, $\dot\psi_2$, $\dot\psi_3$, ... $\dot\psi_r$ are given, while ξ_{r+1}, ξ_{r+2}, ... vanish. Let $\dot\psi_1$, $\dot\psi_2$, ... ξ_1, ξ_2, &c. correspond to the actual motion; and

$$\dot\psi_1 + \Delta\dot\psi_1, \dot\psi_2 + \Delta\dot\psi_2, \ldots \xi_1 + \Delta\xi_1, \xi_2 + \Delta\xi_2, \ldots$$

to another motion satisfying the same *velocity* conditions. For each suffix either $\Delta\dot\psi$ *or* ξ vanishes. Now for the kinetic energy of the supposed motion,

$$2(T + \Delta T) = (\xi_1 + \Delta\xi_1)(\dot\psi_1 + \Delta\dot\psi_1) + \ldots$$
$$= 2T + \xi_1\Delta\dot\psi_1 + \xi_2\Delta\dot\psi_2 + \ldots$$
$$+ \Delta\xi_1 . \dot\psi_1 + \Delta\xi_2 . \dot\psi_2 + \ldots + \Delta\xi_1\Delta\dot\psi_1 + \Delta\xi_2\Delta\dot\psi_2 + \ldots$$

But by the reciprocal relation (4) of § 77

$$\xi_1\Delta\dot\psi_1 + \ldots = \Delta\xi_1 . \dot\psi_1 + \ldots,$$

of which the former by hypothesis is zero; so that

$$2\Delta T = \Delta\xi_1\Delta\dot\psi_1 + \Delta\xi_2\Delta\dot\psi_2 + \ldots\ldots\ldots\ldots\ldots(1),$$

shewing that the energy of the supposed motion exceeds that of the actual motion by the energy of that motion which would have to be compounded with the latter to produce the former. The motion actually induced in the system has thus less energy than any other satisfying the same velocity conditions. In a subsequent chapter we shall make use of this property to find a superior limit to the energy of a system set in motion with prescribed velocities.

If any diminution be made in the inertia of any of the parts of a system, the motion corresponding to prescribed velocity conditions will in general undergo a change. The value of T will necessarily be less than before; for there would be a decrease even if the motion remained unchanged, and therefore *a fortiori* when the motion is such as to make T an absolute minimum. Conversely any increase in the inertia increases the initial value of T.

This theorem is analogous to that of § 74. The analogue for initial motions of the theorem of § 75, relating to the potential energy of a system displaced by given forces, is that of Bertrand, and may be thus stated:—If a system start from rest under the operation of given impulses, the kinetic energy of the actual motion exceeds that of any other motion which the system might have been guided to take with the assistance of mere constraints, by the kinetic energy of the difference of the motions[1].

[The theorems of Kelvin and Bertrand represent different aspects of the same truth. Let us suppose that the prescribed impulse is entirely of the first type ξ_1. Then $T = \frac{1}{2}\xi_1\dot{\psi}_1$, whether the motion be free or be subjected to any constraint. Further, under any given circumstances as to constraint, $\dot{\psi}_1$ is proportional to ξ_1, and the ratio $\xi_1 : \dot{\psi}_1$ may be regarded as the moment of inertia; so that

$$T = \tfrac{1}{2}\xi_1\dot{\psi}_1 = \tfrac{1}{2}m\dot{\psi}_1^2 = \tfrac{1}{2}\xi_1^2/m.$$

Kelvin's theorem asserts that the introduction of a constraint can v increase the value of T when $\dot{\psi}_1$ is given. Hence whether $\dot{\psi}_1$ be given or not, the constraint can only increase the ratio of $2T$ to $\dot{\psi}_1^2$ or of ξ_1 to $\dot{\psi}_1$. Both theorems are included in the statement that the moment of inertia is increased by the introduction of a constraint.]

80. We will not dwell at any greater length on the mechanics of a system subject to impulses, but pass on to investigate Lagrange's equations for continuous motion. We shall suppose that the connections binding together the parts of the system are not explicit functions of the time; such cases of forced motion as we shall have to consider will be specially shewn to be within the scope of the investigation.

By D'Alembert's Principle in combination with that of Virtual Velocities,

$$\Sigma m\,(\ddot{x}\,\delta x + \ddot{y}\,\delta y + \ddot{z}\,\delta z) = \Sigma\,(X\,\delta x + Y\,\delta y + Z\,\delta z)\ldots\ldots(1),$$

[1] Thomson and Tait, § 311. *Phil. Mag.* March, 1875.

where δx, δy, δz denote a displacement of the system of the most general kind possible without violating the connections of its parts. Since the displacements of the individual particles of the system are mutually related, $\delta x, \ldots$ are not independent. The object now is to transform to other variables $\psi_1, \psi_2, \ldots$, which shall be independent. We have

$$\ddot{x}\,\delta x = \frac{d}{dt}(\dot{x}\,\delta x) - \tfrac{1}{2}\delta\dot{x}^2,$$

so that

$$\Sigma m(\ddot{x}\,\delta x + \ddot{y}\,\delta y + \ddot{z}\,\delta z) = \frac{d}{dt}.\,\Sigma m(\dot{x}\,\delta x + \dot{y}\,\delta y + \dot{z}\,\delta z) - \delta T.$$

But (§ 76) we have already found that

$$\Sigma m(\dot{x}\,\delta x + \dot{y}\,\delta y + \dot{z}\,\delta z) = \frac{dT}{d\dot{\psi}_1}\delta\psi_1 + \frac{dT}{d\dot{\psi}_2}\delta\psi_2 + \ldots,$$

while
$$\delta T = \frac{dT}{d\psi_1}\delta\psi_1 + \frac{dT}{d\dot{\psi}_1}\delta\dot{\psi}_1 + \ldots,$$

if T be expressed as a quadratic function of $\dot{\psi}_1, \dot{\psi}_2, \ldots$, whose coefficients are in general functions of $\psi_1, \psi_2, \ldots$. Also

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}_1}\delta\psi_1\right) = \frac{d}{dt}\left(\frac{dT}{d\dot{\psi}_1}\right).\,\delta\psi_1 + \frac{dT}{d\dot{\psi}_1}\delta\dot{\psi}_1,$$

inasmuch as
$$\frac{d}{dt}\delta\psi_1 = \delta\frac{d}{dt}\psi_1.$$

Accordingly

$$\Sigma m(\ddot{x}\,\delta x + \ddot{y}\,\delta y + \ddot{z}\,\delta z) = \left\{\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}_1}\right) - \frac{dT}{d\psi_1}\right\}\delta\psi_1 + \left\{\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}_2}\right) - \frac{dT}{d\psi_2}\right\}\delta\psi_2 + \ldots\ldots\ldots(2).$$

Thus, if the transformation of the second side of (1) be

$$\Sigma(X\,\delta x + Y\,\delta y + Z\,\delta z) = \Psi_1\delta\psi_1 + \Psi_2\delta\psi_2 + \ldots\ldots\ldots(3),$$

we have equations of motion of the form

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}}\right) - \frac{dT}{d\psi} = \Psi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Since $\Psi\,\delta\psi$ denotes the work done on the system during a displacement $\delta\psi$, Ψ may be regarded as the generalized component of force.

In the case of a conservative system it is convenient to separate from Ψ those parts which depend only on the configuration of the system. Thus, if V denote the potential energy, we may write

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}}\right)-\frac{dT}{d\psi}+\frac{dV}{d\psi}=\Psi \quad\text{.....................} (5),$$

where Ψ is now limited to the forces acting on the system which are not already taken account of in the term $dV/d\psi$.

81. There is also another group of forces whose existence it is often advantageous to recognize specially, namely those arising from friction or viscosity. If we suppose that each particle of the system is retarded by forces proportional to its component velocities, the effect will be shewn in the fundamental equation (1) § 80 by the addition to the left-hand member of the terms

$$\Sigma\,(\kappa_x\dot{x}\,\delta x+\kappa_y\dot{y}\,\delta y+\kappa_z\dot{z}\,\delta z),$$

where κ_x, κ_y, κ_z are coefficients independent of the velocities, but possibly dependent on the configuration of the system. The transformation to the independent co-ordinates ψ_1, ψ_2, &c. is effected in a similar manner to that of

$$\Sigma m\,(\dot{x}\,\delta x+\dot{y}\,\delta y+\dot{z}\,\delta z)$$

considered above (§ 80), and gives

$$\frac{dF}{d\dot{\psi}_1}\,\delta\psi_1+\frac{dF}{d\dot{\psi}_2}\,\delta\psi_2+\text{.....................} (1),$$

where
$$F=\tfrac{1}{2}\Sigma\,(\kappa_x\dot{x}^2+\kappa_y\dot{y}^2+\kappa_z\dot{z}^2)$$
$$=\tfrac{1}{2}b_{11}\dot{\psi}_1^2+\tfrac{1}{2}b_{22}\dot{\psi}_2^2+\ldots+b_{12}\dot{\psi}_1\dot{\psi}_2+b_{23}\dot{\psi}_2\dot{\psi}_3+\text{......} (2).$$

F, it will be observed, is like T a homogeneous quadratic function of the velocities, positive for all real values of the variables. It represents half the rate at which energy is dissipated.

The above investigation refers to retarding forces proportional to the absolute velocities; but it is equally important to consider such as depend on the *relative* velocities of the parts of the system, and fortunately this can be done without any increase of complication. For example, if a force act on the particle x_1 proportional to $(\dot{x}_1-\dot{x}_2)$, there will be at the same moment an equal and opposite force acting on the particle x_2. The additional terms in the fundamental equation will be of the form

$$\kappa_x(\dot{x}_1-\dot{x}_2)\,\delta x_1+\kappa_x(\dot{x}_2-\dot{x}_1)\,\delta x_2,$$

which may be written

$$\kappa_x(\dot{x}_1-\dot{x}_2)\,\delta(x_1-x_2)=\delta\psi_1\,\frac{d}{d\dot{\psi}_1}\{\tfrac{1}{2}\kappa_x(\dot{x}_1-\dot{x}_2)^2\}+\ldots,$$

and so on for any number of pairs of mutually influencing particles. The only effect is the addition of new terms to F, which still appears in the form (2)[1]. We shall see presently that the existence of the function F, which may be called the Dissipation Function, implies certain relations among the coefficients of the generalized equations of vibration, which carry with them important consequences[2].

The equations of motion may now be written in the form

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}}\right) - \frac{dT}{d\psi} + \frac{dF}{d\dot{\psi}} + \frac{dV}{d\psi} = \Psi \quad \text{..............} \quad (3).$$

82. We may now introduce the condition that the motion takes place in the immediate neighbourhood of a configuration of thoroughly stable equilibrium; T and F are then homogeneous quadratic functions of the velocities with coefficients which are to be treated as constant, and V is a similar function of the co-ordinates themselves, provided that (as we suppose to be the case) the origin of each co-ordinate is taken to correspond with the configuration of equilibrium. Moreover all three functions are essentially positive. Since terms of the form $dT/d\psi$ are of the second order of small quantities, the equations of motion become linear, assuming the form

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}}\right) + \frac{dF}{d\dot{\psi}} + \frac{dV}{d\psi} = \Psi \quad \text{.................} \quad (1),$$

where under Ψ are to be included all forces acting on the system not already provided for by the differential coefficients of F and V.

The three quadratic functions will be expressed as follows:—

$$\left.\begin{aligned} T &= \tfrac{1}{2}a_{11}\dot{\psi}_1^2 + \tfrac{1}{2}a_{22}\dot{\psi}_2^2 + \ldots + a_{12}\dot{\psi}_1\dot{\psi}_2 + \ldots \\ F &= \tfrac{1}{2}b_{11}\dot{\psi}_1^2 + \tfrac{1}{2}b_{22}\dot{\psi}_2^2 + \ldots + b_{12}\dot{\psi}_1\dot{\psi}_2 + \ldots \\ V &= \tfrac{1}{2}c_{11}\psi_1^2 + \tfrac{1}{2}c_{22}\psi_2^2 + \ldots + c_{12}\psi_1\psi_2 + \ldots \end{aligned}\right\} \quad \text{........} (2),$$

where the coefficients a, b, c are constants.

From equation (1) we may of course fall back on previous results by supposing F and V, or F and T, to vanish.

A third set of theorems of interest in the application to Elec-

[1] The differences referred to in the text may of course pass into differential coefficients in the case of a body continuously deformed.

[2] The Dissipation Function appears for the first time, so far as I am aware, in a paper on General Theorems relating to Vibrations, published in the *Proceedings of the Mathematical Society* for June, 1873.

tricity may be obtained by omitting T and V, while F is retained, but it is unnecessary to pursue the subject here.

If we substitute the values of T, F and V, and write D for d/dt, we obtain a system of equations which may be put into the form

$$\left.\begin{array}{l} e_{11}\psi_1 + e_{12}\psi_2 + e_{13}\psi_3 + \ldots = \Psi_1 \\ e_{21}\psi_1 + e_{22}\psi_2 + e_{23}\psi_3 + \ldots = \Psi_2 \\ e_{31}\psi_1 + e_{32}\psi_2 + e_{33}\psi_3 + \ldots = \Psi_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \quad\ldots\ldots\ldots\ldots(3).$$

where e_{rs} denotes the quadratic operator

$$e_{rs} = a_{rs}D^2 + b_{rs}D + c_{rs} \quad\ldots\ldots\ldots\ldots (4).$$

It must be particularly remarked that since

$$a_{rs} = a_{sr}, \quad b_{rs} = b_{sr}, \quad c_{rs} = c_{sr},$$

it follows that

$$e_{rs} = e_{sr} \quad\ldots\ldots\ldots\ldots(5).$$

[The theory of *motional* forces, i.e. forces proportional to the velocities, has been further developed in the second edition of Thomson and Tait's *Natural Philosophy* (1879). In the most general case the equations may be written

$$\left.\begin{array}{l} \dfrac{d}{dt}\left(\dfrac{dT}{d\dot\psi_1}\right) + \dfrac{dV}{d\psi_1} + b_{11}\dot\psi_1 + (b_{12} + \beta_{12})\,\dot\psi_2 + (b_{13} + \beta_{13})\,\dot\psi_3 + \ldots = \Psi_1 \\ \dfrac{d}{dt}\left(\dfrac{dT}{d\dot\psi_2}\right) + \dfrac{dV}{d\psi_2} + (b_{21} - \beta_{21})\,\dot\psi_1 + b_{22}\dot\psi_2 + (b_{23} - \beta_{23})\,\dot\psi_3 + \ldots = \Psi_2 \end{array}\right\}(6),$$

where

$$b_{rs} = b_{sr}, \quad \beta_{rs} = -\beta_{sr} \quad\ldots\ldots\ldots\ldots(7).$$

Of these the terms with the coefficients b can be derived from the dissipation function

$$F = \tfrac{1}{2}b_{11}\dot\psi_1^2 + \tfrac{1}{2}b_{22}\dot\psi_2^2 + \ldots + b_{12}\dot\psi_1\dot\psi_2 + \ldots.$$

The terms in β on the other hand do not represent dissipation, and are called the gyrostatic terms.

If we multiply the first of equations (6) by $\dot\psi_1$, the second by $\dot\psi_2$, &c., and then add, we obtain

$$\frac{d\,(T+V)}{dt} + 2F = \Psi_1\dot\psi_1 + \Psi_2\dot\psi_2 + \ldots\ldots\ldots\ldots(8).$$

In this the first term represents the rate at which energy is being stored in the system; $2F$ is the rate of dissipation; and the two together account for the work done upon the system by the external forces.]

83. Before proceeding further, we may draw an important inference from the *linearity* of our equations. If corresponding respectively to the two sets of forces $\Psi_1, \Psi_2, \ldots, \Psi_1', \Psi_2', \ldots$ two motions denoted by $\psi_1, \psi_2, \ldots, \psi_1', \psi_2', \ldots$ be possible, then must also be possible the motion $\psi_1 + \psi_1', \psi_2 + \psi_2', \ldots$ in conjunction with the forces $\Psi_1 + \Psi_1', \Psi_2 + \Psi_2', \ldots$. Or, as a particular case, when there are no impressed forces, the superposition of any two natural vibrations constitutes also a natural vibration. This is the celebrated principle of the Coexistence of Small Motions, first clearly enunciated by Daniel Bernoulli. It will be understood that its truth depends in general on the justice of the assumption that the motion is so small that its square may be neglected.

[Again, if a system be under the influence of constant forces Ψ_1, &c., which displace it into a new position of equilibrium, the vibrations which may occur about the new position are the same as those which might before have occurred about the old position.]

84. To investigate the free vibrations, we must put $\Psi_1, \Psi_2, \ldots$ equal to zero; and we will commence with a system on which no frictional forces act, for which therefore the coefficients e_{rs}, &c. are *even* functions of the symbol D. We have

$$\left.\begin{array}{l} e_{11}\psi_1 + e_{12}\psi_2 + \ldots = 0 \\ e_{21}\psi_1 + e_{22}\psi_2 + \ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

From these equations, of which there are as many (m) as the system possesses degrees of liberty, let all but one of the variables be eliminated. The result, which is of the same form whichever be the co-ordinate retained, may be written

$$\nabla\psi = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where ∇ denotes the determinant

$$\begin{vmatrix} e_{11}, & e_{12}, & e_{13}, & \ldots \\ e_{21}, & e_{22}, & e_{23}, & \ldots \\ e_{31}, & e_{32}, & e_{33}, & \ldots \\ \vdots & \ldots & \ldots & \ldots \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and is (if there be no friction) an even function of D of degree $2m$. Let $\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_m$ be the roots of $\nabla = 0$ considered as an equation in D. Then by the theory of differential equations the most general value of ψ is

$$\psi = A\epsilon^{\lambda_1 t} + A'\epsilon^{-\lambda_1 t} + B\epsilon^{\lambda_2 t} + B'\epsilon^{-\lambda_2 t} + \ldots\ldots\ldots(4),$$

where the $2m$ quantities A, A', B, B', &c. are arbitrary constants. This form holds good for each of the co-ordinates, but the constants in the different expressions are not independent. In fact if a particular solution be

$$\psi_1 = A_1 \epsilon^{\lambda_1 t}, \quad \psi_2 = A_2 \epsilon^{\lambda_1 t}, \text{ \&c.,}$$

the *ratios* $A_1 : A_2 : A_3 \ldots$ are completely determined by the equations

$$\left.\begin{array}{l} e_{11}A_1 + e_{12}A_2 + e_{13}A_3 + \ldots\ldots = 0 \\ e_{21}A_1 + e_{22}A_2 + e_{23}A_3 + \ldots\ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots\ldots\ldots\ldots(5),$$

where in each of the coefficients such as e_{rs}, λ_1 is substituted for D. Equations (5) are necessarily compatible, by the condition that λ_1 is a root of $\nabla = 0$. The ratios $A_1' : A_2' : A_3' \ldots$ corresponding to the root $-\lambda_1$ are the same as the ratios $A_1 : A_2 : A_3 \ldots$, but for the other pairs of roots λ_2, $-\lambda_2$, &c. there are distinct systems of ratios.

85. The nature of the system with which we are dealing imposes an important restriction on the possible values of λ. If λ_1 were real, either λ_1 or $-\lambda_1$ would be real and positive, and we should obtain a particular solution for which the co-ordinates, and with them the kinetic energy denoted by

$$\lambda_1^2 \{\tfrac{1}{2}a_{11}A_1^2 + \ldots a_{12}A_1A_2 + \ldots\} \epsilon^{\pm 2\lambda_1 t},$$

increase without limit. Such a motion is obviously impossible for a conservative system, whose whole energy can never differ from the sum of the potential and kinetic energies with which it was animated at starting. This conclusion is not evaded by taking λ_1 negative; because we are as much at liberty to trace the motion backwards as forwards. It is as certain that the motion never *was* infinite, as that it never *will be*. The same argument excludes the possibility of a complex value of λ.

We infer that all the values of λ are purely imaginary, corresponding to *real negative* values of λ^2. Analytically, the fact that the roots of $\nabla = 0$, considered as an equation in D^2, are all real and negative, must be a consequence of the relations subsisting between the coefficients $a_{11}, a_{12}, \ldots, c_{11}, c_{12}, \ldots$ in virtue of the fact that for all real values of the variables T and V are positive. The case of two degrees of liberty will be afterwards worked out in full.

86. The form of the solution may now be advantageously changed by writing in_1 for λ_1, &c. (where $i=\sqrt{-1}$), and taking new arbitrary constants. Thus

$$\left.\begin{aligned}\psi_1 &= A_1\cos(n_1t-\alpha)+B_1\cos(n_2t-\beta)+C_1\cos(n_3t-\gamma)+\dots\\ \psi_2 &= A_2\cos(n_1t-\alpha)+B_2\cos(n_2t-\beta)+C_2\cos(n_3t-\gamma)+\dots\\ \psi_3 &= A_3\cos(n_1t-\alpha)+B_3\cos(n_2t-\beta)+C_3\cos(n_3t-\gamma)+\dots\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\end{aligned}\right\}\dots(1),$$

where n_1^2, n_2^2, &c. are the m roots of the equation of m^{th} degree in n^2 found by writing $-n^2$ for D^2 in $\nabla=0$. For each value of n the ratios $A_1:A_2:A_3\dots$ are determinate and real.

This is the complete solution of the problem of the free vibrations of a conservative system. We see that the whole motion may be resolved into m normal harmonic vibrations of (in general) different periods, each of which is entirely independent of the others. If the motion, depending on the original disturbance, be such as to reduce itself to one of these (n_1) we have

$$\psi_1=A_1\cos(n_1t-\alpha),\quad \psi_2=A_2\cos(n_1t-\alpha),\ \&\text{c.}\ \dots\dots(2),$$

where the ratios $A_1:A_2:A_3\dots$ depend on the constitution of the system, and only the absolute amplitude and phase are arbitrary. The several co-ordinates are always in similar (or opposite) phases of vibration, and the whole system is to be found in the configuration of equilibrium at the same moment.

We perceive here the mechanical foundation of the supremacy of harmonic vibrations. If the motion be sufficiently small, the differential equations become linear with constant coefficients; while circular (and exponential) functions are the only ones which retain their type on differentiation.

87. The m periods of vibration, determined by the equation $\nabla=0$, are quantities intrinsic to the system, and must come out the same whatever co-ordinates may be chosen to define the configuration. But there is one system of co-ordinates, which is especially suitable, that namely in which the normal types of vibration are defined by the vanishing of all the co-ordinates but one. In the first type the original co-ordinates ψ_1, ψ_2, &c. have given ratios; let the quantity fixing the absolute values be ϕ_1, so that in this type each co-ordinate is a known multiple of ϕ_1. So in the second type each co-ordinate may be regarded as a known multiple of a second quantity ϕ_2, and so on. By a suitable determination of the m quantities ϕ_1, ϕ_2, &c., *any* configuration of the system may be represented as compounded of the m configurations of these types, and thus the quantities ϕ themselves may be looked upon as co-ordinates defining the configuration of the system. They are called the *normal* co-ordinates[1].

When expressed in terms of the normal co-ordinates, T and V are reduced to sums of squares; for it is easily seen that if the products also appeared, the resulting equations of vibration would not be satisfied by putting any $m-1$ of the co-ordinates equal to zero, while the remaining one was finite.

We might have commenced with this transformation, assuming from Algebra that any two homogeneous quadratic functions can be reduced by linear transformations to sums of squares.[2] Thus

$$\left.\begin{aligned} T &= \tfrac{1}{2}a_1\dot{\phi}_1^2 + \tfrac{1}{2}a_2\dot{\phi}_2^2 + \dots \\ V &= \tfrac{1}{2}c_1\phi_1^2 + \tfrac{1}{2}c_2\phi_2^2 + \dots \end{aligned}\right\} \qquad (1),$$

where the coefficients (in which the double suffixes are no longer required) are necessarily positive if the equilibrium be stable.

Lagrange's equations now become

$$a_1\ddot{\phi}_1 + c_1\phi_1 = 0, \quad a_2\ddot{\phi}_2 + c_2\phi_2 = 0, \text{ &c.} \qquad (2),$$

of which the solution is

$$\phi_1 = A\cos(n_1t - \alpha), \quad \phi_2 = B\cos(n_2t - \beta), \text{ &c.} \qquad (3),$$

where A, B..., α, β... are arbitrary constants, and

$$n_1^2 = c_1 \div a_1, \quad n_2^2 = c_2 \div a_2, \text{ &c.} \qquad (4).$$

[The vibrations expressed by the various normal co-ordinates are completely independent of one another, and the energy of the whole motion is the simple sum of the parts corresponding to the several normal vibrations taken separately. In fact by (1)

$$T + V = \tfrac{1}{2}c_1A_1^2 + \tfrac{1}{2}c_2A_2^2 + \dots \qquad (5).$$

By the nature of the case the coefficients a are necessarily positive. But if the equilibrium be unstable, some of the coefficients c may be negative. Corresponding to any negative c, n becomes imaginary and the circular functions of the time are replaced by exponentials.

In any motion proportional to $e^{\lambda t}$ the disturbance is equally multiplied in equal times, and the degree of instability may be considered to be measured by λ. If there be more than one

[1] Thomson and Tait's *Natural Philosophy*, first edition 1867, § 337.

[2] See Routh's *Rigid Dynamics*, p. 405.

unstable mode, the relative importance is largely determined bv the corresponding values of λ. Thus, if

$$\psi = Ae^{\lambda_1 t} + Be^{\lambda_2 t},$$

in which $\lambda_1 > \lambda_2$, then whatever may be the finite ratio of $A : B$, the first term ultimately acquires the preponderance, inasmuch as

$$Ae^{\lambda_1 t} : Be^{\lambda_2 t} = (A/B)\, e^{(\lambda_1 - \lambda_2)t}.$$

In general, unstable equilibrium when disturbed infinitesimally will be departed from according to that mode which is *most unstable*, viz. for which λ is greatest. In a later chapter we shall meet with interesting applications of this principle.

The reduction to normal co-ordinates allows us readily to trace what occurs when two of the values of n^2 become equal. It is evident that there is no change of form. The spherical pendulum may be referred to as a simple example of equal roots. It is remarkable that both Lagrange and Laplace fell into the error of supposing that equality among roots necessarily implies terms containing t as a factor[1]. The analytical theory of the general case (where the co-ordinates are not normal) has been discussed by Somof[2] and by Routh[3].]

88. The interpretation of the equations of motion leads to a theorem of considerable importance, which may be thus stated[4]. The period of a conservative system vibrating in a constrained type about a position of stable equilibrium is stationary in value when the type is normal. We might prove this from the original equations of vibration, but it will be more convenient to employ the normal co-ordinates. The constraint, which may be supposed to be of such a character as to leave only one degree of freedom, is represented by taking the quantities ϕ in given ratios.

If we put

$$\phi_1 = A_1\theta, \quad \phi_2 = A_2\theta, \text{ \&c.} \qquad\qquad (1),$$

θ is a variable quantity, and A_1, A_2, &c. are given for a given constraint.

The expressions for T and V become

$$T = \{\tfrac{1}{2}a_1A_1^2 + \tfrac{1}{2}a_2A_2^2 + \ldots\ldots\}\,\dot{\theta}^2,$$
$$V = \{\tfrac{1}{2}c_1A_1^2 + \tfrac{1}{2}c_2A_2^2 + \ldots\ldots\}\,\theta^2,$$

[1] Thomson and Tait, 2nd edition, § 343 *m*.

[2] *St Petersb. Acad. Sci. Mém.* I. 1859.

[3] *Stability of Motion* (Adams Prize Essay for 1877). See also Routh's *Rigid Dynamics*, 5th edition, 1892.

[4] *Proceedings of the Mathematical Society*, June, 1873.

whence, if θ varies as $\cos pt$,

$$p^2 = \frac{c_1 A_1^2 + c_2 A_2^2 + \ldots + c_m A_m^2}{a_1 A_1^2 + a_2 A_2^2 + \ldots + a_m A_m^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

This gives the period of the vibration of the constrained type; and it is evident that the period is stationary, when all but one of the coefficients A_1, A_2, ... vanish, that is to say, when the type coincides with one of those natural to the system, and no constraint is needed.

[In the foregoing statement the equilibrium is supposed to be thoroughly stable, so that all the quantities c are positive. But the theorem applies equally even though any or all of the c's be negative. Only if p^2 itself be negative, the period becomes imaginary. In this case the stationary character attaches to the coefficients of t in the exponential terms, quantities which measure the *degree* of instability.

Corresponding theorems, of importance in other branches of science, may be stated for systems such that only T and F, or only V and F, are sensible[1].

The stationary property of the roots of Lagrange's determinant (3) § 84, suggests a general method of approximating to their values. Beginning with assumed rough approximations to the ratios $A_1 : A_2 : A_3 \ldots\ldots$ we may calculate a first approximation to p^2 from

$$p^2 = \frac{\frac{1}{2}\, c_{11} A_1^2 + \frac{1}{2}\, c_{22} A_2^2 + \ldots + c_{12} A_1 A_2 + \ldots}{\frac{1}{2}\, a_{11} A_1^2 + \frac{1}{2}\, a_{22} A_2^2 + \ldots + a_{12} A_1 A_2 + \ldots} \quad \ldots\ldots (3).$$

With this value of p^2 we may recalculate the ratios $A_1 : A_2 \ldots$ from any $(m-1)$ of equations (5) § 84, then again by application of (3) determine an improved value of p^2, and so on.]

By means of the same theorem we may prove that an increase in the mass of any part of a vibrating system is attended by a prolongation of all the natural periods, or at any rate that no period can be diminished. Suppose the increment of mass to be infinitesimal. After the alteration, the types of free vibration will in general be changed; but, by a suitable constraint, the system may be made to retain any one of the former types. If this be done, it is certain that any vibration which involves a motion of the part whose mass has been increased will have its period prolonged. Only as a particular case (as, for example, when a load is placed at the node of a vibrating string) can the period

[1] *Brit. Ass. Rep.* for 1885, p. 911.

remain unchanged. The theorem now allows us to assert that the removal of the constraint, and the consequent change of type, can only affect the period by a quantity of the second order; and that therefore in the limit the free period cannot be less than before the change. By integration we infer that a finite increase of mass must prolong the period of every vibration which involves a motion of the part affected, and that in no case can the period be diminished; but in order to see the correspondence of the two sets of periods, it may be necessary to suppose the alterations made by steps. Conversely, the effect of a removal of part of the mass of a vibrating system must be to shorten the periods of all the free vibrations.

In like manner we may prove that if the system undergo such a change that the potential energy of a given configuration is diminished, while the kinetic energy of a given motion is unaltered, the periods of the free vibrations are all increased, and conversely. This proposition may sometimes be used for tracing the effects of a constraint; for if we suppose that the potential energy of any configuration violating the condition of constraint gradually increases, we shall approach a state of things in which the condition is observed with any desired degree of completeness. During each step of the process every free vibration becomes (in general) more rapid, and a number of the free periods (equal to the degrees of liberty lost) become infinitely small. The same practical result may be reached without altering the potential energy by supposing the *kinetic* energy of any *motion* violating the condition to increase without limit. In this case one or more periods become infinitely large, but the finite periods are ultimately the same as those arrived at when the potential energy is increased, although in one case the periods have been throughout increasing, and in the other diminishing. This example shews the necessity of making the alterations by steps; otherwise we should not understand the correspondence of the two sets of periods. Further illustrations will be given under the head of two degrees of freedom.

By means of the principle that the value of the free periods is stationary, we may easily calculate corrections due to any deviation in the system from theoretical simplicity. If we take as a hypothetical type of vibration that proper to the simple system, the period so found will differ from the truth by quantities depending on the squares of the irregularities. Several examples of such calculations will be given in the course of this work.

89. Another point of importance relating to the period of a system vibrating in an arbitrary type remains to be noticed. It appears from (2) § 88, that the period of the vibration corresponding to any hypothetical type is included between the greatest and least of those natural to the system. In the case of systems like strings and plates which are treated as capable of continuous deformation, there is no least natural period; but we may still assert that the period calculated from any hypothetical type cannot exceed that belonging to the gravest normal type. When therefore the object is to estimate the longest proper period of a system by means of calculations founded on an assumed type, we know *a priori* that the result will come out too small.

In the choice of a hypothetical type judgment must be used, the object being to approach the truth as nearly as can be done without too great a sacrifice of simplicity. Thus the type for a string heavily weighted at one point might suitably be taken from the extreme case of an infinite load, when the two parts of the string would be straight. As an example of a calculation of this kind, of which the result is known, we will take the case of a uniform string of length l, stretched with tension T_1, and inquire what the period would be on certain suppositions as to the type of vibration.

Taking the origin of x at the middle of the string, let the curve of vibration on the positive side be

$$y = \cos pt \left\{1 - \left(\frac{2x}{l}\right)^n\right\} \quad \text{..................(1)},$$

and on the negative side the image of this in the axis of y, n being not less than unity. This form satisfies the condition that y vanishes when $x = \pm \frac{1}{2}l$. We have now to form the expressions for T and V, and it will be sufficient to consider the positive half of the string only. Thus, ρ being the longitudinal density,

$$T = \tfrac{1}{2} \int_0^{\frac{1}{2}l} \rho \dot{y}^2 dx = \frac{\rho n^2 l p^2 \sin^2 pt}{2(n+1)(2n+1)},$$

and

$$V = \tfrac{1}{2} T_1 \int_0^{\frac{1}{2}l} \left(\frac{dy}{dx}\right)^2 dx = \frac{n^2 T_1 \cos^2 pt}{(2n-1)\,l}.$$

Hence $$p^2 = \frac{2(n+1)(2n+1)}{2n-1} \cdot \frac{T_1}{\rho l^2} \qquad\qquad (2).$$

If $n=1$, the string vibrates as if the mass were concentrated in its middle point, and

$$p^2 = \frac{12\, T_1}{\rho l^2}.$$

If $n = 2$, the form is parabolic, and

$$p^2 = \frac{10\, T_1}{\rho l^2}.$$

The true value of p^2 for the gravest type is $\frac{\pi^2 T_1}{\rho l^2}$, so that the assumption of a parabolic form gives a period which is too small in the ratio $\pi : \sqrt{10}$ or $\cdot 9936 : 1$. The minimum of p^2, as given by (2), occurs when $n = \frac{1}{2}(\sqrt{6}+1) = 1{\cdot}72474$, and gives

$$p^2 = 9{\cdot}8990\, \frac{T_1}{\rho l^2}.$$

The period is now too small in the ratio

$$\pi : \sqrt{9{\cdot}8990} = {\cdot}99851 : 1.$$

It will be seen that there is considerable latitude in the choice of a type, even the violent supposition that the string vibrates as two straight pieces giving a period less than ten per cent. in error. And whatever type we choose to take, the period calculated from it cannot be greater than the truth.

[In the above applications it is assumed that there are no unstable modes. When unstable modes exist, the statement is that a constrained mode if stable possesses a frequency of vibration less than that of the highest normal mode, and if unstable has a degree of instability less than that of the most unstable normal mode.]

90. The rigorous determination of the periods and types of vibration of a given system is usually a matter of great difficulty, arising from the fact that the functions necessary to express the modes of vibration of most continuous bodies are not as yet recognised in analysis. It is therefore often necessary to fall back on methods of approximation, referring the proposed system to some other of a character more amenable to analysis, and calculating corrections depending on the supposition that the difference between the two systems is small. The problem of approximately simple systems is thus one of great importance, more especially as it is impossible in practice actually to realise the simple forms about which we can most easily reason.

Let us suppose then that the vibrations of a simple system are thoroughly known, and that it is required to investigate those of a system derived from it by introducing small variations in the mechanical functions. If ϕ_1, ϕ_2, &c. be the normal co-ordinates of the original system,

$$T = \tfrac{1}{2} a_1 \dot{\phi}_1^2 + \tfrac{1}{2} a_2 \dot{\phi}_2^2 + \ldots,$$
$$V = \tfrac{1}{2} c_1 \phi_1^2 + \tfrac{1}{2} c_2 \phi_2^2 + \ldots,$$

and for the varied system, referred to the same co-ordinates, which are now only approximately normal,

$$\left.\begin{aligned} T + \delta T &= \tfrac{1}{2}(a_1 + \delta a_{11})\dot{\phi}_1^2 + \ldots + \delta a_{12}\dot{\phi}_1\dot{\phi}_2 + \ldots \\ V + \delta V &= \tfrac{1}{2}(c_1 + \delta c_{11})\phi_1^2 + \ldots + \delta c_{12}\phi_1\phi_2 + \ldots \end{aligned}\right\} \ldots\ldots (1),$$

in which δa_{11}, δa_{12}, δc_{11}, δc_{12}, &c. are to be regarded as small quantities. In certain cases new co-ordinates may appear, but if so their coefficients must be small. From (1) we obtain for the Lagrangian equations of motion,

$$\left.\begin{aligned} &(a_1 + \delta a_{11} D^2 + c_1 + \delta c_{11})\phi_1 + (\delta a_{12} D^2 + \delta c_{12})\phi_2 \\ &\qquad\qquad + (\delta a_{13} D^2 + \delta c_{13})\phi_3 + \ldots = 0 \\ &(\delta a_{21} D^2 + \delta c_{21})\phi_1 + (a_2 + \delta a_{22} D^2 + c_2 + \delta c_{22})\phi_2 \\ &\qquad\qquad + (\delta a_{23} D^2 + \delta c_{23})\phi_3 + \ldots = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots (2).$$

In the original system the fundamental types of vibration are those which correspond to the variation of but a single co-ordinate at a time. Let us fix our attention on one of them, involving say a variation of ϕ_r, while all the remaining co-ordinates vanish. The change in the system will in general entail an alteration in the fundamental or normal types; but under the circumstances contemplated the alteration is small. The new normal type is expressed by the synchronous variation of the other co-ordinates in addition to ϕ_r; but the ratio of any other ϕ_s to ϕ_r is small. When these ratios are known, the normal mode of the altered system will be determined.

Since the whole motion is simple harmonic, we may suppose that each co-ordinate varies as $\cos p_r t$, and substitute in the differential equations $-p_r^2$ for D^2. In the s^{th} equation ϕ_s occurs with the finite coefficient

$$-a_s p_r^2 - \delta a_{ss} p_r^2 + c_s + \delta c_{ss}.$$

The coefficient of ϕ_r is

$$- \delta a_{rs} p_r^2 + \delta c_{rs}.$$

The other terms are to be neglected in a first approximation, since both the co-ordinate (relatively to ϕ_r) and its coefficient are small quantities. Hence

$$\phi_s : \phi_r = - \frac{\delta c_{rs} - p_r^2 \delta a_{rs}}{c_s - p_r^2 a_s} \qquad (3).$$

Now $$- a_s p_s^2 + c_s = 0,$$

and thus $$\phi_s : \phi_r = \frac{p_r^2 \delta a_{rs} - \delta c_{rs}}{a_s (p_s^2 - p_r^2)} \qquad (4),$$

the required result.

If the kinetic energy alone undergo variation,

$$\phi_s : \phi_r = \frac{p_r^2}{p_s^2 - p_r^2} \frac{\delta a_{rs}}{a_s} \qquad (5).$$

The corrected value of the period is determined by the rth equation of (2), not hitherto used. We may write it,

$$\phi_r \{ - p_r^2 a_r - p_r^2 \delta a_{rr} + c_r + \delta c_{rr} \} + \Sigma \phi_s (- p_r^2 \delta a_{rs} + \delta c_{rs}) = 0.$$

Substituting for $\phi_s : \phi_r$ from (4), we get

$$p_r^2 = \frac{c_r + \delta c_{rr}}{a_r + \delta a_{rr}} - \Sigma \frac{(\delta c_{rs} - p_r^2 \delta a_{rs})^2}{a_s a_r (p_s^2 - p_r^2)} \qquad (6).$$

The first term gives the value of p_r^2 calculated without allowance for the change of type, and is sufficient, as we have already proved, when the square of the alteration in the system may be neglected. The terms included under the symbol Σ, in which the summation extends to all values of s other than r, give the correction due to the change of type and are of the second order. Since a_s and a_r are positive, the sign of any term depends upon that of $p_s^2 - p_r^2$. If $p_s^2 > p_r^2$, that is, if the mode s be more acute than the mode r, the correction is negative, and makes the calculated note graver than before; but if the mode s be the graver, the correction raises the note. If r refer to the gravest mode of the system, the whole correction is negative; and if r refer to the acutest mode, the whole correction is positive, as we have already seen by another method.

91. As an example of the use of these formulæ, we may take the case of a stretched string, whose longitudinal density ρ is not quite constant. If x be measured from one end, and y

be the transverse displacement, the configuration at any time t will be expressed by

$$y=\phi_1 \sin\frac{\pi x}{l}+\phi_2 \sin\frac{2\pi x}{l}+\phi_3 \sin\frac{3\pi x}{l}+ \ldots\ldots\ldots (1),$$

l being the length of the string. ϕ_1, $\phi_2, \ldots$ are the normal co-ordinates for $\rho=$ constant, and though here ρ is not strictly constant, the configuration of the system may still be expressed by means of the same quantities. Since the potential energy of any configuration is the same as if $\rho=$ constant, $\delta V=0$. For the kinetic energy we have

$$T+\delta T=\tfrac{1}{2}\int_0^l \rho\left(\dot{\phi}_1 \sin\frac{\pi x}{l}+\dot{\phi}_2 \sin\frac{2\pi x}{l}+\ldots\right)^2 dx$$

$$=\tfrac{1}{2}\dot{\phi}_1^2\int_0^l \rho \sin^2\frac{\pi x}{l}\,dx+\tfrac{1}{2}\dot{\phi}_2^2\int_0^l \rho\sin^2\frac{2\pi x}{l}\,dx+\ldots$$

$$+\dot{\phi}_1\dot{\phi}_2\int_0^l \rho\sin\frac{\pi x}{l}\sin\frac{2\pi x}{l}\,dx+\ldots.$$

If ρ were constant, the products of the velocities would disappear, since ϕ_1, ϕ_2, &c. are, on that supposition, the normal co-ordinates. As it is, the integral coefficients, though not actually evanescent, are small quantities. Let $\rho=\rho_0+\delta\rho$; then in our previous notation

$$a_r=\tfrac{1}{2}l\rho_0,\quad \delta a_{rr}=\int_0^l \delta\rho\sin^2\frac{r\pi x}{l}\,dx,\quad \delta a_{rs}=\int_0^l \delta\rho\sin\frac{r\pi x}{l}\sin\frac{s\pi x}{l}\,dx.$$

Thus the type of vibration is expressed by

$$\phi_s : \phi_r=\frac{p_r^2}{p_s^2-p_r^2}\cdot\frac{2}{l\rho_0}\int_0^l \delta\rho\sin\frac{r\pi x}{l}\sin\frac{s\pi x}{l}\,dx;$$

or, since $p_r^2 : p_s^2=r^2 : s^2$,

$$\phi_s : \phi_r=\frac{r^2}{s^2-r^2}\int_0^l \frac{2\delta\rho}{l\rho_0}\sin\frac{r\pi x}{l}\sin\frac{s\pi x}{l}\,dx \ldots\ldots\ldots (2).$$

Let us apply this result to calculate the displacement of the nodal point of the second mode ($r=2$), which would be in the middle, if the string were uniform. In the neighbourhood of this point, if $x=\tfrac{1}{2}l+\delta x$, the approximate value of y is

$$y=\phi_1\sin\frac{\pi}{2}+\phi_2\sin\frac{2\pi}{2}+\phi_3\sin\frac{3\pi}{2}+\ldots$$

$$+\delta x\left\{\frac{\pi}{l}\phi_1\cos\frac{\pi}{2}+\frac{2\pi}{l}\phi_2\cos\frac{2\pi}{2}+\ldots\right\}$$

$$=\phi_1-\phi_3+\phi_5-\ldots+\frac{\pi}{l}\delta x\{-2\phi_2+2\phi_4+\ldots\}.$$

Hence when $y=0$,

$$\delta x = \frac{l}{2\pi\phi_2}\{\phi_1 - \phi_3 + \phi_5 - \ldots\} \ldots\ldots\ldots\ldots\ldots (3)$$

approximately, where

$$\phi_s : \phi_2 = \frac{4}{s^2-4}\int_0^l \frac{2\delta\rho}{l\rho_0}\sin\frac{2\pi x}{l}\sin\frac{s\pi x}{l}\,dx \ldots\ldots\ldots (4).$$

To shew the application of these formulæ, we may suppose the irregularity to consist in a small load of mass $\rho_0\lambda$ situated at $x=\frac{1}{4}l$, though the result might be obtained much more easily directly. We have

$$\delta x = \frac{2\lambda}{\pi\sqrt{2}}\left\{\frac{2}{1^2-4} - \frac{2}{3^2-4} - \frac{2}{5^2-4} + \frac{2}{7^2-4} + \ldots\ldots\ldots\right\},$$

from which the value of δx may be calculated by approximation. The real value of δx is, however, very simple. The series within brackets may be written

$$-\left\{1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \&c.\right\},$$

which is equal to

$$-\int_0^1 \frac{1+x^2}{1+x^4}\,dx.$$

The value of the definite integral is

$$\pi \div 4\sin\frac{\pi}{4}\,[1],$$

and thus

$$\delta x = -\frac{2\lambda}{\pi\sqrt{2}}\cdot\frac{\pi\sqrt{2}}{4} = -\frac{\lambda}{2},$$

as may also be readily proved by equating the periods of vibration of the two parts of the string, that of the loaded part being calculated approximately on the assumption of unchanged type.

As an example of the formula (6) § 90 for the period, we may take the case of a string carrying a small load $\rho_0\lambda$ at its middle point. We have

$$a_r = \tfrac{1}{2}l\rho_0, \quad \delta a_{rr} = \rho_0\lambda\sin^2\frac{r\pi}{2}, \quad \delta a_{rs} = \rho_0\lambda\sin\frac{r\pi}{2}\sin\frac{s\pi}{2},$$

and thus, if P_r be the value corresponding to $\lambda=0$, we get when r is even, $p_r = P_r$, and when r is odd,

$$p_r^2 = P_r^2\left\{\frac{1}{1+2\lambda/l} - \Sigma\frac{4r^2}{s^2-r^2}\frac{\lambda^2}{l^2}\right\} \ldots\ldots\ldots\ldots\ldots (5),$$

[1] Todhunter's *Int. Calc.* § 255.

where the summation is to be extended to all the odd values of s other than r. If $r = 1$,

$$p_1^2 = P_1^2 \left\{1 - \frac{2\lambda}{l} + \frac{4\lambda^2}{l^2} - \Sigma \frac{4}{s^2 - 1} \frac{\lambda^2}{l^2}\right\}.$$

Now
$$2\Sigma \frac{1}{s^2 - 1} = \Sigma \frac{1}{s - 1} - \Sigma \frac{1}{s + 1},$$

in which the values of s are 3, 5, 7, 9.... Accordingly

$$\Sigma \frac{1}{s^2 - 1} = \frac{1}{4},$$

and
$$p_1^2 = P_1^2 \left\{1 - \frac{2\lambda}{l} + \frac{3\lambda^2}{l^2} + \ldots\ldots\right\} \ldots\ldots\ldots\ldots\ldots (6),$$

giving the pitch of the gravest tone accurately as far as the square of the ratio $\lambda : l$.

In the general case the value of p_r^2, correct as far as the first order in $\delta\rho$, will be

$$p_r^2 = P_r^2 \left\{1 - \frac{\delta a_{rr}}{a_r}\right\} = P_r^2 \left\{1 - \frac{2}{l} \int_0^l \frac{\delta\rho}{\rho_0} \sin^2 \frac{r\pi x}{l}\, dx\right\} \ldots (7).$$

92. The theory of vibrations throws great light on expansions of arbitrary functions in series of other functions of specified types. The best known example of such expansions is that generally called after Fourier, in which an arbitrary periodic function is resolved into a series of harmonics, whose periods are submultiples of that of the given function. It is well known that the difficulty of the question is confined to the proof of the *possibility* of the expansion; if this be assumed, the determination of the coefficients is easy enough. What I wish now to draw attention to is, that in this, and an immense variety of similar cases, the possibility of the expansion may be inferred from physical considerations.

To fix our ideas, let us consider the small vibrations of a uniform string stretched between fixed points. We know from the general theory that the whole motion, whatever it may be, can be analysed into a series of component motions, each represented by a harmonic function of the time, and capable of existing by itself. If we can discover these normal types, we shall be in a position to represent the most general vibration possible by combining them, assigning to each an arbitrary amplitude and phase.

Assuming that a motion is harmonic with respect to time, we get to determine the type an equation of the form

$$\frac{d^2y}{dx^2} + k^2y = 0,$$

whence it appears that the normal functions are

$$y = \sin\frac{\pi x}{l}, \quad y = \sin\frac{2\pi x}{l}, \quad y = \sin\frac{3\pi x}{l}, \text{ \&c.}$$

We infer that the most general position which the string can assume is capable of representation by a series of the form

$$A_1 \sin\frac{\pi x}{l} + A_2 \sin\frac{2\pi x}{l} + A_3 \sin\frac{3\pi x}{l} + \ldots\ldots,$$

which is a particular case of Fourier's theorem. There would be no difficulty in proving the theorem in its most general form.

So far the string has been supposed uniform. But we have only to introduce a variable density, or even a single load at any point of the string, in order to alter completely the expansion whose possibility may be inferred from the dynamical theory. It is unnecessary to dwell here on this subject, as we shall have further examples in the chapters on the vibrations of particular systems, such as bars, membranes, and confined masses of air.

92 *a*. In § 88 we have a formula for the frequency of vibration applicable when by the imposition of given constraints the original system is left with only one degree of freedom. It is of interest to trace also the effect of less complete constraints, such as may be expressed by linear relations among the normal co-ordinates of number less by at least two than that of the (original) degrees of freedom. Thus we may suppose that

$$\left.\begin{array}{l} f_1\phi_1 + f_2\phi_2 + f_3\phi_3 + \ldots = 0 \\ g_1\phi_1 + g_2\phi_2 + g_3\phi_3 + \ldots = 0 \\ h_1\phi_1 + h_2\phi_2 + h_3\phi_3 + \ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (1).$$

If the number of equations (r) fall short of the number of the degrees of freedom by unity, the ratios $\phi_1 : \phi_2 : \phi_3 \ldots$ are fully determined, and the case is that of but one outstanding degree of freedom discussed in § 88.

This problem may be treated in more than one way, but the most instructive procedure is to trace the effect of additions to T and V. We will suppose that equations (1) § 87 are altered to

$$T = \tfrac{1}{2}a_1\dot{\phi}_1^2 + \tfrac{1}{2}a_2\dot{\phi}_2^2 + \ldots + \tfrac{1}{2}\alpha\,(f_1\dot{\phi}_1 + f_2\dot{\phi}_2 + \ldots)^2 \quad\text{.........(2)},$$

$$V = \tfrac{1}{2}c_1\phi_1^2 + \tfrac{1}{2}c_2\phi_2^2 + \ldots + \tfrac{1}{2}\gamma\,(f_1\phi_1 + f_2\phi_2 + \ldots)^2 \quad\text{.........(3)},$$

and that F, not previously existent, is now

$$F = \tfrac{1}{2}\beta\,(f_1\dot{\phi}_1 + f_2\dot{\phi}_2 + \ldots)^2 \quad\text{....................(4)}.$$

The connection with the proposed problem will be understood by supposing for instance that $\alpha = 0$, $\beta = 0$, while $\gamma = \infty$. By (3) the potential energy of any displacement violating the condition

$$f_1\phi_1 + f_2\phi_2 + \ldots = 0 \quad\text{.......................(5)}$$

is then infinite, and this is tantamount to the imposition of the constraint represented by (5).

Lagrange's equations with λ written for D now become

$$\left.\begin{array}{l}(a_1\lambda^2 + c_1)\,\phi_1 + f_1\,(\alpha\lambda^2 + \beta\lambda + \gamma)\,(f_1\phi_1 + f_2\phi_2 + \ldots) = 0\\ (a_2\lambda^2 + c_2)\,\phi_2 + f_2\,(\alpha\lambda^2 + \beta\lambda + \gamma)\,(f_1\phi_1 + f_2\phi_2 + \ldots) = 0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\}\ldots(6).$$

If we multiply the first of these by $f_1/(a_1\lambda^2 + c_1)$, the second by $f_2/(a_2\lambda^2 + c_2)$, and so on, and add the results together, the factor $(f_1\phi_1 + f_2\phi_2 + \ldots)$ will divide out, and the determinant takes the form

$$\frac{f_1^2}{a_1\lambda^2 + c_1} + \frac{f_2^2}{a_2\lambda^2 + c_2} + \ldots\ldots\ldots + \frac{1}{\alpha\lambda^2 + \beta\lambda + \gamma} = 0 \quad\text{...... (7)}.$$

If any one of the quantities α, β, γ become infinite while the others remain finite, the effect is equivalent to the imposition of the constraint (5), and the result may be written

$$\Sigma f^2/(a\lambda^2 + c) = 0 \quad\text{.......................(8)}[1].$$

When multiplied out this equation is of degree $(m-1)$ in λ^2, one degree of freedom having been lost.

If we put $\beta = 0$, (7) is an equation of the mth degree in λ^2, and the coefficients α, γ enter in the same way as do a_1, c_1; a_2, c_2; &c.

In order to refer more directly to the case of vibrations about stable equilibrium, we will write p^2 for $-\lambda^2$. The values of p^2 belonging to the unaltered system, viz. n_1^2, n_2^2,..., are given as before by

$$c_1 - a_1n_1^2 = 0; \qquad c_2 - a_2n_2^2 = 0, \ \&\text{c.}, \quad\text{................. (9)};$$

and we will also write

$$\gamma - \alpha\nu^2 = 0 \quad\text{........................(10)},$$

[1] Routh's *Rigid Dynamics*, 5th edition, 1892, § 67.

where ν^2 relates to the supposed additions to T and V considered as belonging to an independent vibrator. Let the order of magnitude of these quantities be

$$n_1^2,\ n_2^2, \ldots\ldots\ldots n_r^2,\ \nu^2,\ n_{r+1}^2, \ldots\ldots\ldots n_m^2 \ldots\ldots\ldots\ldots (11).$$

We shall see that there is a root of (7) between each consecutive pair of the quantities (11).

Our equation may be written

$$\begin{aligned} &f_1^2(\gamma - \alpha p^2)(c_2 - a_2p^2)(c_3 - a_3p^2)\ldots\ldots \\ &+ f_2^2(\gamma - \alpha p^2)(c_1 - a_1p^2)(c_3 - a_3p^2)\ldots\ldots \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ (c_1 - a_1p^2)(c_2 - a_2p^2)\ldots\ldots\ldots = 0 \ \ldots\ldots\ldots\ldots\ldots (12). \end{aligned}$$

When p^2 coincides with any of the quantities (11), all but one of the terms in (12) vanish, and the sign of the expression is the same as that of the term which remains over. When $p^2 < n_1^2$, all the terms are positive, so that there is no root less than n_1^2. When $p^2 = n_1^2$, the expression (12) reduces to the positive quantity

$$f_1^2(\gamma - \alpha n_1^2)(c_2 - a_2n_1^2)(c_3 - a_3n_1^2)\ldots\ldots$$

When p^2 rises to n_2^2, (12) becomes

$$f_2^2(\gamma - \alpha n_2^2)(c_1 - a_1n_2^2)(c_3 - a_3n_2^2)\ldots\ldots;$$

and this is *negative*, since the factor $(c_1 - a_1n_2^2)$ is now negative. Hence there is a root of (12) between n_1^2 and n_2^2. When $p^2 = n_3^2$, the expression is again positive, and thus there is a root between n_2^2 and n_3^2. This argument may be continued, and it proves that there is a root of (12) between any consecutive two of the $(m+1)$ quantities (11). The m roots of (12) are now accounted for, and there is none greater than n_m^2. If we compare the values of the roots before and after the change, we see that the effect is to cause a movement which is in every case *towards* ν^2.[1] Considered absolutely the movement is in one direction for those roots that are greater than ν^2 and in the opposite direction for those that are less than ν^2. Accordingly the interval from n_r^2 to n_{r+1}^2, in which ν^2 lies, contains after the change *two* roots, one on either side of ν^2.

If ν^2 be less than any of the quantities n^2, as happens when $\gamma = 0$, one root lies between ν^2 and n_1^2, one between n_1^2 and n_2^2, and so on. Thus every root is depressed. On the other hand if $\nu^2 > n_m^2$, every root is increased. This happens if $\alpha = 0$. (§ 88.)

[1] It will be understood that in particular cases the movement may vanish.

The results now arrived at are of course independent of the special machinery of normal co-ordinates used in the investigation. If to any part of a system (n_1^2, n_2^2......) be attached a vibrator (ν^2) having a single degree of freedom, the effect is to displace all the quantities n_1^2,... in the direction of ν^2. Let us now suppose that a second change is made in the vibrator whereby α becomes $\alpha+\alpha'$, and γ becomes $\gamma+\gamma'$. Every root of the determinantal equation moves towards ν'^2, where $\gamma'-\alpha'\nu'^2=0$. If we suppose that $\nu'^2=\nu^2$, the movements are in all cases in the same directions as before. Going back now to the original system, and supposing that α, γ grow from zero to their actual values in such a manner that ν^2 remains constant, we see that during this process the roots move without regression in the direction of closer agreement with ν^2.

As α and γ become infinite, one root of (12) moves to coincidence with ν^2, while the remaining $(m-1)$ roots, corresponding to the constrained system, are given by

$$\Sigma f^2/(c-ap^2)=0 \quad\text{......................(13)},$$

and are independent of the value of ν^2.

Particular cases are obtained by supposing either $\nu^2=0$, or $\nu^2=\infty$. Whether the constraint is effected by making infinite the kinetic energy of any motion, or the potential energy of any displacement, which violates it, makes no difference to the vibrations which remain. In the first case one vibration becomes infinitely slow, and in the second case one becomes infinitely quick. However the constraint be arrived at, the $(m-1)$ frequencies of vibration of the constrained system *separate*[1] the m frequencies of the original system.

Any number of examples of this theorem may be invented without difficulty. Consider the case of a uniform stretched string, held at both ends and vibrating transversely. This is the original system. Now introduce a constraint by holding at rest a point which divides the length in the proportion (say) of 3 : 2. The two parts vibrate independently, and the frequencies for each part form an arithmetical progression. If the frequencies proper to the undivided string be 1, 2, 3, 4......; those for the parts are

[1] But in particular cases the "separation" may vanish. The theorem in the text was proved for two degrees of freedom in the first edition of this work. In its generality it appears to be due to Routh.

$\frac{5}{2}(1, 2, 3, \ldots)$ and $\frac{5}{3}(1, 2, 3, \ldots)$. The beginning of each series is shewn in the accompanying scheme;

1		2		3		4		5		6		7		8		9		10		11		12
	$1\frac{2}{3}$				$3\frac{1}{3}$			5			$6\frac{2}{3}$				$8\frac{1}{3}$			10			$11\frac{2}{3}$	
			$2\frac{1}{2}$					5					$7\frac{1}{2}$					10				

and it will be seen that between any consecutive numbers in the first row there is a number to be found *either* in the second *or* in the third row. In the case of 5 and 10 we have an extreme condition of things; but the slightest displacement of the point at which the constraint is applied will displace one of the fives, tens &c. to the left and the other to the right.

The coincidences may be avoided by dividing the string incommensurably. Thus, if x be an incommensurable number less than unity, one of the series of quantities m/x, $m/(1-x)$, where m is a whole number, can be found which shall lie between any given consecutive integers, and but one such quantity can be found.

Again, let us suppose that a system is referred to co-ordinates which are not normal (§ 84), and let the constraint represented by $\psi_1 = 0$ be imposed. The determinant of the altered system is formed from that of the original system by erasing the first row and the first column. It may be called ∇_1, and from this again may be formed in like manner a new determinant ∇_2, and so on. These determinants form a series of functions of p^2, regularly decreasing in degree; and we conclude that the roots of each separate the roots of that immediately preceding[1].

It may be remarked that while for the sake of simplicity of statement we have supposed that the equilibrium of the original system was thoroughly stable, as also that of the vibration brought into connection therewith, these restrictions may easily be dispensed with. In any case the series of positive and negative quantities, $n_1^2, n_2^2, \ldots\ldots$ and ν^2, may be arranged in algebraic order, and the effect of the vibrator is to cause a movement of every value of p^2 in the direction of ν^2.

In order to extend the above theory we will now suppose that the addition to T is

$$\tfrac{1}{2}\alpha_f(f_1\dot{\phi}_1 + f_2\dot{\phi}_2 + \ldots)^2 + \tfrac{1}{2}\alpha_g(g_1\dot{\phi}_1 + g_2\dot{\phi}_2 + \ldots)^2 + \tfrac{1}{2}\alpha_h(h_1\dot{\phi}_1 + h_2\dot{\phi}_2 + \ldots)^2 + \ldots\ldots\ldots \quad (14)$$

[1] Routh's *Rigid Dynamics*, 5th edition, Part II. § 58.

and the addition to V

$$\tfrac{1}{2}\gamma_f(f_1\phi_1+f_2\phi_2+\dots)^2+\tfrac{1}{2}\gamma_g(g_1\phi_1+g_2\phi_2+\dots)^2+\dots\dots(15).$$

If we set

$$\alpha_f\lambda^2+\gamma_f=F',\ \alpha_g\lambda^2+\gamma_g=G',\ \dots\dots\dots\dots(16),$$

and so on, Lagrange's equations become

$$(a_1\lambda^2+c_1)\,\phi_1+F'f_1(f_1\phi_1+f_2\phi_2+\dots)$$
$$+G'g_1(g_1\phi_1+g_2\phi_2+\dots)+H'h_1(h_1\phi_1+h_2\phi_2+\dots)+\dots=0\dots(17)$$
$$(a_2\lambda^2+c_2)\,\phi_2+F'f_2(f_1\phi_1+f_2\phi_2+\dots)$$
$$+G'g_2(g_1\phi_1+g_2\phi_2+\dots)+H'h_2(h_1\phi_1+h_2\phi_2+\dots)+\dots=0\dots(18),$$

and so on, the number of equations being equal to the number (m) of co-ordinates ϕ_1, ϕ_2.... The number of additions (r), corresponding to the letters f, g, h, ..., is supposed to be less than m.

From the above m equations let r new ones be formed, as follows. For the first multiply (17) by $f_1/(a_1\lambda^2+c_1)$, (18) by $f_2/(a_2\lambda^2+c_2)$, and so on, and add the results together. For the second proceed in the same manner, using the multipliers $g_1/(a_1\lambda^2+c_1)$, $g_2/(a_2\lambda^2+c_2)$, &c. In like manner for the third equation use h instead of g, and so on. In this way we obtain r equations which may be written

$$F'(f_1\phi_1+f_2\phi_2+\dots)\{1/F'+F_1^2+F_2^2+F_3^2+\dots\}$$
$$+G'(g_1\phi_1+g_2\phi_2+\dots)\{F_1G_1+F_2G_2+\dots\}$$
$$+H'(h_1\phi_1+h_2\phi_2+\dots)\{F_1H_1+F_2H_2+\dots\}+\dots\dots=0\ \dots(19),$$

$$F'(f_1\phi_1+f_2\phi_2+\dots)\{G_1F_1+G_2F_2+\dots\}$$
$$+G'(g_1\phi_1+g_2\phi_2+\dots)\{1/G'+G_1^2+G_2^2+\dots\}$$
$$+H'(h_1\phi_1+h_2\phi_2+\dots)\{G_1H_1+G_2H_2+\dots\}+\dots\dots=0\dots(20),$$

and so on, where for brevity

$$\left.\begin{array}{l} F_1^2=f_1^2/(a_1\lambda^2+c_1),\ F_2^2=f_2^2/(a_2\lambda^2+c_2),\ \&c.,\\ G_1^2=g_1^2/(a_1\lambda^2+c_1),\ G_2^2=g_2^2/(a_2\lambda^2+c_2),\ \&c.\\ F_1G_1=f_1g_1/(a_1\lambda^2+c_1),\ \&c. \end{array}\right\}\dots\dots(21).$$

The determinantal equation, of the rth order, is thus

$$\begin{vmatrix} 1/F'+\Sigma F^2, & \Sigma FG, & \Sigma FH,\dots \\ \Sigma FG, & 1/G'+\Sigma G^2, & \Sigma GH, \\ \Sigma FH, & \Sigma GH, & 1/H'+\Sigma H^2,\dots \\ \dots & \dots & \dots \end{vmatrix}=0\dots\dots(22).$$

If, for example, there be two additions to T and V of the kind prescribed, the equation is

$$\frac{1}{F'G'} + \frac{\Sigma F^2}{G'} + \frac{\Sigma G^2}{F'} + \Sigma F^2 . \Sigma G^2 - \{\Sigma FG\}^2 = 0 \quad \text{.........(23)},$$

and herein

$$(F_1^2 + F_2^2 + \ldots)(G_1^2 + G_2^2 + \ldots) - (F_1G_1 + F_2G_2 + \ldots)^2$$
$$= \Sigma\Sigma (F_1G_2 - F_2G_1)^2 \text{............(24)}.$$

Equation (23) is in general of the mth degree in λ^2, and determines the frequencies of vibration. In the extreme case where F' and G' are made infinite, the system is subject to the two constraints

$$\left.\begin{aligned} f_1\phi_1 + f_2\phi_2 + \ldots &= 0 \\ g_1\phi_1 + g_2\phi_2 + \ldots &= 0 \end{aligned}\right\} \text{..................(25)},$$

and the equation[1] giving the $(m-2)$ outstanding roots is

$$\frac{(f_1g_2 - f_2g_1)^2}{(a_1\lambda^2 + c_1)(a_2\lambda^2 + c_2)} + \frac{(f_1g_3 - f_3g_1)^2}{(a_1\lambda^2 + c_1)(a_3\lambda^2 + c_3)} + \ldots\ldots = 0 \text{......(26)}.$$

In general if the system be subject to the r constraints (1), the determinantal equation is

$$\begin{vmatrix} \Sigma FF, & \Sigma FG, & \Sigma FH, \ldots \\ \Sigma FG, & \Sigma GG, & \Sigma GH, \ldots \\ \Sigma FH, & \Sigma GH, & \Sigma HH, \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} = 0 \text{ .... (27)}.$$

If r be less than m, this determinant can be resolved[2] into a sum of squares of determinants of the same order (r). Thus if there be three constraints, the first of these squares is

$$\begin{vmatrix} F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \\ H_1 & H_2 & H_3 \end{vmatrix}^2 \text{......................(28)},$$

and the others are to be found by including every combination of the m suffixes taken three together. To fall back upon the original notation we have merely in (28) to replace the capital letters $F, G, \ldots$ by $f, g, \ldots$, and to introduce the denominator

$$(a_1\lambda^2 + c_1)(a_2\lambda^2 + c_2)(a_3\lambda^2 + c_3).$$

The determinantal equation for a system originally of m degrees of freedom and subjected to r constraints is thus found. Its form

[1] This result is due to Routh, *loc. cit.* § 67.

[2] Salmon, *Lessons on Higher Algebra*, § 24.

is largely determined by the consideration that it must remain unaffected by interchanges either of the letters or of the suffixes. That it would become nugatory if two of the conditions of constraint coincided, could also have been foreseen. If $r = m - 1$, the system is reduced to one degree of freedom, and the equation is

$$\begin{vmatrix} f_2 & f_3 & f_4 \ldots \\ g_2 & g_3 & g_4 \ldots \\ h_2 & h_3 & h_4 \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}^2 (a_1\lambda^2 + c_1) + \begin{vmatrix} f_1 & f_3 & f_4 \ldots \\ g_1 & g_3 & g_4 \ldots \\ h_1 & h_3 & h_4 \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}^2 (a_2\lambda^2 + c_2) + \ldots = 0 \ldots\ldots (29),$$

in agreement with § (88).

There are theories, parallel to the foregoing, for systems in which T and F, or V and F, are alone sensible. In these cases, if the functions be intrinsically positive, the normal motions are proportional to exponential functions of the time such as $e^{-t/\tau}$. The quantities τ_1, τ_2,... are called the time-constants, or persistences, of the motions, being the times occupied by the motions in subsiding in the ratio of $e : 1$. The new persistences, after the introduction of a constraint, will separate the original values.

The best illustrations of this theory are electrical, where the motions are not restricted to be small. Suppose (to take an electro-magnetic example) that in one branch of a net-work of conductors there is introduced a coil of persistence (when closed upon itself) equal to τ', the original persistences being τ_1, τ_2,.... Then the new persistences lie in all cases nearer to τ', and they separate the quantities τ', τ_1, τ_2.... If τ' be made infinite as by increasing the self-induction of the additional coil without limit, or be made to vanish as by breaking the contact in the branch, the result is a constraint, and the new values of the persistences separate the former ones.

93. The determination of the coefficients to suit arbitrary initial conditions may always be readily effected by the fundamental property of the normal functions, and it may be convenient to sketch the process here for systems like strings, bars, membranes, plates, &c. in which there is only one dependent variable ζ to be considered. If u_1, u_2... be the normal functions, and ϕ_1, ϕ_2... the corresponding co-ordinates,

$$\zeta = \phi_1 u_1 + \phi_2 u_2 + \phi_3 u_3 + \ldots\ldots\ldots\ldots (1).$$

The equations of free motion are

$$\ddot{\phi}_1 + n_1^2\phi_1 = 0, \quad \ddot{\phi}_2 + n_2^2\phi_2 = 0, \text{ \&c.} \quad\ldots\ldots\ldots\ldots(2),$$

of which the solutions are

$$\left.\begin{array}{l}\phi_1 = A_1 \sin n_1 t + B_1 \cos n_1 t \\ \phi_2 = A_2 \sin n_2 t + B_2 \cos n_2 t \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\} \quad\ldots\ldots\ldots\ldots(3).$$

The initial values of ζ and $\dot{\zeta}$ are therefore

$$\left.\begin{array}{l}\zeta_0 = B_1 u_1 + B_2 u_2 + B_3 u_3 + \ldots \\ \dot{\zeta}_0 = n_1 A_1 u_1 + n_2 A_2 u_2 + n_3 A_3 u_3 + \ldots\end{array}\right\} \quad\ldots\ldots\ldots\ldots(4).$$

and the problem is to determine A_1, A_2, ... B_1, B_2 ... so as to correspond with arbitrary values of ζ_0 and $\dot{\zeta}_0$.

If $\rho\, dx$ be the mass of the element dx, we have from (1)

$$T = \tfrac{1}{2}\int \rho\, \dot{\zeta}^2 dx$$

$$= \tfrac{1}{2}\dot{\phi}_1^2 \int \rho\, u_1^2 dx + \tfrac{1}{2}\dot{\phi}_2^2 \int \rho\, u_2^2 dx + \ldots + \dot{\phi}_1\dot{\phi}_2 \int \rho u_1 u_2 dx + \ldots.$$

But the expression for T in terms of $\dot{\phi}_1$, $\dot{\phi}_2$, &c. cannot contain the products of the normal generalized velocities, and therefore every integral of the form

$$\int \rho\, u_r u_s dx = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Hence to determine B_r we have only to multiple the first of equations (4) by ρu_r and integrate over the system. We thus obtain

$$B_r \int \rho\, u_r^2 dx = \int \rho\, u_r \zeta_0 dx \quad\ldots\ldots\ldots\ldots\ldots(6).$$

Similarly,
$$n_r A_r \int \rho\, u_r^2 dx = \int \rho\, u_r \dot{\zeta}_0 dx \quad\ldots\ldots\ldots\ldots\ldots(7).$$

The process is just the same whether the element dx be a line area, or volume.

The conjugate property, expressed by (5), depends upon the fact that the functions u are normal. As soon as this is known by the solution of a differential equation or otherwise, we may infer the conjugate property without further proof, but the property itself is most intimately connected with the fundamental variational equation of motion § 94.

94. If V be the potential energy of deformation, ζ the displacement, and ρ the density of the (line, area, or volume) element dx, the equation of virtual velocities gives immediately

$$\delta V + \int \rho \ddot{\zeta} \delta\zeta dx = 0 \qquad\qquad (1).$$

In this equation δV is a symmetrical function of ζ and $\delta\zeta$, as may be readily proved from the expression for V in terms of generalized co-ordinates. In fact if

$$V = \tfrac{1}{2} c_{11}\psi_1^2 + \ldots + c_{12}\psi_1\psi_2 + \ldots,$$

$$\delta V = c_{11}\psi_1\delta\psi_1 + c_{22}\psi_2\delta\psi_2 + \ldots$$
$$+ c_{12}(\psi_1\delta\psi_2 + \psi_2\delta\psi_1) + \ldots.$$

Suppose now that ζ refers to the motion corresponding to a normal function u_r, so that $\ddot{\zeta} + n_r^2\zeta = 0$, while $\delta\zeta$ is identified with another normal function u_s; then

$$\delta V = n_r^2 \int \rho u_r u_s dx.$$

Again, if we suppose, as we are equally entitled to do, that ζ varies as u_s and $\delta\zeta$ as u_r, we get for the same quantity δV,

$$\delta V = n_s^2 \int \rho\, u_r u_s dx;$$

and therefore
$$(n_r^2 - n_s^2) \int \rho\, u_r u_s dx = 0 \qquad\qquad (2),$$

from which the conjugate property follows, if the motions represented respectively by u_r and u_s have different periods.

A good example of the connection of the two methods of treatment will be found in the chapter on the transverse vibrations of bars.

95. Professor Stokes[1] has drawn attention to a very general law connecting those parts of the free motion which depend on the initial *displacements* of a system not subject to frictional forces, with those which depend on the initial *velocities*. If a velocity of any type be communicated to a system at rest, and then after a small interval of time the opposite velocity be communicated, the effect in the limit will be to start the system without velocity, but with a displacement of the corresponding type. We may readily prove from this that in order

[1] *Dynamical Theory of Diffraction*, *Cambridge Trans.* Vol. IX. p. 1, 1856.

to deduce the motion depending on initial displacements from that depending on the initial velocities, it is only necessary to differentiate with respect to the time, and to replace the arbitrary constants (or functions) which express the initial velocities by those which express the corresponding initial displacements.

Thus, if ϕ be any normal co-ordinate satisfying the equation

$$\ddot{\phi} + n^2\phi = 0,$$

the solution in terms of the initial values of ϕ and $\dot{\phi}$ is

$$\phi = \phi_0 \cos nt + \frac{1}{n}\dot{\phi}_0 \sin nt \qquad (1),$$

of which the first term may be obtained from the second by Stokes' rule.

CHAPTER V.

VIBRATING SYSTEMS IN GENERAL CONTINUED.

96. WHEN dissipative forces act upon a system, the character of the motion is in general more complicated. If two only of the functions T, F, and V be finite, we may by a suitable linear transformation rid ourselves of the products of the co-ordinates, and obtain the normal types of motion. In the preceding chapter we have considered the case of $F = 0$. The same theory with obvious modifications will apply when $T = 0$, or $V = 0$, but these cases though of importance in other parts of Physics, such as Heat and Electricity, scarcely belong to our present subject.

The presence of friction will not interfere with the reduction of T and V to sums of squares; but the transformation proper for them will not in general suit also the requirements of F. The general equation can then only be reduced to the form

$$a_1\ddot{\phi}_1 + b_{11}\dot{\phi}_1 + b_{12}\dot{\phi}_2 + \ldots + c_1\phi_1 = \Phi_1, \quad \&c. \ldots\ldots\ldots (1),$$

and not to the simpler form applicable to a system of one degree of freedom, viz.

$$a_1\phi_1 + b_1\dot{\phi}_1 + c_1\phi_1 = \Phi_1, \quad \&c. \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

We may, however, choose which pair of functions we shall reduce, though in Acoustics the choice would almost always fall on T and V.

97. There is, however, a not unimportant class of cases in which the reduction of all three functions may be effected; and the theory then assumes an exceptional simplicity. Under this head the most important are probably those when F is of the same form as T or V. The first case occurs frequently, in books at any rate, when the motion of each part of the system is resisted by a retarding force, proportional both to the mass and velocity of the part. The same exceptional reduction is possible when F is a linear function of T and V, or when T is itself of the same form as V. In any of these cases, the equations of motion are of the same form as for a system of one degree of freedom, and the theory possesses certain peculiarities which make it worthy of separate consideration.

The equations of motion are obtained at once from T, F and V:—

$$\left.\begin{array}{l} a_1\ddot{\phi}_1 + b_1\dot{\phi}_1 + c_1\phi_1 = \Phi_1, \\ a_2\ddot{\phi}_2 + b_2\dot{\phi}_2 + c_2\phi_2 = \Phi_2, \text{ \&c.} \end{array}\right\} \quad\text{.................}(1),$$

in which the co-ordinates are separated.

For the free vibrations we have only to put $\Phi_1 = 0$, &c., and the solution is of the form

$$\phi = e^{-\frac{1}{2}\kappa t}\left\{\dot{\phi}_0\,\frac{\sin n't}{n'} + \phi_0\left(\cos n't + \frac{\kappa}{2n'}\sin n't\right)\right\} \text{......}(2),$$

where $\kappa = b/a, \quad n^2 = c/a, \quad n' = \sqrt{(n^2 - \frac{1}{4}\kappa^2)}$,

and ϕ_0 and $\dot{\phi}_0$ are the initial values of ϕ and $\dot{\phi}$.

The whole motion may therefore be analysed into component motions, each of which corresponds to the variation of but one normal co-ordinate at a time. And the vibration in each of these modes is altogether similar to that of a system with only one degree of liberty. After a certain time, greater or less according to the amount of dissipation, the free vibrations become insignificant, and the system returns sensibly to rest.

[If F be of the same form as T, all the values of κ are equal, viz. all vibrations die out at the same rate.]

Simultaneously with the free vibrations, but in perfect independence of them, there may exist forced vibrations depending on the quantities Φ. Precisely as in the case of one degree of freedom, the solution of

$$a\ddot{\phi} + b\dot{\phi} + c\phi = \Phi \text{ ....}(3)$$

may be written

$$\phi = \frac{1}{n'}\int_0^t e^{-\frac{1}{2}\kappa(t-t')}\sin n'(t-t')\,\Phi\,dt' \text{ ....} (4),$$

where as above

$$\kappa = b/a, \quad n^2 = c/a, \quad n' = \sqrt{(n^2 - \tfrac{1}{4}\kappa^2)}.$$

To obtain the complete expression for ϕ we must add to the right-hand member of (4), which makes the initial values of ϕ and $\dot{\phi}$ vanish, the terms given in (2) which represent the residue at time t of the initial values ϕ_0 and $\dot{\phi}_0$. If there be no friction, the value of ϕ in (4) reduces to

$$\phi = \frac{1}{n}\int_0^t \sin n(t-t')\,\Phi\,dt' \quad\text{.................}\quad (5).$$

98. The complete independence of the normal co-ordinates leads to an interesting theorem concerning the relation of the subsequent motion to the initial disturbance. For if the forces which act upon the system be of such a character that they do no work on the displacement indicated by $\delta\phi_1$, then $\Phi_1 = 0$. No such forces, however long continued, can produce any effect on the motion ϕ_1. If it exist, they cannot destroy it; if it do not exist, they cannot generate it. The most important application of the theorem is when the forces applied to the system act at a node of the normal component ϕ_1, that is, at a point which the component vibration in question does not tend to set in motion. Two extreme cases of such forces may be specially noted, (1) when the force is an impulse, starting the system from rest, (2) when it has acted so long that the system is again at rest under its influence in a disturbed position. So soon as the force ceases, natural vibrations set in, and in the absence of friction would continue for an indefinite time. We infer that whatever in other respects their character may be, they contain no component of the type ϕ_1. This conclusion is limited to cases where T, F, V admit of simultaneous reduction, including of course the case of no friction.

99. The formulæ quoted in § 97 are applicable to any kind of force, but it will often happen that we have to deal only with the effects of impressed forces of the harmonic type, and we may then advantageously employ the more special formulæ applicable to such forces. In using normal co-ordinates, we have first to calculate the forces Φ_1, Φ_2, &c. corresponding to each period, and thence deduce the values of the co-ordinates themselves. If among the natural periods (calculated without allowance for friction) there be any nearly agreeing in magnitude with the period of an impressed force, the corresponding component vibrations will be abnormally large, unless indeed the force itself be greatly attenuated in the preliminary resolution. Suppose, for example, that a transverse force of harmonic type and given period acts at a single point of a stretched string. All the normal modes of vibration will, in general, be excited, not however in their own proper periods, but in the period of the impressed force; but any normal component, which has a node at the point of application, will not be excited. The magnitude of each component thus depends on two things: (1) on the situation of its nodes with respect to the point at which the force is applied, and (2) on the degree of agreement between its own proper period and that of the force. It is important to remember that in response to a simple harmonic force, the system will vibrate in general in *all* its modes, although in particular cases it may sometimes be sufficient to attend to only one of them as being of paramount importance.

100. When the periods of the forces operating are very long relatively to the free periods of the system, an equilibrium theory is sometimes adequate, but in such a case the solution could generally be found more easily without the use of the normal co-ordinates. Bernoulli's theory of the Tides is of this class, and proceeds on the assumption that the free periods of the masses of water found on the globe are small relatively to the periods of the operative forces, in which case the inertia of the water might be left out of account. As a matter of fact this supposition is only very roughly and partially applicable, and we are consequently still in the dark on many important points relating to the tides. The principal forces have a semi-diurnal period, which is not sufficiently long in relation to the natural periods concerned, to allow of the inertia of the water being neglected. But if the rotation of the earth had been much slower, the equilibrium theory of the tides might have been adequate.

A corrected equilibrium theory is sometimes useful, when the period of the impressed force is sufficiently long in comparison with most of the natural periods of a system, but not so in the case of one or two of them. It will be sufficient to take the case where there is no friction. In the equation

$$a\ddot{\phi} + c\phi = \Phi, \quad \text{or} \quad \ddot{\phi} + n^2\phi = \Phi/a,$$

suppose that the impressed force varies as $\cos pt$. Then

$$\phi = \Phi \div a\,(n^2 - p^2) \quad \text{..................... (1).}$$

The equilibrium theory neglects p^2 in comparison with n^2, and takes

$$\phi = \Phi \div an^2 \quad \text{..........................(2).}$$

Suppose now that this course is justifiable, except in respect of the single normal co-ordinate ϕ_1. We have then only to add to the result of the equilibrium theory, the difference between the true and the there assumed value of ϕ_1, viz.

$$\phi_1 = \frac{\Phi_1}{a_1(n_1^2 - p^2)} - \frac{\Phi_1}{a_1 n_1^2} = \frac{p^2}{n_1^2 - p^2} \cdot \frac{\Phi_1}{a} \quad \text{.............(3).}$$

The other extreme case ought also to be noticed. If the forced vibrations be extremely rapid, they may become nearly independent of the potential energy of the system. Instead of neglecting p^2 in comparison with n^2, we have then to neglect n^2 in comparison with p^2, which gives

$$\phi = -\Phi \div ap^2 \quad \text{......................(4).}$$

If there be one or two co-ordinates to which this treatment is not applicable, we may supplement the result, calculated on the hypothesis that V is altogether negligible, with corrections for these particular co-ordinates.

101. Before passing on to the general theory of the vibrations of systems subject to dissipation, it may be well to point out some peculiarities of the free vibrations of continuous systems, started by a force applied at a single point. On the suppositions and notations of § 93, the configuration at any time is determined by

$$\zeta = \phi_1 u_1 + \phi_2 u_2 + \phi_3 u_3 + \quad \text{...................(1),}$$

where the normal co-ordinates satisfy equations of the form

$$a_r \ddot{\phi}_r + c_r \phi_r = \Phi_r \quad \text{.......................(2).}$$

Suppose now that the system is held at rest by a force applied at the point Q. The value of Φ_r is determined by the consideration that $\Phi_r \delta\phi_r$ represents the work done upon the system by the impressed forces during a hypothetical displacement $\delta\zeta = \delta\phi_r u_r$, that is

$$\delta\phi_r \int Z u_r \, dx;$$

thus
$$\Phi_r = \int Z u_r \, dx = u_r(Q) \int Z \, dx;$$

so that initially by (2)

$$c_r \phi_r = u_r(Q) \int Z \, dx \quad \text{....................(3).}$$

If the system be let go from this configuration at $t=0$, we have at any subsequent time t,

$$\phi_r = \cos n_r t \frac{u_r(Q)\int Z\,dx}{c_r} = \cos n_r t \frac{u_r(Q)\int Z\,dx}{n_r^2 \int \rho\, u_r^2 dx} \qquad \text{(4)},$$

and at the point P

$$\zeta = \Sigma \cos n_r t \frac{u_r(P)\, u_r(Q) \int Z\,dx}{n_r^2 \int \rho\, u_r^2 dx} \qquad \text{(5)}.$$

At particular points $u_r(P)$ and $u_r(Q)$ vanish, but on the whole

$$u_r(P)\, u_r(Q) \div \int \rho\, u_r^2 dx$$

neither converges, nor diverges, with r. The series for ζ therefore converges with n_r^{-2}.

Again, suppose that the system is started by an impulse from the configuration of equilibrium. In this case initially

$$a_r \dot{\phi}_r = \int \Phi_r dt = u_r(Q) \int Z_1 dx,$$

whence at time t

$$\phi_r = \frac{\sin n_r t}{a_r n_r} . u_r(Q) . \int Z_1 dx = \frac{\sin n_r t . u_r(Q)}{n_r \int \rho u_r^2 dx} \int Z_1 dx \qquad \text{(6)}.$$

This gives

$$\zeta = \Sigma \sin n_r t \frac{u_r(P)\, u_r(Q) \int Z_1 dx}{n_r \int \rho u_r^2 dx} \qquad \text{(7)},$$

shewing that in this case the series converges with n_r^{-1}, that is more slowly than in the previous case.

In both cases it may be observed that the value of ζ is symmetrical with respect to P and Q, proving that the displacement at time t for the point P when the force or impulse is applied at Q, is the same as it would be at Q if the force or impulse had been applied at P. This is an example of a very general reciprocal theorem, which we shall consider at length presently.

As a third case we may suppose the body to start from rest as deformed by a force *uniformly distributed*, over its length, area, or volume. We readily find

$$\zeta = \Sigma \cos n_r t \frac{u_r(P) \,.\, Z \,.\, \int u_r dx}{n_r^2 \int \rho u_r^2 dx} \quad\text{.....................(8).}$$

The series for ζ will be more convergent than when the force is concentrated in a single point.

In exactly the same way we may treat the case of a continuous body whose motion is subject to dissipation, provided that the three functions T, F, V be simultaneously reducible, but it is not necessary to write down the formulæ.

102. If the three mechanical functions T, F and V of any system be not simultaneously reducible, the natural vibrations (as has already been observed) are more complicated in their character. When, however, the dissipation is small, the method of reduction is still useful; and this class of cases besides being of some importance in itself will form a good introduction to the more general theory. We suppose then that T and V are expressed as sums of squares

$$\left.\begin{aligned} T &= \tfrac{1}{2}\, a_1 \dot{\phi}_1^2 + \tfrac{1}{2}\, a_2 \dot{\phi}_2^2 + \ldots \\ V &= \tfrac{1}{2}\, c_1 \phi_1^2 + \tfrac{1}{2}\, c_2 \phi_2^2 + \ldots \end{aligned}\right\} \quad\text{.................. (1),}$$

while F still appears in the more general form

$$F = \tfrac{1}{2}\, b_{11} \dot{\phi}_1^2 + \tfrac{1}{2}\, b_{22} \dot{\phi}_2^2 + \ldots + b_{12} \dot{\phi}_1 \dot{\phi}_2 + \quad\text{...........(2).}$$

The equations of motion are accordingly

$$\left.\begin{aligned} a_1 \ddot{\phi}_1 + b_{11} \dot{\phi}_1 + b_{12} \dot{\phi}_2 + b_{13} \dot{\phi}_3 + \ldots + c_1 \phi_1 = 0 \\ a_2 \ddot{\phi}_2 + b_{21} \dot{\phi}_1 + b_{22} \dot{\phi}_2 + b_{23} \dot{\phi}_3 + \ldots + c_2 \phi_2 = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad\text{...........(3),}$$

in which the coefficients b_{11}, b_{12}, &c. are to be treated as small. If there were no friction, the above system of equations would be satisfied by supposing one co-ordinate ϕ_r to vary suitably, while the other co-ordinates vanish. In the actual case there will be a corresponding solution in which the value of any other co-ordinate ϕ_s will be small relatively to ϕ_r.

Hence, if we omit terms of the second order, the r^{th} equation becomes,

$$a_r \ddot{\phi}_r + b_{rr} \dot{\phi}_r + c_r \phi_r = 0 \quad\text{....................(4),}$$

from which we infer that ϕ_r varies approximately as if there were no change due to friction in the type of vibration. If ϕ_r vary as $e^{p_r t}$, we obtain to determine p_r

$$a_r p_r^2 + b_{rr} p_r + c_r = 0 \quad\text{.................... (5).}$$

The roots of this equation are complex, but the real part is small in comparison with the imaginary part. [The character of the motion represented by (5) has already been discussed (§ 45). The rate at which the vibrations die down is proportional to b_{rr}, and the period, if the term be still admitted, is approximately the same as if there were no dissipation.]

From the s^{th} equation, if we introduce the supposition that all the co-ordinates vary as $e^{p_r t}$, we get

$$(p_r^2 a_s + c_s)\,\phi_s + b_{rs} p_r \phi_r = 0,$$

terms of the second order being omitted; whence

$$\phi_s : \phi_r = -\frac{b_{rs} p_r}{p_r^2 a_s + c_s} = \frac{b_{rs} p_r}{a_s (p_s^2 - p_r^2)} \quad\text{.........(6).}$$

This equation determines approximately the altered type of vibration. Since the chief part of p_r is imaginary, we see that the co-ordinates ϕ_s are approximately in the same phase, *but that that phase differs by a quarter period from the phase of* ϕ_r. Hence when the function F does not reduce to a sum of squares, the character of the elementary modes of vibration is less simple than otherwise, and the various parts of the system are no longer simultaneously in the same phase.

We proved above that, when the friction is small, the value of p_r may be calculated approximately without allowance for the change of type; but by means of (6) we may obtain a still closer approximation, in which the squares of the small quantities are retained. The r^{th} equation (3) gives

$$a_r p_r^2 + c_r + b_{rr} p_r + \Sigma \frac{p_r^2 b^2_{rs}}{a_s (p_s^2 - p_r^2)} = 0 \quad\text{.............. (7).}$$

The leading part of the terms included under Σ being real, the correction has no effect on the real part of p_r on which the rate of decay depends.

102 *a*. Following the electrical analogy we may conveniently describe the forces expressed by F as forces of *resistance*. In § 102 we have seen that if the resistances be small, the periods are independent of them. We may therefore extend to this case the application of the theorems with regard to the effect upon the periods of additions to T and V, which have been already proved when there are no resistances.

By (5) § 102, if the forces of resistance be increased, the rates of subsidence of all the normal motions are in general increased with them; but in particular cases it may happen that there is no change in a rate of subsidence.

It is natural to inquire whether this conclusion is limited to *small* resistances, for at first sight it would appear likely to hold good generally. An argument sufficient to decide this question may be founded upon a particular case. Consider a system formed by attaching two loads at any points of a stretched string vibrating transversely. If the mass of the string itself be neglected, there are two degrees of freedom and two periods of vibration corresponding to two normal modes. In each of these modes both loads in general vibrate. Now suppose that a force of resistance is introduced retarding the motion of *one* of the loads, and that this force gradually increases. At first the effect is to cause both kinds of vibration to die out and that at an increasing rate, but afterwards the law changes. For when the resistance becomes infinite, it is equivalent to a *constraint*, holding at rest the load upon which it acts. The remaining vibration is then unaffected by resistance, and maintains itself indefinitely. Thus the rate of subsidence of one of the normal modes has decreased to evanescence in spite of a continual increase in the forces of resistance F. This case is of course sufficient to disprove the suggested general theorem.

103. We now return to the consideration of the general equations of § 84.

If ψ_1, ψ_2, &c. be the co-ordinates and Ψ_1, Ψ_2, &c. the forces, we have

$$\left.\begin{array}{l} e_{11}\psi_1 + e_{12}\psi_2 + \ldots = \Psi_1 \\ e_{21}\psi_1 + e_{22}\psi_2 + \ldots = \Psi_2, \text{\&c.} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $$e_{rs} = a_{rs}D^2 + b_{rs}D + c_{rs} \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

For the free vibrations Ψ_1, &c. vanish. If ∇ be the determinant

$$\nabla = \begin{vmatrix} e_{11}, & e_{12}, \ldots \\ e_{21}, & e_{22}, \ldots \\ \ldots & \ldots \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

the result of eliminating from (1) all the co-ordinates but one, is

$$\nabla\psi = 0 \quad\ldots\ldots\ldots\ldots(4).$$

Since ∇ now contains odd powers of D, the $2m$ roots of the equation $\nabla = 0$ no longer occur in equal positive and negative pairs, but contain a real as well as an imaginary part. The complete integral may however still be written

$$\psi = Ae^{\mu_1 t} + A'e^{\mu_1' t} + Be^{\mu_2 t} + B'e^{\mu_2' t} + \ldots\ldots\ldots(5),$$

where the pairs of conjugate roots are μ_1, μ_1'; μ_2, μ_2'; &c. Corresponding to each root, there is a particular solution such as

$$\psi_1 = A_1 e^{\mu_1 t}, \quad \psi_2 = A_2 e^{\mu_1 t}, \quad \psi_3 = A_3 e^{\mu_1 t}, \text{ \&c.},$$

in which the *ratios* $A_1 : A_2 : A_3 \ldots$ are determined by the equations of motion, and only the absolute value remains arbitrary. In the present case however (where ∇ contains odd powers of D) these ratios are not in general real, and therefore the variations of the co-ordinates ψ_1, ψ_2, &c. are not synchronous in phase. If we put $\mu_1 = \alpha_1 + i\beta_1$, $\mu_1' = \alpha_1 - i\beta_1$, &c., we see that none of the quantities α can be positive, since in that case the energy of the motion would increase with the time, as we know it cannot do.

103 *a*. The general argument (§§ 85, 103) from considerations of energy as to the nature of the roots of the determinantal equation (Thomson and Tait's *Natural Philosophy*, 1st edition 1867) has been put into a more mathematical form by Routh[1]. His investigation relates to the most general form of the equation in which the relations § 82

$$a_{rs} = a_{sr}, \qquad b_{rs} = b_{sr}, \qquad c_{rs} = c_{sr} \quad\ldots\ldots\ldots (1),$$

are not assumed. But for the sake of brevity and as sufficient for almost all acoustical problems, these relations will here be supposed to hold.

We shall have occasion to consider two solutions corresponding to two roots μ, ν of the equation. For the first we have

$$\psi_1 = M_1 e^{\mu t}, \; \psi_2 = M_2 e^{\mu t}, \; \psi_3 = M_3 e^{\mu t}, \text{ \&c.}\ldots\ldots\ldots(2),$$

and for the second

$$\psi_1 = N_1 e^{\nu t}, \; \psi_2 = N_2 e^{\nu t}, \; \psi_3 = N_3 e^{\nu t}, \text{ \&c.}\ldots\ldots\ldots(3).$$

In either of these solutions, for example (2), the ratios

$$M_1 : M_2 : M_3 : \ldots\ldots$$

[1] *Rigid Dynamics*, 5th edition, Ch. VII.

are determinate when μ has been chosen. They are real when μ is real; and when μ is complex $(\alpha \pm i\beta)$, they take the form $P \pm iQ$.

If now we substitute the values of ψ from (2) in the equations of motion, we get

$$\left.\begin{array}{l}(a_{11}\mu^2 + b_{11}\mu + c_{11}) M_1 + (a_{12}\mu^2 + b_{12}\mu + c_{12}) M_2 + \ldots\ldots = 0 \\ (a_{12}\mu^2 + b_{12}\mu + c_{12}) M_1 + (a_{22}\mu^2 + b_{22}\mu + c_{22}) M_2 + \ldots\ldots = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\} \ldots(4).$$

The first result is obtained by multiplying these equations in order by M_1, M_2, &c. and adding. It may be written

$$A\mu^2 + B\mu + C = 0, \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where

$$A = \tfrac{1}{2}a_{11}M_1^2 + \tfrac{1}{2}a_{22}M_2^2 + a_{12}M_1M_2 + \ldots\ldots\ldots\ldots (6),$$

$$B = \tfrac{1}{2}b_{11}M_1^2 + \tfrac{1}{2}b_{22}M_2^2 + b_{12}M_1M_2 + \ldots\ldots\ldots\ldots (7),$$

$$C = \tfrac{1}{2}c_{11}M_1^2 + \tfrac{1}{2}c_{22}M_2^2 + c_{12}M_1M_2 + \ldots\ldots\ldots\ldots (8).$$

The functions A, B, C, are, it will be seen, the same as we have already denoted by T, F, and V respectively; but the varied notation may be useful as reminding us that there is as yet no limitation upon the nature of these quadratic functions.

The following inferences from (5) are drawn by Routh:—

(α) If A, B, C either be zero, or be one-signed functions of the same sign, the fundamental determinant cannot have a real positive root. For if μ were real, the coefficients M_1, M_2,...... would be real. We should thus have the sum of three positive quantities equal to zero.

(β) If there be no forces of resistance, i.e. if the term B be absent, and if A and C be one-signed and have the same sign, the fundamental determinant cannot have a real root, positive or negative.

(γ) If A, B, C be one-signed functions, but if the sign of B be opposite to that of A and C, the fundamental determinant cannot have a real negative root.

The second equation is obtained as before from (4), except that now the multipliers are N_1, N_2,... appropriate to the root ν. The result may be written

$$A(\mu, \nu)\,\mu^2 + B(\mu, \nu)\,\mu + C(\mu, \nu) = 0 \ldots\ldots\ldots\ldots\ldots (9),$$

where

$$2A(\mu, \nu) = a_{11}M_1N_1 + a_{22}M_2N_2 + \ldots\ldots$$
$$+ a_{12}(M_1N_2 + M_2N_1) + \ldots\ldots\ldots\ldots\ldots (10),$$

with similar suppositions for $B(\mu, \nu)$ and $C(\mu, \nu)$. $A(\mu, \nu)$ is thus a symmetrical function of the M's and N's, so that

$$A(\mu, \nu) = A(\nu, \mu) \qquad \text{.....................(11).}$$

It will be observed that according to this notation $A(\mu, \mu)$ is the same as A in (6).

In like manner

$$A(\mu, \nu)\,\nu^2 + B(\mu, \nu)\,\nu + C(\mu, \nu) = 0 \qquad \text{...............(12),}$$

shewing that μ, ν are both roots of the quadratic, whose coefficients are $A(\mu, \nu)$, $B(\mu, \nu)$, $C(\mu, \nu)$. Accordingly

$$u + \nu = -\frac{B(\mu, \nu)}{A(\mu, \nu)}, \qquad \mu\nu = \frac{C(\mu, \nu)}{A(\mu, \nu)} \qquad \text{..............(13).}$$

We will now suppose that μ, ν are two conjugate complex roots, so that

$$\mu = \alpha + i\beta, \qquad \nu = \alpha - i\beta,$$

where α, β are real. Under these circumstances if M_1, M_2, ... be $P_1 + iQ_1$, $P_2 + iQ_2$, ..., then N_1, N_2, ... will be $P_1 - iQ_1$, $P_2 - iQ_2$, ..., the P's and Q's being real. Thus by (10)

$$\begin{aligned} 2A(\mu, \nu) &= a_{11}(P_1^2 + Q_1^2) + a_{22}(P_2^2 + Q_2^2) + \ldots\ldots \\ &\quad + 2a_{12}(P_1P_2 + Q_1Q_2) + \ldots\ldots \\ &= 2A(P) + 2A(Q) \qquad \text{.......................(14).} \end{aligned}$$

In (14) $A(P)$, $A(Q)$ are functions, such as (6), of real variables. From (13) we now find

$$2\alpha = -\frac{B(P) + B(Q)}{A(P) + A(Q)} \qquad \text{.................(15),}$$

$$\alpha^2 + \beta^2 = \frac{C(P) + C(Q)}{A(P) + A(Q)} \qquad \text{.................(16).}$$

From these Routh deduces the following conclusions:—

(δ) If A and B be one-signed and have the same sign (whether C be a one-signed function or not), then the real part α of every imaginary root must be negative and not zero. But if B be absent, then the real part of every imaginary root is zero.

(ϵ) If A and C be one-signed and have opposite signs, then whatever may be the character of B, there can be no imaginary roots.

It may be remarked that if B do not occur, and if μ^2 and ν^2 be different roots of the determinant, it follows from (9), (12) that

$$A(\mu, \nu) = C(\mu, \nu) = 0 \qquad \text{....................(17).}$$

When the number of degrees of freedom is finite, the fundamental determinant may be expanded in powers of μ, giving an equation $f(\mu) = 0$ of degree $2m$. The condition of stability is that all the real roots and the real parts of all the complex roots should be negative. If, as usual, complex quantities $x + iy$ be represented by points whose co-ordinates are x, y, the condition is that all points representing roots should lie to the left of the axis of y. The application of Cauchy's rule relative to the number of roots within any contour, by taking as the contour the infinite semi-circle on the positive side of the axis of y, is very fully discussed by Routh[1], who has thrown the results into forms convenient for practical application to particular cases.

103 *b*. The theorems of § 103 *a* do not exhaust all that general mechanical principles would lead us to expect as to the character of the roots of the fundamental determinant, and it may be well to pursue the question a little further. We will suppose throughout that A is one-signed and *positive.*

If B and C be both one-signed and positive, we see that the equilibrium is thoroughly stable; for from (α) it follows that there can be no positive root, and from (δ) that no complex root can have its real part positive.

In like manner the equations of § 103 *a* suffice for the case where C is one-signed and positive, B one-signed and negative. By (5) every real root is positive, and by (15) the real part of every complex root. Hence the equilibrium is unstable in *every* mode.

When C is one-signed and negative, all the roots are real (ϵ); but (5) does not tell us whether they are positive or negative. When $B = 0$, we know (§ 87) that the roots occur in pairs of equal numerical value and of opposite sign. In this case therefore there are m positive and m negative roots. We will prove that this state of things cannot be disturbed by B. For if the determinant be expanded, the coefficient of μ^{2m} is the discriminant of A, and the coefficient of μ^0 is the discriminant of C. By supposition neither of these quantities is zero, and thus no root of the equation can be other than finite. Hence as B increases from zero to its actual magnitude as a function of the variables, no root of the equation can change sign, and accordingly there remain m

[1] Adams Prize Essay 1877; *Rigid Dynamics* § 290.

positive and m negative roots. It should be noticed that in this argument there is no restriction upon the character of B.

In the case of a real root the values of M_1, M_2, ... are real, and thus the motion is such as might take place under a constraint reducing the system to one degree of freedom. But if this constraint were actually imposed, there would be *two* corresponding values of μ, being the values given by (5). In general only one of these is applicable to the question in hand. Otherwise it would be possible to define m kinds of constraint, one or other of which would be consistent with any of the $2m$ roots. But this could only happen when the *three* functions A, B, C are simultaneously reducible to sums of squares (§ 97).

When $B = 0$, there are m modes of motion, and two roots for each mode. In the present application to the case where C is one-signed and negative, each of the m modes for $B = 0$ gives one positive and one negative root. The positive root denotes instability, and although the negative root gives a motion which diminishes without limit, the character of instability is considered to attach to the mode as a whole, and all the m modes are said to be unstable. But when B is finite, there are in general $2m$ distinct modes with one root corresponding to each. Of the $2m$ modes m are unstable, but the remaining m modes must be reckoned as stable. On the whole, however, the equilibrium is unstable, so that the influence of B, even when positive, is insufficient to obviate the instability due to the character of C.

We must not prolong much further our discussion of unstable systems, but there is one theorem respecting real roots too fundamental to be passed over. It may be regarded as an extension of that of § 88.

The value of μ corresponding to a given constraint $M_1 : M_2 : \ldots$ is one of the roots of (5): and it follows from (4) that the value of μ is stationary when the imposed constraint coincides with one of the modes of free motion. The effect of small changes in A, B, C may thus be calculated from (5) without allowance for the accompanying change of type.

Let C, being negative for the mode under consideration, undergo numerical increase, while A and B remain unchanged as functions of the co-ordinates. The latter condition requires that the roots of (5), one of which is positive and one negative, should move either both towards zero or both away from zero; and the first condition excludes the former alternative. Whether it be the positive or the negative root of (5) which is the root of the determinant, we infer that the change in question causes the latter to move away from zero.

In like manner if A increase, while B and C remain unchanged, the movement of the root, whether positive or negative, is necessarily towards zero.

Again, if A and C be given, while B increases algebraically as a function of the variables, the movement of the root of the determinant must be in the negative direction.

An algebraic increase in B thus increases the stability, or decreases the instability, in every mode. A numerical increase in C or decrease in A on the other hand promotes the stability of the stable modes and the instability of the unstable modes.

We can do little more than allude to the theorem relating to the effect of a single constraint upon a system for which C is one-signed and negative. Whatever be the nature of B, the $(m-1)$ positive roots of the determinant, appropriate to the system after the constraint has been applied, will separate the m positive roots of the original determinant, and a like proposition will hold for the negative roots. Upon this we may found a generalization of the foregoing conclusions analogous to that of § 92 *a*. Consider an independent vibrator of one degree of freedom for which C is positive, and let the roots of the frequency equation be ν_1, ν_2, one negative and one positive. If we regard this as forming part of the system, we have in all $(2m+2)$ roots. The effect of a constraint by which the two parts of the system are connected will be to reduce the $(2m+2)$ back to $2m$. Of these the m positive will separate the $(m+1)$ quantities formed of the m positive roots of the original equation together with (the positive) ν_2, and a similar proposition will hold for the negative roots. The effect of the vibrator upon the original system is thus to cause a movement of the positive roots towards ν_2, and a movement of the negative roots towards ν_1. This conclusion covers all the previous statements as to the effect of changes in A, B, C upon the values of the roots.

Enough has now been said on the subject of the free vibrations of a system in general. Any further illustration that it may require will be afforded by the discussion of the case of two degrees of freedom, § 112, and by the vibrations of strings and other special bodies with which we shall soon be occupied. We resume the equations (1) § 103, with the view of investigating further the nature of *forced vibrations.*

104. In order to eliminate from the equations all the co-ordinates but one (ψ_1), operate on them in succession with the minor determinants

$$\frac{d\nabla}{de_{11}},\quad \frac{d\nabla}{de_{21}},\quad \frac{d\nabla}{de_{31}},\ \&c.,$$

and add the results together; and in like manner for the other co-ordinates. We thus obtain as the equivalent of the original system of equations

$$\left.\begin{aligned}
\nabla\psi_1 &= \frac{d\nabla}{de_{11}}\Psi_1 + \frac{d\nabla}{de_{21}}\Psi_2 + \frac{d\nabla}{de_{31}}\Psi_3 + \ldots \\
\nabla\psi_2 &= \frac{d\nabla}{de_{12}}\Psi_1 + \frac{d\nabla}{de_{22}}\Psi_2 + \frac{d\nabla}{de_{32}}\Psi_3 + \ldots \\
\nabla\psi_3 &= \frac{d\nabla}{de_{13}}\Psi_1 + \frac{d\nabla}{de_{23}}\Psi_2 + \frac{d\nabla}{de_{33}}\Psi_3 + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\}\ \ldots\ldots\ldots\ldots (1),$$

in which the differentiations of ∇ are to be made without recognition of the equality subsisting between e_{rs} and e_{sr}.

The forces Ψ_1, Ψ_2, &c. are any whatever, subject, of course, to the condition of not producing so great a displacement or motion that the squares of the small quantities become sensible. If, as is often the case, the forces operating be made up of two parts, one constant with respect to time, and the other periodic, it is convenient to separate in imagination the two classes of effects produced. The effect due to the constant forces is exactly the same as if they acted alone, and is found by the solution of a statical problem. It will therefore generally be sufficient to suppose the forces periodic, the effects of any constant forces, such as gravity, being merely to alter the configuration about which the vibrations proper are executed. We may thus without any real loss of generality confine ourselves to periodic, and therefore by Fourier's theorem to harmonic forces.

We might therefore assume as expressions for Ψ_1, &c. circular functions of the time; but, as we shall have frequent occasion to recognise in the course of this work, it is usually more convenient to employ an imaginary exponential function, such as Ee^{ipt}, where E is a constant which may be complex. When the corresponding symbolical solution is obtained, its real and imaginary parts may be separated, and belong respectively to the real and imaginary parts of the data. In this way the analysis gains considerably in brevity, inasmuch as differentiations and alterations of phase are expressed by merely modifying the complex coefficient without changing the form of the function. We therefore write

$$\Psi_1 = E_1 e^{ipt}, \quad \Psi_2 = E_2 e^{ipt}, \text{ \&c.}$$

The minor determinants of the type $\dfrac{d\nabla}{de_{rs}}$ are rational integral functions of the symbol D, and operate on Ψ_1, &c. according to the law

$$f(D)\, e^{ipt} = f(ip)\, e^{ipt} \qquad\qquad (2).$$

Our equations therefore assume the form

$$\nabla\psi_1 = A_1 e^{ipt}, \quad \nabla\psi_2 = A_2 e^{ipt}, \text{ \&c.} \qquad\qquad (3),$$

where A_1, A_2, &c. are certain complex constants. And the symbolical solutions are

$$\psi_1 = A_1 \nabla^{-1} e^{ipt}, \text{ \&c.},$$

or by (2),

$$\psi_1 = A_1 \frac{e^{ipt}}{\nabla(ip)}, \text{ \&c.} \qquad\qquad (4),$$

where $\nabla(ip)$ denotes the result of substituting ip for D in ∇.

Consider first the case of a system exempt from friction.

∇ and its differential coefficients are then *even* functions of D, so that $\nabla(ip)$ is real. Throwing away the imaginary part of the solution, writing $R_1 e^{i\theta_1}$ for A_1, &c., we have

$$\psi_1 = \frac{R_1}{\nabla(ip)} \cos(pt + \theta_1), \text{ \&c.} \qquad\qquad (5).$$

If we suppose that the forces Ψ_1, &c. (in the case of more than one generalized component) have all the same phase, they may be expressed by

$$E_1 \cos(pt + \alpha), \quad E_2 \cos(pt + \alpha), \text{ \&c.};$$

and then, as is easily seen, the co-ordinates themselves agree in phase with the forces:

$$\psi_1 = \frac{R_1}{\nabla(ip)} \cos(pt + \alpha) \qquad\qquad (6).$$

The amplitudes of the vibrations depend among other things on the magnitude of $\nabla(ip)$. Now, if the period of the forces be the same as one of those belonging to the free vibrations, $\nabla(ip) = 0$, and the amplitude becomes infinite. This is, of course, just the case in which it is essential to introduce the consideration of friction, from which no natural system is really exempt.

If there be friction, $\nabla(ip)$ is complex; but it may be divided into two parts—one real and the other purely imaginary, of which the latter depends entirely on the friction. Thus, if we put

$$\nabla(ip) = \nabla_1(ip) + ip\,\nabla_2(ip) \quad\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

∇_1, ∇_2 are even functions of ip, and therefore real. If as before $A_1 = R_1 e^{i\theta_1}$, our solution takes the form

$$\psi_1 = \frac{R_1 e^{i\theta_1} e^{i\gamma} e^{ipt}}{\{\nabla_1(ip)|^2 + p^2 \nabla_2(ip)|^2\}^{\frac{1}{2}}},$$

or, on throwing away the imaginary part,

$$\psi_1 = \frac{R_1 \cos(pt + \theta_1 + \gamma)}{\{\nabla_1(ip)|^2 + p^2 \nabla_2(ip)|^2\}^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

where

$$\tan\gamma = -\frac{p\,\nabla_2(ip)}{\nabla_1(ip)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

We have said that $\nabla_2(ip)$ depends entirely on the friction; but it is not true, on the other hand, that $\nabla_1(ip)$ is exactly the same, as if there had been no friction. However, this is approximately the case, if the friction be small; because any part of $\nabla(ip)$, which depends on the first power of the coefficients of friction, is necessarily imaginary. Whenever there is a coincidence between the period of the force and that of one of the free vibrations, $\nabla_1(ip)$ vanishes, and we have $\tan\gamma = -\infty$, and therefore

$$\psi_1 = \frac{R_1 \sin(pt + \theta_1)}{p\,\nabla_2(ip)} \quad\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

indicating a vibration of large amplitude, only limited by the friction.

On the hypothesis of small friction, θ is in general small, and so also is γ, except in case of approximate equality of periods. With certain exceptions, therefore, the motion has nearly the same (or opposite) phase with the force that excites it.

When a force expressed by a harmonic term acts on a system, the resulting motion is everywhere harmonic, and retains the original period, provided always that the squares of the displacements and velocities may be neglected. This important principle was enunciated by Laplace and applied by him to the theory of the tides. Its great generality was also recognised by Sir John Herschel, to whom we owe a formal demonstration of its truth[1].

If the force be not a harmonic function of the time, the types of vibration in different parts of the system are in general different from each other and from that of the force. The harmonic functions are thus the only ones which preserve their type unchanged, which, as was remarked in the Introduction, is a strong reason for anticipating that they correspond to simple tones.

105. We now turn to a somewhat different kind of forced vibration, where, instead of given *forces* as hitherto, given inexorable *motions* are prescribed.

If we suppose that the co-ordinates $\psi_1, \psi_2, \ldots \psi_r$ are given functions of the time, while the forces of the remaining types $\Psi_{r+1}, \Psi_{r+2}, \ldots \Psi_m$ vanish, the equations of motion divide themselves into two groups, viz.

$$\left.\begin{array}{l} e_{11}\psi_1 + e_{12}\psi_2 + \ldots + e_{1m}\psi_m = \Psi_1 \\ e_{21}\psi_1 + e_{22}\psi_2 + \ldots + e_{2m}\psi_m = \Psi_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ e_{r1}\psi_1 + e_{r2}\psi_2 + \ldots + e_{rm}\psi_m = \Psi_r \end{array}\right\} \ldots\ldots\ldots\ldots (1);$$

and

$$\left.\begin{array}{l} e_{r+1,1}\psi_1 + e_{r+1,2}\psi_2 + \ldots + e_{r+1,m}\psi_m = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ e_{m1}\quad \psi_1 + e_{m2}\quad \psi_2 + \ldots + e_{mm}\quad \psi_m = 0 \end{array}\right\} \ldots\ldots\ldots (2).$$

In each of the $m - r$ equations of the latter group, the first r terms are known explicit functions of the time, and have the same effect as known forces acting on the system. The equations of this group are therefore sufficient to determine the unknown quantities; after which, if required, the forces necessary to maintain the prescribed motion may be determined from the first group. It is obvious that there is no essential difference between the two classes of problems of forced vibrations.

106. The motion of a system devoid of friction and executing simple harmonic vibrations in consequence of prescribed variations of some of the co-ordinates, possesses a peculiarity parallel to those considered in §§ 74, 79. Let

$$\psi_1 = A_1 \cos pt, \quad \psi_2 = A_2 \cos pt, \quad \&c.,$$

[1] *Encyc. Metrop.* art. 323. Also *Outlines of Astronomy*, § 650.

in which the quantities $A_1 \ldots A_r$ are regarded as given, while the remaining ones are arbitrary. We have from the expressions for T and V, § 82,

$$2(T+V) = \tfrac{1}{2}(c_{11}+p^2 a_{11})A_1^2 + \ldots + (c_{12}+p^2 a_{12})A_1A_2 + \ldots$$
$$+ \{\tfrac{1}{2}(c_{11}-p^2 a_{11})A_1^2 + \ldots + (c_{12}-p^2 a_{12})A_1A_2 + \ldots\}\cos 2pt,$$

from which we see that the equations of motion express the condition that E, the variable part of $T+V$, which is proportional to

$$\tfrac{1}{2}(c_{11}-p^2 a_{11})A_1^2 + \ldots + (c_{12}-p^2 a_{12})A_1A_2 + \ldots \quad \ldots\ldots (1),$$

shall be stationary in value, for all variations of the quantities $A_{r+1} \ldots A_m$. Let p'^2 be the value of p^2 natural to the system when vibrating under the restraint defined by the ratios

$$A_1 : A_2 \ldots A_r : A_{r+1} : \ldots A_m;$$

then

$$p'^2 = \{\tfrac{1}{2}c_{11}A_1^2 + \ldots + c_{12}A_1A_2 + \ldots\} \div \{\tfrac{1}{2}a_{11}A_1^2 + \ldots + a_{12}A_1A_2 + \ldots\},$$

so that

$$E = (p'^2 - p^2)\{\tfrac{1}{2}a_{11}A_1^2 + \ldots + a_{12}A_1A_2 + \ldots\} \ldots\ldots\ldots (2).$$

From this we see that if p^2 be certainly less than p'^2; that is, if the prescribed period be greater than any of those natural to the system under the partial constraint represented by

$$A_1 : A_2 \ldots A_r,$$

then E is necessarily positive, and the stationary value—there can be but one—is an absolute minimum. For a similar reason, if the prescribed period be *less* than any of those natural to the partially constrained system, E is an absolute maximum algebraically, but arithmetically an absolute minimum. But when p^2 lies within the range of possible values of p'^2, E may be positive or negative, and the actual value is not the greatest or least possible. Whenever a natural vibration is consistent with the imposed conditions, that will be the vibration assumed. The variable part of $T+V$ is then zero.

For convenience of treatment we have considered apart the two great classes of forced vibrations and free vibrations; but there is, of course, nothing to prevent their coexistence. After the lapse of a sufficient interval of time, the free vibrations always disappear, however small the friction may be. The case of absolutely no friction is purely ideal.

There is one caution, however, which may not be superfluous in respect to the case where given *motions* are forced on the system. Suppose, as before, that the co-ordinates $\psi_1, \psi_2, \ldots \psi_r$ are given. Then the free vibrations, whose existence or non-existence is a matter of indifference so far as the forced motion is concerned, must be understood to be such as the system is capable of, when the co-ordinates $\psi_1 \ldots \psi_r$ *are not allowed to vary from zero.* In order to prevent their varying, forces of the corresponding types must be introduced; so that from one point of view the motion in question may be regarded as forced. But the applied forces are merely of the nature of a constraint; and their effect is the same as a limitation on the freedom of the motion.

106 *a*. The principles of the last sections shew that if $\psi_1, \psi_2 \ldots \psi_r$ be given harmonic functions of the time $A_1 \cos pt$, $A_2 \cos pt, \ldots$, the forces of the other types vanishing, then the motion is determinate, unless p is so chosen as to coincide with one of the values proper to the system when $\psi_1, \psi_2 \ldots \psi_r$ are maintained at zero. As an example, consider the case of a membrane capable of vibrating transversely. If the displacement ψ at every point of the contour be given (proportional to $\cos pt$), then in general the value in the interior is determinate; but an exception occurs if p have one of the values proper to the membrane when vibrating with the contour held at rest. This problem is considered by M. Duhem[1] on the basis of a special analytical investigation by Schwartz. It will be seen that it may be regarded as a particular case of a vastly more general theorem.

A like result may be stated for an elastic solid of which the surface motion (proportional to $\cos pt$) is given at every point. Of course, the motion at the boundary need not be more than partially given. Thus for a mass of air we may suppose given the motion *normal* to a closed surface. The internal motion is then determinate, unless the frequency chosen is one of those proper to the mass, when the surface is made unyielding.

107. Very remarkable reciprocal relations exist between the forces and motions of different types, which may be regarded as extensions of the corresponding theorems for systems in which only V or T has to be considered (§ 72 and §§ 77, 78). If we suppose that all the component forces, except two—Ψ_1 and Ψ_2—are zero, we obtain from § 104,

$$\left.\begin{aligned} \nabla\psi_1 &= \frac{d\nabla}{de_{11}}\Psi_1 + \frac{d\nabla}{de_{21}}\Psi_2 \\ \nabla\psi_2 &= \frac{d\nabla}{de_{12}}\Psi_1 + \frac{d\nabla}{de_{22}}\Psi_2 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\quad (1).$$

[1] *Cours de Physique Mathématique*, Tome Second, p. 190. Paris 1891.

We now consider two cases of motion for the same system; first when Ψ_2 vanishes, and secondly (with dashed letters) when Ψ_1' vanishes. If $\Psi_2 = 0$,

$$\psi_2 = \nabla^{-1}\frac{d\nabla}{de_{12}}\Psi_1 \qquad (2)$$

Similarly, if $\Psi_1' = 0$,

$$\psi_1' = \nabla^{-1}\frac{d\nabla}{de_{21}}\Psi_2' \qquad (3).$$

In these equations ∇ and its differential coefficients are rational integral functions of the symbol D; and since in every case $e_{rs} = e_{sr}$, ∇ is a symmetrical determinant, and therefore

$$\frac{d\nabla}{de_{rs}} = \frac{d\nabla}{de_{sr}} \qquad (4).$$

Hence we see that if a force Ψ_1 act on the system, the co-ordinate ψ_2 is related to it in the same way as the co-ordinate ψ_1' is related to the force Ψ_2', when this latter force is supposed to act alone.

In addition to the motion here contemplated, there may be free vibrations dependent on a disturbance already existing at the moment subsequent to which all new sources of disturbance are included in Ψ; but these vibrations are themselves the effect of forces which acted previously. However small the dissipation may be, there must be an interval of time after which free vibrations die out, and beyond which it is unnecessary to go in taking account of the forces which have acted on a system. If therefore we include under Ψ forces of sufficient remoteness, there are no independent vibrations to be considered, and in this way the theorem may be extended to cases which would not at first sight appear to come within its scope. Suppose, for example, that the system is at rest in its position of equilibrium, and then begins to be acted on by a force of the first type, gradually increasing in magnitude from zero to a finite value Ψ_1, at which point it ceases to increase. If now at a given epoch of time the force be suddenly destroyed and remain zero ever afterwards, free vibrations of the system will set in, and continue until destroyed by friction. At any time t subsequent to the given epoch, the co-ordinate ψ_2 has a value dependent upon t proportional to Ψ_1. The theorem allows us to assert that this value ψ_2 bears the same relation to Ψ_1 as ψ_1' would at the same moment have borne to Ψ_2', if the original cause of the vibrations had been a force of the second type increasing gradually from zero to Ψ_2', and then suddenly vanishing at the given epoch of time. We have already had an example of this in § 101, and a like result obtains when the cause of the original disturbance is an impulse, or, as in the problem of the pianoforte-string, a variable force of finite though short duration. In these applications of our theorem we obtain results relating to free vibrations, considered as the residual effect of forces whose actual operation may have been long before.

108. In an important class of cases the forces Ψ_1 and Ψ_2' are harmonic, and of the same period. We may represent them by $A_1 e^{ipt}$, $A_2' e^{ipt}$, where A_1 and A_2' may be assumed to be *real*, if the forces be in the same phase at the moments compared. The results may then be written

$$\left.\begin{aligned} \psi_2 &= A_1 \frac{d \log \nabla (ip)}{de_{12}} e^{ipt} \\ \psi_1' &= A_2' \frac{d \log \nabla (ip)}{de_{21}} e^{ipt} \end{aligned}\right\} \quad\dots\dots\dots\dots(1),$$

where ip is written for D. Thus,

$$A_2' \psi_2 = A_1 \psi_1' \quad\dots\dots\dots\dots(2).$$

Since the ratio $A_1 : A_2'$ is by hypothesis real, the same is true of the ratio $\psi_1' : \psi_2$; which signifies that the motions represented by those symbols are in the same phase. Passing to real quantities we may state the theorem thus:—

If a force $\Psi_1 = \mathrm{A}_1 \cos \mathrm{pt}$, *acting on the system give rise to the motion* $\psi_2 = \theta \mathrm{A}_1 \cos (\mathrm{pt} - \epsilon)$; *then will a force* $\Psi_2' = \mathrm{A}_2' \cos \mathrm{pt}$ *produce the motion* $\psi_1' = \theta \mathrm{A}_2' \cos (\mathrm{pt} - \epsilon)$.

If there be no friction, ϵ will be zero.

If $A_1 = A_2'$, then $\psi_1' = \psi_2$. But it must be remembered that the forces Ψ_1 and Ψ_2' are not necessarily comparable, any more than the co-ordinates of corresponding types, one of which for example may represent a linear and another an angular displacement.

The reciprocal theorem may be stated in several ways, but before proceeding to these we will give another investigation, not requiring a knowledge of determinants.

If $\Psi_1, \Psi_2, \dots \psi_1, \psi_2, \dots$ and $\Psi_1', \Psi_2', \dots \psi_1', \psi_2', \dots$ be two sets of forces and corresponding displacements, the equations of motion, § 103, give

$$\Psi_1\psi_1' + \Psi_2\psi_2' + \ldots = \psi_1'(e_{11}\psi_1 + e_{12}\psi_2 + e_{13}\psi_3 + \ldots)$$
$$+ \psi_2'(e_{21}\psi_1 + e_{22}\psi_2 + e_{23}\psi_3 + \ldots) + \ldots.$$

Now, if all the forces vary as e^{ipt}, the effect of a symbolic operator such as e_{rs} on any of the quantities ψ is merely to multiply that quantity by the constant found by substituting ip for D in e_{rs}. Supposing this substitution made, and having regard to the relations $e_{rs} = e_{sr}$, we may write

$$\Psi_1\psi_1' + \Psi_2\psi_2' + \ldots = e_{11}\psi_1\psi_1' + e_{22}\psi_2\psi_2' + \ldots$$
$$+ e_{12}(\psi_1'\psi_2 + \psi_2'\psi_1) + \ldots \qquad (3).$$

Hence by the symmetry

$$\Psi_1\psi_1' + \Psi_2\psi_2' + \ldots = \Psi_1'\psi_1 + \Psi_2'\psi_2 + \ldots \qquad (4),$$

which is the expression of the reciprocal relation.

109. In the applications that we are about to make it will be supposed throughout that the forces of all types but two (which we may as well take as the first and second) are zero. Thus

$$\Psi_1\psi_1' + \Psi_2\psi_2' = \Psi_1'\psi_1 + \Psi_2'\psi_2 \qquad (1).$$

The consequences of this equation may be exhibited in three different ways. In the first we suppose that

$$\Psi_2 = 0, \quad \Psi_1' = 0,$$

whence
$$\psi_2 : \Psi_1 = \psi_1' : \Psi_2' \qquad (2),$$

shewing, as before, that the relation of ψ_2 to Ψ_1 in the first case when $\Psi_2 = 0$ is the same as the relation of ψ_1' to Ψ_2' in the second case, when $\Psi_1 = 0$, the identity of relationship extending to phase as well as amplitude.

A few examples may promote the comprehension of a law, whose extreme generality is not unlikely to convey an impression of vagueness.

If P and Q be two points of a horizontal bar supported in any manner (e.g. with one end clamped and the other free), a given harmonic transverse force applied at P will give at any moment the same vertical deflection at Q as would have been found at P, had the force acted at Q.

If we take angular instead of linear displacements, the theorem will run:—A given harmonic *couple* at P will give the same *rotation* at Q as the couple at Q would give at P.

Or if one displacement be linear and the other angular, the result may be stated thus: Suppose for the first case that a harmonic couple acts at P, and for the second that a vertical force of the same period and phase acts at Q, then the linear displacement at Q in the first case has at every moment the same phase as the rotatory displacement at P in the second, and the amplitudes of the two displacements are so related that the maximum couple at P would do the same work in acting over the maximum rotation at P due to the force at Q, as the maximum force at Q would do in acting through the maximum displacement at Q due to the couple at P. In this case the statement is more complicated, as the forces, being of different kinds, cannot be taken equal.

If we suppose the period of the forces to be excessively long, the momentary position of the system tends to coincide with that in which it would be maintained at rest by the then acting forces, and the equilibrium theory becomes applicable. Our theorem then reduces to the statical one proved in § 72.

As a second example, suppose that in a space occupied by air, and either wholly, or partly, confined by solid boundaries, there are two spheres A and B, whose centres have one degree of freedom. Then a periodic force acting on A will produce the same motion in B, as if the parts were interchanged; and this, whatever membranes, strings, forks on resonance cases, or other bodies capable of being set into vibration, may be present in their neighbourhood.

Or, if A and B denote two points of a solid elastic body of any shape, a force parallel to OX, acting at A, will produce the same motion of the point B parallel to OY as an equal force parallel to OY acting at B would produce in the point A, parallel to OX.

Or again, let A and B be two points of a space occupied by air, between which are situated obstacles of any kind. Then a sound originating at A is perceived at B with the same intensity as that with which an equal sound originating at B would be perceived at A.[1] The obstacle, for instance, might consist of a rigid

[1] Helmholtz, *Crelle*, Bd. LVII., 1859. The sounds must be such as in the absence of obstacles would diffuse themselves equally in all directions.

wall pierced with one or more holes. This example corresponds to the optical law that if by any combination of reflecting or refracting surfaces one point can be seen from a second, the second can also be seen from the first. In Acoustics the sound shadows are usually only partial in consequence of the not insignificant value of the wave-length in comparison with the dimensions of ordinary obstacles: and the reciprocal relation is of considerable interest.

A further example may be taken from electricity. Let there be two circuits of insulated wire A and B, and in their neighbourhood any combination of wire-circuits or solid conductors in communication with condensers. A periodic electro-motive force in the circuit A will give rise to the same current in B as would be excited in A if the electro-motive force operated in B.

Our last example will be taken from the theory of conduction and radiation of heat, Newton's law of cooling being assumed as a basis. The temperature at any point A of a conducting and radiating system due to a steady (or harmonic) source of heat at B is the same as the temperature at B due to an equal source at A. Moreover, if at any time the source at B be removed, the whole subsequent course of temperature at A will be the same as it would be at B if the parts of B and A were interchanged.

110. The second way of stating the reciprocal theorem is arrived at by taking in (1) of § 109,

$$\psi_1 = 0, \quad \psi_2' = 0;$$

whence

$$\Psi_1\psi_1' = \Psi_2'\psi_2 \quad \text{.......................... (1)},$$

or

$$\Psi_1 : \psi_2 = \Psi_2' : \psi_1' \quad \text{..................... (2)},$$

shewing that the relation of Ψ_1 to ψ_2 in the first case, when $\psi_1 = 0$, is the same as the relation of Ψ_2' to ψ_1' in the second case, when $\psi_2' = 0$.

Thus in the example of the rod, if the point P be held at rest while a given vibration is imposed upon Q (by a force there applied), the reaction at P is the same both in amplitude and phase as it would be at Q if that point were held at rest and the given vibration were imposed upon P.

So if A and B be two electric circuits in the neighbourhood of any number of others, C, D, ... whether closed or terminating in condensers, and a given periodic current be excited in A by the necessary electro-motive force, the induced electro-motive force in B is the same as it would be in A, if the parts of A and B were interchanged.

The third form of statement is obtained by putting in (1) of § 109,

$$\Psi_1 = 0, \quad \psi_2' = 0;$$

whence
$$\Psi_1'\psi_1 + \Psi_2'\psi_2 = 0 \qquad (3),$$

or
$$\psi_1 : \psi_2 = -\Psi_2' : \Psi_1' \qquad (4),$$

proving that the ratio of ψ_1 to ψ_2 in the first case, when Ψ_2 acts alone, is the negative of the ratio of Ψ_2' to Ψ_1' in the second case, when the forces are so related as to keep ψ_2' equal to zero.

Thus if the point P of the rod be held at rest while a periodic force acts at Q, the reaction at P bears the same numerical ratio to the force at Q as the displacement at Q would bear to the displacement at P, if the rod were caused to vibrate by a force applied at P.

111. The reciprocal theorem has been proved for all systems in which the frictional forces can be represented by the function F, but it is susceptible of a further and an important generalization. We have indeed proved the existence of the function F for a large class of cases where the motion is resisted by forces proportional to the absolute or relative velocities, but there are other sources of dissipation not to be brought under this head, whose effects it is equally important to include; for example, the dissipation due to the conduction or radiation of heat. Now although it be true that the forces in these cases are not *for all possible motions* in a constant ratio to the velocities or displacements, yet in any actual case of periodic motion (τ) they are necessarily periodic, and therefore, whatever their phase, expressible by a sum of two terms, one proportional to the displacement (absolute or relative) and the other proportional to the velocity of the part of the system affected. If the coefficients be the same, not necessarily for all motions whatever, *but for all motions of the period* τ, the function F exists in the only sense required for our present purpose. In fact since it is exclusively with motions of period τ that the theorem is concerned, it is plainly a matter of indifference whether the functions T, F, V are dependent upon τ or not. Thus extended, the theorem is perhaps sufficiently general to cover the whole field of dissipative forces.

It is important to remember that the Principle of Reciprocity is limited to systems which vibrate about a configuration of *equilibrium*, and is therefore not to be applied without reservation to such a problem as that presented by the transmission of sonorous waves through the atmosphere when disturbed by wind. The vibrations must also be of such a character that the square of the motion can be neglected throughout; otherwise our demonstration would not hold good. Other apparent exceptions depend on a misunderstanding of the principle itself. Care must be taken to observe a proper correspondence between the forces and displacements, the rule being that the action of the force over the displacement is to represent *work done*. Thus *couples* correspond to *rotations*, *pressures* to increments of *volume*, and so on.

111 *a*. The substance of the preceding sections is taken from a paper by the Author[1], in which the action of dissipative forces appears first to have been included. Reciprocal theorems of a special character, and with exclusion of dissipation, had been previously given by other writers. One, due to von Helmholtz, has already been quoted. Reference may also be made to the reciprocal theorem of Betti[2], relating to a uniform isotropic elastic solid, upon which bodily and surface forces act. Lamb[3] has shewn that these results and more recent ones of von Helmholtz[4] may be deduced from a very general equation established by Lagrange in the *Mécanique Analytique*.

111 *b*. In many cases of practical interest the external force, in response to which a system vibrates harmonically, is applied at a single point. This may be called the driving-point, and it becomes important to estimate the reaction of the system upon it. When T and F only are sensible, or F and V only, certain general conclusions may be stated, of which a specimen will here be given. For further details reference must be made to a paper by the Author[5].

1 "Some General Theorems relating to Vibrations," *Proc. Math. Soc.*, 1873.

2 *Il Nuovo Cimento*, 1872.

3 *Proc. Math. Soc.*, Vol. xix., p. 144, Jan. 1888.

4 *Crelle*, t. 100, pp. 137, 213. 1886.

5 "The Reaction upon the Driving-point of a System executing Forced Harmonic Oscillations of various Periods." *Phil. Mag.*, May, 1886.

Consider a system, devoid of potential energy, in which the co-ordinate ψ_1 is made to vary by the operation of the harmonic force Ψ_1, proportional to e^{ipt}. The other co-ordinates may be chosen arbitrarily, and it will be very convenient to choose them so that no product of them enters into the expressions for T and F. They would be in fact the normal co-ordinates of the system on the supposition that ψ_1 is constrained (by a suitable force of its own type) to remain zero. The expressions for T and F thus take the following forms:—

$$T = \tfrac{1}{2}a_{11}\dot{\psi}_1^2 + \tfrac{1}{2}a_{22}\dot{\psi}_2^2 + \tfrac{1}{2}a_{33}\dot{\psi}_3^2 + \dots$$
$$+ a_{12}\dot{\psi}_1\dot{\psi}_2 + a_{13}\dot{\psi}_1\dot{\psi}_3 + a_{14}\dot{\psi}_1\dot{\psi}_4 + \dots\dots\dots \quad (1).$$
$$F = \tfrac{1}{2}b_{11}\dot{\psi}_1^2 + \tfrac{1}{2}b_{22}\dot{\psi}_2^2 + \tfrac{1}{2}b_{33}\dot{\psi}_3^2 + \dots$$
$$+ b_{12}\dot{\psi}_1\dot{\psi}_2 + b_{13}\dot{\psi}_1\dot{\psi}_3 + b_{14}\dot{\psi}_1\dot{\psi}_4 + \dots\dots\dots \quad (2).$$

The equations for a force Ψ_1, proportional to e^{ipt}, are accordingly

$$(ipa_{11} + b_{11})\,\dot{\psi}_1 + (ipa_{12} + b_{12})\,\dot{\psi}_2 + (ipa_{13} + b_{13})\,\dot{\psi}_3 + \dots = \Psi_1,$$
$$(ipa_{12} + b_{12})\,\dot{\psi}_1 + (ipa_{22} + b_{22})\,\dot{\psi}_2 = 0,$$
$$(ipa_{13} + b_{13})\,\dot{\psi}_1 + (ipa_{33} + b_{33})\,\dot{\psi}_3 = 0,$$
$$\dots\dots\dots\dots\dots\dots\dots$$

By means of the second and following equations $\dot{\psi}_2$, $\dot{\psi}_3 \dots$ are expressed in terms of $\dot{\psi}_1$. Introducing these values into the first equation, we get

$$\Psi_1/\dot{\psi}_1 = ipa_{11} + b_{11} - \frac{(ipa_{12} + b_{12})^2}{ipa_{22} + b_{22}} - \frac{(ipa_{13} + b_{13})^2}{ipa_{33} + b_{33}} - \dots \quad\dots\dots\dots (3).$$

The ratio $\Psi_1/\dot{\psi}_1$ is a complex quantity, of which the real part corresponds to the work done by the force in a complete period and dissipated in the system. By an extension of electrical language we may call it the *resistance* of the system and denote it by the letter R'. The other part of the ratio is imaginary. If we denote it by $ipL'\dot{\psi}_1$, or $L'\ddot{\psi}_1$, L' will be the moment of inertia, or self-induction of electrical theory. We write therefore

$$\Psi_1 = (R' + ipL')\,\dot{\psi}_1 \dots\dots\dots\dots (4);$$

and the values of R' and L' are to be deduced by separation of the real and the imaginary parts of the right-hand member of (3). In this way we get

$$R' = b_{11} - \Sigma\frac{b_{12}^2}{b_{22}} + p^2\,\Sigma\frac{(a_{12}b_{22} - a_{22}b_{12})^2}{b_{22}\,(b_{22}^2 + p^2a_{22}^2)} \dots\dots\dots (5).$$

This is the value of the resistance as determined by the constitution of the system, and by the frequency of the imposed vibration. Each component of the latter series (which alone involves p) is of the form $\alpha p^2/(\beta + \gamma p^2)$, where α, β, γ are all positive, and (as may be seen most easily by considering its reciprocal) increases continually as p^2 increases from zero to infinity. We conclude that as the frequency of vibration increases, the value of R' increases continuously with it. At the lower limit the motion is determined sensibly by the quantities b (the resistances) only, and the corresponding resultant resistance R' is an absolute minimum, whose value is

$$b_{11} - \Sigma\,(b_{12}^2/b_{22}) \quad\dots\dots\dots\dots\dots\dots\dots\; (6).$$

At the upper limit the motion is determined by the inertia of the component parts without regard to resistances, and the value of R' is

$$b_{11} - \Sigma\,\frac{b_{12}^2}{b_{22}} + \Sigma\,\frac{(a_{12}b_{22} - a_{22}b_{12})^2}{b_{22}a_{22}^2},$$

or

$$b_{11} + \Sigma\left(b_{22}\frac{a_{12}^2}{a_{22}^2} - 2b_{12}\frac{a_{12}}{a_{22}}\right) \dots\dots\dots\dots\dots\dots\; (7).$$

When p is either very large or very small, all the co-ordinates are in the same phase, and (6), (7) may be identified with $2F/\dot{\psi}_1^2$.

Also
$$L' = a_{11} - \Sigma\,\frac{a_{12}^2}{a_{22}} + \Sigma\,\frac{(a_{12}b_{22} - a_{22}b_{12})^2}{a_{22}\,(b_{22}^2 + p^2a_{22}^2)} \quad\dots\dots\dots\; (8).$$

In the latter series every term is positive, and continually diminishes as p^2 increases. Hence every increase of frequency is attended by a diminution of the moment of inertia, which tends ultimately to the minimum corresponding to the disappearance of the dissipative terms.

If p be either very large or very small, (8) identifies itself with $2T/\dot{\psi}_1^2$.

As a simple example take the problem of the reaction upon the primary circuit of the electric currents generated in a neighbouring secondary circuit. In this case the co-ordinates (or rather their rates of increase) are naturally taken to be the currents themselves, so that $\dot{\psi}_1$ is the primary, and $\dot{\psi}_2$ the secondary current. In usual electrical notation we represent the coefficients of self-induction by L, N, and of mutual induction by M, so that

$$T = \tfrac{1}{2}L\dot{\psi}_1^2 + M\dot{\psi}_1\dot{\psi}_2 + \tfrac{1}{2}N\dot{\psi}_2^2,$$

and the resistances by R and S. Thus

$$a_{11} = L, \quad a_{12} = M, \quad a_{22} = N;$$
$$b_{11} = R, \quad b_{12} = 0, \quad b_{22} = S;$$

and (5) and (8) become at once

$$R' = R + \frac{p^2M^2S}{S^2 + p^2N^2} \quad\text{.....................(9)},$$

$$L' = L - \frac{p^2M^2N}{S^2 + p^2N^2} \quad\text{.................. (10)}.$$

These formulæ were given originally by Maxwell, who remarked that the reaction of the currents in the secondary has the effect of increasing the effective resistance and diminishing the effective self-induction of the primary circuit.

If the rate of alternation be very slow, the secondary circuit is without influence. If, on the other hand, the rate be very rapid,

$$R' = R + M^2S/N^2, \qquad L' = L - M^2/N.$$

112. In Chapter III. we considered the vibrations of a system with one degree of freedom. The remainder of the present Chapter will be devoted to some details of the case where the degrees of freedom are two.

If x and y denote the two co-ordinates, the expressions for T and V are of the form

$$\left.\begin{aligned} 2T &= L\dot{x}^2 + 2M\dot{x}\dot{y} + N\dot{y}^2 \\ 2V &= Ax^2 + 2Bxy + Cy^2 \end{aligned}\right\} \quad\text{............... (1)};$$

so that, in the absence of friction, the equations of motion are

$$\left.\begin{aligned} L\ddot{x} + M\ddot{y} + Ax + By &= X \\ M\ddot{x} + N\ddot{y} + Bx + Cy &= Y \end{aligned}\right\} \text{.................. (2)}.$$

When there are no impressed forces, we have for the natural vibrations

$$\left.\begin{aligned} (LD^2 + A)\,x + (MD^2 + B)\,y &= 0 \\ (MD^2 + B)\,x + (ND^2 + C)\,y &= 0 \end{aligned}\right\} \quad\text{............ (3)},$$

D being the symbol of differentiation with respect to time.

If a solution of (3) be $x = l\,e^{\lambda t}$, $y = m\,e^{\lambda t}$, λ^2 is one of the roots of

$$(L\lambda^2 + A)(N\lambda^2 + C) - (M\lambda^2 + B)^2 = 0 \quad\text{......... (4)},$$

or

$$\lambda^4(LN - M^2) + \lambda^2(LC + NA - 2MB) + AC - B^2 = 0 \text{ ...... (5)}.$$

The constants L, M, N; A, B, C, are not entirely arbitrary. Since T and V are essentially positive, the following inequalities must be satisfied:—

$$LN > M^2, \quad AC > B^2 \text{ .... (6)}.$$

Moreover, L, N, A, C must themselves be positive.

We proceed to examine the effect of these restrictions on the roots of (5).

In the first place the three coefficients in the equation are positive. For the first and third, this is obvious from (6). The coefficient of λ^2

$$= (\sqrt{LC} - \sqrt{NA})^2 + 2\sqrt{LNAC} - 2MB,$$

in which, as is seen from (6), $\sqrt{LNAC}$ is necessarily greater than MB. We conclude that the values of λ^2, if real, are both negative.

It remains to prove that the roots are in fact real. The condition to be satisfied is that the following quantity be not negative:—

$$(LC + NA - 2MB)^2 - 4(LN - M^2)(AC - B^2).$$

After reduction this may be brought into the form

$$4(\sqrt{LN}\,.\,B - \sqrt{AC}\,.\,M)^2$$
$$+ (\sqrt{LC} - \sqrt{NA})^2\{(\sqrt{LC} - \sqrt{NA})^2 + 4(\sqrt{LNAC} - MB)\},$$

which shews that the condition is satisfied, since $\sqrt{LNAC} - MB$ is positive. This is the analytical proof that the values of λ^2 are both real and negative; a fact that might have been anticipated without any analysis from the physical constitution of the system whose vibrations they serve to express.

The two values of λ^2 are different, unless *both*

$$\left.\begin{aligned}\sqrt{LN}\,.\,B - \sqrt{AC}\,.\,M &= 0\\ \sqrt{LC} - \sqrt{NA} &= 0\end{aligned}\right\},$$

which require that

$$L : M : N = A : B : C \quad\text{.................}(7).$$

The common spherical pendulum is an example of this case.

By means of a suitable force Y the co-ordinate y may be prevented from varying. The system then loses one degree of freedom, and the period corresponding to the remaining one is in general different from either of those possible before the introduction of Y. Suppose that the types of the motions obtained by thus preventing in turn the variation of y and x are respectively $e^{\mu_1 t}$, $e^{\mu_2 t}$. Then μ_1^2, μ_2^2 are the roots of the equation

$$(L\lambda^2 + A)(N\lambda^2 + C) = 0,$$

being that obtained from (4) by suppressing M and B. Hence (4) may itself be put into the form

$$LN(\lambda^2-\mu_1^2)(\lambda^2-\mu_2^2)=(M\lambda^2+B)^2 \quad\text{..........(8)},$$

which shews at once that neither of the roots of λ^2 can be intermediate in value between μ_1^2 and μ_2^2. A little further examination will prove that one of the roots is greater than both the quantities μ_1^2, μ_2^2, and the other less than both. For if we put

$$f(\lambda^2)=LN(\lambda^2-\mu_1^2)(\lambda^2-\mu_2^2)-(M\lambda^2+B)^2,$$

we see that when λ^2 is very small, f is positive $(AC-B^2)$; when λ^2 decreases (algebraically) to μ_1^2, f changes sign and becomes negative. Between 0 and μ_1^2 there is therefore a root; and also by similar reasoning between μ_2^2 and $-\infty$. We conclude that the tones obtained by subjecting the system to the two kinds of constraint in question are both intermediate in pitch between the tones given by the natural vibrations of the system. In particular cases μ_1^2, μ_2^2 may be equal, and then

$$\lambda^2=\frac{\sqrt{LN}\mu^2\pm B}{\sqrt{LN}\mp M}=\frac{-\sqrt{AC}\pm B}{\sqrt{LN}\mp M} \quad\text{..............(9)}.$$

This proposition may be generalized. *Any* kind of constraint which leaves the system still in possession of one degree of freedom may be regarded as the imposition of a forced relation between the co-ordinates, such as

$$\alpha x+\beta y=0 \quad\text{........................(10)}.$$

Now if $\alpha x+\beta y$, and any other homogeneous linear function of x and y, be taken as new variables, the same argument proves that the single period possible to the system after the introduction of the constraint, is intermediate in value between those two in which the natural vibrations were previously performed. Conversely, the two periods which become possible when a constraint is removed, lie one on each side of the original period.

If the values of λ^2 be equal, which can only happen when

$$L:M:N=A:B:C,$$

the introduction of a constraint has no effect on the period; for instance, the limitation of a spherical pendulum to one vertical plane.

113. As a simple example of a system with two degrees of freedom, we may take a stretched string of length l, itself without inertia, but carrying two equal masses m at distances a and b from one end (Fig. 17). Tension $= T_1$.

Fig. 17.

If x and y denote the displacements,

$$2T = m(\dot{x}^2 + \dot{y}^2),$$

$$2V = T_1 \left\{ \frac{x^2}{a} + \frac{(x-y)^2}{b-a} + \frac{y^2}{l-b} \right\}.$$

Since T and V are not of the same form, it follows that the two periods of vibration are in every case unequal.

If the loads be symmetrically attached, the character of the two component vibrations is evident. In the first, which will have the longer period, the two weights move together, so that x and y remain equal throughout the vibration. In the second x and y are numerically equal, but opposed in sign. The middle point of the string then remains at rest, and the two masses are always to be found on a straight line passing through it. In the first case $x - y = 0$, and in the second $x + y = 0$; so that $x - y$ and $x + y$ are the new variables which must be assumed in order to reduce the functions T and V simultaneously to a sum of squares.

For example, if the masses be so attached as to divide the string into three equal parts,

$$\left.\begin{aligned} 2T &= \frac{m}{2} \{(\dot{x}+\dot{y})^2 + (\dot{x}-\dot{y})^2\} \\ 2V &= \frac{3T_1}{2l} \{(x+y)^2 + 3(x-y)^2\} \end{aligned}\right\} \text{..............(1)},$$

from which we obtain as the complete solution,

$$\left.\begin{aligned} x + y &= A \cos\left(\sqrt{\frac{3T_1}{lm}} \, . \, t + \alpha\right) \\ x - y &= B \cos\left(\sqrt{\frac{9T_1}{lm}} \, . \, t + \beta\right) \end{aligned}\right\} \text{..............(2)},$$

where, as usual, the constants A, α, B, β are to be determined by the initial circumstances.

114. When the two natural periods of a system are nearly equal, the phenomenon of intermittent vibration sometimes presents itself in a very curious manner. In order to illustrate this, we may recur to the string loaded, we will now suppose, with two equal masses at distances from its ends equal to one-fourth of the length. If the middle point of the string were absolutely fixed, the two similar systems on either side of it would be completely independent, or, if the whole be considered as one system, the two periods of vibration would be equal. We now suppose that instead of being absolutely fixed, the middle point is attached to springs, or other machinery, destitute of inertia, so that it is capable of yielding *slightly*. The reservation as to inertia is to avoid the introduction of a third degree of freedom.

From the symmetry it is evident that the fundamental vibrations of the system are those represented by $x+y$ and $x-y$. Their periods are slightly different, because, on account of the yielding of the centre, the potential energy of a displacement when x and y are equal, is less than that of a displacement when x and y are opposite; whereas the kinetic energies are the same for the two kinds of vibration. In the solution

$$\left.\begin{aligned} x+y&=A\cos(n_1t+\alpha)\\ x-y&=B\cos(n_2t+\beta)\end{aligned}\right\}\text{.....................(1)},$$

we are therefore to regard n_1 and n_2 as nearly, but not quite, equal. Now let us suppose that initially x and $\dot{x}$ vanish. The conditions are

$$\left.\begin{aligned} A\cos\alpha+\quad B\cos\beta&=0\\ n_1A\sin\alpha+n_2B\sin\beta&=0\end{aligned}\right\},$$

which give approximately

$$A+B=0,\quad \alpha=\beta.$$

Thus
$$\left.\begin{aligned} x&=A\sin\frac{n_2-n_1}{2}t\quad\sin\left(\frac{n_1+n_2}{2}t+\alpha\right)\\ y&=A\cos\frac{n_2-n_1}{2}t\quad\cos\left(\frac{n_1+n_2}{2}t+\alpha\right)\end{aligned}\right\}\text{.........(2)}.$$

The value of the co-ordinate x is here approximately expressed by a harmonic term, whose amplitude, being proportional to $\sin\frac{1}{2}(n_2-n_1)t$, is a slowly varying harmonic function of the time. The vibrations of the co-ordinates are therefore intermittent, and so adjusted that each amplitude vanishes at the moment that the other is at its maximum.

This phenomenon may be prettily shewn by a tuning fork of very low pitch, heavily weighted at the ends, and firmly held by screwing the stalk into a massive support. When the fork vibrates in the normal manner, the rigidity, or want of rigidity, of the stalk does not come into play; but if the displacements of the two prongs be in the same direction, the slight yielding of the stalk entails a small change of period. If the fork be excited by striking one prong, the vibrations are intermittent, and appear to transfer themselves backwards and forwards between the prongs. Unless, however, the support be very firm, the abnormal vibration, which involves a motion of the centre of inertia, is soon dissipated; and then, of course, the vibration appears to become steady. If the fork be merely held in the hand, the phenomenon of intermittence cannot be obtained at all.

115. The stretched string with two attached masses may be used to illustrate some general principles. For example, the period of the vibration which remains possible when one mass is held at rest, is intermediate between the two free periods. Any increase in either load depresses the pitch of both the natural vibrations, and conversely. If the new load be situated at a point of the string not coinciding with the places where the other loads are attached, nor with the node of one of the two previously possible free vibrations (the other has no node), the effect is still to prolong both the periods already present. With regard to the third finite period, which becomes possible for the first time after the addition of the new load, it must be regarded as derived from one of infinitely small magnitude, of which an indefinite number may be supposed to form part of the system. It is instructive to trace the effect of the introduction of a new load and its gradual increase from zero to infinity, but for this purpose it will be simpler to take the case where there is but one other. At the commencement there is one finite period τ_1, and another of infinitesimal magnitude τ_2. As the load increases τ_2 becomes finite, and both τ_1 and τ_2 continually increase. Let us now consider what happens when the load becomes very great. One of the periods is necessarily large and capable of growing beyond all limit. The other must approach a fixed finite limit. The first belongs to a motion in which the larger mass vibrates nearly as if the other were absent; the second is the period of the vibration of the smaller mass, taking place much as if the larger were fixed. Now since τ_1 and τ_2 can never be equal, τ_1 must be always the greater; and we infer, that as the load becomes continually larger, it is τ_1 that increases indefinitely, and τ_2 that approaches a finite limit.

We now pass to the consideration of forced vibrations.

116. The general equations for a system of two degrees of freedom including friction are

$$\left.\begin{array}{l}(LD^2 + \alpha D + A)\,x + (MD^2 + \beta D + B)\,y = X \\ (MD^2 + \beta D + B)\,x + (ND^2 + \gamma D + C)\,y = Y\end{array}\right\} \dots\dots\dots(1).$$

In what follows we shall suppose that $Y=0$, and that $X = e^{ipt}$. The solution for y is

$$y = -\frac{(B - p^2M + i\beta p)\,e^{ipt}}{(A - p^2L + i\alpha p)(C - p^2N + i\gamma p) - (B - p^2M + i\beta p)^2} \dots(2).$$

If the connection between x and y be of a loose character, the constants M, β, B are small, so that the term $(B - p^2M + i\beta p)^2$ in the denominator may in general be neglected. When this is permissible, the co-ordinate y is the same as if x had been prevented from varying, and a force Y had been introduced whose magnitude is independent of N, γ, and C. But if, in consequence of an approximate isochronism between the force and one of the motions which become possible when x or y is constrained to be zero, either $A - p^2L + i\alpha p$ or $C - p^2N + i\gamma p$ be small, then the term in the denominator containing the coefficients of mutual influence must be retained, being no longer *relatively* unimportant; and the solution is accordingly of a more complicated character.

Symmetry shews that if we had assumed $X=0$, $Y = e^{ipt}$, we should have found the same value for x as now obtains for y. This is the Reciprocal Theorem of § 108 applied to a system capable of two independent motions. The string and two loads may again be referred to as an example.

117. So far for an imposed force. We shall next suppose that it is a *motion* of one co-ordinate ($x = e^{ipt}$) that is prescribed, while $Y=0$; and for greater simplicity we shall confine ourselves to the case where $\beta = 0$. The value of y is

$$y = -\frac{(B - Mp^2)\,e^{ipt}}{C - Np^2 + i\gamma p} \dots\dots\dots\dots\dots\dots\dots (1).$$

Let us now inquire into the reaction of this motion on x. We have

$$(MD^2 + B)\,y = -\frac{(B - Mp^2)^2\,e^{ipt}}{C - Np^2 + i\gamma p} \dots\dots\dots\dots\dots(2).$$

If the real and imaginary parts of the coefficient of e^{ipt} be respectively A' and $i\alpha' p$, we may put

$$(MD^2 + B)\, y = A'x + \alpha'\dot{x} \quad \text{.................(3)},$$

and

$$A' = -\frac{(B - Mp^2)^2\,(C - Np^2)}{(C - Np^2)^2 + \gamma^2 p^2} \quad \text{..............(4)},$$

$$\alpha' = \frac{(B - Mp^2)^2 \gamma}{(C - Np^2)^2 + \gamma^2 p^2} \quad \text{......................(5)}.$$

It appears that the effect of the reaction of y (over and above what would be caused by holding $y = 0$) is represented by changing A into $A + A'$, and α into $\alpha + \alpha'$, where A' and α' have the above values, and is therefore equivalent to the effect of an alteration in the coefficients of spring and friction. These alterations, however, are not constants, *but functions of the period of the motion contemplated*, whose character we now proceed to consider.

Let n be the value of p corresponding to the natural frictionless period of y (x being maintained at zero); so that $C - n^2 N = 0$. Then

$$\left.\begin{aligned} A' &= (B - Mp^2)^2 \frac{N(p^2 - n^2)}{N^2 (p^2 - n^2)^2 + \gamma^2 p^2} \\ \alpha' &= (B - Mp^2)^2 \frac{\gamma}{N^2 (p^2 - n^2)^2 + \gamma^2 p^2} \end{aligned}\right\} \quad \text{...........(6)}.$$

In most cases with which we are practically concerned γ is small, and interest centres mainly on values of p not much differing from n. We shall accordingly leave out of account the variations of the positive factoı $(B - Mp^2)^2$, and in the small term $\gamma^2 p^2$, substitute for p its approximate value n. When p is not nearly equal to n, the term in question is of no importance.

As might be anticipated from the general principle of work, α' is always positive. Its maximum value occurs when $p = n$ nearly, and is then proportional to $1/\gamma n^2$, which varies *inversely* with γ. This might not have been expected on a superficial view of the matter, for it seems rather a paradox that, the greater the friction, the less should be its result. But it must be remembered that γ is only the *coefficient* of friction, and that when γ is small the maximum motion is so much increased that the whole work spent against friction is greater than if γ were more considerable.

But the point of most interest is the dependence of A' on p. If p be less than n, A' is negative. As p passes through the value

n, A' vanishes, and changes sign. When A' is negative, the influence of y is to diminish the recovering power of the vibration x, and we see that this happens when the forced vibration is slower than that natural to y. The tendency of the vibration y is thus to retard the vibration x, if the latter be already the slower, but to accelerate it, if it be already the more rapid, only vanishing in the critical case of perfect isochronism. The attempt to make x vibrate at the rate determined by n is beset with a peculiar difficulty, analogous to that met with in balancing a heavy body with the centre of gravity above the support. On whichever side a slight departure from precision of adjustment may occur the influence of the dependent vibration is always to increase the error. Examples of the instability of pitch accompanying a strong resonance will come across us hereafter; but undoubtedly the most interesting application of the results of this section is to the explanation of the anomalous refraction, by substances possessing a very marked selective absorption, of the two kinds of light situated (in a normal spectrum) immediately on either side of the absorption band[1]. It was observed by Christiansen and Kundt, the discoverers of this remarkable phenomenon, that media of the kind in question (for example, *fuchsine* in alcoholic solution) refract the ray immediately *below* the absorption-band abnormally *in excess*, and that *above* it *in defect*. If we suppose, as on other grounds it would be natural to do, that the intense absorption is the result of an agreement between the vibrations of the kind of light affected, and some vibration proper to the molecules of the absorbing agent, our theory would indicate that for light of somewhat greater period the effect must be the same as a relaxation of the natural elasticity of the ether, manifesting itself by a slower propagation and increased refraction. On the other side of the absorption-band its influence must be in the opposite direction.

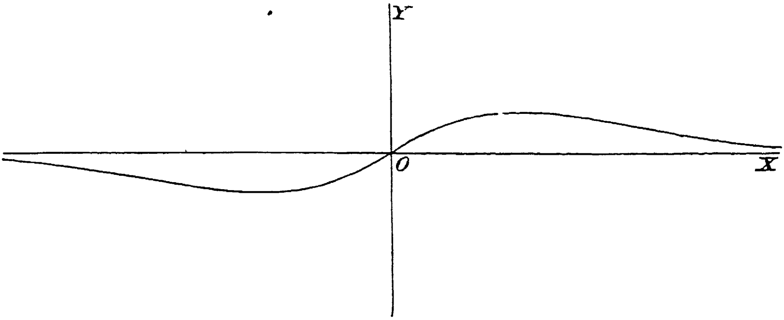
In order to trace the law of connection between A' and p, take for brevity, $\gamma n = a$, $N(p^2 - n^2) = x$, so that

$$A' \propto \frac{x}{x^2 + a^2}.$$

When the sign of x is changed, A' is reversed with it, but preserves its numerical value. When $x = 0$, or $\pm\infty$, A' vanishes.

[1] *Phil. Mag.*, May, 1872. Also Sellmeier, *Pogg. Ann.* t. cxliii. p. 272, 1871.

Fig. 18.



Hence the origin is on the representative curve (Fig. 18), and the axis of x is an asymptote. The maximum and minimum values of A' occur when x is respectively equal to $+a$, or $-a$; and then

$$\frac{x}{x^2+a^2}=\pm\frac{1}{2a}.$$

The corresponding values of p are given by

$$p^2=n^2\pm\frac{\gamma n}{N} \quad\text{.........................}\quad (7).$$

Hence, the smaller the value of a or γ, the greater will be the maximum alteration of A, and the corresponding value of p will approach nearer and nearer to n. It may be well to repeat, that in the optical application a diminished γ is attended by an *increased* maximum absorption. When the adjustment of periods is such as to favour A' as much as possible, the corresponding value of α' is one half of *its* maximum.

CHAPTER VI.

TRANSVERSE VIBRATIONS OF STRINGS.

118. AMONG vibrating bodies there are none that occupy a more prominent position than Stretched Strings. From the earliest times they have been employed for musical purposes, and in the present day they still form the essential parts of such important instruments as the pianoforte and the violin. To the mathematician they must always possess a peculiar interest as the battle-field on which were fought out the controversies of D'Alembert, Euler, Bernoulli and Lagrange, relating to the nature of the solutions of partial differential equations. To the student of Acoustics they are doubly important. In consequence of the comparative simplicity of their theory, they are the ground on which difficult or doubtful questions, such as those relating to the nature of simple tones, can be most advantageously faced; while in the form of a Monochord or Sonometer, they afford the most generally available means for the comparison of pitch.

The 'string' of Acoustics is a perfectly uniform and flexible filament of solid matter stretched between two fixed points—in fact an ideal body, never actually realized in practice, though closely approximated to by most of the strings employed in music. We shall afterwards see how to take account of any small deviations from complete flexibility and uniformity.

The vibrations of a string may be divided into two distinct classes, which are practically independent of one another, if the amplitudes do not exceed certain limits. In the first class the displacements and motions of the particles are *longitudinal*, so that the string always retains its straightness. The potential energy of a displacement depends, not on the whole tension, but on the *changes* of tension which occur in the various parts of the string, due to the increased or diminished extension. In order to calculate it we must know the relation between the extension of a string and the stretching force. The approximate law (given by Hooke) may be expressed by saying that the extension varies as the tension, so that if l and l' denote the natural and the stretched lengths of a string, and T the tension,

$$\frac{l'-l}{l}=\frac{T}{E} \qquad \text{.........................} \quad (1),$$

where E is a constant, depending on the material and the section, which may be interpreted to mean the tension that would be necessary to stretch the string to twice its natural length, if the law applied to so great extensions, which, in general, it is far from doing.

119. The vibrations of the second kind are *transverse*; that is to say, the particles of the string move sensibly in planes perpendicular to the line of the string. In this case the potential energy of a displacement depends upon the general tension, and the small variations of tension accompanying the additional stretching due to the displacement may be left out of account. It is here assumed that the stretching due to the motion may be neglected in comparison with that to which the string is already subject in its position of equilibrium. Once assured of the fulfilment of this condition, we do not, in the investigation of transverse vibrations, require to know anything further of the law of extension.

The most general vibration of the transverse, or lateral, kind may be resolved, as we shall presently prove, into two sets of component normal vibrations, executed in perpendicular planes. Since it is only in the initial circumstances that there can be any distinction, pertinent to the question, between one plane and another, it is sufficient for most purposes to regard the motion as entirely confined to a single plane passing through the line of the string.

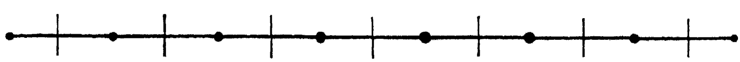
In treating of the theory of strings it is usual to commence with two particular solutions of the partial differential equation, representing the transmission of waves in the positive and negative directions, and to combine these in such a manner as to suit the case of a finite string, whose ends are maintained at rest; neither of the solutions taken by itself being consistent with the existence of *nodes*, or places of permanent rest. This aspect of the question is very important, and we shall fully consider it; but it seems scarcely desirable to found the solution in the first instance on a property so peculiar to a *uniform* string as the undisturbed transmission of waves. We will proceed by the more general method of assuming (in conformity with what was proved in the last chapter) that the motion may be resolved into normal components of the harmonic type, and determining their periods and character by the special conditions of the system.

Towards carrying out this design the first step would naturally be the investigation of the partial differential equation, to which the motion of a continuous string is subject. But in order to throw light on a point, which it is most important to understand clearly,—the connection between finite and infinite freedom, and the passage corresponding thereto between arbitrary constants and arbitrary functions, we will commence by following a somewhat different course.

120. In Chapter III. it was pointed out that the fundamental vibration of a string would not be entirely altered in character, if the mass were concentrated at the middle point. Following out this idea, we see that if the whole string were divided into a number of small parts and the mass of each concentrated at its centre, we might by sufficiently multiplying the number of parts arrive at a system, still of finite freedom, but capable of representing the continuous string with any desired accuracy, so far at least as the lower component vibrations are concerned. If the analytical solution for any number of divisions can be obtained, its limit will give the result corresponding to a uniform string. This is the method followed by Lagrange.

Let l be the length, ρl the whole mass of the string, so that ρ denotes the mass per unit length, T_1 the tension.

Fig. 19.



The length of the string is divided into $m+1$ equal parts (a), so that

$$(m+1)\,a=l \quad\quad\quad (1).$$

At the m points of division equal masses (μ) are supposed concentrated, which are the representatives of the mass of the portions (a) of the string, which they severally bisect. The mass of each terminal portion of length $\frac{1}{2}a$ is supposed to be concentrated at the final points. On this understanding, we have

$$(m+1)\,\mu = \rho l \quad\text{.......................}\quad (2).$$

We proceed to investigate the vibrations of a string, itself devoid of inertia, but loaded at each of m points equidistant (a) from themselves and from the ends, with a mass μ.

If $\psi_1, \psi_2 \ldots\ldots \psi_{m+2}$ denote the lateral displacements of the loaded points, including the initial and final points, we have the following expressions for T and V,

$$T = \tfrac{1}{2}\mu\{\dot\psi_1^2 + \dot\psi_2^2 + \ldots + \dot\psi^2_{m+1} + \dot\psi^2_{m+2}\} \quad\text{..................}\quad (3)$$

$$V = \frac{T_1}{2a}\{(\psi_2-\psi_1)^2 + (\psi_3-\psi_2)^2 + \ldots + (\psi_{m+2}-\psi_{m+1})^2\} \ldots (4),$$

with the conditions that ψ_1 and ψ_{m+2} vanish. These give by Lagrange's Method the m equations of motion,

$$\left.\begin{array}{l} B\psi_1 + A\psi_2 + B\psi_3 = 0 \\ B\psi_2 + A\psi_3 + B\psi_4 = 0 \\ B\psi_3 + A\psi_4 + B\psi_5 = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ B\psi_m + A\psi_{m+1} + B\psi_{m+2} = 0 \end{array}\right\} \quad\text{..............}\quad (5),$$

where $$A = \mu D^2 + \frac{2T_1}{a}, \qquad B = -\frac{T_1}{a} \quad\text{.....................}\quad (6).$$

Supposing now that the vibration under consideration is one of normal type, we assume that ψ_1, ψ_2, &c. are all proportional to $\cos(nt-\epsilon)$, where n remains to be determined. A and B may then be regarded as constants, with a substitution of $-n^2$ for D^2.

If for the sake of brevity we put

$$C = A \div B = -2 + \frac{\mu a n^2}{T_1} \quad\text{.................}\quad (7),$$

the determinantal equation, which gives the values of n^2, assumes the form

$$\begin{vmatrix} C, & 1, & 0, & 0, & 0\ldots\ldots \\ 1, & C, & 1, & 0, & 0\ldots\ldots \\ 0, & 1, & C, & 1, & 0\ldots\ldots \\ 0, & 0, & 1, & C, & 1\ldots\ldots \\ 0, & 0, & 0, & 1, & C\ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \; m \text{ rows} \quad = 0 \qquad \ldots\ldots\ldots\ldots (8).$$

From this equation the values of the roots might be found. It may be proved that, if $C = 2\cos\theta$, the determinant is equivalent to $\sin(m+1)\theta \div \sin\theta$; but we shall attain our object with greater ease directly from (5) by acting on a hint derived from the known results relating to a continuous string, and assuming for trial a particular type of vibration. Thus let a solution be

$$\psi_r = P \sin(r-1)\beta \cos(nt - \epsilon) \qquad \ldots\ldots\ldots\ldots (9),$$

a form which secures that $\psi_1 = 0$. In order that ψ_{m+2} may vanish,

$$(m+1)\beta = s\pi \qquad \ldots\ldots\ldots\ldots (10),$$

where s is an integer. Substituting the assumed values of ψ in the equations (5), we find that they are satisfied, provided that

$$2B\cos\beta + A = 0 \qquad \ldots\ldots\ldots\ldots (11);$$

so that the value of n in terms of β is

$$n = 2\sin\frac{\beta}{2}\sqrt{\frac{T_1}{\mu a}} \qquad \ldots\ldots\ldots\ldots (12).$$

A normal vibration is thus represented by

$$\psi_r = P_s \sin\frac{(r-1)s\pi}{m+1}\cos(n_s t - \epsilon_s) \qquad \ldots\ldots\ldots\ldots (13),$$

where

$$n_s = 2\sqrt{\frac{T_1}{\mu a}}\sin\frac{s\pi}{2(m+1)} \qquad \ldots\ldots\ldots\ldots (14),$$

and P_s, ϵ_s denote arbitrary constants independent of the general constitution of the system. The m admissible values of n are found from (14) by ascribing to s in succession the values 1, 2, 3...m, and are all different. If we take $s = m+1$, ψ_r vanishes, so that this does not correspond to a possible vibration. Greater values of s give only the same periods over again. If $m+1$ be even, one of the values of n—that, namely, corresponding to

$s = \frac{1}{2}(m+1)$,—is the same as would be found in the case of only a single load ($m = 1$). The interpretation is obvious. In the kind of vibration considered every alternate particle remains at rest, so that the intermediate ones really move as though they were attached to the centres of strings of length $2a$, fastened at the ends.

The most general solution is found by putting together all the possible particular solutions of normal type

$$\psi_r = \sum_{s=1}^{s=m} P_s \sin \frac{(r-1)s\pi}{m+1} \cos(n_s t - \epsilon_s) \quad \text{.........}(15),$$

and, by ascribing suitable values to the arbitrary constants, can be identified with the vibration resulting from arbitrary initial circumstances.

Let x denote the distance of the particle r from the end of the string, so that $(r-1)a = x$; then by substituting for μ and a from (1) and (2), our solution may be written,

$$\psi(x) = P_s \sin s\frac{\pi x}{l} \cos(n_s t - \epsilon_s) \quad \text{...........}(16),$$

$$n_s = \frac{2(m+1)}{l}\sqrt{\frac{T_1}{\rho}} \sin \frac{s\pi}{2(m+1)} \quad \text{...........}(17).$$

In order to pass to the case of a continuous string, we have only to put m infinite. The first equation retains its form, and specifies the displacement at any point x. The limiting form of the second is simply

$$n = \frac{s\pi}{l}\sqrt{\frac{T_1}{\rho}} \quad \text{......................}(18),$$

whence for the periodic time,

$$\tau = \frac{2\pi}{n} = \frac{2l}{s}\sqrt{\frac{\rho}{T_1}} \quad \text{......................}(19).$$

The periods of the component tones are thus aliquot parts of that of the gravest of the series, found by putting $s = 1$. The whole motion is in all cases periodic; and the period is $2l\sqrt{(\rho/T_1)}$. This statement, however, must not be understood as excluding a shorter period; for in particular cases any number of the lower components may be absent. All that is asserted is that the above-mentioned interval of time is *sufficient* to bring about a complete recurrence. We defer for the present any further discussion of the important formula (19), but it is interesting to observe the approach to a limit in (17), as m is made successively greater and greater. For this purpose it will be sufficient to take the gravest tone for which $s=1$, and accordingly to trace the variation of $\frac{2(m+1)}{\pi}\sin\frac{\pi}{2(m+1)}$.

The following are a series of simultaneous values of the function and variable :—

m	1	2	3	4	9	19	39
$\frac{2(m+1)}{\pi}\sin\frac{\pi}{2(m+1)}$	·9003	·9549	·9745	·9836	·9959	·9990	·9997

It will be seen that for very moderate values of m the limit is closely approached. Since m is the number of (moveable) loads, the case $m=1$ corresponds to the problem investigated in Chapter III., but in comparing the results we must remember that we there supposed the *whole* mass of the string to be concentrated at the centre. In the present case the load at the centre is only half as great; the remainder being supposed concentrated at the ends, where it is without effect.

From the fact that our solution is general, it follows that any initial form of the string can be represented by

$$\psi(x)=\Sigma_{s=1}^{s=\infty}(P\cos\epsilon)_s\sin s\frac{\pi x}{l} \quad\ldots\ldots\ldots\ldots (20).$$

And, since any form possible for the string at all may be regarded as initial, we infer that any finite single valued function of x, which vanishes at $x=0$ and $x=l$, can be expanded within those limits in a series of sines of $\pi x/l$ and its multiples,—which is a case of Fourier's theorem. We shall presently shew how the more general form can be deduced.

121. We might now determine the constants for a continuous string by integration as in § 93, but it is instructive to solve the problem first in the general case (m finite), and afterwards to proceed to the limit. The initial conditions are

$$\psi(a) = A_1 \sin\frac{\pi a}{l} \quad + A_2 \sin 2\frac{\pi a}{l} \quad + \ldots + A_m \sin m\frac{\pi a}{l},$$

$$\psi(2a) = A_1 \sin 2\frac{\pi a}{l} + A_2 \sin 4\frac{\pi a}{l} \quad + \ldots + A_m \sin 2m\frac{\pi a}{l},$$

....

$$\psi(ma) = A_1 \sin m\frac{\pi a}{l} + A_2 \sin 2m\frac{\pi a}{l} + \ldots + A_m \sin mm\frac{\pi a}{l};$$

where, for brevity, $A_s = P_s \cos\epsilon_s$, and $\psi(a)$, $\psi(2a)$ ...... $\psi(ma)$ are the initial displacements of the m particles.

To determine any constant A_s, multiply the first equation by $\sin(s\pi a/l)$, the second by $\sin(2s\pi a/l)$, &c., and add the results. Then, by Trigonometry, the coefficients of all the constants, except A_s, vanish, while that of $A_s = \frac{1}{2}(m+1)$[1]. Hence

$$A_s = \frac{2}{m+1}\sum_{r=1}^{r=m}\psi(ra)\sin rs\frac{\pi a}{l} \text{..............(1).}$$

We need not stay here to write down the values of B_s (equal to $P_s \sin\epsilon_s$) as depending on the initial velocities. When a becomes infinitely small, ra under the sign of summation ranges by infinitesimal steps from zero to l. At the same time $\frac{1}{m+1} = \frac{a}{l}$, so that writing $ra = x$, $a = dx$, we have ultimately

$$A_s = \frac{2}{l}\int_0^l \psi(x)\sin\left(\frac{s\pi x}{l}\right)dx \text{ .... (2),}$$

expressing A_s in terms of the initial displacements.

122. We will now investigate independently the partial differential equation governing the transverse motion of a perfectly flexible string, on the suppositions (1) that the magnitude of the tension may be considered constant, (2) that the square of the inclination of any part of the string to its initial direction may be neglected. As before, ρ denotes the linear density at any point, and T_1 is the constant tension. Let rectangular co-ordinates be taken parallel, and perpendicular to the string, so that x gives the equilibrium and x, y, z the displaced position of any particle at time t. The forces acting on the element dx are the tensions at

[1] Todhunter's *Int. Calc.*, p. 267.

its two ends, and any impressed forces $Y\rho\, dx$, $Z\rho\, dx$. By D'Alembert's Principle these form an equilibrating system with the reactions against acceleration, $-\rho\, d^2y/dt^2$, $-\rho\, d^2z/dt^2$. At the point x the components of tension are

$$T_1 \frac{dy}{dx}, \quad T_1 \frac{dz}{dx},$$

if the squares of dy/dx, dz/dx be neglected; so that the forces acting on the element dx arising out of the tension are

$$T_1 \frac{d}{dx}\left(\frac{dy}{dx}\right) dx, \quad T_1 \frac{d}{dx}\left(\frac{dz}{dx}\right) dx.$$

Hence for the equations of motion,

$$\left.\begin{aligned} \frac{d^2y}{dt^2} &= \frac{T_1}{\rho}\frac{d^2y}{dx^2} + Y \\ \frac{d^2z}{dt^2} &= \frac{T_1}{\rho}\frac{d^2z}{dx^2} + Z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

from which it appears that the dependent variables y and z are altogether independent of one another.

The student should compare these equations with the corresponding equations of finite differences in § 120. The latter may be written

$$\mu \frac{d^2}{dt^2} \psi(x) = \frac{T_1}{a}\{\psi(x-a) + \psi(x+a) - 2\psi(x)\}.$$

Now in the limit, when a becomes infinitely small,

$$\psi(x-a) + \psi(x+a) - 2\psi(x) = \psi''(x)\, a^2,$$

while $\mu = \rho a$; and the equation assumes ultimately the form

$$\frac{d^2}{dt^2}\psi(x) = \frac{T_1}{\rho}\frac{d^2}{dx^2}\psi(x),$$

agreeing with (1).

In like manner the limiting forms of (3) and (4) of § 120 are

$$T = \tfrac{1}{2}\int \rho \left(\frac{dy}{dt}\right)^2 dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$V = \tfrac{1}{2}T_1 \int \left(\frac{dy}{dx}\right)^2 dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

which may also be proved directly.

The first is obvious from the definition of T. To prove the second, it is sufficient to notice that the potential energy in any configuration is the work required to produce the necessary stretching against the tension T_1. Reckoning from the configuration of equilibrium, we have

$$V = T_1 \int \left(\frac{ds}{dx} - 1\right) dx ;$$

and, so far as the third power of $\frac{dy}{dx}$,

$$\frac{ds}{dx} - 1 = \tfrac{1}{2} \left(\frac{dy}{dx}\right)^2 .$$

123. In most of the applications that we shall have to make, the density ρ is constant, there are no impressed forces, and the motion may be supposed to take place in one plane. We may then conveniently write

$$\frac{T_1}{\rho} = a^2 \quad \text{(1)},$$

and the differential equation is expressed by

$$\frac{d^2y}{d\,(at)^2} = \frac{d^2y}{dx^2} \quad \text{(2)}.$$

If we now assume that y varies as $\cos mat$, our equation becomes

$$\frac{d^2y}{dx^2} + m^2 y = 0 \quad \text{(3)},$$

of which the most general solution is

$$y = (A \sin mx + C \cos mx) \cos mat \quad \text{(4)},$$

This, however, is not the most general harmonic motion of the period in question. In order to obtain the latter, we must assume

$$y = y_1 \cos mat + y_2 \sin mat \quad \text{(5)},$$

where y_1, y_2 are functions of x, not necessarily the same. On substitution in (2) it appears that y_1 and y_2 are subject to equations of the form (3), so that finally

$$\left.\begin{aligned} y = (A \sin mx + C \cos mx) \cos mat \\ + (B \sin mx + D \cos mx) \sin mat \end{aligned}\right\} \quad \text{(6)},$$

an expression containing four arbitrary constants. For any continuous length of string satisfying without interruption the differential equation, this is the most general solution possible, under the condition that the motion at every point shall be simple harmonic. But whenever the string forms part of a system vibrating freely and without dissipation, we know from former chapters that all parts are simultaneously in the same phase, which requires that

$$A : B = C : D \qquad (7);$$

and then the most general vibration of simple harmonic type is

$$y = \{\alpha \sin mx + \beta \cos mx\} \cos (mat - \epsilon) \qquad (8).$$

124. The most simple as well as the most important problem connected with our present subject is the investigation of the free vibrations of a finite string of length l held fast at both its ends. If we take the origin of x at one end, the terminal conditions are that when $x = 0$, and when $x = l$, y vanishes for all values of t. The first requires that in (6) of § 123

$$C = 0, \quad D = 0 \qquad (1);$$

and the second that

$$\sin ml = 0 \qquad (2),$$

or that $ml = s\pi$, where s is an integer. We learn that the only harmonic vibrations possible are such as make

$$m = \frac{s\pi}{l} \qquad (3),$$

and then

$$y = \sin \frac{s\pi x}{l}\left(A \cos \frac{s\pi at}{l} + B \sin \frac{s\pi at}{l}\right) \qquad (4).$$

Now we know *a priori* that whatever the motion may be, it can be represented as a sum of simple harmonic vibrations, and we therefore conclude that the most general solution for a string, fixed at 0 and l, is

$$y = \Sigma_{s=1}^{s=\infty} \sin \frac{s\pi x}{l}\left(A_s \cos \frac{s\pi at}{l} + B_s \sin \frac{s\pi at}{l}\right) \qquad (5).$$

The slowest vibration is that corresponding to $s = 1$. Its period (τ_1) is given by

$$\tau_1 = \frac{2l}{a} = 2l\sqrt{\frac{\rho}{T_1}} \qquad (6).$$

The other components have periods which are aliquot parts of τ_1 :—

$$\tau_s = \tau_1 \div s \qquad (7);$$

so that, as has been already stated, the whole motion is under all circumstances periodic in the time τ_1. The sound emitted constitutes in general a musical *note*, according to our definition of that term, whose pitch is fixed by τ_1, the period of its gravest component. It may happen, however, in special cases that the gravest vibration is absent, and yet that the whole motion is not periodic in any shorter time. This condition of things occurs, if $A_1^2 + B_1^2$ vanish, while, for example, $A_2^2 + B_2^2$ and $A_3^2 + B_3^2$ are finite. In such cases the sound could hardly be called a note; but it usually happens in practice that, when the gravest tone is absent, some other takes its place in the character of fundamental, and the sound still constitutes a note in the ordinary sense, though, of course, of elevated pitch. A simple case is when all the odd components beginning with the first are missing. The whole motion is then periodic in the time $\frac{1}{2}\tau_1$, and if the second component be present, the sound presents nothing unusual.

The pitch of the note yielded by a string (6), and the character of the fundamental vibration, were first investigated on mechanical principles by Brook Taylor in 1715; but it is to Daniel Bernoulli (1755) that we owe the general solution contained in (5). He obtained it, as we have done, by the synthesis of particular solutions, permissible in accordance with his Principle of the Coexistence of Small Motions. In his time the generality of the result so arrived at was open to question; in fact, it was the opinion of Euler, and also, strangely enough, of Lagrange[1], that the series of sines in (5) was not capable of representing an arbitrary function; and Bernoulli's argument on the other side, drawn from the infinite number of the disposable constants, was certainly inadequate[2].

Most of the laws embodied in Taylor's formula (6) had been discovered experimentally long before (1636) by Mersenne. They may be stated thus:—

[1] See Riemann's *Partielle Differential Gleichungen*, § 78.

[2] Dr Young, in his memoir of 1800, seems to have understood this matter quite correctly. He says, "At the same time, as M. Bernoulli has justly observed, since every figure may be infinitely approximated, by considering its ordinates as composed of the ordinates of an infinite number of trochoids of different magnitudes, it may be demonstrated that all these constituent curves would revert to their initial state, in the same time that a similar chord bent into a trochoidal curve would perform a single vibration; and this is in some respects a convenient and compendious method of considering the problem."

(1) For a given string and a given tension, the time varies as the length.

This is the fundamental principle of the monochord, and appears to have been understood by the ancients[1].

(2) When the length of the string is given, the time varies inversely as the square root of the tension.

(3) Strings of the same length and tension vibrate in times, which are proportional to the square roots of the linear density.

These important results may all be obtained by the method of dimensions, if it be assumed that τ depends only on l, ρ, and T_1.

For, if the units of length, time and mass be denoted respectively by $[L]$, $[T]$, $[M]$, the dimensions of these symbols are given by

$$l=[L], \quad \rho=[ML^{-1}], \quad T_1=[MLT^{-2}],$$

and thus (see § 52) the only combination of them capable of representing a time is $T_1^{-\frac{1}{2}}.\rho^{\frac{1}{2}}.l$. The only thing left undetermined is the numerical factor.

125. Mersenne's laws are exemplified in all stringed instruments. In playing the violin different notes are obtained from the same string by shortening its efficient length. In tuning the violin or the pianoforte, an adjustment of pitch is effected with a constant length by varying the tension; but it must be remembered that ρ is not quite invariable.

To secure a prescribed pitch with a string of given material, it is requisite that one relation only be satisfied between the length, the thickness, and the tension; but in practice there is usually no great latitude. The length is often limited by considerations of convenience, and its curtailment cannot always be compensated by an increase of thickness, because, if the tension be not increased proportionally to the section, there is a loss of flexibility, while if the tension be so increased, nothing is effected towards lowering the pitch. The difficulty is avoided in the lower strings of the pianoforte and violin by the addition of a coil of fine wire, whose effect is to impart inertia without too much impairing flexibility.

[1] Aristotle "knew that a pipe or a chord of double length produced a sound of which the vibrations occupied a double time; and that the properties of concords depended on the proportions of the times occupied by the vibrations of the separate sounds."—Young's *Lectures on Natural Philosophy*, Vol. I. p. 404.

For quantitative investigations into the laws of strings, the sonometer is employed. By means of a weight hanging over a pulley, a catgut, or a metallic wire, is stretched across two bridges mounted on a resonance case. A moveable bridge, whose position is estimated by a scale running parallel to the wire, gives the means of shortening the efficient portion of the wire to any desired extent. The vibrations may be excited by plucking, as in the harp, or with a bow (well supplied with rosin), as in the violin.

If the moveable bridge be placed half-way between the fixed ones, the note is raised an octave; when the string is reduced to one-third the note obtained is the twelfth.

By means of the law of lengths, Mersenne determined for the first time the frequencies of known musical notes. He adjusted the length of a string until its note was one of assured position in the musical scale, and then prolonged it under the same tension until the vibrations were slow enough to be counted.

For experimental purposes it is convenient to have two, or more, strings mounted side by side, and to vary in turn their lengths, their masses, and the tensions to which they are subjected. Thus in order that two strings of equal length may yield the interval of the octave, their tensions must be in the ratio of 1 : 4, if the masses be the same; or, if the tensions be the same the masses must be in the reciprocal ratio.

The sonometer is very useful for the numerical determination of pitch. By varying the tension, the string is tuned to unison with a fork, or other standard of known frequency, and then by adjustment of the moveable bridge, the length of the string is determined, which vibrates in unison with any note proposed for measurement. The law of lengths then gives the means of effecting the desired comparison of frequencies.

Another application by Scheibler to the determination of absolute pitch is important. The principle is the same as that explained in Chapter III., and the method depends on deducing the absolute pitch of two notes from a knowledge of both the *ratio* and the *difference* of their frequencies. The lengths of the sonometer string when in unison with a fork, and when giving with it four beats per second, are carefully measured. The ratio of the lengths is the inverse ratio of the frequencies, and the difference of the frequencies is four. From these data the absolute pitch of the fork can be calculated.

The pitch of a string may be calculated also by Taylor's formula from the mechanical elements of the system, but great precautions are necessary to secure accuracy. The tension is produced by a weight, whose mass (expressed with the same unit as ρ) may be called P; so that $T_1 = gP$, where $g = 32{\cdot}2$, if the units of length and time be the foot and the second. In order to secure that the whole tension acts on the vibrating segment, no bridge must be interposed, a condition only to be satisfied by suspending the string vertically. After the weight is attached, a portion of the string is isolated by clamping it firmly at two points, and the length is measured. The mass of the unit of length ρ refers to the stretched state of the string, and may be found indirectly by observing the elongation due to a tension of the same order of magnitude as T_1, and calculating what would be produced by T_1 according to Hooke's law, and by weighing a known length of the string in its normal state. After the clamps have been secured great care is required to avoid fluctuations of temperature, which would seriously influence the tension. In this way Seebeck obtained very accurate results.

126. When a string vibrates in its gravest normal mode, the excursion is at any moment proportional to $\sin(\pi x/l)$, increasing numerically from either end towards the centre; no intermediate point of the string remains permanently at rest. But it is otherwise in the case of the higher normal components. Thus, if the vibration be of the mode expressed by

$$y = \sin\frac{s\pi x}{l}\left(A_s \cos\frac{s\pi at}{l} + B_s \sin\frac{s\pi at}{l}\right),$$

the excursion is proportional to $\sin(s\pi x/l)$, which vanishes at $s-1$ points, dividing the string into s equal parts. These points of no motion are called nodes, and may evidently be touched or held fast without in any way disturbing the vibration. The production of 'harmonics' by lightly touching the string at the points of aliquot division is a well-known resource of the violinist. All component modes are excluded which have not a node at the point touched; so that, as regards pitch, the effect is the same as if the string were securely fastened there.

127. The constants, which occur in the general value of y, § 124, depend on the special circumstances of the vibration, and may be expressed in terms of the initial values of y and $\dot{y}$.

Putting $t = 0$, we find

$$y_0 = \Sigma_{s=1}^{s=\infty} A_s \sin \frac{s\pi x}{l}; \quad \dot{y}_0 = \frac{\pi a}{l} \Sigma_{s=1}^{s=\infty} sB_s \sin \frac{s\pi x}{l} \quad \text{......} (1).$$

Multiplying by $\sin \frac{s\pi x}{l}$, and integrating from 0 to l, we obtain

$$A_s = \frac{2}{l}\int_0^l y_0 \sin \frac{s\pi x}{l} dx; \quad B_s = \frac{2}{\pi a s}\int_0^l \dot{y}_0 \sin \frac{s\pi x}{l} dx \quad \text{......} (2).$$

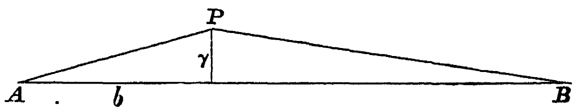
These results exemplify Stokes' law, § 95; for that part of y, which depends on the initial velocities, is

$$y = \Sigma_{s=1}^{s=\infty} \frac{2}{\pi a s} \sin \frac{s\pi x}{l} \sin \frac{s\pi a t}{l} \int_0^l \dot{y}_0 \sin \frac{s\pi x}{l} dx,$$

and from this the part depending on initial displacements may be inferred, by differentiating with respect to the time, and substituting y_0 for $\dot{y}_0$.

When the condition of the string at some one moment is thoroughly known, these formulæ allow us to calculate the motion for all subsequent time. For example, let the string be initially at rest, and so displaced that it forms two sides of a triangle. Then $B_s = 0$; and

Fig. 20.



$$A_s = \frac{2\gamma}{l}\left\{\int_0^b \frac{x}{b} \sin \frac{s\pi x}{l} dx + \int_b^l \frac{l-x}{l-b} \sin \frac{s\pi x}{l} dx\right\}$$

$$= \frac{2\gamma l^2}{\pi^2 s^2 b (l-b)} \sin \frac{s\pi b}{l} \quad \text{............................} (3),$$

on integration.

We see that A_s vanishes, if $\sin (s\pi b/l) = 0$, that is, if there be a node of the component in question situated at P. A more comprehensive view of the subject will be afforded by another mode of solution to be given presently.

128. In the expression for y the coefficients of $\sin(s\pi x/l)$ are the normal co-ordinates of Chapters IV. and V. We will denote them therefore by ϕ_s, so that the configuration and motion of the system at any instant are defined by the values of ϕ_s and $\dot{\phi}_s$ according to the equations

$$\left.\begin{aligned} y &= \phi_1 \sin\frac{\pi x}{l} + \phi_2 \sin\frac{2\pi x}{l} + \ldots + \phi_s \sin\frac{s\pi x}{l} + \ldots \\ \dot{y} &= \dot{\phi}_1 \sin\frac{\pi x}{l} + \dot{\phi}_2 \sin\frac{2\pi x}{l} + \ldots + \dot{\phi}_s \sin\frac{s\pi x}{l} + \ldots \end{aligned}\right\} \quad \ldots\ldots (1).$$

We proceed to form the expressions for T and V, and thence to deduce the normal equations of vibration.

For the kinetic energy,

$$T = \tfrac{1}{2}\rho \int_0^l \dot{y}^2 dx = \tfrac{1}{2}\rho \int_0^l \left\{\Sigma_{s=1}^{s=\infty} \dot{\phi}_s \sin\frac{s\pi x}{l}\right\}^2 dx$$

$$= \tfrac{1}{2}\rho \int_0^l \Sigma_{s=1}^{s=\infty} \dot{\phi}_s^2 \sin^2\frac{s\pi x}{l}\, dx,$$

the product of every pair of terms vanishing by the general property of normal co-ordinates. Hence

$$T = \tfrac{1}{4}\rho l\, \Sigma_{s=1}^{s=\infty} \dot{\phi}_s^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In like manner,

$$V = \tfrac{1}{2}T_1 \int_0^l \left(\frac{dy}{dx}\right)^2 dx = \tfrac{1}{2}T_1 \int_0^l \left\{\Sigma_{s=1}^{s=\infty} \phi_s \frac{s\pi}{l} \cos\frac{s\pi x}{l}\right\}^2 dx$$

$$= \tfrac{1}{4}T_1 l \,.\, \Sigma_{s=1}^{s=\infty} \frac{s^2\pi^2}{l^2} \phi_s^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

These expressions do not presuppose any particular motion, either natural, or otherwise; but we may apply them to calculate the whole energy of a string vibrating naturally, as follows:—If M be the whole mass of the string (ρl), and its equivalent ($a^2\rho$) be substituted for T_1, we find for the sum of the energies,

$$T + V = \tfrac{1}{4}M \,.\, \Sigma_{s=1}^{s=\infty} \left\{\dot{\phi}_s^2 + \frac{s^2\pi^2 a^2}{l^2} \phi_s^2\right\} \quad \ldots\ldots\ldots\ldots (4),$$

or, in terms of A_s and B_s of § 126,

$$T + V = \pi^2 M \,.\, \Sigma_{s=1}^{s=\infty} \frac{A_s^2 + B_s^2}{\tau_s^2} \quad \ldots\ldots\ldots\ldots\ldots (5).$$

If the motion be not confined to the plane of xy, we have merely to add the energy of the vibrations in the perpendicular plane.

Lagrange's method gives immediately the equation of motion

$$\ddot{\phi}_s + \left(\frac{s\pi a}{l}\right)^2 \phi_s = \frac{2}{l\rho}\Phi_s \quad \text{.................. (6),}$$

which has been already considered in § 66. If ϕ_0 and $\dot{\phi}_0$ be the initial values of ϕ and $\dot{\phi}$, the general solution is

$$\phi = \dot{\phi}_0 \frac{\sin nt}{n} + \phi_0 \cos nt$$

$$+ \frac{2}{l\rho n}\int_0^t \sin n(t-t')\,\Phi\,dt' \quad \text{....................(7),}$$

where n is written for $s\pi a/l$.

By definition Φ_s is such that $\Phi_s\,\delta\phi_s$ represents the work done by the impressed forces on the displacement $\delta\phi_s$. Hence, if the force acting at time t on an element of the string $\rho\,dx$ be $\rho\,Y dx$,

$$\Phi_s = \int_0^l \rho\,Y \sin\frac{s\pi x}{l}\,dx \quad \text{....................(8).}$$

In these equations ϕ_s is a linear quantity, as we see from (1); and Φ_s is therefore a force of the ordinary kind.

129. In the applications that we have to make, the only impressed force will be supposed to act in the immediate neighbourhood of one point $x=b$, and may usually be reckoned as a whole, so that

$$\Phi_s = \sin\frac{s\pi b}{l}\int \rho\,Y dx \quad \text{....................(1).}$$

If the point of application of the force coincide with a node of the mode (s), $\Phi_s = 0$, and we learn that the force is altogether without influence on the component in question. This principle is of great importance; it shews, for example, that if a string be at rest in its position of equilibrium, no force applied at its centre, whether in the form of plucking, striking, or bowing, can generate any of the even normal components[1]. If after the operation of the force, its point of application be damped, as by touching it

[1] The observation that a harmonic is not generated, when one of its nodal points is plucked, is due to Young.

with the finger, all motion must forthwith cease; for those components which have not a node at the point in question are stopped by the damping, and those which have, are absent from the beginning[1]. More generally, by damping any point of a sounding string, we stop all the component vibrations which have not, and leave entirely unaffected those which have a node at the point touched.

The case of a string pulled aside at one point and afterwards let go from rest may be regarded as included in the preceding statements. The complete solution may be obtained thus. Let the motion commence at the time $t=0$; from which moment $\Phi_s=0$. The value of ϕ_s at time t is

$$\phi_s = (\phi_s)_0 \cos nt + \frac{1}{n}(\dot{\phi}_s)_0 \sin nt \qquad\qquad (2),$$

where $(\phi_s)_0$, $(\dot{\phi}_s)_0$ denote the initial values of the quantities affected with the suffix s. Now in the problem in hand $(\dot{\phi}_s)_0 = 0$, and $(\phi_s)_0$ is determined by

$$n^2 (\phi_s)_0 = \frac{2}{l\rho}\Phi_s = \frac{2}{l\rho} Y' \sin\frac{s\pi b}{l} \qquad\qquad (3),$$

if Y' denote the force with which the string is held aside at the point b. Hence at time t

$$\phi_s = \frac{2}{l\rho n^2} Y' \sin\frac{s\pi b}{l} \cos nt \qquad\qquad (4),$$

and by (1) of § 128

$$y = \frac{2}{l\rho} Y' . \Sigma_{s=1}^{s=\infty} \sin\frac{s\pi b}{l} \sin\frac{s\pi x}{l} \frac{\cos nt}{n^2} \qquad\qquad (5),$$

where $n = s\pi a/l$.

The symmetry of the expression (5) in x and b is an example of the principle of § 107.

The problem of determining the subsequent motion of a string set into vibration by an impulse acting at the point b, may be treated in a similar manner. Integrating (6) of § 128 over the duration of the impulse, we find ultimately, with the same notation as before,

$$(\dot{\phi}_s)_0 = \frac{2}{l\rho} \sin\frac{s\pi b}{l} Y_1,$$

[1] A like result ensues when the point which is damped is at the same distance from one end of the string as the point of excitation is from the other end.

if $\int Y'dt$ be denoted by Y_1. At the same time $(\phi_s)_0 = 0$, so that by (2) at time t

$$y = \frac{2Y_1}{l\rho} \sum_{s=1}^{s=\infty} \sin\frac{s\pi b}{l} \sin\frac{s\pi x}{l} \frac{\sin nt}{n} \qquad\qquad (6).$$

The series of component vibrations is less convergent for a struck than for a plucked string, as the preceding expressions shew. The reason is that in the latter case the initial value of y is continuous, and only dy/dx discontinuous, while in the former it is $\dot{y}$ itself that makes a sudden spring. See §§ 32, 101.

The problem of a string set in motion by an impulse may also be solved by the general formulæ (7) and (8) of § 128. The force finds the string at rest at $t = 0$, and acts for an infinitely short time from $t = 0$ to $t = \tau'$. Thus $(\phi_s)_0$ and $(\dot{\phi}_s)_0$ vanish, and (7) of § 128 reduces to

$$\phi_s = \frac{2}{l\rho n} \sin nt \int_0^{\tau'} \Phi_s dt',$$

while by (8) of § 128

$$\int_0^{\tau'} \Phi_s dt' = \sin\frac{s\pi b}{l} \int_0^{\tau'} Y'dt' = \sin\frac{s\pi b}{l} Y_1.$$

Hence, as before,

$$\phi_s = \frac{2}{l\rho n} Y_1 \sin\frac{s\pi b}{l} \sin nt \qquad\qquad (7).$$

Hitherto we have supposed the disturbing force to be concentrated at a single point. If it be distributed over a distance β on either side of b, we have only to integrate the expressions (6) and (7) with respect to b, substituting, for example, in (7) in place of $Y_1 \sin(s\pi b/l)$,

$$\int_{b-\beta}^{b+\beta} Y_1' \sin\frac{s\pi b}{l} db.$$

If Y_1' be constant between the limits, this reduces to

$$Y_1' \frac{2l}{s\pi} \sin\frac{s\pi\beta}{l} \sin\frac{s\pi b}{l} \qquad\qquad (8).$$

The principal effect of the distribution of the force is to render the series for y more convergent.

130. The problem which will next engage our attention is that of the pianoforte wire. The cause of the vibration is here the blow of a hammer, which is projected against the string, and after the impact rebounds. But we should not be justified in assuming, as in the last section, that the mutual action occupies so short a time that its duration may be neglected. Measured by the standards of ordinary life the duration of the contact is indeed very small, but here the proper comparison is with the natural periods of the string. Now the hammers used to strike the wires of a pianoforte are covered with several layers of cloth for the express purpose of making them more yielding, with the effect of prolonging the contact. The rigorous treatment of the problem would be difficult, and the solution, when obtained, probably too complicated to be of use; but by introducing a certain simplification Helmholtz has obtained a solution representing all the essential features of the case. He remarks that since the actual yielding of the string must be slight in comparison with that of the covering of the hammer, the law of the force called into play during the contact must be nearly the same as if the string were absolutely fixed, in which case the force would vary very nearly as a circular function. We shall therefore suppose that at the time $t=0$, when there are neither velocities nor displacements, a force $F\sin pt$ begins to act on the string at $x=b$, and continues through half a period of the circular function, that is, until $t=\pi/p$, after which the string is once more free. The magnitude of p will depend on the mass and elasticity of the hammer, but not to any great extent on the velocity with which it strikes the string.

The required solution is at once obtained by substituting for Φ_s in the general formula (7) of § 128 its value given by

$$\Phi_s = F\sin\frac{s\pi b}{l}\sin pt' \quad\text{.....................(1)},$$

the range of the integration being from 0 to π/p. We find $(t>\pi/p)$

$$\phi_s = \frac{2F}{lnp}\sin\frac{s\pi b}{l}\int_0^{\frac{\pi}{p}}\sin n(t-t')\sin pt'\,dt'$$

$$=\frac{4p\cos\frac{n\pi}{2p}}{lpn(p^2-n^2)}\,.\,F\sin\frac{s\pi b}{l}\,.\sin n\left(t-\frac{\pi}{2p}\right)\quad\text{...........(2)},$$

and the final solution for y becomes, if we substitute tor n and ρ their values,

$$y=\frac{4apl^2F}{\pi T_1}\Sigma_{s=1}^{s=\infty}\frac{\cos\frac{s\pi^2a}{2pl}\,.\,\sin\frac{s\pi b}{l}}{s\,(l^2p^2-s^2a^2\pi^2)}\sin\frac{s\pi x}{l}\sin\frac{s\pi a}{l}\left(t-\frac{\pi}{2p}\right)\ldots(3).$$

We see that all components vanish which have a node at the point of excitement, but this conclusion does not depend on any particular law of force. The interest of the present solution lies in the information that may be elicited from it as to the dependence of the resulting vibrations on the duration of contact. If we denote the ratio of this quantity to the fundamental period of the string by ν, so that $\nu = \pi a : 2pl$, the expression for the amplitude of the component s is

$$\frac{8Fl}{\pi^2 T_1} \cdot \frac{\nu \cos (s\pi\nu)}{s(1-4s^2\nu^2)} \sin \frac{s\pi b}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

We fall back on the case of an impulse by putting $\nu = 0$, and

$$Y_1 = \int_0^{\pi/p} F \sin pt \, dt = \frac{2F}{p}.$$

When ν is finite, those components disappear, whose periods are $\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \ldots$ of the duration of contact; and when s is very great, the series converges with s^{-3}. Some allowance must also be made for the finite breadth of the hammer, the effect of which will also be to favour the convergence of the series.

The laws of the vibration of strings may be verified, at least in their main features, by optical methods of observation—either with the vibration-microscope, or by a tracing point recording the character of the vibration on a revolving drum. This character depends on two things,—the mode of excitement, and the point whose motion is selected for observation. Those components do not appear which have nodes either at the point of excitement, or at the point of observation. The former are not generated, and the latter do not manifest themselves. Thus the simplest motion is obtained by plucking the string at the centre, and observing one of the points of trisection, or *vice versa*. In this case the first harmonic which contaminates the purity of the principal vibration is the fifth component, whose intensity is usually insufficient to produce much disturbance.

[The dynamical theory of the vibration of strings may be employed to test the laws of hearing, and the necessary experiments are easily carried out upon a grand pianoforte. Having freed a string, say c, from its damper by pressing the digital, pluck it at one-third of its length. According to Young's theorem the third component vibration is not excited then, and in correspondence with that fact the ear fails to detect the component g'. A slight displacement of the point plucked brings g' in again; and if a resonator (g') be used to assist the ear, it is only with difficulty that the point can be hit with such precision as entirely to extinguish the tone. Experiments of this kind shew that the ear analyses the sound of a string into precisely the same constituents as are found by sympathetic resonance, that is, into simple tones, according to Ohm's definition of this conception. Such experiments are also well adapted to shew that it is not a mere play of imagination when we hear overtones, as some people believe it is on hearing them for the first time[1].

If, after the string has been sounded loudly by striking the digital, it be touched with the finger at one of the points of trisection, all components are stopped except the 3rd, 6th, &c., so that these are left isolated. The inexperienced observer is usually surprised by the loudness of the residual sound, and begins to appreciate the large part played by overtones.]

131. The case of a periodic force is included in the general solution of § 128, but we prefer to follow a somewhat different method, in order to make an extension in another direction. We have hitherto taken no account of dissipative forces, but we will now suppose that the motion of each element of the string is resisted by a force proportional to its velocity. The partial differential equation becomes

$$\frac{d^2y}{dt^2} + \kappa \frac{dy}{dt} = a^2 \frac{d^2y}{dx^2} + Y \quad\quad (1),$$

by means of which the subject may be treated. But it is still simpler to avail ourselves of the results of the last chapter, remarking that in the present case the dissipation-function F is of the same form as T. In fact

$$F = \tfrac{1}{4}\rho\kappa l \,.\, \Sigma_{s=1}^{s=\infty} \dot{\phi}_s^2 \quad\quad (2),$$

where ϕ_1, ϕ_2,... are the normal co-ordinates, by means of which T and V are reduced to sums of squares. The equations of motion are therefore simply

$$\ddot{\phi}_s + \kappa\dot{\phi}_s + n^2\phi_s = \frac{2}{l\rho}\Phi_s \quad\quad (3),$$

[1] Helmholtz, Ch. IV.; Brandt, *Pogg. Ann.*, Vol. CXII. p. 324, 1861.

of the same form as obtains for systems with but one degree of freedom. It is only necessary to add to what was said in Chapter III., that since κ is independent of s, the natural vibrations subside in such a manner that the amplitudes maintain their relative values.

If a periodic force $F\cos pt$ act at a single point, we have

$$\Phi_s = F\sin\frac{s\pi b}{l}\cos pt \quad\text{......................}(4),$$

and § 46
$$\phi_s = \frac{2F\sin\epsilon}{l\rho p\kappa}\sin\frac{s\pi b}{l}\cos(pt-\epsilon) \quad\text{..............}(5),$$

where
$$\tan\epsilon = \frac{p\kappa}{n^2-p^2} \quad\text{..............................}(6).$$

If among the natural vibrations there be any one nearly isochronous with $\cos pt$, then a large vibration of that type will be forced, unless indeed the point of excitement should happen to fall near a node. In the case of exact coincidence, the component vibration in question vanishes; for no force applied at a node can generate it, under the present law of friction, which however, it may be remarked, is very special in character. If there be no friction, $\kappa = 0$, and

$$l\rho\phi_s = \frac{2F}{n^2-p^2}\sin\frac{s\pi b}{l}\cos pt \quad\text{.................}(7),$$

which would make the vibration infinite, in the case of perfect isochronism, unless $\sin(s\pi b/l) = 0$.

The value of y is here, as usual

$$y = \phi_1\sin\frac{\pi x}{l} + \phi_2\sin\frac{2\pi x}{l} + \phi_3\sin\frac{3\pi x}{l} + \quad\text{..............}(8).$$

132. The preceding solution is an example of the use of normal co-ordinates in a problem of forced vibrations. It is of course to free vibrations that they are more especially applicable, and they may generally be used with advantage throughout, whenever the system after the operation of various forces is ultimately left to itself. Of this application we have already had examples.

In the case of vibrations due to periodic forces, one advantage of the use of normal co-ordinates is the facility of comparison with the *equilibrium theory*, which it will be remembered is the theory of the motion on the supposition that the inertia of the system may be left out of account. If the value of the normal co-ordinate ϕ_s on the equilibrium theory be $A_s \cos pt$, then the actual value will be given by the equation

$$\phi_s = \frac{n^2 A_s}{n^2 - p^2} \cos pt \qquad (1),$$

so that, when the result of the equilibrium theory is known and can readily be expressed in terms of the normal co-ordinates, the true solution with the effects of inertia included can at once be written down.

In the present instance, if a force $F \cos pt$ of very long period act at the point b of the string, the result of the equilibrium theory, in accordance with which the string would at any moment consist of two straight portions, will be

$$l\rho\phi_s = \frac{2F}{n^2} \sin \frac{s\pi b}{l} \cos pt \qquad (2).$$

from which the actual result for all values of p is derived by simply writing $(n^2 - p^2)$ in place of n^2.

The value of y in this and similar cases may however be expressed in finite terms, and the difficulty of obtaining the finite expression is usually no greater than that of finding the form of the normal functions when the system is free. Thus in the equation of motion

$$\frac{d^2y}{dt^2} = a^2 \frac{d^2y}{dx^2} + Y,$$

suppose that Y varies as $\cos mat$. The forced vibration will then satisfy

$$\frac{d^2y}{dx^2} + m^2 y = -\frac{1}{a^2} Y \qquad (3).$$

If $Y = 0$, the investigation of the normal functions requires the solution of

$$\frac{d^2y}{dx^2} + m^2 y = 0,$$

and a subsequent determination of m to suit the boundary conditions. In the problem of forced vibrations m is given, and we have only to supplement any particular solution of (3) with the complementary function containing two arbitrary constants. This function, apart from the value of m and the ratio of the constants, is of the same form as the normal functions; and all that remains to be effected is the determination of the two constants in accordance with the prescribed boundary conditions which the complete solution must satisfy. Similar considerations apply in the case of any continuous system.

133. If a periodic force be applied at a single point, there are two distinct problems to be considered; the first, when at the point $x = b$, a given periodic force acts; the second, when it is the actual motion of the point b that is obligatory. But it will be convenient to treat them together.

The usual differential equation

$$\frac{d^2y}{dt^2} + \kappa \frac{dy}{dt} = a^2 \frac{d^2y}{dx^2} \quad \text{.......................(1)},$$

is satisfied over both the parts into which the string is divided at b, but is violated in crossing from one to the other.

In order to allow for a change in the arbitrary constants, we must therefore assume distinct expressions for y, and afterwards introduce the two conditions which must be satisfied at the point of junction. These are

(1) That there is no discontinuous change in the value of y;

(2) That the resultant of the tensions acting at b balances the impressed force.

Thus, if $F \cos pt$ be the force, the second condition gives

$$T_1 \Delta \left(\frac{dy}{dx}\right) + F \cos pt = 0 \quad \text{....................(2)},$$

where $\Delta (dy/dx)$ denotes the alteration in the value of dy/dx incurred in crossing the point $x = b$ in the positive direction.

We shall, however, find it advantageous to replace $\cos pt$ by the complex exponential e^{ipt}, and finally discard the imaginary part, when the symbolical solution is completed. On the assumption that y varies as e^{ipt}, the differential equation becomes

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \quad \text{.........................(3)};$$

where λ^2 is the complex constant,

$$\lambda^2 = \frac{1}{a^2} (p^2 - ip\kappa) \quad \text{......................(4)}.$$

The most general solution of (3) consists of two terms, proportional respectively to $\sin\lambda x$, and $\cos\lambda x$; but the condition to be satisfied at $x=0$ shews that the second does not occur here. Hence if γe^{ipt} be the value of y at $x=b$,

$$y=\gamma\frac{\sin\lambda x}{\sin\lambda b}\cdot e^{ipt}\qquad\qquad(5),$$

is the solution applying to the first part of the string from $x=0$ to $x=b$. In like manner it is evident that for the second part we shall have

$$y=\gamma\frac{\sin\lambda(l-x)}{\sin\lambda(l-b)}e^{ipt}\qquad\qquad(6).$$

If γ be given, these equations constitute the symbolical solution of the problem; but if it be the force that is given, we require further to know the relation between it and γ.

Differentiation of (5) and (6) and substitution in the equation analogous to (2) gives

$$\gamma=\frac{F}{T_1}\frac{\sin\lambda b\ \sin\lambda(l-b)}{\lambda\sin\lambda l}\qquad\qquad(7).$$

Thus

$$\left.\begin{aligned}y&=\frac{F}{T_1}\frac{\sin\lambda x\ \sin\lambda(l-b)}{\lambda\sin\lambda l}e^{ipt}\\&\qquad\text{from } x=0 \text{ to } x=b\\y&=\frac{F}{T_1}\frac{\sin\lambda(l-x)\ \sin\lambda b}{\lambda\sin\lambda l}e^{ipt}\\&\qquad\text{from } x=b \text{ to } x=l\end{aligned}\right\}\ldots(8)^1.$$

These equations exemplify the general law of reciprocity proved in the last chapter; for it appears that the motion at x due to the force at b is the same as would have been found at b, had the force acted at x.

In discussing the solution we will take first the case in which there is no friction. The coefficient κ is then zero; while λ is real, and equal to p/a. The real part of the solution, corresponding to the force $F\cos pt$, is found by simply putting $\cos pt$ for e^{ipt} in (8), but it seems scarcely necessary to write the equations again for the sake of so small a change. The same remark applies to the forced motion given in terms of γ.

It appears that the motion becomes infinite in case the force

[1] Donkin's *Acoustics*, p. 121.

is isochronous with one of the natural vibrations of the entire string, unless the point of application be a node; but in practice it is not easy to arrange that a string shall be subject to a force of given magnitude. Perhaps the best method would be to attach a small mass of iron, attracted periodically by an electro-magnet, whose coils are traversed by an intermittent current. But unless some means of compensation were devised, the mass would have to be very small in order to avoid its inertia introducing a new complication.

A better approximation may be obtained to the imposition of an obligatory motion. A massive fork of low pitch, excited by a bow or sustained in permanent operation by electro-magnetism, executes its vibrations in approximate independence of the reactions of any light bodies which may be connected with it. In order therefore to subject any point of a string to an obligatory transverse motion, it is only necessary to attach it to the extremity of one prong of such a fork, whose plane of vibration is perpendicular to the length of the string. This method of exhibiting the forced vibrations of a string appears to have been first used by Melde[1].

Another arrangement, better adapted for aural observation, has been employed by Helmholtz. The end of the stalk of a powerful tuning-fork, set into vibration with a bow, or otherwise, is pressed against the string. It is advisable to file the surface, which comes into contact with the string, into a suitable (saddle-shaped) form, the better to prevent slipping and jarring.

Referring to (5) we see that, if $\sin \lambda b$ vanished, the motion (according to this equation) would become infinite, which may be taken to prove that in the case contemplated, the motion would really become great,—so great that corrections, previously insignificant, rise into importance. Now $\sin \lambda b$ vanishes, when the force is isochronous with one of the natural vibrations of the first part of the string, supposed to be held fixed at 0 and b.

When a fork is placed on the string of a monochord, or other instrument properly provided with a sound-board, it is easy to find by trial the places of maximum resonance. A very slight displacement on either side entails a considerable falling off in the volume of the sound. The points thus determined divide the string into a number of equal parts, of such length that the natural note of any one of them (when fixed at both ends) is

[1] Pogg. *Ann.* cix. p. 193, 1859.

the same as the note of the fork, as may readily be verified. The important applications of resonance which Helmholtz has made to purify a simple tone from extraneous accompaniment will occupy our attention later.

134. Returning now to the general case where λ is complex, we have to extract the real parts from (5), (6), (8) of § 133. For this purpose the sines which occur as factors, must be reduced to the form $Re^{i\epsilon}$. Thus let

$$\sin \lambda x = R_x e^{i\epsilon_x} \qquad\qquad (1),$$

with a like notation for the others. From (5) § 133 we shall thus obtain

$$y = \gamma \frac{R_x}{R_b} \cos (pt + \epsilon_x - \epsilon_b) \qquad\qquad (2),$$

from $x = 0$ to $x = b$,

and from (6) § 133

$$y = \gamma \frac{R_{l-x}}{R_{l-b}} \cos (pt + \epsilon_{l-x} - \epsilon_{l-b}),$$

from $x = b$ to $x = l$,

corresponding to the obligatory motion $y = \gamma \cos pt$ at b.

By a similar process from (8) § 133, if

$$\lambda = \alpha + i\beta \qquad\qquad (3),$$

we should obtain

$$\left.\begin{aligned} y &= \frac{F}{T_1} \frac{R_x \,.\, R_{l-b}}{\sqrt{(\alpha^2 + \beta^2)} \,.\, R_l} \cos \left(pt + \epsilon_x + \epsilon_{l-b} - \epsilon_l - \tan^{-1} (\beta/\alpha) \right) \\ &\qquad\qquad \text{from } x = 0 \text{ to } x = b \\ y &= \frac{F}{T_1} \frac{R_{l-x} \,.\, R_b}{\sqrt{(\alpha^2 + \beta^2)} \,.\, R_l} \cos \left(pt + \epsilon_{l-x} + \epsilon_b - \epsilon_l - \tan^{-1} (\beta/\alpha) \right) \\ &\qquad\qquad \text{from } x = b \text{ to } x = \iota \end{aligned}\right\} \ldots (4),$$

corresponding to the impressed force $F \cos pt$ at b. It remains to obtain the forms of R_x, ϵ_x, &c.

The values of α and β are determined by

$$\alpha^2 - \beta^2 = \frac{p^2}{a^2}, \quad 2\alpha\beta = -\frac{p\kappa}{a^2} \qquad\qquad (5),$$

and $\sin \lambda x = \sin \alpha x \cos i\beta x + \cos \alpha x \sin i\beta x$

$$= \sin \alpha x \, \frac{e^{\beta x} + e^{-\beta x}}{2} + i \cos \alpha x \, \frac{e^{\beta x} - e^{-\beta x}}{2},$$

so that

$$R_x^2 = \sin^2 \alpha x \left(\frac{e^{\beta x} + e^{-\beta x}}{2}\right)^2 + \cos^2 \alpha x \left(\frac{e^{\beta x} - e^{-\beta x}}{2}\right)^2 \ldots (6),$$

$$\tan \epsilon_x = \frac{e^{\beta x} - e^{-\beta x}}{e^{\beta x} + e^{-\beta x}} \cot \alpha x \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

while

$$\sqrt{(\alpha^2 + \beta^2)} = \frac{1}{a} \sqrt[4]{(p^4 + p^2\kappa^2)} \ \ldots\ldots\ldots\ldots\ldots (8).$$

This completes the solution.

If the friction be very small, the expressions may be simplified. For instance in this case, to a sufficient approximation.

$$\alpha = p/a, \quad \beta = -\kappa/2a, \quad \sqrt{(\alpha^2 + \beta^2)} = p/a,$$

$$\tfrac{1}{2}(e^{\beta x} + e^{-\beta x}) = 1, \quad \tfrac{1}{2}(e^{\beta x} - e^{-\beta x}) = -\kappa x/2a\,;$$

so that, corresponding to the obligatory motion at b $y = \gamma \cos pt$, the amplitude of the motion between $x = 0$ and $x = b$ is, approximately

$$\gamma \left\{ \frac{\sin^2 \dfrac{px}{a} + \dfrac{\kappa^2 x^2}{4a^2} \cos^2 \dfrac{px}{a}}{\sin^2 \dfrac{pb}{a} + \dfrac{\kappa^2 b^2}{4a^2} \cos^2 \dfrac{pb}{a}} \right\}^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots (9),[1]$$

which becomes great, but not infinite, when $\sin(pb/a) = 0$, or the point of application is a node.

If the imposed force, or motion, be not expressed by a single harmonic term, it must first be resolved into such. The preceding solution may then be applied to each component separately, and the results added together. The extension to the case of more than one point of application of the impressed forces is also obvious. To obtain the most general solution satisfying the conditions, the expression for the natural vibrations must also be added; but these become reduced to insignificance after the motion has been in progress for a sufficient time.

The law of friction assumed in the preceding investigation is the only one whose results can be easily followed deductively, and it is sufficient to give a general idea of the effects of dissipative forces on the motion of a string. But in other respects the conclusions drawn from it possess a fictitious simplicity, depending on the fact that F—the dissipation-function—is similar in form to T, which makes the normal co-ordinates independent of each other.

1 Reference may be made to a paper by Morton & Vinycomb, *Phil. Mag.* Nov. 1904. Editor.

In almost any other case (for example, when but a single point of the string is retarded by friction) there are no normal co-ordinates properly so called. There exist indeed elementary types of vibration into which the motion may be resolved, and which are perfectly independent, but these are essentially different in character from those with which we have been concerned hitherto, for the various parts of the system (as affected by one elementary vibration) are not simultaneously in the same phase. Special cases excepted, no linear transformation of the co-ordinates (with real coefficients) can reduce T, F, and V together to a sum of squares.

If we suppose that the string has no inertia, so that $T=0$, F and V may then be reduced to sums of squares. This problem is of no acoustical importance, but it is interesting as being mathematically analogous to that of the conduction and radiation of heat in a bar whose ends are maintained at a constant temperature.

135. Thus far we have supposed that at two fixed points, $x=0$ and $x=l$, the string is held at rest. Since absolute fixity cannot be attained in practice, it is not without interest to inquire in what manner the vibrations of a string are liable to be modified by a yielding of the points of attachment; and the problem will furnish occasion for one or two remarks of importance. For the sake of simplicity we shall suppose that the system is symmetrical with reference to the centre of the string, and that each extremity is attached to a mass M (treated as unextended in space), and is urged by a spring (μ) towards the position of equilibrium. If no frictional forces act, the motion is necessarily resolvable into normal vibrations. Assume

$$y = \{\alpha \sin mx + \beta \cos mx\} \cos (mat - \epsilon) \qquad \text{.........(1)}.$$

The conditions at the ends are that

$$\left.\begin{array}{lll} \text{when} & x=0, & M\ddot{y}+\mu y = T_1 \dfrac{dy}{dx} \\ \text{when} & x=l, & M\ddot{y}+\mu y = -T_1 \dfrac{dy}{dx} \end{array}\right\} \qquad \text{..............(2)},$$

which give

$$\frac{\alpha}{\beta} = \frac{\beta \tan ml - \alpha}{\alpha \tan ml + \beta} = \frac{\mu - Ma^2m^2}{mT_1} \qquad \text{............(3)},$$

two equations, sufficient to determine m, and the ratio of β to α. Eliminating the latter ratio, we find

$$\tan ml = \frac{2\nu}{1-\nu^2} \quad\text{.......................(4)},$$

if for brevity we write ν for $\dfrac{\mu - Ma^2m^2}{mT_1}$.

Equation (3) has an infinite number of roots, which may be found by writing $\tan\theta$ for ν, so that $\tan ml = \tan 2\theta$, and the result of adding together *all* the corresponding particular solutions, each with its two arbitrary constants α and ϵ, is necessarily the most general solution of which the problem is capable, and is therefore adequate to represent the motion due to an arbitrary initial distribution of displacement and velocity. We infer that any function of x may be expanded between $x=0$ and $x=l$ in a series of terms

$$\phi_1(\nu_1 \sin m_1x + \cos m_1x) + \phi_2(\nu_2 \sin m_2x + \cos m_2x) + \text{......(5)},$$

m_1, m_2, &c. being the roots of (3) and ν_1, ν_2, &c. the corresponding values of ν. The quantities ϕ_1, ϕ_2, &c. are the *normal* co-ordinates of the system.

From the symmetry of the system it follows that in each normal vibration the value of y is numerically the same at points equally distant from the middle of the string, for example, at the two ends, where $x=0$ and $x=l$. Hence $\nu_s \sin m_sl + \cos m_sl = \pm 1$, as may be proved also from (4).

The kinetic energy T of the whole motion is made up of the energy of the string, and that of the masses M. Thus

$$T = \tfrac{1}{2}\rho\int_0^l \{\Sigma\,\dot\phi\,(\nu\sin mx + \cos mx)\}^2\,dx$$
$$+\tfrac{1}{2}M\{\dot\phi_1 + \dot\phi_2 + \ldots\}^2 + \tfrac{1}{2}M\{\dot\phi_1(\nu_1\sin m_1l + \cos m_1l) + \ldots\}^2.$$

But by the characteristic property of normal co-ordinates, terms containing their products cannot be really present in the expression for T, so that

$$\rho\int_0^l(\nu_r\sin m_rx + \cos m_rx)(\nu_s \sin m_sx + \cos m_sx)\,dx$$
$$+M + M(\nu_r\sin m_rl + \cos m_rl)(\nu_s\sin m_sl + \cos m_sl) = 0\text{......(6)},$$

if r and s be different.

This theorem suggests how to determine the arbitrary constants, so that the series (5) may represent an arbitrary function y. Take the expression

$$\rho\int_0^l y\,(\nu_s \sin m_s x + \cos m_s x)\,dx + My_0 + My_l\,(\nu_s \sin m_s l + \cos m_s l) \ldots (7),$$

and substitute in it the series (5) expressing y. The result is a series of terms of the type

$$\rho\int_0^l \phi_r\,(\nu_r \sin m_r x + \cos m_r x)\,(\nu_s \sin m_s x + \cos m_s x)\,dx$$

$$+ M\phi_r + M\phi_r\,(\nu_r \sin m_r l + \cos m_r l)\,(\nu_s \sin m_s l + \cos m_s l),$$

all of which vanish by (6), except the one for which $r = s$. Hence ϕ_s is equal to the expression (7) divided by

$$\rho\int_0^l (\nu_s \sin m_s x + \cos m_s x)^2\,dx + M + M\,(\nu_s \sin m_s l + \cos m_s l)^2 \ldots (8),$$

and thus the coefficients of the series are determined. If $M = 0$, even although μ be finite, the process is of course much simpler, but the unrestricted problem is instructive. So much stress is often laid on special proofs of Fourier's and Laplace's series, that the student is apt to acquire too contracted a view of the nature of those important results of analysis.

We shall now shew how Fourier's theorem in its general form can be deduced from our present investigation. Let $M = 0$; then if $\mu = \infty$, the ends of the string are fast, and the equation determining m becomes $\tan ml = 0$, or $ml = s\pi$, as we know it must be. In this case the series for y becomes

$$y = A_1 \sin\frac{\pi x}{l} + A_2 \sin\frac{2\pi x}{l} + A_3 \sin\frac{3\pi x}{l} + \ldots\ldots\ldots (9),$$

which must be general enough to represent any arbitrary functions of x, vanishing at 0 and l, between those limits. But now suppose that μ is zero, M still vanishing. The ends of the string may be supposed capable of sliding on two smooth rails perpendicular to its length, and the terminal condition is the vanishing of dy/dx. The equation in m is the *same as before*; and we learn that any function y' whose rates of variation vanish at $x = 0$ and $x = l$, can be expanded in a series

$$y' = B_1 \cos\frac{\pi x}{l} + B_2 \cos\frac{2\pi x}{l} + B_3 \cos\frac{3\pi x}{l} + \ldots\ldots (10).$$

This series remains unaffected when the sign of x is changed, and the first series merely changes sign without altering its numerical magnitude. If therefore y' be an even function of x, (10) represents it from $-l$ to $+l$. And in the same way, if y be an odd function of x, (9) represents it between the same limits.

Now, whatever function of x $\phi(x)$ may be, it can be divided into two parts, one of which is even, and the other odd, thus:

$$\phi(x) = \frac{\phi(x) + \phi(-x)}{2} + \frac{\phi(x) - \phi(-x)}{2};$$

so that, if $\phi(x)$ be such that $\phi(-l) = \phi(+l)$ and $\phi'(-l) = \phi'(+l)$, it can be represented between the limits $\pm l$ by the mixed series

$$A_1 \sin\frac{\pi x}{l} + B_1 \cos\frac{\pi x}{l} + A_2 \sin\frac{2\pi x}{l} + B_2 \cos\frac{2\pi x}{l} + \ldots\ldots \quad (11).$$

This series is periodic, with the period $2l$. If therefore $\phi(x)$ possess the same property, no matter what in other respects its character may be, the series is its complete equivalent. This is Fourier's theorem[1].

We now proceed to examine the effects of a slight yielding of the supports, in the case of a string whose ends are approximately fixed. The quantity ν may be great, either through μ or through M. We shall confine ourselves to the two principal cases, (1) when μ is great and M vanishes, (2) when μ vanishes and M is great.

In the first case $$\nu = \frac{\mu}{T_1 m},$$

and the equation in m is approximately

$$\tan ml = -\frac{2}{\nu} = -\frac{2T_1 m}{\mu}.$$

Assume $ml = s\pi + x$, where x is small; then

$$x = \tan x = -\frac{2T_1 . s\pi}{\mu l} \text{ approximately,}$$

and $$ml = s\pi\left(1 - \frac{2T_1}{\mu l}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

[1] The best 'system' for proving Fourier's theorem from dynamical considerations is an endless chain stretched round a smooth cylinder (§ 139), or a thin re-entrant column of air enclosed in a ring-shaped tube.

To this order of approximation the tones do not cease to form a harmonic scale, but the pitch of the whole is slightly lowered. The effect of the yielding is in fact the same as that of an increase in the length of the string in the ratio $1 : 1 + \frac{2T_1}{\mu l}$, as might have been anticipated.

The result is otherwise if μ vanish, while M is great. Here

$$\nu = -\frac{Ma^2 m}{T_1},$$

and
$$\tan ml = \frac{2T_1}{Ma^2 m} \text{ approximately.}$$

Hence
$$ml = s\pi + \frac{2T_1 l}{Ma^2 . s\pi} \quad\text{.....................(13).}$$

The effect is thus equivalent to a decrease in l in the ratio

$$1 : 1 - \frac{2T_1 l}{Ma^2 . s^2\pi^2},$$

and consequently there is a rise in pitch, the rise being the greater the lower the component tone. It might be thought that any kind of yielding would depress the pitch of the string, but the preceding investigation shews that this is not the case. Whether the pitch will be raised or lowered, depends on the sign of ν, and this again depends on whether the natural note of the mass M urged by the spring μ is lower or higher than that of the component vibration in question.

136. The problem of an otherwise uniform string carrying a finite load M at $x = b$ can be solved by the formulæ investigated in § 133. For, if the force $F \cos pt$ be due to the reaction against acceleration of the mass M,

$$F = \gamma p^2 M \quad\text{..............................(1),}$$

which combined with equation (7) of § 133 gives, to determine the possible values of λ (or $p : a$),

$$a^2 M\lambda \sin \lambda b \sin \lambda (l - b) = T_1 \sin \lambda l \quad\text{............(2).}$$

The value of y for any normal vibration corresponding to λ is

$$\left.\begin{aligned} y &= P \sin \lambda x \sin \lambda (l - b) \cos (a\lambda t - \epsilon) \\ &\qquad \text{from } x = 0 \text{ to } x = b \\ y &= P \sin \lambda (l - x) \sin \lambda b \cos (a\lambda t - \epsilon) \\ &\qquad \text{from } x = b \text{ to } x = l \end{aligned}\right\} \quad\text{......... (3),}$$

where P and ϵ are arbitrary constants.

It does not require analysis to prove that any normal components which have a node at the point of attachment are unaffected by the presence of the load. For instance, if a string be weighted at the centre, its component vibrations of even orders remain unchanged, while all the odd components are depressed in pitch. Advantage may sometimes be taken of this effect of a load, when it is desired for any purpose to disturb the harmonic relation of the component tones.

If M be very great, the gravest component is widely separated in pitch from all the others. We will take the case when the load is at the centre, so that $b = l - b = \frac{1}{2} l$. The equation in λ then becomes

$$\sin\frac{\lambda l}{2} \cdot \left\{\frac{\lambda l}{2}\tan\frac{\lambda l}{2} - \frac{\rho l}{M}\right\} = 0 \qquad (4),$$

where $\rho l : M$, denoting the ratio of the masses of the string and the load, is a small quantity which may be called α^2. The first root corresponding to the tone of lowest pitch occurs when $\frac{1}{2}\lambda l$ is small, and such that

$$(\tfrac{1}{2}\lambda l)^2 \{1 + \tfrac{1}{3}(\tfrac{1}{2}\lambda l)^2\} = \alpha^2 \text{ nearly},$$

whence $$\tfrac{1}{2}\lambda l = \alpha(1 - \tfrac{1}{6}\alpha^2),$$

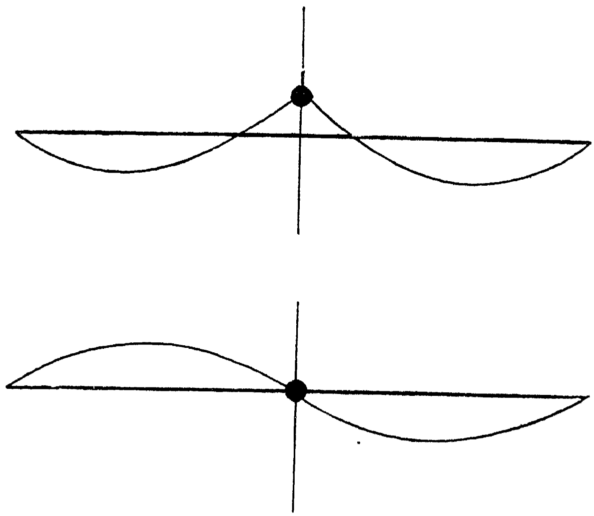
and the periodic time is given by

$$\tau = \pi\sqrt{\frac{Ml}{T_1}}\left(1 + \frac{\rho l}{6M}\right) \qquad (5).$$

The second term constitutes a correction to the rough value obtained in a previous chapter (§ 52), by neglecting the inertia of the string altogether. That it would be additive might have been expected, and indeed the formula as it stands may be obtained from the consideration that in the actual vibration the two parts of the string are nearly straight, and may be assumed to be exactly so in computing the kinetic and potential energies, without entailing any appreciable error in the calculated period. On this supposition the retention of the inertia of the string increases the kinetic energy corresponding to a given velocity of the load in the ratio of $M : M + \frac{1}{3}\rho l$, which leads to the above result. This method has indeed the advantage in one respect, as it might be applied when ρ is not uniform, or nearly uniform. All that is necessary is that the load M should be sufficiently predominant.

There is no other root of (4), until $\sin\frac{1}{2}\lambda l = 0$, which gives the second component of the string,—a vibration independent of the load. The roots after the first occur in closely contiguous pairs: for one set is given by $\frac{1}{2}\lambda l = s\pi$, and the other approximately by $\frac{1}{2}\lambda l = s\pi + \frac{\rho l}{s\pi M}$, in which the second term is small. The two types of vibration for $s = 1$ are shewn in the figure.

Fig 21.



The general formula (2) may also be applied to find the effect of a small load on the pitch of the various components.

137. Actual strings and wires are not perfectly flexible. They oppose a certain resistance to bending, which may be divided into two parts, producing two distinct effects. The first is called viscosity, and shews itself by damping the vibrations. This part produces no sensible effect on the periods. The second is conservative in its character, and contributes to the potential energy of the system, with the effect of shortening the periods. A complete investigation cannot conveniently be given here, but the case which is most interesting in its application to musical instruments, admits of a sufficiently simple treatment.

When rigidity is taken into account, something more must be specified with respect to the terminal conditions than that y vanishes. Two cases may be particularly noted:—

(i) When the ends are clamped, so that $dy/dx = 0$ at the ends.

(ii) When the terminal directions are perfectly free, in which case $d^2y/dx^2 = 0$.

It is the latter which we propose now to consider.

If there were no rigidity, the type of vibration would be

$$y \propto \sin \frac{s\pi x}{l}, \text{ satisfying the second condition.}$$

The effect of the rigidity might be slightly to disturb the type; but whether such a result occur or not, the period calculated from the potential and kinetic energies on the supposition that the type remains unaltered is necessarily correct as far as the first order of small quantities (§ 88).

Now the potential energy due to the stiffness is expressed by

$$\delta V = \tfrac{1}{2} B \int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx \quad\quad\quad (1),$$

where B is a quantity depending on the nature of the material and on the form of the section in a manner that we are not now prepared to examine. The *form* of δV is evident, because the force required to bend any element ds is proportional to ds, and to the amount of bending already effected, that is to ds/ρ. The whole work which must be done to produce a curvature $1/\rho$ in ds is therefore proportional to ds/ρ^2; while to the approximation to which we work $ds = dx$, and $1/\rho = d^2y/dx^2$.

Thus, if $y = \phi \sin \dfrac{s\pi x}{l}$,

$$T = \tfrac{1}{4}\rho l \dot{\phi}^2; \quad V = \tfrac{1}{4} T_1 l \,.\, \frac{s^2\pi^2}{l^2} \phi^2 \left(1 + \frac{B}{T_1}\frac{s^2\pi^2}{l^2}\right),$$

and the period of ϕ is given by

$$\tau = \tau_0 \left(1 - \frac{B}{2T_1}\frac{s^2\pi^2}{l^2}\right) \quad\quad\quad (2),$$

if τ_0 denote what the period would become if the string were endowed with perfect flexibility. It appears that the effect of the stiffness increases rapidly with the order of the component vibrations, which cease to belong to a harmonic scale. However, in the strings employed in music, the tension is usually sufficient to reduce the influence of rigidity to insignificance.

The method of this section cannot be applied without modification to the other case of terminal condition, namely, when the ends are clamped. In their immediate neighbourhood the type of vibration must differ from that assumed by a perfectly flexible string by a quantity, which is no longer small, and whose square therefore cannot be neglected. We shall return to this subject, when treating of the transverse vibrations of rods.

138. There is one problem relating to the vibrations of strings which we have not yet considered, but which is of some practical interest, namely, the character of the motion of a violin (or cello) string under the action of the bow. In this problem the *modus operandi* of the bow is not sufficiently understood to allow us to follow exclusively the *a priori* method: the indications of theory must be supplemented by special observation. By a dexterous combination of evidence drawn from both sources Helmholtz has succeeded in determining the principal features of the case, but some of the details are still obscure.

Since the note of a good instrument, well handled, is musical, we infer that the vibrations are strictly periodic, or at least that strict periodicity is the ideal. Moreover—and this is very important—the note elicited by the bow has nearly, or quite, the same pitch as the natural note of the string. The vibrations, although forcèd, are thus in some sense free. They are wholly dependent for their maintenance on the energy drawn from the bow, and yet the bow does not determine, or even sensibly modify, their periods. We are reminded of the self-acting electrical interrupter, whose motion is indeed forced in the technical sense, but has that kind of freedom which consists in determining (wholly, or in part) under what influences it shall come.

But it does not at once follow from the fact that the string vibrates with its natural periods, that it conforms to its natural types. If the coefficients of the Fourier expansion

$$y = \phi_1 \sin\frac{\pi x}{l} + \phi_2 \sin\frac{2\pi x}{l} + \ldots\ldots$$

be taken as the independent co-ordinates by which the configuration of the system is at any moment defined, we know that when there is no friction, or friction such that $F \propto T$, the natural vibrations are expressed by making each co-ordinate a *simple* harmonic (or quasi-harmonic) function of the time; while, for all that has hitherto appeared to the contrary, each co-ordinate in the present case might be *any* function of the time periodic in time τ. But a little examination will shew that the vibrations must be sensibly natural in their types as well as in their periods.

The force exercised by the bow at its point of application may be expressed by

$$Y = \Sigma A_r \cos\left(\frac{2r\pi t}{\tau} - \epsilon_r\right);$$

so that the equation of motion for the co-ordinate ϕ_s is

$$\ddot{\phi}_s + \kappa\dot{\phi}_s + \frac{s^2\pi^2a^2}{l^2}\phi_s = \frac{2}{l\rho}\sin\frac{s\pi b}{l}\,.\,\Sigma A_r \cos\left(\frac{2r\pi t}{\tau} - \epsilon_r\right),$$

b being the point of application. Each of the component parts of Φ_s will give a corresponding term of its own period in the solution, but the one whose period is the same as the natural period of ϕ_s will rise enormously in relative importance. Practically then, if the damping be small, we need only retain that part of ϕ_s which depends on $A_s \cos\left(\frac{2s\pi t}{\tau} - \epsilon_s\right)$, that is to say, we may regard the vibrations as natural in their types.

Another material fact, supported by evidence drawn both from theory and aural observation, is this. All component vibrations are absent which have a node at the point of excitation. "In order, however, to extinguish these tones, it is necessary that the coincidence of the point of application of the bow with the node should be very *exact*. A very small deviation reproduces the missing tones with considerable strength[1]."

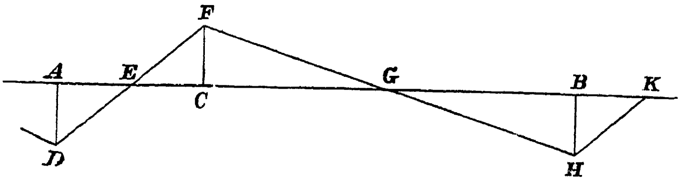
The remainder of the evidence on which Helmholtz' theory rests, was derived from direct observation with the vibration-microscope. As explained in Chapter II., this instrument affords a view of the curve representing the motion of the point under observation, as it would be seen traced on the surface of a transparent cylinder. In order to deduce the representative curve in its ordinary form, the imaginary cylinder must be conceived to be unrolled, or developed, into a plane.

The simplest results are obtained when the bow is applied at a node of one of the higher components, and the point observed is one of the other nodes of the same system. If the bow work fairly so as to draw out the fundamental tone clearly and strongly, the representative curve is that shewn in figure 22; where the

[1] Donkin's *Acoustics*, p. 131.

abscissæ correspond to the time (AB being a complete period), and the ordinates represent the displacement. The remarkable

Fig. 22.



fact is disclosed that the whole period τ may be divided into two parts τ_0 and $\tau - \tau_0$, during each of which the velocity of the observed point is constant; but the velocities to and fro are in general unequal.

We have now to represent this curve by a series of harmonic terms. If the origin of time correspond to the point A, and $AD = FC = \gamma$, Fourier's theorem gives

$$y = \frac{2\gamma\tau^2}{\pi^2\tau_0(\tau - \tau_0)} \Sigma_{s=1}^{s=\infty} \frac{1}{s^2} \sin \frac{s\pi\tau_0}{\tau} \sin \frac{2s\pi}{\tau}\left(t - \frac{\tau_0}{2}\right) \text{......(1)}.$$

With respect to the value of τ_0, we know that all those components of y must vanish for which $\sin(s\pi x_0/l) = 0$ (x_0 being the point of observation), because under the circumstances of the case the bow cannot generate them. There is therefore reason to suppose that $\tau_0 : \tau = x_0 : l$; and in fact observation proves that $AC : CB$ (in the figure) is equal to the ratio of the two parts into which the string is divided by the point of observation.

Now the free vibrations of the string are represented in general by

$$y = \Sigma_{s=1}^{s=\infty} \sin\frac{s\pi x}{l}\left\{A_s \cos\frac{2s\pi t}{\tau} + B_s \sin\frac{2s\pi t}{\tau}\right\};$$

and this at the point $x = x_0$ must agree with (1). For convenience of comparison, we may write

$$A_s \cos\frac{2s\pi t}{\tau} + B_s \sin\frac{2s\pi t}{\tau} = C_s \cos\frac{2s\pi}{\tau}\left(t - \frac{\tau_0}{2}\right) + D_s \sin\frac{2s\pi}{\tau}\left(t - \frac{\tau_0}{2}\right),$$

and it then appears that $C_s = 0$.

We find also to determine D_s

$$\sin\frac{s\pi x_0}{l}\,.\,D_s=\frac{2\gamma\tau^2}{\pi^2\tau_0(\tau-\tau_0)}\frac{1}{s^2}\sin\frac{s\pi x_0}{l},$$

whence

$$D_s=\frac{2\gamma\tau^2}{\pi^2\tau_0(\tau-\tau_0)}\frac{1}{s^2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (2),$$

unless $\sin(s\pi x_0/l)=0$.

In the case reserved, the comparison leaves D_s undetermined, but we know on other grounds that D_s then vanishes. However, for the sake of simplicity, we shall suppose for the present that D_s is always given by (2). If the point of application of the bow do not coincide with a node of any of the lower components, the error committed will be of no great consequence.

On this understanding the complete solution of the problem is

$$y=\frac{2\gamma\tau^2}{\pi^2\tau_0(\tau-\tau_0)}\sum_{s=1}^{s=\infty}\frac{1}{s^2}\sin\frac{s\pi x}{l}\sin\frac{2s\pi}{\tau}\left(t-\frac{\tau_0}{2}\right)\ldots\ldots(3).$$

The amplitudes of the components are therefore proportional to s^{-2}. In the case of a plucked string we found for the corresponding function $s^{-2}\sin(s\pi b/l)$, which is somewhat similar. If the string be plucked at the middle, the even components vanish, but the odd ones follow the same law as obtains for a violin string. The equation (3) indicates that the string is always in the form of two straight lines meeting at an angle. In order more conveniently to shew this, let us change the origin of the time, and the constant multiplier so that

$$y=\frac{8P}{\pi^2}\Sigma\frac{1}{s^2}\sin\frac{s\pi x}{l}\sin\frac{2s\pi t}{\tau}\ \ldots\ldots\ldots\ldots\ldots\ (4)$$

will be the equation expressing the form of the string at any time.

Now we know (§ 127) that the equation of the pair of lines proceeding from the fixed ends of the string, and meeting at a point whose co-ordinates are α, β, is

$$y=\frac{2\beta l^2}{\pi^2\alpha(l-\alpha)}\Sigma\frac{1}{s^2}\sin\frac{s\pi\alpha}{l}\sin\frac{s\pi x}{l}$$

Thus at the time t, (4) represents such a pair of lines, meeting at the point whose co-ordinates are given by

$$\frac{\beta l^2}{\alpha(l-\alpha)}=\pm 4P,$$

$$\sin\frac{s\pi\alpha}{l}=\pm\sin\frac{2s\pi t}{\tau}.$$

These equations indicate that the projection on the axis of x of the point of intersection moves uniformly backwards and forwards between $x=0$ and $x=l$, and that the point of intersection itself is situated on one or other of two parabolic arcs, of which the equilibrium position of the string is a common chord.

Since the motion of the string as thus defined by that of the point of intersection of its two straight parts, has no especial relation to x_0 (the point of observation), it follows that, according to these equations, the same kind of motion might be observed at any other point. And this is approximately true. But the theoretical result, it will be remembered, was only obtained by assuming the presence in certain proportions of component vibrations having nodes at x_0, though in fact their absence is required by mechanical laws. The presence or absence of these components is a matter of indifference when a node is the point of observation, but not in any other case. When the node is departed from, the vibration curve shews a series of ripples, due to the absence of the components in question. Some further details will be found in Helmholtz and Donkin.

The sustaining power of the bow depends upon the fact that solid friction is less at moderate than at small velocities, so that when the part of the string acted upon is moving with the bow (not improbably at the same velocity), the mutual action is greater than when the string is moving in the opposite direction with a greater relative velocity. The accelerating effect in the first part of the motion is thus not entirely neutralised by the subsequent retardation, and an outstanding acceleration remains capable of maintaining the vibration in spite of other losses of energy. A curious effect of the same peculiarity of solid friction has been observed by W. Froude, who found that the vibrations of a pendulum swinging from a shaft might be maintained or even increased by causing the shaft to rotate.

[Another case in which the vibrations of a string are maintained is that of the Aeolian Harp. It has often been suggested that the action of the wind is analogous to that of a bow; but the analogy is disproved by the observation[1] that the vibrations are executed in a plane *transverse* to the direction of the wind. The true explanation involves hydrodynamical theory not yet developed.]

[1] *Phil. Mag.*, March, 1879, p. 161.

139. A string stretched on a smooth curved surface will in equilibrium lie along a geodesic line, and, subject to certain conditions of stability, will vibrate about this configuration, if displaced. The simplest case that can be proposed is when the surface is a cylinder of any form, and the equilibrium position of the string is perpendicular to the generating lines. The student will easily prove that the motion is independent of the curvature of the cylinder, and that the vibrations are in all essential respects the same as if the surface were developed into a plane. The case of an endless string, forming a necklace round the cylinder, is worthy of notice.

In order to illustrate the characteristic features of this class of problems, we will take the comparatively simple example of a string stretched on the surface of a smooth sphere, and lying, when in equilibrium, along a great circle. The co-ordinates to which it will be most convenient to refer the system are the latitude θ measured from the great circle as equator, and the longitude ϕ measured along it. If the radius of the sphere be a, we have

$$T=\frac{1}{2}\int\rho\,(a\dot{\theta})^2 a\,d\phi=\frac{a^3\rho}{2}\int\dot{\theta}^2 d\phi\;\ldots\ldots\ldots\ldots\ldots(1).$$

The extension of the string is denoted by

$$\int(ds-a\,d\phi)=a\int\left(\frac{ds}{a\,d\phi}-1\right)d\phi.$$

Now

$$ds^2=(a\,d\theta)^2+(a\cos\theta\,d\phi)^2;$$

so that

$$\frac{ds}{a\,d\phi}-1=\left\{\left(\frac{d\theta}{d\phi}\right)^2+\cos^2\theta\right\}^{\frac{1}{2}}-1=\frac{1}{2}\left(\frac{d\theta}{d\phi}\right)^2-\frac{\theta^2}{2},\text{ approximately.}$$

Thus

$$V=\tfrac{1}{2}aT_1\int\left\{\left(\frac{d\theta}{d\phi}\right)^2-\theta^2\right\}d\phi\;\ldots\ldots\ldots\ldots\ldots(2);$$[1]

and

$$\delta V=aT_1\,.\,\delta\theta\left[\frac{d\theta}{d\phi}\right]_0^l-aT_1\int_0^l\delta\theta\left(\frac{d^2\theta}{d\phi^2}+\theta\right)d\phi.$$

If the ends be fixed,

$$\delta\theta\left[\frac{d\theta}{d\phi}\right]_0^l=0,$$

[1] Cambridge Mathematical Tripos Examination, 1876.

and the equation of virtual velocities is

$$a^3\rho\int_0^l \ddot{\theta}\,\delta\theta\,d\phi - aT_1\int_0^l \delta\theta\left(\frac{d^2\theta}{d\phi^2}+\theta\right)d\phi = 0,$$

whence, since $\delta\theta$ is arbitrary,

$$a^2\rho\,\ddot{\theta} = T_1\left(\frac{d^2\theta}{d\phi^2}+\theta\right) \quad\text{..................... (3).}$$

This is the equation of motion.

If we assume $\theta \propto \cos pt$, we get

$$\frac{d^2\theta}{d\phi^2}+\theta+\frac{a^2\rho}{T_1}p^2\theta = 0 \quad\text{..................... (4),}$$

of which the solution, subject to the condition that θ vanishes with ϕ, is

$$\theta = A\sin\left\{\frac{a^2\rho}{T_1}p^2+1\right\}^{\frac{1}{2}}\phi\,.\cos pt \quad\text{............... (5).}$$

The remaining condition to be satisfied is that θ vanishes when $a\phi = l$, or $\phi = \alpha$, if $\alpha = l/a$.

This gives

$$p^2 = \frac{T_1}{a^2\rho}\left(\frac{m^2\pi^2}{\alpha^2}-1\right) = \frac{T_1}{\rho}\left(\frac{m^2\pi^2}{l^2}-\frac{1}{a^2}\right) \quad\text{........... (6),}$$

where m is an integer.

The normal functions are thus of the same form as for a straight string, viz.

$$\theta = A\sin\frac{m\pi\phi}{\alpha}\cos pt. \quad\text{.................. (7),}$$

but the series of periods is different. The effect of the curvature is to make each tone graver than the corresponding tone of a straight string. If $\alpha > \pi$, one at least of the values of p^2 is negative, indicating that the corresponding modes are unstable. If $\alpha = \pi$, p_1 is zero, the string being of the same length in the displaced position, as when $\theta = 0$.

A similar method might be applied to calculate the motion of a string stretched round the equator of any surface of revolution[1].

140. The approximate solution of the problem for a vibrating string of nearly but not quite uniform longitudinal density has been fully considered in Chapter IV. § 91, as a convenient example of

[1] [For a more general treatment of this question see Michell, *Messenger of Mathematics*, vol. XIX. p. 87, 1890.]

the general theory of approximately simple systems. It will be sufficient here to repeat the result. If the density be $\rho_0 + \delta\rho$, the period τ_r of the r^{th} component vibration is given by

$$\tau_r{}^2 = \frac{4l^2\rho_0}{T_1}\left\{1 + \frac{2}{l}\int_0^l \frac{\delta\rho}{\rho_0}\sin^2\frac{r\pi x}{l}\,dx\right\} \quad\ldots\ldots\ldots\ldots (1).$$

If the irregularity take the form of a small load of mass m at the point $x = b$, the formula may be written

$$\tau_r{}^2 = \frac{4l^2\rho_0}{T_1}\left\{1 + \frac{2m}{l\rho_0}\sin^2\frac{r\pi b}{l}\right\} \quad\ldots\ldots\ldots\ldots (2).$$

These values of τ^2 are correct as far as the first power of the small quantities $\delta\rho$ and m, and give the means of calculating a correction for such slight departures from uniformity as must always occur in practice.

As might be expected, the effect of a small load vanishes at nodes, and rises to a maximum at the points midway between consecutive nodes. When it is desired merely to make a rough estimate of the effective density of a nearly uniform string, the formula indicates that attention is to be given to the neighbourhood of loops rather than to that of nodes.

[The effect of a small variation of density upon the period is the same whether it occur at a distance x from one end of the string, or at an equal distance from the other end. The *mean* variation at points equidistant from the centre is all that we need regard, and thus no generality will be lost if we suppose that the density remains symmetrically distributed with respect to the centre. Thus we may write

$$\tau_r{}^2 = \frac{4l^2\rho_0}{T_1}(1 + \alpha_r) \quad\ldots\ldots\ldots\ldots (3)$$

where
$$\alpha_r = \frac{2}{l}\int_0^{\frac{1}{2}l} \frac{\delta\rho}{\rho_0}\left(1 - \cos\frac{2\pi r x}{l}\right)dx \quad\ldots\ldots\ldots\ldots (4).$$

In this equation $\delta\rho$ may be expanded from 0 to $\frac{1}{2}l$ in the series

$$\frac{\delta\rho}{\rho_0} = A_0 + A_1\cos\frac{2\pi x}{l} + \ldots + A_r\cos\frac{2\pi r x}{l} + \ldots\ldots\ldots\ldots (5),$$

where
$$A_0 = \frac{2}{l}\int_0^{\frac{1}{2}l} \frac{\delta\rho}{\rho_0}\,dx \quad\ldots\ldots\ldots\ldots (6),$$

$$A_r = \frac{4}{l}\int_0^{\frac{1}{2}l} \frac{\delta\rho}{\rho_0}\cos\frac{2\pi r x}{l}\,dx \quad\ldots\ldots\ldots\ldots (7).$$

Accordingly,

$$\alpha_r = A_0 - \tfrac{1}{2}A_r \text{ ....(8).}$$

This equation, as it stands, gives the changes in period in terms of the changes of density supposed to be known. And it shews conversely that a variation of density may always be found which will give prescribed arbitrary displacements to all the periods. This is a point of some interest.

In order to secure a reasonable continuity in the density, it is necessary to suppose that $\alpha_1, \alpha_2 \ldots$ are so prescribed that α_r assumes ultimately a constant value when r is increased indefinitely. If this condition be satisfied, we may take $A_0 = \alpha_\infty$, and then A_r tends to zero as r increases.

As a simple example, suppose that it be required so to vary the density of a string that, while the pitch of the fundamental tone is displaced, all other tones shall remain unaltered. The conditions give

$$\alpha_2 = \alpha_3 = \alpha_4 \ldots\ldots = \alpha_\infty = 0.$$

Accordingly

$$A_0 = A_2 = A_3 = \ldots\ldots = 0,$$

and

$$A_1 = -2\alpha_1.$$

Thus by (5)

$$\delta\rho/\rho_0 = -2\alpha_1 \cos(2\pi x/l).]$$

141. The differential equation determining the motion of a string, whose longitudinal density ρ is variable, is

$$\rho \frac{d^2y}{dt^2} = T_1 \frac{d^2y}{dx^2} \text{ .... (1),}$$

from which, if we assume $y \propto \cos pt$, we obtain to determine the normal functions

$$\frac{d^2y}{dx^2} + \nu^2 \rho y = 0 \text{ .... (2),}$$

where ν^2 is written for p^2/T_1. This equation is of the second order and linear, but has not hitherto been solved in finite terms. Considered as defining the curve assumed by the string in the normal mode under consideration, it determines the *curvature* at any point, and accordingly embodies a rule by which the curve can be constructed graphically. Thus in the application to a string fixed at both ends, if we start from either end at an arbitrary inclination, and with zero curvature, we are always directed by the equation with what curvature to proceed, and in this way we may trace out the entire curve.

If the assumed value of ν^2 be right, the curve will cross the axis of x at the required distance, and the law of vibration will be completely determined. If ν^2 be not known, different values may be tried until the curve ends rightly; a sufficient approximation to the value of ν^2 may usually be arrived at by a calculation founded on an assumed type (§§ 88, 90).

Whether the longitudinal density be uniform or not, the periodic time of any simple vibration varies *cæteris paribus* as the square root of the density and inversely as the square root of the tension under which the motion takes place.

The converse problem of determining the density, when the period and the type of vibration are given, is always soluble. For this purpose it is only necessary to substitute the given value of y, and of its second differential coefficient in equation (2). Unless the density be infinite, the extremities of a string are points of zero curvature.

When a given string is shortened, every component tone is raised in pitch. For the new state of things may be regarded as derived from the old by introduction, at the proposed point of fixture, of a spring (without inertia), whose stiffness is gradually increased without limit. At each step of the process the potential energy of a given deformation is augmented, and therefore (§ 88) the pitch of every tone is raised. In like manner an addition to the length of a string depresses the pitch, even though the added part be destitute of inertia.

142. Although a general integration of equation (2) of § 141 is beyond our powers, we may apply to the problem some of the many interesting properties of the solution of the linear equation of the second order, which have been demonstrated by MM. Sturm and Liouville[1]. It is impossible in this work to give anything like a complete account of their investigations; but a sketch, in which the leading features are included, may be found interesting, and will throw light on some points connected with the general

[1] The memoirs referred to in the text are contained in the first volume of Liouville's *Journal* (1836).

theory of the vibrations of continuous bodies. I have not thought it necessary to adhere very closely to the methods adopted in the original memoirs.

At no point of the curve satisfying the equation

$$\frac{d^2y}{dx^2} + \nu^2\rho\, y = 0 \qquad (1),$$

can both y and dy/dx vanish together. By successive differentiations of (1) it is easy to prove that, if y and dy/dx vanish simultaneously, all the higher differential coefficients d^2y/dx^2, d^3y/dx^3; &c. must also vanish at the same point, and therefore by Taylor's theorem the curve must coincide with the axis of x.

Whatever value be ascribed to ν^2, the curve satisfying (1) is *sinuous*, being concave throughout towards the axis of x, since ρ is everywhere positive. If at the origin y vanish, and dy/dx be positive, the ordinate will remain positive for all values of x below a certain limit dependent on the value ascribed to ν^2. If ν^2 be very small, the curvature is slight, and the curve will remain on the positive side of the axis for a great distance. We have now to prove that as ν^2 increases, all the values of x which satisfy the equation $y = 0$ gradually diminish in magnitude.

Let y' be the ordinate of a second curve satisfying the equation

$$\frac{d^2y'}{dx^2} + \nu'^2\rho\, y' = 0 \qquad (2),$$

as well as the condition that y' vanishes at the origin, and let us suppose that ν'^2 is somewhat greater than ν^2. Multiplying (2) by y, and (1) by y', subtracting, and integrating with respect to x between the limits 0 and x, we obtain, since y and y' both vanish with x,

$$y'\frac{dy}{dx} - y\frac{dy'}{dx} = (\nu'^2 - \nu^2)\int_0^x \rho\, y\, y'\, dx \qquad (3).$$

If we further suppose that x corresponds to a point at which y vanishes, and that the difference between ν'^2 and ν^2 is very small, we get ultimately

$$y'\frac{dy}{dx} \doteq \delta\nu^2 \int_0^x \rho\, y^2\, dx \qquad (4).$$

The right-hand member of (4) being essentially positive, we earn that y' and dy/dx are of the same sign, and therefore that, whether dy/dx be positive or negative, y' is already of the same sign as that to which y is changing, or in other words, the value of x for which y' vanishes is less than that for which y vanishes.

If we fix our attention on the portion of the curve lying between $x=0$ and $x=l$, the ordinate continues positive throughout as the value of ν^2 increases, until a certain value is attained, which we will call ν_1^2. The function y is now identical in form with the first normal function u_1 of a string of density ρ fixed at 0 and l, and has no root except at those points. As ν^2 again increases, the first root moves inwards from $x=l$ until, when a second special value ν_2^2 is attained, the curve again crosses the axis at the point $x=l$, and then represents the second normal function u_2. This function has thus one internal root, and one only. In like manner corresponding to a higher value ν_3^2 we obtain the third normal function u_3 with two internal roots, and so on. The n^{th} function u_n has thus exactly $n-1$ internal roots, and since its first differential coefficient never vanishes simultaneously with the function, it changes sign each time a root is passed.

From equation (3) it appears that if u_r and u_s be two different normal functions,

$$\int_0^l \rho\, u_r u_s\, dx = 0 \qquad (5).$$

A beautiful theorem has been discovered by Sturm relating to the number of the roots of a function derived by addition from a finite number of normal functions. If u_m be the component of lowest order, and u_n the component of highest order, the function

$$f(x) = \phi_m u_m + \phi_{m+1} u_{m+1} + \ldots\ldots + \phi_n u_n \qquad (6),$$

where ϕ_m, ϕ_{m+1}, &c. are arbitrary coefficients, has *at least* $m-1$ internal roots, and *at most* $n-1$ internal roots. The extremities at $x=0$ and at $x=l$ correspond of course to roots in all cases. The following demonstration bears some resemblance to that given by Liouville, but is considerably simpler, and, I believe, not less rigorous.

If we suppose that $f(x)$ has exactly μ internal roots (any number of which may be equal), the derived function $f'(x)$ cannot have less than $\mu+1$ internal roots, since there must be at least one root of $f'(x)$ between each pair of consecutive roots of $f(x)$, and the whole number of roots of $f(x)$ concerned is $\mu+2$. In like manner, we see that there must be at least μ roots of $f''(x)$, besides the extremities, which themselves necessarily correspond to roots; so that in passing from $f(x)$ to $f''(x)$ it is impossible that any roots can be lost. Now

$$f''(x) = \phi_m u_m'' + \phi_{m+1} u''_{m+1} + \ldots\ldots + \phi_n u_n''$$
$$= -\rho\,(\nu_m^2 \phi_m u_m + \nu^2_{m+1} \phi_{m+1} u_{m+1} + \ldots\ldots + \nu_n^2 \phi_n u_n) \ldots (7),$$

as we see by (1); and therefore, since ρ is always positive, we infer that

$$\nu_m^2 \phi_m u_m + \nu^2_{m+1} \phi_{m+1} u_{m+1} + \ldots\ldots + \nu_n^2 \phi_n u_n \ \ldots\ldots (8),$$

has at least μ roots.

Again, since (8) is an expression of the same form as $f(x)$, similar reasoning proves that

$$\nu_m^4 \phi_m u_m + \nu^4_{m+1} \phi_{m+1} u_{m+1} + \ldots\ldots + \nu_n^4 \phi_n u_n$$

has at least μ internal roots; and the process may be continued to any extent. In this way we obtain a series of functions, all with μ internal roots at least, which differ from the original function $f(x)$ by the continually increasing relative importance of the components of the higher orders. When the process has been carried sufficiently far, we shall arrive at a function, whose form differs as little as we please from that of the normal function of highest order, viz. u_n, and which has therefore $n-1$ internal roots. It follows that, since no roots can be lost in passing down the series of functions, the number of internal roots of $f(x)$ cannot exceed $n-1$.

The other half of the theorem is proved in a similar manner by continuing the series of functions backwards from $f(x)$. In this way we obtain

$$\phi_m u_m + \phi_{m+1} u_{m+1} + \ldots\ldots + \phi_n u_n$$
$$\nu_m^{-2} \phi_m u_m + \nu^{-2}_{m+1} \phi_{m+1} u_{m+1} + \ldots\ldots + \nu_n^{-2} \phi_n u_n$$
$$\nu_m^{-4} \phi_m u_m + \nu^{-4}_{m+1} \phi_{m+1} u_{m+1} + \ldots\ldots + \nu_n^{-4} \phi_n u_n$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

arriving at last at a function sensibly coincident in form with the normal function of *lowest* order, viz. u_m, and having therefore $m-1$ internal roots. Since no roots can be lost in passing up the series from this function to $f(x)$, it follows that $f(x)$ cannot have fewer internal roots than $m-1$; but it must be understood that any number of the $m-1$ roots may be equal.

We will now prove that $f(x)$ cannot be identically zero, unless all the coefficients ϕ vanish. Suppose that ϕ_r is not zero. Multiply (6) by ρu_r, and integrate with respect to x between the limits 0 and l. Then by (5)

$$\int_0^l \rho u_r f(x)\, dx = \phi_r \int_0^l \rho u_r^2 dx \quad \text{.........} \quad (9);$$

from which, since the integral on the right-hand side is finite, we see that $f(x)$ cannot vanish for all values of x included within the range of integration.

Liouville has made use of Sturm's theorem to shew how a series of normal functions may be compounded so as to have an arbitrary sign at all points lying between $x=0$ and $x=l$. His method is somewhat as follows.

The values of x for which the function is to change sign being $a, b, c, \ldots$, quantities which without loss of generality we may suppose to be all different, let us consider the series of determinants,

$$\begin{vmatrix} u_1(a), & u_1(x) \\ u_2(a), & u_2(x) \end{vmatrix}, \quad \begin{vmatrix} u_1(a), & u_1(b), & u_1(x) \\ u_2(a), & u_2(b), & u_2(x) \\ u_3(a), & u_3(b), & u_3(x) \end{vmatrix}, \text{ \&c.}$$

The first is a linear function of $u_1(x)$ and $u_2(x)$, and by Sturm's theorem has therefore one internal root at most, which root is evidently a. Moreover the determinant is not identically zero, since the coefficient of $u_2(x)$, viz. $u_1(a)$, does not vanish, whatever be the value of a. We have thus obtained a function, which changes sign at an arbitrary point a, and there only internally.

The second determinant vanishes when $x=a$, and when $x=b$, and, since it cannot have more than two internal roots, it changes sign, when x passes through these values, and there only. The coefficient of $u_3(x)$ is the value assumed by the first determinant when $x=b$, and is therefore finite. Hence the second determinant is not identically zero.

Similarly the third determinant in the series vanishes and changes sign when $x=a$, when $x=b$, and when $x=c$, and at these internal points only. The coefficient of $u_4(x)$ is finite, being the value of the second determinant when $x=c$.

It is evident that by continuing this process we can form functions compounded of the normal functions, which shall vanish and change sign for any arbitrary values of x, and not elsewhere internally; or, in other words, we can form a function whose sign is arbitrary over the whole range from $x=0$ to $x=l$.

On this theorem Liouville founds his demonstration of the possibility of representing an arbitrary function between $x=0$ and $x=l$ by a series of normal functions. If we assume the possibility of the expansion and take

$$f(x)=\phi_1 u_1(x)+\phi_2 u_2(x)+\phi_3 u_3(x)+\dots\dots\dots\dots \quad (10),$$

the necessary values of ϕ_1, ϕ_2, &c. are determined by (9), and we find

$$f(x)=\Sigma\left\{u_r(x)\int_0^l \rho\, u_r(x) f(x)\, dx \div \int_0^l \rho\, u_r^2(x)\, dx\right\}\dots\dots(11).$$

If the series on the right be denoted by $F(x)$, it remains to establish the identity of $f(x)$ and $F(x)$.

If the right-hand member of (11) be multiplied by $\rho u_r(x)$ and integrated with respect to x from $x=0$ to $x=l$, we see that

$$\int_0^l \rho\, u_r(x) F(x)\, dx=\int_0^l \rho\, u_r(x) f(x)\, dx,$$

or, as we may also write it,

$$\int_0^l \{F(x)-f(x)\}\, \rho\, u_r(x)\, dx=0\dots\dots\dots\dots\dots\dots \quad (12),$$

where $u_r(x)$ is *any* normal function. From (12) it follows that

$$\int_0^l \{F(x)-f(x)\}\{A_1 u_1(x)+A_2 u_2(x)+A_3 u_3(x)+\dots\}\, \rho\, dx=0\dots(13),$$

where the coefficients A_1, A_2, &c. are arbitrary.

Now if $F(x)-f(x)$ be not identically zero, it will be possible so to choose the constants A_1, A_2, &c. that $A_1 u_1(x)+A_2 u_2(x)+\dots$ has throughout the same sign as $F(x)-f(x)$, in which case every element of the integral would be positive, and equation (13) could not be true. It follows that $F(x)-f(x)$ cannot differ from zero, or that the series of normal functions forming the right-hand member of (11) is identical with $f(x)$ for all values of x from $x=0$ to $x=l$.

The arguments and results of this section are of course applicable to the particular case of a uniform string for which the normal functions are circular.

[As a particular case of variable density the supposition that

$\rho = \sigma x^{-2}$ is worthy of notice, § 148 *b*. In the notation there adopted

$$m^2 + \tfrac{1}{4} = n^2 = p^2\sigma/T_1 \quad\text{.....................(14)},$$

and the general solution is

$$y = Ax^{\frac{1}{2}+im} + Bx^{\frac{1}{2}-im} \quad\text{.....................(15)}.$$

If the string be fixed at two points, whose abscissæ x_1, x_2 are as r to 1, the frequency equation is $r^{2im} = 1$, or

$$n^2 = \tfrac{1}{4} + \frac{s^2\pi^2}{(\log r)^2} \quad\text{.......................(16)},$$

where s denotes an integer. The proper frequencies thus depend only upon the *ratio* of the terminal abscissæ. By supposing r nearly equal to unity we may fall back upon the usual formula (§ 124) applicable to a uniform string.

The general form of the normal function is

$$y = x^{\frac{1}{2}} \sin \frac{s\pi \log (x/x_1)}{\log (x_2/x_1)} \quad\text{.................(17).]}$$

142 *a*. The points where the string remains at rest, or nodes, are of course determined by the roots of the normal functions, when the vibrations are free. In this case the frequency is limited to certain definite values; but when the vibrations are forced, they may be of any frequency, and it becomes possible to trace the motion of the nodal points as the frequency increases continuously.

For example, suppose that the imposed force acts at a single point P of a string AB, whose density may be variable. So long as the frequency is less than that of either of the two parts AP, PB (supposed to be held at rest at both extremities) into which the string is divided, there can be no (interior) node (Q). Otherwise, that part of the string AQ between the node Q and one extremity (A), which does not include P, would be vibrating freely, and more slowly than is possible for the longer length AP, included between the point P and the same extremity. When the frequency is raised, so as to coincide with the smaller of those proper to AP, PB, say AP, a node enters at P and then advances towards A. At each coincidence of the frequency with one of those proper to the whole string AB, the vibration identifies itself with the corresponding free vibration, and at each coincidence with a frequency proper to AP, or BP, a new node appears at P, and advances in the first case towards A and in the second towards B. And throughout the whole sequence of events all the nodes move *outwards* from P towards A or B.

Thus, if the string be uniform and be bisected at P, there is no node until the pitch rises to the octave (c') of the note (c) of the string. At this stage two nodes enter at P, and move outwards symmetrically. When g' is reached, the mode of vibration is that of the free vibration of the same pitch, and the nodes are at the two points of trisection. At c'' these nodes have moved outwards so far as to bisect AP, BP, and two new nodes enter at P.

143. When the vibrations of a string are not confined to one plane, it is usually most convenient to resolve them into two sets executed in perpendicular planes, which may be treated independently. There is, however, one case of this description worth a passing notice, in which the motion is most easily conceived and treated without resolution.

Suppose that

$$\left.\begin{aligned} y &= \sin\frac{s\pi x}{l}\cos\frac{2s\pi t}{\tau} \\ z &= \sin\frac{s\pi x}{l}\sin\frac{2s\pi t}{\tau} \end{aligned}\right\}\qquad\qquad(1).$$

Then

$$r = \sqrt{(y^2 + z^2)} = \sin\frac{s\pi x}{l}\qquad\qquad(2),$$

and

$$z : y = \tan(2s\pi t/\tau)\qquad\qquad(3),$$

shewing that the whole string is at any moment in one plane, which revolves uniformly, and that each particle describes a circle with radius $\sin(s\pi x/l)$. In fact, the whole system turns without relative displacement about its position of equilibrium, completing each revolution in the time τ/s. The mechanics of this case is quite as simple as when the motion is confined to one plane, the resultant of the tensions acting at the extremities of any small portion of the string's length being balanced by the centrifugal force.

144. The general differential equation for a uniform string, viz.

$$\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2}\qquad\qquad(1),$$

may be transformed by a change of variables into

$$\frac{d^2 y}{du\,dv} = 0 \quad\text{.......................... (2),}$$

where $u = x - at$, $v = x + at$. The general solution of (2) is

$$y = f(u) + F(v) = f(x - at) + F(x + at) \quad\text{......... (3)}[1],$$

f, F being two arbitrary functions.

Let us consider first the case in which F vanishes. When t has any particular value, the equation

$$y = f(x - at) \quad\text{.......................... (4),}$$

expressing the relation between x and y, represents the form of the string. A change in the value of t is merely equivalent to an alteration in the origin of x, so that (4) indicates that a certain *form* is propagated along the string with uniform velocity a in the positive direction. Whatever the value of y may be at the point x and at the time t, the same value of y will obtain at the point $x + a\,\Delta t$ at the time $t + \Delta t$.

The form thus perpetuated may be any whatever, so long as it does not violate the restrictions on which (1) depends.

When the motion consists of the propagation of a wave in the positive direction, a certain relation subsists between the inclination and the velocity at any point. Differentiating (4) we find

$$\frac{dy}{dt} = -a\,\frac{dy}{dx} \quad\text{.......................... (5).}$$

Initially, dy/dt and dy/dx may both be given arbitrarily, but if the above relation be not satisfied, the motion cannot be represented by (4).

In a similar manner the equation

$$y = F(x + at) \quad\text{...................... (6)}$$

denotes the propagation of a wave in the *negative* direction, and the relation between dy/dt and dy/dx corresponding to (5) is

$$\frac{dy}{dt} = a\,\frac{dy}{dx} \quad\text{.......................... (7).}$$

In the general case the motion consists of the simultaneous propagation of two waves with velocity a, the one in the positive.

[1] [Equations (1) and (3) are due to D'Alembert (1750).]

and the other in the negative direction; and these waves are entirely independent of one another. In the first $dy/dt = -a\,dy/dx$, and in the second $dy/dt = a\,dy/dx$. The initial values of dy/dt and dy/dx must be conceived to be divided into two parts, which satisfy respectively the relations (5) and (7). The first constitutes the wave which will advance in the positive direction without change of form; the second, the negative wave. Thus, initially,

$$\left.\begin{aligned} f'(x) + F'(x) &= \frac{dy}{dx} \\ f'(x) - F'(x) &= -\frac{1}{a}\frac{dy}{dt} \end{aligned}\right\},$$

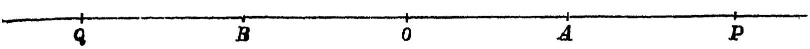
whence

$$\left.\begin{aligned} f'(x) &= \tfrac{1}{2}\left(\frac{dy}{dx} - \frac{1}{a}\frac{dy}{dt}\right) \\ F'(x) &= \tfrac{1}{2}\left(\frac{dy}{dx} + \frac{1}{a}\frac{dy}{dt}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

equations which determine the functions f' and F' for all values of the argument from $x = -\infty$ to $x = \infty$, if the initial values of dy/dx and dy/dt be known.

If the disturbance be originally confined to a finite portion of the string, the positive and negative waves separate after the interval of time required for each to traverse half the disturbed portion.

Fig. 23.



Suppose, for example, that AB is the part initially disturbed. A point P on the positive side remains at rest until the positive wave has travelled from A to P, is disturbed during the passage of the wave, and ever after remains at rest. The negative wave never affects P at all. Similar statements apply, *mutatis mutandis*, to a point Q on the negative side of AB. If the character of the original disturbance be such that $a\,dy/dx - dy/dt$ vanishes initially, there is no positive wave, and the point P is never disturbed at all; and if $a\,dy/dx + dy/dt$ vanish initially, there is no negative wave. If dy/dt vanish initially, the positive and the negative waves are similar and equal, and then neither can vanish. In cases where either wave vanishes, its evanescence may be considered to be due to the mutual destruction of two component waves, one depending on the initial displacements, and the other on the initial velocities. On the one side these two waves conspire, and on the other they destroy one another. This explains the apparent paradox, that P can fail to be affected sooner or later after AB has been disturbed.

The subsequent motion of a string that is initially displaced without velocity, may be readily traced by graphical methods. Since the positive and the negative waves are equal, it is only necessary to divide the original disturbance into two equal parts, to displace these, one to the right, and the other to the left, through a space equal to at, and then to recompound them. We shall presently apply this method to the case of a plucked string of finite length.

145. Vibrations are called *stationary*, when the motion of each particle of the system is proportional to some function of the time, the same for all the particles. If we endeavour to satisfy

$$\frac{d^2y}{dt^2} = a^2 \frac{d^2y}{dx^2} \quad \text{.............................(1)},$$

by assuming $y = XT$, where X denotes a function of x only, and T a function of t only, we find

$$\frac{1}{T}\frac{d^2T}{d\,(at)^2} = \frac{1}{X}\frac{d^2X}{dx^2} = m^2 \quad \text{(a constant)},$$

so that

$$\left.\begin{aligned} T &= A \cos mat + B \sin mat \\ X &= C \cos mx + D \sin mx \end{aligned}\right\} \quad \text{..............(2)},$$

proving that the vibrations must be simple harmonic, though of arbitrary period. The value of y may be written

$$y = P \cos (mat - \epsilon) \cos (mx - \alpha)$$

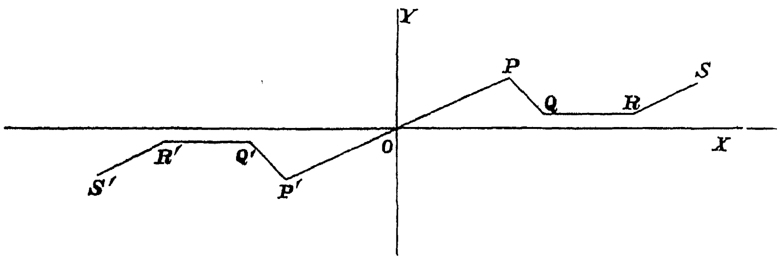
$$= \tfrac{1}{2} P \cos (mat + mx - \epsilon - \alpha) + \tfrac{1}{2} P \cos (mat - mx - \epsilon + \alpha) \text{...(3)},$$

shewing that the most general kind of stationary vibration may be regarded as due to the superposition of equal progressive vibrations, whose directions of propagation are opposed. Conversely, two stationary vibrations may combine into a progressive one.

The solution $y = f(x - at) + F(x + at)$ applies in the first instance to an infinite string, but may be interpreted so as to give the solution of the problem for a finite string in certain cases. Let us suppose, for example, that the string terminates at $x=0$, and is held fast there, while it extends to infinity in the positive direction only. Now so long as the point $x=0$ actually remains at rest, it is a matter of indifference whether the string be prolonged on the negative side or not. We are thus led to regard the given string as forming part of one doubly infinite, and to seek whether and how the initial displacements and velocities on the negative side can be taken, so that on the whole there shall be no displacement at $x=0$ throughout the subsequent motion. The initial values of y and $\dot{y}$ on the positive side determine the corresponding parts of the positive and negative waves, into which we know that the whole motion can be resolved. The former has no influence at the point $x=0$. On the negative side the positive and the negative waves are initially at our disposal, but with the latter we are not concerned. The problem is to determine the positive wave on the negative side, so that in conjunction with the given negative wave on the positive side of the origin, it shall leave that point undisturbed.

Let $OPQRS$... be the line (of any form) representing the wave in OX, which advances in the negative direction. It is

Fig. 24.

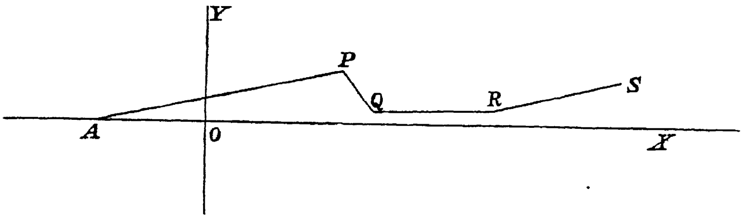


evident that the requirements of the case are met by taking on the other side of O what may be called the *contrary* wave, so that O is the geometrical centre, bisecting every chord (such as PP') which passes through it. Analytically, if $y=f(x)$ is the equation of $OPQRS$......, $-y=f(-x)$ is the equation of $OP'Q'R'S'$...... When after a time t the curves are shifted to the left and to the right respectively through a distance at, the co-ordinates corresponding to $x=0$ are necessarily equal and opposite, and therefore when compounded give zero resultant displacement.

The effect of the constraint at O may therefore be represented by supposing that the negative wave moves through undisturbed, but that a positive wave at the same time emerges from O. This reflected wave may at any time be found from its parent by the following rule:

Let $APQRS$... be the position of the parent wave. Then the reflected wave is the position which this would assume, if it were

Fig. 25.



turned through two right angles, first about OX as an axis of rotation, and then through the same angle about OY. In other words, the return wave is the image of $APQRS$ formed by successive optical reflection in OX and OY, regarded as plane mirrors.

The same result may also be obtained by a more analytical process. In the general solution

$$y = f(x - at) + F(x + at),$$

the functions $f(z)$, $F(z)$ are determined by the initial circumstances for all positive values of z. The condition at $x = 0$ requires that

$$f(-at) + F(at) = 0$$

for all positive values of t, or

$$f(-z) = -F(z)$$

for positive values of z. The functions f and F are thus determined for all positive values of x and t.

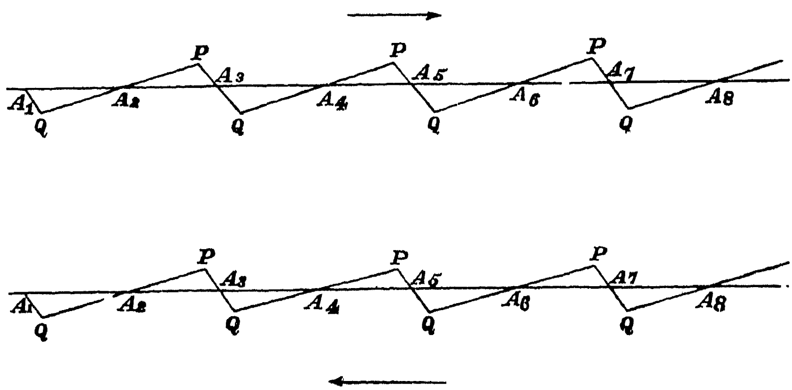
There is now no difficulty in tracing the course of events when *two* points of the string A and B are held fast. The initial disturbance in AB divides itself into positive and negative waves, which are reflected backwards and forwards between the fixed points, changing their character from positive to negative, and *vice versâ*, at each reflection. After an even number of reflections in each case the original form and motion is completely recovered. The process is most easily followed in imagination when the initial disturbance is confined to a small part of the string, more particularly when its character is such as to give rise to a wave propagated in one direction only. The *pulse* travels with uniform velocity (a) to and fro along the length of the string, and after it has returned *a second time* to its starting point the original condition of things is exactly restored. The period of the motion is thus the time required for the pulse to traverse the length of the string twice, or

$$\tau = 2l/a \qquad \text{.............................(1).}$$

The same law evidently holds good whatever may be the character of the original disturbance, only in the general case it may happen that the *shortest* period of recurrence is some aliquot part of τ.

146. The method of the last few sections may be advantageously applied to the case of a plucked string. Since the initial velocity vanishes, half of the displacement belongs to the positive and half to the negative wave. The manner in which the wave must be completed so as to produce the same effect as the constraint, is shewn in the figure, where the upper curve represents

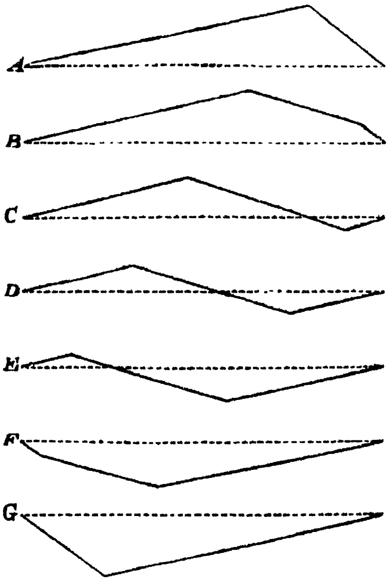
Fig. 26.



the positive, and the lower the negative wave in their initial positions. In order to find the configuration of the string at any future time, the two curves must be superposed, after the upper has been shifted to the right and the lower to the left through a space equal to at.

The resultant curve, like its components, is made up of straight pieces. A succession of six at intervals of a twelfth of the period,

Fig. 27.



shewing the course of the vibration, is given in the figure (Fig. 27), taken from Helmholtz. From G the string goes back again to A through the same stages[1].

It will be observed that the inclination of the string at the points of support alternates between two constant values.

147. If a small disturbance be made at the time t at the point x of an infinite stretched string, the effect will not be felt at O until after the lapse of the time x/a, and will be in all respects the same as if a like disturbance had been made at the point $x + \Delta x$ at time $t - \Delta x/a$. Suppose that similar disturbances are communicated to the string at intervals of time τ at points whose distances from O increase each time by $a\,\delta\tau$, then it is evident that the result at O will be the same as if the disturbances were all made at the same point, provided that the time-intervals be increased from τ to $\tau + \delta\tau$. This remark con-

[1] This method of treating the vibration of a plucked string is due to Young. *Phil. Trans.*, 1800. The student is recommended to make himself familiar with it by actually constructing the forms of Fig. 27.

tains the theory of the alteration of pitch due to motion of the source of disturbance; a subject which will come under our notice again in connection with aerial vibrations.

148. When one point of an infinite string is subject to a forced vibration, trains of waves proceed from it in both directions according to laws, which are readily investigated. We shall suppose that the origin is the point of excitation, the string being there subject to the forced motion $y = Ae^{ipt}$; and it will be sufficient to consider the positive side. If the motion of each element ds be resisted by the frictional force $\kappa\rho\dot{y}\,ds$, the differential equation is

$$\frac{d^2y}{dt^2} + \kappa\frac{dy}{dt} = a^2\frac{d^2y}{dx^2} \text{ ....(1)};$$

or since $y \propto e^{ipt}$,

$$\frac{d^2y}{dx^2} = \left(\frac{i\kappa p}{a^2} - \frac{p^2}{a^2}\right) y = \lambda^2 y \text{ ....(2)},$$

if for brevity we write λ^2 for the coefficient of y.

The general solution is

$$y = \{Ce^{-\lambda x} + D\,e^{+\lambda x}\}\,e^{ipt} \text{ ....(3)}.$$

Now since y is supposed to vanish at an infinite distance, D must vanish, if the real part of λ be taken positive. Let

$$\lambda = \alpha + i\beta,$$

where α is positive.

Then the solution is

$$y = Ae^{-(\alpha+i\beta)x + ipt} \text{ ....(4)},$$

or, on throwing away the imaginary part,

$$y = Ae^{-\alpha x}\cos(pt - \beta x) \text{ ....(5)},$$

corresponding to the forced motion at the origin

$$y = A\cos pt \text{ ....(6)}.$$

An arbitrary constant may, of course, be added to t.

To determine α and β, we have

$$\alpha^2 - \beta^2 = -\frac{p^2}{a^2}; \qquad 2\alpha\beta = \frac{\kappa p}{a^2} \text{ ....(7)}.$$

If we suppose that κ is small,

$$\beta = p/a, \qquad \alpha = \kappa/2a \quad \text{nearly},$$

and

$$y = Ae^{-\kappa x/2a}\cos\left(pt - \frac{p}{a}x\right) \text{ ....(8)}.$$

This solution shews that there is propagated along the string a wave, whose amplitude slowly diminishes on account of the exponential factor. If $\kappa = 0$, this factor disappears, and we have simply

$$y = A \cos\left(pt - \frac{px}{a}\right) \qquad\qquad (9).$$

This result stands in contradiction to the general law that, when there is no friction, the forced vibrations of a system (due to a single simple harmonic force) must be synchronous in phase throughout. According to (9), on the contrary, the phase varies continuously in passing from one point to another along the string. The fact is, that we are not at liberty to suppose $\kappa = 0$ in (8), inasmuch as that equation was obtained on the assumption that the real part of λ in (3) is positive, and not zero. However long a finite string may be, the coefficient of friction may be taken so small that the vibrations are not damped before reaching the further end. After this point of smallness, reflected waves begin to complicate the result, and when the friction is diminished indefinitely, an infinite series of such must be taken into account, and would give a resultant motion of the same phase throughout.

This problem may be solved for a string whose mass is supposed to be concentrated at equidistant points, by the method of § 120. The co-ordinate ψ_1 may be supposed to be given ($= He^{ipt}$), and it will be found that the system of equations (5) of § 120 may all be satisfied by taking

$$\psi_r = \theta^{r-1}\psi_1 \qquad\qquad (10),$$

where θ is a complex constant determined by a quadratic equation. The result for a continuous string may be afterwards deduced.

[In the notation of § 120 the quadratic equation is

$$B\theta^2 + A\theta + B = 0 \qquad\qquad (11),$$

where

$$A = -\mu p^2 + \frac{2T_1}{a}, \quad B = -\frac{T_1}{a} \qquad\qquad (12).$$

The roots of (11) are

$$\theta = \frac{-A \pm \sqrt{(A^2 - 4B^2)}}{2B} \qquad\qquad (13),$$

and are imaginary if $4B^2 > A^2$, that is, if

$$p^2 < \frac{4T_1}{\mu a} \qquad\qquad (14),$$

a condition always satisfied in passing to the limit where α and μ are infinitely small. In any case when (14) is satisfied the modulus of θ is unity, so that (10) represents wave propagation.

If, however, (14) be not satisfied, the values of θ are real. In this case all the motions are in the same phase, and no wave is propagated. The vibration impressed upon ψ_1 is imitated upon a reduced scale by ψ_2, ψ_3......, with amplitudes which form a geometrical progression. In the first case the motion is propagated to an infinite distance, but in the second it is practically confined to a limited region round the source.]

148 *a*. So long as the conditions of § 144 are satisfied, a positive, or a negative, wave is propagated undisturbed. If however there be any want of uniformity, such (for example) as that caused by a load attached at a particular point, reflection will ensue when that point is reached. The most interesting problem under this head is that of two strings of different longitudinal densities, attached to one another, and vibrating transversely under the common tension T_1. Or, if we regard the string as single, the density may be supposed to vary discontinuously from one uniform value (ρ_1) to another (ρ_2). If a_1, a_2 denote the corresponding velocities of propagation,

$$a_1^2 = T_1/\rho_1, \qquad a_2^2 = T_1/\rho_2 \quad\text{..........}(1),$$

and

$$\mu = a_1/a_2 = \sqrt{(\rho_2/\rho_1)} \quad\text{..........}(2).$$

The conditions to be satisfied at the junction of the two parts are (i) the continuity of the displacement y, and (ii) the continuity of dy/dx. If the two parts met at a finite angle, an infinitely small element at the junction would be subject to a finite force.

Let us suppose that a positive wave of harmonic type, travelling in the first part (ρ_1), impinges upon the second (ρ_2). In the latter the motion will be adequately represented by a positive wave, but in the former we must provide for a negative reflected wave. Thus we may take for the two parts respectively

$$y = He^{ik_1(a_1t-x)} + Ke^{ik_1(a_1t+x)} \quad\text{..........}(3),$$

$$y = Le^{ik_2(a_2t-x)} \quad\text{..........}(4),$$

where

$$k_1 = 2\pi/\lambda_1, \qquad k_2 = 2\pi/\lambda_2,$$

so that

$$k_1a_1 = k_2a_2 \quad\text{..........}(5).$$

The conditions at the junction ($x = 0$) give

$$H + K = L \quad\text{.........................(6)},$$

$$k_1 H - k_1 K = k_2 L \quad\text{.........................(7)}$$

whence
$$\frac{K}{H} = \frac{k_1 - k_2}{k_1 + k_2} = -\frac{\mu - 1}{\mu + 1} \quad\text{....................(8)}.$$

Since the ratio K/H is real, we may suppose that both quantities are real; and if we throw away the imaginary parts from (3) and (4) we get as the solution in terms of real quantities

$$y = H \cos k_1 (a_1 t - x) + K \cos k_1 (a_1 t + x) \text{..............(9)};$$

$$y = (H + K) \cos k_2 (a_2 t - x) \quad\text{.........................(10)}.$$

The ratio of amplitudes of the reflected and the incident waves expressed by (8) is that first obtained by T. Young for the corresponding problem in Optics.

148 *b*. The expression for the intensity of reflection established in § 148 *a* depends upon the assumption that the transition from the one density to the other is sudden, that is occupies a distance which is small in comparison with a wave length. If the transition be gradual, the reflection may be expected to fall off, and in the limit to disappear altogether.

The problem of gradual transition includes, of course, that of a variable medium, and would in general be encumbered with great difficulties. There is, however, one case for which the solution may be readily expressed, and this it is proposed to consider in the present section. The longitudinal density is supposed to vary as the inverse square of the abscissa. If y, denoting the transverse displacement be proportional to e^{ipt}, the equation which it must satisfy as a function of x, is (§ 141),

$$\frac{d^2 y}{dx^2} + n^2 x^{-2} y = 0 \text{.......................(1)},$$

where n^2 is some positive constant, of the nature of an abstract number.

The solution of (1) is $y = A x^{\frac{1}{2} + im} + B x^{\frac{1}{2} - im}$ ....(2),

where
$$m^2 = n^2 - \tfrac{1}{4} \quad\text{............................(3)}.$$

If m be real, that is, if $n > \frac{1}{2}$, we may obtain, by supposing $A = 0$, as a final solution in real quantities,

$$y = C x^{\frac{1}{2}} \cos (pt - m \log x + \epsilon) \quad\text{.................(4)},$$

which represents a positive progressive wave, in many respects similar to those propagated in uniform media.

Let us now suppose that, to the left of the point $x = x_1$, the variable medium is replaced by one of uniform constitution, such that there is no discontinuity of density at the point of transition; and let us inquire what reflection a positive progressive wave in the uniform medium will undergo on arrival at the variable medium. It will be sufficient to consider the case where m is real, that is, where the change of density is but moderately rapid.

By supposition, there is no negative wave in the variable medium, so that $A = 0$ in (2). Thus

$$y = Bx^{\frac{1}{2}-im}, \qquad \frac{dy}{dx} = (\tfrac{1}{2} - im)Bx^{-\frac{1}{2}-im};$$

and, when $x = x_1$,

$$\frac{dy}{y\,dx} = \frac{\frac{1}{2} - im}{x_1} \quad \text{.....................(5).}$$

The general solution for the uniform medium, satisfying the equation $d^2y/dx^2 + n^2x_1^{-2}y = 0$, may be written

$$y = He^{-in\frac{x-x_1}{x_1}} + Ke^{+in\frac{x-x_1}{x_1}} \quad \text{.......................(6),}$$

from which, when $x = x_1$,

$$\frac{dy}{y\,dx} = -\frac{in}{x_1}\frac{H-K}{H+K} \quad \text{.......................(7).}$$

In equation (6), H represents the amplitude of the incident positive wave, and K the amplitude of the reflected negative wave. The condition to be satisfied at $x = x_1$ is expressed by equating the values of $\dfrac{dy}{y\,dx}$ given by (5) and (7). Thus

$$\frac{K}{H} = \frac{i(n-m) + \frac{1}{2}}{i(n+m) - \frac{1}{2}} \quad \text{.......................(8),}$$

which gives, in symbolical form, the ratio of the reflected to the incident vibration.

Having regard to (3), we may write (8) in the form

$$\frac{K}{H} = \frac{-i}{2(n+m)} \quad \text{.......................(9);}$$

so that the amplitude of the reflected wave is $\frac{1}{2}(n+m)^{-1}$ of that of the incident. Thus, as was to be expected, when n and m are great, *i.e.*, when the density changes slowly in the variable medium, there is but little reflection. As regards phase, the result embodied in (9) may be represented by supposing that the reflection occurs at $x=x_1$, and involves a change of phase amounting to a quarter period.

Passing on now to the more important problem, we will suppose that the variable medium extends only so far as the point $x=x_2$, beyond which the density retains uniformly its value at that point. A positive wave travelling at first in a uniform medium of density proportional to x_1^{-2}, passes at the point $x=x_1$ into a variable medium of density proportional to x^{-2}, and again, at the point $x=x_2$, into a uniform medium of density proportional to x_2^{-2}. The velocities of propagation are inversely proportional to the square roots of the densities, so that, if μ be the refractive index between the extreme media,

$$\mu=\frac{x_1}{x_2} \qquad\qquad (10).$$

The thickness (d) of the layer of transition is

$$d=x_2-x_1 \qquad\qquad (11).$$

The wave-lengths in the two media are given by

$$\lambda_1=\frac{2\pi x_1}{n}, \qquad \lambda_2=\frac{2\pi x_2}{n},$$

so that
$$n=\frac{2\pi d}{\lambda_2-\lambda_1}=\frac{2\pi d}{(\mu^{-1}-1)\lambda_1} \qquad\qquad (12).$$

For the first medium we take, as before,

$$y=He^{-in\frac{x-x_1}{x_1}}+Ke^{+in\frac{x-x_1}{x_1}} \qquad\qquad (6).$$

giving, when $x=x_1$,

$$\frac{dy}{y\,dx}=-\frac{in}{x_1}\frac{H-K}{H+K}=-\frac{in\theta}{x_1} \qquad\qquad (7),$$

if, for brevity, we write θ for $\dfrac{H-K}{H+K}$.

For the variable medium,

$$y=Ax^{\frac{1}{2}+im}+Bx^{\frac{1}{2}-im} \qquad\qquad (2),$$

giving, when $x=x_1$,

$$\frac{dy}{y\,dx}=x_1^{-1}\frac{(\frac{1}{2}+im)Ax_1^{im}+(\frac{1}{2}-im)Bx_1^{-im}}{Ax_1^{im}+Bx_1^{-im}} \qquad (13)$$

Hence the condition to be satisfied at $x = x_1$ gives

$$\tfrac{1}{2} + im \frac{Ax_1^{im} - Bx_1^{-im}}{Ax_1^{im} + Bx_1^{-im}} = -in\theta;$$

whence

$$\frac{A}{B} = x_1^{-2im} \frac{im - in\theta - \frac{1}{2}}{im + in\theta + \frac{1}{2}} \quad\text{.....................(14).}$$

The condition to be satisfied at $x = x_2$ may be deduced from (14), by substituting x_2 for x_1, putting at the same time $\theta = 1$ in virtue of the supposition that in the second medium there is no negative wave. Hence, equating the two values of $A : B$, we get

$$x_1^{-2im} \frac{im - in\theta - \frac{1}{2}}{im + in\theta + \frac{1}{2}} = x_2^{-2im} \frac{im - in - \frac{1}{2}}{im + in + \frac{1}{2}} \quad\text{.........(15),}$$

as the equation from which the reflected wave in the first medium is to be found. Having regard to (3), we get

$$\theta = \frac{H - K}{H + K} = \frac{m + n + \frac{1}{2}i + \mu^{2im}(m - n - \frac{1}{2}i)}{m + n - \frac{1}{2}i + \mu^{2im}(m - n + \frac{1}{2}i)},$$

so that

$$\frac{K}{H} = \frac{-i + \mu^{2im}\, i}{2(m + n) + 2\mu^{2im}(m - n)} \quad\text{............(16).}$$

This is the symbolical solution. To interpret it in real quantities, we must distinguish the cases of m real and m imaginary. If the transition be not too sudden, m is real, and (16) may be written

$$\frac{K}{H} = \frac{i}{2} \frac{-1 + \cos(2m \log \mu) + i \sin(2m \log \mu)}{m + n + (m - n)\cos(2m \log \mu) + i(m - n)\sin(2m \log \mu)}.$$

Thus the expression for the ratio of the *intensities* of the reflected and the incident waves is, after reduction,

$$\frac{\sin^2(m \log \mu)}{4m^2 + \sin^2(m \log \mu)} \quad\text{..................(17).}$$

If m be imaginary, we may write $im = m'$; (16) then gives for the ratio of intensities,

$$\frac{(\mu^{m'} - \mu^{-m'})^2}{(\mu^{m'} - \mu^{-m'})^2 + 16m'^2} \quad\text{..................(18);}$$

or, if we introduce the notation of hyperbolic trigonometry § 170,

$$\frac{\sinh^2(m' \log \mu)}{\sinh^2(m' \log \mu) + 4m'^2} \quad\text{..................(19).}$$

For the critical value $m = 0$, we get, from (17) or (19),

$$\frac{(\log \mu)^2}{4 + (\log \mu)^2} \quad\text{..........................(20).}$$

These expressions allow us to trace the effect of a more or less gradual transition between media of given indices. If the transition be absolutely abrupt, $n=0$, by (12); so that $m'=\frac{1}{2}$. In this case, (18) gives us (§ 148 *a*) Young's well-known formula

$$\left(\frac{\mu-1}{\mu+1}\right)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21).$$

Since $\frac{\sinh x}{x}$ increases continually from $x=0$, the ratio (19) increases continually from $m'=0$ to $m'=\frac{1}{2}$, *i.e.*, diminishes continually from the case of sudden transition $m'=\frac{1}{2}$, when its value is (21), to the critical case $m'=0$, when its value is (20), after which this form no longer holds good. When $m'=0$, $n=\frac{1}{2}$, and, by (12), $d=(\lambda_2-\lambda_1)/4\pi$.

When $n>\frac{1}{2}$, (17) is the appropriate form. We see from it that with increasing n the reflection diminishes, until it vanishes, when $m\log\mu=\pi$, *i.e.* when

$$n^2=\tfrac{1}{4}+\frac{\pi^2}{(\log\mu)^2}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22).$$

With a still more gradual transition the reflection revives, reaches a maximum, again vanishes when $m\log\mu=2\pi$, and so on[1].

148 *c*. In the problem of connected strings, vibrating under the influence of tension alone, the velocity in each uniform part is independent of wave length, and there is nothing corresponding to optical *dispersion*. This state of things will be departed from if we introduce the consideration of stiffness, and it may be of interest to examine in a simple case how far the problem of reflection is thereby modified. As in § 148 *a*, we will suppose that at $x=0$ the density changes discontinuously from ρ_1 to ρ_2, but that now the vibrations of the second part occur under the influence of sensible stiffness. The differential equation applicable in this case is, § 188,

$$\beta^2\frac{d^4y}{dx^4}-a_2^2\frac{d^2y}{dx^2}+\frac{d^2y}{dt^2}=0,$$

or, if y vary as e^{int},

$$-\beta^2\frac{d^4y}{dx^4}+a_2^2\frac{d^2y}{dx^2}+n^2y=0\quad\ldots\ldots\ldots\ldots(1),$$

so that, if y vary as e^{ikx},

$$\beta^2k^4+a_2^2k^2-n^2=0\quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

[1] *Proc. Math. Soc.*, vol. XI. February, 1880; where will also be found a numerical example illustrative of optical conditions.

In consequence of the stiffness represented by β^2 the velocity of propagation deviates from a_2, and must be found from (2). The two values of k^2 given by this equation are real, one being positive and the other negative. The four admissible values of k may thus be written $\pm k_2$, $\pm ih_2$, so that the complete solution of (1) will be

$$y = Ae^{ik_2x} + Be^{-ik_2x} + Ce^{-h_2x} + De^{h_2x} \quad \text{.........} \quad (3),$$

h_2, k_2 being real and positive. The velocity of propagation is n/k_2

In the application which we have to make the disturbance of the imperfectly flexible second part is due to a positive wave entering it from the first part. When x is great and positive, (3) must reduce to its second term. Thus

$$A = 0, \quad D = 0;$$

and we are left with

$$y = Be^{-ik_2x} + Ce^{-h_2x} \quad \text{...................} \quad (4).$$

This holds when x is positive. When x is negative, corresponding to the perfectly flexible first part, we have

$$y = He^{-ik_1x} + Ke^{ik_1x} \quad \text{....................} \quad (5),$$

in which

$$k_1 = n/a_1 \quad \text{.............................} \quad (6).$$

The "refractive index" is given by

$$\mu = k_2/k_1 \quad \text{.............................} \quad (7).$$

The conditions at the junction are first the continuity of y and dy/dx. Further, d^2y/dx^2 in (4) must vanish at this place, inasmuch as curvature implies a couple (§ 162), and this could not be transmitted by the first part. Hence

$$H + K = B + C \quad \text{.........................} \quad (8),$$

$$k_1(H - K) = k_2B - ih_2C \quad \text{.................} \quad (9),$$

$$-k_2^2B + h_2^2C = 0 \quad \text{....................} \quad (10).$$

From these we deduce

$$\frac{H+K}{H-K} = \frac{k_1(h_2 + ik_2)}{k_2h_2} \quad \text{.................} \quad (11),$$

$$\frac{K}{H} = \frac{h_2(k_1 - k_2) + ik_1k_2}{h\,(k_1 + k_2) + ik_1k_2} \quad \text{..............} \quad (12);$$

and thence for the *intensity* of reflection, equal to Mod². (K/H),

$$\frac{(k_1 - k_2)^2 + k_1^2 k_2^2/h_2^2}{(k_1 + k_2)^2 + k_1^2 k_2^2/h_2^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(13)}.$$

If the second part, as well as the first, be perfectly flexible, $\beta = 0$, $h_2 = \infty$, and we fall back on Young's formula. In general, the intensity of reflection is not accurately given by this formula, even though we employ therein the value of the refractive index appropriate to the waves actually under propagation.

CHAPTER VII.

LONGITUDINAL AND TORSIONAL VIBRATIONS OF BARS.

149. THE next system to the string in order of simplicity is the bar, by which term is usually understood in Acoustics a mass of matter of uniform substance and elongated cylindrical form. At the ends the cylinder is cut off by planes perpendicular to the generating lines. The centres of inertia of the transverse sections lie on a straight line which is called the *axis*.

The vibrations of a bar are of three kinds—longitudinal, torsional, and lateral. Of these the last are the most important, but at the same time the most difficult in theory. They are considered by themselves in the next chapter, and will only be referred to here so far as is necessary for comparison and contrast with the other two kinds of vibrations.

Longitudinal vibrations are those in which the axis remains unmoved, while the transverse sections vibrate to and fro in the direction perpendicular to their planes. The moving power is the resistance offered by the rod to extension or compression.

One peculiarity of this class of vibrations is at once evident. Since the force necessary to produce a given extension in a bar is proportional to the area of the section, while the mass to be moved is also in the same proportion, it follows that for a bar of given length and material the periodic times and the modes of vibration are independent of the area and of the form of the transverse section. A similar law obtains, as we shall presently see, in the case of torsional vibrations.

It is otherwise when the vibrations are lateral. The periodic times are indeed independent of the thickness of the bar in the direction perpendicular to the plane of flexure, but the motive power in this case, viz. the resistance to bending, increases more rapidly than the thickness in that plane, and therefore an increase in thickness is accompanied by a rise of pitch.

In the case of longitudinal and lateral vibrations, the mechanical constants concerned are the density of the material and the value of Young's modulus. For small extensions (or compressions) Hooke's law, according to which the tension varies as the extension, holds good. If the extension, viz. $\frac{\text{actual length} - \text{natural length}}{\text{natural length}}$, be called ϵ, we have $T = q\epsilon$, where q is Young's modulus, and T is the tension per unit area necessary to produce the extension ϵ. Young's modulus may therefore be defined as the force which would have to be applied to a bar of unit section, in order to double its length, if Hooke's law continued to hold good for so great extensions; its dimensions are accordingly those of a force divided by an area.

The torsional vibrations depend also on a second elastic constant μ, whose interpretation will be considered in the proper place.

Although in theory the three classes of vibrations, depending respectively on resistance to extension, to torsion, and to flexure are quite distinct, and independent of one another so long as the squares of the strains may be neglected, yet in actual experiments with bars which are neither uniform in material nor accurately cylindrical in figure it is often found impossible to excite longitudinal or torsional vibrations without the accompaniment of some measure of lateral motion. In bars of ordinary dimensions the gravest lateral motion is far graver than the gravest longitudinal or torsional motion, and consequently it will generally happen that the principal tone of either of the latter kinds agrees more or less perfectly in pitch with some overtone of the former kind. Under such circumstances the regular modes of vibrations become unstable, and a small irregularity may produce a great effect. The difficulty of exciting purely longitudinal vibrations in a bar is similar to that of getting a string to vibrate in one plane.

With this explanation we may proceed to consider the three classes of vibrations independently, commencing with longitudinal vibrations, which will in fact raise no mathematical questions beyond those already disposed of in the previous chapters.

150. When a rod is stretched by a force parallel to its length, the stretching is in general accompanied by lateral contraction in such a manner that the augmentation of volume is less than if the displacement of every particle were parallel to the axis. In the case of a short rod and of a particle situated near the cylindrical boundary, this lateral motion would be comparable in magnitude with the longitudinal motion, and could not be overlooked without risk of considerable error. But where a rod, whose length is great in proportion to the linear dimensions of its section, is subject to a stretching of one sign throughout, the longitudinal motion accumulates, and thus in the case of ordinary rods vibrating longitudinally in the graver modes, the inertia of the lateral motion may be neglected. Moreover we shall see later how a correction may be introduced, if necessary.

Let x be the distance of the layer of particles composing any section from the equilibrium position of one end, when the rod is unstretched, either by permanent tension or as the result of vibrations, and let ξ be the displacement, so that the actual position is given by $x+\xi$. The equilibrium and actual position of a neighbouring layer being $x+\delta x$, $x+\delta x+\xi+\dfrac{d\xi}{dx}\delta x$ respectively, the *elongation* is $d\xi/dx$, and thus, if T be the tension per unit area acting across the section,

$$T=q\frac{d\xi}{dx}\text{..........................(1).}$$

Consider now the forces acting on the slice bounded by x and $x+\delta x$. If the area of the section be ω, the tension at x is by (1) $q\omega\, d\xi/dx$, acting in the negative direction, and at $x+\delta x$ the tension is

$$q\omega\left(\frac{d\xi}{dx}+\frac{d^2\xi}{dx^2}\delta x\right),$$

acting in the positive direction; and thus the force on the slice due to the action of the adjoining parts is on the whole

$$q\omega\frac{d^2\xi}{dx^2}\delta x.$$

The mass of the element is $\rho\omega\,\delta x$, if ρ be the original density, and therefore if X be the accelerating force acting on it, the equation of equilibrium is

$$X+\frac{q}{\rho}\frac{d^2\xi}{dx^2}=0\text{..........................(2).}$$

In what follows we shall not require to consider the operation of an impressed force. To find the equation of motion we have only to replace X by the reaction against acceleration $-\ddot{\xi}$, and thus if $q : \rho = a^2$, we have

$$\frac{d^2\xi}{dt^2} = a^2 \frac{d^2\xi}{dx^2} \qquad \text{(3)}.$$

This equation is of the same form as that applicable to the transverse displacements of a stretched string, and indicates the undisturbed propagation of waves of any type in the positive and negative directions. The velocity a is relative to the *unstretched* condition of the bar; the apparent velocity with which a disturbance is propagated in space will be greater in the ratio of the stretched and unstretched lengths of any portion of the bar. The distinction is material only in the case of permanent tension.

151. For the actual magnitude of the velocity of propagation, we have

$$a^2 = q : \rho = q\omega : \rho\omega,$$

which is the ratio of the whole tension necessary (according to Hooke's law) to double the length of the bar and the longitudinal density. If the same bar were stretched with total tension T, and were flexible, the velocity of propagation of waves along it would be $\sqrt{(T : \rho\omega)}$. In order then that the velocity might be the same in the two cases, T must be $q\omega$, or, in other words, the tension would have to be that theoretically necessary in order to double the length. The tones of longitudinally vibrating rods are thus very high in comparison with those obtainable from strings of comparable length.

In the case of steel the value of q is about 22×10^8 grammes weight per square centimetre. To express this in absolute units of force on the C. G. S.[1] system, we must multiply by 980. In the same system the density of steel (identical with its specific gravity referred to water) is 7·8. Hence for steel

$$a = \sqrt{\frac{980 \times 22 \times 10^8}{7 \cdot 8}} = 530{,}000$$

approximately, which shews that the velocity of sound in steel is about 530,000 centimetres per second, or about 16 times greater

[1] Centimetre, Gramme, Second. This system is recommended by a Committee of the British Association. *Brit. Ass. Report*, 1873.

than the velocity of sound in air. In glass the velocity is about the same as in steel.

It ought to be mentioned that in strictness the value of q determined by statical experiments is not that which ought to be used here. As in the case of gases, which will be treated in a subsequent chapter, the rapid alterations of state concerned in the propagation of sound are attended with thermal effects, one result of which is to increase the effective value of q beyond that obtained from observations on extension conducted at a constant temperature. But the data are not precise enough to make this correction of any consequence in the case of solids.

152. The solution of the general equation for the longitudinal vibrations of an unlimited bar, namely

$$\xi = f(x - at) + F(x + at),$$

being the same as that applicable to a string, need not be further considered here.

When both ends of a bar are free, there is of course no permanent tension, and at the ends themselves there is no temporary tension. The condition for a free end is therefore

$$\frac{d\xi}{dx} = 0 \quad\text{.............................(1).}$$

To determine the normal modes of vibration, we must assume that ξ varies as a harmonic function of the time—$\cos nat$. Then as a function of x, ξ must satisfy

$$\frac{d^2\xi}{dx^2} + n^2\xi = 0 \quad\text{..........................(2),}$$

of which the complete integral i.

$$\xi = A \cos nx + B \sin nx \quad\text{....................(3),}$$

where A and B are independent of x.

Now since $d\xi/dx$ vanishes always when $x = 0$, we get $B = 0$; and again since $d\xi/dx$ vanishes when $x = l$—the natural length of the bar, $\sin nl = 0$, which shews that n is of the form

$$n = \frac{i\pi}{l} \quad\text{.............................(4),}$$

i being integral.

Accordingly, the normal modes are given by equations of the form

$$\xi = A \cos \frac{i\pi x}{l} \cos \frac{i\pi at}{l} \quad\text{.....................}(5),$$

in which of course an arbitrary constant may be added to t, if desired.

The complete solution for a bar with both ends free is therefore expressed by

$$\xi = \Sigma_{i=0}^{i=\infty} \cos \frac{i\pi x}{l}\left\{A_i \cos \frac{i\pi at}{l} + B_i \sin \frac{i\pi at}{l}\right\} \quad\text{......}(6),$$

where A_i and B_i are arbitrary constants, which may be determined in the usual manner, when the initial values of ξ and $\dot{\xi}$ are given.

A zero value of i is admissible; it gives a term representing a displacement ξ constant with respect both to space and time, and amounting in fact only to an alteration of the origin.

The period of the gravest component in (6) corresponding to $i = 1$, is $2l/a$, which is the time occupied by a disturbance in travelling twice the length of the rod. The other tones found by ascribing integral values to i form a complete harmonic scale; so that according to this theory the note given by a rod in longitudinal vibration would be in all cases musical.

In the gravest mode the centre of the rod, where $x = \frac{1}{2}l$, is a place of no motion, or node; but the periodic elongation or compression $d\xi/dx$ is there a maximum.

153. The case of a bar with one end free and the other fixed may be deduced from the general solution for a bar with both ends free, and of twice the length. For whatever may be the initial state of the bar free at $x = 0$ and fixed at $x = l$, such displacements and velocities may always be ascribed to the sections of a bar extending from 0 to $2l$ and free at both ends as shall make the motions of the parts from 0 to l identical in the two cases. It is only necessary to suppose that from l to $2l$ the displacements and velocities are initially equal and opposite to those found in the portion from 0 to l at an equal distance from the centre $x = l$. Under these circumstances the centre must by the symmetry remain at rest throughout the motion, and then the portion from 0 to l satisfies all the required conditions. We conclude that the vibrations of a bar free at one end and fixed at the other are identical with those of one half of a bar of twice the length of which both ends are free, the latter vibrating only in the uneven modes, obtained by making i in succession all *odd* integers. The tones of the bar still belong to a harmonic scale, but the even tones (octave, &c. of the fundamental) are wanting.

The period of the gravest tone is the time occupied by a pulse in travelling *four* times the length of the bar.

154. When both ends of a bar are fixed, the conditions to be satisfied at the ends are that the value of ξ is to be invariable. At $x=0$, we may suppose that $\xi=0$. At $x=l$, ξ is a small constant α, which is zero if there be no permanent tension. Independently of the vibrations we have evidently $\xi = x\alpha \div l$, and we should obtain our result most simply by assuming this term at once. But it may be instructive to proceed by the general method.

Assuming that as a function of the time ξ varies as

$$A \cos nat + B \sin nat,$$

we see that as a function of x it must satisfy

$$\frac{d^2\xi}{dx^2} + n^2\xi = 0,$$

of which the general solution is

$$\xi = C \cos nx + D \sin nx \quad \text{.................(1)}.$$

But since ξ vanishes with x for all values of t, $C=0$, and thus we may write

$$\xi = \Sigma \sin nx \,\{A \cos nat + B \sin nat\}.$$

The condition at $x=l$ now gives

$$\Sigma \sin nl \,\{A \cos nat + B \sin nat\} = \alpha,$$

from which it follows that for every finite admissible value of n

$$\sin nl = 0, \quad \text{or} \quad n = \frac{i\pi}{l}.$$

But for the zero value of n, we get

$$A_0 \sin nl = \alpha,$$

and the corresponding term in ξ is

$$\xi = A_0 \sin nx = \alpha \frac{\sin nx}{\sin nl} = \alpha \frac{x}{l}.$$

The complete value of ξ is accordingly

$$\xi = \alpha \frac{x}{l} + \Sigma_{\iota=1}^{\iota=\infty} \sin \frac{\iota\pi x}{l} \left\{A_\iota \cos \frac{\iota\pi at}{l} + B_\iota \sin \frac{\iota\pi at}{l}\right\} \ldots(2).$$

The series of tones form a complete harmonic scale (from which however any of the members may be missing in any actual case of vibration), and the period of the gravest component is the time taken by a pulse to travel twice the length of the rod, the same therefore as if both ends were free. It must be observed that we have here to do with the *unstretched* length of the rod, and that the period for a given natural length is independent of the permanent tension.

The solution of the problem of the doubly fixed bar in the case of no permanent tension might also be derived from that of a doubly free bar by mere differentiation with respect to x. For in the latter problem $d\xi/dx$ satisfies the necessary differential equation, viz.

$$\frac{d^2}{dt^2}\left(\frac{d\xi}{dx}\right) = a^2 \frac{d^2}{dx^2}\left(\frac{d\xi}{dx}\right),$$

inasmuch as ξ satisfies

$$\frac{d^2\xi}{dt^2} = a^2 \frac{d^2\xi}{dx^2};$$

and at both ends $d\xi/dx$ vanishes. Accordingly $d\xi/dx$ in this problem satisfies all the conditions prescribed for ξ in the case when both ends are fixed. The two series of tones are thus identical.

155. The effect of a small load M attached to any point of the rod is readily calculated approximately, as it is sufficient to assume the type of vibration to be unaltered (§ 88). We will take the case of a rod fixed at $x = 0$, and free at $x = l$. The kinetic energy is proportional to

$$\tfrac{1}{2}\int_0^l \rho\omega \sin^2 \frac{\iota\pi x}{2l}\, dx + \tfrac{1}{2} M \sin^2 \frac{\iota\pi x}{2l},$$

or to

$$\frac{\rho\omega l}{4}\left(1 + \frac{2M}{\rho\omega l} \sin^2 \frac{\iota\pi x}{2l}\right).$$

Since the potential energy is unaltered, we see by the principles of Chapter IV., that the effect of the small load M at a distance x from the fixed end is to increase the period of the component tones in the ratio

$$1 : 1 + \frac{M}{\rho\omega l} \sin^2 \frac{i\pi x}{2l}.$$

The small quantity $M : \rho\omega l$ is the ratio of the load to the whole mass of the rod.

If the load be attached at the free end, $\sin^2(i\pi x/2l) = 1$, and the effect is to depress the pitch of every tone by the same small interval. It will be remembered that i is here an *uneven* integer.

If the point of attachment of M be a node of any component, the pitch of that component remains unaltered by the addition.

156 Another problem worth notice occurs when the load at the free end is great in comparison with the mass of the rod. In this case we may assume as the type of vibration, a condition of uniform extension along the length of the rod.

If ξ be the displacement of the load M, the kinetic energy is

$$T = \tfrac{1}{2} M\dot{\xi}^2 + \tfrac{1}{2}\dot{\xi}^2 \int_0^l \rho\omega \frac{x^2}{l^2} dx = \tfrac{1}{2}\dot{\xi}^2 (M + \tfrac{1}{3}\rho\omega l) \;\;\ldots\ldots\ldots(1).$$

The tension corresponding to the displacement ξ is $q\omega\,\xi/l$, and thus the potential energy of the displacement is

$$V = \frac{q\omega\xi^2}{2l} \;\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The equation of motion is

$$(M + \tfrac{1}{3}\rho\omega l)\,\ddot{\xi} + \frac{q\omega}{l}\,\xi = 0,$$

and if $\xi \propto \cos pt$

$$p^2 = \frac{q\omega}{l} \div (M + \tfrac{1}{3}\rho\omega l) \;\;\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The correction due to the inertia of the rod is thus equivalent to the addition to M of one-third of the mass of the rod.

156 *a*. So long as a rod or a wire is uniform, waves of longitudinal vibration are propagated along it without change of type, but any interruption, or alteration of mechanical properties, will in general give rise to reflection. If two uniform wires be joined, the problem of determining the reflection at the junction may be conducted as in § 148 *a*. The conditions to be satisfied at the junction are (i) the continuity of ξ, and (ii) the continuity of $q\omega\, d\xi/dx$, measuring the tension. If ρ_1, ρ_2, ω_1, ω_2, a_1, a_2 denote the volume densities, the sections, and the velocities in the two wires, the ratio of the reflected to the incident amplitude is given by

$$\frac{K}{H}=\frac{\rho_1\omega_1 a_1-\rho_2\omega_2 a_2}{\rho_1\omega_1 a_1+\rho_2\omega_2 a_2}\qquad\qquad(1).$$

The reflection vanishes, or the incident wave is propagated through the junction without loss, if

$$\rho_1\omega_1 a_1=\rho_2\omega_2 a_2\qquad\qquad(2).$$

This result illustrates the difficulty which is met with in obtaining effective transmission of sound from air to metal, or from metal to air, in the mechanical telephone. Thus the value of ρa is about 100,000 times greater in the case of steel than in the case of air.

157. Our mathematical discussion of longitudinal vibrations may close with an estimate of the error involved in neglecting the inertia of the lateral motion of the parts of the rod not situated on the axis. If the ratio of lateral contraction to longitudinal extension be denoted by μ, the lateral displacement of a particle distant r from the axis will be $\mu r\epsilon$ in the case of equilibrium, where ϵ is the extension. Although in strictness this relation will be modified by the inertia of the lateral motion, yet for the present purpose it may be supposed to hold good, § 88.

The constant μ is a numerical quantity, lying between 0 and $\frac{1}{2}$. If μ were negative, a longitudinal tension would produce a lateral swelling, and if μ were greater than $\frac{1}{2}$, the lateral contraction would be great enough to overbalance the elongation, and cause a diminution of volume on the whole. The latter state of things would be inconsistent with stability, and the former can scarcely be possible in ordinary solids. At one time it was supposed that μ was necessarily equal to $\frac{1}{4}$, so that there was only one independent elastic constant, but experiments have since shewn that μ is variable. For glass and brass Wertheim found experimentally $\mu=\frac{1}{3}$.

If η denote the lateral displacement of the particle distant r

from the axis, and if the section be circular, the kinetic energy due to the lateral motion is

$$\delta T = \pi\rho \int_0^l \int_0^r \dot{\eta}^2 dx \, . \, r dr = \frac{\rho\omega\mu^2 r^2}{4} \, . \int_0^l \left(\frac{d\dot{\xi}}{dx}\right)^2 dx.$$

Thus the whole kinetic energy is

$$T + \delta T = \frac{\rho\omega}{2} \int_0^l \dot{\xi}^2 dx + \frac{\rho\omega\mu^2 r^2}{4} \int_0^l \left(\frac{d\dot{\xi}}{dx}\right)^2 dx.$$

In the case of a bar free at both ends, we have

$$\xi \propto \cos \frac{i\pi x}{l}, \quad \frac{d\xi}{dx} \propto -\frac{i\pi}{l} \sin \frac{i\pi x}{l},$$

and thus

$$T + \delta T : T = 1 + \frac{i^2 \mu^2 \pi^2}{2} \frac{r^2}{l^2}.$$

The effect of the inertia of the lateral motion is therefore to increase the period in the ratio

$$1 : 1 + \frac{i^2 \mu^2 \pi^2}{4} \frac{r^2}{l^2}.$$

This correction will be nearly insensible for the graver modes of bars of ordinary proportions of length to thickness.

[A more complete solution of the problem of the present section has been given by Pochhammer[1], who applies the general equations for an elastic solid to the case of an infinitely extended cylinder of circular section. The result for longitudinal vibrations, so far as the term in r^2/l^2, is in agreement with that above determined. A similar investigation has also been published by Chree[2], who has also treated the more general question[3] in which the cylindrical section is not restricted to be circular.]

158. Experiments on longitudinal vibrations may be made with rods of deal or of glass. The vibrations are excited by friction § 138, with a wet cloth in the case of glass; but for metal or wooden rods it is necessary to use leather charged with powdered rosin. "The longitudinal vibrations of a pianoforte string may be excited by gently rubbing it longitudinally with a piece of india rubber, and those of a violin string by placing the bow obliquely across the string, and moving it along the string longitudinally, keeping the same point of the bow upon the string. The note is unpleasantly shrill in both cases."

[1] *Crelle*, Bd. 81, 1876. [2] *Quart. Math. Journ.*, Vol. 21, p. 287, 1886.
[3] *Ibid*, Vol. 23, p. 317, 1889.

"If the peg of the violin be turned so as to alter the pitch of the lateral vibrations very considerably, it will be found that the pitch of the longitudinal vibrations has altered very slightly. The reason of this is that in the case of the lateral vibrations the change of velocity of wave-transmission depends chiefly on the change of tension, which is considerable. But in the case of the longitudinal vibrations, the change of velocity of wave-transmission depends upon the change of extension, which is comparatively slight[1]."

In Savart's experiments on longitudinal vibrations, a peculiar sound, called by him a "son rauque," was occasionally observed, whose pitch was an octave below that of the longitudinal vibration. According to Terquem[2] the cause of this sound is a transverse vibration, whose appearance is due to an approximate agreement between its own period and that of the sub-octave of the longitudinal vibration § 68 *b*. If this view be correct, the phenomenon would be one of the second order, probably referable to the fact that longitudinal compression of a bar tends to produce curvature.

159. The second class of vibrations, called torsional, which depend on the resistance opposed to twisting, is of very small importance. A solid or hollow cylindrical rod of circular section may be twisted by suitable forces, applied at the ends, in such a manner that each transverse section remains in its own plane. But if the section be not circular, the effect of a twist is of a more complicated character, the twist being necessarily attended by a warping of the layers of matter originally composing the normal sections. Although the effects of the warping might probably be determined in any particular case if it were worth while, we shall confine ourselves here to the case of a circular section, when there is no motion parallel to the axis of the rod.

The force with which twisting is resisted depends upon an elastic constant different from q, called the rigidity. If we denote it by n, the relation between q, n, and μ may be written

$$n = \frac{q}{2(\mu+1)} \quad\text{.......................}(1)^{3},$$

[1] Donkin's *Acoustics*, p. 154.

[2] *Ann. de Chimie*, LVII. 129—190.

[3] Thomson and Tait, § 683. This, it should be remarked, applies to isotropic material only.

shewing that n lies between $\frac{1}{2}q$ and $\frac{1}{3}q$. In the case of $\mu=\frac{1}{3}$, $n=\frac{3}{8}q$.

Let us now suppose that we have to do with a rod in the form of a thin tube of radius r and thickness dr, and let θ denote the angular displacement of any section, distant x from the origin. The rate of twist at x is represented by $d\theta/dx$, and the shear of the material composing the pipe by $r\,d\theta/dx$. The opposing force per unit of area is $nr\,d\theta/dx$; and since the area is $2\pi r\,dr$, the moment round the axis is

$$2n\pi r^3\,dr\,\frac{d\theta}{dx}.$$

Thus the force of restitution acting on the slice dx has the moment

$$2n\pi r^3\,dr\,dx\,\frac{d^2\theta}{dx^2}.$$

Now the moment of inertia of the slice under consideration is $2\pi r\,dr\,.\,dx\,.\,\rho\,.\,r^2$, and therefore the equation of motion assumes the form

$$\rho\,\frac{d^2\theta}{dt^2}=n\,\frac{d^2\theta}{dx^2}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Since this is independent of r, the same equation applies to a cylinder of finite thickness or to one solid throughout.

The velocity of wave propagation is $\sqrt{(n/\rho)}$, and the whole theory is precisely similar to that of longitudinal vibrations, the condition for a free end being $d\theta/dx=0$, and for a fixed end $\theta=0$, or, if a permanent twist be contemplated, $\theta=$ constant.

The velocity of longitudinal vibrations is to that of torsional vibrations in the ratio $\sqrt{q}:\sqrt{n}$ or $\sqrt{(2+2\mu)}:1$. The same ratio applies to the frequencies of vibration for bars of equal length vibrating in corresponding modes under corresponding terminal conditions. If $\mu=\frac{1}{3}$, the ratio of frequencies would be

$$\sqrt{q}:\sqrt{n}=\sqrt{8}:\sqrt{3}=1\cdot63,$$

corresponding to an interval rather greater than a fifth.

In any case the ratio of frequencies must lie between

$$\sqrt{2}:1=1\cdot414,\quad\text{and}\quad\sqrt{3}:1=1\cdot732.$$

Longitudinal and torsional vibrations were first investigated by Chladni.

CHAPTER VIII.

LATERAL VIBRATIONS OF BARS.

160. In the present chapter we shall consider the lateral vibrations of thin elastic rods, which in their natural condition are straight. Next to those of strings, this class of vibrations is perhaps the most amenable to theoretical and experimental treatment. There is difficulty sufficient to bring into prominence some important points connected with the general theory, which the familiarity of the reader with circular functions may lead him to pass over too lightly in the application to strings; while at the same time the difficulties of analysis are not such as to engross attention which should be devoted to general mathematical and physical principles.

Daniel Bernoulli[1] seems to have been the first who attacked the problem. Euler, Riccati, Poisson, Cauchy, and more recently Strehlke[2], Lissajous[3], and A. Seebeck[4] are foremost among those who have advanced our knowledge of it.

161. The problem divides itself into two parts, according to the presence, or absence, of a permanent longitudinal tension. The consideration of permanent tension entails additional complication, and is of interest only in its application to stretched strings, whose stiffness, though small, cannot be neglected altogether. Our attention will therefore be given principally to the two extreme cases, (1) when there is no permanent tension, (2) when the tension is the chief agent in the vibration.

[1] *Comment. Acad. Petrop.* t. XIII.
[2] Pogg. *Ann.* Bd. XXVII. p. 505, 1833.
[3] *Ann. d. Chimie* (3), XXX. 385, 1850.
[4] *Abhandlungen d. Math. Phys. Classe d. K. Sächs. Gesellschaft d. Wissenschaften.* Leipzig, 1852.

With respect to the section of the rod, we shall suppose that one principal axis lies in the plane of vibration, so that the bending at every part takes place in a direction of maximum or minimum (or stationary) flexural rigidity. For example, the surface of the rod may be one of revolution, each section being circular, though not necessarily of constant radius. Under these circumstances the potential energy of the bending for each element of length is proportional to the square of the curvature multiplied by a quantity depending on the material of the rod, and on the moment of inertia of the transverse section about an axis through its centre of inertia perpendicular to the plane of bending. If ω be the area of the section, $\kappa^2\omega$ its moment of inertia, q Young's modulus, ds the element of length, and dV the corresponding potential energy for a curvature $1 \div R$ of the axis of the rod,

$$dV = \tfrac{1}{2} q \kappa^2 \omega \frac{ds}{R^2} \quad \text{..................(1).}$$

This result is readily obtained by considering the extension of the various filaments of which the bar may be supposed to be made up. Let η be the distance from the axis of the projection on the plane of bending of a filament of section $d\omega$. Then the length of the filament is altered by the bending in the ratio

$$1 : 1 + \frac{\eta}{R},$$

R being the radius of curvature. Thus on the side of the axis for which η is positive, viz. on the *outward* side, a filament is extended, while on the other side of the axis there is compression. The force necessary to produce the extension η/R is $q\,\eta/R\,.\,d\omega$ by the definition of Young's modulus; and thus the whole couple by which the bending is resisted amounts to

$$\int q\,\frac{\eta}{R}\,.\,\eta\,.\,d\omega = \frac{q}{R}\,\kappa^2\omega,$$

if ω be the area of the section and κ its radius of gyration about a line through the axis, and perpendicular to the plane of bending. The angle of bending corresponding to a length of axis ds is $ds \div R$, and thus the work required to bend ds to curvature $1 \div R$ is

$$\tfrac{1}{2} q\,\kappa^2\omega\,\frac{ds}{R^2},$$

since the *mean* is half the *final* value of the couple.

[For a more complete discussion of the legitimacy of the foregoing method of calculation the reader must be referred to works upon the Theory of Elasticity. The question of lateral vibrations has been specially treated by Pochhammer[1] on the basis of the general equations.]

For a circular section κ is one-half the radius.

That the potential energy of the bending would be proportional, *cæteris paribus*, to the square of the curvature, is evident beforehand. If we call the coefficient B, we may take

$$V = \tfrac{1}{2}\int B \frac{ds}{R^2},$$

or, in view of the approximate straightness,

$$V = \tfrac{1}{2}\int B \left(\frac{d^2y}{dx^2}\right)^2 dx \qquad (2),$$

in which y is the lateral displacement of that point on the axis of the rod whose abscissa, measured parallel to the undisturbed position, is x. In the case of a rod whose sections are similar and similarly situated B is a constant, and may be removed from under the integral sign.

The kinetic energy of the moving rod is derived partly from the motion of translation, parallel to y, of the elements composing it, and partly from the rotation of the same elements about axes through their centres of inertia perpendicular to the plane of vibration. The former part is expressed by

$$\tfrac{1}{2}\int \rho\omega\, \dot{y}^2 dx \qquad (3),$$

if ρ denote the volume-density. To express the latter part, we have only to observe that the angular displacement of the element dx is dy/dx, and therefore its angular velocity $d^2y/dt\,dx$. The square of this quantity must be multiplied by half the moment of inertia of the element, that is, by $\tfrac{1}{2}\kappa^2\rho\omega\, dx$. We thus obtain

$$T = \tfrac{1}{2}\int \rho\omega\, \dot{y}^2 dx + \tfrac{1}{2}\int \kappa^2\rho\omega \left(\frac{d}{dt}\frac{dy}{dx}\right)^2 dx \qquad (4).$$

[1] *Crelle*, Bd. 81, 1876.

162. In order to form the equation of motion we may avail ourselves of the principle of virtual velocities. If for simplicity we confine ourselves to the case of uniform section, we have

$$\delta V = B\int \frac{d^2y}{dx^2}\frac{d^2\delta y}{dx^2}\,dx$$

$$= B\frac{d^2y}{dx^2}\frac{d\delta y}{dx} - B\frac{d^3y}{dx^3}\delta y + B\int\frac{d^4y}{dx^4}\delta y\,dx \quad\text{.........(1)},$$

where the terms free from the integral sign are to be taken between the limits. This expression includes only the internal forces due to the bending. In what follows we shall suppose that there are no forces acting from without, or rather none that do work upon the system. A force of constraint, such as that necessary to hold any point of the bar at rest, need not be regarded, as it does no work and therefore cannot appear in the equation of virtual velocities.

The virtual moment of the accelerations is

$$\int\rho\omega\frac{d^2y}{dt^2}\delta y\,dx + \int\rho\omega\kappa^2\frac{d^2}{dt^2}\left(\frac{dy}{dx}\right)\delta\left(\frac{dy}{dx}\right)dx$$

$$= \int\rho\omega\left(\frac{d^2y}{dt^2} - \kappa^2\frac{d^4y}{dx^2dt^2}\right)\delta y\,dx + \rho\omega\kappa^2\,\delta y\frac{d^3y}{dt^2dx} \quad\text{......... (2)}.$$

Thus the variational equation of motion is

$$\int\left\{B\frac{d^4y}{dx^4} + \rho\omega\left(\frac{d^2y}{dt^2} - \kappa^2\frac{d^4y}{dx^2dt^2}\right)\right\}\delta y\,dx$$

$$+ B\frac{d^2y}{dx^2}\delta\left(\frac{dy}{dx}\right) + \left\{\rho\omega\kappa^2\frac{d^3y}{dt^2dx} - B\frac{d^3y}{dx^3}\right\}\delta y = 0 \quad\text{....... (3)},$$

in which the terms free from the integral sign are to be taken between the limits. From this we derive as the equation to be satisfied at all points of the length of the bar

$$B\frac{d^4y}{dx^4} + \rho\omega\left(\frac{d^2y}{dt^2} - \kappa^2\frac{d^4y}{dx^2dt^2}\right) = 0,$$

while at each end

$$B\frac{d^2y}{dx^2}\delta\left(\frac{dy}{dx}\right) + \left\{\rho\omega\kappa^2\frac{d^3y}{dt^2dx} - B\frac{d^3y}{dx^3}\right\}\delta y = 0\,;$$

or, if we introduce the value of B viz. $q\kappa^2\omega$, and write $q/\rho = b^2$,

$$\frac{d^2y}{dt^2} + b^2\kappa^2\frac{d^4y}{dx^4} - \kappa^2\frac{d^4y}{dx^2dt^2} = 0 \quad\text{.................. (4)},$$

and for each end

$$b^2\frac{d^2y}{dx^2}\,\delta\left(\frac{dy}{dx}\right)+\left\{\frac{d^3y}{dt^2dx}-b^2\frac{d^3y}{dx^3}\right\}\delta y=0 \;\;\ldots\ldots\ldots(5).$$

In these equations b expresses the velocity of transmission of longitudinal waves.

The condition (5) to be satisfied at the ends assumes different forms according to the circumstances of the case. It is possible to conceive a constraint of such a nature that the ratio $\delta\,(dy/dx) : \delta y$ has a prescribed finite value. The second boundary condition is then obtained from (5) by introduction of this ratio. But in all the cases that we shall have to consider, there is either no constraint or the constraint is such that either $\delta\,(dy/dx)$ or δy vanishes, and then the boundary conditions take the form

$$\frac{d^2y}{dx^2}\,\delta\left(\frac{dy}{dx}\right)=0, \qquad \left\{\frac{d^3y}{dt^2dx}-b^2\frac{d^3y}{dx^3}\right\}\delta y=0 \;\ldots\ldots\ldots\ldots(6).$$

We must now distinguish the special cases that may arise. If an end be free, δy and $\delta\,(dy/dx)$ are both arbitrary, and the conditions become

$$\frac{d^2y}{dx^2}=0, \qquad \frac{d^3y}{dt^2dx}-b^2\frac{d^3y}{dx^3}=0\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

the first of which may be regarded as expressing that no couple acts at the free end, and the second that no force acts.

If the direction at the end be free, but the end itself be constrained to remain at rest by the action of an applied force of the necessary magnitude, in which case for want of a better word the rod is said to be *supported*, the conditions are

$$\frac{d^2y}{dx^2}=0, \qquad \delta y=0 \;\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

by which (5) is satisfied.

A third case arises when an extremity is constrained to maintain its direction by an applied couple of the necessary magnitude, but is free to take any position. We have then

$$\delta\left(\frac{dy}{dx}\right)=0, \qquad \frac{d^3y}{dt^2dx}-b^2\frac{d^3y}{dx^3}=0 \;\ldots\ldots\ldots\ldots\;(9).$$

Fourthly, the extremity may be constrained both as to position and direction, in which case the rod is said to be *clamped*. The conditions are plainly

$$\delta\left(\frac{dy}{dx}\right) = 0, \qquad \delta y = 0 \quad \ldots\ldots\ldots\ldots\ldots (10).$$

Of these four cases the first and the last are the more important; the third we shall omit to consider, as there are no experimental means by which the contemplated constraint could be realized. Even with this simplification a considerable variety of problems remain for discussion, as either end of the bar may be free, clamped or supported, but the complication thence arising is not so great as might have been expected. We shall find that different cases may be treated together and that the solution for one case may sometimes be derived immediately from that of another.

In experimenting on the vibrations of bars, the condition for a clamped end may be realized with the aid of a vice of massive construction. In the case of a free end there is of course no difficulty so far as the end itself is concerned; but, when both ends are free, a question arises as to how the weight of the bar is to be supported. In order to interfere with the vibration as little as possible, the supports must be confined to the neighbourhood of the nodal points. It is sometimes sufficient merely to lay the bar on bridges, or to pass a loop of string round the bar and draw it tight by screws attached to its ends. For more exact purposes it would perhaps be preferable to carry the weight of the bar on a pin traversing a hole drilled through the middle of the thickness in the plane of vibration.

When an end is to be 'supported,' it may be pressed into contact with a fixed plate whose plane is perpendicular to the length of the bar.

163. Before proceeding further we shall introduce a supposition, which will greatly simplify the analysis, without seriously interfering with the value of the solution. We shall assume that the terms depending on the angular motion of the sections of the bar may be neglected, which amounts to supposing the *inertia* of each section concentrated at its centre. We shall afterwards (§ 186) investigate a correction for the rotatory inertia, and shall prove that under ordinary circumstances it is small. The equation of motion now becomes

$$\frac{d^2y}{dt^2} + \kappa^2 b^2 \frac{d^4y}{dx^4} = 0 \qquad (1),$$

and the boundary conditions for a free end

$$\frac{d^2y}{dx^2} = 0, \qquad \frac{d^3y}{dx^3} = 0 \qquad (2).$$

The next step in conformity with the general plan will be the assumption of the harmonic form of y. We may conveniently take

$$y = u \cos\left(\frac{\kappa b}{l^2} m^2 t\right) \qquad (3),$$

where l is the length of the bar, and m is an abstract number, whose value has to be determined. Substituting in (1), we obtain

$$\frac{d^4u}{dx^4} = \frac{m^4}{l^4} u \qquad (4).$$

If $u = e^{pmx/l}$ be a solution, we see that p is one of the fourth roots of unity, viz. $+1$, -1, $+i$, $-i$; so that the complete solution is

$$u = A \cos m\frac{x}{l} + B \sin m\frac{x}{l} + C\, e^{mx/l} + D\, e^{-mx/l} \qquad (4a),$$

containing four arbitrary constants.

[The simplest case occurs when the motion is strictly periodic with respect to x, C and D vanishing. If λ be the wave-length and τ the period of the vibration, we have

$$\frac{2\pi}{\lambda} = \frac{m}{l}, \qquad \frac{2\pi}{\tau} = \kappa b \frac{m^2}{l^2},$$

so that

$$\tau = \frac{\lambda^2}{2\pi\kappa b} \qquad (4b).]$$

In the case of a finite rod we have still to satisfy the four boundary conditions,—two for each end. These determine the ratios $A : B : C : D$, and furnish besides an equation which m must satisfy. Thus a series of particular values of m are alone admissible, and for each m the corresponding u is determined in everything except a constant multiplier. We shall distinguish the different functions u belonging to the same system by suffixes.

The value of y at any time may be expanded in a series of the functions u (§§ 92, 93). If ϕ_1, ϕ_2, &c. be the normal co-ordinates, we have

$$y = \phi_1 u_1 + \phi_2 u_2 + \dots \quad\text{.....................(5)},$$

and

$$T = \tfrac{1}{2}\rho\omega \int (\dot{\phi}_1 u_1 + \dot{\phi}_2 u_2 + \dots)^2 \, dx$$

$$= \tfrac{1}{2}\rho\omega \left\{ \dot{\phi}_1^2 \int u_1^2 dx + \dot{\phi}_2^2 \int u_2^2 dx + \dots \right\} \quad\text{.............. (6)}.$$

We are fully justified in asserting at this stage that each integrated product of the functions vanishes, and therefore the process of the following section need not be regarded as more than a *verification*. It is however required in order to determine the value of the integrated squares.

164. Let u_m, $u_{m'}$ denote two of the normal functions corresponding respectively to m and m'. Then

$$\frac{d^4 u_m}{dx^4} = \frac{m^4}{l^4} u_m, \qquad \frac{d^4 u_{m'}}{dx^4} = \frac{m'^4}{l^4} u_{m'} \quad\text{..............(1)};$$

or, if dashes indicate differentiation with respect to (mx/l), $(m'x/l)$,

$$u_m'''' = u_m, \qquad u_{m'}'''' = u_{m'} \quad\text{.................. (2)}.$$

If we subtract equations (1) after multiplying them by $u_{m'}$, u_m respectively, and then integrate over the length of the bar, we have

$$\frac{m'^4 - m^4}{l^4} \int u_m u_{m'} \, dx = \int \left(u_m \frac{d^4 u_{m'}}{dx^4} - u_{m'} \frac{d^4 u_m}{dx^4} \right) dx$$

$$= u_m \frac{d^3 u_{m'}}{dx^3} - u_{m'} \frac{d^3 u_m}{dx^3} + \frac{du_{m'}}{dx} \frac{d^2 u_m}{dx^2} - \frac{du_m}{dx} \frac{d^2 u_{m'}}{dx^2} \quad\text{..... (3)},$$

the integrated terms being taken between the limits.

Now whether the end in question be clamped, supported, or free[1], each term vanishes on account of one or other of its

[1] The reader should observe that the cases here specified are particular, and that the right-hand member of (3) vanishes, provided that

$$u_m : \frac{d^3 u_m}{dx^3} = u_{m'} : \frac{d^3 u_{m'}}{dx^3},$$

and

$$\frac{du_m}{dx} : \frac{d^2 u_m}{dx^2} = \frac{du_{m'}}{dx} : \frac{d^2 u_{m'}}{dx^2}.$$

These conditions include, for instance, the case of a rod whose end is urged towards its position of equilibrium by a force proportional to the displacement, as by a spring without inertia.

factors. We may therefore conclude that, if u_m, $u_{m'}$ refer to two modes of vibration (corresponding of course to the same terminal conditions) of which a rod is capable, then

$$\int u_m u_{m'} dx = 0 \text{ ....(4)},$$

provided m and m' be different.

The attentive reader will perceive that in the process just followed, we have in fact retraced the steps by which the fundamental differential equation was itself proved in § 162. It is the original *variational* equation that has the most immediate connection with the conjugate property. If we denote y by u and δy by v,

$$\delta V = B \int \frac{d^2u}{dx^2}\frac{d^2v}{dx^2} dx,$$

and the equation in question is

$$B \int \frac{d^2u}{dx^2}\frac{d^2v}{dx^2} dx + \rho\omega \int \ddot{u} v \, dx = 0 \text{ ....(5)}.$$

Suppose now that u relates to a normal component vibration, so that $\ddot{u} + n^2 u = 0$, where n is some constant; then

$$n^2 \rho\omega \int uv \, dx = B \int \frac{d^2u}{dx^2}\frac{d^2v}{dx^2} dx.$$

By similar reasoning, if v be a normal function, and u represent any displacement possible to the system,

$$n'^2 \rho\omega \int uv \, dx = B \int \frac{d^2u}{dx^2}\frac{d^2v}{dx^2} dx.$$

We conclude that if u and v be both normal functions, *which have different periods*,

$$\int uv \, dx = 0 \text{.........................(6)};$$

and this proof is evidently as direct and general as could be desired.

The reader may investigate the formula corresponding to (6), when the term representing the rotatory inertia is retained.

By means of (6) we may verify that the admissible values of n^2 are real. For if n^2 were complex, and $u = \alpha + i\beta$ were a normal function, then $\alpha - i\beta$, the conjugate of u, would be a normal function also, corresponding to the conjugate of n^2, and then the product of the two functions, being a sum of squares, would not vanish, when integrated[1].

If in (3) m and m' be the same, the equation becomes identically true, and we cannot at once infer the value of $\int u_m{}^2 dx$. We must take m' equal to $m + \delta m$, and trace the limiting form of the equation as δm tends to vanish. [It should be observed that the function $u_{m+\delta m}$ is not a normal function of the system; it is supposed to be derived from u_m by variation of m in (4a) § 163, the coefficients A, B, C, D being retained constant.] In this way we find

$$\frac{4m^3}{l^4}\int u_m{}^2 dx = u\frac{d}{dm}\frac{d^3u}{dx^3} - \frac{du}{dm}\frac{d^3u}{dx^3} + \frac{d^2u}{dx^2}\frac{d}{dm}\frac{du}{dx} - \frac{du}{dx}\frac{d}{dm}\frac{d^2u}{dx^2},$$

the right-hand side being taken between the limits.

Now $$\frac{du}{dx} = \frac{m}{l}u', \text{ \&c.,} \qquad \frac{du}{dm} = \frac{x}{l}u', \text{ \&c.,}$$

and thus

$$\frac{4m^3}{l^4}\int u_m{}^2 dx = \frac{3m^2}{l^3}uu''' + \frac{m^3x}{l^4}uu'''' - \frac{m^3x}{l^4}u'u'''$$
$$+ \frac{m^2}{l^3}u'u'' + \frac{m^3x}{l^4}(u'')^2 - \frac{2m^2}{l^3}u'u'' - \frac{m^3x}{l^4}u'u''',$$

in which $u'''' = u$, so that

$$\frac{4m}{l}\int u_m{}^2 dx = 3uu''' + \frac{mx}{l}u^2 - \frac{2mx}{l}u'u''' - u'u'' + \frac{mx}{l}(u'')^2 \quad \text{...(7)},$$

between the limits.

Now whether an end be clamped, supported, or free,

$$uu''' = 0, \qquad u'u'' = 0,$$

and thus, if we take the origin of x at one end of the rod,

$$\int_0^l u^2 dx = \left\{\frac{x}{4}(u^2 - 2u'u''' + u''^2)\right\}_0^l$$
$$= \tfrac{1}{4}l(u^2 - 2u'u''' + u''^2)_{x=l} \quad \text{..............(8)}.$$

The form of our integral is independent of the terminal condition at $x = 0$. If the end $x = l$ be free, u'' and u''' vanish, and accordingly

$$\int_0^l u^2 dx = \tfrac{1}{4}l\,u^2(l) \quad \text{.......................(9)},$$

[1] This method is, I believe, due to Poisson.

that is to say, for a rod with one end free the mean value of u^2 is one-fourth of the terminal value, and that whether the other end be clamped, supported, or free.

Again, if we suppose that the rod is clamped at $x=l$, u and u' vanish, and (8) gives

$$\int_0^l u^2 dx = \tfrac{1}{4} l \left[u''(l)\right]^2.$$

Since this must hold good whatever be the terminal condition at the other end, we see that for a rod, one end of which is fixed and the other free,

$$\int_0^l u^2 dx = \tfrac{1}{4} l u^2 \text{(free end)} = \tfrac{1}{4} l u''^2 \text{(fixed end)},$$

shewing that in this case u^2 at the free end is the same as u''^2 at the clamped end

The annexed table gives the values of four times the mean of u^2 in the different cases.

clamped, free.........	u^2 (free end), or u''^2 (clamped end)
free, free	u^2 (free end)
clamped, clamped ...	u''^2 (clamped end)
supported, supported	$-2u'u'''$ (supported end) $=2u'^2$
supported, free	u^2 (free end), or $-2u'u'''$ (supported end)
supported, clamped	u''^2 (clamped end), or $-2u'u'''$ (supported end)

By the introduction of these values the expression for T assumes a simpler form. In the case, for example, of a clamped-free or a free-free rod,

$$T = \frac{\rho l \omega}{8}\{\dot{\phi}_1^2 u_1^2(l) + \dot{\phi}_2^2 u_2^2(l) + \dots\} \text{ ....(10)},$$

where the end $x=l$ is supposed to be free.

165. A similar method may be applied to investigate the values of $\int u'^2 dx$, and $\int u''^2 dx$. In the derivation of equation (7) of the preceding section nothing was assumed beyond the truth of the equation $u''''=u$, and since this equation is equally true of any of the derived functions, we are at liberty to replace u by u' or u''. Thus

$$\frac{4m}{l}\int_0^l u'^2 dx = 3u'u + \frac{mx}{l}u^2 - 2\frac{mx}{l}u''u - u''u''' + \frac{mx}{l}u'''^2$$

$$= 3uu' + \frac{mu}{l}u'^2 - u''u''' + \frac{mx}{l}u'''^2,$$

taken between the limits, since the term uu'' vanishes in all three cases.

For a free-free rod

$$\frac{4m}{l}\int_0^l u'^2 dx = 3(uu')_l - 3(uu')_0 + m(u'^2)_l$$

$$= 6(uu')_l + m(u'^2)_l \qquad (1),$$

for, as we shall see, the values of uu' must be equal and opposite at the two ends. Whether u be positive or negative at $x = l$, uu' is positive.

For a rod which is clamped at $x = 0$ and free at $x = l$

$$\frac{4m}{l}\int_0^l u'^2 dx = 3(uu')_l + mu_l'^2 + (u''u''')_0$$

[We have already seen that $u_0'' = \pm u_l$; and it may be proved from the formulæ of § 173 that

$$-\frac{u_0'''}{u_l'} = \frac{u_0''}{u_l} = \frac{\cos m + \cosh m}{\sin m \sinh m},$$

so that $$\frac{(u''u''')_0}{(u'u)_l} = -\frac{(\cos m + \cosh m)^2}{\sin^2 m \sinh^2 m} = -1.]$$

Thus $$\frac{4m}{l}\int_0^l u'^2 dx = 2(uu')_l + mu_l'^2 \qquad (2),$$

a result that we shall have occasion to use later.

By applying the same equation to the evaluation of $\int u''^2 dx$, we find

$$\frac{4m}{l}\int u''^2 dx = 3u''u' + \frac{mx}{l}u''^2 - 2\frac{mx}{l}u'''u' - u'''u + \frac{mx}{l}u^2$$

$$= m(u''^2 - 2u'u''' + u^2)_l,$$

since $u'u''$ and uu''' vanish.

Comparing this with (8) § 164, we see that

$$\int u''^2 dx = \int u^2 dx \qquad\qquad (3),$$

whatever the terminal conditions may be.

The same result may be arrived at more directly by integrating by parts the equation

$$\frac{m^4}{l^4} u^2 = u \frac{d^4 u}{dx^4}.$$

166. We may now form the expression for V in terms of the normal co-ordinates.

$$V = \frac{b^2 \kappa^2 \rho\omega}{2} \int \left\{ \phi_1 \frac{d^2 u_1}{dx^2} + \phi_2 \frac{d^2 u_2}{dx^2} + \dots \right\}^2 dx$$

$$= \frac{b^2 \kappa^2 \rho\omega}{2} \left\{ \phi_1^2 \int \left(\frac{d^2 u_1}{dx^2}\right)^2 dx + \phi_2^2 \int \left(\frac{d^2 u_2}{dx^2}\right)^2 dx + \dots \right\}$$

$$= \frac{b^2 \kappa^2 \rho\omega}{2 l^4} \left\{ m_1^4 \phi_1^2 \int u_1^2 dx + m_2^4 \phi_2^2 \int u_2^2 dx + \dots \right\} \qquad (1).$$

If the functions u be those proper to a rod free at $x = l$, this expression reduces to

$$V = \frac{b^2 \kappa^2 \rho\omega}{8 l^3} \left\{ m_1^4 [u_1(l)]^2 \phi_1^2 + m_2^4 [u_2(l)]^2 \phi_2^2 + \dots \right\} \qquad (2).$$

In any case the equations of motion are of the form

$$\rho\omega \int u_1^2 dx \, \ddot{\phi}_1 + \frac{b^2 \kappa^2 \rho\omega}{l^4} m_1^4 \int u_1^2 dx \, \phi_1 = \Phi_1 \qquad (3),$$

and, since $\Phi_1 \delta\phi_1$ is by definition the work done by the impressed forces during the displacement $\delta\phi_1$,

$$\Phi_1 = \int Y u_1 \rho\omega \, dx \qquad (4),$$

if $Y\rho\omega dx$ be the lateral force acting on the element of mass $\rho\omega dx$. If there be no impressed forces, the equation reduces to

$$\ddot{\phi}_1 + \frac{b^2 \kappa^2 m_1^4}{l^4} \phi_1 = 0 \qquad (5),$$

as we know it ought to do.

167. The significance of the reduction of the integrals $\int u^2 dx$ to dependence on the terminal values of the function and its derivatives may be placed in a clearer light by the following line of argument. To fix the ideas, consider the case of a rod clamped at $x=0$, and free at $x=l$, vibrating in the normal mode expressed by u. If a small addition Δl be made to the rod at the free end, the form of u (considered as a function of x) is changed, but in accordance with the general principle established in Chapter IV. (§ 88), we may calculate the period under the altered circumstances without allowance for the change of type, if we are content to neglect the square of the change. In consequence of the straightness of the rod at the place where the addition is made, there is no alteration in the potential energy, and therefore the alteration of period depends entirely on the variation of T. This quantity is increased in the ratio

$$\int_0^l u^2 dx : \int_0^{l+\Delta l} u^2 dx,$$

$$\text{or} \qquad 1 : 1 + \frac{u_l^2 \Delta l}{\int_0^l u^2 dx},$$

which is also the ratio in which the square of the period is augmented. Now, as we shall see presently, the actual period varies as l^2, and therefore the change in the square of the period is in the ratio

$$1 : 1 + 4\Delta l / l.$$

A comparison of the two ratios shews that

$$u_l^2 : \int u^2 dx = 4 : l.$$

The above reasoning is not insisted upon as a demonstration, but it serves at least to explain the reduction of which the integral is susceptible. Other cases in which such integrals occur may be treated in a similar manner, but it would often require care to predict with certainty what amount of discontinuity in the varied type might be admitted without passing out of the range of the principle on which the argument depends. The reader may, if he pleases, examine the case of a string in the middle of which a small piece is interpolated.

168. In treating problems relating to vibrations the usual course has been to determine in the first place the forms of the normal functions, viz. the functions representing the normal types, and afterwards to investigate the integral formulæ by means of which the particular solutions may be combined to suit arbitrary initial circumstances. I have preferred to follow a different order, the better to bring out the generality of the method, *which does not depend upon a knowledge of the normal functions.* In pursuance of the same plan, I shall now investigate the connection of the arbitrary constants with the initial circumstances, and solve one or two problems analogous to those treated under the head of Strings.

The general value of y may be written

$$y = \left(A_1 \cos \frac{\kappa b}{l^2} m_1^2 t + B_1 \sin \frac{\kappa b}{l^2} m_1^2 t\right) u_1$$

$$+ \left(A_2 \cos \frac{\kappa b}{l^2} m_2^2 t + B_2 \sin \frac{\kappa b}{l^2} m_2^2 t\right) u_2$$

$$+ \dots \dots \dots \dots \dots \dots \dots \dots \quad (1),$$

so that initially

$$y_0 = A_1 u_1 + A_2 u_2 + \dots \quad \dots\dots\dots\dots \quad (2),$$

$$\dot{y}_0 = \frac{\kappa b}{l^2} \{m_1^2 B_1 u_1 + m_2^2 B_2 u_2 + \dots\} \quad \dots\dots\dots\dots \quad (3).$$

If we multiply (2) by u_r and integrate over the length of the rod, we get

$$\int y_0 u_r dx = A_r \int u_r^2 dx \quad \dots\dots\dots\dots \quad (4),$$

and similarly from (3)

$$\frac{l^2}{\kappa b} \int \dot{y}_0 u_r dx = m_r^2 B_r \int u_r^2 dx \quad \dots\dots\dots\dots \quad (5),$$

formulæ which determine the arbitrary constants A_r, B_r.

It must be observed that we do not need to prove analytically the possibility of the expansion expressed by (1). If *all* the particular solutions are included, (1) necessarily represents the most general vibration possible, and may therefore be adapted to represent any admissible initial state.

Let us now suppose that the rod is originally at rest, in its position of equilibrium, and is set in motion by a blow which imparts velocity to a small portion of it. Initially, that is, at the moment when the rod becomes free, $y_0 = 0$, and $\dot{y}_0$ differs from zero only in the neighbourhood of one point ($x = c$).

From (4) it appears that the coefficients A vanish, and from (5) that

$$m_r^2 B_r \int u_r^2 dx = \frac{l^2}{\kappa b} u_r(c) \int \dot{y}_0 dx.$$

Calling $\int \dot{y}_0 \rho\omega dx$, the whole momentum of the blow, Y, we have

$$B_r = \frac{l^2 Y}{\kappa b \rho\omega} \frac{u_r(c)}{m_r^2 \int u_r^2 dx} \qquad\qquad (6),$$

and for the final solution

$$y = \frac{l^2 Y}{\kappa b \rho\omega} \left\{ \frac{u_1(c)\, u_1(x)}{m_1^2 \int u_1^2 dx} \sin\left(\frac{\kappa b}{l^2} m_1^2 t\right) + \ldots \right.$$

$$\left. + \frac{u_r(c)\, u_r(x)}{m_r^2 \int u_r^2 dx} \sin\left(\frac{\kappa b}{l^2} m_r^2 t\right) + \ldots\ldots \right\} \qquad (7).$$

In adapting this result to the case of a rod free at $x = l$, we may replace

$$\int u_r^2 dx \quad \text{by} \quad \tfrac{1}{4} l\, [u_r(l)]^2.$$

If the blow be applied at a node of one of the normal components, that component is missing in the resulting motion. The present calculation is but a particular case of the investigation of § 101.

169. As another example we may take the case of a bar, which is initially at rest but deflected from its natural position by a lateral force acting at $x = c$. Under these circumstances the coefficients B vanish, and the others are given by (4), § 168.

Now

$$\int_0^l y_0 u_r dx = \frac{l^4}{m_r^4} \int_0^l y_0 \frac{d^4 u_r}{dx^4} dx,$$

and on integrating by parts

$$\int_0^l y_0 \frac{d^4 u_r}{dx^4} dx = y_0 \frac{d^3 u_r}{dx^3} - \frac{dy_0}{dx} \frac{d^2 u_r}{dx^2}$$

$$+ \frac{d^2 y_0}{dx^2} \frac{du_r}{dx} - \frac{d^3 y_0}{dx^3} u_r + \int_0^l \frac{d^4 y_0}{dx^4} u_r dx,$$

in which the terms free from the integral sign are to be taken between the limits; by the nature of the case y_0 satisfies the same terminal conditions as does u_r, and thus all these terms vanish at both limits. If the external force initially applied to the element dx be Ydx, the equation of equilibrium of the bar gives

$$\rho\omega\,\kappa^2 b^2 \frac{d^4 y_0}{dx^4} = Y \quad \text{......................} (1)$$

and accordingly

$$\int_0^l y_0 u_r\, dx = \frac{l^4}{\rho\omega\kappa^2 b^2 m_r^4} \int_0^l Y u_r(x)\, dx.$$

If we now suppose that the initial displacement is due to a force applied in the immediate neighbourhood of the point $x = c$, we have

$$\int_0^l y_0 u_r\, dx = \frac{l^4 u_r(c)}{\rho\omega\kappa^2 b^2 m_r^4} \int Y dx,$$

and for the complete value of y at time t,

$$y = \Sigma \left\{ \frac{l^4 u_r(c)\, u_r(x)}{m_r^4 \kappa^2 b^2 \int \rho\omega\, u_r^2\, dx} \cos \frac{\kappa b}{l^2} m_r^2 t \right\} \int Y dx \quad \text{......} (2).$$

In deriving the above expression we have not hitherto made any special assumptions as to the conditions at the ends, but if we now confine ourselves to the case of a bar which is clamped at $x = 0$ and free at $x = l$, we may replace

$$\int u_r^2 dx \quad \text{by} \quad \tfrac{1}{4} l\, [u_r(l)]^2.$$

If we suppose further that the force to which the initial deflection is due acts at the end, so that $c = l$, we get

$$y = 4\, \Sigma \left\{ \frac{l^3 u_r(x)}{m_r^4 \kappa^2 b^2 \rho\omega\, u_r(l)} \cos \frac{\kappa b}{l^2} m_r^2 t \right\} \int Y dx \quad \text{......} (3).$$

When $t = 0$, this equation must represent the initial displacement. In cases of this kind a difficulty may present itself as to how it is possible for the series, every term of which satisfies the condition $y''' = 0$, to represent an initial displacement in which this condition is violated. The fact is, that after triple differentiation with respect to x, the series no longer converges for $x = l$, and accordingly the value of y''' is not to be arrived at by making the differentiations first and summing the terms afterwards. The truth of this statement will be apparent if we consider a point distant dl from the end, and replace

$$u'''(l - dl) \quad \text{by} \quad u'''(l) - u^{\text{IV}}(l)\, dl,$$

in which $u^{\text{IV}}(l)$ is equal to

$$\frac{m^4}{l^4}u(l).$$

For the solution of the present problem by normal co-ordinates the reader is referred to § 101.

170. The forms of the normal functions in the various particular cases are to be obtained by determining the ratios of the four constants in the general solution of

$$\frac{d^4u}{dx^4}=\frac{m^4}{l^4}u.$$

If for the sake of brevity x' be written for (mx/l), the solution may be put into the form

$$u=A(\cos x'+\cosh x')+B(\cos x'-\cosh x)$$
$$+C(\sin x'+\sinh x')+D(\sin x'-\sinh x') \ldots\ldots (1),$$

where $\cosh x$ and $\sinh x$ are the hyperbolic cosine and sine of x, defined by the equations

$$\cosh x=\tfrac{1}{2}(e^x+e^{-x}),\quad \sinh x=\tfrac{1}{2}(e^x-e^{-x}) \ldots\ldots\ldots\ldots (2).$$

I have followed the usual notation, though the introduction of a special symbol might very well be dispensed with, since

$$\cosh x=\cos ix,\quad \sinh x=-i\sin ix \ldots\ldots\ldots\ldots (3),$$

where $i=\sqrt{(-1)}$; and then the connection between the formulæ of circular and hyperbolic trigonometry would be more apparent. The rules for differentiation are expressed in the equations

$$\frac{d}{dx}\cosh x=\sinh x,\qquad \frac{d}{dx}\sinh x=\cosh x$$

$$\frac{d^2}{dx^2}\cosh x=\cosh x,\qquad \frac{d^2}{dx^2}\sinh x=\sinh x.$$

In differentiating (1) any number of times, the same four compound functions as there occur are continually reproduced. The only one of them which does not vanish with x' is $\cos x'+\cosh x'$, whose value is then 2.

Let us take first the case in which both ends are free. Since d^2u/dx^2 and d^3u/dx^3 vanish with x, it follows that $B=0$, $D=0$, so that

$$u=A(\cos x'+\cosh x')+C(\sin x'+\sinh x') \ldots\ldots\ldots (4).$$

We have still to satisfy the necessary conditions when $x = l$, or $x' = m$. These give

$$\left.\begin{aligned} A\,(-\cos m + \cosh m) + C\,(-\sin m + \sinh m) = 0 \\ A\,(\ \ \sin m + \sinh m) + C\,(-\cos m + \cosh m) = 0 \end{aligned}\right\} \text{......(5)},$$

equations whose compatibility requires that

$$(\cosh m - \cos m)^2 = \sinh^2 m - \sin^2 m,$$

or in virtue of the relation

$$\cosh^2 m - \sinh^2 m = 1 \text{....................(6)},$$

$$\cos m \cosh m = 1 \text{....................(7)}.$$

This is the equation whose roots are the admissible values of m. If (7) be satisfied, the two ratios of $A : C$ given in (5) are equal, and either of them may be substituted in (4). The constant multiplier being omitted, we have for the normal function

$$u = (\sin m - \sinh m)\left\{\cos\frac{mx}{l} + \cosh\frac{mx}{l}\right\}$$

$$-(\cos m - \cosh m)\left\{\sin\frac{mx}{l} + \sinh\frac{mx}{l}\right\} \text{......... (8)},$$

or, if we prefer it

$$u = (\cos m - \cosh m)\left\{\cos\frac{mx}{l} + \cosh\frac{mx}{l}\right\}$$

$$+(\sin m + \sinh m)\left\{\sin\frac{mx}{l} + \sinh\frac{mx}{l}\right\} \text{.........(9)};$$

and the simple harmonic component of this type is expressed by

$$y = Pu \cos\left(\frac{\kappa b}{l^2} m^2 t + \epsilon\right) \text{ ....(10)}.$$

171. The frequency of the vibration is $\frac{\kappa b}{2\pi l^2} m^2$, in which b is a velocity depending only on the material of which the bar is formed, and m is an abstract number. Hence for a given material and mode of vibration the frequency varies directly as κ—the radius of gyration of the section about an axis perpendicular to the plane of bending—and inversely as the *square* of the length. These results might have been anticipated by the argument from dimensions, if it were considered that the frequency is necessarily determined by the value of l, together with that of κb—the only quantity depending on space, time and mass, which occurs in the differential equation. If everything concerning a bar be given, except its absolute magnitude, the frequency varies inversely as the linear dimension.

These laws find an important application in the case of tuning-forks, whose prongs vibrate as rods, fixed at the ends where they join the stalk, and free at the other ends. Thus the period of vibration of forks of the same material and shape varies as the linear dimension. The period will be approximately independent of the thickness perpendicular to the plane of bending, but will vary inversely with the thickness in the plane of bending. When the thickness is given, the period is as the square of the length.

In order to lower the pitch of a fork we may, for temporary purposes, load the ends of the prongs with soft wax, or file away the metal near the base, thereby weakening the spring. To raise the pitch, the ends of the prongs, which act by inertia, may be filed.

The value of b attains its maximum in the case of steel, for which it amounts to about 5237 metres per second. For brass the velocity would be less in about the ratio 1·5 : 1, so that a tuning-fork made of brass would be about a fifth lower in pitch than if the material were steel.

[For the design of steel vibrators and for rough determinations of frequency, especially when below the limit of hearing, the theoretical formula is often convenient. If the section of the bar be rectangular and of thickness t in the plane of vibration, $\kappa^2 = \frac{1}{12}t^2$; and then with the above value of b, and the values of m given later, we get as applicable to the gravest mode

$$\text{(clamped-free) frequency} = 84590\, t/l^2,$$

$$\text{(free-free) frequency} = 538400\, t/l^2.$$

l and t being expressed in centimetres.

The first of these may be used to calculate the pitch of steel tuning-forks.

The lateral vibrations of a bar may be excited by a blow, as when a tuning-fork is struck against a pad. This method is also employed for the harmonicon, in which strips of metal or glass are supported at the nodes, in such a manner that the free vibrations are but little impeded. A frictional maintenance may be obtained with a bow, or by the action of the wetted fingers upon a slender rod of glass suitably attached. The electro-magnetic maintenance of forks has been already considered, § 64. It may be applied with equal facility to the case of metal bars, or even to that of wooden planks carrying iron armatures, free at both ends and supported at the nodes. The maintenance by a stream of wind of the vibrations of harmonium and organ reeds may also be referred to

The sound of a bar vibrating laterally may be reinforced by a suitably tuned resonator, which may be placed under the middle portion or under one end. On this principle dinner gongs have been constructed, embracing one octave or more of the diatonic scale.]

172. The solution for the case when both ends are clamped may be immediately derived from the preceding by a double differentiation. Since y satisfies at both ends the terminal conditions

$$\frac{d^2y}{dx^2} = 0, \qquad \frac{d^3y}{dx^3} = 0,$$

it is clear that y'' satisfies

$$y'' = 0, \qquad \frac{dy'}{dx} = 0,$$

which are the conditions for a clamped end. Moreover the general differential equation is also satisfied by y''. Thus we may take, omitting a constant multiplier, as before,

$$u = (\sin m - \sinh m)\{\cos x' - \cosh x'\}$$
$$- (\cos m - \cosh m)\{\sin x' - \sinh x'\} \;\ldots\ldots\ldots\ldots(1),$$

while m is given by the same equation as before, namely,

$$\cos m \cosh m = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

We conclude that the component tones have the same pitch in the two cases.

In each case there are four systems of points determined by the evanescence of y and its derivatives. Where y vanishes, there is a node; where y' vanishes, a loop, or place of maximum displacement; where y'' vanishes, a point of inflection; and where y''' vanishes, a place of maximum curvature. Where there are in the first case (free-free) points of inflection and of maximum curvature, there are in the second (clamped-clamped) nodes and loops respectively; and *vice versâ*, points of inflection and of maximum curvature for a doubly-clamped rod correspond to nodes and loops of a rod whose ends are free.

173. We will now consider the vibrations of a rod clamped at $x=0$, and free at $x=l$. Reverting to the general integral (1) § 170, we see that A and C vanish in virtue of the conditions at $x=0$, so that

$$u = B(\cos x' - \cosh x') + D(\sin x' - \sinh x') \qquad (1).$$

The remaining conditions at $x=l$ give

$$\left.\begin{array}{l} B(\ \ \cos m + \cosh m) + D(\sin m + \sinh m) = 0 \\ B(-\sin m + \sinh m) + D(\cos m + \cosh m) = 0 \end{array}\right\},$$

whence, omitting the constant multiplier,

$$u = (\sin m + \sinh m)\left\{\cos\frac{mx}{l} - \cosh\frac{mx}{l}\right\}$$
$$- (\cos m + \cosh m)\left\{\sin\frac{mx}{l} - \sinh\frac{mx}{l}\right\} \qquad (2),$$

or

$$u = (\cos m + \cosh m)\left\{\cos\frac{mx}{l} - \cosh\frac{mx}{l}\right\}$$
$$+ (\sin m - \sinh m)\left\{\sin\frac{mx}{l} - \sinh\frac{mx}{l}\right\} \qquad (3),$$

where m must be a root of

$$\cos m \cosh m + 1 = 0 \qquad (4).$$

The periods of the component tones in the present problem are thus different from, though, as we shall see presently, nearly related to, those of a rod both whose ends are clamped, or free.

If the value of u in (2) or (3) be differentiated twice, the result (u'') satisfies of course the fundamental differential equation. At $x=0$, d^2u''/dx^2, d^3u''/dx^3 vanish, but at $x=l$ u'' and du''/dx vanish. The function u'' is therefore applicable to a rod clamped at l and free at 0, proving that the points of inflection and of maximum curvature in the original curve are at the same distances from the clamped end, as the nodes and loops respectively are from the free end.

174. In default of tables of the hyperbolic cosine or its logarithm, the admissible values of m may be calculated as follows. Taking first the equation

$$\cos m \cosh m = 1 \quad\text{.......................}(1),$$

we see that m, when large, must approximate in value to $\frac{1}{2}(2i+1)\pi$, i being an integer. If we assume

$$m = \tfrac{1}{2}(2i+1)\pi - (-1)^i\beta \quad\text{.................}(2),$$

β will be positive and comparatively small in magnitude.

Substituting in (1), we find

$$\cot \tfrac{1}{2}\beta = e^m = e^{\frac{1}{2}(2i+1)\pi}\, e^{-(-1)^i\beta};$$

or, if $e^{\frac{1}{2}(2i+1)\pi}$ be called a,

$$a \tan \tfrac{1}{2}\beta = e^{(-1)^i\beta} \quad\text{..........................}(3),$$

an equation which may be solved by successive approximation after expanding $\tan\frac{1}{2}\beta$ and $e^{(-1)^i\beta}$ in ascending powers of the small quantity β. The result is

$$\beta_i = \frac{2}{a} + (-1)^i\frac{4}{a^2} + \frac{34}{3a^3} + (-1)^i\frac{112}{3a^4} + \text{...........}(4)^1,$$

which is sufficiently accurate, even when $i = 1$.

By calculation

$$\beta_1 = \cdot 0179666 - \cdot 0003228 + \cdot 0000082 - \cdot 0000002 = \cdot 0176518.$$

β_2, β_3, β_4, β_5 are found still more easily. After β_5 the first term of the series gives β correctly as far as six significant figures. The table contains the value of β, the angle whose circular measure is β, and the value of $\sin\frac{1}{2}\beta$, which will be required further on.

Free-Free Bar.

	β.	β expressed in degrees, minutes, and seconds.	$\sin\frac{\beta}{2}$.
1	$10^{-1}\times\cdot 176518$	1° 0′ 40″·94	$10^{-2}\times\cdot 88258$
2	$10^{-3}\times\cdot 777010$	2′ 40″·2699	$10^{-3}\times\cdot 38850$
3	$10^{-4}\times\cdot 335505$	6″·92029	$10^{-4}\times\cdot 16775$
4	$10^{-5}\times\cdot 144989$	·299062	$10^{-6}\times\cdot 72494$
5	$10^{-7}\times\cdot 626556$	·0129237	$10^{-7}\times\cdot 31328$

[1] This process is somewhat similar to that adopted by Strehlke.

The values of m which satisfy (1) are

$$m_1 = 4{\cdot}7123890 + \beta_1 = 4{\cdot}7300408$$
$$m_2 = 7{\cdot}8539816 - \beta_2 = 7{\cdot}8532046$$
$$m_3 = 10{\cdot}9955743 + \beta_3 = 10{\cdot}9956078$$
$$m_4 = 14{\cdot}1371669 - \beta_4 = 14{\cdot}1371655$$
$$m_5 = 17{\cdot}2787596 + \beta_5 = 17{\cdot}2787596$$

after which $m = \frac{1}{2}(2i+1)\pi$ to seven decimal places.

We will now consider the roots of the equation

$$\cos m \cosh m = -1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)^1.$$

[Assuming

$$m_i = \tfrac{1}{2}(2i-1)\pi - (-1)^i \alpha_i \quad\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

we have $$e^{m_i} = \cot \tfrac{1}{2}\alpha_i = e^{\frac{1}{2}(2i-1)\pi} \,.\, e^{-(-1)^i \alpha_i},$$

or $$a \tan \tfrac{1}{2}\alpha_{i+1} = e^{-(-1)^i \alpha_{i+1}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

a having the value previously defined.

Thus, as in (4),

$$\alpha_{i+1} = \frac{2}{a} - (-1)^i \frac{4}{a^2} + \frac{34}{3a^3} - (-1)^i \frac{112}{3a^4} + \ldots\ldots\ldots(8),$$

α_{i+1} being *approximately* equal to β_i.

The values calculated from (8) are

$$\alpha_2 = 10^{-1} \times 182979 \qquad \alpha_4 = 10^{-4} \times {\cdot}335527,$$
$$\alpha_3 = 10^{-3} \times {\cdot}775804, \qquad \alpha_5 = 10^{-5} \times {\cdot}144989,$$

after which the difference between α_{i+1} and β_i does not appear.]

The value of α_1 may be obtained by trial and error from the equation

$$\log_{10} \cot \tfrac{1}{2}\alpha_1 - {\cdot}6821882 - {\cdot}43429448\,\alpha_1 = 0,$$

and will be found to be

$$\alpha_1 = {\cdot}3043077.$$

Another method by which m_1 may be obtained directly will be given presently.

The values of m, which satisfy (5), are

$$m_1 = 1{\cdot}5707963 + \alpha_1 = 1{\cdot}875104$$
$$m_2 = 4{\cdot}7123890 - \alpha_2 = 4{\cdot}694091$$
$$m_3 = 7{\cdot}8539816 + \alpha_3 = 7{\cdot}854757$$
$$m_4 = 10{\cdot}9955743 - \alpha_4 = 10{\cdot}995541$$
$$m_5 = 14{\cdot}1371669 + \alpha_5 = 14{\cdot}137168$$
$$m_6 = 17{\cdot}2787596 - \alpha_6 = 17{\cdot}278759\,,$$

[1] The calculation of the roots of (5) given in the first edition was affected by an error, which has been pointed out by Greenhill (*Math. Mess.*, Dec. 1886).

after which $m = \frac{1}{2}(2i-1)\pi$ sensibly. The frequencies are proportional to m^2, and are therefore for the higher tones nearly in the ratio of the squares of the odd numbers. However, in the case of overtones of very high order, the pitch may be slightly disturbed by the rotatory inertia, whose effect is here neglected.

175. Since the component vibrations of a system, not subject to dissipation, are necessarily of the harmonic type, all the values of m^2, which satisfy

$$\cos m \cosh m = \pm 1 \text{......................} (1),$$

must be real. We see further that, if m be a root, so are also $-m$, $m\sqrt{(-1)}$, $-m\sqrt{(-1)}$. Hence, taking first the lower sign, we have

$$\frac{1}{2}(\cos m \cosh m + 1) = 1 - \frac{m^4}{12} + \frac{m^8}{12^2.35} - \text{......}$$

$$= \left(1 - \frac{m^4}{m_1^4}\right)\left(1 - \frac{m^4}{m_2^4}\right) \text{ \&c.................} (2).$$

If we take the logarithms of both sides, expand, and equate coefficients, we get

$$\Sigma \frac{1}{m^4} = \frac{1}{12}; \quad \Sigma \frac{1}{m^8} = \frac{1}{12^2} \cdot \frac{33}{35}; \text{ \&c............} (3).$$

This is for a clamped-free rod.

From the known value of Σm^{-8}, the value of m_1 may be derived with the aid of approximate values of m_2, m_3,...... We find

$$\Sigma m^{-8} = \cdot 006547621,$$

and $$m_2^{-8} = \cdot 000004242$$

$$m_3^{-8} = \cdot 000000069$$

$$m_4^{-8} = \cdot 000000005,$$

whence $$m_1^{-8} = \cdot 006543305$$

giving $$m_1 = \cdot 1875104, \text{ as before.}$$

In like manner, if both ends of the bar be clamped or free,

$$1 - \frac{m^4}{12.35} + \dots = \left(1 - \frac{m^4}{m_1^4}\right)\left(1 - \frac{m^4}{m_2^4}\right) \text{ \&c. ....} (4),$$

whence $\Sigma \frac{1}{m^4} = \frac{1}{12.35}$ &c., where of course the summation is exclusive of the zero value of m.

176. The frequencies of the series of tones are proportional to m^2. The interval between any tone and the gravest of the series may conveniently be expressed in octaves and fractions of an octave. This is effected by dividing the difference of the logarithms of m^2 by the logarithm of 2. The results are as follows:

1·4629	2·6478
2·4358	4·1332
3·1590	5·1036
3·7382, &c	5·8288, &c.

where the first column relates to the tones of a rod both whose ends are clamped, or free; and the second column to the case of a rod clamped at one end but free at the other. Thus from the second column we find that the first overtone is 2·6478 octaves higher than the gravest tone. The fractional part may be reduced to mean semitones by multiplication by 12. The interval is then two octaves + 7·7736 mean semitones. It will be seen that the rise of pitch is much more rapid than in the case of strings.

If a rod be clamped at one end and free at the other, the pitch of the gravest tone is $2\,(\log 4{\cdot}7300 - \log 1{\cdot}8751) \div \log 2$ or 2·6698 octaves lower than if both ends were clamped, or both free.

177. In order to examine more closely the curve in which the rod vibrates, we will transform the expression for u into a form more convenient for numerical calculation, taking first the case when both ends are free. Since $m = \frac{1}{2}(2i+1)\pi - (-1)^i \beta$, $\cos m = \sin\beta$, $\sin m = \cos i\pi \times \cos\beta$; and therefore, m being a root of $\cos m \cosh m = 1$, $\cosh m = \operatorname{cosec}\beta$.

Also

$$\sinh^2 m = \cosh^2 m - 1 = \tan^2 m = \cot^2 \beta,$$

or, since $\cot\beta$ is positive,

$$\sinh m = \cot\beta.$$

Thus

$$\frac{\sin m - \sinh m}{\cos m - \cosh m} = \frac{1 - \cos i\pi \sin\beta}{\cos\beta}$$

$$= \frac{(\cos\tfrac{1}{2}\beta - \cos i\pi \sin\tfrac{1}{2}\beta)^2}{(\cos\tfrac{1}{2}\beta - \cos i\pi \sin\tfrac{1}{2}\beta)(\cos\tfrac{1}{2}\beta + \cos i\pi \sin\tfrac{1}{2}\beta)}$$

$$= \frac{\cos\tfrac{1}{2}\beta \cos i\pi - \sin\tfrac{1}{2}\beta}{\cos\tfrac{1}{2}\beta \cos i\pi + \sin\tfrac{1}{2}\beta}.$$

We may therefore take, omitting the constant multiplier,

$$u = (\cos \tfrac{1}{2}\beta \cos i\pi + \sin \tfrac{1}{2}\beta) \left\{\sin \frac{mx}{l} + \sinh \frac{mx}{l}\right\}$$

$$- (\cos \tfrac{1}{2}\beta \cos i\pi - \sin \tfrac{1}{2}\beta) \left\{\cos \frac{mx}{l} + \cosh \frac{mx}{l}\right\}$$

$$= \sqrt{2} \cos i\pi \ \sin \left\{\frac{mx}{l} - \frac{\pi}{4} + (-1)^i \frac{\beta}{2}\right\}$$

$$+ \sin \tfrac{1}{2}\beta\, e^{mx/l} - \cos i\pi \ \cos \tfrac{1}{2}\beta\, e^{-mx/l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If we further throw out the factor $\sqrt{2}$, and put $l = 1$, we may take

$$u = F_1 + F_2 + F_3,$$

where

$$\left.\begin{aligned} F_1 &= \cos i\pi \sin \{mx - \tfrac{1}{4}\pi + \tfrac{1}{2}(-1)^i \beta\} \\ \log F_2 &= \quad mx \log e + \log \sin \tfrac{1}{2}\beta - \log \sqrt{2} \\ \log \pm F_3 &= - mx \log e + \log \cos \tfrac{1}{2}\beta - \log \sqrt{2} \end{aligned}\right\} \ldots\ldots\ldots\ldots (2),$$

from which u may be calculated for different values of i and x.

At the centre of the bar, $x = \frac{1}{2}$, and F_2, F_3 are numerically equal in virtue of $e^m = \cot \frac{1}{2}\beta$. When i is *even*, these terms cancel. For F_1, we have $F_1 = (-1)^i \sin \frac{1}{2} i\pi$, which is equal to zero when i is even, and to ± 1 when i is odd. When i is even, therefore, the sum of the three terms vanishes, and there is accordingly a node in the middle.

When $x = 0$, u reduces to $-2(-1)^i \sin \{\frac{1}{4}\pi - \frac{1}{2}(-1)^i \beta\}$, which (since β is always small) shews that for no value of i is there a node at the end. If a long bar of steel (held, for example, at the centre) be gently tapped with a hammer while varying points of its length are damped with the fingers, an unusual deadness in the sound will be noticed. as the end is closely approached.

178. We will now take some particular cases.

Vibration with two nodes. $i = 1$.

If $i = 1$, the vibration is the gravest of which the rod is capable. Our formulæ become

$$F_1 = - \sin \{x(270^\circ + 1^\circ\ 0'\ 40''{\cdot}94) - 45^\circ - 30'\ 20''{\cdot}47\}$$

$$\log F_2 = \quad 2{\cdot}054231\, x + 3{\cdot}7952391$$

$$\log F_3 = - 2{\cdot}054231\, x + \bar{1}{\cdot}8494681,$$

from which is calculated the following table, giving the values of u for x equal to ·00, ·05, ·10, &c.

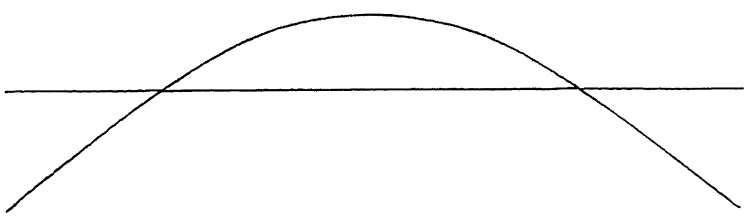
The values of $u : u(\cdot 5)$ for the intermediate values of x (in the last column) were found by interpolation formulæ. If o, p, q, r, s, t be six consecutive terms, that intermediate between q and r is

$$\frac{q+r}{2}+\frac{q+r-(p+s)}{4^2}+\frac{3}{4^4}\left\{2\left[q+r-(p+s)\right]-(p+s)\ +o+t\right\}.$$

x	F_1	F_2	F_3	u	$u : u(\cdot 5)$
·000	+ ·7133200	+ ·0062408	+ ·7070793	+ 1·4266401	+ 1·645219
·025	...	...	...	...	1·454176
·050	·5292548	·0079059	·5581572	1·0953179	1·263134
·075	...	...	.	...	1·072162
·100	·3157243	·0100153	·4406005	·7663401	·8837528
·125	...	...	...	...	·6969004
·150	+ ·0846166	·0126874	·3478031	·4451071	·5133028
·175	...	...	...	...	·3341625
·200	− ·1512020	·0160726	·2745503	+ ·1394209	+ ·1607819
·225	...	...	...	...	− ·0054711
·250	·3786027	·0203609	·2167256	− ·1415162	·1631982
·275	...	...	...	...	·3109982
·300	·5849255	·0257934	·1710798	·3880523	·4475066
·325	...	...	...	...	·5714137
·350	·7586838	·0326753	·1350477	·5909608	·6815032
·375	...	...	...	...	·7766629
·400	·8902038	·0413934	·1066045	·7422059	·8559210
·425	...	...	...	...	·9184491
·450	·9721635	·0524376	·0841519	·8355740	·9635940
·475	...	..	...	...	·9908730
·500	− 1·000000	+ ·0664285	·0664282	− ·8671433	− 1·0000000

Since the vibration curve is symmetrical with respect to the middle of the rod, it is unnecessary to continue the table beyond $x=\cdot 5$. The curve itself is shewn in fig. 28.

Fig. 28.



To find the position of the node, we have by interpolation

$$x = \cdot 200 + \frac{\cdot 1607819}{\cdot 1662530} \times \cdot 025 = \cdot 22418,$$

which is the fraction of the whole length by which the node is distant from the nearer end.

Vibration with three nodes. $i = 2$.

$$F_1 = \sin\{(450^\circ - 2' 40'' \cdot 27)\, x - 45^\circ + 1' 20'' \cdot 135\}$$

$$\log F_2 = 3\cdot 410604\, x + \bar{4}\cdot 4388816$$

$$\log(-F_3) = -3\cdot 410604\, x + \bar{1}\cdot 8494850.$$

x	$u : -u(0)$	x	$u : -u(0)$
·000	− 1·0000	·250	+ ·5847
·025	·8040	·275	·6374
·050	·6079	·300	·6620
·075	·4147	·325	·6569
·100	·2274	·350	·6245
·125	− ·0487	·375	·5652
·150	+ ·1175	·400	·4830
·175	·2672	·425	·3805
·200	·3972	·450	·2627
·225	·5037	·475	·1340
		·500	·0000

In this table, as in the preceding, the values of u were calculated directly for $x = \cdot 000$, ·050, ·100 &c, and interpolated for the intermediate values. For the position of the node the table gives by ordinary interpolation $x = \cdot 132$. Calculating from the above formulæ, we find

$$u(\cdot 1321) = -\cdot 000076,$$

$$u(\cdot 1322) = +\cdot 000881,$$

whence $x = \cdot 132108$, agreeing with the result obtained by Strehlke. The place of maximum excursion mav be found from the derived function. We get

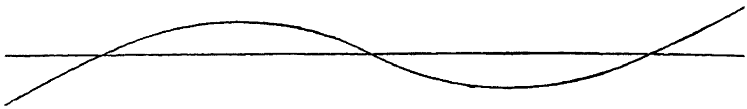
$$u'(\cdot 3083) = +\cdot 0006077, \qquad u'(3084) = -\cdot 0002227,$$

whence
$$u'(\cdot 308373) = 0.$$

Hence u is a maximum, when $x = \cdot 308373$; it then attains the value $\cdot 6636$, which, it should be observed, is much less than the excursion at the end.

The curve is shewn in fig. 29.

Fig. 29.



Vibration with four nodes. $i = 3$.

$$F_1 = -\sin\{(630^\circ + 6''\cdot 92)x - 45^\circ - 3''\cdot 46\},$$

$$\log F_2 = 4\cdot 775332\, x + \bar{5}\cdot 0741527,$$

$$\log F_3 = -4\cdot 775332\, x + \bar{1}\cdot 8494850.$$

From this $u(0) = 1\cdot 41424$, $u(\tfrac{1}{2}) = 1\cdot 00579$. The positions of the nodes are readily found by trial and error. Thus

$$u(\cdot 3558) = -\cdot 000037 \qquad u(\cdot 3559) = +\cdot 001047,$$

whence $u(\cdot 355803) = 0$. The value of x for the node near the end is $\cdot 0944$, (Seebeck).

The position of the loop is best found from the derived function. It appears that $u' = 0$, when $x = \cdot 2200$, and then $u = -\cdot 9349$. There is also a loop at the centre, where however the excursion is not so great as at the two others.

Fig. 30.

We saw that at the centre of the bar F_2 and F_3 are numerically equal. In the neighbourhood of the middle, F_3 is evidently very small, if i be moderately great, and thus the equation for the nodes reduces approximately to

$$\frac{mx}{l} - \frac{\pi}{4} + (-1)^i \frac{\beta}{2} = \pm n\pi,$$

n being an integer. If we transform the origin to the centre of the rod, and replace m by its approximate value $\tfrac{1}{2}(2i+1)\pi$, we find

$$\frac{x}{l} = \frac{\pm 2n - i}{2i + 1},$$

shewing that near the middle of the bar the nodes are uniformly spaced, the interval between consecutive nodes being $2l \div (2i+1)$. This theoretical result has been verified by the measurements of Strehlke and Lissajous.

For methods of approximation applicable to the nodes near the ends, when i is greater than 3, the reader is referred to the memoir by Seebeck already mentioned § 160, and to Donkin's *Acoustics* (p. 194).

179. The calculations are very similar for the case of a bar clamped at one end and free at the other. If $u \propto F$, and $F = F_1 + F_2 + F_3$, we have in general

$$F_1 = \cos\{mx + \tfrac{1}{4}\pi + \tfrac{1}{2}(-1)^i\alpha\},$$

$$F_2 = \frac{(-1)^i}{\sqrt{2}} \sin \tfrac{1}{2}\alpha\, e^{mx}; \quad F_3 = -\frac{1}{\sqrt{2}} \cos \tfrac{1}{2}\alpha\, e^{-mx}.$$

If $i = 1$, we obtain for the calculation of the gravest vibration-curve

$$F_1 = \cos\left\{\frac{180}{\pi} mx^0 + 45^0 - 8^0\, 43' \cdot 0665\right\},$$

$$\log(-F_2) = \quad mx \log e + \bar{1}\cdot 0300909.$$

$$\log(-F_3) = -mx \log e + \bar{1}\cdot 8444383.$$

These give on calculation

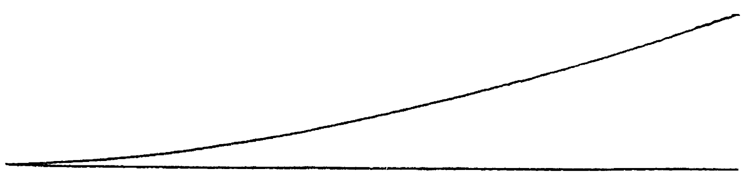
$$F(0) = \cdot 000000, \qquad F(\cdot 6) = \cdot 743452,$$

$$F(\cdot 2) = \cdot 102974, \qquad F(\cdot 8) = 1\cdot 169632,$$

$$F(\cdot 4) = \cdot 370625, \qquad F(1\cdot 0) = 1\cdot 612224,$$

from which fig. 31 was constructed.

Fig. 31.



The distances of the nodes from the free end in the case of a rod clamped at the other end are given by Seebeck and by Donkin

2nd tone ·2261.

3rd tone ·1321, ·4999.

4th tone ·0944, ·3558, ·6439.

i^{th} tone $\dfrac{1\cdot3222}{4i-2}$, $\dfrac{4\cdot9820}{4i-2}$, $\dfrac{9\cdot0007}{4i-2}$, $\dfrac{4j-3}{4i-2}$, $\dfrac{4i-10\cdot9993}{4i-2}$, $\dfrac{4i-7\cdot0175}{4i-2}$.

"The last row in this table must be understood as meaning that $\dfrac{4j-3}{4i-2}$ may be taken as the distance of the j^{th} node from the free end, except for the first three and the last two nodes."

When both ends are free, the distances of the nodes from the nearer end are

1st tone ·2242.

2nd tone ·1321 ·5.

3rd tone ·0944 ·3558.

i^{th} tone $\dfrac{1\cdot3222}{4i+2}$ $\dfrac{4\cdot9820}{4i+2}$ $\dfrac{9\cdot0007}{4i+2}$ $\dfrac{4j-3}{4i+2}$.

The points of inflection for a free-free rod (corresponding to the nodes of a clamped-clamped rod) are also given by Seebeck;—

	1st point.	2nd point.	κ^{th} point.
1st tone	No inflection point.		
2nd tone......	·5000		
3rd tone......	·3593		
i^{th} tone	$\dfrac{5\cdot0175}{4i+2}$	$\dfrac{8\cdot9993}{4i+2}$	$\dfrac{4\kappa+1}{4i+2}$

Except in the case of the extreme nodes (which have no corresponding inflection-point), the nodes and inflection-points always occur in close proximity.

180. The case where one end of a rod is free and the other *supported* does not need an independent investigation, as it may be referred to that of a rod with both ends free *vibrating in an even mode*, that is, with a node in the middle. For at the central node y and y'' vanish, which are precisely the conditions for a supported end. In like manner the vibrations of a clamped-supported rod are the same as those of one-half of a rod both whose ends are clamped, vibrating with a central node.

181. The last of the six combinations of terminal conditions occurs when both ends are supported. Referring to (1) § 170, we see that the conditions at $x = 0$, give $A = 0$, $B = 0$; so that

$$u = (C + D) \sin x' + (C - D) \sinh x'.$$

Since u and u'' vanish when $x' = m$, $C - D = 0$, and $\sin m = 0$.

Hence the solution is

$$y = \sin \frac{i\pi x}{l} \cos \frac{i^2 \pi^2 \kappa b}{l^2} t \quad \text{.................. (1)},$$

where i is an integer. An arbitrary constant multiplier may of course be prefixed, and a constant may be added to t.

It appears that the normal curves are the same as in the case of a string stretched between two fixed points, but the sequence of tone is altogether different, the frequency varying as the *square* of i. The nodes and inflection-points coincide, and the loops (which are also the points of maximum curvature) bisect the distances between the nodes.

182. The theory of a vibrating rod may be applied to illustrate the general principle that the natural periods of a system fulfil the maximum-minimum condition, and that the greatest of the natural periods exceeds any that can be obtained by a variation of type. Suppose that the vibration curve of a clamped-free rod is that in which the rod would dispose itself if deflected by a force applied at its free extremity. The equation of the curve may be taken to be

$$y = -3lx^2 + x^3,$$

which satisfies $d^4y/dx^4 = 0$ throughout, and makes y and y' vanish at 0, and y'' at l. Thus, if the configuration of the rod at time t be

$$y = (-3lx^2 + x^3) \cos pt \quad \text{..................... (1)},$$

the potential energy is by (1) § 161, $6 q\kappa^2 \omega l^3 \cos^2 pt$, while the kinetic energy is $\frac{33}{70}\rho\omega\, l^7\, p^2 \sin^2 pt$; and thus $p^2 = \frac{140}{11}\frac{\kappa^2 b^2}{l^4}$. Now p_1 (the true value of p for the gravest tone) is equal to

$$\frac{\kappa b}{l^2} \times (1\cdot 8751)^2;$$

so that

$$p_1 : p = (1\cdot 8751)^2 \sqrt{\frac{11}{140}} = \cdot 98556,$$

shewing that the real pitch of the gravest tone is rather (but comparatively little) lower than that calculated from the hypothetical type. It is to be observed that the hypothetical type in question violates the terminal condition $y''' = 0$. This circumstance, however, does not interfere with the application of the principle, for the assumed type may be any which would be admissible as an initial configuration; but it tends to prevent a very close agreement of periods.

We may expect a better approximation, if we found our calculation on the curve in which the rod would be deflected by a force acting at some little distance from the free end, between which and the point of action of the force ($x = c$) the rod would be straight, and therefore without potential energy. Thus

$$\text{potential energy} = 6\, q\kappa^2 \omega c^3 \cos^2 pt.$$

The kinetic energy can be readily found by integration from the value of y.

From 0 to c $\qquad y = -3cx^2 + x^3$;

and from c to l $\qquad y = c^2(c - 3x)$,

as may be seen from the consideration that y and y' must not suddenly change at $x = c$. The result is

$$\text{kinetic energy} = \rho\omega\, p^2 \sin^2 pt \left[\frac{33}{70} c^7 + \tfrac{1}{2} c^4 (l - c)(c^2 + 3l^2)\right],$$

whence

$$\frac{1}{p^2} = \frac{1}{6\kappa^2 b^2}\left[\frac{33}{70} c^4 + \frac{c}{2}(l - c)(c^2 + 3l^2)\right] \text{..............} (2).$$

The maximum value of $1/p^2$ will occur when the point of application of the force is in the neighbourhood of the node of the second normal component vibration. If we take $c = \frac{3}{4} l$, we obtain a result which is too high in the musical scale by the interval expressed by the ratio 1 : ·9977, and is accordingly extremely near the truth. This example may give an idea how nearly the period of a vibrating system may be calculated by simple means without the solution of differential or transcendental equations.

The type of vibration just considered would be that actually assumed by a bar which is itself devoid of inertia, but carries a load M at its free end, provided that the rotatory inertia of M could be neglected. We should have, in fact,

$$V = 6q\kappa^2\omega l^3 \cos^2 pt, \qquad T = 2Ml^6 p^2 \sin^2 pt,$$

so that

$$p^2 = \frac{3q\kappa^2\omega}{Ml^3} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Even if the inertia of the bar be not altogether negligible in comparison with M, we may still take the same type as the basis of an approximate calculation:

$$V = 6q\kappa^2\omega l^3 \cos^2 pt,$$

$$T = \left(2Ml^6 + \frac{33}{70}\rho\omega l^7\right) p^2 \sin^2 pt,$$

whence

$$\frac{1}{p^2} = \frac{l^3}{3q\kappa^2\omega}\left(M + \frac{33}{140}\rho\omega l\right) \quad\ldots\ldots\ldots\ldots\ldots (4),$$

that is, M is to be increased by about one quarter of the mass of the rod. Since this result is accurate when M is infinite, and does not differ much from the truth, even when $M = 0$, it may be regarded as generally applicable as an approximation. The error will always be on the side of estimating the pitch too high.

183. But the neglect of the rotatory inertia of M could not be justified under the ordinary conditions of experiment. It is as easy to imagine, though not to construct, a case in which the inertia of translation should be negligible in comparison with the inertia of rotation, as the opposite extreme which has just been considered. If both kinds of inertia in the mass M be included, even though that of the bar be neglected altogether, the system possesses two distinct and independent periods of vibration.

Let z and θ denote the values of y and dy/dx at $x = l$. Then the equation of the curve of the bar is

$$y = \frac{3z - l\theta}{l^2}\, x^2 + \frac{l\theta - 2z}{l^3}\, x^3,$$

and

$$V = \frac{2q\kappa^2\omega}{l^3}\{3z^2 - 3zl\theta + l^2\theta^2\} \quad\text{.............. (1);}$$

while for the kinetic energy

$$T = \tfrac{1}{2}M\dot{z}^2 + \tfrac{1}{2}M\kappa'^2\dot{\theta}^2 \quad\text{.......................... (2),}$$

if κ' be the radius of gyration of M about an axis perpendicular to the plane of vibration.

The equations of motion are therefore

$$\left.\begin{aligned} M\ddot{z} \quad + \frac{2q\kappa^2\omega}{l^3} \quad (6z - 3l\theta) = 0 \\ M\kappa'^2\ddot{\theta} + \frac{2q\kappa^2\omega}{l^3}(-3lz + 2l^2\theta) = 0 \end{aligned}\right\} \quad\text{............ (3);}$$

whence, if z and θ vary as $\cos pt$, we find

$$p^2 = \frac{2q\kappa^2\omega}{Ml\kappa'^2}\left\{1 + \frac{3\kappa'^2}{l^2} \pm \sqrt{1 + \frac{3\kappa'^2}{l^2} + \frac{9\kappa'^4}{l^4}}\right\} \quad\text{......... (4),}$$

corresponding to the two periods, which are always different.

If we neglect the rotatory inertia by putting $\kappa' = 0$, we fall back on our previous result

$$p^2 = \frac{3q\kappa^2\omega}{Ml^3}.$$

The other value of p^2 is then infinite.

If $\kappa' : l$ be merely small, so that its higher powers may be neglected,

$$\left.\begin{aligned} p^2 = \frac{4q\kappa^2\omega}{Ml\kappa'^2}\left(1 + \frac{9}{4}\frac{\kappa'^2}{l^2}\right) \\ p^2 = \frac{3q\kappa^2\omega}{Ml^3}\left(1 - \frac{9}{4}\frac{\kappa'^2}{l^2}\right) \end{aligned}\right\} \quad\text{.................. (5).}$$

If on the other hand κ'^2 be very great, so that rotation is prevented,

$$p^2 = \frac{12\,q\kappa^2\omega}{Ml^3} \quad\text{or}\quad \frac{q\kappa^2\omega}{Ml\kappa'^2} \quad\text{...... .... (6),}$$

the latter of which is very small. It appears that when rotation is prevented, the pitch is an octave higher than if there were no rotatory inertia at all. These conclusions might also be derived directly from the differential equations; for if $\kappa' = \infty$, $\theta = 0$, and then

$$M\ddot{z} + \frac{12\, q\kappa^2 \omega}{l^3} z = 0;$$

but if $\kappa' = 0$, $\theta = 3z/2l$, by the second of equations (3), and in that case

$$M\ddot{z} + \frac{3q\kappa^2 \omega}{l^3} z = 0.$$

184. If any addition to a bar be made at the end, the period of vibration is prolonged. If the end in question be free, suppose first that the piece added is without inertia. Since there would be no alteration in either the potential or kinetic energies, the pitch would be unchanged; but in proportion as the additional part acquires inertia, the pitch falls (§ 88).

In the same way a small continuation of a bar beyond a clamped end would be without effect, as it would acquire no motion. No change will ensue if the new end be also clamped; but as the first clamping is relaxed, the pitch falls, in consequence of the diminution in the potential energy of a given deformation.

The case of a 'supported' end is not quite so simple. Let the original end of the rod be A, and let the added piece which is at first supposed to have no inertia, be AB. Initially the end A is fixed, or held, if we like so to regard it, by a spring of infinite stiffness. Suppose that this spring, which has no inertia, is gradually relaxed. During this process the motion of the new end B diminishes, and at a certain point of relaxation, B comes to rest. During this process the pitch falls. B, being now at rest, may be supposed to become fixed, and the abolition of the spring at A entails another fall of pitch, to be further increased as AB acquires inertia.

185. The case of a rod which is not quite uniform may be treated by the general method of § 90. We have in the notation there adopted

$$c_r = \int B_0 \left(\frac{d^2 u_r}{dx^2}\right)^2 dx, \qquad \delta c_r = \int \delta B \left(\frac{d^2 u_r}{dx^2}\right)^2 dx$$

$$a_r = \int \rho\omega_0 u_r^2 dx, \qquad \delta a_r = \int \delta\overline{\rho\omega}\, u_r^2 dx,$$

whence, P_r being the uncorrected value of p_r,

$$p_r^2 = P_r^2 \left\{ 1 + \frac{\int \delta B \left(\frac{d^2 u_r}{dx^2}\right)^2 dx}{\int B_0 \left(\frac{d^2 u_r}{dx^2}\right)^2 dx} - \frac{\int \delta \overline{\rho\omega}\, u_r^2 dx}{\int \overline{\rho\omega_0}\, u_r^2 dx} \right\}$$

$$= P_r^2 \left\{ 1 + \frac{\int \delta B\, u_r''^2 dx}{B_0 \int u_r^2 dx} - \frac{\int \delta\rho\omega\, u_r\, dx}{\rho\omega_0 \int u_r^2 dx} \right\} \quad \text{..............(1).}$$

[If the motion be strictly periodic with respect to x, u_r'' is proportional to u_r, and both quantities vanish at a node. Accordingly an irregularity situated at a node of this kind of motion has no effect upon the period. A similar conclusion will hold good approximately for the interior nodes of a bar vibrating with numerous subdivisions, even though, as when the terminals are clamped or free, the mode of motion be not strictly periodic with respect to x.]

If the rod be clamped at 0 and free at l,

$$p_r^2 = \frac{B_0 m^4}{\rho\omega_0 l^4} \left\{ 1 + \frac{4}{l u_l^2} \int_0^l \frac{\delta B}{B_0} u''^2 dx - \frac{4}{l u_l^2} \int_0^l \frac{\delta \overline{\rho\omega}}{\overline{\rho\omega_0}} u^2 dx \right\}.$$

The same formula applies to a doubly free bar.

The effect of a small load dM is thus given by

$$p^2 = \frac{B_0 m^4}{\rho\omega_0 l^4} \left\{ 1 - 4 \frac{u^2 dM}{u_l^2 M} \right\} \quad \text{.................. (2),}$$

where M denotes the mass of the whole bar. If the load be at the end, its effect is the same as a lengthening of the bar in the ratio $M : M + dM$. (Compare § 167.)

[In (2) dM is supposed to act by inertia only; but a similar formula may conveniently be employed when an irregularity of mass dM depends upon a variation of section, without a change of mechanical properties. Since $B = q\kappa^2\omega$,

$$\delta B / B_0 = \delta(\kappa^2\omega)/(\kappa^2\omega)_0 ;$$

so that the effect of a local excrescence is given by

$$p^2/P^2 = 1 + \frac{4u''^2}{l u_l^2} \int \frac{\delta(\kappa^2\omega)}{(\kappa^2\omega)_0} dx - \frac{4u^2}{l u_l^2} \int \frac{\delta\omega}{\omega_0} dx \quad \text{...........(3).}$$

If the thickness in the plane of bending be constant, $\delta\kappa^2 = 0$, and

$$\delta(\kappa^2\omega)/(\kappa^2\omega)_0 = \delta\omega/\omega_0.$$

Further,
$$\int \frac{\delta\omega\, dx}{l\omega_0} = \frac{dM}{M};$$

and thus
$$p^2/P^2 = 1 + 4\frac{dM}{M}\frac{u''^2 - u^2}{u_l^2} \qquad (4).$$

If, however, the thickness in the plane perpendicular to that of bending be constant, and in the plane of bending variable (2γ), then
$$\delta(\kappa^2\omega)/(\kappa^2\omega)_0 = \delta\gamma^3/\gamma_0^3 = 3\,\delta\gamma/\gamma_0 = 3\,\delta\omega/\omega_0;$$
and in place of (4)
$$p^2/P^2 = 1 + 4\frac{dM}{M}\frac{3u''^2 - u^2}{u_l^2} \qquad (5).$$

If a tuning-fork be filed (dM negative) near the stalk (clamped end), the pitch is lowered; and if it be filed near the free end, the pitch is raised. Since $u_0''^2 = u_l^2$, the effects of a given stroke of the file are equal and opposite in the circumstances of (4), but in the circumstances of (5) the effect at the stalk is three times as great as at the free end.]

186. The same principle may be applied to estimate the correction due to the rotatory inertia of a uniform rod. We have only to find what addition to make to the kinetic energy, supposing that the bar vibrates according to the same law as would obtain, were there no rotatory inertia.

Let us take, for example, the case of a bar clamped at 0 and free at l, and assume that the vibration is of the type,
$$y = u\cos pt,$$
where u is one of the functions investigated in § 179. The kinetic energy of the rotation is
$$\tfrac{1}{2}\int \rho\omega\kappa^2\left(\frac{d^2y}{dx\,dt}\right)^2 dx = \frac{\rho\omega\kappa^2 m^2 p^2}{2l^2}\sin^2 pt\int_0^l u'^2\,dx$$
$$= \frac{\rho\omega\kappa^2 m p^2}{8l}\sin^2 pt\,(2uu' + mu'^2)_l,$$

by (2) § 165.

To this must be added
$$\frac{\rho\omega}{2}p^2\sin^2 pt\int_0^l u^2\,dx, \quad \text{or} \quad \frac{\rho\omega l}{8}p^2\sin^2 pt\, u_l^2;$$

so that the kinetic energy is increased in the ratio
$$1 : 1 + \frac{m\kappa^2}{l^2}\left(2\frac{u'}{u} + m\frac{u'^2}{u^2}\right)_l.$$

The altered frequency bears to that calculated without allowance for rotatory inertia a ratio which is the square root of the reciprocal of the preceding. Thus

$$p:P=1-\tfrac{1}{2}\frac{m\kappa^2}{l^2}\left(2\frac{u'}{u}+m\frac{u'^2}{u^2}\right)_l \quad\text{..............(1).}$$

By use of the relations $\cosh m=-\sec m$, $\sinh m=\cos i\pi.\tan m$, we may express $u':u$ when $x=l$ in the form

$$\frac{u'}{u}=\frac{-\sin m}{\cos i\pi+\cos m}=\frac{\cos\alpha}{1-\cos i\pi\sin\alpha},$$

if we substitute for m from

$$m=\tfrac{1}{2}(2i-1)\pi-(-1)^i\alpha.$$

In the case of the gravest tone, $\alpha=\cdot3043$, or, in degrees and minutes, $\alpha=17^\circ\,26'$, whence

$$\frac{u'}{u}=\cdot73413,\qquad 2\frac{u'}{u}+m\frac{u'^2}{u^2}=2\cdot4789.$$

Thus

$$p:P=1-2\cdot3241\frac{\kappa^2}{l^2}\quad\text{....................(2),}$$

which gives the correction for rotatory inertia in the case of the gravest tone.

When the order of the tone is moderate, α is very small, and then

$$u':u=1\quad\text{sensibly,}$$

and

$$p:P=1-\left(1+\frac{m}{2}\right)\frac{m\kappa^2}{l^2}\quad\text{.................(3),}$$

shewing that the correction increases in importance with the order of the component.

In all ordinary bars $\kappa:l$ is very small, and the term depending on its square may be neglected without sensible error.

187. When the rigidity and density of a bar are variable from point to point along it, the normal functions cannot in general be expressed analytically, but their nature may be investigated by the methods of Sturm and Liouville explained in § 142.

If, as in § 162, B denote the variable flexural rigidity at any point of the bar, and $\rho\omega\, dx$ the mass of the element, whose length is dx, we find as the general differential equation

$$\frac{d^2}{dx^2}\left(B\frac{d^2y}{dx^2}\right)+\rho\omega\frac{d^2y}{dt^2}=0 \text{ .... } (1),$$

the effects of rotatory inertia being omitted. If we assume that $y \propto \cos \nu t$, we obtain as the equation to determine the form of the normal functions

$$\frac{d^2}{dx^2}\left(B\frac{d^2y}{dx^2}\right)=\nu^2\rho\omega y \text{ .... } (2),$$

in which ν^2 is limited by the terminal conditions to be one of an infinite series of definite quantities ν_1^2, ν_2^2, ν_3^2......

Let us suppose, for example, that the bar is clamped at both ends, so that the terminal values of y and dy/dx vanish. The first normal function, for which ν^2 has its lowest value ν_1^2, has no internal root, so that the vibration-curve lies entirely on one side of the equilibrium-position. The second normal function has one internal root, the third function has two internal roots, and, generally, the r^{th} function has $r-1$ internal roots.

Any two different normal functions are conjugate, that is to say, their product will vanish when multiplied by $\rho\omega\, dx$, and integrated over the length of the bar.

Let us examine the number of roots of a function $f(x)$ of the form

$$f(x)=\phi_m u_m(x)+\phi_{m+1}u_{m+1}(x)+\ldots+\phi_n u_n(x) \text{......} (3),$$

compounded of a finite number of normal functions, of which the function of lowest order is $u_m(x)$ and that of highest order is $u_n(x)$. If the number of internal roots of $f(x)$ be μ, so that there are $\mu+4$ roots in all, the derived function $f'(x)$ cannot have less than $\mu+1$ internal roots besides two roots at the extremities, and the second derived function cannot have less than $\mu+2$ roots. No roots can be lost when the latter function is multiplied by B, and another double differentiation with respect to x will leave at least μ internal roots. Hence by (2) and (3) we conclude that

$$\nu_m^2\phi_m u_m(x)+\nu_{m+1}^2\phi_{m+1}u_{m+1}(x)+\ldots+\nu_n^2\phi_n u_n(x) \text{...} (4)$$

has at least as many roots as $f(x)$. Since (4) is a function of the same form as $f(x)$, the same argument may be repeated, and a series of functions obtained every member of which has at least as many roots as $f(x)$ has. When the operation by which (4) was derived from (3) has been repeated sufficiently often, a function is arrived at whose form differs as little as we please from that of the component normal function of highest order $u_n(x)$; and we conclude that $f(x)$ cannot have more than $n-1$ internal roots. In like manner we may prove that $f(x)$ cannot have less than $m-1$ internal roots.

The application of this theorem to demonstrate the possibility of expanding an arbitrary function in an infinite series of normal functions would proceed exactly as in § 142.

[An analytical investigation of certain cases where the section of a rod is supposed to be variable, will be found in a memoir by Kirchhoff[1]].

188. When the bar, whose lateral vibrations are to be considered, is subject to longitudinal tension, the potential energy of any configuration is composed of two parts, the first depending on the stiffness by which the bending is directly opposed, and the second on the reaction against the extension, which is a necessary accompaniment of the bending, when the ends are nodes. The second part is similar to the potential energy of a deflected string; the first is of the same nature as that with which we have been occupied hitherto in this Chapter, though it is not entirely independent of the permanent tension.

Consider the extension of a filament of the bar of section $d\omega$, whose distance from the axis projected on the plane of vibration is η. Since the sections, which were normal to the axis originally, remain normal during the bending, the length of the filament bears to the corresponding element of the axis the ratio $R+\eta : R$, R being the radius of curvature. Now the axis itself is extended in the ratio $q : q+T$, reckoning from the unstretched state, if $T\omega$ denote the whole tension to which the bar is subjected. Hence the actual tension on the filament is $\{T+\eta(T+q)/R\}\,d\omega$, from which we find for the moment of the couple acting across the section

$$\int\left\{T+\frac{\eta}{R}(T+q)\right\}\eta\,d\omega=\frac{q+T}{R}\kappa^2\omega,$$

[1] *Berlin Monatsber.*, 1879; *Collected Works*, p. 339. See also Todhunter and Pearson's *History of the Theory of Elasticity*, Vol. II., Part ii., § 1302.

and for the whole potential energy due to stiffness

$$\tfrac{1}{2}(q+T)\kappa^2\omega\int\left(\frac{d^2y}{dx^2}\right)^2 dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

an expression differing from that previously used (§ 162) by the substitution of $(q+T)$ for q.

Since q is the tension required to stretch a bar of unit area to twice its natural length, it is evident that in most practical cases T would be negligible in comparison with q.

The expression (1) denotes the work that would be gained during the straightening of the bar, if the length of each element of the axis were preserved constant during the process. But when a stretched bar or string is allowed to pass from a displaced to the natural position, the length of the axis is decreased. The amount of the decrease is $\tfrac{1}{2}\int(dy/dx)^2\,dx$, and the corresponding gain of work is

$$\tfrac{1}{2}T\omega\int\left(\frac{dy}{dx}\right)^2 dx.$$

Thus

$$V=\tfrac{1}{2}(q+T)\kappa^2\omega\int\left(\frac{d^2y}{dx^2}\right)^2 dx+\tfrac{1}{2}T\omega\int\left(\frac{dy}{dx}\right)^2 dx \ldots\ldots (2).$$

The variation of the first part due to a hypothetical displacement is given in § 162. For the second part, we have

$$\tfrac{1}{2}\delta\int\left(\frac{dy}{dx}\right)^2 dx=\int\frac{dy}{dx}\frac{d\delta y}{dx}\,dx=\left\{\frac{dy}{dx}\delta y\right\}-\int\frac{d^2y}{dx^2}\delta y\,dx \ldots\ldots (3).$$

In all the cases that we have to consider, δy vanishes at the limits. The general differential equation is accordingly

$$\kappa^2(q+T)\frac{d^4y}{dx^4}-T\frac{d^2y}{dx^2}+\rho\frac{d^2y}{dt^2}-\kappa^2\rho\frac{d^4y}{dx^2\,dt^2}=0,$$

or, if we put $q+T=b^2\rho$, $T=a^2\rho$,

$$\kappa^2\left(b^2\frac{d^4y}{dx^4}-\frac{d^4y}{dx^2\,dt^2}\right)-a^2\frac{d^2y}{dx^2}+\frac{d^2y}{dt^2}=0 \ldots\ldots\ldots (4).$$

For a more detailed investigation of this equation the reader is referred to the writings of Clebsch[1] and Donkin.

189. If the ends of the rod, or wire, be clamped, $dy/dx=0$, and the terminal conditions are satisfied. If the nature of the support be such that, while the extremity is constrained to be a node, there

[1] *Theorie der Elasticität fester Körper.* Leipzig, 1862.

is no couple acting on the bar, d^2y/dx^2 must vanish, that is to say, the end must be straight. This supposition is usually taken to represent the case of a string stretched over bridges, as in many musical instruments; but it is evident that the part beyond the bridge must partake of the vibration, and that therefore its length cannot be altogether a matter of indifference.

If in the general differential equation we take y proportional to $\cos nt$, we get

$$\kappa^2\left(b^2\frac{d^4y}{dx^4}+n^2\frac{d^2y}{dx^2}\right)-a^2\frac{d^2y}{dx^2}-n^2y=0\ \ldots\ldots\ldots\ldots(1),$$

which is evidently satisfied by

$$y=\sin i\frac{\pi x}{l}\cos nt\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2),$$

if n be suitably determined. The same solution also makes y and y'' vanish at the extremities. By substitution we obtain for n,

$$n^2=\frac{i^2\pi^2}{l^2}\,\frac{a^2l^2+i^2\pi^2\kappa^2b^2}{l^2+i^2\pi^2\kappa^2}\ \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

which determines the frequency.

If we suppose the wire infinitely thin, $n^2=i^2\pi^2a^2\div l^2$, the same as was found in Chapter VI., by starting from the supposition of perfect flexibility. If we treat $\kappa:l$ as a very small quantity, the approximate value of n is

$$n=\frac{i\pi a}{l}\left\{1+i^2\frac{\pi^2\kappa^2}{2l^2}\left(\frac{b^2}{a^2}-1\right)\right\}.$$

For a wire of circular section of radius r, $\kappa^2=\frac{1}{4}r^2$, and if we replace b and a by their values in terms of q, T, and ρ,

$$n=\frac{i\pi a}{l}\left\{1+\frac{i^2\pi^2}{8}\frac{r^2}{l^2}\frac{q}{T}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

which gives the correction for rigidity[1]. Since the expression within brackets involves i, it appears that the harmonic relation of the component tones is disturbed by the stiffness.

190. The investigation of the correction for stiffness when the ends of the wire are clamped is not so simple, in consequence of the change of type which occurs near the ends. In order to pass from the case of the preceding section to that now under con-

[1] Donkin's *Acoustics*, Art. 184.

sideration an additional constraint must be introduced, with the effect of still further raising the pitch. The following is, in the main, the investigation of Seebeck and Donkin.

If the rotatory inertia be neglected, the differential equation becomes

$$\left(D^4 - \frac{a^2}{\kappa^2 b^2} D^2 - \frac{n^2}{b^2 \kappa^2}\right) y = 0 \quad\quad (1),$$

where D stands for $\frac{d}{dx}$. In the equation

$$D^4 - \frac{a^2}{\kappa^2 b^2} D^2 - \frac{n^2}{b^2 \kappa^2} = 0,$$

one of the values of D^2 must be positive, and the other negative. We may therefore take

$$D^4 - \frac{a^2}{\kappa^2 b^2} D^2 - \frac{n^2}{b^2 \kappa^2} = (D^2 - \alpha^2)(D^2 + \beta^2) \quad\quad (2),$$

and for the complete integral of (1)

$$y = A \cosh \alpha x + B \sinh \alpha x + C \cos \beta x + D \sin \beta x \quad\quad (3),$$

where α and β are functions of n determined by (2).

The solution must now be made to satisfy the four boundary conditions, which, as there are only three disposable ratios, lead to an equation connecting α, β, l. This may be put into the form

$$\frac{\sinh \alpha l \, \sin \beta l}{1 - \cosh \alpha l \, \cos \beta l} + \frac{2\alpha\beta}{\alpha^2 - \beta^2} = 0 \quad\quad (4).$$

The value of $\frac{2\alpha\beta}{\alpha^2 - \beta^2}$, determined by (2), is $\frac{2nb\kappa}{a^2}$, so that

$$\frac{\sinh \alpha l \, \sin \beta l}{1 - \cosh \alpha l \, \cos \beta l} + \frac{2nb\kappa}{a^2} = 0 \quad\quad (5).$$

From (2) we find also that

$$\left.\begin{aligned} \alpha^2 &= \frac{a^2}{2b^2\kappa^2}\left\{\sqrt{1 + 4\frac{n^2 b^2 \kappa^2}{a^4}} + 1\right\} \\ \beta^2 &= \frac{a^2}{2b^2\kappa^2}\left\{\sqrt{1 + 4\frac{n^2 b^2 \kappa^2}{a^4}} - 1\right\} \end{aligned}\right\} \quad\quad (6).$$

Thus far our equations are rigorous, or rather as rigorous as the differential equation on which they are founded; but we shall now introduce the supposition that the vibration considered is but slightly affected by the existence of rigidity. This being the case, the approximate expression for y is

$$y = \sin \frac{i\pi x}{l} \cos\left(\frac{i\pi}{l} at\right),$$

and therefore

$$\beta = i\pi/l, \qquad n = i\pi a/l \text{ ....(7)},$$

nearly.

The introduction of these values into the second of equations (6) proves that $n^2b^2\kappa^2/a^4$ or $b^2\kappa^2/a^2l^2$ is a small quantity under the circumstances contemplated, and therefore that α^2l^2 is a large quantity. Since $\cosh \alpha l$, $\sinh \alpha l$ are both large, equation (5) reduces to

$$\tan \beta l = \frac{2nb\kappa}{a^2},$$

or, on substitution of the approximate value for β derived from (6),

$$\tan \frac{nl}{a} = 2\frac{nb\kappa}{a^2}.$$

The approximate value of nl/a is $i\pi$. If we take $nl/a = i\pi + \theta$ we get

$$\tan(i\pi + \theta) = \tan\theta = \theta = 2\frac{nb\kappa}{a^2} = 2i\pi \frac{b}{a}\frac{\kappa}{l},$$

so that

$$n = i\frac{\pi a}{l}\left(1 + 2\frac{b}{a}\frac{\kappa}{l}\right) \text{ ....(8)}.$$

According to this equation the component tones are all raised in pitch by the same small interval, and therefore the harmonic relation is not disturbed by the rigidity. It would probably be otherwise if terms involving $\kappa^2 : l^2$ were retained; it does not therefore follow that the harmonic relation is better preserved in spite of rigidity when the ends are clamped than when they are free, but only that there is no additional disturbance in the former case, though the absolute alteration of pitch is much greater. It should be remarked that $b : a$ or $\sqrt{(q + T)} : \sqrt{T}$, is a large quantity, and that, if our result is to be correct, $\kappa : l$ must be small enough to bear multiplication by $b : a$ and yet remain small.

The theoretical result embodied in (8) has been compared with experiment by Seebeck, who found a satisfactory agreement. The constant of stiffness was deduced from observations of the rapidity of the vibrations of a small piece of the wire, when one end was clamped in a vice.

[As the result of a second approximation Seebeck gives (*loc. cit.*)

$$n^2 = n_0^2 \left\{1 + 4\frac{b\kappa}{al} + (12 + i^2\pi^2)\frac{b^2\kappa^2}{a^2l^2}\right\} \quad\text{............(9)}].$$

191. It has been shewn in this chapter that the theory of bars, even when simplified to the utmost by the omission of unimportant quantities, is decidedly more complicated than that of perfectly flexible strings. The reason of the extreme simplicity of the vibrations of strings is to be found in the fact that waves of the harmonic type are propagated with a velocity independent of the wave length, so that an arbitrary wave is allowed to travel without decomposition. But when we pass from strings to bars, the constant in the differential equation, viz. $d^2y/dt^2 + \kappa^2 b^2 d^4y/dx^4 = 0$, is no longer expressible as a velocity, and therefore the velocity of transmission of a train of harmonic waves cannot depend on the differential equation alone, but must vary with the wave length. Indeed, if it be admitted that the train of harmonic waves can be propagated at all, this consideration is sufficient by itself to prove that the velocity must vary inversely as the wave length. The same thing may be seen from the solution applicable to waves propagated in one direction, viz. $y = \cos\frac{2\pi}{\lambda}(Vt - x)$, which satisfies the differential equation if

$$V = \frac{2\pi\kappa b}{\lambda} \quad\text{..........................(1).}$$

Let us suppose that there are two trains of waves of equal amplitudes, but of different wave lengths, travelling in the same direction. Thus

$$y = \cos 2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right) + \cos 2\pi\left(\frac{t}{\tau'} - \frac{x}{\lambda'}\right)$$

$$= 2\cos\pi\left\{t\left(\frac{1}{\tau} - \frac{1}{\tau'}\right) - x\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)\right\}\cos\pi\left\{t\left(\frac{1}{\tau} + \frac{1}{\tau'}\right) - x\left(\frac{1}{\lambda} + \frac{1}{\lambda'}\right)\right\} \quad\text{...(2).}$$

If $\tau' - \tau$, $\lambda' - \lambda$ be small, we have a train of waves, whose amplitude slowly varies from one point to another between the values 0 and 2, forming a series of groups separated from one another by regions comparatively free from disturbance. In the case of a string or of a column of air, λ varies as τ, and then the groups move forward with the same velocity as the component trains, and there is no change of type. It is otherwise when, as in the case of a bar vibrating transversely, the velocity of propagation is a function of the wave length. The position at time t of the middle of the group which was initially at the origin is given by

$$t\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)-x\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)=0,$$

which shews that the velocity of the group is

$$\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)\div\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)=\delta\left(\frac{1}{\tau}\right)\div\delta\left(\frac{1}{\lambda}\right).$$

If we suppose that the velocity V of a train of waves varies as λ^n, we find

$$\frac{d\,(1/\tau)}{d\,(1/\lambda)}=\frac{d\,(V/\lambda)}{d\,(1/\lambda)}=-(n-1)\,V\;\ldots\ldots\ldots\ldots\ldots(3).$$

In the present case $n=-1$, and accordingly the velocity of the groups is *twice* that of the component waves[1].

192. On account of the dependence of the velocity of propagation on the wave length, the condition of an infinite bar at any time subsequent to an initial disturbance confined to a limited portion, will have none of the simplicity which characterises the corresponding problem for a string; but nevertheless Fourier's investigation of this problem may properly find a place here.

It is required to determine a function of x and t, so as to satisfy

$$\frac{d^2y}{dt^2}+\frac{d^4y}{dx^4}=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(1),$$

and make initially $y=\phi(x)$, $\dot{y}=\psi(x)$.

A solution of (1) is

$$y=\cos q^2t\;\cos q\,(x-\alpha)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where q and α are constants, from which we conclude that

$$y=\int_{-\infty}^{+\infty}d\alpha\,F(\alpha)\int_{-\infty}^{+\infty}dq\cos q^2t\;\cos q\,(x-\alpha)$$

[1] In the corresponding problem for waves on the surface of deep water, the velocity of propagation varies directly as the square root of the wave length, so that $n=\frac{1}{2}$. The velocity of a group of such waves is therefore *one half* of that of the component trains. [See note on Progressive Waves, appended to this volume.]

is also a solution, where $F(\alpha)$ is an arbitrary function of α. If now we put $t=0$,

$$y_0=\int_{-\infty}^{+\infty} d\alpha F(\alpha)\int_{-\infty}^{+\infty} dq \cos q(x-\alpha),$$

which shews that $F(\alpha)$ must be taken to be $\frac{1}{2\pi}\phi(\alpha)$, for then by Fourier's double integral theorem $y_0=\phi(x)$. Moreover, $\dot{y}=0$; hence

$$y=\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\alpha\,\phi(\alpha)\int_{-\infty}^{+\infty} dq \cos q^2 t\ \cos q(x-\alpha)\ldots\ldots(3)$$

satisfies the differential equation, and makes initially

$$y=\phi(x),\qquad \dot{y}=0.$$

By Stokes' theorem (§ 95), or independently, we may now supply the remaining part of the solution, which has to satisfy the differential equation while it makes initially $y=0$, $\dot{y}=\psi(x)$; it is

$$y=\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\alpha\,\psi(\alpha)\int_{-\infty}^{+\infty} dq\frac{1}{q^2}\sin q^2 t\ \cos q(x-\alpha)\ldots\ldots(4).$$

The final result is obtained by adding the right-hand members of (3) and (4).

In (3) the integration with respect to q may be effected by means of the formula

$$\int_{-\infty}^{+\infty} dq \cos q^2 t \cos qz=\sqrt{\frac{\pi}{t}}\sin\left(\frac{\pi}{4}+\frac{z^2}{4t}\right)\ldots\ldots\ldots\ldots(5),$$

which may be proved as follows. If in the well-known integral formula

$$\int_{-\infty}^{+\infty} e^{-a^2x^2}\,dx=\frac{\sqrt{\pi}}{a},$$

we put $x+b$ for x, we get

$$\int_{-\infty}^{+\infty} e^{-a^2(x^2+2bx)}\,dx=\frac{\sqrt{\pi}}{a}e^{a^2b^2}.$$

Now suppose that $a^2=i=e^{\frac{1}{2}i\pi}$, where $i=\sqrt{(-1)}$, and retain only the real part of the equation. Thus

$$\int_{-\infty}^{+\infty}\cos(x^2+2bx)\,dx=\sqrt{\pi}\sin(b^2+\tfrac{1}{4}\pi),$$

whence

$$\int_{-\infty}^{+\infty} \cos x^2 \cos 2bx\, dx = \sqrt{\pi} \sin (b^2 + \tfrac{1}{4}\pi),$$

from which (5) follows by a simple change of variable. Thus equation (3) may be written

$$y = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} d\alpha\, \phi(\alpha) \sin \left\{\frac{\pi}{4} + \frac{(x-\alpha)^2}{4t}\right\},$$

or, if $\dfrac{\alpha - x}{2\sqrt{t}} = \mu$,

$$y = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\mu\, (\cos \mu^2 + \sin \mu^2)\, \phi(x + 2\mu\sqrt{t}) \quad \ldots\ldots (6).$$

192 *a*. If the axis of the rod be curved instead of straight, we obtain problems which may be regarded as extensions of those of the present and of the last chapters. The most important case under this head is that of a circular ring, whose section we will regard as also circular, and of radius (c) small in comparison with the radius (a) of the circular axis.

The investigation of the flexural modes of vibration, executed in the plane of the ring, is analogous to the case of a cylinder (see § 233), and was first effected by Hoppe[1]. If s be the number of periods in the circumference, the coefficient p of the time in the expression for the vibrations is given by

$$p^2 = \frac{1}{4} \frac{s^2 (s^2 - 1)^2}{1 + s^2} \frac{q}{\rho} \frac{c^2}{a^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where q is Young's modulus and ρ the density of the material. This may be compared with equation (9) § 233. To fall back upon the case of a straight axis we have only to suppose s and a to be infinite in such a manner that $2\pi a/s$ is equal to the proposed linear period. The vibrations in question are then purely transverse.

In the class of vibrations considered above the circular axis remains unextended, and (§ 232) the periods are comparatively long. For the other class of vibrations in the plane of the ring, Hoppe found

$$p^2 = (1 + s^2) \frac{q}{\rho} \frac{1}{a^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

[1] *Crelle*, Bd. 63, p. 158, 1871.

The frequencies are here independent of c, and the vibrations are analogous to the longitudinal vibrations of straight rods.

If $s=0$ in (2), we have the solution for vibrations which are purely radial.

For flexural vibrations perpendicular to the plane of the ring, the result[1] corresponding to (1) is

$$p^2 = \frac{1}{4}\frac{s^2(s^2-1)^2}{1+\mu+s^2}\frac{q}{\rho}\frac{c^2}{a^4} \qquad \text{.....................(3)},$$

the difference consisting only in the occurrence of Poisson's ratio (μ) in the denominator.

Our limits will not allow of our dwelling further upon the problem of this section. A complete investigation will be found in Love's *Treatise on Elasticity*, Chapter XVIII. The effect of a small curvature upon the lateral vibrations of a limited bar has been especially considered by Lamb[2].

[1] Michell, *Messenger of Mathematics*, XIX., 1889.

[2] *Proc. Lond. Math. Soc.*, XIX., p. 365, 1888.

CHAPTER IX.

VIBRATIONS OF MEMBRANES.

193. THE theoretical membrane is a perfectly flexible and infinitely thin lamina of solid matter, of uniform material and thickness, which is stretched in all directions by a tension so great as to remain sensibly unaltered during the vibrations and displacements contemplated. If an imaginary line be drawn across the membrane in any direction, the mutual action between the two portions separated by an element of the line is proportional to the length of the element and perpendicular to its direction. If the force in question be $T_1 \, ds$, T_1 may be called the *tension of the membrane;* it is a quantity of one dimension in mass and -2 in time.

The principal problem in connection with this subject is the investigation of the transverse vibrations of membranes of different shapes, whose boundaries are fixed. Other questions indeed may be proposed, but they are of comparatively little interest; and, moreover, the methods proper for solving them will be sufficiently illustrated in other parts of this work. We may therefore proceed at once to the consideration of a membrane stretched over the area included within a fixed, closed, plane boundary.

194. Taking the plane of the boundary as that of xy, let w denote the small displacement therefrom of any point P of the membrane. Round P take a small area S, and consider the forces acting upon it parallel to z. The resolved part of the tension is expressed by

$$T_1 \int \frac{dw}{dn} \, ds,$$

where ds denotes an element of the boundary of S, and dn an element of the normal to the curve drawn outwards. This is balanced by the reaction against acceleration measured by $\rho S \ddot{w}$,

ρ being a symbol of one dimension in mass and -2 in length denoting the superficial density. Now by Green's theorem, if $\nabla^2 = d^2/dx^2 + d^2/dy^2$,

$$\int \frac{dw}{dn}\, ds = \iint \nabla^2 w\, dS = \nabla^2 w \,.\, S \quad \text{ultimately,}$$

and thus the equation of motion is

$$\frac{d^2w}{dt^2} = \frac{T_1}{\rho}\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right) \quad \text{..................(1).}$$

The condition to be satisfied at the boundary is of course $w = 0$.

The differential equation may also be investigated from the expression for the potential energy, which is found by multiplying the tension by the superficial stretching. The altered area is

$$\iint \sqrt{1 + \left(\frac{dw}{dx}\right)^2 + \left(\frac{dw}{dy}\right)^2}\, dx\, dy\,;$$

and thus

$$V = \tfrac{1}{2} T_1 \iint \left\{\left(\frac{dw}{dx}\right)^2 + \left(\frac{dw}{dy}\right)^2\right\} dx\, dy \quad \text{..............(2),}$$

from which δV is easily found by an integration by parts.

If we write $T_1 \div \rho = c^2$, then c is of the nature of a velocity, and the differential equation is

$$\frac{d^2w}{dt^2} = c^2\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right) \quad \text{....................(3).}$$

195. We shall now suppose that the boundary of the membrane is the rectangle formed by the coordinate axes and the lines $x = a$, $y = b$. For every point within the area (3) § 194 is satisfied, and for every point on the boundary $w = 0$.

A particular integral is evidently

$$w = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos pt \quad \text{.................(1),}$$

where

$$p^2 = c^2\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \quad \text{..........................(2),}$$

and m and n are integers; and from this the general solution may be derived. Thus

$$w = \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\{A_{mn}\cos pt + B_{mn}\sin pt\} \quad \text{...... (3).}$$

That this result is really general may be proved *a posteriori*, by shewing that it may be adapted to express arbitrary initial circumstances.

Whatever function of the co-ordinates w may be, it can be expressed for all values of x between the limits 0 and a by the series

$$Y_1 \sin\frac{\pi x}{a} + Y_2 \sin\frac{2\pi x}{a} + \ldots\ldots,$$

where the coefficients Y_1, Y_2, &c. are independent of x. Again whatever function of y any one of the coefficients Y may be, it can be expanded between 0 and b in the series

$$C_1 \sin\frac{\pi y}{b} + C_2 \sin\frac{2\pi y}{b} + \ldots\ldots,$$

where C_1 &c. are constants. From this we conclude that any function of x and y can be expressed within the limits of the rectangle by the double series

$$\sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b};$$

and therefore that the expression for w in (3) can be adapted to arbitrary initial values of w and $\dot{w}$. In fact

$$\left.\begin{aligned} A_{mn} &= \frac{4}{ab}\int_0^a\int_0^b w_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy, \\ B_{mn} &= \frac{4}{abp}\int_0^a\int_0^b \dot{w}_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy, \end{aligned}\right\}\ldots\ldots(4).$$

The character of the normal functions of a given rectangle,

$$\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b},$$

as depending on m and n, is easily understood. If m and n be both unity, w retains the same sign over the whole of the rectangle, vanishing at the edge only; but in any other case there are nodal lines running parallel to the axes of coordinates. The number of the nodal lines parallel to x is $n-1$, their equations being

$$y = \frac{b}{n}, \ \frac{2b}{n}, \ldots\ldots \frac{(n-1)\,b}{n}.$$

In the same way the equations of the nodal lines parallel to y are

$$x=\frac{a}{m},\ \frac{2a}{m},\ldots\ldots\frac{(m-1)a}{m},$$

being $m-1$ in number. The nodal system divides the rectangle into mn equal parts, in each of which the numerical value of w is repeated.

196. The expression for w in terms of the normal functions is

$$w=\Sigma\Sigma\,\phi_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\ \ldots\ldots\ldots\ldots\ldots\ (1),$$

where ϕ_{mn} &c. are the normal coordinates. We proceed to form the expression for V in terms of ϕ_{mn}. We have

$$\left(\frac{dw}{dx}\right)^2=\pi^2\left\{\Sigma\Sigma\,\phi_{mn}\frac{m}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right\}^2,$$

$$\left(\frac{dw}{dy}\right)^2=\pi^2\left\{\Sigma\Sigma\,\phi_{mn}\frac{n}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right\}^2.$$

In integrating these expressions over the area of the rectangle the products of the normal coordinates disappear, and we find

$$V=\frac{T_1}{2}\iint\left\{\left(\frac{dw}{dx}\right)^2+\left(\frac{dw}{dy}\right)^2\right\}dx\,dy$$

$$=\frac{T_1}{2}\frac{ab\pi^2}{4}\Sigma\Sigma\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\phi_{mn}^2\ \ldots\ldots\ldots\ldots\ldots\ldots\ (2),$$

the summation being extended to all integral values of m and n.

The expression for the kinetic energy is proved in the same way to be

$$T=\frac{\rho}{2}\frac{ab}{4}\Sigma\Sigma\,\dot{\phi}_{mn}^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

from which we deduce as the normal equation of motion

$$\ddot{\phi}_{mn}+c^2\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\phi_{mn}=\frac{4}{ab\rho}\,\Phi_{mn}\ldots\ldots\ldots\ldots(4).$$

In this equation

$$\Phi_{mn}=\int_0^a\int_0^b Z\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy\ldots\ldots\ldots\ldots(5),$$

if $Z\,dx\,dy$ denote the transverse force acting on the element $dx\,dy$.

Let us suppose that the initial condition is one of rest under the operation of a constant force Z, such as may be supposed to arise from gaseous pressure. At the time $t=0$, the impressed force is removed, and the membrane left to itself. Initially the equation of equilibrium is

$$c^2\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)(\phi_{mn})_0=\frac{4}{ab\rho}\,\Phi_{mn}\qquad\qquad(6),$$

whence $(\phi_{mn})_0$ is to be found. The position of the system at time t is then given by

$$\phi_{mn}=(\phi_{mn})_0\cos\left(\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\,.\,c\pi t\right)\qquad\qquad(7),$$

in conjunction with (1).

In order to express Φ_{mn}, we have merely to substitute for Z its value in (5), or in this case simply to remove Z from under the integral sign. Thus

$$\Phi_{mn}=Z\int_0^a\int_0^b\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy,$$

$$=Z\frac{ab}{mn\pi^2}(1-\cos m\pi)(1-\cos n\pi).$$

We conclude that Φ_{mn} vanishes, unless m and n are *both* odd, and that then

$$\Phi_{mn}=\frac{4ab}{mn\pi^2}Z.$$

Accordingly, m and n being both odd,

$$\phi_{mn}=\frac{16Z}{\pi^2\rho}\,\frac{\cos pt}{mnp^2}\qquad\qquad(8),$$

where

$$p^2=c^2\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\qquad\qquad(9).$$

This is an example of (8), § 101.

If the membrane, previously at rest in its position of equilibrium, be set in motion by a blow applied at the point (α, β), the solution is

$$\phi_{mn}=\frac{4}{abp}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}\iint\dot{w}_0\,dx\,dy\,.\sin pt\ldots(10).$$

[As an example of forced vibrations, suppose that a harmonic force acts at the centre. Unless m and n are both odd, $\Phi_{mn}=0$, and in the case reserved

$$\Phi_{mn}=\pm Z_1\cos qt\qquad\qquad(11),$$

where Z_1 is the whole force acting at time t, and $\pm$ represents $\sin \frac{1}{2}m\pi \sin \frac{1}{2}n\pi$. From (4) and (9) we have

$$\phi_{mn} = \frac{\pm 4 Z_1 \cos qt}{ab\rho\,(p_{mn}^2 - q^2)} \quad\text{.................} \quad (12)$$

and w is then given by (1).

In the case of a *square* membrane, p is a symmetrical function of m and n. When m and n are unequal, the terms occur in pairs, such as

$$\frac{\pm 4 Z_1 \cos qt}{a^2\rho\,(p_{mn}^2 - q^2)} \left\{ \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} + \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{a} \right\} \text{...} (13),$$

a combination symmetrical as between x and y. The vibration is of course similarly related as well to the four sides as to the four corners of the square.

In the neighbourhood of the centre, where the force is applied, the series loses its convergency, and the displacement w tends to become (logarithmically) infinite.]

197. The frequency of the natural vibrations is found by ascribing different integral values to m and n in the expression

$$\frac{p}{2\pi} = \frac{c}{2} \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \quad\text{......................} \quad (1).$$

For a given mode of vibration the pitch falls when either side of the rectangle is increased. In the case of the gravest mode, when $m = 1$, $n = 1$, additions to the shorter side are the more effective; and when the form is very elongated, additions to the longer side are almost without effect.

When a^2 and b^2 are incommensurable, no two pairs of values of m and n can give the same frequency, and each fundamental mode of vibration has its own characteristic period. But when a^2 and b^2 are commensurable, two or more fundamental modes may have the same periodic time, and may then coexist in any proportions, while the motion still retains its simple harmonic character. In such cases the specification of the period does not completely determine the type. The full consideration of the problem now presenting itself requires the aid of the theory of numbers; but it will be sufficient for the purposes of this work to consider a few of the simpler cases, which arise when the membrane is square. The reader will find fuller information in Riemann's lectures on partial differential equations.

If $a=b$,

$$\frac{p}{2\pi}=\frac{c}{2a}\sqrt{m^2+n^2}\ \dots\dots\dots\dots\dots\dots\dots\ (2).$$

The lowest tone is found by putting m and n equal to unity, which gives only one fundamental mode:—

$$w=\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\cos pt\ \dots\dots\dots\dots\dots\dots\dots(3).$$

Next suppose that one of the numbers m, n is equal to 2, and the other to unity. In this way two distinct types of vibration are obtained, whose periods are the same. If the two vibrations be synchronous in phase, the whole motion is expressed by

$$w=\left\{C\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}+D\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a}\right\}\cos pt\dots(4);$$

so that, although every part vibrates synchronously with a harmonic motion, the type of vibration is to some extent arbitrary. Four particular cases may be especially noted. First, if $D=0$,

$$w=C\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\cos pt\ \dots\dots\ \dots\dots\dots\ (5),$$

which indicates a vibration with one node along the line $x=\frac{1}{2}a$. Similarly if $C=0$, we have a node parallel to the other pair of edges. Next, however, suppose that C and D are finite and equal. Then w is proportional to

$$\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}+\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a},$$

which may be put into the form

$$2\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\left(\cos\frac{\pi x}{a}+\cos\frac{\pi y}{a}\right).$$

This expression vanishes, when

$$\sin\pi x/a=0,\quad\text{or}\quad\sin\pi y/a=0$$

or again, when

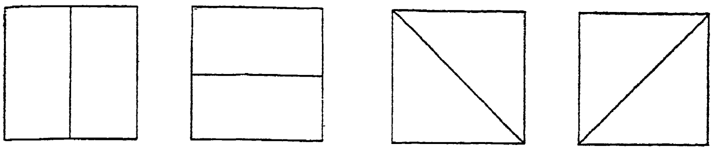
$$\cos\pi x/a+\cos\pi y/a=0.$$

The first two equations give the edges, which were originally assumed to be nodal; while the third gives $y+x=a$, representing one diagonal of the square.

In the fourth case, when $C=-D$, we obtain for the nodal lines, the edges of the square together with the diagonal $y=x$. The figures represent the four cases.

Fig. 32.

$D = 0.$ $C = 0.$ $C - D = 0.$ $C + D = 0.$



[Frequency (referred to gravest) = 1·58.]

For other relative values of C and D the interior nodal line is curved, but is always analytically expressed by

$$C \cos \frac{\pi x}{a} + D \cos \frac{\pi y}{a} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and may be easily constructed with the help of a table of logarithmic cosines.

The next case in order of pitch occurs when $m = 2$, $n = 2$. The values of m and n being equal, no alteration is caused by their interchange, while no other pair of values gives the same frequency of vibration. The only type to be considered is accordingly

$$w = \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{a} \cos pt,$$

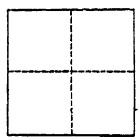
whose nodes, determined by the equation

$$\sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} = 0,$$

are (in addition to the edges) the straight lines Fig. (33)

$$x = \tfrac{1}{2}a \qquad y = \tfrac{1}{2}a.$$

Fig. 33.



[Frequency = 2·00.]

The next case which we shall consider is obtained by ascribing to m, n the values 3, 1, and 1, 3 successively. We have

$$w = \left\{C \sin \frac{3\pi x}{a} \sin \frac{\pi y}{a} + D \sin \frac{\pi x}{a} \sin \frac{3\pi y}{a}\right\} \cos pt.$$

The nodes are given by

$$\sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \left\{C\left(4\cos^2 \frac{\pi x}{a} - 1\right) + D\left(4\cos^2 \frac{\pi y}{a} - 1\right)\right\} = 0,$$

or, if we reject the first two factors, which correspond to the edges,

$$C\left(4\cos^2 \frac{\pi x}{a} - 1\right) + D\left(4\cos^2 \frac{\pi y}{a} - 1\right) = 0 \ldots\ldots\ldots (7).$$

If $C=0$, we have $\quad y=\frac{1}{3}a, \quad y=\frac{2}{3}a.$

If $D=0$, $\quad x=\frac{1}{3}a, \quad x=\frac{2}{3}a.$

If $C=-D$,
$$\cos\frac{\pi x}{a}=\pm\cos\frac{\pi y}{a},$$

whence,
$$y=x, \quad y=a-x,$$

which represent the two diagonals.

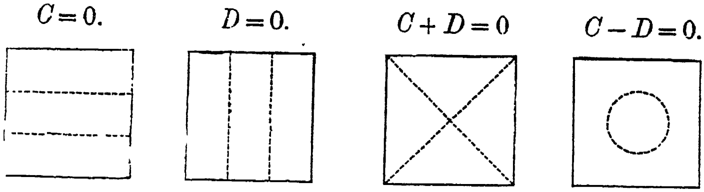
Lastly, if $C=D$, the equation of the node is

$$\cos^2\frac{\pi x}{a}+\cos^2\frac{\pi y}{a}=\frac{1}{2},$$

or

$$1+\cos\frac{2\pi x}{a}+\cos\frac{2\pi y}{a}=0 \quad\text{.................}\quad (8),$$

Fig. 34.



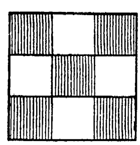
[Frequency = 2·24.]

In case (4) when $x=\frac{1}{2}a$, $y=\frac{1}{4}a$, or $\frac{3}{4}a$; and similarly when $y=\frac{1}{2}a$, $x=\frac{1}{4}x$, or $\frac{3}{4}a$. Thus one half of each of the lines joining the middle points of opposite edges is intercepted by the curve.

[The diameters of the nodal curve parallel to the sides of the square are thus equal to $\frac{1}{2}a$. Those measured along the diagonals are sensibly smaller, equal to $\frac{1}{3}\sqrt{2}\,.\,a$, or $\cdot 471\,a$.]

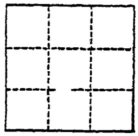
It should be noticed that in whatever ratio to one another C and D may be taken, the nodal curve always passes through the four points of intersection of the nodal lines of the first two cases, $C=0$, $D=0$. If the vibrations of these cases be compounded with corresponding phases, it is evident that in the shaded compartments of Fig. (35) the directions of displacement are the same, and that therefore no part of the nodal curve is to be found there; whatever the ratio of amplitudes, the curve must be drawn through the unshaded portions. When on the other hand the phases are opposed, the nodal curve will pass exclusively through the shaded portions.

Fig. 35.



When $m=3$, $n=3$, the nodes are the straight lines parallel to the edges shewn in Fig. (36).

Fig. 36.



[Frequency = 3·00.]

The last case [Frequency = 2·55] which we shall consider is obtained by putting

$$m=3,\ n=2, \quad \text{or} \quad m=2,\ n=3.$$

The nodal system is

$$C\sin\frac{3\pi x}{a}\sin\frac{2\pi y}{a}+D\sin\frac{2\pi x}{a}\sin\frac{3\pi y}{a}=0,$$

or, if the factors corresponding to the edges be rejected,

$$C\left(4\cos^2\frac{\pi x}{a}-1\right)\cos\frac{\pi y}{a}+D\cos\frac{\pi x}{a}\left(4\cos^2\frac{\pi y}{a}-1\right)=0 \quad\ldots\ldots(9).$$

If C or D vanish, we fall back on the nodal systems of the component vibrations, consisting of straight lines parallel to the edges. If $C=D$, our equation may be written

$$\left(\cos\frac{\pi x}{a}+\cos\frac{\pi y}{a}\right)\left(4\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}-1\right)=0 \quad\ldots\ldots(10),$$

of which the first factor represents the diagonal $y+x=a$, and the second a hyperbolic curve.

If $C=-D$, we obtain the same figure relatively to the other diagonal[1].

198. The pitch of the natural modes of a square membrane, which is nearly, but not quite uniform, may be investigated by the general method of § 90.

We will suppose in the first place that m and n are equal. In this case, when the pitch of a uniform membrane is given, the mode of its vibration is completely determined. If we now conceive a variation of density to ensue, the natural type of vibration is in general modified, but the period may be calculated approximately without allowance for the change of type.

We have

$$T=\tfrac{1}{2}\iint(\rho_0+\delta\rho)\,\dot{\phi}_{mm}{}^2\sin^2\frac{m\pi x}{a}\sin^2\frac{m\pi y}{a}\,dx\,dy$$

$$=\tfrac{1}{2}\dot{\phi}_{mm}{}^2\left\{\rho_0\frac{a^2}{4}+\iint\delta\rho\sin^2\frac{m\pi x}{a}\sin^2\frac{m\pi y}{a}\,dx\,dy\right\},$$

[1] Lamé, *Leçons sur l'élasticité*, p. 129.

of which the second term is the increment of T due to $\delta\rho$. Hence if $w \propto \cos pt$, and P denote the value of p previously to variation, we have

$$p_{mm}^2 : P_{mm}^2 = 1 - \frac{4}{a^2}\int_0^a\int_0^a \frac{\delta\rho}{\rho_0}\sin^2\frac{m\pi x}{a}\sin^2\frac{m\pi y}{a}\,dx\,dy \;\;\ldots\ldots (1),$$

where
$$P_{mm}^2 = \frac{2\,c^2\pi^2 m^2}{a^2}, \quad \text{and} \quad c^2 = T_1 \div \rho_0.$$

For example, if there be a small load M attached to the middle of the square,

$$p_{mm}^2 : P_{mm}^2 = 1 - \frac{4M}{a^2\rho_0}\sin^4 m\frac{\pi}{2} \;\;\ldots\ldots\ldots\ldots\ldots (2),$$

in which $\sin^4 \frac{1}{2}m\pi$ vanishes, if m be even, and is equal to unity, if m be odd. In the former case the centre is on the nodal line of the unloaded membrane, and thus the addition of the load produces no result.

When, however, m and n are unequal, the problem, though remaining subject to the same general principles, presents a peculiarity different from anything we have hitherto met with. The natural type for the unloaded membrane corresponding to a specified period is now to some extent arbitrary; but the introduction of the load will in general remove the indeterminate element. In attempting to calculate the period on the assumption of the undisturbed type, the question will arise how the selection of the undisturbed type is to be made, seeing that there are an indefinite number, which in the uniform condition of the membrane give identical periods. The answer is that those types must be chosen which differ infinitely little from the actual types assumed under the operation of the load, and such a type will be known by the criterion of its making the period calculated from it a maximum or minimum.

As a simple example, let us suppose that a small load M is attached to the membrane at a point lying on the line $x = \frac{1}{2}a$, and that we wish to know what periods are to be substituted for the two equal periods of the unloaded membrane, found by making

$$m = 2,\; n = 1, \quad \text{or} \quad m = 1,\; n = 2.$$

It is clear that the normal types to be chosen, are those whose nodes are represented in the first two cases of Fig. (32). In the first case the increase in the period due to the load is zero, which is the least that it can be; and in the second case the increase is the greatest possible. If β be the ordinate of M, the kinetic energy is altered in the ratio

$$\frac{\rho}{2}\frac{a^2}{4} : \frac{\rho}{2}\frac{a^2}{4} + \frac{M}{2}\sin^2\frac{2\pi\beta}{a};$$

and thus
$$p_{12}^2 : P_{12}^2 = 1 - \frac{4M}{a^2\rho}\sin^2\frac{2\pi\beta}{a} \quad\text{.................(3)}$$

while
$$p_{21}^2 = P_{21}^2 = P_{12}^2.$$

The ratio characteristic of the interval between the two natural tones of the loaded membrane is thus approximately

$$1 + \frac{2M}{a^2\rho}\sin^2\frac{2\pi\beta}{a} \quad\text{.......................(4).}$$

If $\beta = \frac{1}{2}a$, neither period is affected by the load.

As another example, the case where the values of m and n are 3 and 1, considered in § 197, may be referred to. With a load in the middle, the two normal types to be selected are those corresponding to the last two cases of Fig. (34), in the former of which the load has no effect on the period.

The problem of determining the vibration of a square membrane which carries a relatively heavy load is more difficult, and we shall not attempt its solution. But it may be worth while to recall to memory the fact that the actual period is greater than any that can be calculated from a hypothetical type, which differs from the actual one.

199. The preceding theory of square membranes includes a good deal more than was at first intended. Whenever in a vibrating system certain parts remain at rest, they may be supposed to be absolutely fixed, and we thus obtain solutions of other questions than those originally proposed. For example, in the present case, wherever a diagonal of the square is nodal, we obtain a solution applicable to a membrane whose fixed boundary is an isosceles right-angled triangle. Moreover, any mode of vibration possible to the triangle corresponds to some natural mode of the square, as may be seen by supposing two triangles put together, the vibrations being equal and opposite at points which are the images of each other in the common hypothenuse. Under these circumstances it is evident that the hypothenuse would remain at rest without constraint, and therefore the vibration in question is included among those of which a complete square is capable.

The frequency of the gravest tone of the triangle is found by putting $m = 1$, $n = 2$ in the formula

$$\frac{p}{2\pi} = \frac{c}{2a}\sqrt{(m^2 + n^2)} \qquad\qquad (1),$$

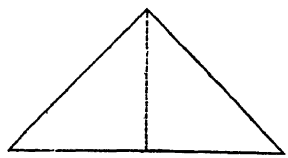
and is therefore equal to $c\sqrt{5}/2a$.

The next tone occurs, when $m = 3$, $n = 1$. In this case

$$\frac{p}{2\pi} = \frac{c\sqrt{10}}{2a} \qquad\qquad (2),$$

as might also be seen by noticing that the triangle divides itself into two, Fig. (37), whose sides are less than those of the whole triangle in the ratio $\sqrt{2} : 1$.

Fig. 37.



For the theory of the vibrations of a membrane whose boundary is in the form of an equilateral triangle, the reader is referred to Lamé's *Leçons sur l'élasticité.* It is proved that the frequency of the gravest tone is $c \div h$, where h is the height of the triangle, which is the same as the frequency of the gravest tone of a square whose diagonal is h.

200. When the fixed boundary of the membrane is circular, the first step towards a solution of the problem is the expression of the general differential equation in polar co-ordinates. This may be effected analytically; but it is simpler to form the polar equation *de novo* by considering the forces which act on the polar element of area $r\,d\theta\,dr$. As in § 194 the force of restitution acting on a small area of the membrane is

$$-T_1\int\frac{dw}{dn}\,ds = -T_1\left\{\frac{d}{dr}\left(\frac{dw}{dr}\,r\,d\theta\right)dr + \frac{d}{d\theta}\left(\frac{dw}{r\,d\theta}\,dr\right)d\theta\right\}$$

$$= -T_1\,.\,r\,d\theta\,dr\left\{\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \frac{1}{r^2}\frac{d^2w}{d\theta^2}\right\};$$

and thus, if $T_1/\rho = c^2$ as before, the equation of motion is

$$\frac{d^2w}{dt^2} = c^2\left\{\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \frac{1}{r^2}\frac{d^2w}{d\theta^2}\right\} \qquad\qquad (1).$$

The subsidiary condition to be satisfied at the boundary is that $w = 0$, when $r = a$.

In order to investigate the normal component vibrations we have now to assume that w is a harmonic function of the time.

Thus, if $w \propto \cos(pt-\epsilon)$, and for the sake of brevity we write $p/c=k$, the differential equation appears in the form

$$\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}+\frac{1}{r^2}\frac{d^2w}{d\theta^2}+k^2w=0\ldots\ldots\ldots\ldots\ldots(2),$$

in which k is the reciprocal of a linear quantity.

Now whatever may be the nature of w as a function of r and θ, it can be expanded in Fourier's series

$$w=w_0+w_1\cos(\theta+\alpha_1)+w_2\cos 2(\theta+\alpha_2)+\ldots\ldots(3),$$

in which w_0, w_1, &c. are functions of r, but not of θ. The result of substituting from (3) in (2) may be written

$$\Sigma\left\{\frac{d^2w_n}{dr^2}+\frac{1}{r}\frac{dw_n}{dr}+\left(k^2-\frac{n^2}{r^2}\right)w_n\right\}\cos n(\theta+\alpha_n)=0,$$

the summation extending to all integral values of n. If we multiply this equation by $\cos n(\theta+\alpha_n)$, and integrate with respect to θ between the limits 0 and 2π, we see that each term must vanish separately, and we thus obtain to determine w_n as a function of r

$$\frac{d^2w_n}{dr^2}+\frac{1}{r}\frac{dw_n}{dr}+\left(k^2-\frac{n^2}{r^2}\right)w_n=0\ldots\ldots\ldots\ldots\ldots(4),$$

in which it is a matter of indifference whether the factor $\cos n(\theta+\alpha_n)$ be supposed to be included in w_n or not.

The solution of (4) involves two distinct functions of r, each multiplied by an arbitrary constant. But one of these functions becomes infinite when r vanishes, and the corresponding particular solution must be excluded as not satisfying the prescribed conditions at the origin of co-ordinates. This point may be illustrated by a reference to the simpler equation derived from (4) by making k and n vanish, when the solution in question reduces to $w=\log r$, which, however, does not at the origin satisfy $\nabla^2w=0$, as may be seen from the value of $\int(dw/dn)\,ds$, integrated round a small circle with the origin for centre. In like manner the complete integral of (4) is too general for our present purpose, since it covers the case in which the centre of the membrane is subjected to an external force.

The other function of r, which satisfies (4), is the Bessel's function of the n^{th} order, denoted by $J_n(kr)$, and may be expressed in several ways. The ascending series (obtained immediately from the differential equation) is

$$J_n(z) = \frac{z^n}{2^n \Gamma(n+1)} \left\{ 1 - \frac{z^2}{2 \,.\, 2n+2} + \frac{z^4}{2 \,.\, 4 \,.\, 2n+2 \,.\, 2n+4} - \frac{z^6}{2 \,.\, 4 \,.\, 6 \,.\, 2n+2 \,.\, 2n+4 \,.\, 2n+6} + \ldots\ldots \right\} \ldots\ldots\ldots (5),$$

from which the following relations between functions of consecutive orders may readily be deduced:

$$J_0'(z) = -J_1(z) \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$2J_n'(z) = J_{n-1}(z) - J_{n+1}(z) \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$\frac{2n}{z} J_n(z) = J_{n-1}(z) + J_{n+1}(z) \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

When n is an integer, $J_n(z)$ may be expressed by the definite integral

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \omega - n\omega)\, d\omega \ldots\ldots\ldots\ldots\ldots (9),$$

which is Bessel's original form. From this expression it is evident that J_n and its differential coefficients with respect to z are always less than unity.

The ascending series (5), though infinite, is convergent for all values of n and z; but, when z is great, the convergence does not begin for a long time, and then the series becomes useless as a basis for numerical calculation. In such cases another series proceeding by descending powers of z may be substituted with advantage. This series is

$$J_n(z) = \sqrt{\frac{2}{\pi z}} \left\{ 1 - \frac{(1^2 - 4n^2)(3^2 - 4n^2)}{1 \,.\, 2 \,.\, (8z)^2} + \ldots\ldots \right\} \cos\left(z - \frac{\pi}{4} - n\frac{\pi}{2}\right)$$
$$+ \sqrt{\frac{2}{\pi z}} \left\{ \frac{1^2 - 4n^2}{1 \,.\, 8z} - \frac{(1^2 - 4n^2)(3^2 - 4n^2)(5^2 - 4n^2)}{1 \,.\, 2 \,.\, 3 \,.\, (8z)^3} + \ldots\ldots \right\}$$
$$\times \sin\left(z - \frac{\pi}{4} - n\frac{\pi}{2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (10);$$

it terminates, if $2n$ be equal to an odd integer, but otherwise, it runs on to infinity, and becomes ultimately divergent. Nevertheless when z is great, the convergent part may be employed in calculation; for it can be proved that the sum of any number of terms differs from the true value of the function by less than the last term included. We shall have occasion later, in connection with another problem, to consider the derivation of this descending series.

As Bessel's functions are of considerable importance in theoretical acoustics, I have thought it advisable to give a table for the functions J_0 and J_1, extracted from Lommel's[1] work, and due

[1] Lommel, *Studien über die Bessel'schen Functionen.* Leipzig, 1863.

originally to Hansen. The functions J_0 and J_1 are connected by the relation

$$J_0' = -J_1.$$

z	$J_0(z)$	$J_1(z)$	z	$J_0(z)$	$J_1(z)$	z	$J_0(z)$	$J_1(z)$
0·0	1·0000	0·0000	4·5	·3205	·2311	9·0	·0903	·2453
0·1	·9975	·0499	4·6	·2961	·2566	9·1	·1142	·2324
0·2	·9900	·0995	4·7	·2693	·2791	9·2	·1367	·2174
0·3	·9776	·1483	4·8	·2404	·2985	9·3	·1577	·2004
0·4	·9604	·1960	4·9	·2097	·3147	9·4	·1768	·1816
0·5	·9385	·2423	5·0	·1776	·3276	9·5	·1939	·1613
0·6	·9120	·2867	5·1	·1443	·3371	9·6	·2090	·1395
0·7	·8812	·3290	5·2	·1103	·3432	9·7	·2218	·1166
0·8	·8463	·3688	5·3	·0758	·3460	9·8	·2323	·0928
0·9	·8075	·4060	5·4	·0412	·3453	9·9	·2403	·0684
1·0	·7652	·4401	5·5	− ·0068	·3414	10·0	·2459	·0435
1·1	·7196	·4709	5·6	+ ·0270	·3343	10·1	·2490	+ ·0184
1·2	·6711	·4983	5·7	·0599	·3241	10·2	·2496	− ·0066
1·3	·6201	·5220	5·8	·0917	·3110	10·3	·2477	·0313
1·4	·5669	·5419	5·9	·1220	·2951	10·4	·2434	·0555
1·5	·5118	·5579	6·0	·1506	·2767	10·5	·2366	·0789
1·6	·4554	·5699	6·1	·1773	·2559	10·6	·2276	·1012
1·7	·3980	·5778	6·2	·2017	·2329	10·7	·2164	·1224
1·8	·3400	·5815	6·3	·2238	·2081	10·8	·2032	·1422
1·9	·2818	·5812	6·4	·2433	·1816	10·9	·1881	·1604
2·0	·2239	·5767	6·5	·2601	·1538	11·0	·1712	·1768
2·1	·1666	·5683	6·6	·2740	·1250	11·1	·1528	·1913
2·2	·1104	·5560	6·7	·2851	·0953	11·2	·1330	·2039
2·3	·0555	·5399	6·8	·2931	·0652	11·3	·1121	·2143
2·4	+ ·0025	·5202	6·9	·2981	·0349	11·4	·0902	·2225
2·5	− ·0484	·4971	7·0	·3001	− ·0047	11·5	·0677	·2284
2·6	·0968	·4708	7·1	·2991	+ ·0252	11·6	·0446	·2320
2·7	·1424	·4416	7·2	·2951	·0543	11·7	− ·0213	·2333
2·8	·1850	·4097	7·3	·2882	·0826	11·8	+ ·0020	·2323
2·9	·2243	·3754	7·4	·2786	·1096	11·9	·0250	·2290
3·0	·2601	·3391	7·5	·2663	·1352	12·0	·0477	·2234
3·1	·2921	·3009	7·6	·2516	·1592	12·1	·0697	·2157
3·2	·3202	·2613	7·7	·2346	·1813	12·2	·0908	·2060
3·3	·3443	·2207	7·8	·2154	·2014	12·3	·1108	·1943
3·4	·3643	·1792	7·9	·1944	·2192	12·4	·1296	·1807
3·5	·3801	·1374	8·0	·1717	·2346	12·5	·1469	·1655
3·6	·3918	·0955	8·1	·1475	·2476	12·6	·1626	·1487
3·7	·3992	·0538	8·2	·1222	·2580	12·7	·1766	·1307
3·8	·4026	+ ·0128	8·3	·0960	·2657	12·8	·1887	·1114
3·9	·4018	− ·0272	8·4	·0692	·2708	12·9	·1988	·0912
4·0	·3972	·0660	8·5	·0419	·2731	13·0	·2069	·0703
4·1	·3887	·1033	8·6	+ ·0146	·2728	13·1	·2129	·0489
4·2	·3766	·1386	8·7	− ·0125	·2697	13·2	·2167	·0271
4·3	·3610	·1719	8·8	·0392	·2641	13·3	·2183	− ·0052
4·4	·3423	·2028	8·9	·0653	·2559	13·4	·2177	+ ·0166

201. In accordance with the notation for Bessel's functions the expression for a normal component vibration may therefore be written

$$w = P\,J_n(kr)\,\cos n(\theta+\alpha)\,\cos(pt+\epsilon)\,\ldots\ldots\ldots\ldots(1);$$

and the boundary condition requires that

$$J_n(ka) = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

an equation whose roots give the admissible values of k, and therefore of p.

The complete expression for w is obtained by combining the particular solutions embodied in (1) with all admissible values of k and n, and is necessarily general enough to cover any initial circumstances that may be imagined. We conclude that any function of r and θ may be expanded within the limits of the circle $r = a$ in the series

$$w = \Sigma\Sigma\, J_n(kr)\,\{\phi\cos n\theta + \psi\sin n\theta\}\ldots\ldots\ldots(3).$$

For every integral value of n there are a series of values of k, given by (2); and for each of these the constants ϕ and ψ are arbitrary.

The determination of the constants is effected in the usual way. Since the energy of the motion is equal to

$$\tfrac{1}{2}\rho\int_0^a\int_0^{2\pi}\dot{w}^2 r\,d\theta\,dr\,\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and when expressed by means of the normal co-ordinates can only involve their squares, it follows that the product of any two of the terms in (3) vanishes, when integrated over the area of the circle. Thus, if we multiply (3) by $J_n(kr)\cos n\theta$, and integrate, we find

$$\int_0^a\int_0^{2\pi} w\,J_n(kr)\cos n\theta\,r dr\,d\theta$$

$$= \phi\iint[J_n(kr)]^2\cos^2 n\theta\,r dr\,d\theta = \phi\,.\,\pi\int_0^a[J_n(kr)]^2\,r dr\ldots\ldots(5),$$

by which ϕ is determined. The corresponding formula for ψ is obtained by writing $\sin n\theta$ for $\cos n\theta$. A method of evaluating the integral on the right will be given presently. Since ϕ and ψ each contain two terms, one varying as $\cos pt$ and the other as $\sin pt$, it is now evident how the solution may be adapted so as to agree with arbitrary initial values of w and $\dot{w}$.

202. Let us now examine more particularly the character of the fundamental vibrations. If $n=0$, w is a function of r only, that is to say, the motion is symmetrical with respect to the centre of the membrane. The nodes, if any, are the concentric circles, whose equation is

$$J_0(kr)=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

When n has an integral value different from zero, w is a function of θ as well as of r, and the equation of the nodal system takes the form

$$J_n(kr)\ \cos n(\theta-\alpha)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The nodal system is thus divisible into two parts, the first consisting of the concentric circles represented by

$$J_n(kr)=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and the second of the diameters

$$\theta=\alpha+(2m+1)\,\pi/2n \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

where m is an integer. These diameters are n in number, and are ranged uniformly round the centre; in other respects their position is arbitrary. The radii of the circular nodes will be investigated further on.

203. The important integral formula

$$\int_0^a J_n(kr)\,J_n(k'r)\,rdr=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where k and k' are different roots of

$$J_n(ka)=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

may be verified analytically by means of the differential equations satisfied by $J_n(kr)$, $J_n(k'r)$; but it is both simpler and more instructive to begin with the more general problem, where the boundary of the membrane is not restricted to be circular.

The variational equation of motion is

$$\delta V+\rho\iint \ddot{w}\,\delta w\,dx\,dy=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where

$$V=\tfrac{1}{2}T_1\iint\left\{\left(\frac{dw}{dx}\right)^2+\left(\frac{dw}{dy}\right)^2\right\}dx\,dy \quad\ldots\ldots\ldots\ldots (4),$$

and therefore

$$\delta V=T_1\iint\left\{\frac{dw}{dx}\frac{d\delta w}{dx}+\frac{dw}{dy}\frac{d\delta w}{dy}\right\}dx\,dy \ldots\ldots\ldots\ldots (5).$$

In these equations w refers to the actual motion, and δw to a hypothetical displacement consistent with the conditions to which the system is subjected. Let us now suppose that the system is executing one of its normal component vibrations, so that $w = u$, and

$$\ddot{u} + p^2 u = 0 \qquad \text{.........................(6)},$$

while δw is proportional to another normal function v.

Since $k = p/c$, we get from (3)

$$k^2 \iint uv\,dx\,dy = \iint \left\{\frac{du}{dx}\frac{dv}{dx} + \frac{du}{dy}\frac{dv}{dy}\right\} dx\,dy \qquad \text{........(7)}.$$

The integral on the right is symmetrical with respect to u and v and thus

$$(k'^2 - k^2) \iint u\,v\,dx\,dy = 0 \qquad \text{..................(8)},$$

where k'^2 bears the same relation to v that k^2 bears to u.

Accordingly, if the normal vibrations represented by u and v have different periods,

$$\iint u\,v\,dx\,dy = 0 \qquad \text{.......................(9)}.$$

In obtaining this result, we have made no assumption as to the boundary conditions beyond what is implied in the absence of reactions against acceleration, which, if they existed, would appear in the fundamental equation (3).

If in (8) we suppose $k' = k$, the equation is satisfied identically, and we cannot infer the value of $\iint u^2 dx\,dy$. In order to evaluate this integral we must follow a rather different course.

If u and v be functions satisfying within a certain contour the equations $\nabla^2 u + k^2 u = 0$, $\nabla^2 v + k'^2 v = 0$, we have

$$(k'^2 - k^2) \iint u\,v\,dx\,dy = \iint (v\,\nabla^2 u - u\,\nabla^2 v)\,dx\,dy$$

$$= \int \left(v\frac{du}{dn} - u\frac{dv}{dn}\right) ds \qquad \text{...........(10)},$$

by Green's theorem. Let us now suppose that v is derived from u by slightly varying k, so that

$$v = u + \frac{du}{dk}\,\delta k, \quad k' = k + \delta k\,;$$

substituting in (10), we find

$$2k \iint u^2 dx\,dy = \int \left(\frac{du}{dk}\frac{du}{dn} - u\,\frac{d^2 u}{dn\,dk}\right) ds \qquad \text{................(11)};$$

or, if u vanish on the boundary,

$$2k\iint u^2 dx\, dy = \int \frac{du}{dk}\frac{du}{dn}\, ds \quad\text{.....................} (12).$$

For the application to a circular area of radius r, we have

$$\left.\begin{aligned} u &= \cos n\theta\ J_n(kr) \\ v &= \cos n\theta\ J_n(k'r) \end{aligned}\right\}\quad\text{....................}(13),$$

and thus from (10) on substitution of polar co-ordinates and integration with respect to θ,

$$(k'^2 - k^2)\int_0^r J_n(kr)\, J_n(k'r)\, rdr$$
$$= rJ_n(k'r)\frac{d}{dr}J_n(kr) - rJ_n(kr)\frac{d}{dr}J_n(k'r) \quad\text{...........}(14).$$

Accordingly, if

$$\frac{d}{dr}J_n(k'r) \ :\ J_n(k'r) = \frac{d}{dr}J_n(kr) \ :\ J_n(kr),$$

and k and k' be different,

$$\int_0^r J_n(kr)\, J_n(k'r)\, rdr = 0 \quad\text{...............}(15),$$

an equation first proved by Fourier for the case when

$$J_n(kr) = J_n(k'r) = 0.$$

Again from (11)

$$2k\int_0^r J_n^2(kr)\, rdr = r\,\frac{dJ}{dk}\frac{dJ}{dr} - rJ\frac{d^2J}{dr\, dk}$$
$$= kr^2J'^2 - kr^2J\left(J'' + \frac{1}{kr}J'\right),$$

dashes denoting differentiation with respect to kr. Now

$$J'' + \frac{1}{kr}J' + \left(1 - \frac{n^2}{k^2r^2}\right)J = 0,$$

and thus

$$2\int_0^r J_n^2(kr)\, rdr = r^2 J_n'^2(kr) + r^2\left(1 - \frac{n^2}{k^2r^2}\right)J_n^2(kr) \quad\text{......} (16).$$

This result is general; but if, as in the application to membranes with fixed boundaries, $J_n(kr) = 0$,

then

$$2\int_0^r J_n^2(kr)\, rdr = r^2 J_n'^2(kr) \quad\text{.................} (17).$$

204. We may use the result just arrived at to simplify the expressions for T and V. From

$$w = \Sigma\Sigma \{\phi_{mn} J_n (k_{mn} r) \cos n\theta + \psi_{mn} J_n (k_{mn} r) \sin n\theta\} \text{........(1)},$$

we find

$$T = \tfrac{1}{4} \rho\pi a^2 \Sigma\Sigma J_n'^2 (k_{mn} a) \{\dot\phi_{mn}^2 + \dot\psi_{mn}^2\} \text{...........(2)},$$

$$V = \tfrac{1}{4} \rho\pi a^2 \Sigma\Sigma p_{mn}^2 J_n'^2 (k_{mn} a) \{\phi_{mn}^2 + \psi_{mn}^2\} \text{.....(3)};$$

whence is derived the normal equation of motion

$$\ddot\phi_{mn} + p_{mn}^2 \phi_{mn} = \frac{2\,\Phi_{mn}}{\rho\pi a^2 J_n'^2 (k_{mn} a)} \text{..............(4)},$$

and a similar equation for ψ_{mn}. The value of Φ_{mn} is to be found from the consideration that $\Phi_{mn} \delta\phi_{mn}$ denotes the work done by the impressed forces during a hypothetical displacement $\delta\phi_{mn}$; so that if Z be the impressed force, reckoned per unit of area,

$$\Phi_{mn} = \iint Z J_n (k_{mn} r) \cos n\theta \, r dr \, d\theta \text{........ (5)}.$$

These expressions and equations do not apply to the case $n = 0$, when ϕ and ψ are amalgamated. We then have

$$\left.\begin{aligned} T &= \tfrac{1}{2} \rho\pi a^2 J_0'^2 (k_{m0} a)\, \dot\phi_{m0}^2 \\ V &= \tfrac{1}{2} \rho\pi a^2 p_{m0}^2 J_0'^2 (k_{m0} a)\, \phi_{m0}^2 \end{aligned}\right\} \text{.... ....(6)},$$

$$\ddot\phi_{m0} + p_{m0}^2 \phi_{m0} = \frac{\Phi_{m0}}{\rho\pi a^2 J_0'^2 (k_{m0} a)} \text{................(7)}.$$

As an example, let us suppose that the initial velocities are zero, and the initial configuration that assumed under the influence of a constant pressure Z; thus

$$\Phi_{m0} = Z \,.\, 2\pi \int_0^a J_0 (k_{m0} r)\, r dr.$$

Now by the differential equation,

$$r J_0 (kr) = - \{r J_0'' (kr) + \frac{1}{k} J_0' (kr)\},$$

and thus

$$\int_0^a J_0 (kr)\, r dr = - \frac{a}{k} J_0' (ka) \text{........ ....(8)};$$

so that

$$\Phi_{m0} = - \frac{2\pi a}{k_{m0}} Z J_0' (k_{m0} a).$$

Substituting this in (7), we see that the initial value of ϕ_{m0} is

$$(\phi_{m0})_{t=0} = \frac{-2Z}{k_{m0} p_{m0}^2 \rho a J_0' (k_{m0} a)} \text{.............. (9)}$$

For values of n other than zero, Φ and the initial value of ϕ_{mn} vanish. The state of the system at time t is expressed by

$$\phi_{m0} = (\phi_{m0})_{t=0} \cdot \cos p_{m0} t \quad \ldots\ldots\ldots\ldots\ldots\ldots(10),$$

$$w = \Sigma \phi_{m0} J_0 (k_{m0} r) \ldots\ldots\ldots\ldots\ldots\ldots(11),$$

the summation extending to all the admissible values of k_{m0}.

As an example of *forced* vibrations, we may suppose that Z, still constant with respect to space, varies as a harmonic function of the time. This may be taken to represent roughly the circumstances of a small membrane set in vibration by a train of aerial waves. If $Z = \cos qt$, we find, nearly as before.

$$w = \frac{2}{\rho a} \cos qt \, \Sigma \frac{J_0 (k_{m0} r)}{k_{m0} (q^2 - p_{m0}^2) J_0' (k_{m0} a)} \ldots\ldots\ldots\ldots(12).$$

The forced vibration is of course independent of θ. It will be seen that, while none of the symmetrical normal components are missing, their relative importance may vary greatly, especially if there be a near approach in value between q and one of the series of quantities p_{m0}. If the approach be very close, the effect of dissipative forces must be included.

[Again, suppose that the force is applied locally at the centre. By (5)

$$\Phi_{m0} = Z_1 \cos qt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

if $Z_1 \cos qt$ denote the whole force at time t. From (7)

$$\phi_{m0} = \frac{Z_1 \cos qt}{\rho \pi a^2 (p_{m0}^2 - q^2) J_0'^2 (k_{m0} a)} \ldots\ldots\ldots\ldots(14),$$

and w is then given by (11). The series is convergent, unless $r = 0$.

But this problem would be more naturally attacked by including in the solutions of (4) § 200 the second Bessel's function § 341. In this method $k = q/c$; and the ratio of constants by which the two functions of r are multiplied is determined by the boundary condition. When q coincides with one of the values of p, the second function disappears from the solution.]

205. The pitches of the various simple tones and the radii of the nodal circles depend on the roots of the equation

$$J_n (ka) = J_n (z) = 0.$$

If these (exclusive of zero) taken in order of magnitude be called $z_n^{(1)}, z_n^{(2)}, z_n^{(3)} \ldots\ldots z_n^{(s)} \ldots\ldots$, then the admissible values of p are to be found by multiplying the quantities $z_n^{(s)}$ by c/a. The particular solution may then be written

$$w = J_n\left(z_n^{(s)}\frac{r}{a}\right)\{A_n^{(s)}\cos n\theta + B_n^{(s)}\sin n\theta\}\cos\left\{\frac{c}{a}z_n^{(s)}t - \epsilon_n^{(s)}\right\}\ldots(1).$$

The lowest tone of the group n corresponds to $z_n^{(1)}$; and since in this case $J_n(z_n^{(1)}r/a)$ does not vanish for any value of r less than a, there is no interior nodal circle. If we put $s=2$, J_n will vanish, when

$$z_n^{(2)}\frac{r}{a} = z_n^{(1)},$$

that is, when

$$r = a\frac{z_n^{(1)}}{z_n^{(2)}},$$

which is the radius of the one interior nodal circle. Similarly if we take the root $z_n^{(s)}$, we obtain a vibration with $s-1$ nodal circles (exclusive of the boundary) whose radii are

$$a\frac{z_n^{(1)}}{z_n^{(s)}},\quad a\frac{z_n^{(2)}}{z_n^{(s)}}, \ldots\ldots a\frac{z_n^{(s-1)}}{z_n^{(s)}}.$$

All the roots of the equation $J_n(ka)=0$ are *real*. For, if possible, let $ka = \lambda + i\mu$ be a root; then $k'a = \lambda - i\mu$ is also a root, and thus by (14) § 203,

$$4i\lambda\mu\int_0^a J_n(kr)\,J_n(k'r)\,rdr = 0.$$

Now $J_n(kr)$, $J_n(k'r)$ are conjugate complex quantities, whose product is necessarily positive; so that the above equation requires that either λ or μ vanish. That λ cannot vanish appears from the consideration that if ka were a pure imaginary, each term of the ascending series for J_n would be positive, and therefore the sum of the series incapable of vanishing. We conclude that $\mu = 0$, or that k is real[1]. The same result might be arrived at from the consideration that only circular functions of the time can enter into the analytical expression for a normal component vibration.

The equation $J_n(z)=0$ has no equal roots (except zero). From equations (7) and (8) § 200 we get

$$J_n' = \frac{n}{z}J_n - J_{n+1},$$

[1] Riemann, *Partielle Differentialgleichungen*, Braunschweig, 1869, p. 260.

whence we see that if J_n, J_n' vanished for the same value of z, J_{n+1} would also vanish for that value. But in virtue of (8) § 200 this would require that *all* the functions J_n vanish for the value of z in question[1].

206. The actual values of z_n may be found by interpolation from Hansen's tables so far as these extend; or formulæ may be calculated from the descending series by the method of successive approximation, expressing the roots directly. For the important case of the symmetrical vibrations ($n = 0$), the values of z_0 may be found from the following, given by Stokes[2]:

$$\frac{z_0^{(s)}}{\pi} = s - \cdot 25 + \frac{\cdot 050661}{4s-1} - \frac{\cdot 053041}{(4s-1)^3} + \frac{\cdot 262051}{(4s-1)^5} \ldots\ldots (1).$$

For $n = 1$, the formula is

$$\frac{z_1^{(s)}}{\pi} = s + \cdot 25 - \frac{\cdot 151982}{4s+1} + \frac{\cdot 015399}{(4s+1)^3} - \frac{\cdot 245270}{(4s+1)^5} \ldots\ldots (2).$$

The latter series is convergent enough, even for the first root, corresponding to $s = 1$. The series (1) will suffice for values of s greater than unity; but the first root must be calculated independently. The accompanying table (A) is taken from Stokes' paper, with a slight difference of notation.

It will be seen either from the formulæ, or the table, that the difference of successive roots of high order is approximately π. This is true for all values of n, as is evident from the descending series (10) § 200.

[The general formula, analogous to (1) and (2), for the roots of $J_n(z)$ has been investigated by Prof. McMahon. If $m = 4n^2$, and

$$a = \tfrac{1}{4}\pi(2n - 1 + 4s) \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

we have

$$z_n^{(s)} = a - \frac{m-1}{8a} - \frac{4(m-1)(7m-31)}{3(8a)^3}$$
$$- \frac{32(m-1)(83m^2 - 982m + 3779)}{15(8a)^5} + \ldots\ldots\ldots (4).$$

[1] Bourget, "Mémoire sur le mouvement vibratoire des membranes circulaires," *Ann. de l'école normale*, t. III., 1866. In one passage M. Bourget implies that he has proved that no two Bessel's functions of integral order can have the same root, but I cannot find that he has done so. The theorem, however, is probably true; in the case of functions, whose orders differ by 1 or 2, it may be easily proved from the formulæ of § 200.

[2] *Camb. Phil. Trans.* Vol. IX. "On the numerical calculation of a class of definite integrals and infinite series." [In accordance with the calculation of Prof. McMahon the numerator of the last term in (2) has been altered from ·245835 to ·245270.]

This formula may be applied not only to integral values of n as in (1) and (2), but also when n is fractional. The cases of $n=\frac{1}{2}$, and $n=\frac{3}{2}$ are considered in § 207.]

M. Bourget has given in his memoir very elaborate tables of the frequencies of the different simple tones and of the radii of the nodal circles. Table B includes the values of z, which satisfy $J_n(z)$, for $n=0, 1, \ldots 5$, $s=1, 2, \ldots 9$.

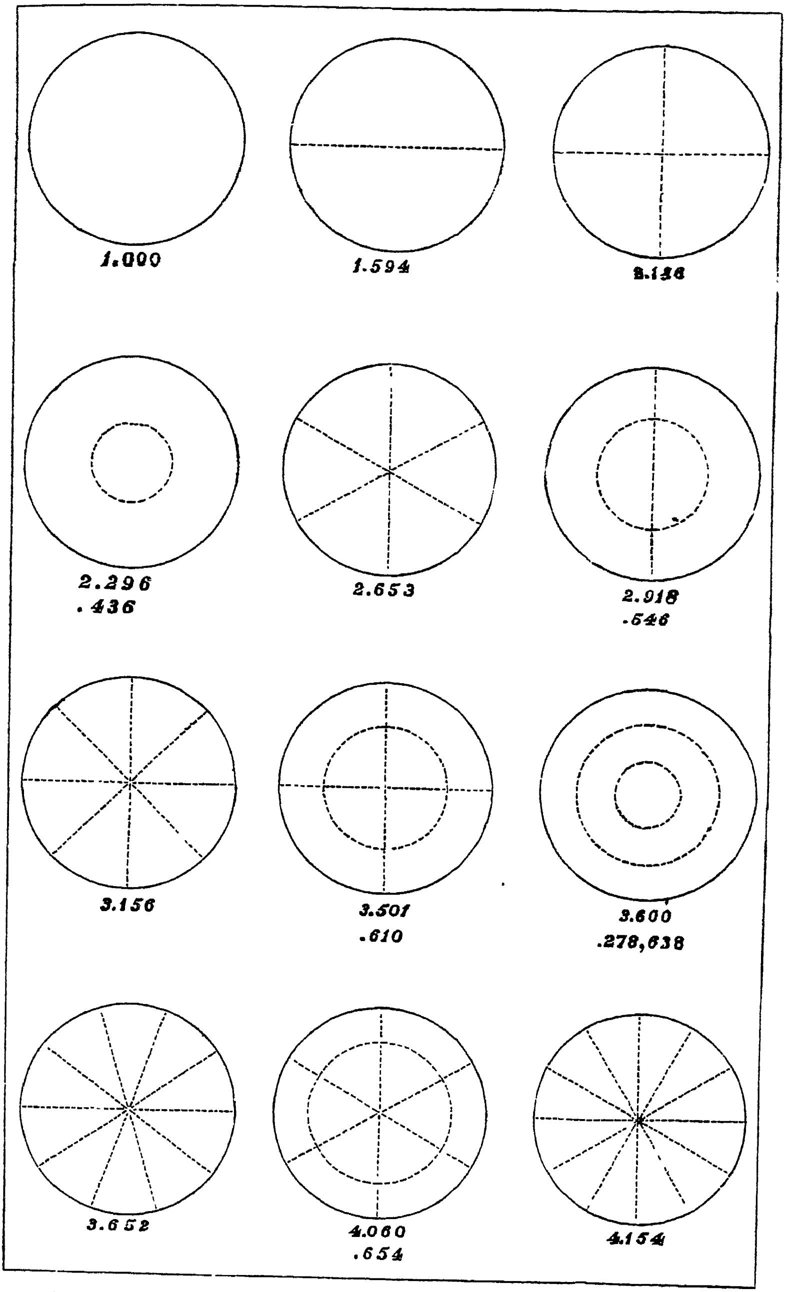
TABLE A.

s	$\frac{z}{\pi}$ for $J_0(z)=0$.	Diff.	$\frac{z}{\pi}$ for $J_1(z)=0$.	Diff.
1	·7655	·9916	1·2197	1·0133
2	1·7571	·9975	2·2330	1·0053
3	2·7546	·9988	3·2383	1·0028
4	3·7534	·9993	4·2411	1·0017
5	4·7527	·9995	5·2428	1·0011
6	5·7522	·9997	6·2439	1·0009
7	6·7519	9997	7·2448	1·0006
8	7·7516	9998	8·2454	1·0005
9	8·7514	·9999	9·2459	1·0004
10	9·7513	·9999	10·2463	1·0003
11	10·7512	·9999	11·2466	1·0003
12	11·7511		12·2469	

When n is considerable the calculation of the earlier roots becomes troublesome. For very high values of n, $z_n^{(1)}/n$ approximates to a ratio of equality, as may be seen from the consideration that the pitch of the gravest tone of a very acute sector must tend to coincide with that of a long parallel strip, whose width is equal to the greatest width of the sector.

TABLE B.

s	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
1	2·404	3·832	5·135	6·379	7·586	8·780
2	5·520	7·016	8·417	9·760	11·064	12·339
3	8·654	10·173	11·620	13·017	14·373	15·700
4	11·792	13·323	14·796	16·224	17·616	18·982
5	14·931	16·470	17·960	19·410	20·827	22·220
6	18·071	19·616	21·117	22·583	24·018	25·431
7	21·212	22·760	24·270	25·749	27·200	28·628
8	24·353	25·903	27·421	28·909	30·371	31·813
9	27·494	29·047	30·571	32·050	33·512	34·983



The figures represent the more important normal modes of vibration, and the numbers affixed give the frequency referred to the gravest as unity, together with the radii of the circular nodes expressed as fractions of the radius of the membrane. In the case of six nodal diameters the frequency stated is the result of a rough calculation by myself.

The tones corresponding to the various fundamental modes of the circular membrane do not belong to a harmonic scale, but there are one or two approximately harmonic relations which may be worth notice. Thus

$$\tfrac{4}{3} \times 1{\cdot}594 = 2{\cdot}125 = 2{\cdot}136 \text{ nearly},$$

$$\tfrac{5}{3} \times 1{\cdot}594 = 2{\cdot}657 = 2{\cdot}653 \text{ nearly},$$

$$2 \times 1{\cdot}594 = 3{\cdot}188 = 3{\cdot}156 \text{ nearly};$$

so that the four gravest modes with nodal diameters only would give a consonant chord.

The area of the membrane is divided into segments by the nodal system in such a manner that the sign of the vibration changes whenever a node is crossed. In those modes of vibration which have nodal diameters there is evidently no displacement of the centre of inertia of the membrane. In the case of symmetrical vibrations the displacement of the centre of inertia is proportional to

$$\int_0^a J_0(kr)\, rdr = -\int_0^a \left\{ J_0''(kr) + \frac{1}{kr} J_0'(kr) \right\} rdr = -\frac{a}{k} J_0'(ka),$$

an expression which does not vanish for any of the admissible values of k, since $J_0'(z)$ and $J_0(z)$ cannot vanish simultaneously. In all the symmetrical modes there is therefore a displacement of the centre of inertia of the membrane.

207. Hitherto we have supposed the circular area of the membrane to be complete, and the circumference only to be fixed; but it is evident that our theory virtually includes the solution of other problems, for example—some cases of a membrane bounded by two concentric circles. The *complete* theory for a membrane in the form of a ring requires the second Bessel's function.

The problem of the membrane in the form of a semi-circle may be regarded as already solved, since any mode of vibration of which the semi-circle is capable must be applicable to the complete circle also. In order to see this, it is only necessary to attribute to any point in the complementary semi-circle the opposite motion to that which obtains at its optical image in the bounding diameter. This line will then require no constraint to keep it nodal. Similar considerations apply to any sector whose angle is an aliquot part of two right angles.

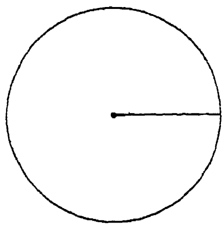
When the opening of the sector is arbitrary, the problem may be solved in terms of Bessel's functions of fractional order. If the fixed radii are $\theta = 0$, $\theta = \beta$, the particular solution is

$$w = P\, J_{\nu\pi/\beta}(kr)\, \sin\frac{\nu\pi\theta}{\beta} \cos(pt - \epsilon) \quad \ldots\ldots\ldots (1),$$

where ν is an integer. We see that if β be an aliquot part of π, $\nu\pi/\beta$ is integral, and the solution is included among those already used for the complete circle.

An interesting case is when $\beta = 2\pi$, which corresponds to the problem of a complete circle, of which the radius $\theta = 0$ is constrained to be nodal.

Fig. 38.



We have

$$w = P\, J_{\frac{1}{2}\nu}(kr)\, \sin \tfrac{1}{2}\nu\theta \cos(pt - \epsilon).$$

When ν is even, this gives, as might be expected, modes of vibration possible without the constraint; but, when ν is odd, new modes make their appearance. In fact, in the latter case the descending series for J terminates, so that the solution is expressible in finite terms. Thus, when $\nu = 1$,

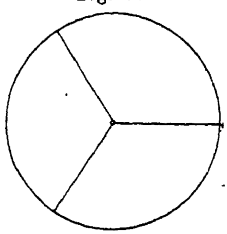
$$w = P\frac{\sin kr}{\sqrt{(kr)}} \sin \tfrac{1}{2}\theta \cos(pt - \epsilon) \quad \ldots\ldots\ldots\ldots (2).$$

The values of k are given by

$$\sin ka = 0, \quad \text{or } ka = s\pi.$$

Thus the circular nodes divide the fixed radius into equal parts, and the series of tones form a harmonic scale. In the case of the gravest mode, the whole of the membrane is at any moment deflected on the same side of its equilibrium position. It is remarkable that the application of the constraint to the radius $\theta = 0$ makes the problem easier than before.

Fig. 39.



If we take $\nu = 3$, the solution is

$$w = P\frac{1}{\sqrt{(kr)}}\left(\frac{\sin kr}{kr} - \cos kr\right)\sin\tfrac{3}{2}\theta\,\cos(pt - \epsilon)\ldots\ldots(3).$$

In this case the nodal radii are Fig. (39)

$$\theta = \tfrac{2}{3}\pi, \quad \theta = \tfrac{4}{3}\pi;$$

and the possible tones are given by the equation

$$\tan ka = ka \ldots\ldots\ldots\ldots(4).$$

To calculate the roots of $\tan x = x$ we may assume

$$x = (s + \tfrac{1}{2})\pi - y = X - y,$$

where y is a positive quantity, which is small when x is large.

Substituting this, we find $\cot y = X - y$,
whence

$$y = \frac{1}{X}\left(1 + \frac{y}{X} + \frac{y^2}{X^2} + \ldots\right) - \frac{y^3}{3} - \frac{2y^5}{15} - \frac{17y^7}{315} - \ldots$$

This equation is to be solved by successive approximation. It will readily be found that

$$y = X^{-1} + \frac{2}{3}X^{-3} + \frac{13}{15}X^{-5} + \frac{146}{105}X^{-7} + \ldots,$$

so that the roots of $\tan x = x$ are given by

$$x = X - X^{-1} - \frac{2}{3}X^{-3} - \frac{13}{15}X^{-5} - \frac{146}{105}X^{-7} - \ldots\ldots(5),$$

where $$X = (s + \tfrac{1}{2})\pi.$$

In the first quadrant there is no root after zero since $\tan x > x$, and in the second quadrant there is none because the signs of x and $\tan x$ are opposite. The first root after zero is thus in the third quadrant, corresponding to $s = 1$. Even in this case the series converges sufficiently to give the value of the root with considerable accuracy, while for higher values of s it is all that could be desired. The actual values of x/π are 1·4303, 2·4590, 3·4709, 4·4747, 5·4818, 6·4844, &c.

208. The effect on the periods of a slight inequality in the density of the circular membrane may be investigated by the general method § 90, of which several examples have already been given. It will be sufficient here to consider the case of a small load M attached to the membrane at a point whose radius vector is r'.

We will take first the symmetrical types ($n=0$), which may still be supposed to apply notwithstanding the presence of M. The kinetic energy T is (6) § 204 altered from

$\frac{1}{2}\rho\pi a^2 J_0'^2(k_{m0}a)\,\dot\phi_{m0}^2$ to $\frac{1}{2}\rho\pi a^2 J_0'^2(k_{m0}a)\,\dot\phi_{m0}^2+\frac{1}{2}M\dot\phi_{m0}^2 J_0^2(k_{m0}r')$,

and therefore

$$p_{m0}^2 : P_{m0}^2 = 1 - \frac{M}{\rho\pi a^2}\frac{J_0^2(k_{m0}r')}{J_0'^2(k_{m0}a)}\quad\text{.................(1)},$$

where P_{m0}^2 denotes the value of p_{m0}^2, when there is no load.

The unsymmetrical normal types are not fully determinate for the unloaded membrane; but for the present purpose they must be taken so as to make the resulting periods a maximum or minimum, that is to say, so that the effect of the load is the greatest and least possible. Now, since a load can never raise the pitch, it is clear that the influence of the load is the least possible, viz. zero, when the type is such that a nodal diameter (it is indifferent which) passes through the point at which the load is attached. The unloaded membrane must be supposed to have two coincident periods, of which *one* is unaltered by the addition of the load. The other type is to be chosen, so that the alteration of period is as great as possible, which will evidently be the case when the radius vector r' bisects the angle between two adjacent nodal diameters. Thus, if r' correspond to $\theta=0$, we are to take

$$w=\phi_{mn}J_n(k_{mn}r)\cos n\theta;$$

so that (2) § 204

$$T=\tfrac{1}{4}\rho\pi a^2\dot\phi_{mn}^2 J_n'^2(k_{mn}a)+\tfrac{1}{2}M\dot\phi_{mn}^2 J_n^2(k_{mn}r').$$

The *altered* p_{mn}^2 is therefore given by

$$p_{mn}^2 : P_{mn}^2 = 1 - \frac{2M}{\rho\pi a^2}\frac{J_n^2(k_{mn}r')}{J_n'^2(k_{mn}a)}\quad\text{.................(2)}.$$

Of course, if r' be such that the load lies on one of the nodal circles, neither period is affected.

For example, let M be at the centre of the membrane. $J_n(0)$ vanishes, except when $n=0$; and $J_0(0)=1$. It is only the symmetrical vibrations whose pitch is influenced by a central load, and for them by (1)

$$p_{m0}^2 : P_{m0}^2 = 1 - \frac{M}{J_0'^2(k_{m0}a)\,\rho\pi a^2}\quad\text{.................(3)}.$$

By (6) § 200 $J_0'(z) = -J_1(z)$,

so that the application of the formula requires only a knowledge of the values of $J_1(z)$, when $J_0(z)$ vanishes, § 200. For the gravest mode the value of $J_0'(k_{m0}a)$ is ·51903[1]. When $k_{m0}a$ is considerable,

$$J_1^2(k_{m0}a) = 2 \div \pi k_{m0}a$$

approximately; so that for the higher components the influence of the load in altering the pitch increases.

The influence of a small irregularity in disturbing the nodal system may be calculated from the formulæ of § 90. The most obvious effect is the breaking up of nodal diameters into curves of hyperbolic form due to the introduction of subsidiary symmetrical vibrations. In many cases the disturbance is favoured by close agreement between some of the natural periods.

209. We will next investigate how the natural vibrations of a uniform membrane are affected by a slight departure from the exact circular form.

Whatever may be the nature of the boundary, w satisfies the equation

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \frac{1}{r^2}\frac{d^2w}{d\theta^2} + k^2w = 0 \quad\text{.................(1)},$$

where k is a constant to be determined. By Fourier's theorem w may be expanded in the series

$$w = w_0 + w_1\cos(\theta + \alpha_1) + w_2\cos 2(\theta + \alpha_2) + \ldots\ldots$$
$$+ w_n \cos n(\theta + \alpha_n) + \ldots\ldots,$$

where w_0, w_1, &c. are functions of r only. Substituting in (1), we see that w_n must satisfy

$$\frac{d^2w_n}{dr^2} + \frac{1}{r}\frac{dw_n}{dr} + \left(k^2 - \frac{n^2}{r^2}\right)w_n = 0,$$

of which the solution is

$$w_n \propto J_n(kr);$$

for, as in § 200, the other function of r cannot appear.

The general expression for w may thus be written

$$w = A_0 J_0(kr) + J_1(kr)(A_1\cos\theta + B_1\sin\theta)$$
$$+ \ldots + J_n(kr)(A_n\cos n\theta + B_n \sin n\theta) + \quad\text{........(2)}.$$

For all points on the boundary w is to vanish.

[1] The succeeding values are approximately ·341, ·271, ·232, ·206, ·187, &c.

In the case of a nearly circular membrane the radius vector is nearly constant. We may take $r = a + \delta r$, δr being a small function of θ. Hence the boundary condition is

$$0 = A_0 [J_0 (ka) + k\delta r J_0' (ka)] + \ldots\ldots$$
$$+ [J_n (ka) + k\delta r J_n' (ka)] [A_n \cos n\theta + B_n \sin n\theta]$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

which is to hold good for all values of θ.

Let us consider first those modes of vibration which are nearly symmetrical, for which therefore approximately

$$w = A_0 J_0 (kr).$$

All the remaining coefficients are small relatively to A_0, since the type of vibration can only differ a little from what it would be, were the boundary an exact circle. Hence if the squares of the small quantities be omitted, (3) becomes

$$A_0 [J_0 (ka) + k\delta r J_0' (ka)] + J_1 (ka) [A_1 \cos\theta + B_1 \sin\theta]$$
$$+ \ldots + J_n (ka) [A_n \cos n\theta + B_n \sin n\theta] + \ldots = 0 \ldots\ldots (4).$$

If we integrate this equation with respect to θ between the limits 0 and 2π, we obtain

$$2\pi J_0 (ka) + J_0' (ka) \int_0^{2\pi} k\delta r \, d\theta = 0,$$

or

$$J_0 \left\{ka + k \int_0^{2\pi} \delta r \frac{d\theta}{2\pi}\right\} = 0 \ldots\ldots\ldots\ldots\ldots (5),$$

which shews that the pitch of the vibration is approximately the same as if the radius vector had uniformly its *mean value*.

This result allows us to form a rough estimate of the pitch of any membrane whose boundary is not extravagantly elongated. If σ denote the area, so that $\rho\sigma$ is the mass of the whole membrane, the frequency of the gravest tone is approximately

$$(2\pi)^{-1} \times 2{\cdot}404 \times \sqrt{\frac{\pi T_1}{\sigma\rho}} \ldots\ldots\ldots\ldots\ldots (6)[1].$$

In order to investigate the altered type of vibration, we may

[1] [A numerical error is here corrected.]

multiply (4) by $\cos n\theta$, or $\sin n\theta$, and then integrate as before. Thus

$$\left.\begin{aligned} A_0 J_0'(ka)\int_0^{2\pi} k\,\delta r \cos n\theta\, d\theta + \pi A_n J_n(ka) = 0 \\ A_0 J_0'(ka)\int_0^{2\pi} k\,\delta r \sin n\theta\, d\theta + \pi B_n J_n(ka) = 0 \end{aligned}\right\} \quad \text{.........(7)},$$

which determine the ratios $A_n : A_0$ and $B_n : A_0$.

If $$\delta r = \delta r_0 + \delta r_1 + \ldots + \delta r_n + \ldots$$

be Fourier's expansion, the final expression for w may be written,

$$w : A_0 = J_0(kr) - k J_0'(ka)\left\{\frac{J_1(kr)\,\delta r_1}{J_1(ka)} + \ldots + \frac{J_n(kr)\,\delta r_n}{J_n(ka)} + \ldots\right\} \quad \text{......(8)}.$$

When the vibration is not approximately symmetrical, the question becomes more complicated. The normal modes for the truly circular membrane are to some extent indeterminate, but the irregularity in the boundary will, in general, remove the indeterminateness. The position of the nodal diameters must be taken, so that the resulting periods may have maximum or minimum values. Let us, however, suppose that the approximate type is

$$w = A_\nu J_\nu(kr) \cos \nu\theta \quad \text{....................(9)},$$

and afterwards investigate how the initial line must be taken in order that this form may hold good.

All the remaining coefficients being treated as small in comparison with A_ν, we get from (4)

$$\begin{aligned} A_0 J_0(ka) + \ldots + A_\nu[J_\nu(ka) + k\,\delta r J_\nu'(ka)] \cos \nu\theta \\ + B_\nu J_\nu(ka) \sin \nu\theta + \ldots\ldots \\ + J_n(ka)\,[A_n \cos n\theta + B_n \cos n\theta] + \ldots = 0 \quad \text{...... (10)}. \end{aligned}$$

Multiplying by $\cos \nu\theta$ and integrating,

$$\pi J_\nu(ka) + k J_\nu'(ka)\int_0^{2\pi} \delta r \cos^2 \nu\theta\, d\theta = 0,$$

or

$$J_\nu\left[ka + k\int_0^{2\pi} \delta r \cos^2 \nu\theta \frac{d\theta}{\pi}\right] = 0,$$

which shews that the effective radius of the membrane is

$$a + \int_0^{2\pi} \delta r \cos^2 \nu\theta \frac{d\theta}{\pi} \quad \text{....................(11)}.$$

The ratios of A_n and B_n to A_ν may be found as before by integrating equation (10) after multiplication by $\cos n\theta$, $\sin n\theta$.

But the point of greatest interest is the pitch. The initial line is to be so taken as to make the expression (11) a maximum or minimum. If we refer to a line fixed in space by putting $\theta - \alpha$ instead of θ, we have to consider the dependence on α of the quantity

$$\int_0^{2\pi} \delta r \cos^2 \nu (\theta - \alpha)\, d\theta,$$

which may also be written

$$\cos^2 \nu\alpha \int_0^{2\pi} \delta r \cos^2 \nu\theta\, d\theta + 2 \cos \nu\alpha \sin \nu\alpha \int_0^{2\pi} \delta r \cos \nu\theta \sin \nu\theta\, d\theta$$
$$+ \sin^2 \nu\alpha \int_0^{2\pi} \delta r \sin^2 \nu\theta\, d\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

and is of the form

$$A \cos^2 \nu\alpha + 2B \cos \nu\alpha \sin \nu\alpha + C \sin^2 \nu\alpha,$$

A, B, C being independent of α. There are accordingly two admissible positions for the nodal diameters, one of which makes the period a maximum, and the other a minimum. The diameters of one set bisect the angles between the diameters of the other set.

There are, however, cases where the normal modes remain indeterminate, which happens when the expression (12) is independent of α. This is the case when δr is constant, or when δr is proportional to $\cos \nu\theta$. For example, if δr were proportional to $\cos 2\theta$, or in other words the boundary were slightly elliptical, the nodal system corresponding to $n = 2$ (that consisting of a pair of perpendicular diameters) would be arbitrary in position, at least to this order of approximation. But the single diameter, corresponding to $n = 1$, must coincide with one of the principal axes of the ellipse, and the periods will be different for the two axes

210. We have seen that the gravest tone of a membrane, whose boundary is approximately circular, is nearly the same as that of a mechanically similar membrane in the form of a circle of the same mean radius or area. If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle. In the case of approximate circularity an analytical demonstration may be given, of which the following is an outline.

The general value of w being

$$w = A_0 J_0(kr) + \ldots + J_n(kr)(A_n \cos n\theta + B \sin n\theta) + \ldots\ldots(1),$$

in which for the present purpose the coefficients $A_1, B_1, \ldots$ are small relatively to A_0, we find from the condition that w vanishes when $r = a + \delta r$,

$$A_0 J_0(ka) + kA_0 J_0'(ka)\,\delta r + \tfrac{1}{2}k^2 A_0 J_0''(ka)\,.\,(\delta r)^2 + \ldots\ldots$$
$$+ \Sigma\left[\{J_n(ka) + kJ_n'(ka)\,\delta r + \ldots\}\{A_n \cos n\theta + B_n \sin n\theta\}\right] = 0 \ldots(2).$$

Hence, if

$$\delta r = \alpha_1 \cos\theta + \beta_1 \sin\theta + \ldots + \alpha_n \cos n\theta + \beta_n \sin n\theta + \ldots\ldots(3),$$

we obtain on integration with respect to θ from 0 to 2π,

$$2A_0 J_0 + \tfrac{1}{2}k^2 A_0 J_0'' \sum_{n=1}^{n=\infty} (\alpha_n^2 + \beta_n^2)$$
$$+ k \sum_{n=1}^{n=\infty} \left[(\alpha_n A_n + \beta_n B_n) J_n'\right] = 0 \ldots\ldots\ldots\ldots (4),$$

from which we see, as before, that if the squares of the small quantities be neglected, $J_0(ka) = 0$, or that to this order of approximation the mean radius is also the effective radius. In order to obtain a closer approximation we first determine $A_n : A_0$ and $B_n : A_0$ by multiplying (2) by $\cos n\theta$, $\sin n\theta$, and then integrating between the limits 0 and 2π. Thus

$$A_n J_n = -k\alpha_n A_0 J_0', \quad B_n J_n = -k\beta_n A_0 J_0' \ldots\ldots\ldots(5).$$

Substituting these values in (4), we get

$$J_0(ka) = \tfrac{1}{2}k^2 \sum_{n=1}^{n=\infty}\left[(\alpha_n^2 + \beta_n^2)\left\{\frac{J_n' J_0'}{J_n} - \tfrac{1}{2}J_0''\right\}\right] \ldots\ldots(6).$$

Since J_0 satisfies the fundamental equation

$$J_0'' + \frac{1}{ka}J_0' + J_0 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

and in the present case $J_0 = 0$ approximately, we may replace J_0'' by $-\frac{1}{ka}J_0'$. Equation (6) then becomes

$$J_0(ka) = \tfrac{1}{2}k^2 J_0' \sum_{n=1}^{n=\infty}\left[(\alpha_n^2 + \beta_n^2)\left\{\frac{J_n'}{J_n} + \frac{1}{2ka}\right\}\right] \ldots\ldots(8).$$

Let us now suppose that $a + da$ is the equivalent radius of the membrane, so that

$$J_0 [k(a+da)] = J_0(ka) + J_0'(ka)\, k\, da = 0.$$

Then by (8) we find

$$da = -\tfrac{1}{2} k \sum_{n=1}^{n=\infty} \left[(\alpha_n^2 + \beta_n^2) \left\{ \frac{J_n'}{J_n} + \frac{1}{2ka} \right\} \right] \text{.........} (9).$$

Again, if $a + da'$ be the radius of the truly circular membrane of equal area,

$$da' = \frac{1}{4a} \sum_{n=1}^{n=\infty} (\alpha_n^2 + \beta_n^2) \text{..................} (10);$$

so that

$$da' - da = \frac{1}{2a} \sum_{n=1}^{n=\infty} \left[(\alpha_n^2 + \beta_n^2) \left\{ 1 + ka \frac{J_n'(ka)}{J_n(ka)} \right\} \right] \text{......} (11).$$

The question is now as to the sign of the right-hand member. If $n = 1$, and z be written for ka,

$$1 + z \frac{J_n'(z)}{J_n(z)}$$

vanishes approximately by (7), since in general $J_1 = -J_0'$, and in the present case $J_0(z) = 0$ nearly. Thus $da' - da = 0$, as should evidently be the case, since the term in question represents merely a displacement of the circle without an alteration in the form of the boundary. When $n = 2$, (8) § 200,

$$J_2 = \frac{2}{z} J_1 - J_0,$$

from which and (7) we find that, when $J_0 = 0$,

$$\frac{J_2'}{J_2} = \frac{z^2 - 4}{2z} \text{........................} (12),$$

whence

$$da' - da = \frac{1}{2a} (\alpha_2^2 + \beta_2^2) \left(\frac{z^2}{2} - 1 \right) \text{...........} (13),$$

which is positive, since $z = 2{\cdot}404$.

We have still to prove that

$$1 + z \frac{J_n'(z)}{J_n(z)}$$

is positive for integral values of n greater than 2, when $z = 2{\cdot}404$.

For this purpose we may avail ourselves of a theorem given in Riemann's *Partielle Differentialgleichungen*, to the effect that neither J_n nor J_n' has a root (other than zero) less than n. The differential equation for J_n may be put into the form

$$\frac{d^2 J_n(z)}{d(\log z)^2} + (z^2 - n^2) J_n(z) = 0;$$

while initially J_n and J_n' (as well as $dJ_n/d\log z$) are positive. Accordingly $dJ_n/d\log z$ begins by increasing and does not cease to do so before $z = n$, from which it is clear that within the range $z = 0$ to $z = n$, neither J_n nor J_n' can vanish. And since J_n and J_n' are both positive until $z = n$, it follows that, when n is an integer greater than 2·404, $da' - da$ is positive. We conclude that, unless α_2, β_2, $\alpha_3, \ldots$ all vanish, da' is greater than da, which shews that in the case of any membrane of approximately circular outline, the circle of equal area exceeds the circle of equal pitch.

We have seen that a good estimate of the pitch of an approximately circular membrane may be obtained from its area alone, but by means of equation (9) a still closer approximation may be effected. We will apply this method to the case of an ellipse, whose semi-axis major is R and eccentricity e.

The polar equation of the boundary is

$$r = R\left\{1 - \tfrac{1}{4}e^2 - \tfrac{7}{64}e^4 + \ldots\ldots + \tfrac{1}{4}e^2\cos 2\theta + \ldots\ldots\right\} \ldots\ldots (14);$$

so that in the notation of this section

$$a = R\left(1 - \tfrac{1}{4}e^2 - \tfrac{7}{64}e^4\right), \quad \alpha_2 = \tfrac{1}{4}e^2 R.$$

Accordingly by (9)

$$da = -\frac{e^4 R}{32} \cdot kR \cdot \left\{\frac{J_2'(z)}{J_2(z)} + \frac{1}{2z}\right\},$$

or by (12), since $kR = z = 2·404$,

$$da = -\frac{2·779}{64} e^4 R.$$

Thus the radius of the circle of equal pitch is

$$a + da = R\left\{1 - \frac{1}{4}e^2 - \frac{9·779\, e^4}{64}\right\} \ldots\ldots\ldots\ldots (15)$$

in which the term containing e^4 should be correct.

The result may also be expressed in terms of e and the area σ. We have

$$R=\sqrt{\frac{\sigma}{\pi}}\left(1+\frac{1}{4}e^2+\frac{5}{32}e^4\right),$$

and thus

$$a+da=\sqrt{\frac{\sigma}{\pi}}\left(1-\frac{3\cdot779}{64}e^4\right)\text{..............} (16),$$

from which we see how small is the influence of a moderate eccentricity, when the area is given.

211. When the fixed boundary of a membrane is neither straight nor circular, the problem of determining its vibrations presents difficulties which in general could not be overcome without the introduction of functions not hitherto discussed or tabulated. A partial exception must be made in favour of an elliptic boundary; but for the purposes of this treatise the importance of the problem is scarcely sufficient to warrant the introduction of complicated analysis. The reader is therefore referred to the original investigation of M. Mathieu[1].

[The method depends upon the use of conjugate functions. If

$$x+iy=e\cos(\xi+i\eta)\text{..................}(1),$$

then the curves $\eta=$ const. are confocal ellipses, and $\xi=$ const. are confocal hyperbolas. In terms of ξ, η the fundamental equation $(\nabla^2+k^2)u=0$ becomes

$$\frac{d^2u}{d\xi^2}+\frac{d^2u}{d\eta^2}+k'^2(\cosh^2\eta-\cos^2\xi)\,u=0\text{ ....} (2),$$

where $k'=ke$.

The solution of (2) may be found in the form

$$u=\Xi(\xi)\,.\,\mathrm{H}(\eta)\text{......................}(3),$$

in which Ξ is a function of ξ only, and H a function of η only, provided

$$\frac{d^2\Xi}{d\xi^2}-(k'^2\cos^2\xi-a)\,\Xi=0\text{ ....} (4),$$

$$\frac{d^2\mathrm{H}}{d\eta^2}+(k'^2\cosh^2\eta-a)\,\mathrm{H}=0\text{ ....} (5),$$

a being an arbitrary constant[2].

[1] Liouville, XIII., 1868; *Cours de physique mathématique*, 1873, p. 122.
[2] Pockels, *Über die partielle Differentialgleichung* $\Delta u+k^2u=0$, p. 114.

Michell[1] has shewn that the elliptic transformation (1) is the only one which yields an equation capable of satisfaction in the form (3).]

Soluble cases may be invented by means of the general solution

$$w = A_0 J_0 (kr) + \ldots + (A_n \cos n\theta + B_n \sin n\theta) J_n (kr) + \ldots\ldots$$

For example we might take

$$w = J_0 (kr) - \lambda J_1 (kr) \cos\theta,$$

and attaching different values to λ, trace the various forms of boundary to which the solution will then apply.

Useful information may sometimes be obtained from the theorem of § 88, which allows us to prove that any contraction of the fixed boundary of a vibrating membrane must cause an elevation of pitch, because the new state of things may be conceived to differ from the old merely by the introduction of an additional constraint. Springs, without inertia, are supposed to urge the line of the proposed boundary towards its equilibrium position, and gradually to become stiffer. At each step the vibrations become more rapid, until they approach a limit, corresponding to infinite stiffness of the springs and absolute fixity of their points of application. It is not necessary that the part cut off should have the same density as the rest, or even any density at all.

For instance, the pitch of a regular polygon is intermediate between those of the inscribed and circumscribed circles. Closer limits would however be obtained by substituting for the circumscribed circle that of equal area according to the result of § 210. In the case of the hexagon, the ratio of the radius of the circle of equal area to that of the circle inscribed is 1·050, so that the mean of the two limits cannot differ from the truth by so much as $2\frac{1}{2}$ per cent. In the same way we might conclude that the sector of a circle of 60° is a graver form than the equilateral triangle obtained by substituting the chord for the arc of the circle.

The following table giving the relative frequency in certain calculable cases for the gravest tone of membranes under similar mechanical conditions and of *equal area* (σ), shews the effect of a greater or less departure from the circular form.

[1] *Messenger of Mathematics*, vol. XIX. p. 86, 1890.

Circle..	$2{\cdot}404 \, . \sqrt{\pi} = 4{\cdot}261.$
Square..	$\sqrt{2} \, . \, \pi = 4{\cdot}443.$
Quadrant of a circle........................	$\dfrac{5{\cdot}135}{2} \, . \sqrt{\pi} = 4{\cdot}551.$
Sector of a circle 60°........................	$6{\cdot}379 \sqrt{\dfrac{\pi}{6}} = 4{\cdot}616.$
Rectangle 3 × 2.............................	$\sqrt{\dfrac{13}{6}} \, . \, \pi = 4{\cdot}624.$
Equilateral triangle........................	$2\pi \, . \sqrt{\tan 30^\circ} = 4{\cdot}774.$
Semicircle.....................................	$3{\cdot}832 \sqrt{\dfrac{\pi}{2}} = 4{\cdot}803.$
Rectangle 2 × 1 .... / Right-angled isosceles triangle...........	$\pi \sqrt{\dfrac{5}{2}} = 4{\cdot}967.$
Rectangle 3 × 1.............................	$\pi \sqrt{\dfrac{10}{3}} = 5{\cdot}736.$

For instance, if a square and a circle have the same area, the former is the more acute in the ratio 4·443 : 4·261, or 1·043 : 1.

For the circle the absolute frequency is

$$(2\pi)^{-1} \times 2{\cdot}404 \, c \sqrt{\frac{\pi}{\sigma}}, \quad \text{where} \quad c = \sqrt{T_1} \div \sqrt{\rho}.$$

In the case of similar forms the frequency is inversely as the linear dimension.

[From the principle that an extension of boundary is always accompanied by a fall of pitch, we may infer that the gravest mode of a membrane of any shape, and of any variable density, is devoid of internal nodal lines.]

212. The theory of the free vibrations of a membrane was first successfully considered by Poisson[1]. His theory in the case of the rectangle left little to be desired, but his treatment of the circular membrane was restricted to the symmetrical vibrations. Kirchhoff's solution of the similar, but much more difficult, problem of the circular plate was published in 1850, and Clebsch's *Theory of Elasticity* (1862) gives the general theory of the circular membrane including the effects of stiffness and

[1] *Mém. de l'Académie*, t. VIII. 1829.

of rotatory inertia[1]. It will therefore be seen that there was not much left to be done in 1866; nevertheless the memoir of Bourget already referred to contains a useful discussion of the problem accompanied by very complete numerical results, the whole of which however were not new.

213. In his experimental investigations M. Bourget made use of various materials, of which paper proved to be as good as any. The paper is immersed in water, and after removal of the superfluous moisture by blotting-paper is placed upon a frame of wood whose edges have been previously coated with glue. The contraction of the paper in drying produces the necessary tension, but many failures may be met with before a satisfactory result is obtained. Even a well stretched membrane requires considerable precautions in use, being liable to great variations in pitch in consequence of the varying moisture of the atmosphere. The vibrations are excited by organ-pipes, of which it is necessary to have a series proceeding by small intervals of pitch, and they are made evident to the eye by means of a little sand scattered on the membrane. If the vibration be sufficiently vigorous, the sand accumulates on the nodal lines, whose form is thus defined with more or less precision. Any inequality in the tension shews itself by the circles becoming elliptic.

The principal results of experiment are the following:—

A circular membrane cannot vibrate in unison with every sound. It can only place itself in unison with sounds more acute than that heard when the membrane is gently tapped.

As theory indicates, these possible sounds are separated by less and less intervals, the higher they become.

The nodal lines are only formed distinctly in response to certain definite sounds. A little above or below confusion ensues, and when the pitch of the pipe is decidedly altered, the membrane remains unmoved. There is not, as Savart supposed, a continuous transition from one system of nodal lines to another.

The nodal lines are circles or diameters or combinations of circles and diameters, as theory requires. However, when the

[1] [The reader who wishes to pursue the subject from a mathematical point of view is referred to an excellent discussion by Pockels (Leipzig, 1891) of the differential equation $\nabla^2 u + k^2 u = 0$.]

number of diameters exceeds two, the sand tends to heap itself confusedly towards the middle of the membrane, and the nodes are not well defined.

The same general laws were verified by MM. Bernard and Bourget in the case of square membranes[1]; and these authors consider that the results of theory are decisively established in opposition to the views of Savart, who held that a membrane was capable of responding to any sound, no matter what its pitch might be. But I must here remark that the distinction between forced and free vibrations does not seem to have been sufficiently borne in mind. When a membrane is set in motion by aerial waves having their origin in an organ-pipe, the vibration is properly speaking *forced*. Theory asserts, not that the membrane is only capable of vibrating with certain defined frequencies, but that it is only capable of so vibrating *freely*. When however the period of the force is not approximately equal to one of the natural periods, the resulting vibration may be insensible.

In Savart's experiments the sound of the pipe was two or three octaves higher than the gravest tone of the membrane, and was accordingly never far from unison with one of the series of overtones. MM. Bourget and Bernard made the experiment under more favourable conditions. When they sounded a pipe somewhat lower in pitch than the gravest tone of the membrane, the sand remained at rest, but was thrown into vehement vibration as unison was approached. So soon as the pipe was decidedly higher than the membrane, the sand returned again to rest. A modification of the experiment was made by first tuning a pipe about a third higher than the membrane when in its natural condition. The membrane was then heated until its tension had increased sufficiently to bring the pitch above that of the pipe. During the process of cooling the pitch gradually fell, and the point of coincidence manifested itself by the violent motion of the sand, which at the beginning and end of the experiment was sensibly at rest.

M. Bourget found a good agreement between theory and observation with respect to the radii of the circular nodes, though the test was not very precise, in consequence of the sensible width of the bands of sand; but the relative pitch of the various simple tones deviated considerably from the theoretical estimates. The

[1] *Ann. de Chim.* LX. 449—479. 1860.

committee of the French Academy appointed to report on M. Bourget's memoir suggest as the explanation the want of perfect fixity of the boundary. It should also be remembered that the theory proceeds on the supposition of perfect flexibility—a condition of things not at all closely approached by an ordinary membrane stretched with a comparatively small force. But perhaps the most important disturbing cause is the resistance of the air, which acts with much greater force on a membrane than on a string or bar in consequence of the large surface exposed. The gravest mode of vibration, during which the displacement is at all points in the same direction, might be affected very differently from the higher modes, which would not require so great a transference of air from one side to the other.

[In the case of kettle-drums the matter is further complicated by the action of the shell, which limits the motion of the air upon one side of the membrane. From the fact that kettle-drums are struck, not in the centre, but at a point about midway between the centre and edge, we may infer that the vibrations which it is desired to excite are not of the symmetrical class. The sound is indeed but little affected when the central point is touched with the finger. Under these circumstances the principal vibration (1) is that with one nodal diameter and no nodal circle, and to this corresponds the greater part of the sound obtained in the normal use of the instrument. Other tones, however, are audible, which correspond with vibrations characterized (2) by two nodal diameters and no nodal circle, (3) by three nodal diameters and no nodal circles, (4) by one nodal diameter and one nodal circle. By observation with resonators upon a large kettle-drum of 25 inches diameter the pitch of (2) was found to be about a fifth above (1), that of (3) about a major seventh above (1), and that of (4) a little higher again, forming an imperfect octave with the principal tone. For the corresponding modes of a uniform perfectly flexible membrane vibrating *in vacuo*, the theoretical intervals are those represented by the ratios 1·34, 1·66, 1·83 respectively[1].

The vibrations of soap films have been investigated by Melde[2]. The frequencies for surfaces of equal area in the form of the circle, the square and the equilateral triangle, were found to be as

[1] *Phil. Mag.*, vol. VII., p. 160, 1879.

[2] *Pogg. Ann.*, 159, p. 275, 1876. *Akustik*, p. 131, 1883.

1·000 : 1·049 : 1·175. In membranes of this kind the tension is due to capillarity, and is independent of the thickness of the film.]

213 *a*. The forced vibrations of square and circular membranes have been further experimentally studied by Elsas[1], who has confirmed the conclusions of Savart as to the responsiveness of a membrane to sounds of arbitrary pitch. In these experiments the vibrations of a fork were communicated to the membrane by means of a light thread, attached normally at the centre; and the position of the nodal curves and of the maxima of disturbance was traced in the usual manner by sand and lycopodium. A series of figures accompanies the memoir, shewing the effect of sounds of progressively rising pitch.

In many instances the curves found do not exhibit the symmetries demanded by the supposed conditions. Thus in the case of the square membrane all the curves should be similarly related to the four corners, and in the case of the circular membrane all the curves should be circles. The explanation is probably to be sought in the difficulty of attaining equality of tension. If there be any irregularity, the effect will be to introduce modes of vibration which should not appear, as having nodes at the point of excitation, and this especially when there is a near agreement of periods. Or again, an irregularity may operate to disturb the balance between two modes of theoretically identical pitch, which should be excited to the same degree. Indeed the passage through such a point of isochronism may be expected to be highly unstable in the absence of moderate dissipative forces.

The theoretical solution of these questions has already (§§ 196, 204) been given, but would need much further development for an accurate determination of the nodal curves relating to periods not included among the natural periods. But the general course of the phenomenon can be traced without difficulty.

If the imposed frequency be less than the lowest natural frequency, the vibration is devoid of (internal) nodes. For a nodal line, if it existed, being of necessity either endless or terminated at the boundary[2], would divide the membrane into two parts. Of

[1] *Nova Acta der Ksl. Leop. Carol. Deutschen Akademie*, Bd. XLV. Nr. 1. Halle, 1882.

[2] Otherwise the extremity would have to remain at rest under the action of component tensions from the surrounding parts which are all in one direction.

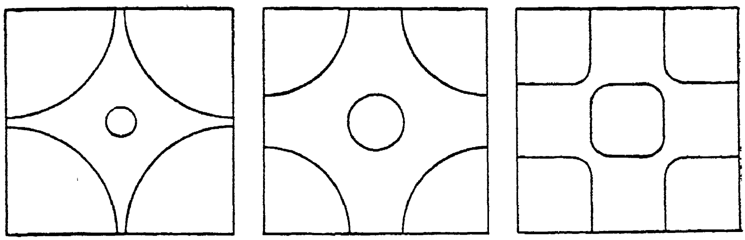
these one part would be vibrating freely with a frequency less than the lowest natural to the whole membrane, an impossible condition of things (§ 211). The absence of nodal curves under the above-mentioned conditions is one of the conclusions drawn by Elsas from his observations.

As the frequency of the imposed vibration rises through the lowest natural frequency, a nodal curve manifests itself round the point of excitation, and gradually extends. The course of things is most easily followed in the case of the circular membrane, excited at the centre. The nodal curves are then of necessity also circles, and it is evident that the first appearance of a nodal circle can take place only at the centre. Otherwise there would be a circular annulus of finite internal diameter, vibrating freely with a frequency only infinitesimally higher than that of the entire circle. At first sight indeed it might appear that even an infinitely small nodal circle would entail a finite elevation of pitch, but a consideration of the solution (§ 204) as expressed by a combination of Bessel's functions of the first and second kinds, shews that this is not the case. At the point of isochronism the second function disappears, and immediately afterwards re-enters with an infinitely small coefficient. But inasmuch as this function is itself infinite when $r=0$, a nodal circle of vanishing radius is possible. Accordingly the fixation of the centre of a vibrating circular membrane does not alter the pitch, a conclusion which may be extended to the fixation of any number of detached points of a membrane of any shape.

The effect of gradually increasing frequency upon the nodal system of a circular membrane may be thus summarized. Below the first proper tone there is no internal node. As this point is reached, the mode of vibration identifies itself with the corresponding free mode, and then an infinitely small nodal circle manifests itself. As the frequency further increases, this circle expands, until when the second proper tone is reached, it coincides with the nodal circle of the free vibration of this frequency. Another infinitely small circle now appears, and it, as well as the first, continually expands, until they coincide with the nodal system of a free vibration in the third proper tone. This process continues as the pitch rises, every circle moving continually outwards. At each coincidence with a natural frequency the nodal system identifies itself with that of the free vibration, and a new circle begins to form itself at the centre.

The behaviour of a square membrane is of course more difficult to follow in detail. The transition from Fig. (34) case (4), corresponding to $m=3$, $n=1$, and $m=1$, $n=3$, to Fig. (36) where $m=3$, $n=3$, can be traced in Elsas's curves through such forms as

Fig. 39 *a*.



CHAPTER X.

VIBRATIONS OF PLATES[1].

214. IN order to form according to Green's method the equations of equilibrium and motion for a thin solid plate of uniform isotropic material and constant thickness, we require the expression for the potential energy of bending. It is easy to see that for each unit of area the potential energy V is a positive homogeneous symmetrical quadratic function of the two principal curvatures. Thus, if ρ_1, ρ_2 be the principal radii of curvature, the expression for V will be

$$A\left(\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}+\frac{2\mu}{\rho_1\rho_2}\right) \qquad (1),$$

where A and μ are constants, of which A must be positive, and μ must be numerically less than unity. Moreover if the material be of such a character that it undergoes no lateral contraction when a bar is pulled out, the constant μ must vanish. This amount of information is almost all that is required for our purpose, and we may therefore content ourselves with a mere statement of the relations of the constants in (1) with those by means of which the elastic properties of bodies are usually defined.

From Thomson and Tait's *Natural Philosophy*, §§ 639, 642, 720, it appears that, if $2h$ be the thickness, q Young's modulus,

[1] [This Chapter deals only with *flexural* vibrations. The extensional vibrations of an infinite plane plate are briefly considered in Chapter X. A, as a particular case of those of an infinite cylindrical shell. They are not of much acoustical importance.]

and μ the ratio of lateral contraction to longitudinal elongation when a bar is pulled out, the expression for V is

$$V=\frac{qh^3}{3(1-\mu^2)}\left\{\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}+\frac{2\mu}{\rho_1\rho_2}\right\}$$

$$=\frac{qh^3}{3(1-\mu^2)}\left\{\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)^2-\frac{2(1-\mu)}{\rho_1\rho_2}\right\} \quad\ldots\ldots\ldots (2)^1.$$

[Equation (2) gives the interpretation of the constants of (1) in its application to a homogeneous plate of isotropic material; but the expression (1) itself is of far wider scope. The material composing the plate may vary from layer to layer, and the elastic character of any layer need not be isotropic, but only symmetrical with respect to the normal. As a particular case, the middle layer, or indeed any other layer, may be supposed to be *physically* inextensible.

Similar remarks apply to the investigations of the following chapter relating to curved shells.]

If w be the small displacement perpendicular to the plane of the plate at the point whose rectangular coordinates in the plane of the plate are x, y,

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\nabla^2 w, \qquad \frac{1}{\rho_1\rho_2}=\frac{d^2w}{dx^2}\frac{d^2w}{dy^2}-\left(\frac{d^2w}{dxdy}\right)^2,$$

and thus for a unit of area, we have

$$V=\frac{qh^3}{3(1-\mu^2)}\left[(\nabla^2 w)^2-2(1-\mu)\left\{\frac{d^2w}{dx^2}\frac{d^2w}{dy^2}-\left(\frac{d^2w}{dxdy}\right)^2\right\}\right] \quad (3),$$

which quantity has to be integrated over the surface (S) of the plate.

[1] The following comparison of the notations used by the principal writers may save trouble to those who wish to consult the original memoirs.

Rigidity $=n$ (Thomson) $=\mu$ (Lamé).

Young's modulus $=E$ (Clebsch) $=M$ (Thomson) $=\frac{9nk}{3k+n}$ (Thomson)

$=\frac{n(3m-n)}{m}$ (Thomson) $=q$ (Kirchhoff and Donkin) $=2K\frac{1+3\theta}{1+\theta}$ (Kirchhoff).

Ratio of lateral contraction to longitudinal elongation $=\mu$ (Clebsch and Donkin)

$=\sigma$ (Thomson) $=\frac{m-n}{2m}$ (Thomson) $=\frac{\theta}{1+2\theta}$ (Kirchhoff) $=\frac{\lambda}{2\lambda+2\mu}$ (Lamé).

Poisson assumed this ratio to be $\frac{1}{4}$, and Wertheim $\frac{1}{3}$.

215. We proceed to find the variation of V, but it should be previously noticed that the second term in V, namely $\iint \frac{dS}{\rho_1\rho_2}$, represents the *total curvature* of the plate, and is therefore dependent only on the state of things at the edge.

$$\delta V = \frac{2qh^3}{3(1-\mu^2)} \iint \left\{ \nabla^2 w \,.\, \nabla^2 \delta w - (1-\mu)\, \delta \frac{1}{\rho_1\rho_2} \right\} dS \;\ldots\ldots\; (1);$$

so that we have to consider the two variations

$$\iint \nabla^2 w \,.\, \nabla^2 \delta w \,.\, dS \quad \text{and} \quad \iint \delta (\rho_1\rho_2)^{-1}\, dS.$$

Now by Green's theorem

$$\iint \nabla^2 w \,.\, \nabla^2 \delta w \,.\, dS = \iint \nabla^4 w \,.\, \delta w \,.\, dS - \int \frac{d\nabla^2 w}{dn} \,.\, \delta w \,.\, ds + \int \nabla^2 w \frac{d\delta w}{dn}\, ds \;\ldots\ldots\ldots\ldots\; (2),$$

in which ds denotes an element of the boundary, and d/dn denotes differentiation with respect to the normal of the boundary drawn outwards.

The transformation of the second part is more difficult. We have

$$\delta \iint \frac{dS}{\rho_1\rho_2} = \iint \left\{ \frac{d^2 w}{dx^2}\frac{d^2 \delta w}{dy^2} + \frac{d^2 w}{dy^2}\frac{d^2 \delta w}{dx^2} - 2\frac{d^2 w}{dx\,dy}\frac{d^2 \delta w}{dx\,dy} \right\} dS.$$

The quantity under the sign of integration may be put into the form

$$\frac{d}{dy}\left(\frac{d\delta w}{dy}\frac{d^2 w}{dx^2} - \frac{d\delta w}{dx}\frac{d^2 w}{dx\,dy}\right) + \frac{d}{dx}\left(\frac{d\delta w}{dx}\frac{d^2 w}{dy^2} - \frac{d\delta w}{dy}\frac{d^2 w}{dx\,dy}\right).$$

Now, if F be any function of x and y,

$$\left.\begin{aligned} \iint \frac{dF}{dy}\, dx\,dy &= \int F \sin\theta\, ds \\ \iint \frac{dF}{dx}\, dx\,dy &= \int F \cos\theta\, ds \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (3),$$

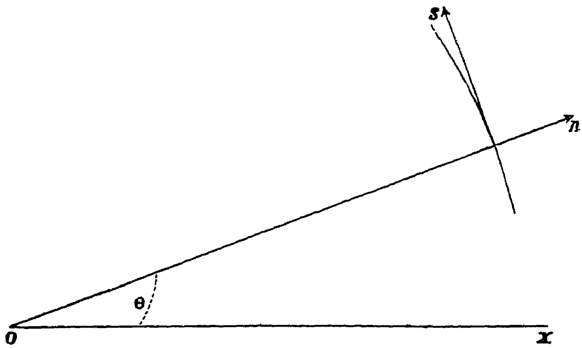
where θ is the angle between x and the normal drawn outwards, and the integration on the right-hand side extends round the boundary. Using these, we find

$$\delta \iint \frac{dS}{\rho_1\rho_2} = \int ds \sin\theta \left\{ \frac{d\delta w}{dy}\frac{d^2 w}{dx^2} - \frac{d\delta w}{dx}\frac{d^2 w}{dx\,dy} \right\} + \int ds \cos\theta \left\{ \frac{d\delta w}{dx}\frac{d^2 w}{dy^2} - \frac{d\delta w}{dy}\frac{d^2 w}{dx\,dy} \right\}$$

If we substitute for $d\delta w/dx$, $d\delta w/dy$ their values in terms $d\delta w/dn$, $d\delta w/ds$, from the equations (see Fig. 40)

$$\left.\begin{aligned}\frac{d\delta w}{dx} &= \frac{d\delta w}{dn}\cos\theta - \frac{d\delta w}{ds}\sin\theta \\ \frac{d\delta w}{dy} &= \frac{d\delta w}{dn}\sin\theta + \frac{d\delta w}{ds}\cos\theta\end{aligned}\right\}\quad\text{..............} (4)$$

Fig. 40.



we obtain

$$\delta\iint\frac{dS}{\rho_1\rho_2} = \int ds\frac{d\delta w}{dn}\left\{\sin^2\theta\frac{d^2w}{dx^2} + \cos^2\theta\frac{d^2w}{dy^2} - 2\sin\theta\cos\theta\frac{d^2w}{dxdy}\right\}$$

$$+\int ds\frac{d\delta w}{ds}\left\{\cos\theta\sin\theta\left(\frac{d^2w}{dx^2} - \frac{d^2w}{dy^2}\right) + (\sin^2\theta - \cos^2\theta)\frac{d^2w}{dxdy}\right\}\text{...(5).}$$

The second integral by a partial integration with respect to s may be put into the form

$$\int\delta w\frac{d}{ds}\left\{\cos\theta\sin\theta\left(\frac{d^2w}{dy^2} - \frac{d^2w}{dx^2}\right) + (\cos^2\theta - \sin^2\theta)\frac{d^2w}{dxdy}\right\}ds.$$

Collecting and rearranging our results, we find

$$\delta V = \frac{2qh^3}{3(1-\mu^2)}\left[\iint\nabla^4 w\,\delta w\,dS\right.$$

$$-\int\delta w\,ds\left\{\frac{d\nabla^2 w}{dn} + (1-\mu)\frac{d}{ds}\left(\cos\theta\sin\theta\left(\frac{d^2w}{dy^2} - \frac{d^2w}{dx^2}\right)\right.\right.$$

$$\left.\left. + (\cos^2\theta - \sin^2\theta)\frac{d^2w}{dxdy}\right)\right\}$$

$$+\int\frac{d\delta w}{dn}ds\left\{\mu\nabla^2 w + (1-\mu)\left(\cos^2\theta\frac{d^2w}{dx^2} + \sin^2\theta\frac{d^2w}{dy^2}\right.\right.$$

$$\left.\left.\left. + 2\cos\theta\sin\theta\frac{d^2w}{dxdy}\right)\right\}\right]\text{...(6).}$$

There will now be no difficulty in forming the equations of motion. If ρ be the volume density, and $Z.\rho.2h.dS$ the transverse force acting on the element dS,

$$\delta V - \iint 2Z\rho h\,\delta w\,dS + \iint 2\rho h\ddot{w}\,\delta w\,dS = 0 \qquad\qquad (7)^1$$

is the general variational equation, which must be true whatever function (consistent with the constitution of the system) δw may be supposed to be. Hence by the principles of the Calculus of Variations

$$\frac{qh^2}{3\rho(1-\mu^2)}\nabla^4 w - Z + \ddot{w} = 0 \qquad\qquad (8),$$

at every point of the plate.

If the edges of the plate be free, there is no restriction on the hypothetical boundary values of δw and $d\delta w/dn$, and therefore the coefficients of these quantities in the expression for δV must vanish. The conditions to be satisfied at a free edge are thus

$$\left.\begin{aligned} &\frac{d\nabla^2 w}{dn} + (1-\mu)\frac{d}{ds}\left\{\cos\theta\sin\theta\left(\frac{d^2w}{dy^2} - \frac{d^2w}{dx^2}\right)\right. \\ &\qquad\qquad\left. + (\cos^2\theta - \sin^2\theta)\frac{d^2w}{dx\,dy}\right\} = 0 \\ &\mu\nabla^2 w + (1-\mu)\left\{\cos^2\theta\frac{d^2w}{dx^2} + \sin^2\theta\frac{d^2w}{dy^2}\right. \\ &\qquad\qquad\left. + 2\cos\theta\sin\theta\frac{d^2w}{dx\,dy}\right\} = 0 \end{aligned}\right\} \qquad (9).$$

If the whole circumference of the plate be clamped, $\delta w = 0$, $d\delta w/dn = 0$, and the satisfaction of the boundary conditions is already secured. If the edge be 'supported'[2], $\delta w = 0$, but $d\delta w/dn$ is arbitrary. The second of the equations (9) must in this case be satisfied by w.

216. The boundary equations may be simplified by getting rid of the extrinsic element involved in the use of Cartesian coordinates. Taking the axis of x parallel to the normal of the bounding curve, we see that we may write

$$\cos^2\theta\frac{d^2w}{dx^2} + \sin^2\theta\frac{d^2w}{dy^2} + 2\cos\theta\sin\theta\frac{d^2w}{dx\,dy} = \frac{d^2w}{dn^2}.$$

Also
$$\nabla^2 w = \frac{d^2w}{dn^2} + \frac{d^2w}{d\sigma^2} \qquad\qquad (1),$$

[1] The rotatory inertia is here neglected. [2] Compare § 162.

where σ is a fixed axis coinciding with the tangent at the point under consideration. In general $d^2w/d\sigma^2$ differs from d^2w/ds^2. To obtain the relation between them, we may proceed thus. Expand w by Maclaurin's theorem in ascending powers of the small quantities n and σ, and substitute for n and σ their values in terms of s, the arc of the curve.

Thus in general

$$w = w_0 + \frac{dw}{dn_0} n + \frac{dw}{d\sigma_0} \sigma + \tfrac{1}{2} \frac{d^2w}{dn_0^2} n^2 + \frac{d^2w}{dn_0\, d\sigma_0} n\sigma + \tfrac{1}{2} \frac{d^2w}{d\sigma_0^2} \sigma^2 + \ldots,$$

while on the curve $\sigma = s +$ cubes, $n = -\tfrac{1}{2} s^2/\rho + \ldots$, where ρ is the radius of curvature. Accordingly for points on the curve,

$$w = w_0 - \tfrac{1}{2} \frac{dw}{dn_0} \frac{s^2}{\rho} + \frac{dw}{d\sigma_0} s + \tfrac{1}{2} \frac{d^2w}{d\sigma_0^2} s^2 + \text{cubes of } s,$$

and therefore
$$\frac{d^2w}{ds^2} = \frac{d^2w}{d\sigma^2} - \frac{1}{\rho} \frac{dw}{dn} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

whence from (1)

$$\nabla^2 w = \frac{d^2w}{dn^2} + \frac{1}{\rho} \frac{dw}{dn} + \frac{d^2w}{ds^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

We conclude that the second boundary condition in (9) § 215 may be put into the form

$$\frac{d^2w}{dn^2} + \mu \left(\frac{1}{\rho} \frac{dw}{dn} + \frac{d^2w}{ds^2} \right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

In the same way by putting $\theta = 0$, we see that

$$\cos\theta \sin\theta \left(\frac{d^2w}{dy^2} - \frac{d^2w}{dx^2} \right) + (\cos^2\theta - \sin^2\theta) \frac{d^2w}{dx\,dy}$$

is equivalent to $d^2w/dn\,d\sigma$, where it is to be understood that the axes of n and σ are fixed. The first boundary condition now becomes

$$\frac{d}{dn} \nabla^2 w + (1 - \mu) \frac{d}{ds} \left(\frac{d^2w}{dn\,d\sigma} \right) = 0 \quad \ldots\ldots\ldots\ldots\ldots(5).$$

If we apply these equations to the rectangle whose sides are parallel to the coordinate axes, we obtain as the conditions to be satisfied along the edges parallel to y,

$$\left. \begin{aligned} \frac{d}{dx} \left\{ \frac{d^2w}{dx^2} + (2 - \mu) \frac{d^2w}{dy^2} \right\} = 0 \\ \frac{d^2w}{dx^2} + \mu \frac{d^2w}{dy^2} = 0 \end{aligned} \right\} \quad \ldots\ldots\ldots\ldots\ldots(6).$$

In this case the distinction between σ and s disappears, and ρ, the radius of curvature, is infinitely great. The conditions for the other pair of edges are found by interchanging x and y. These results may be obtained equally well from (9) § 215 directly, without the preliminary transformation.

217. If we suppose $Z=0$, and write

$$\frac{qh^2}{3\rho(1-\mu^2)}=c^4 \quad\text{.......................}(1),$$

the general equation becomes

$$\ddot{w}+c^4\nabla^4 w=0 \quad\text{.......................}(2),$$

or, if $w \propto \cos(pt-\epsilon)$,

$$\nabla^4 w=k^4 w \quad\text{.......................}(3),$$

where

$$k^4=p^2/c^4 \quad\text{.......................}(4).$$

Any two values of w, u and v, corresponding to the same boundary conditions, are *conjugate*, that is to say

$$\iint uv\,dS=0 \quad\text{.......................}(5),$$

provided that the periods be different. In order to prove this from the ordinary differential equation (3), we should have to retrace the steps by which (3) was obtained. This is the method adopted by Kirchhoff for the circular disc, but it is much simpler and more direct to use the variational equation

$$\delta V+2\rho h\iint \ddot{w}\,\delta w\,dS=0 \quad\text{...................}(6),$$

in which w refers to the actual motion, and δw to an arbitrary displacement consistent with the nature of the system. δV is a symmetrical function of w and δw, as may be seen from § 215, or from the general character of V (§ 94).

If we now suppose in the first place that $w=u$, $\delta w=v$, we have

$$\delta V=2\rho hp^2\iint uv\,dS;$$

and in like manner if we put $w=v$, $\delta w=u$, which we are equally entitled to do,

$$\delta V=2\rho hp'^2\iint uv\,dS,$$

whence $$(p^2-p'^2)\iint uv\,dS=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

This demonstration is valid whatever may be the form of the boundary, and whether the edge be clamped, supported, or free, in whole or in part.

As for the case of membranes in the last Chapter, equation (7) may be employed to prove that the admissible values of p^2 are real; but this is evident from physical considerations.

218. For the application to a circular disc, it is necessary to express the equations by means of polar coordinates. Taking the centre of the disc as pole, we have for the general equation to be satisfied at all points of the area

$$(\nabla^4-k^4)w=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where (§ 200) $$\nabla^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{1}{r^2}\frac{d^2}{d\theta^2}.$$

To express the boundary condition (§ 216) for a free edge ($r=a$), we have

$$\frac{d}{dn}\nabla^2w=\frac{d}{dr}\nabla^2w,\quad \frac{d}{ds}\left(\frac{d^2w}{dn\,d\sigma}\right)=\frac{d}{a\,d\theta}\frac{d}{dr}\left(\frac{dw}{r\,d\theta}\right),\quad \frac{d^2w}{ds^2}=\frac{d^2w}{a^2d\theta^2},$$

$\rho=$ radius of curvature $=a$; and thus

$$\left.\begin{aligned}\frac{d}{dr}\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)+\frac{d^2}{d\theta^2}\left(\frac{2-\mu}{a^2}\frac{dw}{dr}-\frac{3-\mu}{a^3}w\right)=0\\ \frac{d^2w}{dr^2}+\mu\left(\frac{1}{a}\frac{dw}{dr}+\frac{1}{a^2}\frac{d^2w}{d\theta^2}\right)=0\end{aligned}\right\}\ldots\ldots\ldots(2).$$

After the differentiations are performed, r is to be made equal to a.

If w be expanded in Fourier's series

$$w=w_0+w_1+\ldots+w_n+\ldots,$$

each term separately must satisfy (2), and thus, since

$$w_n \propto \cos(n\theta-\alpha),$$

$$\left.\begin{aligned}\frac{d}{dr}\left(\frac{d^2w_n}{dr^2}+\frac{1}{r}\frac{dw_w}{dr}\right)-n^2\left(\frac{2-\mu}{a^2}\frac{dw_n}{dr}-\frac{3-\mu}{a^3}w_n\right)=0\\ \frac{d^2w_n}{dr^2}+\mu\left(\frac{1}{a}\frac{dw_n}{dr}-\frac{n^2}{a^2}w_n\right)=0\end{aligned}\right\}\ldots\ldots(3).$$

The superficial differential equation may be written

$$(\nabla^2 + k^2)(\nabla^2 - k^2)\, w = 0,$$

which becomes for the general term of the Fourier expansion

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} + k^2\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} - k^2\right) w_n = 0,$$

shewing that the complete value of w_n will be obtained by adding together, with arbitrary constants prefixed, the general solutions of

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} \pm k^2\right) w_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The equation with the upper sign is the same as that which obtains in the case of the vibrations of circular membranes, and as in the last Chapter we conclude that the solution applicable to the problem in hand is $w_n \propto J_n(kr)$, the second function of r being here inadmissible.

In the same way the solution of the equation with the lower sign is $w_n \propto J_n(ikr)$, where $i = \sqrt{(-1)}$ as usual. [See § 221 *a*.]

The simple vibration is thus

$$w_n = \cos n\theta\, \{\alpha J_n(kr) + \beta J_n(ikr)\} + \sin n\theta\, \{\gamma J_n(kr) + \delta J_n(ikr)\}.$$

The two boundary equations will determine the admissible values of k and the values which must be given to the ratios $\alpha : \beta$ and $\gamma : \delta$. From the form of these equations it is evident that we must have $\alpha : \beta = \gamma : \delta$,

and thus w_n may be expressed in the form

$$w_n = P\cos(n\theta - \alpha)\, \{J_n(kr) + \lambda J_n(ikr)\}\cos(pt - \epsilon) \ldots\ldots(5).$$

As in the case of a membrane the nodal system is composed of the n diameters symmetrically distributed round the centre, but otherwise arbitrary, denoted by

$$\cos(n\theta - \alpha) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

together with the concentric circles, whose equation is

$$J_n(kr) + \lambda J_n(ikr) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

219. In order to determine λ and k we must introduce the boundary conditions. When the edge is free, we obtain from (3) § 218

$$\left.\begin{aligned} -\lambda &= \frac{n^2(\mu - 1)\{ka J_n'(ka) - J_n(ka)\} - k^3a^3 J_n'(ka)}{n^2(\mu - 1)\{ika J_n'(ika) - J_n(ika)\} + ik^3a^3 J_n'(ika)} \\ -\lambda &= \frac{(\mu - 1)\{ka J_n'(ka) - n^2 J_n(ka)\} - k^2a^2 J_n(ka)}{(\mu - 1)\{ika J_n'(ika) - n^2 J_n(ika)\} + k^2a^2 J_n(ika)} \end{aligned}\right\} \ldots(1),$$

in which use has been made of the differential equations satisfied by $J_n(kr)$, $J_n(ikr)$. In each of the fractions on the right the denominator may be derived from the numerator by writing ik in place of k. By elimination of λ the equation is obtained whose roots give the admissible values of k.

When $n=0$, the result assumes a simple form, viz.

$$2(1-\mu)+ika\frac{J_0(ika)}{J_0'(ika)}+ka\frac{J_0(ka)}{J_0'(ka)}=0\ldots\ldots\ldots\ldots(2).$$

This, of course, could have been more easily obtained by neglecting n from the beginning.

The calculation of the lowest root for each value of n is troublesome, and in the absence of appropriate tables must be effected by means of the ascending series for the functions $J_n(kr)$, $J_n(ikr)$. In the case of the higher roots recourse may be had to the semi-convergent descending series for the same functions. Kirchhoff finds

$$\tan(ka-\tfrac{1}{2}n\pi)=\frac{B/(8ka)+C/(8ka)^2-D/(8ka)^3+\ldots}{A+B/(8ka)+D/(8ka)^3+\ldots}\ \ldots\ldots(3),$$

where

$$A=\gamma=(1-\mu)^{-1},$$
$$B=\gamma(1-4n^2)-8,$$
$$C=\gamma(1-4n^2)(9-4n^2)+48(1+4n^2),$$
$$D=-\gamma\tfrac{1}{3}\{(1-4n^2)(9-4n^2)(13-4n^2)\}+8(9+136n^2+80n^4).$$

When ka is great,

$$\tan(ka-\tfrac{1}{2}n\pi)=0\quad\text{approx.};$$

whence

$$ka=\tfrac{1}{2}\pi(n+2h)\ \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where h is an integer.

It appears by a numerical comparison that h is identical with the number of circular nodes, and (4) expresses a law discovered by Chladni, that the frequencies corresponding to figures with a given number of nodal diameters are, with the exception of the lowest, approximately proportional to the squares of consecutive even or uneven numbers, according as the number of the diameters is itself even or odd. Within the limits of application of (4), we see also that the pitch is approximately unaltered, when any number is subtracted from h, provided twice that number be added to n. This law, of which traces appear in the following table, may be expressed by saying that towards raising the pitch nodal circles have twice the effect of nodal diameters. It is probable, however, that, strictly speaking, no two normal components have exactly the same pitch.

h	$n=0$			$n=1$		
	Ch.	P.	W.	Ch.	P.	W.
0	...	...	...	...	...	...
1	Gis	Gis +	A +	b	h −	c −
2	gis′ +	b′ −	b′ +	e″ +	f″ +	fis″ +

h	$n=2$			$n=3$		
	Ch.	P.	W.	Ch.	P.	W.
0	c	c	c	d	dis −	dis −
1	g′.	gis′ +	a′ −	d″. dis″	dis″ +	e″ −

The table, extracted from Kirchhoff's memoir, gives the pitch of the more important overtones of a free circular plate, the gravest being assumed to be C[1]. The three columns under the heads *Ch*, *P*, *W* refer respectively to the results as observed by Chladni and as calculated from theory with Poisson's and Wertheim's values of μ. A *plus* sign denotes that the actual pitch is a little higher, and a *minus* sign that it is a little lower, than that written. The discrepancies between theory and observation are considerable, but perhaps not greater than may be attributed to irregularity in the plate.

220. The radii of the nodal circles in the symmetrical case ($n = 0$) were calculated by Poisson, and compared by him with results obtained experimentally by Savart. The following numbers are taken from a paper by Strehlke[2], who made some careful measurements. The radius of the disc is taken as unity.

	Observation.	Calculation.
One circle ...	0·67815	0·68062.
Two circles...	0·39133	0·39151.
	0·84149	0·84200.
Three circles	0·25631	0·25679.
	0·59107	0·59147.
	0·89360	0·89381.

[1] Gis corresponds to G♯ of the English notation, and *h* to *b* natural.

[2] Pogg. *Ann.* xcv. p. 577. 1855.

The calculated results appear to refer to Poisson's value of μ, but would vary very little if Wertheim's value were substituted.

The following table gives a comparison of Kirchhoff's theory (n not zero) with measurements by Strehlke made on less accurate discs.

Radii of Circular Nodes.

	Observation.				Calculation.	
					$\mu=\frac{1}{4}$ (P.).	$\mu=\frac{1}{3}$ (W.).
$n=1,\ h=1$	0·781	0·783	0·781	0·783	0·78136	0·78088
$n=2,\ h=1$	0·79	0·81	0·82		0·82194	0·82274
$n=3,\ h=1$	0·838	0·842			0·84523	0·84681
$n=1,\ h=2$	0·488	0·492			0·49774	0·49715
	0·869	0·869			0·87057	0·87015

The most general motion of the uniform circular plate is expressed by the superposition, with arbitrary amplitudes and phases, of the normal components already investigated. The determination of the amplitude and phase to correspond to arbitrary initial displacements and velocities is effected precisely as in the corresponding problem for the membrane by the aid of the characteristic property of the normal functions proved in § 217.

221. When the plate is truly symmetrical, whether uniform or not, theory indicates, and experiment verifies, that the position of the nodal diameters is arbitrary, or rather dependent only on the manner in which the plate is supported, and excited. By varying the place of support, any desired diameter may be made nodal. It is generally otherwise when there is any sensible departure from exact symmetry. The two modes of vibration, which originally, in consequence of the equality of periods, could be combined in any proportion without ceasing to be simple harmonic, are now separated and affected with different periods. At the same time the position of the nodal diameters becomes determinate, or rather limited to two alternatives. The one set is derived from the other by rotation through half the angle included between two adjacent diameters of the same set. This supposes that the deviation from uniformity is small; otherwise the nodal system will no longer be composed of approximate circles and diameters at all. The cause of the deviation may be an irregularity either in the material or in the thickness or in the form of the boundary. The effect of a small load at any point may be investigated as in the parallel problem of the membrane § 208. If the place at which the load is attached does not lie on a nodal circle, the normal types are made determinate. The diametral system corresponding to one of the types passes through the place in question, and for this type the period is unaltered. The period of the other type is increased.

[The divergence of free periods, which is due to slight inequalities, would seem to afford an explanation of some curious observations by Savart[1]. When a circular plate, vibrating with nodal diameters, is under the influence of the bow applied at any part of the circumference, the nodal diameters indicated by sand are so situated that the bow lies in the middle of a vibrating segment. If, however, the bow be suddenly withdrawn, the nodal system oscillates, or even revolves, during the subsidence of the motion. It is evident that no such displacement could be expected, were the plate absolutely symmetrical. The same would be true, even in the case of asymmetry, if the bow were so applied as to excite one only of the two determinate vibrations then possible. But in general the effect of the bow must be to excite both kinds of vibrations, and then the matter is more complicated. It would seem that so long as the constraining action of the bow lasts, both vibrations are forced to keep the same time, and the effect is much the same as in the case of symmetry. But on withdrawal of the bow the free vibrations which then ensue take place each in its proper frequency, and a phase difference soon arises by which the effects are modified.

Let us suppose that the origin of θ is so chosen in relation to the irregularities that the types of vibration are represented by $\cos n\theta$, $\sin n\theta$. Then in general the free vibrations, resulting from the action of the bow at an arbitrary point of the circumference, may be taken to be

$$\cos n\alpha \sin n\theta \cos pt - \sin n\alpha \cos n\theta \cos (pt + \epsilon) \text{..........} (1),$$

where ϵ is the difference of phase which has accumulated since the commencement of the free vibrations. In the case of symmetry $\epsilon = 0$, and (1) becomes

$$\sin n(\theta - \alpha) \cos pt \text{......................} (2),$$

[1] *Ann. Chim.*, vol. 36, p. 257, 1827.

which represents a fixed nodal system

$$\theta = \alpha + m(\pi/n) \quad\text{.........................(3)},$$

in any arbitrary position depending upon the point of application of the bow. A similar fixity of the nodal system occurs, in spite of the variable ϵ, when α is so chosen that $\cos n\alpha = 0$ or $\sin n\alpha = 0$. But in general there is no fixed nodal system. When ϵ is a multiple of 2π, that is when the two vibrations are restored to the same phase, there is a nodal system represented by (3). And when ϵ is an odd multiple of π, so that the two vibrations are in opposite phases, we have in place of (2)

$$\sin n(\theta + \alpha) \cos pt \quad\text{......................(4)},$$

with a nodal system

$$\theta = -\alpha + m(\pi/n) \quad\text{.........................(5)}.$$

In these cases there is a nodal system, and in a sense the system may be said to oscillate between the positions given by (3) and (5); but it must not be overlooked that at intermediate times there is no true nodal system at all. Thus, when $\epsilon = \frac{1}{2}\pi$, (1) becomes

$$\cos n\alpha \sin n\theta \cos pt + \sin n\alpha \cos n\theta \sin pt.$$

The squared amplitude of this motion is

$$\cos^2 n\alpha \sin^2 n\theta + \sin^2 n\alpha \cos^2 n\theta,$$

a quantity which does not vanish for any value of θ. In general the squared amplitude is

$$\cos^2 n\alpha \sin^2 n\theta + \sin^2 n\alpha \cos^2 n\theta - 2 \cos n\alpha \sin n\alpha \cos n\theta \sin n\theta \cos \epsilon,$$

or, as it may also be written,

$$\tfrac{1}{2} - \tfrac{1}{2} \cos 2n\alpha \cos 2n\theta - \tfrac{1}{2} \sin 2n\alpha \sin 2n\theta \cos \epsilon \quad\text{...........(6)}.$$

This quantity is a maximum or a minimum when

$$\tan 2n\theta = \cos \epsilon \tan 2n\alpha \quad\text{...................(7)}.$$

The minimum of motion thus oscillates backwards and forwards between $\theta = +\alpha$ and $\theta = -\alpha$; but as we have seen, it is only in these extreme positions that the minimum is zero.

A like phenomenon occurs during the free vibrations of a circular membrane, or in fact of any system of revolution such that the position of nodal lines is arbitrary so long as the symmetry is complete.]

The two other cases of a circular plate in which the edge is either clamped or *supported* would be easier than the preceding in their theoretical treatment, but are of less practical interest on account of the difficulty of experimentally realising the conditions assumed. The general result that the nodal system is composed of concentric circles, and diameters symmetrically distributed, is applicable to all the three cases.

221 *a.* The use in the telephone of a thin circular plate clamped at the edge lends a certain interest to the calculation of the periods and modes of vibration of such a plate. It will suffice to consider the symmetrical modes.

By (5) § 218 we may take as representing the motion in this case

$$w = J_0(kr) + \lambda J_0(ikr) = J_0(kr) + \lambda I_0(kr) \quad \text{.........(1)},$$

from which

$$\frac{dw}{kdr} = J_0'(kr) + i\lambda J_0'(ikr) = -J_1(kr) + \lambda I_1(kr) \quad \text{......(2)};$$

where we write

$$I_0(z) = J_0(iz) = 1 + \frac{z^2}{2^2} + \frac{z^4}{2^2 . 4^2} + \dots \quad \text{.........(3)},$$

$$I_1(z) = iJ_0'(iz) = \frac{z}{2} + \frac{z^3}{2^2 . 4} + \frac{z^5}{2^2 . 4^2 . 5} + \dots \quad \text{....(4)}.$$

Since the plate is clamped at $r = a$, both w and dw/dr must there vanish. Hence, writing $ka = z$, we get as the frequency equation

$$\frac{J_1(z)}{J_0(z)} + \frac{I_1(z)}{I_0(z)} = 0 \quad \text{.......................(5)}.$$

In (5) I_1 and I_0 are both positive, so that the signs of J_1 and J_0 must be opposite. Hence by Table B § 206 the first root must lie between 2·4 and 3·8, the second between 5·5 and 7·0, and so on. The values of the earlier roots might be obtained without much difficulty from the series for I_0 and I_1 by using the table § 200 for J_0 and J_1; but it will be convenient for the present and further purposes to give a short table[1] of the functions I_0 and I_1 themselves. For large values of the argument descending series, analogous to (10) § 200, may be employed.

[1] Calculated by A. Lodge, *Brit. Ass. Rep.*, 1889.

z	$I_0(z)$	$I_1(z)$	z	$I_0(z)$	$I_1(z)$
0·0	1·0000	0·0000	3·0	4·8808	3·9534
·2	1·0100	·1005	3·2	5·7472	4·7343
·4	1·0404	·2040	3·4	6·7848	5·6701
·6	1·0920	·3137	3·6	8·0277	6·7927
·8	1·1665	·4329	3·8	9·5169	8·1404
1·0	1·2661	·5652	4·0	11·3019	9·7595
1·2	1·3937	·7147	4·2	13·4425	11·7056
1·4	1·5534	·8861	4·4	16·0104	14·0462
1·6	1·7500	1·0848	4·6	19·0926	16·8626
1·8	1·9896	1·3172	4·8	22·7937	20·2528
2·0	2·2796	1·5906	5·0	27·2399	24·3356
2·2	2·6291	1·9141	5·2	32·5836	29·2543
2·4	3·0493	2·2981	5·4	39·0088	35·1821
2·6	3·5533	2·7554	5·6	46·7376	42·3283
2·8	4·1573	3·3011	5·8	56·0381	50·9462
			6·0	67·2344	61·3419

The first root of (5) is $z = 3·20$. This then is the value of ka for the gravest symmetrical vibration. The next value of z is about 6·3. Since the frequency varies as k^2 (§ 217), the interval between the tones is nearly two octaves.

Returning to the first root, we have for the frequency (n) § 217,

$$n = \frac{p}{2\pi} = \frac{(3·2)^2 c^2}{2\pi a^2} = \frac{(3·2)^2 \sqrt{q}\,.\,h}{2\pi a^2 \sqrt{3\rho\,(1-\mu^2)}} \quad \text{........ (6).}$$

This is the general formula. For rough calculations μ^2 in the denominator may be omitted. If for the case of iron we take

$$\rho = 7·7, \quad q = 2·0 \times 10^{12},$$

we find
$$n = \frac{2·4 \times 10^5\,.\,2h}{a^2} \quad \text{........................(7),}$$

$2h$ and a being expressed in centimetres.

A telephone plate measured by the author gave

$$a = 2·2, \quad 2h = ·020.$$

According to these values

$$n = 991 \text{ vibrations per second.}$$

222. We have seen that in general Chladni's figures as traced by sand agree very closely with the circles and diameters of theory; but in certain cases deviations occur, which are usually attributed to irregularities in the plate. It must however be remembered that the vibrations excited by a bow are not strictly speaking free, and that their periods are therefore liable to a certain modification. It may be that under the action of the bow two or more normal component vibrations coexist. The whole motion may be simple harmonic in virtue of the external force, although the natural periods would be a little different. Such an explanation is suggested by the regular character of the figures obtained in certain cases.

Another cause of deviation may perhaps be found in the manner in which the plates are supported. The requirements of theory are often difficult to meet in actual experiment. When this is so, we may have to be content with an imperfect comparison; but we must remember that a discrepancy may be the fault of the experiment as well as of the theory.

[In the ordinary use of sand to investigate the vibrations of flat plates and membranes the movement to the nodes is irregular in its character. If a grain be situated elsewhere than at a node, it is made to jump by a sufficiently vigorous transverse vibration. The result may be a movement either towards or from a node; but after a succession of such jumps the grain ultimately finds its way to a node as the only place where it can remain undisturbed. Grains which have already arrived at a node remain there, while others are constantly shifting their position.

It was found by Savart that very fine powder, such as lycopodium, behaves differently from sand. Instead of collecting at the nodes, it heaps itself up at the places of greatest motion. This effect was traced by Faraday[1] to the influence of currents of air, themselves the result of the vibration. In a vacuum all powders move to the nodes.

In some cases the movement of sand to the nodes, or to some of them, takes place in a more direct manner as the result of friction. Thus, in his investigation of the longitudinal vibrations of thin narrow strips of glass, held horizontally, Savart[2] observed the delineation of nodes apparently dependent upon an accompaniment of vibrations of a transverse character. The special peculiarity of this phenomenon was the non-correspondence of the lines traced by sand upon the two faces of the glass when tested

[1] On a Peculiar Class of Acoustical Figures, *Phil. Trans.*, 1831, p. 299.

[2] *Ann. d. Chim.*, vol. 14, p. 113, 1820.

in succession, a fact sufficient to shew that the transverse motion was connected with a failure of uniformity. In consequence of this there are developed transverse vibrations of the same (high) pitch as that of the principal longitudinal motion, and therefore attended with many nodes. These nodes are of course the same whichever face of the glass is uppermost, and it might be supposed that they would all be indicated by the sand, as would happen if the transverse vibrations existed alone. But the combination of the two kinds of motion causes a creeping of the sand towards the *alternate* nodes, the movements of the sand at corresponding points on the two sides of the plate being always in opposite directions. On the one side an inwards longitudinal motion (for example) is attended by an upwards transverse motion, but when the plate is reversed the same inwards longitudinal motion is associated with a transverse motion directed downwards. If there were no transverse motion, the longitudinal force upon any particle resulting from friction would vanish in the long run, but in consequence of the transverse motion this balance is upset, and in a manner different upon the two sides of the plate. The above considerations appear to afford sufficient ground for an explanation of the remarkable phenomenon observed by Savart, but an attempt to follow the matter further into detail would lead us too far[1].]

223. The first attempt to solve the problem with which we have just been occupied is due to Sophie Germain, who succeeded in obtaining the correct differential equation, but was led to erroneous boundary conditions. For a free plate the latter part of the problem is indeed of considerable difficulty. In Poisson's memoir 'Sur l'équilibre et le mouvement des corps élastiques[2],' that eminent mathematician gave *three* equations as necessary to be satisfied at all points of a free edge, but Kirchhoff has proved that in general it would be impossible to satisfy them all. It happens, however, that an exception occurs in the case of the symmetrical vibrations of a circular plate, when one of the equations is true identically. Owing to this peculiarity, Poisson's theory of the symmetrical vibrations is correct, notwithstanding the error in his view as to the boundary conditions. In 1850 the subject was

[1] See Terquem, *C. R.*, XLVI., p. 775, 1858.
[2] *Mém. de l'Acad. d. Sc. à Par*, 1829.

resumed by Kirchhoff[1], who first gave the *two* equations appropriate to a free edge, and completed the theory of the vibrations of a circular disc.

224. The correctness of Kirchhoff's boundary equations has been disputed by Mathieu[2], who, without explaining where he considers Kirchhoff's error to lie, has substituted a different set of equations. He proves that if u and u' be two normal functions, so that $w = u \cos pt$, $w = u' \cos p't$ are possible vibrations, then

$$(p^2 - p'^2) \iint uu'\,dx\,dy$$

$$= c^4 \int ds \left\{ u' \frac{d\nabla^2 u}{dn} - \nabla^2 u \frac{du'}{dn} - \nabla^2 u' \frac{du}{dn} + u \frac{d\nabla^2 u'}{dn} \right\} \text{.........(1).}$$

This follows, if it be admitted that u, u' satisfy respectively the equations

$$c^4 \nabla^4 u = p^2 u, \qquad c^4 \nabla^4 u' = p'^2 u'.$$

Since the left-hand member is zero, the same must be true of the right-hand member; and this, according to Mathieu, cannot be the case, unless at all points of the boundary both u and u' satisfy one of the four following pairs of equations:

$$\left.\begin{array}{l} u = 0 \\ \dfrac{du}{dn} = 0 \end{array}\right\}, \qquad \left.\begin{array}{l} \nabla^2 u = 0 \\ \dfrac{d\nabla^2 u}{dn} = 0 \end{array}\right\}, \qquad \left.\begin{array}{l} u = 0 \\ \nabla^2 u = 0 \end{array}\right\}, \qquad \left.\begin{array}{l} \dfrac{du}{dn} = 0 \\ \dfrac{d\nabla^2 u}{dn} = 0 \end{array}\right\}$$

The second pair would seem the most likely for a free edge, but it is found to lead to an impossibility. Since the first and third pairs are obviously inadmissible, Mathieu concludes that the fourth pair of equations must be those which really express the condition of a free edge. In his belief in this result he is not shaken by the fact that the corresponding conditions for the free end of a bar would be $du/dx = 0$, $d^3u/dx^3 = 0$, the first of which is contradicted by the roughest observation of the vibration of a large tuning-fork.

[1] *Crelle*, t. XL. p. 51. Ueber das Gleichgewicht und die Bewegung einer elastischen Scheibe.

[2] *Liouville*, t. XIV. 1869.

The fact is that although any of the four pairs of equations would secure the evanescence of the boundary integral in (1), it does not follow conversely that the integral can be made to vanish in no other way; and such a conclusion is negatived by Kirchhoff's investigation. There are besides innumerable other cases in which the integral in question would vanish, all that is really necessary being that the boundary appliances should be either at rest, or devoid of inertia.

225. The vibrations of a rectangular plate, whose edge is *supported*, may be easily investigated theoretically, the normal functions being identical with those applicable to a membrane of the same shape, whose boundary is fixed. If we assume

$$w = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos pt \qquad (1),$$

we see that at all points of the boundary,

$$w = 0, \qquad d^2w/dx^2 = 0, \qquad d^2w/dy^2 = 0,$$

which secure the fulfilment of the necessary conditions (§ 215). The value of p, found by substitution in $c^4\nabla^4 w = p^2 w$,

is
$$p = c^2\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \qquad (2),$$

shewing that the analogy to the membrane does not extend to the sequence of tones.

It is not necessary to repeat here the discussion of the primary and derived nodal systems given in Chapter IX. It is enough to observe that if two of the fundamental modes (1) have the same period in the case of the membrane, they must also have the same period in the case of the plate. The derived nodal systems are accordingly identical in the two cases.

The generality of the value of w obtained by compounding with arbitrary amplitudes and phases all possible particular solutions of the form (1) requires no fresh discussion.

Unless the contrary assertion had been made, it would have seemed unnecessary to say that the nodes of a *supported* plate have nothing to do with the ordinary Chladni's figures, which belong to a plate whose edges are free.

The realization of the conditions for a supported edge is scarcely attainable in practice. Appliances are required capable of holding the boundary of the plate at rest, and of such a nature that they give rise to no couples about tangential axes. We may conceive the plate to be held in its place by friction against the walls of a cylinder circumscribed closely round it.

226. The problem of a rectangular plate, whose edges are free, is one of great difficulty, and has for the most part resisted attack[1]. If we suppose that the displacement w is independent of y, the general differential equation becomes identical with that with which we were concerned in Chapter VIII. If we take the solution corresponding to the case of a bar whose ends are free, and therefore satisfying $d^2w/dx^2 = 0$, $d^3w/dx^3 = 0$, when $x = 0$ and when $x = a$, we obtain a value of w which satisfies the general fferential equation, as well as the pair of boundary equations

$$\left.\begin{aligned} \frac{d}{dx}\left\{\frac{d^2w}{dx^2} + (2-\mu)\frac{d^2w}{dy^2}\right\} &= 0 \\ \frac{d^2w}{dx^2} + \mu\frac{d^2w}{dy^2} &= 0 \end{aligned}\right\} \qquad (1),$$

which are applicable to the edges parallel to y; but the second boundary condition for the other pair of edges, namely

$$\frac{d^2w}{dy^2} + \mu\frac{d^2w}{dx^2} = 0 \qquad (2),$$

will be violated, unless $\mu = 0$. This shews that, except in the case reserved, it is not possible for a free rectangular plate to vibrate after the manner of a bar; unless indeed as an approximation, when the length parallel to one pair of edges is so great that the conditions to be satisfied at the second pair of edges may be left out of account.

Although the constant μ (which expresses the ratio of lateral contraction to longitudinal extension when a bar is drawn out) is positive for every known substance, in the case of a few substances—cork, for example—it is comparatively very small. There is, so far as we know, nothing absurd in the idea of a substance

[1] [The case where two opposite edges are free while the other two edges are supported, has been discussed by Voigt (*Göttingen Nachrichten*, 1893) p. 225.]

for which μ vanishes. The investigation of the problem under this condition is therefore not devoid of interest, though the results will not be strictly applicable to ordinary glass or metal plates, for which the value of μ is about $\frac{1}{3}$.[1]

If u_1, u_2, &c. denote the normal functions for a free bar investigated in Chapter VIII., corresponding to 2, 3, ... nodes, the vibrations of a rectangular plate will be expressed by

$$w = u_1\,(x/a), \quad w = u_2\,(x/a), \ \&c.,$$

or

$$w = u_1\,(y/b), \quad w = u_2\,(y/b), \ \&c.$$

In each of these primitive modes the nodal system is composed of straight lines parallel to one or other of the edges of the rectangle. When $b = a$, the rectangle becomes a square, and the vibrations

$$w = u_n\,(x/a), \quad w = u_n\,(y/a),$$

having necessarily the same period, may be combined in any proportion, while the whole motion still remains simple harmonic. Whatever the proportion may be, the resulting nodal curve will of necessity pass through the points determined by

$$u_n\,(x/a) = 0, \quad u_n\,(y/a) = 0.$$

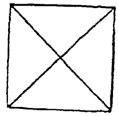
Now let us consider more particularly the case of $n = 1$. The nodal system of the primitive mode, $w = u_1\,(x/a)$, consists of a pair of straight lines parallel to y, whose distance from the nearest edge is $\cdot 2242\,a$. The points in which these lines are met by the corresponding pair for $w = u_1\,(y/a)$, are those through which the nodal curve of the compound vibration must in all cases pass. It is evident that they are symmetrically disposed on the diagonals of the square. If the two primitive vibrations be taken equal, but in opposite phases (or, algebraically, with equal and opposite amplitudes), we have

$$w = u_1\,(x/a) - u_1\,(y/a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

[1] In order to make a plate of material, for which μ is not zero, vibrate in the manner of a bar, it would be necessary to apply constraining couples to the edges parallel to the plane of bending to prevent the assumption of a contrary curvature. The effect of these couples would be to raise the pitch, and therefore the calculation founded on the type proper to $\mu = 0$ would give a result somewhat higher in pitch than the truth.

from which it is evident that w vanishes when $x = y$, that is along the diagonal which passes through the origin. That w will also vanish along the other diagonal follows from the symmetry of the functions, and we conclude that the nodal system of (3) comprises both the diagonals (Fig. 41). This is a well-known mode of vibration of a square plate.

Fig. 41.



A second notable case is when the amplitudes are equal and their phases the same, so that

$$w = u_1 (x/a) + u_1 (y/a) \ldots \qquad (4).$$

The most convenient method of constructing graphically the curves, for which $w = \text{const.}$, is that employed by Maxwell in similar cases. The two systems of curves (in this instance straight lines) represented by $u_1 (x/a) = \text{const.}$, $u_1 (y/a) = \text{const.}$, are first laid down, the values of the constants forming an arithmetical progression with the same common difference in the two cases. In this way a network is obtained which the required curves cross diagonally. The execution of the proposed plan requires an inversion of the table given in Chapter VIII., § 178, expressing the march of the function u_1, of which the result is as follows:—

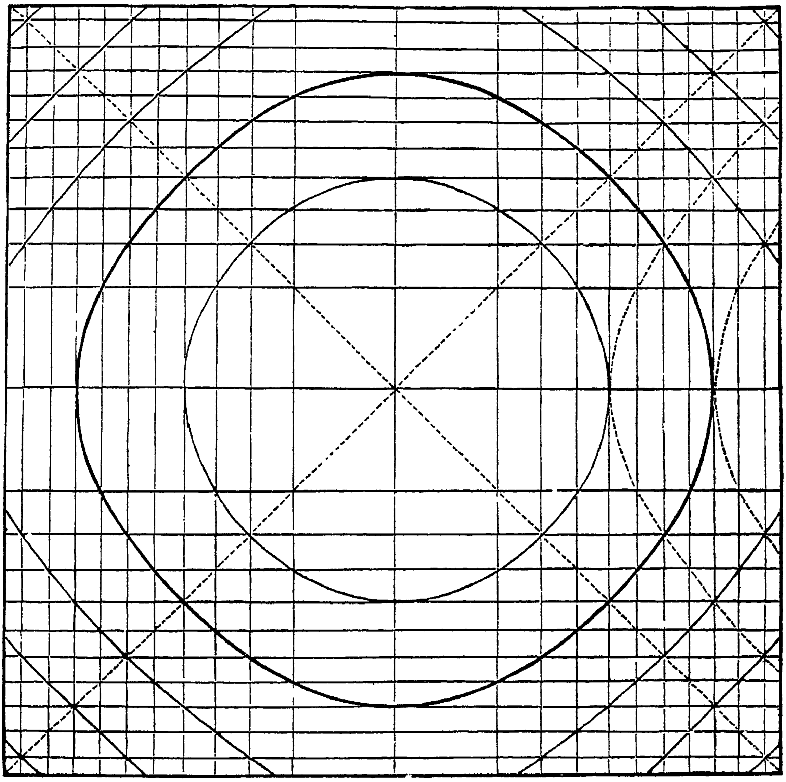
u_1	$x : a$	u_1	$x : a$
+ 1·00	·5000	− ·25	·1871
·75	·3680	·50	·1518
·50	·3106	·75	·1179
·25	·2647	1·00	·0846
·00	·2242	1·25	·0517
		− 1·50	·0190

The system of lines represented by the above values of x (completed symmetrically on the further side of the central line) and the corresponding system for y are laid down in Fig. 42. From these the curves of equal displacement are deduced. At the centre of the square we have w a maximum and equal to 2 on the scale adopted. The first curve proceeding outwards is the locus of points at which $w = 1$. The next is the nodal line, separating the regions of opposite displacement. The remaining curves taken in order give the displacements -1, -2, -3. The numerically greatest negative displacement occurs at the corners of the square, where it amounts to $2 \times 1{\cdot}645 = 3{\cdot}290$.[1]

The nodal curve thus constructed agrees pretty closely with the observations of Strehlke[2]. His results, which refer to three carefully worked plates of glass, are embodied in the following polar equations:

$$r = \begin{matrix} \cdot 40143 \\ \cdot 40143 \\ \cdot 4019 \end{matrix} + \left.\begin{matrix} \cdot 0171 \\ \cdot 0172 \\ \cdot 0168 \end{matrix}\right\} \cos 4t + \left.\begin{matrix} \cdot 00127 \\ \cdot 00127 \\ \cdot 0013 \end{matrix}\right\} \cos 8t,$$

Fig. 42.



the centre of the square being pole. From these we obtain for the radius vector parallel to the sides of the square ($t = 0$) ·41980,

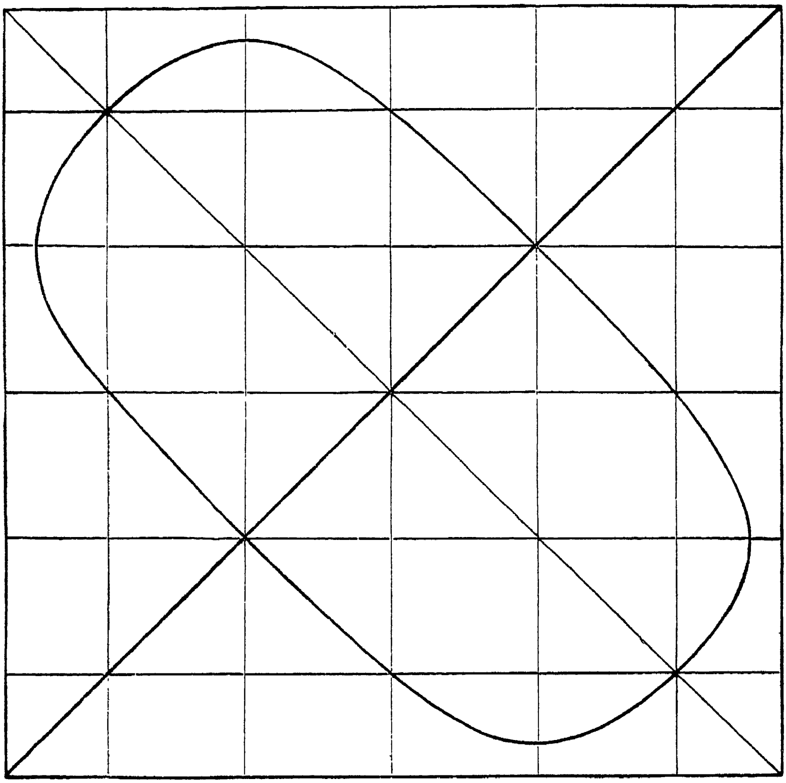
[1] On the nodal lines of a square plate. *Phil. Mag.* August, 1873.

[2] Pogg. *Ann.* Vol. CXLVI. p. 319. 1872.

·41981, ·4200, while the calculated result is ·4154. The radius vector measured along a diagonal is ·3856, ·3855, ·3864, and by calculation ·3900.

By crossing the network in the other direction we obtain the locus of points for which $u_1(x/a) - u_1(y/a)$ is constant, which are the curves of constant displacement for that mode in which the diagonals are nodal. The *pitch* of the vibration is (according to theory) the same in both cases.

Fig. 43.



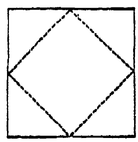
The primitive modes represented by $w = u_2(x/a)$ or $w = u_2(y/a)$ may be combined in like manner. Fig. 43 shews the nodal curve for the vibration

$$w = u_2(x/a) \pm u_2(y/a) \quad\text{.....................(5).}$$

The form of the curve is the same relatively to the other diagonal, if the sign of the ambiguity be altered.

227. The method of superposition does not depend for its application on any particular form of normal function. Whatever the form may be, the mode of vibration, which when $\mu = 0$ passes into that just discussed, must have the same period, whether the approximately straight nodal lines are parallel to x or to y. If the two synchronous vibrations be superposed, the resultant has still the same period, and the general course of its nodal system may be traced by means of the consideration that no point of the plate can be nodal at which the primitive vibrations have the same sign. To determine exactly the line of compensation, a complete knowledge of the primitive normal functions, and not merely of the points at which they vanish, would in general be necessary. Doctor Young and the brothers Weber appear to have had the idea of superposition as capable of giving rise to new varieties of vibration, but it is to Sir Charles Wheatstone[1] that we owe the first systematic application of it to the explanation of Chladni's figures. The results actually obtained by Wheatstone are however only very roughly applicable to a plate, in consequence of the form of normal function implicitly assumed. In place of Fig. 42 (itself, be it remembered, only an approximation) Wheatstone finds for the node of the compound vibration the inscribed square shewn in Fig. 44. This form is really applicable, not to a plate vibrating in virtue of rigidity, but to a stretched membrane, so supported that every point of the circumference is free to move along lines perpendicular to the plane of the membrane. The boundary condition applicable under these circumstances is $\frac{dw}{dn} = 0$; and it is easy to shew that the normal functions which involve only one co-ordinate are

Fig. 44.



$$w = \cos(m\pi x/a), \text{ or } w = \cos(m\pi y/a),$$

the origin being at a corner of the square. Thus the vibration

$$w = \cos\frac{2\pi x}{a} + \cos\frac{2\pi y}{a} \qquad\qquad (1)$$

has its nodes determined by

$$\cos\frac{\pi(x+y)}{a}\cos\frac{\pi(x-y)}{a} = 0,$$

[1] *Phil. Trans.* 1833.

whence $x+y=\frac{1}{2}a$ or $\frac{3}{2}a$, or $x-y=\pm\frac{1}{2}a$, equations which represent the inscribed square.

If
$$w=\cos\frac{2\pi x}{a}-\cos\frac{2\pi y}{a} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

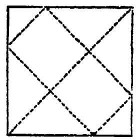
the nodal system is composed of the two diagonals. This result, which depends only on the symmetry of the normal functions, is strictly applicable to a square plate.

When $m=3$,
$$w=\cos\frac{3\pi x}{a}+\cos\frac{3\pi y}{a} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and the equations of the nodal lines are

$$x+y=\frac{a}{3},\ a,\ \frac{5a}{3};\qquad x-y=\pm\frac{a}{3},$$

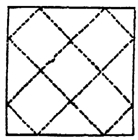
Fig. 45.



shewn in Fig. 45. If the other sign be taken, we obtain a similar figure with reference to the other diagonal.

When $m=4$,
$$w=\cos\frac{4\pi x}{a}+\cos\frac{4\pi y}{a} \ldots\ldots\ldots\ldots (4),$$

Fig. 46.



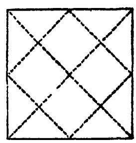
giving the nodal lines

$$x+y=\frac{a}{4},\ \frac{3a}{4},\ \frac{5a}{4},\ \frac{7a}{4},\quad x-y=\pm\frac{a}{4},\ \pm\frac{3a}{4}\ \text{(Fig. 46).}$$

With the other sign
$$w=\cos\frac{4\pi x}{a}-\cos\frac{4\pi y}{a} \quad\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

we obtain

Fig. 47.



$$x+y=\frac{a}{2},\ a,\ \frac{3a}{2},\qquad x-y=0,\ \pm\frac{a}{2}\ \text{(Fig. 47),}$$

representing a system composed of the diagonals, together with the inscribed square.

These forms, which are strictly applicable to the membrane, resemble the figures obtained by means of sand on a square plate more closely than might have been expected. The sequence of tones is however quite different. From § 176 we see that, if μ were zero, the interval between the form (43) derived from three primitive nodes, and (41) or (42) derived from two, would be

1·4629 octaves; and the interval between (41) or (42) and (46) or (47) would be 2·4358 octaves. Whatever may be the value of μ the forms (41) and (42) should have exactly the same pitch, and the same should be true of (46) and (47). With respect to the first-mentioned pair this result is not in agreement with Chladni's observations, who found a difference of more than a whole tone, (42) giving the higher pitch. If however (42) be left out of account, the comparison is more satisfactory. According to theory ($\mu=0$), if (41) gave d, (43) should give $g'-$, and (46), (47) should give $g''+$. Chladni found for (43) $g'\sharp+$, and for (46), (47) $g''\sharp$ and $g''\sharp+$ respectively.

228. The gravest mode of a square plate has yet to be considered. The nodes in this case are the two lines drawn through the middle points of opposite sides. That there must be such a mode will be shewn presently from considerations of symmetry, but neither the form of the normal function, nor the pitch, has yet been determined, even for the particular case of $\mu=0$. A rough calculation however may be founded on an assumed type of vibration.

If we take the nodal lines for axes, the form $w=xy$ satisfies $\nabla^4 w=0$, as well as the boundary conditions proper for a free edge at all points of the perimeter except the actual corners. This is in fact the form which the plate would assume if held at rest by four forces numerically equal, acting at the corners perpendicularly to the plane of the plate, those at the ends of one diagonal being in one direction, and those at the ends of the other diagonal in the opposite direction. From this it follows that $w=xy\cos pt$ would be a possible mode of vibration, if the mass of the plate were concentrated equally in the four corners. By (3) § 214, we see that

$$V=\frac{2qh^3a^2}{3(1+\mu)}\cos^2 pt \quad\text{.....................(1)},$$

inasmuch as

$$d^2w/dx^2=d^2w/dy^2=0, \qquad d^2w/dxdy=\cos pt.$$

For the kinetic energy, if ρ be the volume density, and M the additional mass at each corner,

$$T=\tfrac{1}{2}p^2\sin^2 pt\left\{\int_{-\frac{1}{2}a}^{+\frac{1}{2}a}\int_{-\frac{1}{2}a}^{+\frac{1}{2}a}2\rho h x^2y^2\,dx\,dy+\tfrac{1}{4}Ma^4\right\}$$

$$=\tfrac{1}{2}p^2\sin^2 pt\left\{\frac{\rho h a^6}{8\times 9}+\frac{a^4}{4}M\right\} \quad\text{..................(2)}.$$

Hence

$$\frac{1}{p^2} = \frac{\rho(1+\mu)a^4}{96qh^2}\left(1 + 36\frac{M}{M'}\right) \quad\text{..............}\quad (3),$$

where M' denotes the mass of the plate without the loads. This result tends to become accurate when M is relatively great; otherwise by § 89 it is sensibly less than the truth. But even when $M = 0$, the error is probably not very great. In this case we should have

$$p^2 = \frac{96qh^2}{\rho(1+\mu)a^4} \quad\text{......................}\quad (4),$$

giving a pitch which is somewhat too high. The gravest mode next after this is when the diagonals are nodes, of which the pitch, if $\mu = 0$, would be given by

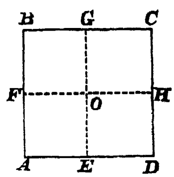
$$p'^2 = \frac{qh^2}{\rho a^4}\,\frac{(4{\cdot}7300)^4}{3} \quad\text{...................}\quad (5),$$

(see § 174).

We may conclude that if the material of the plate were such that $\mu = 0$, the interval between the two gravest tones would be somewhat greater than that expressed by the ratio 1·318. Chladni makes the interval a fifth.

229. That there must exist modes of vibration in which the two shortest diameters are nodes may be inferred from such considerations as the following. In Fig. (48) suppose that GH is a plate of which the edges HO, GO are *supported*, and the edges GC, CH free. This plate, since it tends to a definite position of equilibrium, must be capable of vibrating in certain fundamental modes. Fixing our attention on one of these, let us conceive a distribution of w over the three remaining quadrants, such that in any two that adjoin, the values of w are equal and opposite at points which are the images of each other in the line of separation. If the whole plate vibrate according to the law thus determined, no constraint will be required in order to keep the lines GE, FH fixed, and therefore the whole plate may be regarded as free. The same argument may be used to prove that modes exist in which the diagonals are nodes, or in which both the diagonals and the diameters just considered are together nodal.

Fig. 48.



The principle of symmetry may also be applied to other forms of plate. We might thus infer the possibility of nodal diameters in a circle, or of nodal principal axes in an ellipse. When the

Fig. 49.

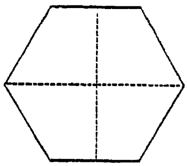


Fig. 50.

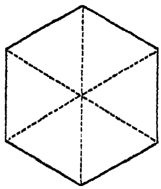
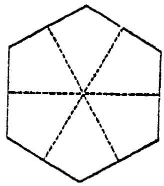


Fig. 51.



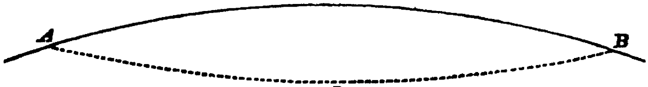
boundary is a regular hexagon, it is easy to see that Figs. (49), (50), (51) represent possible forms.

It is interesting to trace the continuity of Chladni's figures, as the form of the plate is gradually altered. In the circle, for example, when there are two perpendicular nodal diameters, it is a matter of indifference as respects the pitch and the type of vibration, in what position they be taken. As the circle develops into a square by throwing out corners, the position of these diameters becomes definite. In the two alternatives the pitch of the vibration is different, for the projecting corners have not the same efficiency in the two cases. The vibration of a square plate shewn in Fig. (42) corresponds to that of a circle when there is one circular node. The correspondence of the graver modes of a hexagon or an ellipse with those of a circle may be traced in like manner.

230. For plates of uniform material and thickness and of invariable shape, the period of the vibration in any fundamental mode varies as the square of the linear dimension, provided of course that the boundary conditions are the same in all the cases compared. When the edges are clamped, we may go further and assert that the removal of *any* external portion is attended by a rise of pitch, whether the material and the thickness be uniform, or not.

Let AB be a part of a clamped edge (it is of no consequence whether the remainder of the boundary be clamped, or not), and

Fig. 52.

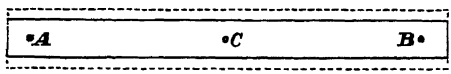


let the piece $ACBD$ be removed, the new edge ADB being also clamped. The pitch of any fundamental vibration is sharper than before the change. This is evident, since the altered vibrations might be obtained from the original system by the introduction of a constraint clamping the edge ADB. The effect of the constraint is to raise the pitch of every component, and the portion $ACBD$ being plane and at rest throughout the motion, may be removed. In order to follow the sequence of changes with greater security from error, it is best to suppose the line of clamping to advance by stages between the two positions ACB, ADB. For example, the pitch of a uniform clamped plate in the form of a regular hexagon is lower than for the inscribed circle and higher than for the circumscribed circle.

When a plate is free, it is not true that an addition to the edge always increases the period. In proof of this it may be sufficient to notice a particular case.

AB is a narrow thin plate, itself without inertia but carrying loads at A, B, C. It is clear that the addition to the breadth

Fig. 53.



indicated by the dotted line would augment the stiffness of the bar, and therefore *lessen* the period of vibration. The same consideration shews that for a uniform free plate of given area there is no lower limit of pitch; for by a sufficient elongation the period of the gravest component may be made to exceed any assignable quantity. When the edges are clamped, the form of gravest pitch is doubtless the circle.

If all the dimensions of a plate, including the thickness, be altered in the same proportion, the period is proportional to the linear dimension, as in every case of a solid body vibrating in virtue of its own elasticity.

The period also varies inversely as the square root of Young's modulus, if μ be constant, and directly as the square root of the mass of unit of volume of the substance.

231. Experimenting with square plates of thin wood whose grain ran parallel to one pair of sides, Wheatstone[1] found that the pitch of the vibrations was different according as the approximately straight nodal lines were parallel or perpendicular to the fibre of the wood. This effect depends on a variation in the flexural rigidity in the two directions. The two sets of vibrations having different periods cannot be combined in the usual manner, and consequently it is not possible to make such a plate of wood vibrate with nodal diagonals. The inequality of periods may however be obviated by altering the ratio of the sides, and then the ordinary mode of superposition giving nodal diagonals is again possible. This was verified by Wheatstone.

A further application of the principle of superposition is due to König[2]. In order that two modes of vibration may combine, it is only necessary that the periods agree. Now it is evident that the sides of a rectangular plate may be taken in such a ratio, that (for instance) the vibration with two nodes parallel to one pair of sides may agree in pitch with the vibration having three nodes parallel to the other pair of sides. In such a case new nodal figures arise by composition of the two primary modes of vibration.

232. When the plate whose vibrations are to be considered is naturally curved, the difficulties of the question are generally much increased. But there is one case in which the complication due to curvature is more than compensated by the absence of a free edge; and this case happens to be of considerable interest, being the best representative of a bell which admits of simple analytical treatment.

A long cylindrical shell of circular section and uniform thickness is evidently capable of vibrations of a flexural character in which the axis remains at rest and the surface cylindrical, while the motion of every part is perpendicular to the generating lines. The problem may thus be treated as one of two dimensions only, and depends upon the consideration of the potential and kinetic energies of the various deformations of which the section is capable. The same analysis also applies to the corresponding vibrations of a ring, formed by the revolution of a small closed area about an external axis (§ 192 a).

[1] *Phil. Trans.* 1833.

[2] Pogg. *Ann.* 1864, cxxii. p. 238.

The cylinder, or ring, is susceptible of two classes of vibrations depending respectively on extensibility and flexural rigidity, and analogous to the longitudinal and lateral vibrations of straight bars. When, however, the cylinder is thin, the forces resisting bending become small in comparison with those by which extension is opposed; and, as in the case of straight bars, the vibrations depending on bending are graver and more important than those which have their origin in longitudinal rigidity. In the limiting case of an infinitely thin shell (or ring), the flexural vibrations become independent of any extension of the circumference as a whole, and may be calculated on the supposition that each part of the circumference retains its natural length throughout the motion.

But although the vibrations about to be considered are analogous to the transverse vibrations of straight bars in respect of depending on the resistance to flexure, we must not fall into the common mistake of supposing that they are exclusively normal. It is indeed easy to see that a motion of a cylinder or ring in which each particle is displaced in the direction of the radius would be incompatible with the condition of no extension. In order to satisfy this condition it is necessary to ascribe to each part of the circumference a tangential as well as a normal motion, whose relative magnitudes must satisfy a certain differential equation. Our first step will be the investigation of this equation.

233. The original radius of the circle being a, let the equilibrium position of any element of the circumference be defined by the vectorial angle θ. During the motion let the polar coordinates of the element become

$$r = a + \delta r, \quad \phi = \theta + \delta\theta.$$

If ds represent the arc of the deformed curve corresponding to $a\,d\theta$, we have

$$(ds)^2 = (a\,d\theta)^2 = (d\delta r)^2 + r^2 (d\theta + d\delta\theta)^2;$$

whence we find, by neglecting the squares of the small quantities δr, $\delta\theta$,

$$\frac{\delta r}{a} + \frac{d\delta\theta}{d\theta} = 0 \quad\text{.........................(1),}$$

as the required relation.

In whatever manner the original circle may be deformed at time t, δr may be expanded by Fourier's theorem in the series

$$\delta r = a\,\{A_1 \cos\theta + B_1 \sin\theta + A_2 \cos 2\theta + B_2 \sin 2\theta + \ldots \\ + A_s \cos s\theta + B_s \sin s\theta + \ldots\}\ldots\ldots\ldots\ldots(2),$$

and the corresponding tangential displacement required by the condition of no extension will be

$$\delta\theta = -A_1 \sin\theta + B_1 \cos\theta + \ldots - \frac{A_s}{s}\sin s\theta + \frac{B_s}{s}\cos s\theta - \ldots\ \ldots\ldots(3),$$

the constant that might be added to $\delta\theta$ being omitted.

If $\sigma a d\theta$ denote the mass of the element $a d\theta$, the kinetic energy T of the whole motion will be

$$T = \tfrac{1}{2}\sigma a \int_0^{2\pi} \left\{\left(\frac{d\delta r}{dt}\right)^2 + a^2\left(\frac{d\delta\theta}{dt}\right)^2\right\} d\theta$$

$$= \tfrac{1}{2}\sigma\pi a^3 \left\{2\,(\dot{A}_1^2 + \dot{B}_1^2) + \frac{5}{4}(\dot{A}_2^2 + \dot{B}_2^2) + \ldots \right. \\ \left. + \left(1 + \frac{1}{s^2}\right)(\dot{A}_s^2 + \dot{B}_s^2) + \ldots\right\}\ldots\ldots\ldots(4),$$

the products of the co-ordinates A_s, B_s disappearing in the integration.

We have now to calculate the form of the potential energy V. Let ρ be the radius of curvature of any element ds; then for the corresponding element of V we may take $\frac{1}{2}B ds\,\{\delta\,(1/\rho)\}^2$, where B is a constant depending on the material and on the thickness. Thus

$$V = \tfrac{1}{2}Ba \int_0^{2\pi} \left(\delta \frac{1}{\rho}\right)^2 d\theta \ \ldots\ldots\ldots\ldots\ldots\ldots\ (5).$$

Now

$$1/\rho = u + d^2u/d\phi^2,$$

and

$$u = \frac{1}{r} = \frac{1}{a}\{1 - A_1 \cos\phi - B_1 \sin\phi - \ldots\},$$

for in the small terms the distinction between ϕ and θ may be neglected.

Hence

$$\delta\frac{1}{\rho} = \frac{1}{a}\Sigma\,\{(s^2 - 1)\,(A_s \cos s\phi + B_s \sin s\phi)\},$$

and

$$V = \frac{B}{2a}\int_0^{2\pi} \{\Sigma (s^2 - 1)(A_s \cos s\theta + B_s \sin s\theta)\}^2 d\theta$$

$$= \pi \frac{B}{2a} \Sigma (s^2 - 1)^2 (A_s^2 + B_s^2) \quad\quad (6),$$

in which the summation extends to all positive integral values of s.

The term for which $s = 1$ contributes nothing to the potential energy, as it corresponds to a displacement of the circle as a whole, without deformation.

We see that when the configuration of the system is defined as above by the co-ordinates A_1, B_1, &c., the expressions for T and V involve only squares; in other words, these are the *normal* co-ordinates, whose independent harmonic variation expresses the vibration of the system.

If we consider only the terms involving $\cos s\theta$, $\sin s\theta$, we have by taking the origin of θ suitably,

$$\delta r = aA_s \cos s\theta, \quad \delta\theta = -\frac{A_s}{s} \sin s\theta \quad\quad (7),$$

while the equation defining the dependence of A_s upon the time is

$$\sigma a^3 \left(1 + \frac{1}{s^2}\right) \ddot{A}_s + \frac{B}{a}(s^2 - 1)^2 A_s = 0 \quad\quad (8),$$

from which we conclude that, if A_s varies as $\cos(pt - \epsilon)$,

$$p^2 = \frac{B}{\sigma a^4} \cdot \frac{s^2 (s^2 - 1)^2}{s^2 + 1} \quad\quad (9).$$

This result was given by Hoppe for a ring in a memoir published in Crelle, Bd. 63, 1871. His method, though more complete than the preceding, is less simple, in consequence of his not recognising explicitly that the motion contemplated corresponds to complete inextensibility of the circumference.

[In the application of (9) to a ring we have, § 192 a,

$$\frac{B}{\sigma} = \frac{c^2}{4} \frac{q}{\rho} \quad\quad (10),$$

where q is Young's modulus, ρ the volume density, and c the radius of the circular section. For the cylindrical shell, (18) § 235 g,

$$\frac{B}{\sigma} = \frac{4mnh^2}{3(m+n)\rho} \quad \text{.....................} \quad (11),$$

$2h$ denoting the thickness, and m, n the elastic constants in Thomson and Tait's notation.]

According to Chladni the frequencies of the tones of a ring are as

$$3^2 : 5^2 : 7^2 : 9^2 \ldots\ldots\ldots$$

If we refer each tone to the gravest of the series, we find for the ratios characteristic of the intervals

2·778, 5·445, 9, 13·44, &c.

The corresponding numbers obtained from the above theoretical formula (9), by making s successively equal to 2, 3, 4. &c.. are

2·828, 5·423, 8·771, 12·87, &c.,

agreeing pretty nearly with those found experimentally.

[Observations upon the tones of thin metallic cylinders, open at one end, have been made by Fenkner[1]. Since the pitch proved to be very nearly independent of the height of the cylinders, the vibrations may be regarded as approximately two-dimensional. In accordance with (9), (11), Fenkner found the frequency proportional to the thickness directly, and to the square of the radius inversely. As regards the sequence of tones from a given cylinder[2], the numbers, referred to the gravest ($s = 2$) as unity, were 2·67, 5·00, 8·00, 12·00, &c. The agreement with (9) would be improved if these numbers were raised by about $\frac{1}{12}$ part, equivalent to an alteration in the pitch of the gravest tone.

The influence of rotation of the shell about its axis has been examined by Bryan[3]. It appears that the nodes are carried round, but with an angular velocity less than that of the rotation. If the latter be denoted by ω, the nodal angular velocity is

$$\frac{s^2-1}{s^2+1}\,\omega.]$$

1 *Wied. Ann.* vol. 8, p. 185, 1879.

2 Melde, *Akustik*, Leipzig, 1883, p. 223.

3 *Proc. Camb. Phil. Soc*, vol. VII. p. 101, 1890.

234. When $s=1$, the frequency is zero, as might have been anticipated. The principal mode of vibration corresponds to $s=2$, and has four nodes, distant from each other by 90^0. These so-called nodes are not, however, places of absolute rest, for the tangential motion is there a maximum. In fact the tangential vibration at these points is half the maximum normal motion. In general for the s^{th} term the maximum tangential motion is $(1/s)$ of the maximum normal motion, and occurs at the nodes of the latter.

When a bell-shaped body is sounded by a blow, the point of application of the blow is a place of maximum normal motion of the resulting vibrations, and the same is true when the vibrations are excited by a violin-bow, as generally in lecture-room experiments. Bells of glass, such as finger-glasses, are however more easily thrown into regular vibration by friction with the wetted finger carried round the circumference. The pitch of the resulting sound is the same as of that elicited by a tap with the soft part of the finger; but inasmuch as the tangential motion of a vibrating bell has been very generally ignored, the production of sound in this manner has been felt as a difficulty. It is now scarcely necessary to point out that the effect of the friction is in the first instance to excite tangential motion, and that the point of application of the friction is the place where the tangential motion is greatest, and therefore where the normal motion vanishes.

235. The existence of tangential vibration in the case of a bell was verified in the following manner. A so-called air-pump receiver was securely fastened to a table, open end uppermost, and set into vibration with the moistened finger. A small chip in the rim, reflecting the light of a candle, gave a bright spot whose motion could be observed with a Coddington lens suitably fixed. As the finger was carried round, the line of vibration was seen to revolve with an angular velocity double that of the finger; and the amount of excursion (indicated by the length of the line of light), though variable, was finite in every position. There was, however, some difficulty in observing the correspondence between the momentary direction of vibration and the situation of the point of excitement. To effect this satisfactorily it was found necessary to apply the friction in the neighbourhood of one point. It then became evident that the spot moved tangentially when the bell was excited at points distant therefrom 0, 90, 180, or 270 degrees.; and normally when the friction was applied at the intermediate points corresponding to 45, 135, 225 and 315 degrees. Care is sometimes required in order to make the bell vibrate in its gravest mode without sensible admixture of overtones.

If there be a small load at any point of the circumference, a slight augmentation of period ensues, which is different according as the loaded point coincides with a node of the normal or of the tangential motion, being greater in the latter case than in the former. The sound produced depends therefore on the place of excitation; in general both tones are heard, and by interference give rise to *beats*, whose frequency is equal to the difference between the frequencies of the two tones. This phenomenon may often be observed in the case of large bells.

235 *a*. In determining the number of nodal meridians ($2s$) corresponding to any particular tone of a bell, advantage may be taken of beats, whether due to accidental irregularities or introduced for the purpose by special loading (compare §§ 208, 209). By tapping cautiously round a circle of *latitude* the places may be investigated where the beats disappear, owing to the absence of one or other of the component tones. But here a decision must not be made too hastily. The inaudibility of the beats may be favoured by an unsuitable position of the ear or of the mouth of the resonator used in connection with the ear. By travelling round, a situation is soon found where the observation can be made to the best advantage. In the neighbourhood of the place where the blow is being tried there is a loop of the vibration which is most excited and a (coincident) node of the vibration which is least excited. When the ear is opposite to a node of the first vibration, and therefore to a loop of the second, the original inequality is redressed, and distinct beats may be heard even though the deviation of the blow from a nodal point may be very small. The accurate determination in this way of two consecutive places where no beats are generated is all that is absolutely necessary for the purpose in view. The ratio of the entire circumference of the circle of latitude to the arc between the points in question is in fact $4s$. Thus, if the arc between consecutive points proved to be 45°, we should infer that we were dealing with the case of $s = 2$, in which the deformation is elliptical. As a greater security against error, it is advisable in practice to determine a larger number of points where no beats occur. Unless the deviation from symmetry be considerable, these points should be uniformly distributed along the circle of latitude[1].

In the above process for determining nodes we are supposed to hear distinctly the tone corresponding to the vibration under investigation. For this purpose the beats are of assistance in directing the attention; but in dealing with the more difficult subjects, such as church bells, it is advisable to have recourse to resonators. A set of v. Helmholtz's pattern, as manufactured by König, are very convenient. The one next higher in pitch to the tone under examination is chosen and tuned by advancing the finger across the aperture. Without the security afforded by resonators, the determination of the octave is very uncertain.

The only class of bells, for which an approximate theory can be given, are those with thin walls, §§ 233, 235 *c*. Of such the following glass bells may be regarded as examples :—

I. *c*′, *e*″♭, *c*‴♯.
II. *a*, *c*″♯, *b*″.
III. *f*′♯, *b*″.

The value of s for the gravest tone was 2, for the second 3, and for the third tone 4.

Similar observations have been made upon a so-called hemispherical bell, of nearly uniform thickness, and weighing about 3 cwt. Four tones could be plainly heard,

e♭, *f*′♯, *e*″, *b*″,

the pitch being taken from a harmonium. The gravest tone has a long duration. When the bell is struck by a hard body, the higher tones are at first predominant, but after a time they die away, and leave *e*♭ in possession of the field. If the striking body be soft, the original preponderance of the higher elements is less marked.

By the method described there was no difficulty in shewing that the four tones correspond respectively to $s = 2, 3, 4, 5$. Thus for the gravest tone the vibration is elliptical with 4 nodal meridians, for the next tone there are 6 nodal meridians, and so on.

[1] The bells, or gongs, as they are sometimes called, of striking clocks often give disagreeable beats. A remedy may be found in a suitable rotation of the bell round its axis.

Tapping along a meridian shewed that the sounds became less clear as the edge was departed from, and this in a continuous manner with no suggestion of a nodal circle of latitude. A question to which we shall recur in connection with church bells here suggests itself. Which of the various coexisting tones characterizes the pitch of the bell as a whole? It would appear to be the third in order, for the founders gave the pitch as E natural.

In church bells there is great concentration of metal at the "sound-bow" where the clapper strikes, indeed to such an extent that we can hardly expect much correspondence with what occurs in the case of thin uniform bells. But the method already described suffices to determine the number of nodal meridians for all the more important tones. From a bell of 6 cwt. by Mears and Stainbank 6 tones could be obtained, viz.:

$$\begin{array}{cccccc} e', & c'', & f''+, & b''\flat, & d''', & f'''. \\ (4) & (4) & (6) & (6) & (8) & \end{array}$$

The pitch of this bell as given by the makers is d'', so that it is the fifth in the above series of tones which characterizes the bell. The number of nodal meridians in the various components is indicated within the parentheses. Thus in the case of the tone e' there are 4 nodal meridians. A similar method of examination along a meridian shewed that there was no nodal circle of latitude. At the same time differences of intensity were observed. This tone is most fully developed when the blow is delivered about midway between the crown and the rim of the bell.

The next tone is c''. Observation shewed that for this vibration also there are four, and but four, nodal meridians. But now there is a well-defined nodal circle of latitude, situated about a quarter of the way up from the rim towards the crown. As heard with a resonator, this tone disappears when the blow is accurately delivered at some point of this circle, but revives with a very small displacement on either side. The nodal circle and the four meridians divide the surface into segments, over each of which the normal motion is of one sign.

To the tone f'' correspond 6 nodal meridians. There is no well-defined nodal circle. The sound is indeed very faint when the tap is much displaced from the sound-bow; it was thought to fall to a minimum when a position about half-way up was reached.

The three graver tones are heard loudly from the sound-bow. But the next in order, $b''\flat$, is there scarcely audible, unless the blow is delivered to the rim itself in a tangential direction. The maximum effect occurs about half-way up. Tapping round the circle revealed 6 nodal meridians.

The fifth tone, d''', is heard loudly from the sound-bow, but soon falls off when the locality of the blow is varied, and in the upper three-fourths of the bell it is very faint. No distinct circular node could be detected. Tapping round the circumference shewed that there were 8 nodal meridians.

The highest tone recorded, f''', was not easy of observation, and the mode of vibration could not be fixed satisfactorily.

Similar results have been obtained from a bell of 4 cwt., cast by Taylor of Loughborough for Ampton church. The nominal pitch (without regard to octave) was *d*, and the following were the tones observed:—

$$\underset{(4)}{e'\flat - 2}, \quad \underset{(4)}{d'' - 6}, \quad \underset{(6)}{f'' + 4}, \quad \underset{(6)}{b''\flat\text{—}b''}, \quad \underset{(8)}{d'''}, \quad g'''.$$

In the specification of pitch the numerals following the note indicate by how much the frequency for the bell differed from that of the harmonium employed as a standard. Thus the gravest tone $e'\flat$ gave 2 beats per second, and was flat. When the number exceeds 3, it is the result of somewhat rough estimation, and cannot be trusted to be quite accurate. Moreover, as has been explained, there are in strictness two frequencies under each head, and these often differ sensibly. In the case of the 4th tone, $b''\flat$—b'' means that, as nearly as could be judged, the pitch of the bell was midway between the two specified notes of the harmonium.

Observations in the laboratory upon the above-mentioned bells having settled the modes of vibration corresponding to the five gravest tones, other bells of the church pattern could be sufficiently investigated by simple determinations of pitch. The results are collected in the following table[1], and include, besides those already given, observations upon a Belgian bell, the property of Mr Haweis, and upon the five bells of the Terling peal. As regards

[1] On Bells, *Phil. Mag.*, vol. 29, p. 1, 1890.

the nominal pitch of the latter bells, several observers concurred in fixing the notes of the peal as

f♯, *g*♯, *a*♯, *b*, *c*♯,

no attention being paid to the question of the octave.

Mears, 1888.	Ampton, 1888.	Belgian Bell.	Terling (5), Osborn, 1783.	Terling (4), Mears, 1810.	Terling (3), Graye, 1623.	Terling (2), Gardner, 1723.	Terling (1), Warner, 1863.
Actual Pitch by Harmonium.							
e'	*e'*♭ − 2	*d'* − 4	*g* − 3	*a* + 3	*a*♯ + 3	*d'* − 6	*d'* + 2
c''	*d''* − 6	*c''*♯ − *d''*	*g'* − 4	*g'*♯ − 4	*a'* + 6	*a'*♯ − 5	*b'* + 2
f'' +	*f''* + 4	*f''* + 1	*a'* + 6	*b'* + 6	*c''*♯ + 4	*d''* + 8	*e''*
b''♭	*b''*♭ − *b''*	*a''* − 6	*d''* − 3	*d''*♯ − *e''*	*e''* + 6	*g'*♯ + (10)	*g''*♯ + 4
d'''	*d'''*	......	*f''*♯ − 2	*g''*♯ − 6	*a''*♯	*b''* + 2	*c'''*♯ + 3
f'''	*g'''*						
Pitch referred to fifth tone as *c*.							
d	*c*♯ − 2		*c*♯ − 3	*c*♯ + 3	*c* + 3	*e*♭ − 6	*c*♯ + 2
b♭	*c* − 6		*c*♯ − 4	*c* − 4	*b*♭ + 6	*b* − 5	*b*♭ + 2
e♭ +	*e*♭ + 4		*e*♭ + 6	*e*♭ + 6	*e*♭ + 4	*e*♭ + 8	*e*♭
a♭	*a*♭ − *a*		*a*♭ − 3	*g* − *g*♯	*f*♯ + 6	*a* + 8	*g* + 4
c	*c*		*c* − 2	*c* − 6	*c*	*c* + 2	*c* + 3

Examination of the table reveals the remarkable fact that in every case of the English bells it is the 5th tone in order which agrees with the nominal pitch, and that, with the exception of Terling (4), no other tone shews such agreement[1]. Moreover, as appeared most clearly in the case of the bell cast by Mears and Stainbank, the nominal pitch, as given by the makers, is *an octave below* the only corresponding tone.

The highly composite, and often discordant, character of the sounds of bells tends to explain the discrepancies sometimes manifested in estimations of pitch. Mr Simpson, who has devoted much attention to the subject, has put forward strong arguments for the opinion that the Belgian makers determine the pitch of their bells by the tone 2nd in order in the above series, so that for instance the pitch of Terling (3) would be *a* and not *a*♯. In subordination to this tone they pay attention also to the next (the 3rd in order), classifying their bells according to the character

[1] In this comparison the gravest tone is disregarded.

of the *third*, whether major or minor, so compounded. Thus in Terling (3) the interval, a' to c'' , is a *major* third. The comparative neglect with which the Belgians treat the 5th tone, regarded almost exclusively by English makers, may perhaps be explained by a less prominent development of this tone in Belgian bells, and by a difference in treatment. When a bell is sounded alone, or with other bells in a comparatively slow succession, attention is likely to concentrate itself upon the graver and more persistent elements of the sound rather than upon the acuter and more evanescent elements, while the contrary may be expected to occur when bells follow one another rapidly in a peal.

In any case the false octaves with which the Table abounds are simple facts of observation, and we may well believe that their correction would improve the general effect. Especially should the octave between the 2nd tone and the 5th tone be made true. Probably the lower octave of the gravest, or *hum-note,* as it is called by English founders, is of less importance. The same may be said of the *fifth,* given by the 4th tone of the series, which is much less prominent. The variations recorded in the Table would seem to shew that no insuperable obstacle stands in the way of obtaining accurate harmonic relations among the various tones.

No adequate explanation has been given of the form adopted for church bells. It appears both from experiment and from the theory of thin shells that this form is especially stiff, as regards the principal mode of deformation ($s = 2$), to forces applied normally and near the rim. Possibly the advantage of this form lies in its rendering less prominent the gravest component of the sound, or the hum-note.

CHAPTER X A.

CURVED PLATES OR SHELLS.

235 b. In the last chapter (§§ 232, 233) we have considered the comparatively simple problem of the vibration in two dimensions of a cylindrical shell, so far at least as relates to vibrations of a *flexural* character. The shell is supposed to be thin, to be composed of isotropic material, and to be bounded by infinite coaxal cylindrical surfaces. It is proposed in the present chapter to treat the problem of the cylindrical shell more generally, and further to give the theory of the flexural vibrations of spherical shells.

In considering the deformation of a thin shell the most important question which presents itself is whether the middle surface, viz. the surface which lies midway between the boundaries, does, or does not, undergo extension. In the former case the deformation may be called *extensional*, and its potential energy is proportional to the thickness of the shell, which will be denoted by $2h$. Since the inertia of the shell, and therefore the kinetic energy of a given motion, is also proportional to h, the frequencies of vibration are in this case *independent* of h, § 44. On the other hand, when no line traced upon the middle surface undergoes extension, the potential energy of a deformation is of a higher order in the small quantity h. If the shell be conceived to be divided into laminæ, the extension in any lamina is proportional to its distance from the middle surface, and the contribution to the potential energy is proportional to the square of that distance. When the integration over the thickness is carried out, the whole potential energy is found to be proportional to h^3. Vibrations of this kind may be called inextensional, or flexural, and (§ 44) their frequencies are proportional to h, so that the sounds become graver without limit as the thickness is reduced.

Vibrations of the one class may thus be considered to depend upon the term of order h, and vibrations of the other class upon the term of order h^3, in the expression for the potential energy. In general both terms occur; and it is only in the limit that the separation into two classes becomes absolute. This is a question which has sometimes presented difficulty. That in the case of extensional vibrations the term in h^3 should be negligible in comparison with the term in h seems reasonable enough. But is it permissible in dealing with the other class of vibrations to omit the term in h while retaining the term in h^3?

The question may be illustrated by consideration of a statical problem. It is a general mechanical principle (§ 74) that, if given displacements (not sufficient by themselves to determine the configuration) be produced in a system originally in equilibrium by forces of corresponding types, the resulting deformation is determined by the condition that the potential energy shall be as small as possible. Apply this principle to the case of an elastic shell, the given displacements being such as not of themselves to involve a stretching of the middle surface. The resulting deformation will, in general, include both stretching and bending, and any expression for the energy will be of the form

$$Ah\,(\text{extension})^2 + Bh^3\,(\text{bending})^2 \quad\ldots\ldots\ldots\ldots\ldots(1).$$

This energy is to be as small as possible. Hence, when the thickness is diminished without limit, the actual displacement will be one of pure bending, if such there be, consistent with the given conditions.

At first sight it may well appear strange that of the two terms the one proportional to the cube of the thickness is to be retained, while that proportional to the first power may be neglected. The fact, however, is that the large potential energy that would accompany any stretching of the middle surface is the very reason why such stretching does not occur. The comparative largeness of the coefficient (proportional to h) is more than neutralized by the smallness of the stretching itself, to the square of which the energy is proportional.

An example may be taken from the case of a rod, clamped at one end *A*, and deflected by a lateral force; it is required to trace the effect of constantly increasing stiffness of the part included between *A* and a neighbouring point *B*. In the limit we may regard the rod as clamped at *B*, and neglect the energy of the part *AB*, in spite of, or rather in consequence of, its infinite stiffness.

It would thus be a mistake to regard the omission of the term in h as especially mysterious. In any case of a constraint which is supposed to be gradually introduced (§ 92 *a*), the vibrations tend to arrange themselves into two classes, in one of which the constraint is observed, while in the other, in which the constraint is violated, the frequencies increase without limit. The analogy with the shell of gradually diminishing thickness is complete if we suppose that at the same time the elastic constants are increased in such a manner that the resistance to *bending* remains unchanged. The resistance to extension then becomes infinite, and in the limit one class of vibrations is purely inextensional, or flexural.

In the investigation which we are about to give of the vibrations of a cylindrical shell, the extensional and the inextensional classes will be considered separately. It would apparently be more direct to establish in the first instance a general expression for the potential energy complete as far as the term in h^3, from which the whole theory might be deduced. Such an expression would involve the extensions and the curvatures of the middle surface. It appears, however, that this method is difficult of application, inasmuch as the potential energy (correct to h^3) does not depend only upon the above-mentioned quantities, but also upon the manner of application of the normal forces, which are in general implied in the existence of middle surface extensions[1]

235 *c*. The first question to be considered is the expression of the conditions that the middle surface remain unextended, or if these conditions be violated, to find the values of the extensions in terms of the displacements of the various points of the surface.

[1] On the Uniform Deformation in Two Dimensions of a Cylindrical Shell, with Application to the General Theory of Deformation of Thin Shells. *Proc. Math. Soc.*, vol. xx. p. 372, 1889.

We will suppose in the first instance merely that the surface is of revolution, and that a point is determined by cylindrical co-ordinates z, r, ϕ. After deformation the co-ordinates of the above point become $z+\delta z$, $r+\delta r$, $\phi+\delta\phi$ respectively. If ds denote an element of arc traced upon the surface,

$$(ds+d\delta s)^2=(dz+d\delta z)^2+(r+\delta r)^2(d\phi+d\delta\phi)^2+(dr+d\delta r)^2,$$

so that

$$ds\,d\delta s=dz\,d\delta z+r^2d\phi\,d\delta\phi+r\delta r\,(d\phi)^2+dr\,d\delta r\ldots\ldots(1).$$

In this we regard z and ϕ as independent variables, so that, for example,

$$d\delta z=\frac{d\delta z}{dz}dz+\frac{d\delta z}{d\phi}d\phi\,;$$

while

$$dr=\frac{dr}{dz}dz+\frac{dr}{d\phi}d\phi,$$

in which by hypothesis $dr/d\phi=0$. Accordingly

$$\frac{d\delta s}{ds}=\frac{(dz)^2}{(ds)^2}\left\{\frac{d\delta z}{dz}+\frac{dr}{dz}\frac{d\delta r}{dz}\right\}+\frac{(d\phi)^2}{(ds)^2}\left\{r^2\frac{d\delta\phi}{d\phi}+r\delta r\right\}$$
$$+\frac{dz\,d\phi}{(ds)^2}\left\{\frac{d\delta z}{d\phi}+r^2\frac{d\delta\phi}{dz}+\frac{dr}{dz}\frac{d\delta r}{d\phi}\right\}\ldots\ldots\ldots\ldots(2),$$

in which $d\delta s/ds$ represents the *extension* of the element ds. If there be no extension of any arc traced upon the surface, (2) must vanish independently of any relations between dz and $d\phi$. Hence

$$\frac{d\delta z}{dz}+\frac{dr}{dz}\frac{d\delta r}{dz}=0\ldots\ldots\ldots\ldots(3),$$

$$r\frac{d\delta\phi}{d\phi}+\delta r=0\ldots\ldots\ldots\ldots(4),$$

$$\frac{d\delta z}{d\phi}+r^2\frac{d\delta\phi}{dz}+\frac{dr}{dz}\frac{d\delta r}{d\phi}=0\ldots\ldots\ldots\ldots(5).$$

From these, by elimination of δr,

$$\frac{d\delta z}{dz}-\frac{dr}{dz}\frac{d}{dz}\left(r\frac{d\delta\phi}{d\phi}\right)=0,$$

$$\frac{d\delta z}{d\phi}+r^2\frac{d\delta\phi}{dz}-r\frac{dr}{dz}\frac{d^2\delta\phi}{d\phi^2}=0\,;$$

and again, by elimination of δz,

$$\frac{d}{dz}\left(r^2\frac{d\delta\phi}{dz}\right)-r\frac{d^2r}{dz^2}\frac{d^2\delta\phi}{d\phi^2}=0\ldots\ldots\ldots\ldots(6).$$

If the distribution of thickness and the form of the boundary or boundaries be symmetrical with respect to the axis, the normal functions of the system are to be found by assuming $\delta\phi$ to be proportional to $\cos s\phi$, or $\sin s\phi$. The equation for $\delta\phi$ may then be put into the form

$$r^2 \frac{d}{dz}\left(r^2 \frac{d\delta\phi}{dz}\right) + s^2 r^3 \frac{d^2 r}{dz^2}\delta\phi = 0 \quad\text{.............. (7).}$$

It will be seen that the conditions of inextension go a long way towards determining the form of the normal functions.

The simplest application is to the case of a *cylinder* for which r is constant, equal say to a. Thus (3), (4), (5), (7) become simply

$$\frac{d\delta z}{dz} = 0, \qquad \delta r + a\frac{d\delta\phi}{d\phi} = 0, \qquad \frac{d\delta z}{d\phi} + a^2\frac{d\delta\phi}{dz} = 0 \quad\text{......(8),}$$

$$\frac{d^2\delta\phi}{dz^2} = 0 \quad\text{.......................... (9).}$$

By (9), if $\delta\phi \propto \cos s\phi$, we may take

$$a\,\delta\phi = (A_s a + B_s z)\cos s\phi \quad\text{... ....(10),}$$

and then, by (8),

$$\delta r = s\,(A_s a + B_s z)\sin s\phi \quad\text{..............(11),}$$

$$\delta z = -s^{-1} B_s a \sin s\phi \quad\text{....................(12).}$$

Corresponding terms, with fresh arbitrary constants, obtained by writing $s\phi + \frac{1}{2}\pi$ for $s\phi$, may of course be added. If $B_s = 0$, the displacement is in two dimensions only (§ 233).

If an inextensible disc be attached to the cylinder at $z = 0$, so as to form a kind of cup, the displacements δr and $\delta\phi$ must vanish for that value of z, exception being made of the case $s = 1$. Hence $A_s = 0$, and

$$a\,\delta\phi = B_s z\cos s\phi, \quad \delta r = sB_s z\sin s\phi, \quad \delta z = -s^{-1}B_s a\sin s\phi \quad\text{...(13).}$$

Again, in the case of a *cone*, for which $r = \tan\gamma\,.\,z$, the equations (3), (4), (5), (7) become

$$\left.\begin{aligned} \frac{d\delta z}{dz} + \tan\gamma\frac{d\delta r}{dz} = 0, \quad z\tan\gamma\frac{d\delta\phi}{d\phi} + \delta r = 0 \\ \frac{d\delta z}{d\phi} + z^2\tan^2\gamma\frac{d\delta\phi}{dz} + \tan\gamma\frac{d\delta r}{d\phi} = 0 \end{aligned}\right\}\quad\text{......(14),}$$

$$\frac{d}{dz}\left(z^2\frac{d\delta\phi}{dz}\right) = 0 \quad\text{......................(15).}$$

If we take, as usual, $\delta\phi \propto \cos s\phi$, we get as the solution of (15)

$$\delta\phi = (A_s + B_s z^{-1}) \cos s\phi \qquad (16),$$

and corresponding thereto

$$\delta r = s \tan\gamma\,(A_s z + B_s) \sin s\phi \qquad (17),$$

$$\delta z = \tan^2\gamma\,[s^{-1} B_s - s\,(A_s z + B_s)] \sin s\phi \qquad (18).$$

If the cone be complete up to the vertex at $z = 0$, $B_s = 0$, so that

$$\delta\phi = A_s \cos s\phi \qquad (19),$$

$$\delta r = s A_s\, r \sin s\phi \qquad (20),$$

$$\delta z = - s A_s \tan\gamma\, r \sin s\phi \qquad (21).$$

For the cone and the cylinder, the second term in the general equation (7) vanishes. We shall obtain a more extensive class of soluble cases by supposing that the surface is such that

$$r^3 \frac{d^2 r}{dz^2} = \text{constant} \qquad (22),$$

an equation which is satisfied by surfaces of the second degree in general. If

$$\frac{z^2}{a^2} + \frac{r^2}{b^2} = 1 \qquad (23),$$

we shall find

$$r^3 \frac{d^2 r}{dz^2} = - \frac{b^4}{a^2} \qquad (24);$$

and thus (7) takes the form

$$\frac{d^2 \delta\phi}{d\alpha^2} - \frac{s^2 b^4}{a^2} \delta\phi = 0 \qquad (25),$$

if $\delta\phi \propto \cos s\phi$, and α is defined by

$$\alpha = \int r^{-2}\, dz \qquad (26),$$

or in the present case

$$\alpha = \frac{a}{2b^2} \log \frac{a+z}{a-z} \qquad (27).$$

The solution of (25) is

$$\delta\phi = \left[A \left(\frac{a+z}{a-z}\right)^{-\frac{1}{2}s} + B \left(\frac{a+z}{a-z}\right)^{+\frac{1}{2}s} \right] \cos s\phi \qquad (28).$$

The corresponding values of δr and δz are to be obtained from (4) and (5).

If the surface be complete through the vertex $z=a$, the term multiplied by B must disappear. Thus, omitting the constant multiplier, we may take

$$\delta\phi=\left(\frac{a-z}{a+z}\right)^{\frac{1}{2}s}\cos s\phi \qquad (29);$$

whence, by (4), (5),

$$\delta r=\frac{sb}{a}\frac{(a-z)^{\frac{1}{2}s+\frac{1}{2}}}{(a+z)^{\frac{1}{2}s-\frac{1}{2}}}\sin s\phi \qquad (30),$$

$$\delta z=(sz+a)\frac{b^2(a-z)^{\frac{1}{2}s}}{a^2(a+z)^{\frac{1}{2}s}}\sin s\phi \qquad (31).$$

If we measure z' from the vertex, $z'=a-z$, and we may write

$$\delta\phi=\left(\frac{z'}{r}\right)^s\cos s\phi \qquad (32),$$

$$\delta r=sr\left(\frac{z'}{r}\right)^s\sin s\phi \qquad (33),$$

$$\delta z=-\delta z'=\frac{b^2}{a^2}\left\{(s+1)\,a-sz'\right\}\left(\frac{z'}{r}\right)^s\sin s\phi \qquad (34).$$

For the parabola, a and b are infinite, while $b^2/a=2a'$, and $r^2=4a'z'$. Thus we may take[1]

$$\delta\phi=r^s\cos s\phi,\quad \delta r=sr^{s+1}\sin s\phi,\quad \delta z'=-2\,(s+1)\,a'r^s\sin s\phi \qquad (35).$$

We will now take into consideration the important case of the sphere, for which in (23) $b=a$. Denoting by θ the angle between the radius vector and the axis, we have $z=a\cos\theta$, $r=a\sin\theta$, and thus from (29), (30), (31)

$$\delta\phi=\cos s\phi\tan^s\tfrac{1}{2}\theta \qquad (36),$$

$$\delta r/a=s\sin s\phi\sin\theta\tan^s\tfrac{1}{2}\theta \qquad (37),$$

$$\delta z/a=(1+s\cos\theta)\sin s\phi\tan^s\tfrac{1}{2}\theta \qquad (38).$$

The other terms of the complete solution, corresponding to (28), are to be obtained by changing the sign of s.

In the above equations the displacements are resolved parallel and perpendicular to the axis $\theta=0$. It would usually be more convenient to resolve along the normal and the meridian. If the components in these directions be denoted by w and $a\,\delta\theta$, we have

$$w=\delta r\sin\theta+\delta z\cos\theta,\quad a\delta\theta=\delta r\cos\theta-\delta z\sin\theta;$$

[1] On the Infinitesimal Bending of Surfaces of Revolution. *Proc. Math. Soc.*, vol. XIII. p. 4, 1881.

so that altogether

$$\delta\phi = \cos s\phi\,[A_s \tan^s \tfrac{1}{2}\theta + B_s \cot^s \tfrac{1}{2}\theta] \qquad (39),$$

$$\delta\theta = -\sin s\phi \sin\theta\,[A_s \tan^s \tfrac{1}{2}\theta - B_s \cot^s \tfrac{1}{2}\theta] \qquad (40),$$

$$w/a = \sin s\phi\,[A_s (s+\cos\theta) \tan^s \tfrac{1}{2}\theta + B_s (s-\cos\theta) \cot^s \tfrac{1}{2}\theta] \qquad (41).$$

To the above may be added terms derived by writing $s\phi + \frac{1}{2}\pi$ for $s\phi$, and changing the arbitrary constants.

235 *d*. We now proceed to apply the equations of § 235 *c* to the principal extensions of a cylindrical surface, with a view to the formation of the expression for the potential energy. The axial and circumferential extensions will be denoted respectively by ϵ_1, ϵ_2, and the shear by ϖ. The first of these is given by (2) § 235 *c*, if we suppose that $d\phi = 0$, $dz/ds = 1$. Since in the case of a cylinder $dr/dz = 0$, we find

$$\epsilon_1 = \frac{d\delta z}{dz} \qquad (1).$$

In like manner

$$\epsilon_2 = \frac{\delta r}{a} + \frac{d\delta\phi}{d\phi} \qquad (2).$$

The value of the shear may be arrived at by considering the difference of extensions for the two diagonals of an infinitesimal square whose sides are dz and $a\,d\phi$. It is

$$\varpi = \frac{1}{a}\frac{d\delta z}{d\phi} + a\,\frac{d\delta\phi}{dz} \qquad (3).$$

The next part of the problem, viz. the expression of the potential energy by means of ϵ_1, ϵ_2, ϖ, appertains to the general theory of elasticity, and can only be treated here in a cursory manner. But it may be convenient to give the leading steps of the investigation, referring for further explanations to the treatises of Thomson and Tait and of Love. In the notation of the former (*Natural Philosophy*, § 694) the general equations in three dimensions are

$$na = S, \quad nb = T, \quad nc = U \qquad (4),$$

$$\left.\begin{aligned} Me &= P - \sigma(Q+R) \\ Mf &= Q - \sigma(R+P) \\ Mg &= R - \sigma(P+Q) \end{aligned}\right\} \qquad (5),$$

where

$$\sigma = \frac{m-n}{2m} \qquad (6)^1.$$

[1] M is Young's modulus, σ is Poisson's ratio, n is the constant of rigidity, and $(m - \frac{1}{3}n)$ that of compressibility.

The energy w, corresponding to unit of volume, is given by

$$2w = (m+n)(e^2+f^2+g^2) + 2(m-n)(fg+ge+ef) + n(a^2+b^2+c^2) \ldots\ldots (7).$$

In the application to a lamina, supposed parallel to the plane xy, we are to take $R=0$, $S=0$, $T=0$, so that

$$g = -\sigma\frac{e+f}{1-\sigma}, \quad a=0, \quad b=0 \ldots\ldots\ldots\ldots\ldots (8).$$

Thus in terms of the extensions e, f, parallel to x, y, and of the shear c, we get

$$w = n\left\{e^2+f^2+\frac{m-n}{m+n}(e+f)^2+\tfrac{1}{2}c^2\right\} \ldots\ldots\ldots\ldots (9).$$

This is the energy reckoned per unit of volume. In order to adapt the expression to our purposes, we must multiply it by the thickness ($2h$). Hence as the energy per unit area of a shell of thickness $2h$, we may take in the notation adopted at the commencement of this section,

$$2nh\left\{\epsilon_1^2+\epsilon_2^2+\tfrac{1}{2}\varpi^2+\frac{m-n}{m+n}(\epsilon_1+\epsilon_2)^2\right\} \ldots\ldots\ldots\ldots (10).$$

This expression may be applied to curved as well as to plane plates, for any modification due to curvature must involve higher powers of h. The same is true of the energy of bending.

235 *e.* We are now prepared for the investigation of the extensional vibrations of an infinite cylindrical shell, assumed to be periodic with respect both to z and to ϕ. It will be convenient to denote by single letters the displacements parallel to z, ϕ, r; we take

$$\delta z = u, \quad a\delta\phi = v, \quad \delta r = w \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

These functions are to be assumed proportional to the sines or cosines of jz/a and $s\phi$. Various combinations may be made, of which an example[1] is

$$u = U\cos s\phi\cos jz/a, \quad v = V\sin s\phi\sin jz/a,$$
$$w = W\cos s\phi\sin jz/a \ldots\ldots (2);$$

so that (1), (2), (3), § 235 *d*

$$a.\epsilon_1 = -jU\cos s\phi\sin jz/a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$
$$a.\epsilon_2 = (W+sV)\cos s\phi\sin jz/a \ldots\ldots\ldots\ldots\ldots\ldots (4),$$
$$a.\varpi = (-sU+jV)\sin s\phi\cos jz/a \ldots\ldots\ldots\ldots\ldots (5).$$

[1] Additions of $\frac{1}{2}\pi$ to $s\phi$, or to jz/a, or to both, may of course be made at pleasure.

The potential energy per unit area is thus (10) § 235 *d*

$$2nha^{-2}\left[\cos^2 s\phi \sin^2 jz/a\left\{j^2U^2+(W+sV)^2+\frac{m-n}{m+n}(W+sV-jU)^2\right\}\right.$$
$$\left.+\tfrac{1}{2}\sin^2 s\phi\cos^2 jz/a\,(-sU+jV)^2\right]\ldots\ldots(6).$$

Again, if ρ be the volume density, the kinetic energy per unit of area is

$$\rho h\left[\left(\frac{dU}{dt}\right)^2\cos^2 s\phi\cos^2 jz/a+\left(\frac{dV}{dt}\right)^2\sin^2 s\phi\sin^2 jz/a\right.$$
$$\left.+\left(\frac{dW}{dt}\right)^2\cos^2 s\phi\sin^2 jz/a\right]\ldots\ldots(7).$$

In the integration of (6), (7) with respect to z and ϕ, $\frac{1}{2}$ is the mean value of the square of each sine or cosine.[1] We may then apply Lagrange's method, regarding U, V, W as independent generalized co-ordinates. If the type of vibration be $\cos pt$, and $p^2\rho/n = k^2$, the resulting equations may be written

$$\{2(N+1)j^2+s^2-k^2a^2\}\,U-(2N+1)jsV-2NjW=0\ldots(8),$$
$$-(2N+1)jsU+\{j^2+2(N+1)s^2-k^2a^2\}\,V+2(N+1)sW=0\ldots(9),$$
$$-2NjU+2(N+1)sV+\{2(N+1)-k^2a^2\}\,W=0\ldots(10),$$

where $$N=\frac{m-n}{m+n}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

The frequency equation is that expressing the evanescence of the determinant of this triad of equations. On reduction it may be written

$$[k^2a^2-j^2-s^2]\,\{k^2a^2[k^2a^2-2(N+1)(j^2+s^2+1)]$$
$$+4(2N+1)j^2\}+4(2N+1)j^2s^2=0\ldots\ldots(12).$$[2]

These equations include of course the theory of the extensional vibrations of a plane plate, for which $a=\infty$. In this application it is convenient to write $a\phi=y$, $s/a=\beta$, $j/a=\gamma$. The displacements are then

$$u=U\cos\beta y\cos\gamma z,\quad v=V\sin\beta y\sin\gamma z,\quad w=W\cos\beta y\sin\gamma z$$
$$\ldots(13).$$

[1] In the physical problem of a simple cylinder the range of integration for ϕ is from 0 to 2π; but mathematically we are not confined to one revolution. We may conceive the shell to consist of several superposed convolutions, and then s is not limited to be a whole number.

[2] Note on the Free Vibrations of an infinitely long Cylindrical Shell. *Proc. Roy. Soc.*, vol. 45, p. 446, 1889.

When a is made infinite while β, γ remain constant, the equations (10), (8), (9) ultimately assume the form $W=0$, and

$$\{2(N+1)\gamma^2+\beta^2-k^2\}U-(2N+1)\gamma\beta V=0\ \ldots(14),$$

$$-(2N+1)\gamma\beta U+\{\gamma^2+2(N+1)\beta^2-k^2\}V=0\ \ldots(15):$$

and the determinantal equation (12) becomes

$$k^2[k^2-\gamma^2-\beta^2][k^2-2(N+1)(\gamma^2+\beta^2)]=0\ldots\ldots(16).$$

In (16), as was to be expected, k^2 appears as a function of $(\beta^2+\gamma^2)$. The first root $k^2=0$ relates to flexural vibrations, not here regarded. The second root is

$$k^2=\beta^2+\gamma^2\ \ldots\ldots\ldots\ldots(17),$$

or

$$p^2=\frac{n}{\rho}(\beta^2+\gamma^2)\ldots\ldots\ldots\ldots(18).$$

At the same time (14) gives

$$\gamma U-\beta V=0\ \ldots\ldots\ldots\ldots(19).$$

These vibrations involve only a shearing of the plate in its own plane. For example, if $\gamma=0$, the vibration may be represented by

$$u=\cos\beta y\cos pt,\quad v=0,\quad w=0\ldots\ldots\ldots\ldots(20).$$

The third root of (16)

$$k^2=2(N+1)(\beta^2+\gamma^2)=\frac{4m}{m+n}(\beta^2+\gamma^2)\ \ldots\ldots\ (21)$$

gives

$$p^2=\frac{4mn}{m+n}\frac{\beta^2+\gamma^2}{\rho}\ \ldots\ldots\ldots\ldots\ (22).$$

The corresponding relation between U and V is

$$\beta U+\gamma V=0\ \ldots\ldots\ldots\ldots(23).$$

A simple example of this case is given by supposing in (13), (23), $\beta=0$. We may take

$$u=\cos\gamma z\ \cos pt,\quad v=0,\quad w=0\ \ldots\ldots\ldots\ (24),$$

the motion being in one dimension.

Reverting to the cylinder we will consider in detail a few particular cases of importance. The first arises when $j=0$, that is, when the vibrations are independent of z. The three equations (8), (9), (10) then reduce to

$$(s^2-k^2a^2)U=0\ldots\ldots\ldots\ldots(25),$$

$$\{2(N+1)s^2-k^2a^2\}V+2(N+1)sW=0\ldots\ldots\ldots(26),$$

$$2(N+1)sV+\{2(N+1)-k^2a^2\}W=0\ldots\ldots\ldots(27);$$

and they may be satisfied in two ways. First let $V = W = 0$: then U may be finite, provided

$$s^2 - k^2a^2 = 0 \quad\text{.......................} (28).$$

The corresponding type for u is

$$u = \cos s\phi \cos pt \quad\text{....................} (29),$$

where

$$p^2 = \frac{ns^2}{\rho a^2} \quad\text{..........................} (30).$$

In this motion the material is sheared without dilatation of area or volume, every generating line of the cylinder moving along its own length. The frequency depends upon the circumferential wave-length, and not upon the curvature of the cylinder.

The second kind of vibrations are those for which $U = 0$, so that the motion is strictly in two dimensions. The elimination of the ratio V/W from (26), (27) gives

$$k^2a^2\{k^2a^2 - 2(N+1)(1+s^2)\} = 0 \quad\text{...........} (31),$$

as the frequency equation. The first root is $k^2 = 0$, indicating infinitely slow motion. The modes in question are flexural, for which, according to our present reckoning, the potential energy is evanescent. The corresponding relation between V and W is by (26)

$$sV + W = 0 \quad\text{......................} (32),$$

giving in (3), (4), (5),

$$\epsilon_1 = 0, \quad \epsilon_2 = 0, \quad \varpi = 0.$$

The other root of (31) is

$$k^2a^2 = 2(N+1)(1+s^2) \quad\text{.................} (33)$$

or

$$p^2 = \frac{4mn}{m+n}\frac{1+s^2}{a^2\rho} \quad\text{.....................} (34);$$

while the relation between V and W is

$$V - sW = 0 \quad\text{......................} (35).$$

The type of the motion may be taken to be

$$u = 0, \quad v = s\sin s\phi \cos pt, \quad w = \cos s\phi \cos pt \quad\text{......} (36).$$

It will be observed that when s is very large, the flexural vibrations (32) tend to become exclusively radial, and the extensional vibrations (35) tend to become exclusively tangential.

Another important class of vibrations are those which are characterized by symmetry round the axis, for which accordingly $s=0$. The general frequency equation (12) reduces in this case to

$$\{k^2a^2-j^2\}\{k^2a^2[k^2a^2-2(N+1)(j^2+1)]+4(2N+1)j^2\}=0 \quad \text{...(37).}$$

Corresponding to the first root we have $U=0$, $W=0$, as is readily proved on reference to the original equations (8), (9), (10) with $s=0$. The vibrations are the purely torsional ones represented by

$$u=0, \quad v=\sin(jz/a)\cos pt, \quad w=0 \quad \text{...........(38),}$$

where

$$p^2=\frac{nj^2}{\rho a^2} \quad \text{........................(39).}$$

The frequency depends upon the wave-length parallel to the axis, and not upon the radius of the cylinder.

The remaining roots of (37) correspond to motions for which $V=0$, or which take place in planes passing through the axis. The general character of these vibrations may be illustrated by the case where j is small, so that the wave-length is a large multiple of the radius of the cylinder. We find approximately from the quadratic which gives the remaining roots

$$\frac{k^2a^2}{N+1}=2+\frac{2N^2j^2}{(N+1)^2} \quad \text{...................(40),}$$

or

$$k^2a^2=\frac{2(2N+1)j^2}{N+1} \quad \text{....................(41).}$$

The vibrations of (40) are almost purely radial. If we suppose that j actually vanishes, we fall back upon

$$k^2a^2=2(N+1) \quad \text{......................(42),}$$

and

$$p^2=\frac{4mn}{m+n}\frac{1}{a^2\rho} \quad \text{..............(43)[1],}$$

obtainable from (33), (34) on introduction of the condition $s=0$. The type of vibration is now

$$u=0, \quad v=0, \quad w=\cos pt \quad \text{.................(44).}$$

[1] This equation was given by Love in a memoir "On the Small Free Vibrations and Deformation of a thin Elastic Shell," *Phil. Trans.*, vol. 179 (1888) p. 523, and also by Chree, Cambridge *Phil. Trans.* vol. xiv, p. 250, 1887.

On the other hand, the vibrations of (41) are ultimately purely axial. As the type we may take

$$u = \cos jz/a \,.\, \cos pt, \quad v = 0, \quad w = \frac{m-n}{2m} j \sin jz/a \,.\, \cos pt \ldots (45),$$

where
$$p^2 = \frac{3m-n}{m} \frac{nj^2}{\rho a^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (46).$$

Now, if q denote Young's modulus, we have, § 214,

$$q = n(3m-n)/m,$$

so that
$$p^2 = \frac{qj^2}{\rho a^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47).$$

Thus u satisfies the equation

$$\frac{d^2u}{dt^2} = \frac{q}{\rho} \frac{d^2u}{dz^2},$$

which is the usual formula (§ 150) for the longitudinal vibrations of a rod, the fact that the section is here a thin annulus not influencing the result to this order of approximation.

Another particular case worthy of notice arises when $s = 1$, so that (12) assumes the form

$$k^2a^2(k^2a^2 - j^2 - 1)[k^2a^2 - 2(N+1)(j^2+2)] + 4j^2(k^2a^2 - j^2)(2N+1) = 0 \ldots (48).$$

As we have already seen, if j be zero, one of the values of k^2 vanishes. If j be small, the corresponding value of k^2 is of the order j^4. Equation (48) gives in this case

$$k^2a^2 = \frac{2N+1}{N+1} j^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (49);$$

or in terms of p and q,

$$p^2 = \frac{qj^4}{2\rho a^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (50).$$

The type of vibration is

$$\left.\begin{aligned} u &= 0 \\ v &= \sin\phi \sin jz/a \,.\, \cos pt \\ w &= -\cos\phi \sin jz/a \,.\, \cos pt \end{aligned}\right\} \ldots\ldots\ldots\ldots (51),$$

and corresponds to the flexural vibrations of a rod (§ 163). In (51) v satisfies the equation

$$\frac{d^2v}{dt^2} + \frac{qa^2}{2\rho} \frac{d^4v}{dz^4} = 0,$$

in which $\frac{1}{2}a^2$ represents the square of the radius of gyration of the section of the cylindrical shell about a diameter.

This discussion of particular cases may suffice. It is scarcely necessary to add, in conclusion, that the most general deformation of the middle surface can be expressed by means of a series of such as are periodic with respect to z and ϕ, so that the problem considered is really the most general small motion of an infinite cylindrical shell.

The extensional vibrations of a cylinder of finite length have been considered by Love in his Theory of Elasticity[1] (1893), where will also be found a full investigation of the general equations of extensional deformation.

235 *f*. When a shell is deformed in such a manner that no line traced upon the middle surface changes in length, the term of order h disappears from the expression for the potential energy, and unless we are content to regard this function as zero, a further approximation is necessary. In proceeding to this the first remark that occurs is that the quality of inextension attaches only to the central lamina. Consider, for example, a portion of a cylindrical shell, which is bent so that the original curvature is increased. It is evident that while the middle lamina remains unextended, those laminæ which lie externally must be stretched, and those that lie internally must be contracted. The amount of these stretchings and contractions is proportional in the first place to the distance from the middle surface, and in the second place to the change of curvature which the middle surface undergoes. The potential energy of bending is thus a question of the *curvatures* of the middle surface. Displacements of translation or rotation, such as a rigid body is capable of, may be disregarded.

In order to take the question in its simplest form, let us refer the original surface to the normal and principal tangents at any point P as axes of co-ordinates, and let us suppose that after deformation the lines in the sheet originally coincident with the principal tangents are brought back (if necessary) so as to occupy the same positions as at first. The possibility of this will be apparent when it is remembered that, in virtue of the inextension of the sheet, the angles of intersections of all lines traced

[1] Also *Phil. Trans.* vol. 179 A, 1888.

upon it remain unaltered. The equation of the original surface in the neighbourhood of the point being

$$z = \tfrac{1}{2}\left(\frac{x^2}{\rho_1} + \frac{y^2}{\rho_2}\right) \text{..........................(1)},$$

that of the deformed surface may be written

$$z = \tfrac{1}{2}\left\{\frac{x^2}{\rho_1 + \delta\rho_1} + \frac{y^2}{\rho_2 + \delta\rho_2} + 2\tau xy\right\} \text{..............(2)}.$$

In strictness $(\rho_1 + \delta\rho_1)^{-1}$, $(\rho_2 + \delta\rho_2)^{-1}$ are the curvatures of the sections made by the planes x, y; but since the principal curvatures are a maximum and a minimum, they represent in general with sufficient accuracy the new principal curvatures, although these are to be found in slightly different planes. The condition of inextension shews that points which have the same x, y in (1) and (2) are corresponding points; and by Gauss's theorem it is further necessary that

$$\frac{\delta\rho_1}{\rho_1} + \frac{\delta\rho_2}{\rho_2} = 0 \text{..........................(3)}.$$

It thus appears that the energy of bending will depend in general upon two quantities, one giving the alterations of principal curvature, and the other τ depending upon the shift (in the material) of the principal planes.

The case of a spherical surface is in some respects exceptional. Previously to the bending there are no planes marked out as principal planes, and thus the position of these planes after bending is of no consequence. The energy depends only upon the alterations of principal curvature, and these by Gauss's theorem are equal and opposite, so that, if a denote the radius of the sphere, the new principal radii are $a + \delta\rho$, $a - \delta\rho$. If the equation of the deformed surface be

$$2z = Ax^2 + 2Bxy + Cy^2 \text{....................(4)},$$

we have
$$(a + \delta\rho)^{-1} + (a - \delta\rho)^{-1} = A + C,$$
$$(a + \delta\rho)^{-1} . (a - \delta\rho)^{-1} = AC - B^2;$$

so that
$$\left(\delta\frac{1}{\rho}\right)^2 = \tfrac{1}{4}(A - C)^2 + B^2 \text{....................(5)}.$$

We have now to express the elongations of the various laminæ of a shell when bent, and we will begin with the case where $\tau = 0$, that is, when the principal planes of curvature remain unchanged. It is evident that in this case the shear c vanishes, and we have to deal only with the elongations e and f parallel to the axes, § 235 *d*. In the section by the plane of zx, let s, s′ denote corresponding infinitely small arcs of the middle surface and of a lamina distant h from it. If ψ be the angle between the terminal normals, $s = \rho_1\psi$, $s' = (\rho_1 + h)\psi$, $s' - s = h\psi$. In the bending, which leaves s unchanged,

$$\delta s' = h\delta\psi = hs\delta(1/\rho_1).$$

Hence

$$e = \delta s'/s' = h\delta(1/\rho_1),$$

and in like manner $f = h\delta(1/\rho_2)$. Thus for the energy U per unit *area* we have

$$dU = nh^2 dh\left\{\left(\delta\frac{1}{\rho_1}\right)^2 + \left(\delta\frac{1}{\rho_2}\right)^2 + \frac{m-n}{m+n}\left(\delta\frac{1}{\rho_1} + \delta\frac{1}{\rho_2}\right)^2\right\},$$

and on integration over the whole thickness of the shell ($2h$)

$$U = \frac{2nh^3}{3}\left\{\left(\delta\frac{1}{\rho_1}\right)^2 + \left(\delta\frac{1}{\rho_2}\right)^2 + \frac{m-n}{m+n}\left(\delta\frac{1}{\rho_1} + \delta\frac{1}{\rho_2}\right)^2\right\} \text{......(6)}.$$

This conclusion may be applied at once, so as to give the result applicable to a spherical shell; for, since the original principal planes are arbitrary, they can be taken so as to coincide with the principal planes after bending. Thus $\tau = 0$; and by Gauss's theorem

$$\delta(1/\rho_1) + \delta(1/\rho_2) = 0,$$

so that

$$U = \frac{4nh^3}{3}\left(\delta\frac{1}{\rho}\right)^2 \text{......................... (7)},$$

where $\delta(1/\rho)$ denotes the change of principal curvature. Since $e = -f$, $g = 0$, the various laminæ are simply sheared, and that in proportion to their distance from the middle surface. The energy is thus a function of the constant of rigidity only.

The result (6) is applicable directly to the plane plate; but this case is peculiar in that, on account of the infinitude of ρ_1, ρ_2 (3) is satisfied without any relation between $\delta\rho_1$ and $\delta\rho_2$. Thus for a plane plate

$$U = \frac{2nh^3}{3}\left\{\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} + \frac{m-n}{m+n}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)^2\right\} \text{.......... (8)},$$

where $1/\rho_1$, $1/\rho_2$, are the two independent principal curvatures after bending[1].

[1] This will be found to agree with the value (2) § 214, expressed in a different notation.

We have thus far considered τ to vanish; and it remains to investigate the effect of the deformations expressed by

$$\delta z = \tau xy = \tfrac{1}{2}\tau(\xi^2 - \eta^2) \qquad\qquad (9),$$

where ξ, η relate to new axes inclined at 45° to those of x, y. The curvatures defined by (9) are in the planes of ξ, η, and are equal in numerical value and opposite in sign. The elongations in these directions for any lamina within the thickness of the shell are $h\tau$, $-h\tau$, and the corresponding energy (as in the case of the sphere just considered) takes the form

$$U' = \frac{4nh^3\tau^2}{3} \qquad\qquad (10).$$

This energy is to be added[1] to that already found in (6); and we get finally

$$U = \frac{2nh^3}{3}\left\{\left(\delta\frac{1}{\rho_1}\right)^2 + \left(\delta\frac{1}{\rho_2}\right)^2 + \frac{m-n}{m+n}\left(\delta\frac{1}{\rho_1} + \delta\frac{1}{\rho_2}\right)^2 + 2\tau^2\right\} \qquad (11),$$

as the complete expression of the energy, when the deformation is such that the middle surface is unextended. We may interpret τ by means of the angle χ, through which the principal planes are shifted; thus

$$\tau = 2\chi\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) \qquad\qquad (12).$$

235 *g*. We will now proceed with the calculation of the potential energy involved in the bending of a cylindrical shell. The problem before us is the expression of the changes of principal curvature and the shifts of the principal planes at any point $P(z, \phi)$ of the cylinder in terms of the displacements u, v, w. As in § 235 *f*, take as fixed co-ordinate axes the principal tangents and normal to the undisturbed cylinder at the point P, the axis of x being parallel to that of the cylinder, that of y tangential to the circular section, and that of ζ normal, measured inwards. If, as it will be convenient to do, we measure z and ϕ from the point P, we may express the undisturbed co-ordinates of a material point Q in the neighbourhood of P, by

$$x = z, \qquad y = a\phi, \qquad \zeta = \tfrac{1}{2}a\phi^2 \qquad\qquad (1).$$

[1] There are clearly no terms involving the products of τ with the changes of principal curvature $\delta(\rho_1^{-1})$, $\delta(\rho_2^{-1})$; for a change in the sign of τ can have no influence upon the energy of the deformation defined by (2).

During the displacement the co-ordinates of Q will receive the increments

$$u, \quad w\sin\phi + v\cos\phi, \quad -w\cos\phi + v\sin\phi;$$

so that after displacement

$$x = z + u, \qquad y = a\phi + w\phi + v(1 - \tfrac{1}{2}\phi^2),$$
$$\zeta = \tfrac{1}{2}a\phi^2 - w(1 - \tfrac{1}{2}\phi^2) + v\phi;$$

or, if u, v, w be expanded in powers of the small quantities z, ϕ,

$$x = z + u_0 + \frac{du}{dz_0}z + \frac{du}{d\phi_0}\phi + \ldots\ldots \quad (2),$$

$$y = a\phi + w_0\phi + v_0 + \frac{dv}{dz_0}z + \frac{dv}{d\phi_0}\phi + \ldots\ldots \quad (3),$$

$$\zeta = \tfrac{1}{2}a\phi^2 - w_0 - \frac{dw}{dz_0}z - \frac{dw}{d\phi_0}\phi + v_0\phi$$
$$+ \tfrac{1}{2}w_0\phi^2 - \tfrac{1}{2}\frac{d^2w}{dz_0^2}z^2 - \frac{d^2w}{dz_0 d\phi_0}z\phi - \tfrac{1}{2}\frac{d^2w}{d\phi_0^2}\phi^2$$
$$+ \frac{dv}{dz_0}z\phi + \frac{dv}{d\phi_0}\phi^2 \ldots\ldots \quad (4),$$

u_0, v_0, ... being the values of u, v at the point P.

These equations give the co-ordinates of the various points of the deformed sheet. We have now to suppose the sheet moved as a rigid body so as to restore the position (as far as the first power of small quantities is concerned) of points infinitely near P. A purely translatory motion by which the displaced P is brought back to its original position will be expressed by the simple omission in (2), (3), (4) of the terms u_0, v_0, w_0 respectively, which are independent of z, ϕ. The effect of an arbitrary rotation is represented by the additions to x, y, ζ respectively of $y\omega_3 - \zeta\omega_2$, $\zeta\omega_1 - x\omega_3$, $x\omega_2 - y\omega_1$; where for the present purpose ω_1, ω_2, ω_3 are small quantities of the order of the deformation, the square of which is to be neglected throughout. If we make these additions to (2), &c., substituting for x, y, ζ in the terms containing θ their approximate values, we find so far as the first powers of z, ϕ

$$x = z + \frac{du}{dz_0}z + \frac{du}{d\phi_0}\phi + a\phi\omega_3,$$

$$y = a\phi + w_0\phi + \frac{dv}{dz_0}z + \frac{dv}{d\phi_0}\phi - z\omega_3,$$

$$\zeta = -\frac{dw}{dz_0}z - \frac{dw}{d\phi_0}\phi + v_0\phi + z\omega_2 - a\phi\omega_1.$$

Now, since the sheet is assumed to be unextended, it must be possible so to determine ω_1, ω_2, ω_3 that to this order $x=z$, $y=a\phi$, $\zeta=0$. Hence

$$\frac{du}{dz_0}=0, \qquad \frac{du}{d\phi_0}+a\omega_3=0,$$

$$\frac{dv}{dz_0}-\omega_3=0, \qquad w_0+\frac{dv}{d\phi_0}=0,$$

$$-\frac{dw}{dz_0}+\omega_2=0, \qquad \frac{dw}{d\phi_0}-v_0+a\omega_1=0.$$

The conditions of inextension are thus (if we drop the suffices as no longer required)

$$\frac{du}{dz}=0, \qquad w+\frac{dv}{d\phi}=0, \qquad \frac{du}{d\phi}+a\frac{dv}{dz}=0 \text{.........(5)},$$

which agree with (8) § 235 *c*.

Returning to (2), &c., as modified by the addition of the translatory and rotatory terms, we get

$$x=z+\text{terms of 2nd order in } z,\ \phi,$$

$$y=a\phi+ \quad \text{,,} \qquad \text{,,}$$

$$\zeta=\tfrac{1}{2}a\phi^2+\tfrac{1}{2}w_0\phi^2-\tfrac{1}{2}\frac{d^2w}{dz_0^2}z^2-\frac{d^2w}{dz_0d\phi_0}z\phi$$

$$-\tfrac{1}{2}\frac{d^2w}{d\phi_0^2}\phi^2+\frac{dv}{dz_0}z\phi+\frac{dv}{d\phi_0}\phi^2;$$

or since by (5) $d^2w/dz^2=0$, and $dv/d\phi=-w$,

$$\zeta=\tfrac{1}{2}a\phi^2-\tfrac{1}{2}w_0\phi^2-\frac{d^2w}{dz_0d\phi_0}z\phi-\tfrac{1}{2}\frac{d^2w}{d\phi_0^2}\phi^2+\frac{dv}{dz_0}z\phi.$$

The equation of the deformed surface after transference is thus

$$\zeta=xy\left\{\frac{1}{a}\frac{dv}{dz_0}-\frac{1}{a}\frac{d^2w}{dz_0d\phi_0}\right\}+y^2\left\{\frac{1}{2a}-\frac{1}{2a^2}w_0-\frac{1}{2a^2}\frac{d^2w}{d\phi_0^2}\right\} \text{....(6)}.$$

Comparing with (2) § 235 *f* we see that

$$\delta\frac{1}{\rho_1}=0, \qquad \delta\frac{1}{\rho_2}=-\frac{1}{a^2}\left(w+\frac{d^2w}{d\phi^2}\right), \qquad \tau=\frac{1}{a}\left(\frac{dv}{dz}-\frac{d^2w}{dzd\phi}\right)\text{...(7)};$$

so that by (11) § 235 *f*

$$U=\frac{4nh^3}{3a^2}\left\{\frac{m}{m+n}\frac{1}{a^2}\left(w+\frac{d^2w}{d\phi^2}\right)^2+\left(\frac{dv}{dz}-\frac{d^2w}{dzd\phi}\right)^2\right\} \text{......(8)}.$$

This is the potential energy of bending reckoned per unit of area. It can, if desired, be expressed by (5) entirely in terms of v[1].

We will now apply (8) to calculate the whole potential energy of a complete cylinder, bounded by plane edges $z = \pm l$, and of thickness which, if variable at all, is a function of z only. Since u, v, w are periodic when ϕ increases by 2π, their most general expression in accordance with (5) is [compare (10), &c., § 235 *c*]

$$v = \Sigma\,[(A_s a + B_s z)\cos s\phi - (A_s' a + B_s' z)\sin s\phi] \quad\text{.........(9)},$$

$$w = \Sigma\,[s\,(A_s a + B_s z)\sin s\phi + s\,(A_s\, a + B_s' z)\cos s\phi] \quad\text{....(10)},$$

$$u = \Sigma\,[-s^{-1} B_s a \sin s\phi - s^{-1} B_s' a \cos s\phi] \quad\text{..................(11)},$$

in which the summation extends to all integral values of s from 0 to ∞. But the displacements corresponding to $s = 0$, $s = 1$ are such as a rigid body might undergo, and involve no absorption of energy. When the values of u, v, w are substituted in (8) all the terms containing products of sines or cosines with different values of s vanish in the integration with respect to ϕ, as do also those which contain $\cos s\phi \sin s\phi$. Accordingly

$$\int_0^{2\pi} U a\, d\phi = \frac{4\pi n h^3}{3a}\left[\frac{m}{m+n}\frac{1}{a^2}\Sigma\,(s^3 - s)^2 \right.$$

$$\left.\{(A_s a + B_s z)^2 + (A_s' a + B_s' z)^2\} + \Sigma\,(s^2 - 1)^2 (B_s^2 + B_s'^2)\right] \quad\text{...(12)}.$$

Thus far we might consider h to be a function of z; but we will now treat it as a constant. In the integration with respect to z the odd powers of z will disappear, and we get as the energy of the whole cylinder of radius a, length $2l$, and thickness $2h$,

$$V = \int_{-l}^{+l}\int_0^{2\pi} U a\, d\phi\, dz = \frac{8\pi n h^3 l}{3a}\,\Sigma\,(s^2 - 1)^2 \left[\frac{m\,.\,s^2}{m+n}\left\{A_s^2 + A_s'^2 \right.\right.$$

$$\left.\left. + \frac{l^2}{3a^2}(B_s^2 + B_s'^2)\right\} + B_s^2 + B_s'^2\right] \quad\text{.................(13)},$$

in which $s = 2, 3, 4, \ldots$.

The expression (13) for the potential energy suffices for the solution of statical problems. As an example we will suppose that the cylinder is compressed along a diameter by equal forces F, applied at the points $z = z_1$, $\phi = 0$, $\phi = \pi$, although it is true that so highly localised a force hardly comes within the scope of

[1] **From the general equations of Mr Love's memoir already cited a concordant result may be obtained on introduction of the appropriate conditions.**

the investigation, in consequence of the stretchings of the middle surface, which will occur in the immediate neighbourhood of the points of application[1].

The work done upon the cylinder by the forces F during the hypothetical displacement indicated by δA_s, &c., will be by (10)

$$-F\Sigma s\,(a\delta A_s' + z_1\delta B_s')\,(1+\cos s\pi),$$

so that the equations of equilibrium are

$$\frac{dv}{dA_s}=0, \qquad \frac{dv}{dB_s}=0.$$

$$\frac{dv}{dA_s'}=-(1+\cos s\pi)\,saF, \qquad \frac{dv}{dB_s'}=-(1+\cos s\pi)\,sz_1F.$$

Thus for all values of s,

$$A_s=B_s=0\,;$$

and for odd values of s, $\quad A_s'=B_s'=0.$

But when s is even,

$$\frac{ms^2}{m+n}A_s'=-\frac{3sa^2F}{8\pi nh^3l\,(s^2-1)^2}\;\dots\dots\dots(14),$$

$$\left\{\frac{ms^2}{m+n}\frac{l^2}{3a^2}+1\right\}B_s'=-\frac{3saz_1F}{8\pi nh^3l\,(s^2-1)^2}\;\dots\dots\dots(15):$$

and the displacement w at any point $(z,\,\phi)$ is given by

$$w=2\,(A_2'a+B_2'z)\cos 2\phi+4\,(A_4'a+B_4'z)\cos 4\phi+\dots(16),$$

where A_2', B_2', A_4',... are determined by (14), (15).

A further discussion of this solution will be found in the memoir[2] from which the preceding results have been taken.

We will now proceed with the calculation for the frequencies of vibration of the complete cylindrical shell of length $2l$. If the volume-density[3] be ρ, we have as the expression of the kinetic energy by means of (9), (10), (11),

$$T=\tfrac{1}{2}\,.\,2h\rho\,.\iint(\dot u^2+\dot v^2+\dot w^2)\,a\,d\phi\,dz$$

$$=2\pi\rho hla\,\Sigma\,\{a^2(1+s^2)\,(\dot A_s+\dot A_s'^2)$$

$$+\left[\tfrac{1}{3}l^2(1+s^2)+s^{-2}a^2\right](\dot B_s^2+\dot B_s'^2)\}\dots\dots\dots(17).$$

[1] Whatever the curvature of the surface, an area upon it may be taken so small as to behave like a plane, and therefore bend, in violation of Gauss's condition, when subjected to a force which is so nearly discontinuous that it varies sensibly within the area.

[2] *Proc. Roy. Soc.* vol. 45, p. 105, 1888.

[3] This can scarcely be confused with the notation for the curvature in the preceding parts of the investigation.

From the expressions for V and T in (13), (17) the types and frequencies of vibration can be at once deduced. The fact that the squares, and not the products, of A_s, B_s, are involved, shews that these quantities are really the normal co-ordinates of the vibrating system. If A_s, or A_s', vary as $\cos p_s t$, we have

$$p_s^2 = \tfrac{4}{3}\,\frac{mn}{m+n}\,\frac{h^2}{\rho a^4}\,\frac{(s^3-s)^2}{s^2+1} \quad\text{.................. (18).}$$

This is the equation for the frequencies of vibration in two dimensions, § 233. For a given material, the frequency is proportional directly to the thickness and inversely to the square on the diameter of the cylinder[1].

In like manner if B_s, or B_s', vary as $\cos p_s' t$, we find

$$p_s'^2 = \tfrac{4}{3}\,\frac{mn}{m+n}\,\frac{h^2}{\rho a^4}\,\frac{(s^3-s)^2}{s^2+1}\,\frac{1+\dfrac{3a^2}{s^2l^2}\dfrac{m+n}{m}}{1+\dfrac{3a^2}{(s^4+s^2)\,l^2}} \quad\text{.......... (19).}$$

If the cylinder be at all long in proportion to its diameter, the difference between p_s' and p_s becomes very small. Approximately in this case

$$p_s'/p_s = 1 + \frac{3a^2}{2s^2l^2}\left(\frac{m+n}{m} - \frac{1}{s^2+1}\right) \quad\text{........... (20);}$$

or, if we take $m = 2n$, $s = 2$,

$$p_2'/p_2 = 1 + \frac{39a^2}{80l^2}.$$

235 *h*. We now pass on to the consideration of spherical shells. The general theory of the extensional vibrations of a complete shell has been given by Lamb[2], but as the subject is of small importance from an acoustical point of view, we shall limit our investigation to the very simple case of symmetrical radial vibrations.

If w be the normal displacement, the lengths of all lines upon the middle surface are altered in the ratio $(a+w):a$. In calculating the potential energy we may take in (10) § 235 *d*

$$\epsilon_1 = \epsilon_2 = w/a, \quad \varpi = 0;$$

[1] There is nothing in these laws special to the cylinder. In the case of similar shells of any form, vibrating by pure bending, the frequency will be as the thicknesses and inversely as corresponding areas. If the similarity extend also to the thickness, then the frequency is inversely as the linear dimension, in accordance with the general law of Cauchy.

[2] *Proc. Lond. Math. Soc.* XIV. p. 50, 1882.

so that the energy per unit area is

$$4nh\frac{3m-n}{m+n}\frac{w^2}{a^2},$$

or for the whole sphere

$$V=4\pi a^2 . 4nh\frac{3m-n}{m+n}\frac{w^2}{a^2} \quad\quad (1).$$

Also for the kinetic energy, if ρ denote the volume density,

$$T=\tfrac{1}{2} . 4\pi a^2 . 2h . \rho . \dot{w}^2 \quad\quad (2).$$

Accordingly if $w=W\cos pt$, we have

$$p^2=\frac{4n}{a^2\rho}\frac{3m-n}{m+n} \quad\quad (3),$$

as the equation for the frequency $(p/2\pi)$.

As regards the general theory Prof. Lamb thus summarizes his results. "The fundamental modes of vibration fall into two classes. In the modes of the *First Class*, the motion at every point of the shell is wholly tangential. In the nth species of this class, the lines of motion are the contour lines of a surface harmonic S_n (Ch. XVII.), and the amplitude of vibration at any point is proportional to the value of $dS_n/d\epsilon$, where $d\epsilon$ is the angle subtended at the centre by a linear element drawn on the surface of the shell at right angles to the contour line passing through the point. The frequency $(p/2\pi)$ is determined by the equation

$$\mathrm{k}^2a^2=(n-1)(n+2) \quad\quad \text{(i)},$$

where a is the radius of the shell, and $\mathrm{k}^2=p^2\rho/\mathbf{n}$, if ρ denote the density, and $\mathbf{n}$ the rigidity, of the substance."

"In the vibrations of the *Second Class*, the motion is partly radial and partly tangential. In the nth species of this class the amplitude of the radial component is proportional to S_n, a surface harmonic of order n. The tangential component is everywhere at right angles to the contour lines of the harmonic S_n on the surface of the shell, and its amplitude is proportional to $\Lambda dS_n/d\epsilon$, where Λ is a certain constant, and $d\epsilon$ has the same meaning as before." Prof. Lamb finds

$$\Lambda=-\frac{\mathrm{k}^2a^2-4\gamma}{2n(n+1)\gamma} \quad\quad \text{(ii)},$$

where k retains its former meaning, and $\gamma=(1+\sigma)/(1-\sigma)$, σ

denoting Poisson's ratio. "Corresponding to each value of n there are *two* values of k^2a^2, given by the equation

$$k^4a^4 - k^2a^2\{(n^2+n+4)\gamma + n^2+n-2\} + 4(n^2+n-2)\gamma = 0 \ldots \text{(iii)}.$$

Of the two roots of this equation, one is < and the other > 4γ. It appears, then, from (ii) that the corresponding fundamental modes are of quite different characters. The mode corresponding to the lower root is always the more important."

"When $n=1$, the values of k^2a^2 are 0 and 6γ. The zero root corresponds to a motion of translation of the shell as a whole parallel to the axis of the harmonic S_1. In the other mode the radial motion is proportional to $\cos\theta$, where θ is the co-latitude measured from the pole of S_1; the tangential motion is along the meridian, and its amplitude (measured in the direction of θ increasing) is proportional to $\frac{1}{2}\sin\theta$.

"When $n=2$, the values of ka corresponding to various values of σ are given by the following table:—

$\sigma=0$	$\sigma=\frac{1}{4}$	$\sigma=\frac{3}{10}$	$\sigma=\frac{1}{3}$	$\sigma=\frac{1}{2}$
1·120	1·176	1·185	1·190	1·215
3·570	4·391	4·601	4·752	5·703

The most interesting variety is that of the zonal harmonic. Making $S=\frac{1}{2}(3\cos^2\theta-1)$, we see that the polar diameter of the shell alternately elongates and contracts, whilst the equator simultaneously contracts and expands respectively. In the mode corresponding to the lower root, the tangential motion is towards the poles when the polar diameter is lengthening, and *vice versâ*. The reverse is the case in the other mode. We can hence understand the great difference in frequency."

Prof. Lamb calculates that a thin glass globe 20 cm. in diameter should, in its gravest mode, make about 5350 vibrations per second.

As regards inextensional modes, their form has already been determined, (39) &c. § 235 *c*, at least for the case where the bounding curve and the thickness are symmetrical with respect to an axis, and it will further appear in the course of the present investigation. What remains to be effected is the calculation of the potential energy of bending corresponding thereto, as dependent upon the alterations of curvature of the middle surface. The process is similar to that followed in § 235 g for the case of the cylinder, and consists in finding the equation of the deformed surface when referred to rectangular axes in and perpendicular to the original surface.

The two systems of co-ordinates to be connected are the usual polar co-ordinates r, θ, ϕ, and rectangular co-ordinates x, y, ζ, measured from the point P, or (a, θ_0, ϕ_0), on the undisturbed sphere. Of these x is measured along the tangent to the meridian, y along the tangent to the circle of latitude, and ζ along the normal inwards.

Since the origin of ϕ is arbitrary, we may take it so that $\phi_0 = 0$. The relation between the two systems is then

$$x = r\{-\sin(\theta - \theta_0) + \sin\theta\cos\theta_0(1 - \cos\phi)\} \quad\quad (4),$$

$$y = r\sin\theta\sin\phi \quad\quad (5),$$

$$\zeta = -r\{\cos(\theta - \theta_0) - \sin\theta_0\sin\theta(1 - \cos\phi)\} + a \quad\quad (6).$$

If we suppose $r = a$, these equations give the rectangular co-ordinates of the point (a, θ, ϕ) on the undisturbed sphere. We have next to imagine this point displaced so that its polar co-ordinates become $a + \delta r$, $\theta + \delta\theta$, $\phi + \delta\phi$, and to substitute these values in (4), (5), (6), retaining only the first power of δr, $\delta\theta$, $\delta\phi$. Thus

$$\begin{aligned} x = (a + \delta r)&\{-\sin(\theta - \theta_0) + \sin\theta\cos\theta_0(1 - \cos\phi)\} \\ &+ a\delta\theta\{-\cos(\theta - \theta_0) + \cos\theta\cos\theta_0(1 - \cos\phi)\} \\ &+ a\delta\phi\sin\theta\cos\theta_0\sin\phi \quad\quad (7), \end{aligned}$$

$$\begin{aligned} y = (a + \delta r)&\sin\theta\sin\phi \\ &+ a\delta\theta\cos\theta\sin\phi + a\delta\phi\sin\theta\cos\phi \quad\quad (8), \end{aligned}$$

$$\begin{aligned} \zeta = a - (a + \delta r)&\{\cos(\theta - \theta_0) - \sin\theta_0\sin\theta(1 - \cos\phi)\} \\ &+ a\delta\theta\{\sin(\theta - \theta_0) + \sin\theta_0\cos\theta(1 - \cos\phi)\} \\ &+ a\delta\phi\sin\theta_0\sin\theta\sin\phi \quad\quad (9). \end{aligned}$$

These equations give the co-ordinates of any point Q of the sphere after displacement; but we shall only need to apply them in the case where Q is in the neighbourhood of P, or $(a, \theta_0, 0)$, and then the higher powers of $(\theta - \theta_0)$ and ϕ may be neglected.

In pursuance of our plan we have now to imagine the displaced and deformed sphere to be brought back as a rigid body so that the parts about P occupy as nearly as possible their former positions. We are thus in the first place to omit from (7), (8), (9) the terms (involving δ) which are independent of $(\theta-\theta_0)$, ϕ. Further we must add to each equation respectively the terms which represent an arbitrary rotation, viz.

$$y\omega_3-\zeta\omega_2,\quad \zeta\omega_1-x\omega_3,\quad x\omega_2-y\omega_1,$$

determining ω_1, ω_2, ω_3 in such a manner that, so far as the first powers of $(\theta-\theta_0)$, ϕ, there shall be coincidence between the original and displaced positions of the point Q.

If we omit all terms of the second order in $(\theta-\theta_0)$ and ϕ, we get from (7) &c.

$$x=-a(\theta-\theta_0)-\delta r_0(\theta-\theta_0)$$
$$-a\left\{[\delta\theta_0]+\frac{d\delta\theta}{d\theta_0}(\theta-\theta_0)+\frac{d\delta\theta}{d\phi_0}\phi\right\}+a\delta\phi_0\sin\theta_0\cos\theta_0\,.\,\phi \;\ldots\; (10),$$

$$y=a\sin\theta_0\,.\,\phi+\delta r_0\sin\theta_0\,.\,\phi+a\delta\theta_0\cos\theta_0\,.\,\phi$$
$$+a\sin\theta_0\left\{[\delta\phi_0]+\frac{d\delta\phi}{d\theta_0}(\theta-\theta_0)+\frac{d\delta\phi}{d\phi_0}\phi\right\}$$
$$+a\delta\phi_0\cos\theta_0(\theta-\theta_0)\;\ldots\ldots\ldots\ldots\;(11),$$

$$\zeta=[-\delta r_0]-\frac{d\delta r}{d\theta_0}(\theta-\theta_0)-\frac{d\delta r}{d\phi_0}\phi$$
$$+a\delta\theta_0(\theta-\theta_0)+a\delta\phi_0\sin^2\theta_0\,.\,\phi\;\ldots\ldots\ldots\ldots\;(12),$$

δr_0 &c. representing the values appropriate to P, where $(\theta-\theta_0)$ and ϕ vanish. The translation of the deformed surface necessary to bring P back to its original position is represented by the omission of the terms included in square brackets. The arbitrary rotation is represented by the additions respectively of

$$a\sin\theta_0\,.\,\phi\,.\,\omega_3,\quad a(\theta-\theta_0)\,\omega_3,\quad -a(\theta-\theta_0)\,\omega_2-a\sin\theta_0\,.\,\phi\,.\,\omega_1;$$

and thus the destruction of the terms of the first order requires that

$$\delta r/a+d\delta\theta/d\theta=0\;\ldots\ldots\ldots\ldots\;(13),$$
$$-d\delta\theta/d\phi+\sin\theta\cos\theta\,\delta\phi+\sin\theta\,\omega_3=0\;\ldots\ldots\;(14);$$
$$\sin\theta\,d\delta\phi/d\theta+\cos\theta\,\delta\phi+\omega_3=0\;\ldots\ldots\;(15),$$
$$(\delta r/a)\sin\theta+\delta\theta\cos\theta+\sin\theta\,d\delta\phi/d\phi=0\;\ldots\ldots\;(16);$$
$$-d\delta(r/a)/d\theta+\delta\theta-\omega_2=0\;\ldots\ldots\;(17),$$
$$-d\delta(r/a)/d\phi+\sin^2\theta\,\delta\phi-\sin\theta\,\omega_1=0\;\ldots\ldots\;(18);$$

the suffixes being omitted.

These six equations determine ω_1, ω_2, ω_3, giving as the three conditions of inextension

$$\delta r/a + d\delta\theta/d\theta = 0 \quad\text{.....................(19)},$$

$$d\delta\theta/d\phi + \sin^2\theta\, d\delta\phi/d\theta = 0 \quad\text{..............(20)},$$

$$\delta r/a + \cot\theta\,\delta\theta + d\delta\phi/d\phi = 0 \quad\text{..............(21)}.$$

From (19), (20), (21), by elimination of δr,

$$\frac{d}{d\phi}\left(\frac{\delta\theta}{\sin\theta}\right) + \sin\theta\frac{d\delta\phi}{d\theta} = 0 \quad\text{..................(22)},$$

$$\frac{d}{d\phi}\delta\phi - \sin\theta\frac{d}{d\theta}\left(\frac{\delta\theta}{\sin\theta}\right) = 0 \quad\text{..............(23)};$$

or, since $\sin\theta\, d/d\theta = d/d\log\tan\tfrac{1}{2}\theta$,

$$\frac{d}{d\phi}\left(\frac{\delta\theta}{\sin\theta}\right) + \frac{d\delta\phi}{d\log\tan\tfrac{1}{2}\theta} = 0 \quad\text{.................(24)},$$

$$\frac{d\delta\phi}{d\phi} - \frac{d}{d\log\tan\tfrac{1}{2}\theta}\left(\frac{\delta\theta}{\sin\theta}\right) = 0 \quad\text{..............(25)}.$$

From (24), (25) we see that both $\delta\phi$ and $\delta\theta/\sin\theta$ satisfy an equation of the second order of the same form, viz.

$$\frac{d^2u}{d(\log\tan\tfrac{1}{2}\theta)^2} + \frac{d^2u}{d\phi^2} = 0 \quad\text{..................(26)}.$$

If the material system be symmetrical about the axis, u is a periodic function of ϕ, and can be expanded by Fourier's theorem in a series of sines and cosines of ϕ and its multiples. Moreover each term of the series must satisfy the equations independently. Thus, if u varies as $\cos s\phi$, (26) becomes

$$\frac{d^2u}{d(\log\tan\tfrac{1}{2}\theta)^2} - s^2u = 0 \quad\text{.................(27)};$$

whence $$u = A'\tan^s\tfrac{1}{2}\theta + B'\cot^s\tfrac{1}{2}\theta \quad\text{.................(28)},$$

where A' and B' are independent of θ. If we take

$$\delta\phi = \cos s\phi\,[A_s\tan^s\tfrac{1}{2}\theta + B_s\cot^s\tfrac{1}{2}\theta] \quad\text{............(29)},$$

we get for the corresponding value of $\delta\theta$ from (24)

$$\delta\theta/\sin\theta = -\sin s\phi\,[A_s\tan^s\tfrac{1}{2}\theta - B_s\cot^s\tfrac{1}{2}\theta] \quad\text{......(30)};$$

and thence from (21)

$$\delta r/a = \sin s\phi\,[A_s(s+\cos\theta)\tan^s\tfrac{1}{2}\theta + B_s(s-\cos\theta)\cot^s\tfrac{1}{2}\theta] \quad\text{...(31)},$$

as in (39), (40), (41) § 235 *c*.

The second solution (in B_s) may be derived from the first (in A_s) in two ways which are both worthy of notice. The manner of derivation from (27) shews that it is sufficient to alter the sign of s, $\tan^s \frac{1}{2}\theta$ becoming $\cot^s \frac{1}{2}\theta$, $\sin s\phi$ becoming $-\sin s\phi$, while $\cos s\phi$ remains unchanged. The other method depends upon the consideration that the general solution must be similarly related to the two poles. It is thus legitimate to alter the first solution by writing throughout $(\pi - \theta)$ in place of θ, changing at the same time the sign of $\delta\theta$.

If we suppose $s = 1$, we get

$$\sin\theta\,\delta\phi = \cos\phi\,[A_1 + B_1 - (A_1 - B_1)\cos\theta],$$

$$\delta\theta = -\sin\phi\,[A_1 - B_1 - (A_1 + B_1)\cos\theta],$$

$$\delta r/a = \sin\phi\,[(A_1 + B_1)\sin\theta].$$

The displacement proportional to $(A_1 - B_1)$ is a rotation of the whole surface as a rigid body round the axis $\theta = \frac{1}{2}\pi$, $\phi = 0$; and that proportional to $(A_1 + B_1)$ represents a translation parallel to the axis $\theta = \frac{1}{2}\pi$, $\phi = \frac{1}{2}\pi$. The complementary translation and rotation with respect to these axes is obtained by substituting $\phi + \frac{1}{2}\pi$ for ϕ.

The two other motions possible without bending correspond to a zero value of s, and are readily obtained from the original equations (19), (20), (21). They are a rotation round the axis $\theta = 0$, represented by

$$\delta\theta = 0, \quad \delta\phi = \text{const.}, \quad \delta r = 0,$$

and a displacement parallel to the same axis represented by

$$\delta\phi = 0, \quad \frac{d}{d\theta}\left(\frac{\delta\theta}{\sin\theta}\right) = 0, \quad \frac{\delta r}{a} = -\cot\theta\,\delta\theta,$$

or

$$\delta\phi = 0, \quad \delta\theta = \gamma\sin\theta, \quad \delta r = -\gamma a\cos\theta.$$

If the sphere be complete, the displacements just considered, and corresponding to $s = 0$, 1, are the only ones possible. For higher values of s we see from (31) that δr is infinite at one or other pole, unless A_s and B_s both vanish. In accordance with Jellet's theorem[1] the complete sphere is incapable of bending.

If neither pole be included in the actual surface, which for example we may suppose bounded by circles of latitude, finite

[1] "On the Properties of Inextensible Surfaces," *Irish Acad. Trans.*, vol. 22, p. 179, 1855.

values of both A and B are admissible, and therefore necessary for a complete solution of the problem. But if, as would more often happen, one of the poles, say $\theta = 0$, is included, the constants B must be considered to vanish. Under these circumstances the solution is

$$\left.\begin{aligned} \delta\phi &= A_s \tan^s \tfrac{1}{2}\theta \cos s\phi \\ \delta\theta &= -A_s \sin\theta \tan^s \tfrac{1}{2}\theta \sin s\phi \\ \delta r &= A_s a (s + \cos\theta) \tan^s \tfrac{1}{2}\theta \sin s\phi \end{aligned}\right\} \text{............(32)},$$

to which is to be added that obtained by writing $s\phi + \frac{1}{2}\pi$ for $s\phi$, and changing the arbitrary c... it.

From (32) we see that, along those meridians for which $\sin s\phi = 0$, the displacement is tangential and in longitude only, while along the intermediate meridians for which $\cos s\phi = 0$, there is no displacement in longitude, but one in latitude, and one normal to the surface of the sphere.

Along the equator $\theta = \frac{1}{2}\pi$,

$$\delta\phi = A_s \cos s\phi, \qquad \delta\theta = -A_s \sin s\phi, \qquad \delta r/a = A_s s \sin s\phi,$$

so that the maximum displacements in latitude and longitude are equal.

Reverting now to the expressions for x, y, ζ in (7), (8), (9), with the addition of the translatory and rotatory terms by which the deformed sphere is brought back as nearly as possible to its original position, we know that so far as the terms of the first order in $(\theta - \theta_0)$ and ϕ are concerned, they are reduced to

$$x = -a(\theta - \theta_0), \qquad y = a \sin\theta_0 \,.\, \phi, \qquad \zeta = 0 \text{.........(33)}.$$

These approximations will suffice for the values of x and y; but in the case of ζ we require the expression complete so as to include all terms of the second order. The calculation is straightforward. For any displacement such as δr in (9) we write

$$\delta r_0 + \frac{d\delta r}{d\theta_0}(\theta - \theta_0) + \frac{d\delta r}{d\phi_0}\phi$$
$$+ \tfrac{1}{2}\frac{d^2\delta r}{d\theta_0^2}(\theta - \theta_0)^2 + \frac{d^2\delta r}{d\theta_0 d\phi_0}(\theta - \theta_0)\phi + \tfrac{1}{2}\frac{d^2\delta r}{d\phi_0^2}\phi^2.$$

The additional rotatory terms are by (17), (18)

$$x\left\{\delta\theta_0 - \frac{1}{a}\frac{d\delta r}{d\theta_0}\right\} + y\left\{\frac{1}{a\sin\theta_0}\frac{d\delta r}{d\phi_0} - \sin\theta_0\,\delta\phi_0\right\}.$$

In these we are to retain only those terms in x, y, which are of the second order and independent of δ, so that we may write

$$x = \tfrac{1}{2}a\phi^2 \sin\theta_0 \cos\theta_0, \quad y = a(\theta - \theta_0)\,\phi\cos\theta_0.$$

In the complete expression for ζ as a quadratic function of $(\theta - \theta_0)$ and ϕ thus obtained, we substitute x and y from (33). The final equation to the deformed surface, after simplification by the aid of (19), (20), (21), may be written

$$\zeta = \frac{x^2}{2a}\left\{1 - \frac{\delta r}{a} - \frac{1}{a}\frac{d^2\delta r}{d\theta^2}\right\} + \frac{xy}{a\sin\theta}\left\{-\frac{1}{a}\frac{d^2\delta r}{d\theta\, d\phi} + \frac{\cot\theta}{a}\frac{d\delta r}{d\phi}\right\}$$
$$+ \frac{y^2}{2a}\left\{1 - \frac{\delta r}{a} - \frac{\cot\theta}{a}\frac{d\delta r}{d\theta} - \frac{1}{a\sin^2\theta}\frac{d^2\delta r}{d\phi^2}\right\} \ldots\ldots(34),$$

the suffixes being now unnecessary.

Taking the value of $\delta r/a$ from (32) we get

$$-\frac{\delta r}{a} - \frac{1}{a}\frac{d^2\delta r}{d\theta^2} = -\frac{s^3 - s}{\sin^2\theta} A_s \tan^s \tfrac{1}{2}\theta \sin s\phi \ldots\ldots\ldots(35),$$

$$-\frac{1}{a\sin\theta}\frac{d^2\delta r}{d\theta\, d\phi} + \frac{\cos\theta}{a\sin^2\theta}\frac{d\delta r}{d\phi} = -\frac{s^3 - s}{\sin^2\theta} A_s \tan^s \tfrac{1}{2}\theta \cos s\phi \ldots.(36),$$

$$-\frac{\delta r}{a} - \frac{\cot\theta}{a}\frac{d\delta r}{d\theta} - \frac{1}{a\sin^2\theta}\frac{d^2\delta r}{d\phi^2} = \frac{s^3 - s}{\sin^2\theta} A_s \tan^s \tfrac{1}{2}\theta \sin s\phi \ldots(37).$$

To obtain the more complete solution corresponding to (31), we have only to add new terms, multiplied by B_s, and derived from the above by *changing the sign of s.* As was to be expected, the values in (35) and (37) are equal and opposite.

Introducing the values now found into (5) § 235 *f*, we obtain as the square of the change of principal curvature at any point

$$\left(\delta\frac{1}{\rho}\right)^2 = \frac{(s^3 - s)^2}{a^2\sin^4\theta}\{A_s^2 \tan^{2s}\tfrac{1}{2}\theta + B_s^2 \cot^{2s}\tfrac{1}{2}\theta - 2A_sB_s\cos 2s\phi\} \ldots(38).$$

It should be remarked that, if either A_s or B_s vanish, (38) is independent of ϕ, so that the change of principal curvature is the same for all points on a circle of latitude, and that in any case (38) becomes independent of the product A_sB_s after integration round the circumference. The change of curvature vanishes if $s = 0$, or $s = 1$, the displacement being that of which a rigid body is capable.

Equations (35) &c. shew that along the meridians where $\delta\phi$ vanishes ($\cos s\phi = 0$) the principal planes of curvature are the meridian and its perpendicular, while along the meridians where δr vanishes, the principal planes are inclined to the meridian at angles of 45°.

The value of the square of the change of curvature obtained in (38) corresponds to that assumed for the displacements in (29) &c., and for some purposes needs to be generalised. We may add terms with coefficients A_s' and B_s' corresponding to a change of $s\phi$ to $(s\phi + \frac{1}{2}\pi)$, and there is further to be considered the summation with respect to s. Putting for brevity t in place of $\tan\frac{1}{2}\theta$, we may take as the complete expression for $[\delta(1/\rho)]^2$,

$$\left[\Sigma\frac{s^3-s}{a\sin^2\theta}\{(A_s t^s + B_s t^{-s})\sin s\phi + (A_s' t^s + B_s' t^{-s})\sin(s\phi+\tfrac{1}{2}\pi)\}\right]^2$$

$$+\left[\Sigma\frac{s^3-s}{a\sin^2\theta}\{(A_s t^s - B_s t^{-s})\cos s\phi + (A_s' t^s - B_s' t^{-s})\cos(s\phi+\tfrac{1}{2}\pi)\}\right]^2$$

When this is integrated with respect to ϕ round the entire circumference, all products of the generalised co-ordinates A_s, B_s, A_s', B_s' disappear, so that (7) § 235 *f* we have as the expression for the potential energy of the surface included between two parallels of latitude

$$V = 2\pi\Sigma(s^3-s)^2\int H\sin^{-3}\theta\{(A_s^2+A_s'^2)t^{2s} + (B_s^2+B_s'^2)t^{-2s}\}\,d\theta \text{......(39)},$$

where
$$H = \tfrac{2}{3}nh^3 \text{..................................(40)}.$$

In the following applications to spherical surfaces where the pole $\theta = 0$ is included, we may omit the terms in B; and, if the thickness be constant, H may be removed from under the integral sign. We have

$$d\theta = \frac{2dt}{1+t^2}, \qquad \sin\theta = \frac{2t}{1+t^2},$$

so that

$$\int_0^\theta \sin^{-3}\theta\, t^{2s}\,d\theta = \tfrac{1}{8}\int(1+t^2)^2 t^{2s-4}\,dt^2 = \tfrac{1}{8}\left(\frac{t^{2s-2}}{s-1} + \frac{2t^{2s}}{s} + \frac{t^{2s+2}}{s+1}\right)\text{...(41)}.$$

In the case of the hemisphere $t = 1$, and (41) assumes the value

$$\frac{2s^2-1}{4(s^3-s)}\text{..........................(42)}.$$

Hence for a hemisphere of uniform thickness

$$V = \tfrac{1}{2}\pi H\Sigma(s^3-s)(2s^2-1)(A_s^2+A_s'^2)\text{.........(43)}.$$

If the extreme value of θ be 60°, instead of 90°, we get in place of (42)

$$\frac{8s^2+4s-3}{4.3^{s+1}(s^3-s)} \quad\text{..................(44)},$$

and $$V=\tfrac{1}{2}\pi H\Sigma 3^{-(s+1)}(s^3-s)(8s^2+4s-3)(A_s^2+A_s'^2)\text{...(45)}.$$

These expressions for V, in conjunction with (32), are sufficien for the solution of statical problems, relative to the deformation of infinitely thin spherical shells under the action of given impressed forces. Suppose, for example, that a string of tension F connects the opposite points on the edge of a hemisphere, represented by $\theta=\frac{1}{2}\pi$, $\phi=\frac{1}{2}\pi$ or $\frac{3}{2}\pi$, and that it is required to find the deformation. It is evident from (32) that all the quantities A_s' vanish, and that the work done by the impressed forces, corresponding to the deformation δA_s, is

$$-\delta A_s as\{\sin\tfrac{1}{2}s\pi+\sin\tfrac{3}{2}s\pi\}F.$$

If s be odd this vanishes, and if s be even it is equal to

$$-2\delta A_s as\sin\tfrac{1}{2}s\pi.F.$$

Hence if s be odd A_s vanishes; and by (43), if s be even,

$$dV/dA_s=\pi H(s^3-s)(2s^2-1)A_s=-2as\sin\tfrac{1}{2}s\pi.F;$$

whence $$A_s=-\frac{2aF\sin\frac{1}{2}s\pi}{\pi H(s^2-1)(2s^2-1)}\quad\text{..............(46)}.$$

By (46) and (32) the deformation is completely determined.

If, to take a case in which the force is tangential, we suppose that the hemisphere rests upon its pole with its edge horizontal, and that a rod of weight W is laid symmetrically along the diameter $\theta=\frac{1}{2}\pi$, we find in like manner

$$A_s=\frac{aW\sin\frac{1}{2}s\pi}{\pi H(s^2-s)(2s^2-1)}\quad\text{................(47)}$$

for all even values of s, and $A_s=0$ for all odd values of s.

We now proceed to evaluate the kinetic energy as defined by the formula

$$T=\tfrac{1}{2}\sigma a^2\iint\left\{\left(\frac{d\delta r}{dt}\right)^2+\left(\frac{a\,d\delta\theta}{dt}\right)^2+\left(\frac{a\sin\theta\,d\delta\phi}{dt}\right)^2\right\}\sin\theta\,d\theta\,d\phi\text{...(48)},$$

in which σ denotes the surface density, supposed to be uniform.

If we take the complete value of $\delta\phi$ from (29), as supplemented by the terms in A_s', B_s', we have

$$\frac{d\delta\phi}{dt} = \Sigma\left[\cos s\phi\,(\dot{A}_s t^s + \dot{B}_s t^{-s}) + \cos(s\phi + \tfrac{1}{2}\pi)\,(\dot{A}_s' t^s + \dot{B}_s' t^{-s})\right].$$

When this expression is squared and integrated with respect to ϕ round the entire circumference, all products of letters with a different suffix, and all products of dashed and undashed letters even with the same suffix, will disappear. Hence replacing $\cos^2 s\phi$ &c. by the mean value $\frac{1}{2}$, we may take

$$\sin^2\theta\left(\frac{d\delta\phi}{dt}\right)^2 = \tfrac{1}{2}\sin^2\theta\,\Sigma\,(\dot{A}_s^2 + \dot{A}_s'^2)\,t^{2s}$$
$$+ \tfrac{1}{2}\sin^2\theta\,\Sigma\,(\dot{B}_s^2 + \dot{B}_s'^2)\,t^{-2s} + \sin^2\theta\,\Sigma\,(\dot{A}_s\dot{B}_s + \dot{A}_s'\dot{B}_s').$$

The mean value (30) of $(d\delta\theta/dt)^2$ is the same as that just written with the substitution throughout of $-B$ for B, so that we may take

$$\left(\frac{d\delta\theta}{dt}\right)^2 + \left(\frac{\sin\theta\, d\delta\phi}{dt}\right)^2 = \sin^2\theta\,\Sigma\,(\dot{A}_s^2 + \dot{A}_s'^2)\,t^{2s}$$
$$+ \sin^2\theta\,\Sigma\,(\dot{B}_s^2 + \dot{B}_s'^2)\,t^{-2s} \ldots\ldots\ldots\ldots(49),$$

as the mean available for our present purpose. In (49) the products of the symbols have disappeared, and if the expression for the kinetic energy were as yet fully formed, the co-ordinates would be shewn to be *normal*. But we have still to include that part of the kinetic energy dependent upon $d\delta r/dt$. As the mean value, applicable for our purpose, we have from (31)

$$\left(\frac{d\delta r}{a\,dt}\right)^2 = \tfrac{1}{2}\Sigma\,(\dot{A}_s^2 + \dot{A}_s'^2)\,(s + \cos\theta)^2\,t^{2s}$$
$$+ \tfrac{1}{2}\Sigma\,(\dot{B}_s^2 + \dot{B}_s'^2)\,(s - \cos\theta)^2\,t^{-2s}$$
$$+ \Sigma\,(\dot{A}_s\dot{B}_s + \dot{A}_s'\dot{B}_s')\,(s^2 - \cos^2\theta) \ldots\ldots\ldots(50).$$

The expressions (49) and (50) have now to be added. If we set for brevity

$$\int \tan^{2s}\tfrac{1}{2}\theta\,\{(s + \cos\theta)^2 + 2\sin^2\theta\}\sin\theta\, d\theta = f(s) \ldots\ldots(51),$$

or putting $x = 1 + \cos\theta$,

$$f(s) = \int^2 \left(\frac{2 - x}{x}\right)^s \{(s-1)^2 + 2x(s+1) - x^2\}\, dx \ldots\ldots(52),$$

we get

$$T = \tfrac{1}{2}\pi\sigma a^4 \{\Sigma f(s)(\dot{A}_s^2 + \dot{A}_s'^2) + \Sigma f(-s)(\dot{B}_s^2 + \dot{B}_s'^2)$$

$$+ 2\Sigma \int (s^2 - \cos^2\theta)\sin\theta d\theta (\dot{A}_s\dot{B}_s + \dot{A}_s'\dot{B}_s')\} \text{ ....(53)}$$

It will be seen that, while V in (39) is expressible by the squares only of the co-ordinates, a like assertion cannot in general be made of T. Hence A_s, B_s &c. are *not* in general the normal co-ordinates. Nor could this have been expected. If, for example, we take the case where the spherical surface is bounded by two circles of latitude equidistant from the equator, symmetry shews that the normal co-ordinates are, not A and B, but $(A+B)$ and $(A-B)$. In this case $f(-s) = f(s)$.

A verification of (53) may readily be obtained in the particular case of $s=1$, the surface under consideration being the entire sphere. Dropping the dashed letters, we get

$$T = \tfrac{1}{2}\pi\sigma a^4 \{\tfrac{20}{3}(\dot{A}_1^2 + \dot{B}_1^2) + \tfrac{8}{3}\dot{A}_1\dot{B}_1\}$$

$$= \tfrac{1}{2}\pi\sigma a^4 \{4(\dot{A}_1 + \dot{B}_1)^2 + \tfrac{8}{3}(\dot{A}_1 - \dot{B}_1)^2\} \text{ ....(54).}$$

In this case the displacements are of the purely translatory and rotatory types already discussed, and the correctness of (54) may be confirmed.

Whatever may be the position of the circles of latitude by which the surface is bounded, the true types and periods of vibration are determined by the application of Lagrange's method to (39), (53).

When one pole, e.g. $\theta = 0$, is included within the surface, the co-ordinates B vanish, and A_s, A_s' become the normal co-ordinates. If we omit the dashed letters, the expression for T becomes simply

$$T = \tfrac{1}{2}\pi\sigma a^4 \Sigma f(s)\dot{A}_s^2 \text{ ....(55).}$$

From (43), (55) the frequencies of free vibrations for a hemisphere are immediately obtainable. The equation for A_s is

$$\sigma a^4 f(s)\ddot{A}_s + H(s^3 - s)(2s^2 - 1)A_s = 0 \text{ ....(56);}$$

so that, if A_s vary as $\cos p_s t$,

$$p_s^2 = \frac{H(s^3 - s)(2s^2 - 1)}{\sigma a^4 f(s)} = \frac{2nh^2}{3\rho a^4} \cdot \frac{(s^3 - s)(2s^2 - 1)}{f(s)}$$

if we introduce the value of H from (40), and express σ by means of the volume density ρ.

In like manner for the saucer of 120°, from (44),

$$p_s^2 = \frac{H(s^3-s)(8s^2+4s-3)}{\sigma a^4 f(s) \,.\, 3^{s+1}} \quad\text{................(58).}$$

The values of $f(s)$ can be calculated without difficulty in the various cases. Thus, for the hemisphere,

$$f(2) = \int_1^2 x^{-2}(4-4x+x^2)(1+6x-x^2)\,dx$$

$$= 20 \log 2 - 12\tfrac{1}{3} = 1{\cdot}52961,$$

$$f(3) = 57\tfrac{1}{3} - 80 \log 2 = 1{\cdot}88156,$$

$$f(4) = 200 \log 2 - 136\tfrac{1}{3} = 2{\cdot}29609, \text{ \&c.};$$

so that

$$p_2 = \frac{\sqrt{H}}{a^2\sqrt{\sigma}} \times 5{\cdot}2400, \quad p_3 = \frac{\sqrt{H}}{a^2\sqrt{\sigma}} \times 14{\cdot}726, \quad p_4 = \frac{\sqrt{H}}{a^2\sqrt{\sigma}} \times 28{\cdot}462.$$

In experiment, it is the *intervals* between the various tones with which we are most concerned. We find

$$p_3/p_2 = 2{\cdot}8102, \quad p_4/p_2 = 5{\cdot}4316 \quad\text{..............(59).}$$

In the case of glass bells, such as are used with air-pumps, the interval between the two gravest tones is usually somewhat smaller; the representative fraction being nearer to 2·5 than 2·8.

For the saucer of 120°, the lower limit of the integral in (52) is $\frac{3}{2}$, and we get on calculation

$$f(2) = {\cdot}12864, \quad f(3) = {\cdot}054884,$$

giving

$$p_2 = \frac{\sqrt{H}}{a^2\sqrt{\sigma}} \times 7{\cdot}9947, \quad p_3 = \frac{\sqrt{H}}{a^2\sqrt{\sigma}} \times 20{\cdot}911,$$

$$p_3 : p_2 = 2{\cdot}6157.$$

The pitch of the two gravest tones is thus decidedly higher than for the hemisphere, and the interval between them is less.

With reference to the theory of tuning bells, it may be worth while to consider the effect of a small change in the angle, for the case of a nearly hemispherical bell. In general

$$p_s^2 = \frac{4H(s^3-s)^2 \int_0^\theta \sin^{-3}\theta \tan^{2s}\tfrac{1}{2}\theta\,d\theta}{a^4\sigma \int_0^\theta \tan^{2s}\tfrac{1}{2}\theta \left\{(s+\cos\theta)^2 + 2\sin^2\theta\right\} \sin\theta\,d\theta} \quad\text{... (60).}$$

If $\theta = \frac{1}{2}\pi + \delta\theta$, and P_s denote the value of p_s for the exact hemisphere, we get from previous results

$$p_s^2 = P_s^2 \left[1 + \delta\theta \left\{\frac{4(s^3 - s)}{2s^2 - 1} - \frac{s^2 + 2}{f(s)}\right\}\right] \text{............} (61).$$

Thus

$$p_2^2 = P_2^2 \left[1 + \delta\theta \left\{\frac{24}{7} - \frac{6}{1 \cdot 52961}\right\}\right] = P_2^2 (1 - \cdot 49\, \delta\theta)$$

$$p_3^2 = P_3^2 \left[1 + \delta\theta \left\{\frac{96}{17} - \frac{11}{1 \cdot 88156}\right\}\right] = P_3^2 (1 - \cdot 20\, \delta\theta),$$

shewing that an increase in the angle depresses the pitch. As to the interval between the two gravest tones, we get

$$\left(\frac{p_3}{p_2}\right)^2 = \left(\frac{P_3}{P_2}\right)^2 \times (1 + \cdot 29\, \delta\theta),$$

shewing that it increases with θ. This agrees with the results given above for $\theta = 60°$.

The fact that the form of the normal functions is independent of the distribution of density and thickness, provided that they vary only with latitude, allows us to calculate a great variety of cases, the difficulties being merely those of simple integration. If we suppose that only a narrow belt in co-latitude θ has sufficient thickness to contribute sensibly to the potential and kinetic energies, we have simply, instead of (60),

$$p_s^2 = \frac{4H(s^3 - s)^2 \sin^{-4}\theta}{a^4\sigma\{(s + \cos\theta)^2 + 2\sin^2\theta\}} \text{............} (62),$$

whence

$$\frac{p_3}{p_2} = 4\sqrt{\left\{\frac{6 + 4\cos\theta - \cos^2\theta}{11 + 6\cos\theta - \cos^2\theta}\right\}} \text{............} (63).$$

The ratio varies very slowly from 3, when $\theta = 0$, to 2·954, when $\theta = \frac{1}{2}\pi$.

If $2h$ denote the thickness at any co-latitude θ, $H \propto h^3$, $\sigma \propto h$. I have calculated the ratio of frequencies of the two gravest tones of a hemisphere on the suppositions (1) that $h \propto \cos\theta$, and (2) that $h \propto (1 + \cos\theta)$. The formula used is that marked (60) with H and σ under the integral signs. In the first case, $p_3 : p_2 = 1{\cdot}7942$, differing greatly from the value for a uniform thickness. On the second more moderate supposition as to the law of thickness,

$$p_3 : p_2 = 2{\cdot}4591, \quad p_4 : p_2 = 4{\cdot}4837.$$

It would appear that the smallness of the interval between the gravest tones of common glass bells is due in great measure to the thickness diminishing with increasing θ.

It is worthy of notice that the curvature of deformation $\delta(\rho^{-1})$, which by (38) varies as $\sin^{-2}\theta \tan^s \frac{1}{2}\theta$, vanishes at the pole for $s = 3$ and higher values, but is finite for $s = 2$.

The present chapter has been derived very largely from various published memoirs by the author[1]. The methods have not escaped criticism, some of which, however, is obviated by the remark that the theory does not profess to be strictly applicable to shells of finite thickness, but only to the limiting case when the thickness is infinitely small. When the thickness increases, it may become necessary to take into account certain "local perturbations" which occur in the immediate neighbourhood of a boundary, and which are of such a nature as to involve extensions of the middle surface. The reader who wishes to pursue this rather difficult question may refer to memoirs by Love[2], Lamb[3], and Basset[4]. From the point of view of the present chapter the matter is perhaps not of great importance. For it seems clear that any extension that may occur must be limited to a region of infinitely small area, and affects neither the types nor the frequencies of vibration. The question of what precisely happens close to a free edge may require further elucidation, but this can hardly be expected from a theory of *thin* shells. At points whose distance from the edge is of the same order as the thickness, the characteristic properties of thin shells are likely to disappear.

[1] *Proc. Lond. Math. Soc.*, XIII. p. 4, 1881; XX. p. 372, 1889; *Proc. Roy. Soc.*, vol. 45, p. 105, 1888; vol. 45, p. 443, 1888.

[2] *Phil. Trans.*, 179 (A), p. 491, 1888; *Proc. Roy. Soc.*, vol. 49, p. 100, 1891; *Theory of Elasticity*, ch. XXI.

[3] *Proc. Lond. Math. Soc.*, vol. XXI. p. 119, 1890.

[4] *Phil. Trans.* 181 (A), p. 433, 1890; *Am. Math. Journ.*, vol. XVI. p. 254, 1894.

CHAPTER X B.

ELECTRICAL VIBRATIONS.

235 *i*. The introduction of the telephone into practical use, and the numerous applications to scientific experiment of which it admits, bring the subject of alternating electric currents within the scope of Acoustics, and impose upon us the obligation of shewing how the general principles expounded in this work may best be brought to bear upon the problems presenting themselves. Indeed Electricity affords such excellent illustrations that the temptation to use some of them has already (§§ 78, 92 *a*, 111 *b*) proved irresistible. It will be necessary, however, to take for granted a knowledge of elementary electrical theory, and to abstain for the most part from pursuing the subject in its application to vibrations of enormously high frequency, such as have in recent years acquired so much importance from the researches initiated by Lodge and by Hertz. In the writings of those physicists and in the works of Prof. J. J. Thomson[1] and of Mr O. Heaviside[2] the reader will find the necessary information on that branch of the subject.

The general idea of including electrical phenomena under those of ordinary mechanics is exemplified in the early writings of Lord Kelvin; and in his "Dynamical Theory of the Electro-magnetic Field[3]" Maxwell gave a systematic exposition of the subject from this point of view.

[1] *Recent Researches in Electricity and Magnetism*, 1893.
[2] *Electrical Papers*, 1892.
[3] *Phil. Trans.* vol. 155, p. 459, 1865; *Collected Works*, vol. 1, p. 526.

235 *j*. We commence with the consideration of a simple electrical circuit, consisting of an electro-magnet whose terminals are connected with the poles of a condenser, or *leyden*[1], of capacity C. The electro-magnet may be a simple coil of insulated wire, of resistance R, and of self-induction or *inductance* L. If there be an iron core, it is necessary to suppose that the metal is divided so as to avoid the interference of internal induced currents, and further that the whole change of magnetism is small[2]. Otherwise the behaviour of the iron is complicated with *hysteresis*, and its effect cannot be represented as a simple augmentation of L. Also for the present we will ignore the hysteresis exhibited by many kinds of leydens.

If x denote the charge of the leyden at time t, $\dot{x}$ is the current, and if $E_1 \cos pt$ be the imposed electro-motive force, the equation is

$$L\ddot{x} + R\dot{x} + x/C = E_1 \cos pt \quad\text{..................}\quad (1).$$

The solution of (1) gives the theory of *forced* electrical vibrations; but we may commence with the consideration of the *free* vibrations corresponding to $E_1 = 0$. This problem has already been treated in § 45, from which it appears that the currents are *oscillatory*, if

$$R < 2\sqrt{(L/C)} \quad\text{..........................}\quad (2).$$

The fact that the discharges of leydens are often oscillatory was suspected by Henry and by v. Helmholtz, but the mathematical theory is due to Kelvin[3].

When R is much smaller than the critical value in (2), a large number of vibrations occur without much loss of amplitude, and the period τ is given by

$$\tau = 2\pi\sqrt{(CL)} \quad\text{......................}\quad (3).$$

In (2), (3) the data may be supposed to be expressed in C. G. S. electro-magnetic measure. If we introduce practical units, so that L', R', C' represent the inductance, resistance and capacity reckoned respectively in earth-quadrants or henrys, ohms, and microfarads[4], we have in place of (2)

$$R' < 2000\sqrt{(L'/C')} \quad\text{....................}\quad (2'),$$

[1] This term has been approved by Lord Kelvin ("On a New Form of Air Leyden &c." *Proc. Roy. Soc.*, vol. 52, p. 6, 1892).

[2] *Phil. Mag.*, vol. 23, p. 225, 1887.

[3] "On Transient Electric Currents," *Phil. Mag.*, June, 1853.

[4] Ohm $= 10^9$, henry $= 10^9$, microfarad $= 10^{-15}$.

and in place of (3)

$$\tau = 2\pi . 10^{-3} \sqrt{(C'L')} \quad \text{.................... (3')}.$$

With ordinary appliances the value of τ is very small; but by including a considerable coil of insulated wire in the discharging circuit of a leyden composed of numerous glass plates Lodge[1] has succeeded in exhibiting oscillatory sparks of periods as long as $\frac{1}{500}$ second.

If the leyden be of infinite capacity or, what comes to the same thing, if it be short-circuited, the equation of free motion reduces to

$$L\ddot{x} + R\dot{x} = 0 \quad \text{........................ (4)};$$

whence

$$\dot{x} = \dot{x}_0 e^{-(R/L)t} \quad \text{........................ (5)[2]},$$

$\dot{x}_0$ representing the value of $\dot{x}$ when $t = 0$. The quantity L/R is sometimes called the time-constant of the circuit, being the time during which free currents fall off in the ratio of $e : 1$.

Returning to equation (1), we see that the problem falls under the general head of vibrations of one degree of freedom, discussed in § 46. In the notation there adopted, $n^2 = (CL)^{-1}$, $\kappa = R/L$, $E = E_1/L$; and the solution is expressed by equations (4) and (5). It is unnecessary to repeat at length the discussion already given, but it may be well to call attention to the case of resonance, where the natural pitch of the electrical vibrator coincides with that of the imposed force ($p^2LC = 1$). The first and third terms then (§ 46) compensate one another, and the equation reduces to

$$R\dot{x} = E_1 \cos pt \quad \text{........................ (6)}.$$

In general, if the leyden be short-circuited ($C = \infty$),

$$\dot{x} = \frac{E_1}{L^2p^2 + R^2} \{R \cos pt + pL \sin pt\} \quad \text{............(7)};$$

so that, if p much exceed R/L, the current is greatly reduced by self-induction. In such a case the introduction of a leyden of suitable capacity, by which the self-induction is compensated, results in a large augmentation of current[3]. The imposed electromotive force may be obtained from a coil forming part of the circuit and revolving in a magnetic field.

[1] *Proc. Roy. Inst.*, March, 1889.

[2] Helmholtz, *Pogg. Ann.*, LXXXIII., p. 505, 1851.

[3] Maxwell, "Experiment in Magneto-Electric Induction," *Phil. Mag.*, May, 1868.

In any circuit, where vibrations, whether forced or free, proportional to $\cos pt$ are in progress, we have $\ddot{x} = -p^2x$, and thus the terms due to self-induction and to the leyden enter into the equation in the same manner. The law is more readily expressed if we use the *stiffness* μ, equal to $1/C$, rather than the capacity itself. We may say that a stiffness μ compensates an inductance L, if $\mu = p^2L$, and that an additional inductance ΔL is compensated by an additional stiffness $\Delta\mu$, provided the above proportionality hold good. This remark allows us to simplify our equations by omitting in the first instance the stiffness of leydens. When the solution has been obtained, we may at any time generalise it by the introduction, in place of L, of $L - \mu p^{-2}$, or $L - (p^2C)^{-1}$. In following this course we must be prepared to admit negative values of L.

235 *k*. We will next suppose that there are two independent circuits with coefficients of self-induction L, N, and of mutual induction M, and examine what will be the effect in the second circuit of the instantaneous establishment and subsequent maintenance of a current $\dot{x}$ in the first circuit. At the first moment the question is one of the function T only, where

$$T = \tfrac{1}{2}L\dot{x}^2 + M\dot{x}\dot{y} + \tfrac{1}{2}N\dot{y}^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and by Kelvin's rule (§ 79) the solution is to be obtained by making (1) a minimum under the condition that $\dot{x}$ has the given value. Thus initially

$$\dot{y}_0 = -\frac{M}{N}\dot{x} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and accordingly (§ 235 *j*) after time t

$$\dot{y} = -\frac{M}{N}\dot{x}\,e^{-(S/N)t} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

if S be the resistance of the circuit. The whole induced current, as measured by a ballistic galvanometer, is given by

$$\int_0^\infty \dot{y}\,dt = -\frac{M\dot{x}}{S} \quad\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

in which N does not appear. The current in the secondary circuit due to the cessation of a previously established steady current $\dot{x}$ in the primary circuit is the opposite of the above.

A curious property of the initial induced current is at once evident from Kelvin's theorem, or from equation (2). It appears that, if M be given, the initial current is greatest when N is least. Further, if the secondary circuit consist mainly of a coil of n turns, the initial current increases with diminishing n. For, although $M \propto n$, $N \propto n^2$; and thus $\dot{y}_0 \propto 1/n$. In fact the small current flowing through the more numerous convolutions has the same effect as regards the energy of the field as the larger current in the fewer convolutions. This peculiar dependence upon n cannot be investigated by the galvanometer, at least without commutators capable of separating one part of the induced current from the rest; for, as we see from (4), the galvanometer reading is affected in the reverse direction. It is possible however to render evident the increased initial current due to a diminished n by observing the magnetizing effect upon steel needles. The magnetization depends mainly upon the initial maximum value of the current, and in a less degree, or scarcely at all, upon its subsequent *duration*.[1]

The general equations for two detached circuits, influencing one another only by induction, may be obtained in the usual manner from (1) and

$$F = \tfrac{1}{2}R\dot{x}^2 + \tfrac{1}{2}S\dot{y}^2 \quad\ldots\ldots\ldots\ldots(5).$$

Thus

$$\left.\begin{aligned} L\ddot{x} + M\ddot{y} + R\dot{x} &= X \\ M\ddot{x} + N\ddot{y} + S\dot{y} &= Y \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(6).$$

These equations, in a more general form, are considered in § 116. If a harmonic force $X = e^{ipt}$ act in the first circuit, and the second circuit be free from imposed force ($Y = 0$), we have on elimination of $\dot{y}$

$$\dot{x}\left\{ip\left(L - p^2\frac{M^2N}{p^2N^2 + S^2}\right) + R + p^2\frac{M^2S}{p^2N^2 + S^2}\right\} = e^{ipt} \ldots (7),$$

shewing that the reaction of the secondary circuit upon the first is to *reduce* the inductance by

$$p^2\frac{M^2N}{p^2N^2 + S^2} \quad\ldots\ldots\ldots\ldots (8)^2,$$

and to *increase* the resistance by

$$p^2\frac{M^2S}{p^2N^2 + S^2} \quad\ldots\ldots\ldots\ldots (9)^2.$$

[1] *Phil. Mag.*, vol. 38, p. 1, 1869; vol. 39, p. 428, 1870.

[2] Maxwell, *Phil. Trans.*, vol. 155, p. 459, 1865, where, however, M^2 is misprinted M.

The formulæ (8) and (9) may be applied to deal with a more general problem of considerable interest, which arises when (as in some of Henry's experiments) the secondary circuit acts upon a third, this upon a fourth, and so on, the only condition being that there must be no mutual induction except between immediate neighbours in the series. For the sake of distinctness we will limit ourselves to four circuits.

In the fourth circuit the current is due *ex hypothesi* only to induction from the third. Its reaction upon the third, for the rate of vibration under contemplation, is given at once by (8) and (9); and if we use the complete values applicable to the third circuit under these conditions, we may thenceforth ignore the fourth circuit. In like manner we can now deduce the reaction upon the secondary, giving the effective resistance and inductance of that circuit under the influence of the third and fourth circuits; and then, by another step of the same kind, we may arrive at the values applicable to the primary circuit, under the influence of all the others. The process is evidently general; and we know by the theorem of § 111 *b* that, however extended the train of circuits, the influence of the others upon the first must be to increase its effective resistance and diminish its effective inertia, in greater and greater degree as the frequency of vibration increases.

In the limit, when the frequency increases indefinitely, the distribution of currents is determined by the induction-coefficients, irrespective of resistance, and, as we shall see presently, it is of such a character that the currents are alternately opposite in sign as we pass along the series.

235 *l*. Whatever may be the number of independent currents, or degrees of freedom, the general equations are always of the kind already discussed §§ 82, 103, 104, viz.

$$\frac{d}{dt}\frac{dT}{d\dot{x}} + \frac{dF}{d\dot{x}} + \frac{dV}{dx} = X \quad \text{.................. (1)},$$

where T, F, V are (§ 82) homogeneous quadratic functions. In (1) the co-ordinates x_1, x_2, ... denote the whole quantity of electricity which has passed at time t, the *currents* being $\dot{x}_1$, $\dot{x}_2$, &c. When $V = 0$, it is simpler to express the phenomena by means of the currents. Thus, in the problem of steady electric flow where all the quantities X, representing electro-motive forces, are constant, the currents are determined directly by the linear equations

$$dF/d\dot{x}_1 = X_1, \quad dF/d\dot{x}_2 = X_2, \text{ \&c.} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

On the other hand when the question under consideration is one of initial impulsive effects, or of forced vibrations of exceedingly high frequency, everything depends upon T, and the equations reduce to

$$\frac{d}{dt}\frac{dT}{d\dot{x}_1} = X_1, \quad \frac{d}{dt}\frac{dT}{d\dot{x}_2} = X_2, \text{ \&c.} \quad \ldots\ldots\ldots\ldots\ldots (3).$$

As an example we may consider the problem, touched upon at the close of § 235 *k*, of a train of circuits where the mutual induction is confined to immediate neighbours, so that

$$T = \tfrac{1}{2}a_{11}x_1^2 + \tfrac{1}{2}a_{22}x_2^2 + \tfrac{1}{2}a_{33}x_3^2 + \ldots$$
$$+ a_{12}x_1x_2 + a_{23}x_2x_3 + a_{34}x_3x_4 + \ldots \;\ldots\; \ldots\ldots (4)^1,$$

coefficients such as a_{13}, a_{14}, a_{24} not appearing. If x_1 be given, either as a current suddenly developed and afterwards maintained constant, or as a harmonic time function of high frequency, while no external forces operate in the other circuits, the problem is to determine x_2, x_3, &c. so as to make T as small as possible, § 79. The equations are easily written down, but the conclusion aimed at is perhaps arrived at more instructively by consideration of the function T itself. For, T being homogeneous in x_1, x_2, &c., we have identically

$$2T = x_1\frac{dT}{dx_1} + x_2\frac{dT}{dx_2} + \ldots \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

And, since when T is a minimum, dT/dx_2, dT/dx_3, &c., all vanish,

$$2T_{\text{min.}} = x_1\frac{dT}{dx_1} = a_{11}x_1^2 + a_{12}x_1x_2.$$

But if x_2, x_3, &c., had all been zero, $2T$ would have been equal to $a_{11}x_1^2$. It is clear therefore that $a_{12}x_1x_2$ is negative; or, as a_{12} is taken positive, the sign of x_2 is the *opposite* to that of x_1.

Again supposing x_1, x_2 both given, we must, when T is a minimum, have dT/dx_3, dT/dx_4, &c., equal to zero, and thus

$$2T_{\text{min.}} = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2a_{23}x_2x_3.$$

As before, $2T$ might have been

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2,$$

1 The dots are omitted as unnecessary.

simply. The minimum value is necessarily less than this, and accordingly the signs of x_2 and x_3 are opposite. This argument may be continued, and it shews that, however long the series may be, the induced currents are alternately opposite in sign[1], a result in harmony with the magnetizations observed by Henry.

In certain cases the minimum value of T may be very nearly zero. This happens when the coils which exercise a mutual inductive influence are so close throughout their entire lengths that they can produce approximately opposite magnetic forces at all points of space. Suppose, for example, that there are two similar coils A and B, each wound with a double wire (A_1, A_2), (B_1, B_2), and combined so that the primary circuit consists of A_1, the secondary of A_2 and B_1 joined by inductionless leads, and the tertiary of B_2 simply closed upon itself. It is evident that T is made approximately zero by taking $x_2 = -x_1$ and $x_3 = -x_2 = x_1$. The argument may be extended to a train of such coils, however long, and also to cases where the number of convolutions in mutually reacting coils is not the same.

In a large class of problems, where leyden effects do not occur sensibly, the course of events is determined by T and F simply. These functions may then be reduced to sums of squares; and the typical equation takes the form

$$a\ddot{x} + b\dot{x} = X \quad \text{.........................}(6).$$

If $X = 0$, that is if there be no imposed electro-motive forces, the solution is

$$\dot{x} = \dot{x}_0 e^{-bt/a} \quad \text{.........................}(7).$$

Thus any system of initial currents flowing whether in detached or connected linear conductors, or in solid conducting masses, may be resolved into "normal" components, each of which dies down exponentially at its own proper rate.

A general property of the "persistences," equal to a/b, is proved in § 92 *a*. For example, any increase in permeability, due to the introduction of iron (regarded as non-conducting), or any diminution of resistance, however local, will in general bring about a rise in the values of all the persistences[2].

In view of the discussions of Chapter V. it is not necessary to dwell upon the solution of equations (1) when X is retained. The

[1] *Phil. Mag.*, vol. 38, p. 13, 1869.

[2] *Brit. Assoc. Report*, 1885, p. 911.

reciprocal theorem of § 109 has many interesting electrical applications; but, after what has there been said, their deduction will present no difficulty.

235 *m.* In § 111 *b* one application of the general formulæ to an electrical system has already been given. As another example, also relating to the case of two degrees of freedom, we may take the problem of two conductors *in parallel*. It is not necessary to include the influence of the leads outside the points of bifurcation; for provided that there be no mutual induction between these parts and the remainder, their inductance and resistance enter into the result by simple addition.

Under the sole operation of resistance, the total current x_1 would divide itself between the two conductors (of resistances R and S) in the parts

$$\frac{S}{R+S}x_1 \text{ and } \frac{R}{R+S}x_1;$$

and we may conveniently so choose the second co-ordinate that the currents in the two conductors are in general

$$\frac{S}{R+S}x_1+x_2 \text{ and } \frac{R}{R+S}x_1-x_2,$$

x_1 still representing the total current in the leads. The dissipation-function, found by multiplying the squares of the above currents by $\frac{1}{2}R$, $\frac{1}{2}S$ respectively, is

$$F=\tfrac{1}{2}\frac{RS}{R+S}x_1^2+\tfrac{1}{2}(R+S)\,x_2^2 \;\ldots\ldots\ldots\ldots\ldots\; (1).$$

Also, L, M, N being the induction coefficients of the two branches,

$$T=\tfrac{1}{2}\,\frac{LS^2+2MRS+NR^2}{(R+S)^2}\,x_1^2 + \frac{(L-M)\,S+(M-N)\,R}{R+S}\,x_1x_2+\tfrac{1}{2}\,(L-2M+N)\,x_2^2\ldots(2).$$

Thus, in the notation of § 111 *b*,

$$a_{11}=\frac{LS^2+2MRS+NR^2}{(R+S)^2},\qquad a_{12}=\frac{(L-M)\,S+(M-N)\,R}{R+S},$$
$$a_{22}=L-2M+N;$$
$$b_{11}=\frac{RS}{R+S},\qquad b_{12}=0,\qquad b_{22}=R+S.$$

Accordingly by (5), (8) § 111 b,

$$R' = \frac{RS}{R+S} + \frac{p^2}{R+S} \frac{\{(L-M)S+(M-N)R\}^2}{(R+S)^2+p^2(L-2M+N)^2} \dots (3).$$

$$L' = \frac{LS^2+2MRS+NR^2}{(R+S)^2} - \frac{\{(L-M)S+(M-N)R\}^2}{(R+S)^2(L-2M+N)}$$
$$+ \frac{\{(L-M)S+(M-N)R\}^2}{(L-2M+N)\{(R+S)^2+p^2(L-2M+N)^2\}} \dots (4).$$

These are respectively the effective resistance and the effective inductance of the combination[1]. It is to be remarked that $(L-2M+N)$ is necessarily positive, representing twice the kinetic energy of the system when the currents in the two conductors are $+1$ and -1.

The expressions for R' and L' may be put into a form[2] which for many purposes is more convenient, by combining the component fractional terms. Thus

$$R' = \frac{RS(R+S)+p^2\{R(M-N)^2+S(L-M)^2\}}{(R+S)^2+p^2(L-2M+N)^2} \dots\dots\dots\dots (3'),$$

$$L' = \frac{LS^2+2MRS+NR^2+p^2(LN-M^2)(L-2M+N)}{(R+S)^2+p^2(L-2M+N)^2} \dots (4'),$$

in which $(LN-M^2)$ is positive by virtue of the nature of T.

As p increases from zero, we know by the general theorem § 111 b, or from the particular expressions (3), (4), that R' continually increases and that L' continually decreases.

When p is very small,

$$R' = \frac{RS}{R+S}, \qquad L' = \frac{LS^2+2MRS+NR^2}{(R+S)^2} \dots\dots (5).$$

In this case the distribution of the main current between the conductors is determined by the resistances, and (§ 111 b) the values of R' and L' coincide respectively with $2F/x_1^2$, $2T/x_1^2$. The resistance is manifestly the same as if the currents were steady.

On the other hand, when p is very great,

$$R' = \frac{R(M-N)^2+S(L-M)^2}{(L-2M+N)^2}, \qquad L' = \frac{LN-M^2}{L-2M+N} \dots (6).$$

In this case the distribution of currents is independent of the resistances, being determined in accordance with Kelvin's theorem

[1] *Phil. Mag.*, vol. 21, p. 377, 1886.

[2] J. J. Thomson, *loc. cit.* § 421.

in such a manner that the ratio of the currents in the two conductors is $(N-M):(L-M)$. As when p is small, the values in (6) coincide with $2F/x_1^2$, $2T/x_1^2$.

When the two wires composing the conductors in parallel are wound closely together, the energy of the field under high frequency may be very small. There is an interesting distinction to be noted here dependent upon the manner in which the connections are made. Consider, for example, the case of a bundle of five contiguous wires wound into a coil, of which three wires, connected in series so as to have maximum inductance, constitute one of the branches in parallel, and the other two, connected similarly in series, constitute the other branch. There is still an alternative as to the manner of connection of the two branches. If steady currents would circulate opposite ways (M negative), the total current is divided into two parts in the ratio 3 : 2, in such a manner that the more powerful current in the double wire nearly neutralises at external points the magnetic effects of the less powerful current in the triple wire, and the total energy of the system is very small. But now suppose that the connections are such that steady currents would circulate the same way in both branches (M positive). It is evident that the condition of minimum energy cannot be satisfied when the currents are in the same direction, but requires that the smaller current in the triple wire should be in the opposite direction to that of the larger current in the double wire. In fact the currents must be as 3 to -2; so that (since on the same scale the total current is unity) the component currents in the branches are both numerically greater than the total current which is algebraically divided between them. And this peculiar feature becomes more and more strongly marked the nearer L and N approach to equality[1].

The unusual development of currents in the branches is, of course, attended by an augmented effective resistance. In the limiting case when the m convolutions of one branch are supposed to coincide geometrically with one another and with the n convolutions of the second branch, we have

$$L : M : N = m^2 : mn : n^2,$$

and from (6)

$$R' = \frac{n^2R + m^2S}{(m-n)^2} \qquad (7),$$

[1] *Phil. Mag.*, vol. 21, p. 376, 1886.

an expression which increases without limit, as m and n approach to equality.

The fact that under certain conditions the currents in both branches of a divided circuit may exceed the current in the mains has been verified by direct experiment[1]. Each of the three currents to be compared traversed short lengths of similar German-silver wire, and the test consisted in finding what lengths of this wire it was necessary in the various cases to include between the terminals of a high resistance telephone in order to obtain sounds of equal intensity. The variable currents were derived from a battery and scraping contact apparatus (§ 235 r), directly included in the main circuit.

The general formulæ (3′), (4′) undergo simplification when the conductors in parallel exercise no mutual induction. Thus, when $M=0$,

$$R'=\frac{RS(R+S)+p^2(RN^2+SL^2)}{(R+S)^2+p^2(L+N)^2} \quad\ldots\ldots\ldots\ldots (8),$$

$$L'=\frac{LS^2+NR^2+p^2LN(L+N)}{(R+S)^2+p^2(L+N)^2} \quad\ldots\ldots\ldots\ldots (9).$$

If further $N=0$, (8) and (9) reduce to

$$R'=\frac{S\{R(R+S)+p^2L^2\}}{(R+S)^2+p^2L^2}, \qquad L'=\frac{LS^2}{(R+S)^2+p^2L^2} \ldots (10).$$

The peculiar features of the combination are brought out most strongly when S, the resistance of the inductionless component, is great in comparison with R. In that case if the current be steady or slowly vibrating, it flows mainly through R, while the resistance and inductance of the combination approximate to R and L respectively; but if on the other hand the current be a rapidly vibrating one, it flows mainly through S, so that the resistance of the combination approximates to S, and the inductance to zero. These conclusions are in agreement with (10).

If the branches in parallel be simple electro-magnets, L and N are necessarily positive, and the numerator in (9) is incapable of vanishing. But, as we have seen, when leydens are admitted, this restriction may be removed. An interesting case arises when the second branch is inductionless, and is interrupted by a leyden of

[1] *Phil. Mag.*, vol. 22, p. 495, 1886.

capacity C, so that $N = -(Cp^2)^{-1}$, while at the same time $R = S$. The latter condition reduces the numerator in (9) to

$$(L + N)\{R^2 + p^2 LN\}.$$

Thus L' vanishes, (i) when $LCp^2 = 1$, and (ii) when $CR^2 = L$. The first alternative is the condition that the loop circuit, considered by itself, should be isochronous with the imposed vibrations. The second expresses the equality of the time-constants of the two branches. If they be equal, the combination behaves like a simple resistance, whatever be the character of the imposed electromotive force[1].

235 *n*. When there are more than two conductors in parallel, the general expressions for the equivalent resistance and inductance of the combination would be very complicated; but a few particular cases are worthy of notice.

The first of these occurs when there is no mutual induction between the members. If the quantities relating to the various branches be distinguished by the suffixes 1, 2, 3, ..., and if E be the difference of potentials at the common terminals, we have

$$E = (ipL_1 + R_1)\,x_1 = (ipL_2 + R_2)\,x_2 = \ldots \quad\ldots\ldots\ldots(1);$$

so that
$$\frac{1}{\Sigma\,(ipL + R)^{-1}} = \frac{E}{x_1 + x_2 + \ldots} = R' + ipL' \quad\ldots\ldots\ldots(2),$$

by which R' and L' are determined. Thus, if we write

$$\Sigma\,\frac{R}{R^2 + p^2L^2} = A, \qquad \Sigma\,\frac{L}{R^2 + p^2L^2} = B \quad\ldots\ldots\ldots(3),$$

we have from (2)

$$R' = \frac{A}{A^2 + p^2B^2}, \qquad L' = \frac{B}{A^2 + p^2B^2} \quad\ldots\ldots\ldots\ldots(4).$$

Equations (3) and (4) contain the solution of the problem[2].

When $p = 0$,

$$R' = \frac{1}{\Sigma\,(R^{-1})}, \qquad L' = \frac{\Sigma\,(LR^{-2})}{\{\Sigma\,(R^{-1})\}^2} \quad\ldots\ldots\ldots\ldots(5).$$

When on the other hand p is very great,

$$R' = \frac{\Sigma\,(RL^{-2})}{\{\Sigma\,(L^{-1})\}^2}, \qquad L' = \frac{1}{\Sigma\,(L^{-1})} \quad\ldots\ldots\ldots\ldots(6).$$

[1] Chrystal, "On the Differential Telephone," *Edin. Trans.*, vol. 29, p. 615, 1880.

[2] *Phil. Mag.*, vol. 21, p. 379, 1886.

Even when the mutual induction between various members cannot be neglected, tolerably simple expressions can be found for the equivalent resistance and inductance in the extreme cases of p infinitely small or infinitely large. As has already been proved, (§ 111 *b*), the above-mentioned quantities then coincide in value with $2F/(\dot{x}_1+\dot{x}_2+\ldots)^2$, and $2T/(\dot{x}_1+\dot{x}_2+\ldots)^2$, and the calculation of these values is easy, inasmuch as the distribution of currents among the branches is determined in the first case entirely by F and in the second case entirely by T. Thus, when p is infinitely small, F is a minimum, and the currents are in proportion to the conductances of the several branches. Accordingly, if the induction coefficients of the branches be denoted, as in § 111 *b*, by a_{11}, a_{22}, ... a_{12}, a_{13}, ..., and the resistances by R_1, R_2, &c., we have

$$R' = \frac{R_1(1/R_1)^2 + R_2(1/R_2)^2 + \ldots}{(1/R_1 + 1/R_2 + \ldots)^2} = \frac{1}{1/R_1 + 1/R_2 + \ldots} \quad \ldots\ldots(7),$$

$$L' = \frac{a_{11}/R_1^2 + a_{22}/R_2^2 + \ldots + 2a_{12}/R_1R_2 + 2a_{13}/R_1R_3 + \ldots}{(1/R_1 + 1/R_2 + 1/R_3 + \ldots)^2} \quad \ldots(8).$$

A similar method applies when $p=\infty$, but the result is less simple on account of the complication in the ratios of currents due to mutual induction[1].

235 *o*. The induction-balance, originally contrived by Dove for use with the galvanometer, has in recent years been adapted to the telephone by Hughes[2], who has described experiments illustrating the marvellous sensibility of the arrangement. The essential features are a primary, or battery, circuit, in which circulates a current rendered intermittent by a make and break interrupter, or by a simple scraping contact, and a secondary circuit containing a telephone. By suitable adjustments the two circuits are rendered conjugate, that is to say the coefficient of mutual induction is caused to vanish, so as to reduce the telephone to silence. The introduction into the neighbourhood of a third circuit, whether composed of a coil of wire, or of a simple conducting mass, such as a coin, will then in general cause a revival of sound.

The destruction of the mutual induction in the case of two flat coils can be arrived at by placing them at a short distance apart,

[1] J. J. Thomson, *loc. cit.* § 422.

[2] *Phil. Mag.* vol. VIII., p. 50, 1879.

in parallel planes, and with accurately adjusted overlapping. But in Hughes' apparatus the balance is obtained more symmetrically by the method of duplication. Four similar coils are employed. Of these two A_1, A_2, mounted at some distance apart with their planes horizontal, and connected in series, constitute the primary induction coil. The secondary induction coil consists in like manner of B_1, B_2, placed symmetrically at short distances from A_1, A_2, and also connected in series, but in such a manner that the induction between A_1 and B_1 tends to balance the induction between A_2 and B_2. If the four coils were perfectly similar, balance would be obtained when the distances were equal. This of course is not to be depended upon, but by a screw motion the distance between one pair, e.g. A_1 and B_1, is rendered adjustable, and in this way a balance between the two inductions is obtained. Wooden cups, fitting into the coils, are provided in such situations that a coin resting in one of them is situated symmetrically between the corresponding primary and secondary coils. The balance, previously adjusted, is of course upset by the introduction of a coin upon one side, but if a perfectly similar coin be introduced upon the other side also, balance may be restored. Hughes found that very minute differences between coins could be rendered evident by outstanding sound in the telephone.

The theory of this apparatus, when the primary currents are harmonic, is simple[1], especially if we suppose that the primary current x_1 is given. If $x_1, x_2, \ldots$ be the currents; $b_1, b_2, \ldots$ the resistances; $a_{11}, a_{22}, a_{12}, \ldots$ the inductances, the equations for the case of three circuits are

$$\left.\begin{aligned} a_{22}\dot{x}_2 + a_{23}\dot{x}_3 + b_2x_2 &= -a_{12}\dot{x}_1 \\ a_{23}\dot{x}_2 + a_{33}\dot{x}_3 + b_3x_3 &= -a_{13}\dot{x}_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

We now assume that x_1, x_2, &c. are proportional to e^{ipt}, where $p/2\pi$ is the frequency of vibration. Thus,

$$ip\,(a_{22}x_2 + a_{23}x_3) + b_2x_2 = -ipa_{12}x_1,$$

$$ip\,(a_{23}x_2 + a_{33}x_3) + b_3x_3 = -ipa_{13}x_1;$$

whence by elimination of x_3

$$x_2\left\{ipa_{22} + b_2 + \frac{p^2a_{23}^2}{ipa_{33} + b_3}\right\} = -ipa_{12}x_1 - \frac{p^2a_{13}a_{23}x_1}{ipa_{33} + b_3} \ldots\ldots (2).$$

[1] *Brit. Assoc. Rep.* 1880, p. 472.

From this it appears that a want of balance depending on a_{12} cannot compensate for the action of the third circuit, so as to produce silence in the secondary circuit, unless b_3 be negligible in comparison with pa_{33}, that is unless the time-constant of the third circuit be very great in comparison with the period of the vibration. Otherwise the effects are in different phases, and therefore incapable of balancing.

We will now introduce a fourth circuit, and suppose that the primary and secondary circuits are accurately conjugate, so that $a_{12}=0$, and also that the mutual induction a_{34} between the third and fourth circuits may be neglected. Then

$$ip\,(a_{22}x_2 + a_{23}x_3 + a_{24}x_4) + b_2x_2 = 0,$$

$$ip\,(a_{32}x_2 + a_{33}x_3) + b_3x_3 = -\,ipa_{13}x_1,$$

$$ip\,(a_{42}x_2 + a_{44}x_4) + b_4x_4 = -\,ipa_{14}x_1\,;$$

whence

$$x_2\left\{ipa_{22} + b_2 + \frac{p^2a_{23}{}^2}{ipa_{33}+b_3} + \frac{p^2a_{24}{}^2}{ipa_{44}+b_4}\right\}$$

$$= -\,p^2x_1\left\{\frac{a_{13}a_{23}}{ipa_{33}+b_3} + \frac{a_{14}a_{24}}{ipa_{44}+b_4}\right\} \;\ldots\ldots (3).$$

It appears that *two* conditions must be satisfied in order to secure a balance, since both the phases and the intensities of the separate effects must be the same. The first condition requires that the time-constants of the third and fourth circuits be equal, unless indeed both be very great, or both be very small, in comparison with the period. If this condition be satisfied, balance ensues when

$$\frac{a_{13}a_{23}}{a_{33}} + \frac{a_{14}a_{24}}{a_{44}} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)\,;$$

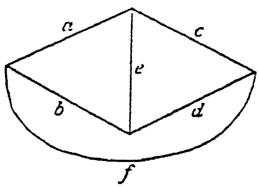
and it is especially to be noted that the adjustment is independent of pitch, so that (by Fourier's theorem) it suffices whatever be the nature of the variable currents operative in the primary.

As regards the position of the third and fourth circuits, usually represented by coins in illustrative experiments, it will be seen from the symmetry of the right-hand member of (3) that the middle position between the primary and secondary coils is suitable, inasmuch as the product $a_{13}a_{23}$ is stationary in value when the coin is moved slightly so as to be nearer say to the primary and further from the secondary[1]. Approximate independence of other displacements is secured by the geometrical symmetry of the coils round the axis.

235 *p*. For the accurate comparison of electrical quantities the "bridge" arrangement of Wheatstone is usually the most convenient, and is equally available with the galvanometer in the case of steady or transitory currents, or with the telephone in the case of periodic currents. Similar effects may be obtained in most cases without a bridge by the employment of the differential galvanometer or the differential telephone[2].

In the ordinary use of the bridge the four members a, b, c, d combined in a quadrilateral Fig. (53 *a*) are simple resistances. The battery branch f joins one pair of opposite corners, and the indicating instrument is in the "bridge" e joining the other pair. "Balance" is obtained, when $ad = bc$. But for our purpose we have to suppose that any member, e.g. a, is not merely a resistance, or even a combination of resistances. It may include an electromagnet, and it may be interrupted by a leyden. But in any case, so long as the current x is strictly harmonic, proportional to e^{ipt}, the general relation between it and the difference of potentials V at the extremities is given by

Fig. 53 *a*.



$$V = (a_1 + ia_2)\, x \qquad (1),$$

where a_1 and ia_2 are the real and imaginary parts of a complex coefficient a, and are functions of the frequency $p/2\pi$. In the particular case of a simple conductor, endowed with inductance L, a_1 represents the resistance, and a_2 is equal to pL. In general, a_1 is positive; but a_2 may be either positive, as in the above example, or negative. The latter case arises when a resistance R is interrupted by a leyden of capacity C. Here $a_1 = R$, $a_2 = -1/pC$. If there be also inductance L,

$$a_1 = R, \qquad a_2 = pL - 1/pC \qquad (2).$$

As we have already seen, § 235 *j*, a_2 may vanish for a particular frequency, and the combination is then equivalent to a simple

[1] See Lodge, *Phil. Mag.*, vol. 9, p. 123, 1880.

[2] Chrystal, *Edin. Trans.*, *loc. cit.*

resistance. But a variation of frequency gives rise to a positive or negative a_2.

In all electrical problems, where there is no mutual induction, the generalized quantities, a, b, &c., combine, just as they do when they represent simple resistances[1]. Thus, if a, a' be two complex quantities representing two conductors in series, the corresponding quantity for the combination is $(a+a')$. Again, if a, a' represent two conductors in parallel, the reciprocal of the resultant is given by addition of the reciprocals of a, a'. For, if the currents be x, x', corresponding to a difference of potentials V at the common terminals,

$$V = ax = a'x',$$

so that

$$x + x' = V(1/a + 1/a').$$

In the application to Wheatstone's combination of the general theory of forced vibrations, we will limit the impressed forces to the battery and the telephone branches. If x, y be the currents in these branches, X, Y the corresponding electro-motive forces, we have, § 107, linear relations between x, y, and X, Y, which may be written

$$\left.\begin{aligned} X &= Ax + By \\ Y &= Bx + Cy \end{aligned}\right\} \qquad (3),$$

the coefficient of y in the first equation being identical with that of x in the second equation, by the reciprocal property. The three constants A, B, C are in general complex quantities, functions of p.

The reciprocal relation may be interpreted as follows. If $Y = 0$, $Bx + Cy = 0$, and

$$y = \frac{BX}{B^2 - AC} \qquad (4).$$

In like manner, if we had supposed $X = 0$, we should have found

$$x = \frac{BY}{B^2 - AC} \qquad (5),$$

shewing that the ratio of the current in one member to the electro-motive force operative in the other is independent of the way in which the parts are assigned to the two members.

[1] For a more complete discussion of this subject see Heaviside "On Resistance and Conductance Operators," *Phil. Mag.*, vol. 24, p. 479, 1887; *Electrical Papers*, vol. II., p. 355.

We have now to determine the constants A, B, C in terms of the electrical properties of the system. If y be maintained zero by a suitable force Y, the relation between x and X is $X = Ax$. A therefore denotes the (generalized) resistance to any electromotive force in the battery member, *when the telephone member is open.* This resistance is made up of f, the resistance in the battery member, and of that of the conductors $a + c$, $b + d$, combined in parallel. Thus

$$A = f + \frac{(a+c)(b+d)}{a+b+c+d} \quad\text{.................. (6).}$$

In like manner

$$C = e + \frac{(a+b)(c+d)}{a+b+c+d} \quad\text{.................. (7).}$$

To determine B let us consider the force Y which must act in e in order that the current through it may be zero, in spite of the operation of X. We have $Y = Bx$. The total current x flows partly along the branch $(a+c)$, and partly along $(b+d)$. The current through $(a+c)$ is

$$\frac{x/(a+c)}{1/(a+c)+1/(b+d)} = \frac{(b+d)\,x}{a+b+c+d} \quad\text{........... (8),}$$

and that through $(b+d)$ is

$$\frac{(a+c)\,x}{a+b+c+d} \quad\text{........................ (9).}$$

The difference of potentials at the terminals of e, supposed to be interrupted, is thus

$$\frac{c\,(b+d)\,x - d\,(a+c)\,x}{a+b+c+d};$$

and accordingly

$$B = \frac{bc - ad}{a+b+c+d} \quad\text{..................... (10).}$$

By (6), (7), (10) the relationship of X, Y to x, y is completely determined.

The problem of the bridge requires the determination of the current y as proportional to X, when $Y = 0$, that is when no electro-motive force acts in the bridge itself; and the solution is given at once by the introduction into (4) of the values of A, B, C from (6), (7), (10).

If there be an approximate "balance," the expression simplifies. For $(bc - ad)$ is then small, and B^2 may be neglected relatively to

AC in the denominator of (4). Thus, as a sufficient approximation in this case, we may write

$$\frac{y}{X} = -\frac{B}{AC} = \frac{(10)}{(6)\times(7)} \qquad\qquad (11).$$

The following interpretation of the process leads very simply to the approximate form (11), and is independent of the general theory. Let us first inquire what electro-motive force is necessary in the telephone member to stop any current through it. If such a force act, the conditions are, externally, the same as if the member were open; and the current x in the battery member due to a force equal to X in that member is X/A, where A is written for brevity as representing the right-hand member of (6). The difference of potentials at the terminals of e, still supposed to be open, is found at once when x is known. It is given by

$$c \times (8) - d \times (9) = Bx,$$

where B is defined by (10). In terms of X the difference of potentials is thus BX/A. If e be now closed, the same fraction expresses the force necessary in e in order to prevent the generation of a current in that member.

The case with which we have to deal is when X acts in f and there is no force in e. We are at liberty, however, to suppose that two opposite forces, each of magnitude BX/A, act in e. One of these, as we have seen, acting in conjunction with X in f, gives no current in e; so that, since electro-motive forces act independently of one another, the actual current in e, closed without internal electro-motive force, is simply that due to the other component. The question is thus reduced to the determination of the current in e due to a given force in that member.

So far the argument is rigorous; but we will now suppose that we have to deal with an approximate balance. In this case a force in e gives rise to very little current in f, and in calculating the current in e, we may suppose f to be broken. The total resistance to the force in e is then given simply by C of equation (7), and the approximate value for y is derived by dividing $-BX/A$ by C, as we found in (11).

A continued application of the foregoing process gives y/X in the form of an infinite geometric series:—

$$\frac{y}{X} = -\frac{B}{AC}\left\{1 + \frac{B^2}{AC} + \frac{B^4}{A^2C^2} + \dots\right\} = \frac{B}{B^2 - AC}.$$

This is the rigorous solution already found in (4); but the first term of the series suffices for practical purposes.

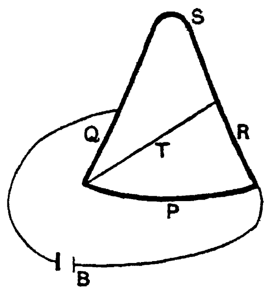
The form of (11) enables us at once to compare the effects of increments of resistance and of inductance in disturbing a balance. For let $ad = bc$, and then change d to $d + d'$, where $d' = d_1' + id_2'$. The value of y/X is proportional to d', and the amplitude of the vibratory current in the bridge is proportional to mod. d', that is, to $\sqrt{(d_1'^2 + d_2'^2)}$. Thus d_1', d_2' are equally efficacious when numerically equal[1]. In most cases where a telephone is employed, the balance is more sensitive to changes of inductance than to changes of resistance.

In the use of the Wheatstone balance for purposes of measurement, it is best to make a equal to c. The equality of b and d can then be tested by interchange of a and c, independently of the exactitude of the equality of these quantities. Another advantage lies in the fact that balance is independent of mutual induction between a and c or between b and d.

235 *q*. In the formulæ of § 235 *p* it has been assumed that there is no mutual induction between the various members of the combination. The more general theory has been considered very fully by Heaviside[2], but to enter upon it would lead us too far. It may be well, however, to sketch the theory of the arrangement adopted by Hughes, which possesses certain advantages in dealing with the electrical properties of wires in short lengths[3].

The apparatus consists of a Wheatstone's quadrilateral, Fig. 53 *b*, with a telephone in the bridge, one of the sides of the quadrilateral being the wire or coil under examination (P), and the other three being the parts into which a single German-silver wire is divided by two sliding contacts. If the battery-branch (B) be closed, and a suitable interrupter be introduced into the telephone-branch (T), balance may be obtained by shifting the contacts. *Provided that the interrupter introduces no electro-motive*

Fig. 53 *b*.



[1] "On the Bridge Method in its Application to Periodic Electric Currents." *Proc. Roy. Soc.*, vol. 49, p. 203, 1891.

[2] "On the Self-Induction of Wires," Part VI.; *Phil. Mag.*, Feb. 1887; *Electrical Papers*, 1892, vol. II., p. 281.

[3] *Journ. Tel. Eng.*, vol. XV. (1886) p. 1; *Proc. Roy. Soc.*, vol. XL. (1886) p. 451.

force of its own[1], the balance indicates the proportionality of the four resistances. If P be the unknown resistance of the conductor under test, Q, R the resistances of the adjacent parts of the divided wire, S that of the opposite part (between the sliding contacts), then, by the ordinary rule, $PS = QR$; while Q, R, S are subject to the relation

$$Q + R + S = W,$$

W being a constant. If now the interrupter be transferred from the telephone to the battery-branch, the balance is usually disturbed on account of induction, and cannot be restored by any mere shifting of the contacts. In order to compensate the induction, another influence of the same kind must be introduced. It is here that the peculiarity of the apparatus lies. A coil (not shewn in the figure) is inserted in the battery and another in the telephone-branch which act inductively upon one another, and are so mounted that the effect may be readily varied. The two coils may be concentric and relatively movable about the common diameter. In this case the action vanishes when the planes are perpendicular. If one coil be very much smaller than the other, the coefficient of mutual induction M is proportional to the cosine of the angle between the planes. By means of the *two* adjustments, the sliding of the contacts and the rotation of the coil, it is usually possible to obtain a fair silence.

Hughes interpreted his observations on the basis of an assumption that the inductance of P was represented by M, irrespective of resistance, and that the resistance to variable currents could (as in the case of steady currents) be equated to QR/S. But the matter is not quite so simple. The true formulæ are, however, readily obtained for the case where the only sensible induction among the sides of the quadrilateral is the inductance L of the conductor P.

Since there is no current through the bridge, there must be the same current (x) in P and in one of the adjacent sides (say) R, and for a like reason the same current y in Q and S. The difference of potentials at time t between the junction of P and R and the junction of Q and S may be expressed by each of the three following equated quantities:—

$$Qy - Px - L\frac{dx}{dt} = -M\frac{d(x+y)}{dt} = Rx - Sy.$$

[1] A condition not always satisfied in practice.

Introducing the assumption that all the quantities vary harmonically with frequency $p/2\pi$, and eliminating the ratio $y:x$, we find as the conditions required for silence in the telephone

$$QR - SP = p^2 ML \quad \text{.................... (1)},$$

$$M(P+Q+R+S) = SL \quad \text{.................. (2)}.$$

It will be seen that the ordinary resistance balance ($SP = QR$) is departed from. The change here considered is peculiar to the apparatus and, so far as its influence is concerned, it does not indicate a real alteration of resistance in the wire. Moreover, since p is involved, the disturbance depends upon the rapidity of vibration, so that in the case of ordinary mixed sounds silence can be attained only approximately. Again, from the second equation we see that M is not in general a correct measure of the value of L[1].

If however, P be known, the application of (2) presents no difficulty. In many cases we may be sure beforehand that P, viz. the effective resistance of the conductor, or combination of conductors, to the variable currents, is the same as if the currents were steady, and then P may be regarded as known. But there are other cases,—some of them will be alluded to below—in which this assumption cannot be made; and it is impossible to determine the unknown quantities L and P from (2) alone. We may then fall back upon (1). By means of the two equations P and L can always be found in terms of the other quantities. But among these is included the frequency of vibration; so that the method is practically applicable only when the interrupter is such as to give an absolute periodicity. A scraping contact, otherwise very convenient, is thus excluded; and this is undoubtedly an objection to the method.

If the member P be without inductance, but be interrupted by a leyden of capacity C, the same formulæ may be employed, with substitution of $-1/p^2C$ for L. Equation (1) then gives a measure of C which is independent of the frequency.

235 *r*. The success of experiments with this kind of apparatus depends very largely upon the action of the interrupter by which the currents are rendered variable. When periodicity is not

1 "Discussion on Prof. Hughes' Address." *Journ. Tel. Eng.*, vol. xv., p. 54. Feb., 1886.

necessary, a scraping contact, actuated by a clock or by a small motor, answers very well; but it is advisable, following Lodge and Hughes, so to arrange matters that the current is suspended altogether at short intervals. The faint scraping sound heard in the neighbourhood of a balance, is more certainly identified when thus rendered intermittent.

But for many of the most interesting experiments a scraping contact is unsuitable. When the inductance and resistance under observation are rapidly varying functions of the frequency, it is evident that no sharp results are possible without an interrupter giving a perfectly regular electrical vibration. With proper appliances an absolute silence, or at least one disturbed only by a slight sensation of the octave of the principal tone, can be arrived at under circumstances where a scraping contact would admit of no approach to a balance at all.

Tuning-forks, driven electromagnetically with liquid or solid contacts (§ 64), answer well so long as the frequency required does not exceed (say) 300 per second; but for experiments with the telephone we desire frequencies of from 500 to 2000 per second. Good results may be obtained with harmonium reed interrupters, the vibrating tongue making contact once during each period with a stop, which can be adjusted exactly to the required position by means of a screw[1].

But perhaps the best interrupter for use with the telephone is obtained by taking advantage of the instability of a jet of fluid. If the diameter and the speed be chosen suitably, the jet may be caused to resolve itself into drops under the action of a tuning-fork in a perfectly regular manner, one drop corresponding to each complete vibration of the fork. Each drop, as it passes, may be made to complete an electric circuit by squeezing itself between the extremities of two fine platinum wires. If the electro-motive force of the battery be pretty high, and if the jet be salted to improve its conductivity, sufficient current passes, especially if the aid of a small step-down transformer be invoked. Finally the apparatus is made self-acting by bringing the fork under the influence of an electro-magnet, itself traversed by the same intermittent current. Such an apparatus may be made to work with frequencies up to 2000 per second, and it possesses many advantages, among which may be mentioned almost absolute

[1] *Phil. Mag.*, vol. 22, p. 472, 1886

constancy of pitch, and the avoidance of loud aerial disturbance. The principles upon which the action of this interrupter depends will be further considered in a subsequent chapter.

235 *s*. Scarcely less important than the interrupter are the arrangements for measuring induction, whether mutual induction, as required in § 235 *q*, or self-induction. Inductometers, as Heaviside calls them, may be conveniently constructed upon the pattern of Hughes. A small coil is mounted so that one diameter coincides with a diameter of a larger coil, and is movable about that diameter. The mutual induction M between the two circuits depends upon the position given to the smaller coil, which is read by a pointer attached to it, and moving over a graduated circle. If the smaller coil were supposed to be infinitely small, the value of M, as has already been stated, would be proportional to the sine of the displacement from the zero position ($M=0$). But an approximation to this state of things is not desirable. If the mean radius of the small coil be increased until it amounts to ·55 of that of the larger, not only is the efficiency much enhanced, but the scale of M is brought to approximate coincidence, over almost the whole practical range, with the scale of degrees[1]. The absolute value of each degree may be arrived at in various ways, perhaps most simply by adjusting the mutual induction of the instrument to balance a standard of mutual induction.

For experiments upon the plan of § 235 *q* the one coil is included in the telephone and the other in the battery branch, but when the object is to secure a variable and measurable inductance, the two coils are connected in series. The inductance of the combination is then $L+2M+N$, of which the first and third terms are independent of the relative position of the coils.

235 *t*. Good results by the method of § 235 *q* have been obtained by Weber[2], and by the author[3] using a reed interrupter of frequency 1050 per second; but the fact that inductance and resistance are mixed up in the measurements is a decided drawback, if it be only because the readings require for their interpretation calculations not readily made upon the spot.

[1] *Phil. Mag.*, vol. 22, p. 498, 1886.

[2] *Electrical Review*, April 9, July 9, 1886.

[3] *Phil. Mag.*, *loc. cit.*

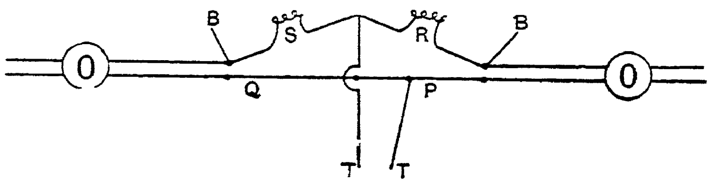
The more obvious arrangement is one in which both the induction and the resistance of the branch containing the subject under examination are in every case brought up to the given totals necessary for a balance. To carry this out conveniently we require to be able to add inductance without altering resistance, and resistance without altering inductance, and both in a measurable degree. The first demand is easily met. If we include in the circuit the *two* coils of an inductometer, connected in series, the inductance of the whole can be varied in a known manner by rotating the smaller coil. On the other hand the introduction, or removal, of resistance without alteration of inductance cannot well be carried out with rigour. But in most cases the object can be sufficiently attained with the aid of a resistance-slide of thin German-silver wire which may be in the form of a nearly close loop.

In the Wheatstone's quadrilateral, as arranged for these experiments, the adjacent sides R, S may be made of similar wires of German silver of equal resistance ($\frac{1}{2}$ ohm). If doubled they give rise to little induction, but the accuracy of the method is independent of this circumstance. The side P includes the conductor, or combination of conductors, under examination, an inductometer, and the resistance-slide. The other side, Q, must possess resistance and inductance greater than any of the conductors to be compared, but need not be susceptible of ready and measurable variations. In order to avoid mutual induction between the branches, P and Q should be placed at some distance away, being connected with the rest of the apparatus by leads of doubled wire.

It will be evident that when the interrupter acts in the battery branch, balance can be obtained at the telephone in the bridge only under the conditions that both the inductance and the resistance in P are equal in the aggregate to the corresponding quantities in Q. Hence when one conductor is substituted for another in P, the alterations demanded at the inductometer and in the slide give respectively the changes of inductance and of resistance. In this arrangement inductance and resistance are well separated, so that the results can be interpreted without calculation; but the movable contacts of the slide appear to introduce uncertainty into the determination of resistance.

In order to get rid of the objectionable movable contacts some sacrifice of theoretical simplicity seems unavoidable. We can no longer keep the total resistances *P* and *Q* constant; but by reverting to the arrangement adopted in a well-known form of Wheatstone's bridge, we cause the resistances taken from *P* to be added to *Q*, and *vice versâ*. The transferable resistance is that of a straight wire of German-silver, with which one telephone terminal makes contact at a point whose position is read off on a divided scale. Any uncertainty in the resistance of *this* contact does not influence the measurements.

Fig. 53 *c*.



The diagram Fig. (53 *c*) shows the connection of the parts. One of the telephone terminals *T* goes to the junction of the ($\frac{1}{2}$ ohm) resistances *R* and *S*, the other to a point upon the divided wire. The branch *P* includes one inductometer (with coils connected in series), the subject of examination, and part of the divided wire. The branch *Q* includes a second inductometer (replaceable by a simple coil possessing suitable inductance), a rheostat, or any resistance roughly adjustable from time to time, and the remainder of the divided wire. The battery branch *B*, in which may also be included the interrupter, has its terminals connected, one to the junction of *P* and *R*, the other to the junction of *Q* and *S*. When it is desired to use steady currents, the telephone can of course be replaced by a galvanometer.

In this arrangement, as in the other, balance requires that the branches *P* and *Q* be similar in respect both of inductance and of resistance. The changes in inductance due to a shift in the movable contact may usually be disregarded, and thus any alteration in the subject (included in *P*) is measured by the rotation necessitated at the inductometer. As for the resistance, it is evident that (*R* and *S* being equal) the value for any additional conductor interposed in *P* is measured by twice the displacement of the sliding contact necessary to regain the balance.

Experimental details of the application of this method to the measurement of various combinations will be found in the paper[1] from which the above sketch is derived. Among these may be mentioned the verification of Maxwell's formulæ, (8), (9) § 235 *k*, as to the influence of a neighbouring circuit, especially in the extreme case where the equivalent inductance is almost destroyed, and of the formula (10) § 235 *m* relating to the behaviour of an electro-magnet shunted by a relatively high simple resistance. But the most interesting in many respects is the application to the phenomena presently to be considered, where the conductors in question are no longer approximately linear but must be regarded as solid masses in which the currents are distributed in a manner that needs to be determined by general electrical theory.

As has already been remarked more than once, a leyden may always be supposed to be included in the circuit, the stiffness thereof having the effect of a negative inductance. If there be no hysteresis in the action of the leyden, the whole effect is thus represented; but when the dielectric employed is solid, it appears that dissipative loss cannot be avoided. The latter effect manifests itself as an augmentation of apparent resistance, indistinguishable, unless the frequency be varied, from the ordinary resistance of the leads. A similar treatment may be applied to an electrolytic cell, the stiffness and resistance being presumably both functions of the frequency.

235 *u*. It was proved by Maxwell[2] that a perfectly conducting sheet, forming a closed or an infinite surface, acts as a magnetic screen, no magnetic actions which may take place on one side of the sheet producing any magnetic effect on the other side. "In practice we cannot use a sheet of perfect conductivity; but the above described state of things may be approximated to in the case of periodic magnetic changes, if the time-constants of the sheet circuits be large in comparison with the periods of the changes."

"The experiment is made by connecting up into a primary circuit a battery, a microphone-clock, and a coil of insulated wire. The secondary circuit includes a parallel coil and a telephone. Under these circumstances the hissing sound is heard almost as well as if the telephone were inserted in the primary circuit

[1] *Phil. Mag., loc. cit.*

[2] *Electricity and Magnetism*, 1873, § 655.

itself. But if a large and stout plate of copper be interposed between the two coils, the sound is greatly enfeebled. By a proper choice of battery and of the distance between the coils, it is not difficult so to adjust the strength that the sound is conspicuous in the one case and inaudible in the other"[1].

One of the simplest applications of Maxwell's principle is to the case of a long cylindrical shell placed within a coaxal magnetizing helix. The condition of minimum energy requires that such currents be developed in the shell as shall neutralize at internal points the action of the coil. Thus, if the conductivity of the shell be sufficiently high, the interior space is screened from the magnetizing force of periodic currents flowing in the outer helix, and conducting circuits situated within the shell must be devoid of induced currents. An obvious deduction is that the currents induced in a solid conducting core will be more and more confined to the neighbourhood of the surface as the frequency of electrical vibration is increased.

The point at which the concentration of current towards the surface becomes important depends upon the relative values of the imposed vibration-period and the principal time-constant of the core circuit. If ρ be the specific resistance of the material, μ its magnetic permeability, a the radius of the cylinder, the expression for the induction (c) parallel to the axis, during the progress of the subsidence of free currents in a normal mode, is

$$c = e^{\lambda t} J_0(kr) \qquad (1),$$

where

$$k^2 = -\frac{4\pi\lambda\mu}{\rho} \qquad (2),$$

and ka is determined by the condition that

$$J_0(ka) = 0 \qquad (3).$$

The roots of (3) are, § 206,

2·404, 5·520, 8·654, 11·792, &c.,

so that for the principal mode of greatest persistence

$$c = e^{\lambda t} J_0(2{\cdot}404\, r/a) \qquad (4),$$

where

$$\lambda = -\frac{(2{\cdot}404)^2 \rho}{4\pi\mu a^2} \qquad (5).$$

[1] Acoustical Observations, *Phil. Mag.*, vol. 13, p. 344, 1882.

For copper in C.G.S. measure $\rho = 1642$, $\mu = 1$, and thus

$$\tau = (-\lambda)^{-1} = \frac{a^2}{800} \text{ nearly}^{1}.$$

In the case of iron we may take as approximate values, $\mu = 100$, $\rho = 10^4$. Thus for an iron wire of diameter $(2a)$ equal to ·33 cm., the value of τ is about $\frac{1}{2000}$ of a second, and is therefore comparable with the periods concerned in telephonic experiments.

Regarded from an analytical point of view the theory of forced vibrations in a conducting core is equally simple, and was worked out almost simultaneously by Lamb[2], Oberbeck[3] and Heaviside[4]. In this case we are to regard λ as given, equal (say) to ip, where $p/2\pi$ is the frequency. If Ie^{ipt} be the imposed magnetizing force, the solution is

$$c = \frac{J_0(kr)}{J_0(ka)} \mu I e^{ipt} \quad \text{.......................(6)},$$

the value of k being given by (2).

"When the period in the field is long in comparison with the time of decay of free currents, we have $J_0(kr) = 1$, nearly, so that c is approximately constant and $= \mu I$ throughout the section of the cylinder. But, in the opposite extreme, when the oscillations in the intensity of the field are rapid in comparison with the decay of free currents, the induced currents extend only to a small depth beneath the surface of the cylinder, the inner strata (so to speak) being almost completely sheltered from electromotive force by the outer ones. Writing $k^2 = (1-i)^2 q^2$, where

$$q^2 = \frac{2\pi\mu p}{\rho},$$

we have, when qr is large,

$$J_0(kr) = \text{const.} \times \frac{e^{qr} \,.\, e^{i(qr - \frac{1}{4}\pi)}}{\sqrt{r}},$$

approximately, and thence

$$c = \mu I \,.\, \sqrt{(a/r)} \,.\, e^{q(r-a) + iq(r-a)} \, [e^{ipt}].$$

This indicates that the electrical disturbance in the cylinder

[1] "On the Duration of Free Electric Currents in an Infinite Conducting Cylinder," *Brit. Assoc. Report* for 1882, p. 446.

[2] *Proc. Math. Soc.*, vol. xv., p. 139, Jan. 1884.

[3] *Wied. Ann.*, vol. xxi., p. 672, Ap. 1884.

[4] *Electrician*, May, 1884. *Electrical Papers*, vol. ii., p. 353.

consists in a series of waves propagated inwards with rapidly diminishing amplitude[1]."

For experimental purposes what we most require to know is the reaction of the core currents upon the helix, in which alone we can directly measure electrical effects. This problem is fully treated by Heaviside[2], but we must confine ourselves here to a mere statement of results. These are most conveniently expressed by the changes of effective inductance L and resistance R due to the core. If m be the number of turns per unit length in the magnetizing helix, and if δL, δR be the apparent alterations of L and R due to the introduction of the core, also reckoned per unit length, we have

$$\left.\begin{aligned}\delta L &= 4m^2\pi^2a^2(\mu P - 1)\\ \delta R &= 4m^2\pi^2a^2\mu . pQ\end{aligned}\right\} \quad\text{(7)},$$

where P and Q are defined by

$$P - iQ = \phi'/\phi \quad\text{(8)},$$

the function ϕ being of the form

$$\phi(x) = J_0(2i\sqrt{x}) = 1 + x + \frac{x^2}{1^2.2^2} + \ldots + \frac{x^n}{1^2.2^2\ldots n^2} + \ldots\ldots \quad\text{(9)},$$

and the argument x being

$$ip\mu.\pi a^2/\rho \quad\text{(10)}.$$

If the material composing the core be non-conducting, $x=0$, and therefore

$$P=1, \qquad Q=0.$$

Accordingly $\quad \delta L = 4m^2\pi^2a^2(\mu - 1), \qquad \delta R = 0 \quad\text{(11)}.$

These values apply also, whatever be the conductivity of the core, if the frequency be sufficiently low.

At the other extreme, when $p=\infty$, we require the ultimate form of ϕ'/ϕ. From the value of J_0 given in (10) § 200, or otherwise, it may be shewn that in the limit

$$\phi'/\phi = x^{-\frac{1}{2}} \quad\text{(12)},$$

so that

$$P = Q = \frac{1}{\sqrt{\{2p\mu.\pi a^2/\rho\}}} \quad\text{(13)}.$$

The introduction of these values into (7) shews that in the limit, when the frequency is exceedingly high,

$$\delta L = -4m^2\pi^2a^2, \qquad \delta R = 0 \quad\text{(14)},$$

[1] Lamb, *loc. cit.*, where is also discussed the problem of the currents induced by the sudden cessation of a previously constant field.

[2] *loc. cit.*

as might also have been inferred from the consideration that the induced currents are then confined to the surface of the core.

An example of the application of these formulæ to an intermediate case and a comparison with experiment will be found in the paper already referred to[1].

235 *v*. The application of Maxwell's principle to the case of a wire, in which a longitudinal electric current is induced, is less obvious; and Heaviside[2] appears to have been the first to state distinctly that the current is to be regarded as propagated inwards from the exterior. The relation between the electromotive force E and the total current C had, however, been given many years earlier by Maxwell[3] in the form of a series. His result is equivalent to

$$\frac{E}{RC} = ipl/R \,.\, A + \frac{\phi\,(ipl\mu/R)}{\phi'\,(ipl\mu/R)} \quad \text{..............} \quad (1),$$

in which R denotes the whole resistance of the length l to steady currents, μ the permeability, and $p/2\pi$ the frequency. The function ϕ is that defined by (9) § 235 *u*, and A is a constant dependent upon the situation of the return current[4].

The most convenient form of the results is that which we have already several times employed. If we write

$$E = R'C + ipL'C \quad \text{......................} \quad (2),$$

in which R' and L' are real, these quantities will represent the effective resistance and inductance of the wire. When the argument in (1) is small, that is when the frequency is relatively low, we thus obtain

$$R' = R\left\{1 + \tfrac{1}{12}\frac{p^2 l^2 \mu^2}{R^2} - \tfrac{1}{180}\frac{p^4 l^4 \mu^4}{R^4} + \ldots\right\} \quad \text{.........} \quad (3),$$

$$L'/l = A + \mu\left\{\tfrac{1}{2} - \tfrac{1}{48}\frac{p^2 l^2 \mu^2}{R^2} + \tfrac{13}{8640}\frac{p^4 l^4 \mu^4}{R^4} + \ldots\right\} \text{......} (4)[5].$$

[1] *Phil. Mag.*, vol. 22, p. 493, 1886.

[2] *Electrician*, Jan., 1885; *Electrical Papers*, vol. i., p. 440.

[3] *Phil. Trans.*, 1865; *Electricity and Magnetism*, vol. ii., § 690.

[4] The simplest case arises when the dielectric, which bounds the cylindrical wire of radius a, is enclosed within a second conducting mass extending outwards to infinity and bounded internally at a cylindrical surface $r=b$. We then have $A = 2\log(b/a)$. See J. J. Thomson, *loc. cit.*, § 272.

[5] *Phil. Mag.*, vol. 21, p. 387, 1886. It is singular that Maxwell (*loc. cit.*) seems to have regarded his solution as conveying a correction to the self-induction only of the wire.

When p is very small, these equations give, as was to be expected,

$$R' = R, \qquad L' = l\,(A + \tfrac{1}{2}\mu) \quad\quad (5).$$

If we include the next terms, we recognise that, in accordance with the general rule, L' begins to diminish and R' to increase.

When p is very great, we have to make use of the limiting form of ϕ'/ϕ. As in § 235 *u*,

$$\phi/\phi' = (1 + i)\,\sqrt{(\tfrac{1}{2}pl\mu/R)} \quad\quad (6);$$

and thus ultimately

$$R' = \sqrt{(\tfrac{1}{2}pl\mu R)} \quad\quad (7),$$

$$L'/l = A + \sqrt{(\mu R/2pl)} \quad\quad (8),$$

the first of which increases without limit with p, while the second tends to the finite limit A, corresponding to the total exclusion of current from the interior of the wire.

Experiments[1] upon an iron wire about 18 metres long and 3·3 millimetres in diameter led to the conclusion that the resistance to variable currents of frequency 1050 was such that $R'/R = 1{\cdot}9$. A calculation based upon (1) shewed that this result is in harmony with theory, if $\mu = 99{\cdot}5$. Such is about the value indicated by other telephonic experiments.

235 *w*. The theory of electric currents in such wires as are commonly employed in laboratory experiments is simple, mainly in consequence of the subordination of electrostatic capacity. When this element can be neglected, the current is necessarily the same at all points along the length of the wire, so that whatever enters a wire at the sending end leaves it unimpaired at the receiving end. In this case the whole electrical character of the wire can be expressed by two quantities, its resistance R and inductance L, and these may usually be treated as constants, independent of the frequency. The relation of the current to the electromotive force under such circumstances has already been discussed (7) § 235 *j*. When we have occasion to consider only the amplitude of the current, irrespective of phase, we may regard it as determined by $\sqrt{[R^2 + p^2L^2]}$, a quantity which is called by Heaviside the *impedance*. Thus in circuits devoid of capacity the impedance is always increased by the existence of L.

[1] *Phil. Mag.*, vol. 22, p. 488, 1886.

Circuits employed for practical telephony may often be regarded as coming under the above description, especially when the wires are suspended and are of but moderate length. But there are other cases in which electrostatic capacity is the dominating feature. The theory of electric cables was established many years ago by Lord Kelvin[1] for telegraphic purposes. If S be the capacity and R the resistance of the cable, reckoned per unit length, V and C the potential and the current at the point z, we have

$$S\,dV/dt = -dC/dz, \qquad RC = -dV/dz \quad\text{......... (1)},$$

whence

$$RS\,dC/dt = d^2C/dz^2 \quad\text{....................... (2)},$$

the well known equation for the conduction of heat discussed by Fourier. On the assumption that C is proportional to e^{ipt}, it reduces to

$$d^2C/dz^2 = \{\sqrt{(\tfrac{1}{2}pRS)}.(1+i)\}^2 C \quad\text{.............. (3)};$$

so that the solution for waves propagated in the positive direction is

$$C = C_0 e^{-\sqrt{(\frac{1}{2}pRS)}.z} \cos\{pt - \sqrt{(\tfrac{1}{2}pRS)}.z\} \quad\text{......... (4)}.$$

The distance in traversing which the current is attenuated in the ratio of e to 1 is thus

$$z = \sqrt{(2/pRS)} \quad\text{....................... (5)}.$$

A very slight consideration of the magnitudes involved is sufficient to give an idea of the difficulty of telephoning through a long cable. If, for example, the frequency $(p/2\pi)$ be that of a note rather more than an octave above middle c, and the cable be such as are used to cross the Atlantic, we have in C.G.S. measure

$$\sqrt{p} = 60, \qquad (RS)^{-1} = 2 \times 10^{16},$$

and accordingly from (5)

$$z = 3 \times 10^6 \text{ cm.} = 20 \text{ miles approximately.}$$

A distance of 20 miles would thus reduce the intensity of sound, measured by the square of the amplitude, to about a tenth, an operation which could not be repeated often without rendering it inaudible. With such a cable the practical limit would not be likely to exceed fifty miles, more especially as the easy intelligibility of speech requires the presence of tones still higher than is supposed in the above numerical example[2].

[1] *Proc. Roy. Soc.*, 1855; *Mathematical and Physical Papers*, vol. II. p. 61.

[2] "On Telephoning through a Cable." *Brit. Ass. Report* for 1884, p. 632.

235 x. In the above theory the insulation is supposed to be perfect and the inductance to be negligible. It is probable that these conditions are sufficiently satisfied in the case of a cable, but in other telephonic lines the inductance is a feature of great importance. The problem has been treated with full generality by Heaviside, but a slight sketch of his investigation is all that our limits permit.

If R, S, L, K be the resistance, capacity or permittance, inductance, and leakage-conductance respectively per unit of length, V and C the potential-difference and current at distance z, the equations, analogous to (1) § 235 w, are

$$KV + S\frac{dV}{dt} = -\frac{dC}{dz}. \qquad RC + L\frac{dC}{dt} = -\frac{dV}{dz} \;\ldots\ldots\; (1).$$

Thus, if the currents are harmonic, proportional to e^{ipt},

$$\frac{d^2C}{dz^2} = (R + ipL)(K + ipS)\,C \;\ldots\ldots\ldots\ldots\; (2),$$

with a similar equation for V.

It might perhaps have been expected that a finite leakage K would always act as a complication; but Heaviside[1] has shewn that it may be so adjusted as to simplify the matter. This case, which is remarkable in itself and also serves to throw light upon the general question, arises when $R/L = K/S$. We will write

$$LSv^2 = 1, \qquad R/L = K/S = q \;\ldots\ldots\ldots\; (2),$$

where v is a velocity of the order of the velocity of light. The equation for V is then by (1)

$$v^2\, d^2V/dz^2 = (d/dt + q)^2 V \;\ldots\ldots\ldots\; (3);$$

or if we take U so that

$$V = e^{-qt}\, U \;\ldots\ldots\ldots\ldots\; (4),$$

$$v^2\, d^2U/dz^2 = d^2U/dt^2 \;\ldots\ldots\ldots\; (5),$$

the well-known equation of undisturbed wave propagation § 144. "Thus, if the wave be positive, or travel in the direction of increasing z, we shall have, if $f_1(z)$ be the state of V initially,

$$V_1 = e^{-qt} f_1(z - vt), \qquad C_1 = V_1/Lv \;\ldots\ldots\ldots\; (6).$$

If V_2, C_2 be a negative wave, travelling the other way,

$$V_2 = e^{-qt} f_2(z + vt), \qquad C_2 = -V_2/Lv \;\ldots\ldots\ldots\; (7).$$

[1] *Electrician*, June 17, 1887. *Electrical Papers*, vol. II. pp. 125, 309.

Thus, any initial state being the sum of V_1 and V_2 to make V, and of C_1 and C_2 to make C, the decomposition of an arbitrarily given initial state of V and C into the waves is effected by

$$V_1 = \tfrac{1}{2}(V + vLC), \qquad V_2 = \tfrac{1}{2}(V - vLC) \quad \text{.........} (8).$$

We have now merely to move V_1 bodily to the right at speed v, and V_2 bodily to the left at speed v, and attenuate them to the extent e^{-qt}, to obtain the state at time t later, provided no changes of condition have occurred. The solution is therefore true for all future time in an infinitely long circuit. But when the end of a circuit is reached, a reflected wave usually results, which must be added on to obtain the real result."

As in § 144, the precise character of the reflection depends upon the terminal conditions. "One case is uniquely simple. Let there be a resistance inserted of amount vL. It introduces the condition $V = vLC$ if at say B, the positive end of the circuit, and $V = -vLC$ if at the negative end, or beginning. These are the characteristics of a positive and of a negative wave respectively; it follows that any disturbance arriving at the resistance is at once absorbed. Thus, if the circuit be given in any state whatever, without impressed force, it is wholly cleared of electrification and current in the time l/v at the most, if l be the length of the circuit, by the complete absorption of the two waves into which the initial state may be decomposed."

"But let the resistance be of amount R_1 at say B; and let V_1 and V_2 be corresponding elements in the incident and reflected waves. Since we have

$$V_1 = vLC_1, \qquad V_2 = -vLC_2, \qquad V_1 + V_2 = R_1(C_1 + C_2) \text{...} (9),$$

we have the reflected wave given by

$$\frac{V_2}{V_1} = \frac{R_1 - vL}{R_1 + vL} \quad \text{......................} (10).$$

If R_1 be greater than the critical resistance of complete absorption, the current is negatived by reflection, whilst the electrification does not change sign. If it be less, the electrification is negatived, whilst the current does not reverse."

"Two cases are specially notable. They are those in which there is no absorption of energy. If $R_1 = 0$, meaning a short circuit, the reflected wave of V is a perverted and inverted copy of the incident. But if $R = \infty$, representing insulation, it is C that is inverted and perverted[1]."

The cases last mentioned are evidently analogous to the reflection of a sonorous aerial wave travelling in a pipe. If the end of the pipe be closed, the reflection is of one character, and if it be open of another character. In both cases the whole energy is reflected, § 257. The waves reflected at the two ends of an electric circuit complicate the general solution, especially when the simplifying condition (2) does not hold. But in many cases of practical interest they may be omitted without much loss of accuracy. One passage over a long line usually introduces considerable attenuation, and then the effect of the reflected wave, which must traverse the line three times in all, becomes insignificant.

In proceeding to the general solution of (2) for a positive wave, we will introduce, after Heaviside, the following abbreviations,

$$v^2LS = 1, \qquad R/Lp = f, \qquad K/Sp = g \text{ ....(11).}$$

In terms of these quantities (2) may be written

$$d^2C/dz^2 = (P + iQ)^2\, C \text{ .... (12),}$$

where

$$P^2 \text{ or } Q^2 = \tfrac{1}{2}\,(p/v)^2\, \{(1+f^2)^{\frac{1}{2}}\,(1+g^2)^{\frac{1}{2}} \pm (fg - 1)\} \text{ ... (13).}$$

Thus, if P and Q be taken positively, the solution for a wave travelling in the positive direction is

$$C = C_0 e^{-Pz} \cos(pt - Qz) \text{ ....(14),}$$

the current at the origin being $C_0 \cos pt$.

The cable formula, § 235 w, is the particular case arrived at by supposing in (13) $f = \infty$, $g = 0$, which then reduces to

$$P^2 = Q^2 = \tfrac{1}{2} pRS \text{ .... (15).}$$

Again, the special case of equation (3) is derivable by putting $f = g = q/p$. The result is

$$P = q/v, \qquad Q = p/v \text{ ....(16).}$$

If the insulation be perfect, $g = 0$, and (13) becomes

$$P^2 \text{ or } Q^2 = \tfrac{1}{2}\,(p/v)^2\, \{(1+f^2)^{\frac{1}{2}} \mp 1\} \text{ .... (17).}$$

[1] Heaviside, *Collected Works*, vol. II. p. 312.

In certain examples of long copper lines of high conductivity, f may be regarded as small so far as telephonic frequencies are concerned. Equation (17) then gives

$$P = pf/2v = R/2vL, \qquad Q = p/v \text{ .... (18).}$$

For a further discussion of the various cases that may arise the reader must be referred to the writings of Heaviside already cited. The object is to secure, as far as may be, the propagation of waves without alteration of type. And here it is desirable to distinguish between simple attenuation and distortion. If, as in (16) and (18), P is independent of p, the amplitudes of all components are reduced in the same ratio, and thus a complex wave travels without *distortion*. The cable formula (15) is an example of the opposite state of things, where waves of high frequency are attenuated out of proportion to waves of low frequency. It appears from Heaviside's calculations that the distortion is lessened by even a moderate inductance.

The effectiveness of the line requires that neither the attenuation nor the distortion exceed certain limits, which however it is hard to lay down precisely. A considerable amount of distortion is consistent with the intelligibility of speech, much that is imperfectly rendered being supplied by the imagination of the hearer.

235 y. It remains to consider the transmitting and receiving appliances. In the early days of telephony, as rendered practical by Graham Bell, similar instruments were employed for both purposes. Bell's telephone consists of a bar magnet, or battery of bar magnets, provided at one end with a short pole-piece which serves as the core of a coil of fine insulated wire. In close proximity to the outer end of the pole-piece is placed a circular disc of thin iron, held at the circumference. Under the influence of the permanent magnet the disc is magnetized radially, the polarity at the centre being of course opposite to that of the neighbouring end of the steel magnet.

The operation of the instrument as a transmitter is readily traced. When sonorous waves impinge upon the disc, it responds with a symmetrical transverse vibration by which its distance from the pole-piece is alternately increased and diminished. When the interval is diminished, more induction passes through the pole-piece, and a corresponding electro-motive force acts in the enveloping coil. The periodic movement of the disc thus gives rise to a periodic current in any circuit connected with the telephone coil.

The electro-motive force is in the first instance proportional to the permanent magnetism to which it is due; and this law would continue to hold, were the behaviour of the pole-piece and of the disc conformable to that of the "soft iron" of approximate theory. But as the magnetism rises, and the state of saturation is more nearly approached, the response to periodic changes of force becomes feebler, and thus the efficiency falls below that indicated by the law of proportionality. If we could imagine the state of saturation in the pole-piece to be actually attained, the induction through the coil would become almost incapable of variation, being reduced to such as might occur were the iron removed. There is thus a point, dependent upon the properties of magnetic matter, beyond which it is pernicious to raise the amount of the permanent magnetism; and this point marks the maximum efficiency of the transmitter. It is probable that the most favourable condition is not fully reached in instruments provided with steel magnets; but the considerations above advanced may serve to explain why an electro-magnet is not substituted.

The action of the receiving instrument may be explained on the same principles. The periodic current in the coil alternately opposes and cooperates with the permanent magnet, and thus the iron disc is subjected to a periodic force acting at its centre. The vibrations are thence communicated to the air, and so reach the ear of the observer. As in the case of the transmitter, the efficiency attains a maximum when the magnetism of the pole-piece is still far short of saturation.

The explanation of the receiver in terms of magnetic forces pulling at the disc is sometimes regarded as inadequate or even as altogether wide of the mark, the sound being attributed to "molecular disturbances" in the pole-piece and disc. There is indeed every reason to suppose that molecular movements accompany the change of magnetic state, but the question is how do these movements influence the ear. It would appear that they can do so only by causing a transverse motion of the surface of the disc, a motion from which nodal subdivisions are not excluded.

In support of the "push and pull theory" it may be useful to cite an experiment tried upon a bipolar telephone. In this instrument each end of a horse-shoe magnet is provided with a pole-piece and coil, and the two pole-pieces are brought into proximity with the disc at places symmetrically situated with regard to the centre. In the normal use of the instrument the two coils are permanently connected as in an ordinary horse-shoe electro-magnet, but for the purposes of the experiment provision was made whereby one of the coils could be reversed at pleasure by means of a reversing key. The sensitiveness of the telephone in the two conditions was tested by including it in the circuit of a Daniell cell and a scraping contact apparatus, resistance from a box being added until the sound was but just easily audible. The resistances employed were such as to dominate the self-induction of the circuit, and the comparison shewed that the reversal of the coil from its normal connection lowered the sensitiveness to current in the ratio of 11 : 1. That the reduction was not still greater is readily explained by outstanding failures of symmetry; but on the "molecular disturbance" theory it is not evident why there should be any reduction at all.

Dissatisfaction with the ordinary theory of the action of a receiving telephone may have arisen from the difficulty of understanding how such very minute motions of the plate could be audible. This is, however, a question of the sensitiveness of the ear, which has been proved capable of appreciating an amplitude of less than 8×10^{-8} cm.[1]. The subject of the audible minimum will be further considered in the second volume of this work.

The calculation *a priori* of the minimum current that should be audible in the telephone is a matter of considerable difficulty; and even the determination by direct experiment has led to widely discrepant numbers. In some recent experiments by the author a unipolar Bell telephone of 70 ohms resistance was employed. The circuit included also a resistance box and an induction coil of known construction, in which acted an electromotive force capable of calculation. Up to a frequency of 307 this could be obtained from a revolving magnet of known moment and situated at a measured distance from the induction coil. For the higher frequencies magnetized tuning-forks, vibrating with measured amplitudes, were substituted. In either case the

[1] *Proc. Roy. Soc.* vol. xxvi. p. 248, 1877.

resistance of the circuit was increased until the residual sound was but just easily audible. Care having been taken so to arrange matters that the self-induction of the circuit was negligible, the current could then be deduced from the resistance and the calculated electro-motive force operating in the induction coil. The following are the results, in which it is to be understood that the currents recorded might have been halved without the sounds being altogether lost:

Pitch	Source	Current in 10^{-8} amperes
128	Fork	2800
192	Revolving Magnet	250
256	Fork	83
307	Revolving Magnet	49
320	Fork	32
384	....	15
512	....	7
640	....	4·4
768	....	10

The effect of a given current depends, of course, upon the manner in which the telephone is wound. If the same space be occupied by the copper in the various cases, the current capable of producing a particular effect is inversely as the square root of the resistance.

The numbers in the above table giving the results of the author's experiments are of the same order of magnitude as those found by Ferraris[1], whose observations, however, related to sounds that were not pure tones. But much lower estimates have been put forward. Thus Tait[2] gives 2×10^{-12} amperes, and Preece a still lower figure, 6×10^{-13}. These discrepancies, enormous as they stand, would be still further increased were the comparison made to refer to the amounts of energy absorbed.

According to the calculations of the author the above tabulated sensitiveness to a periodic current of frequency 256 is about what might reasonably be expected on the push and pull theory[3]. At

[1] *Atti della Accad. d. Sci. Di Torino*, vol. XIII. p. 1024, 1877.
[2] *Edin. Proc.* vol. IX. p. 551, 1878.
[3] I propose shortly to publish these calculations.

this frequency, which is below those proper to the telephone plate (§ 221 *a*), the motion of the plate is governed by elasticity rather than by inertia, and an equilibrium theory (§ 100) is applicable as a rough approximation. The greater sensitiveness of the telephone at frequencies in the neighbourhood of 512 would appear to depend upon resonance (§ 46). It is doubtful whether the much higher sensitiveness claimed by Tait and Preece could be reconciled with theory.

It appears to be established that the iron plate of a telephone may be replaced by one of copper, or even of non-conducting material, without absolute loss of sound; but these effects are probably of a different order of magnitude. In the case of copper induced currents may confer the necessary magnetic properties. For a description of the ingenious receiver invented by Edison and for other information upon telephonic appliances the reader may consult Preece and Stubbs' *Manual of Telephony*.

In existing practice the transmitting instrument depends upon a variable contact. The first carbon transmitter was constructed by Edison in 1877, but the instruments now in use are modifications of Hughes' microphone[1]. A battery current is led into the line through pieces of metal or of carbon in loose juxtaposition, carbon being almost universally employed in practice. Under the influence of sonorous vibration the electrical resistance of the contacts varies, and thus the current in the line is rendered representative of the sound to be reproduced at the receiving end.

That the resistance of the contact should vary with the pressure is not surprising. If two clean convex pieces of metal are forced together, the conductivity between them is represented by the diameter of the circle of contact (§ 306). The relation between the circle of contact and the pressure with which the masses are forced together has been investigated in detail by Hertz[2]. His conclusion for the case of two equal spheres is that the cube of the radius of the circle of contact is proportional to the pressure and to the radii of the spheres. But it has not yet been shewn that the action of the microphone can be adequately explained upon this principle.

[1] *Proc. Roy. Soc.*, vol. XXVII. p. 362, 1878.
[2] Crelle, *Journ. Math.* XCII. p. 156, 1882.

APPENDIX.

ON PROGRESSIVE WAVES.

From the Proceedings of the London Mathematical Society, Vol. IX., p. 21, 1877.

It has often been remarked that, when a group of waves advances into still water, the velocity of the group is less than that of the individual waves of which it is composed; the waves appear to advance through the group, dying away as they approach its anterior limit. This phenomenon was, I believe, first explained by Stokes, who regarded the group as formed by the superposition of two infinite trains of waves, of equal amplitudes and of nearly equal wave-lengths, advancing in the same direction. My attention was called to the subject about two years since by Mr Froude, and the same explanation then occurred to me independently*. In my book on the "Theory of Sound" (§ 191), I have considered the question more generally, and have shewn that, if V be the velocity of propagation of any kind of waves whose wave-length is λ, and $k = 2\pi/\lambda$, then U, the velocity of a group composed of a great number of waves, and moving into an undisturbed part of the medium, is expressed by

$$U = \frac{d\,(kV)}{dk} \quad\text{.................................}(1),$$

* Another phenomenon, also mentioned to me by Mr Froude, admits of a similar explanation. A steam-launch moving quickly through the water is accompanied by a peculiar system of diverging waves, of which the most striking feature is the obliquity of the line containing the greatest elevations of successive waves to the wave-fronts. This wave pattern may be explained by the superposition of two (or more) infinite trains of waves, of slightly differing wave-lengths, whose directions and velocities of propagation are so related in each case that there is no change of position relatively to the boat. The mode of composition will be best understood by drawing on paper two sets of parallel and equidistant lines, subject to the above condition, to represent the crests of the component trains. In the case of two trains of slightly different wave-lengths, it may be proved that the tangent of the angle between the line of maxima and the wave-fronts is half the tangent of the angle between the wave-fronts and the boat's course.

or, as we may also write it,

$$U : V = 1 + \frac{d \log V}{d \log k} \quad \text{.........................(2).}$$

Thus, if $V \propto \lambda^n$, $\qquad U = (1 - n) V$............................(3).

In fact, if the two infinite trains be represented by $\cos k (Vt - x)$ and $\cos k' (V't - x)$, their resultant is represented by

$$\cos k (Vt - x) + \cos k' (V't - x),$$

which is equal to

$$2 \cos \left\{\frac{k'V' - kV}{2} t - \frac{k' - k}{2} x\right\} . \cos \left\{\frac{k'V' + kV}{2} t - \frac{k' + k}{2} x\right\}.$$

If $k' - k$, $V' - V$ be small, we have a train of waves whose amplitude varies slowly from one point to another between the limits 0 and 2, forming a series of groups separated from one another by regions comparatively free from disturbance. The position at time t of the middle of that group, which was initially at the origin, is given by

$$(k'V' - kV) t - (k' - k) x = 0,$$

which shews that the velocity of the group is $(k'V' - kV) \div (k' - k)$. In the limit, when the number of waves in each group is indefinitely great, this result coincides with (1).

The following particular cases are worth notice, and are here tabu lated for convenience of comparison:—

$V \propto \lambda$,	$U = 0$,	Reynolds' disconnected pendulums.
$V \propto \lambda^{\frac{1}{2}}$,	$U = \frac{1}{2} V$,	Deep-water gravity waves.
$V \propto \lambda^0$,	$U = V$,	Aërial waves, &c.
$V \propto \lambda^{-\frac{1}{2}}$,	$U = \frac{3}{2} V$,	Capillary water waves.
$V \propto \lambda^{-1}$,	$U = 2V$,	Flexural waves.

The capillary water waves are those whose wave-length is so small that the force of restitution due to capillarity largely exceeds that due to gravity. Their theory has been given by Thomson (Phil. Mag., Nov. 1871). The flexural waves, for which $U = 2V$, are those corresponding to the bending of an elastic rod or plate ("Theory of Sound," § 191).

In a paper read at the Plymouth meeting of the British Association (afterwards printed in "Nature," Aug. 23, 1877), Prof. Osborne Reynolds gave a dynamical explanation of the fact that a group of deep-water waves advances with only half the rapidity of the individual waves. It appears that the energy propagated across any point, when a train of waves is passing, is only one-half of the energy necessary to supply the waves which pass in the same time, so that, if the train of waves be limited, it is impossible that its front can be propagated with the full velocity of the waves, because this would imply the acquisition of more energy than can in fact be supplied. Prof. Reynolds did not contemplate the cases where *more* energy is propagated than corresponds to the waves passing in the same time; but his argument, applied conversely to the results already given, shews that such cases must exist. The ratio of the energy propagated to that of the passing waves is $U : V$; thus the energy propagated in the unit time is $U : V$ of that existing in a length V, or U times that existing in the unit length. Accordingly

Energy propagated in unit time : Energy contained (on an average) in unit length $= d(kV) : dk$, by (1).

As an example, I will take the case of small irrotational waves in water of finite depth l*. If z be measured downwards from the surface, and the elevation (h) of the wave be denoted by

$$h = H\cos(nt - kx) \quad\text{......................}(4),$$

in which $n = kV$, the corresponding velocity-potential (ϕ) is

$$\phi = -VH\frac{e^{k(z-l)} + e^{-k(z-l)}}{e^{kl} - e^{-kl}}\sin(nt - kx) \quad\text{..............}(5).$$

This value of ϕ satisfies the general differential equation for irrotational motion $(\nabla^2\phi = 0)$, makes the vertical velocity $d\phi/dz$ zero when $z = l$, and $-dh/dt$ when $z = 0$. The velocity of propagation is given by

$$V^2 = \frac{g}{k}\frac{e^{kl} - e^{-kl}}{e^{kl} + e^{-kl}} \quad\text{.........................}(6).$$

We may now calculate the energy contained in a length x, which is supposed to include so great a number of waves that fractional parts may be left out of account.

For the potential energy we have

$$V_1 = g\rho\iint_0^h z\,dz\,dx = \tfrac{1}{2}g\rho\int h^2\,dx = \tfrac{1}{4}g\rho H^2 . x \quad\text{..........}(7).$$

For the kinetic energy,

$$T = \tfrac{1}{2}\rho\iint\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right\}dx\,dz$$

$$= \tfrac{1}{2}\rho\int\left(\phi\frac{d\phi}{dz}\right)_{z=0}dx = \tfrac{1}{4}g\rho H^2 . x \quad\text{...................}(8),$$

by (1) and (6). If, in accordance with the argument advanced at the

* Prof. Reynolds considers the trochoidal wave of Rankine and Froude, which involves molecular rotation.

end of this paper, the equality of V_1 and T be assumed, the value of the velocity of propagation follows from the present expressions. The whole energy in the waves occupying a length x is therefore (for each unit of breadth) $$V_1 + T = \tfrac{1}{2} g\rho H^2 . x \quad\text{.........................(9)},$$

H denoting the maximum elevation.

We have next to calculate the energy propagated in time t across a plane for which x is constant, or, in other words, the work (W) that must be done in order to sustain the motion of the plane (considered as a flexible lamina) in the face of the fluid pressures acting upon the front of it. The variable part of the pressure (δp), at depth z, is given by

$$\delta p = -\rho \frac{d\phi}{dt} = -nVH \frac{e^{k(z-l)} + e^{-k(z-l)}}{e^{kl} - e^{-kl}} \cos(nt - kx),$$

while for the horizontal velocity

$$\frac{d\phi}{dx} = kVH \frac{e^{k(z-l)} + e^{-k(z-l)}}{e^{kl} - e^{-kl}} \cos(nt - kx);$$

so that $$W = \iint \delta p \frac{d\phi}{dx}\, dz\, dt = \tfrac{1}{4} g\rho H^2 . Vt . \left[1 + \frac{4kl}{e^{2kl} - e^{-2kl}}\right] \quad\text{......(10)},$$

on integration. From the value of V in (6) it may be proved that

$$\frac{d(kV)}{dk} = \tfrac{1}{2} V \left\{1 + \frac{1}{V^2} \frac{d(kV^2)}{dk}\right\} = \tfrac{1}{2} V \left\{1 + \frac{4kl}{e^{2kl} - e^{-2kl}}\right\};$$

and it is thus verified that the value of W for a unit time

$$= \frac{d(kV)}{dk} \times \text{energy in unit length.}$$

As an example of the direct calculation of U, we may take the case of waves moving under the joint influence of gravity and cohesion.

It is proved by Thomson that

$$V^2 = \frac{g}{k} + T'k \quad\text{........................ .. (11)},$$

where T' is the cohesive tension. Hence

$$U = \tfrac{1}{2} V \left\{1 + \frac{1}{V^2} \frac{d(kV^2)}{dk}\right\} = \tfrac{1}{2} V \frac{g + 3k^2 T}{g + k^2 T} \quad\text{...........(12)}.$$

When k is small, the surface tension is negligible, and then $U = \tfrac{1}{2} V$; but when, on the contrary, k is large, $U = \tfrac{3}{2} V$, as has already been stated. When $Tk^2 = g$, $U = V$. This corresponds to the minimum velocity of propagation investigated by Thomson.

Although the argument from interference groups seems satisfactory, an independent investigation is desirable of the relation between energy existing and energy propagated. For some time I was at a loss for a method applicable to all kinds of waves, not seeing in particular why the comparison of energies should introduce the consideration of a variation of wave-length. The following investigation, in which the increment of wave-length is *imaginary*, may perhaps be considered to meet the want:—

Let us suppose that the motion of every part of the medium is resisted by a force of very small magnitude proportional to the mass and to the velocity of the part, the effect of which will be that waves generated at the origin gradually die away as x increases. The motion, which in the absence of friction would be represented by $\cos(nt - kx)$, under the influence of friction is represented by $e^{-\mu x}\cos(nt - kx)$, where μ is a small positive coefficient. In strictness the value of k is also altered by the friction; but the alteration is of the second order as regards the frictional forces and may be omitted under the circumstances here supposed. The energy of the waves per unit length at any stage of degradation is proportional to the square of the amplitude, and thus the whole energy on the positive side of the origin is to the energy of so much of the waves at their greatest value, *i.e.*, at the origin, as would be contained in the unit of length, as $\int_0^\infty e^{-2\mu x}\,dx : 1$, or as $(2\mu)^{-1} : 1$. The energy transmitted through the origin in the unit time is the same as the energy dissipated; and, if the frictional force acting on the element of mass m be hmv, where v is the velocity of the element and h is constant, the energy dissipated in unit time is $h\Sigma mv^2$ or $2hT$, T being the kinetic energy. Thus, on the assumption that the kinetic energy is half the whole energy, we find that the energy transmitted in the unit time is to the greatest energy existing in the unit length as $h : 2\mu$. It remains to find the connection between h and μ.

For this purpose it will be convenient to regard $\cos(nt - kx)$ as the real part of $e^{int}e^{ikx}$, and to inquire how k is affected, when n is given, by the introduction of friction. Now the effect of friction is represented in the differential equations of motion by the substitution of $d^2/dt^2 + h\,d/dt$ in place of d^2/dt^2, or, since the whole motion is proportional to e^{int}, by substituting $-n^2 + ihn$ for $-n^2$. Hence the introduction of friction corresponds to an alteration of n from n to $n - \frac{1}{2}ih$ (the square of h being neglected); and accordingly k is altered from k to $k - \frac{1}{2}ih\,dk/dn$. The solution thus becomes $e^{-\frac{1}{2}hx\,dk/dn}e^{i(nt-kx)}$, or, when the imaginary part is rejected, $e^{-\frac{1}{2}hx\,dk/dn}\cos(nt - kx)$; so that $\mu = \frac{1}{2}h\,dk/dn$, and $h : 2\mu = dn/dk$. The ratio of the energy transmitted in the unit time to the energy existing in the unit length is therefore expressed by dn/dk or $d(kV)/dk$, as was to be proved.

It has often been noticed, in particular cases of progressive waves, that the potential and kinetic energies are equal; but I do not call to mind any general treatment of the question. The theorem is not usually true for the individual parts of the medium*, but must be understood to refer either to an integral number of wave-lengths, or to a space so considerable that the outstanding fractional parts of waves may be left out of account. As an example well adapted to give insight into the question, I will take the case of a uniform stretched circular membrane ("Theory of Sound," § 200) vibrating with a given number of nodal circles and diameters. The fundamental modes are not quite determinate in consequence of the symmetry, for any diameter may be made nodal. In order to get rid of this indeterminateness, we may suppose the membrane to carry a small load attached to it anywhere except on a nodal circle. There are then two definite fundamental modes, in one of which the load lies on a nodal diameter, thus producing no effect, and in the other midway between nodal diameters, where it produces a maximum effect ("Theory of Sound," § 208). If vibrations of both modes are going on simultaneously, the potential and kinetic energies of the whole motion may be calculated by *simple addition* of those of the components. Let us now, supposing the load to diminish without limit, imagine that the vibrations are of equal amplitude and differ in phase by a quarter of a period. The result is a *progressive* wave, whose potential and kinetic energies are the sums of those of the stationary waves of which it is composed. For the first component we have $V_1 = E\cos^2 nt$, $T_1 = E\sin^2 nt$; and for the second component, $V_2 = E\sin^2 nt$, $T_2 = E\cos^2 nt$; so that $V_1 + V_2 = T_1 + T_2 = E$, or the potential and kinetic energies of the progressive wave are equal, being the same as the whole energy of either of the components. The method of proof here employed appears to be sufficiently general, though it is rather difficult to express it in language which is appropriate to all kinds of waves.

* Aërial waves are an important exception.

END OF VOL. I.

THE
THEORY OF SOUND

THE

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PREFACE.

THE appearance of this second and concluding volume has been delayed by pressure of other work that could not well be postponed. As in Vol. I. the additions down to § 348 are indicated by square brackets, or by letters following the number of the section. From that point onwards the matter is new with the exception of § 381, which appeared in the first edition as § 348.

The additions to Chapter XIX. deal with aerial vibrations in narrow tubes where the influence of viscosity and heat conduction are important, and with certain phenomena of the second order dependent upon viscosity. Chapter XX. is devoted to capillary vibrations, and the explanation thereby of many beautiful observations due to Savart and other physicists. The sensitiveness of flames and smoke jets, a very interesting department of acoustics, is considered in Chapter XXI., and an attempt is made to lay the foundations of a theoretical treatment by the solution of problems respecting the stability, or otherwise, of stratified fluid motion. §§ 371, 372 deal with "bird-calls," investigated by Sondhauss, and with aeolian tones. In Chapter XXII. a slight sketch is given of the theory of the vibrations of elastic solids, especially as regards the propagation of plane waves, and the disturbance due to a harmonic force operative at one point of an infinite solid. The important problems of the vibrations of plates, cylinders and spheres, are perhaps best dealt with in works devoted specially to the theory of elasticity.

The concluding chapter on the facts and theories of audition could not well have been omitted, but it has entailed labour out of proportion to the results. A large part of our knowledge upon this subject is due to Helmholtz, but most of the workers who have since published their researches entertain divergent views, in some cases, it would seem, without recognizing how fundamental their objections really are. And on several points the observations recorded by well qualified observers are so discrepant that no satisfactory conclusion can be drawn at the present time. The future may possibly shew that the differences are more nominal than real. In any case I would desire to impress upon the student of this part of our subject the importance of studying Helmholtz's views at first hand. In such a book as the present an imperfect outline of them is all that can be attempted. Only one thoroughly familiar with the *Tonempfindungen* is in a position to appreciate many of the observations and criticisms of subsequent writers.

TERLING PLACE, WITHAM.
February, 1896

EDITORIAL NOTE.

THE present re-issue has a few small corrections noted in the author's copy, and an addition [§ 335a] on the maximum disturbance that can be produced by an infinitesimal resonator exposed to plane waves. The author had written this out for inclusion.

May, 1926.

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CHAPTER XXI.

[1] Appears now for the first time.
[2] Appeared in the First Edition.

ERRATUM.

Vol. I. p. 407, footnote. Add reference to Chree, *Camb. Phil. Trans.*, Vol. XIV. p. 250, 1887.

CHAPTER XI.

AERIAL VIBRATIONS.

236. SINCE the atmosphere is the almost universal vehicle of sound, the investigation of the vibrations of a gaseous medium has always been considered the peculiar problem of Physical Acoustics; but in all, except a few specially simple questions, chiefly relating to the propagation of sound in one dimension, the mathematical difficulties are such that progress has been very slow. Even when a theoretical result is obtained, it often happens that it cannot be submitted to the test of experiment, in default of accurate methods of measuring the intensity of vibrations. In some parts of the subject all that we can do is to solve those problems whose mathematical conditions are sufficiently simple to admit of solution, and to trust to them and to general principles not to leave us quite in the dark with respect to other questions in which we may be interested.

In the present chapter we shall regard fluids as perfect, that is to say, we shall assume that the mutual action between any two portions separated by an ideal surface is *normal to that surface.* Hereafter we shall say something about fluid friction; but, in general, acoustical phenomena are not materially disturbed by such deviation from perfect fluidity as exists in the case of air and other gases.

The equality of pressure in all directions about a given point is a necessary consequence of perfect fluidity, whether there be rest or motion, as is proved by considering the equilibrium of a small tetrahedron under the operation of the fluid pressures, the impressed forces, and the reactions against acceleration. In the limit, when the tetrahedron is taken indefinitely small, the fluid pressures on its sides become paramount, and equilibrium requires that their whole magnitudes be proportional to the areas of the faces over which they act. The pressure at the point x, y, z will be denoted by p.

237. If $\rho X dV$, $\rho Y dV$, $\rho Z dV$, denote the impressed forces acting on the element of mass ρdV, the equation of equilibrium is

$$dp = \rho\,(X dx + Y dy + Z dz),$$

where dp denotes the variation of pressure corresponding to changes dx, dy, dz in the co-ordinates of the point at which the pressure is estimated. This equation is readily established by considering the equilibrium of a small cylinder with flat ends, the projections of whose axis on those of co-ordinates are respectively dx, dy, dz. To obtain the equations of motion we have, in accordance with D'Alembert's Principle, merely to replace X, &c. by $X - Du/Dt$, &c., where Du/Dt, &c. denote the accelerations of the particle of fluid considered. Thus

$$\left.\begin{aligned} \frac{dp}{dx} &= \rho\left(X - \frac{Du}{Dt}\right) \\ \frac{dp}{dy} &= \rho\left(Y - \frac{Dv}{Dt}\right) \\ \frac{dp}{dz} &= \rho\left(Z - \frac{Dw}{Dt}\right) \end{aligned}\right\} \qquad (1).$$

In hydrodynamical investigations it is usual to express the velocities of the fluid u, v, w in terms of x, y, z and t. They then denote the velocities of the particle, whichever it may be, that at the time t is found at the point x, y, z. After a small interval of time dt, a new particle has reached x, y, z; $du/dt \,.\, dt$ expresses the excess of its velocity over that of the first particle, while $Du/Dt \,.\, dt$ on the other hand expresses the change in the velocity of the *original* particle in the same time, or the change of velocity at a point, which is not fixed in space, but moves with the fluid. To this notation we shall adhere. In the change contemplated in d/dt, the position in space (determined by the values of x, y, z) is retained invariable, while in D/Dt it is a certain particle of the fluid on which attention is fixed. The relation between the two kinds of differentiation with respect to time is expressed by

$$\frac{D}{Dt} = \frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz} \quad\text{.................(2)},$$

and must be clearly conceived, though in a large class of important problems with which we shall be occupied in the sequel, the distinction practically disappears. Whenever the motion is very small, the terms $u\, d/dx$, &c. diminish in relative importance, and ultimately $D/Dt = d/dt$.

238. We have further to express the condition that there is no creation or annihilation of matter in the interior of the fluid. If α, β, γ be the edges of a small rectangular parallelepiped parallel to the axes of co-ordinates, the quantity of matter which passes out of the included space in time dt in excess of that which enters is

$$\left\{\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz}\right\}\alpha\beta\gamma\, dt;$$

and this must be equal to the actual loss sustained, or

$$-\frac{d\rho}{dt}\alpha\beta\gamma\, dt.$$

Hence

$$\frac{d\rho}{dt} + \frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz} = 0 \quad\text{..............(1)},$$

the so-called equation of continuity. When ρ is constant (with respect to both time and space), the equation assumes the simple form

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \quad\text{......................(2)}.$$

In problems connected with sound, the velocities and the variation of density are usually treated as small quantities. Putting $\rho = \rho_0(1+s)$, where s, called the *condensation*, is small, and neglecting the products $u\, ds/dx$, &c., we find

$$\frac{ds}{dt} + \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \quad\text{...................(3)}.$$

In special cases these equations take even simpler forms. In the case of an incompressible fluid whose motion is entirely parallel to the plane of xy,

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \quad\text{.........................(4)},$$

from which we infer that the expression $u\,dy - v\,dx$ is a perfect differential. Calling it $d\psi$, we have as the equivalent of (4)

$$u = \frac{d\psi}{dy}, \qquad v = -\frac{d\psi}{dx} \quad\text{.................. (5),}$$

where ψ is a function of the co-ordinates which so far is perfectly arbitrary. The function ψ is called the *stream*-function, since the motion of the fluid is everywhere in the direction of the curves ψ = constant. When the motion is steady, that is, always the same at the same point of space, the curves ψ = constant mark out a system of pipes or channels in which the fluid may be supposed to flow. Analytically, the substitution of *one* function ψ for the *two* functions u and v is often a step of great consequence.

Another case of importance is when there is symmetry round an axis, for example, that of x. Everything is then expressible in terms of x and r, where $r = \sqrt{(y^2 + z^2)}$, and the motion takes place in planes passing through the axis of symmetry. If the velocities respectively parallel and perpendicular to the axis of symmetry be u and q, the equation of continuity is

$$\frac{d\,(ru)}{dx} + \frac{d\,(rq)}{dr} = 0 \quad\text{..................... (6),}$$

which, as before, is equivalent to

$$ru = \frac{d\psi}{dr}, \qquad rq = -\frac{d\psi}{dx} \quad\text{.................(7),}$$

ψ being the stream-function.

239. In almost all the cases with which we shall have to deal, the hydrodynamical equations undergo a remarkable simplification in virtue of a proposition first enunciated by Lagrange. If for any part of a fluid mass $u\,dx + v\,dy + w\,dz$ be at one moment a perfect differential $d\phi$, it will remain so for all subsequent time. In particular, if a fluid be originally at rest, and be then set in motion by conservative forces and pressures transmitted from the exterior, the quantities

$$\frac{dv}{dz} - \frac{dw}{dy}, \quad \frac{dw}{dx} - \frac{du}{dz}, \quad \frac{du}{dy} - \frac{dv}{dx},$$

(which we shall denote by ξ, η, ζ) can never depart from zero.

We assume that ρ is a function of p, and we shall write for brevity

$$\varpi = \int \frac{dp}{\rho} \qquad \text{(1)}.$$

The equations of motion obtained from (1), (2), § 237, are

$$\frac{d\varpi}{dx} = X - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz} \qquad \text{(2)},$$

with two others of the same form relating to y and z. By hypothesis,

$$\frac{dX}{dy} = \frac{dY}{dx};$$

so that by differentiating the first of the above equations with respect to y and the second with respect to x, and subtracting, we eliminate ϖ and the impressed forces, obtaining equations which may be put into the form

$$\frac{D\zeta}{Dt} = \frac{du}{dz}\xi + \frac{dv}{dz}\eta - \left(\frac{du}{dx} + \frac{dv}{dy}\right)\zeta \qquad \text{(3)},$$

with two others of the same form giving $D\xi/Dt$, $D\eta/Dt$.

In the case of an incompressible fluid, we may substitute for $du/dx + dv/dy$ its equivalent $-dw/dz$, and thus obtain

$$\frac{D\zeta}{Dt} = \frac{du}{dz}\xi + \frac{dv}{dz}\eta + \frac{dw}{dz}\zeta, \text{ \&c.} \qquad \text{(4)},$$

which are the equations used by Helmholtz as the foundation of his theorems respecting vortices.

If the motion be continuous, the coefficients of ξ, η, ζ in the above equations are all finite. Let L denote their greatest numerical value, and Ω the sum of the numerical values of ξ, η, ζ. By hypothesis, Ω is initially zero; the question is whether in the course of time it can become finite. The preceding equations shew that it cannot; for its rate of increase for a given particle is at any time less than $3L\Omega$, all the quantities concerned being positive. Now even if its rate of increase were as great as $3L\Omega$, Ω would never become finite, as appears from the solution of the equation

$$\frac{D\Omega}{Dt} = 3L\Omega \qquad \text{(5)}.$$

A fortiori in the actual case, Ω cannot depart from zero, and the same must be true of ξ, η, ζ.

It is worth notice that this conclusion would not be disturbed by the presence of frictional forces acting on each particle proportional to its velocity, as may be seen by substituting $X - \kappa u$, $Y - \kappa v$, $Z - \kappa w$, for X, Y, Z in (2)[1]. But it is otherwise with the frictional forces which actually exist in fluids, and are dependent on the *relative* velocities of their parts.

The first satisfactory demonstration of the important proposition now under discussion was given by Cauchy; but that sketched above is due to Stokes[2]. It is not sufficient merely to shew that if, and whenever, ξ, η, ζ vanish, their differential coefficients $D\xi/Dt$, &c. vanish also, though this is a point that is often overlooked. When a body falls from rest under the action of gravity, $\dot{s} \propto s^{\frac{1}{2}}$; but it does not follow that s never becomes finite. To justify that conclusion it would be necessary to prove that s vanishes in the limit, not merely to the first order, but to all orders of the small quantity t; which, of course, cannot be done in the case of a falling body. If, however, the equation had been $\dot{s} \propto s$, all the differential coefficients of s with respect to t would vanish with t, if s did so, and then it might be inferred legitimately that s could never vary from zero.

By a theorem due to Stokes, the moments of momentum about the axes of co-ordinates of any infinitesimal spherical portion of fluid are equal to ξ, η, ζ, multiplied by the moment of inertia of the mass; and thus these quantities may be regarded as the component rotatory velocities of the fluid at the point to which they refer.

If ξ, η, ζ vanish throughout a space occupied by moving fluid, any small spherical portion of the fluid if suddenly solidified would retain only a motion of translation. A proof of this proposition in a generalised form will be given a little later. Lagrange's theorem thus consists in the assertion that particles of fluid at any time destitute of rotation can never acquire it.

[1] By introducing such forces and neglecting the terms dependent on inertia, we should obtain equations applicable to the motion of electricity through uniform conductors.

[2] *Cambridge Trans.* Vol. VIII. p. 307, 1845. B. A. Report on Hydrodynamics, 1847.

240. A somewhat different mode of investigation has been adopted by Thomson, which affords a highly instructive view of the whole subject[1].

By the fundamental equations

$$d\varpi = X\,dx + Y\,dy + Z\,dz - \frac{Du}{Dt}dx - \frac{Dv}{Dt}dy - \frac{Dw}{Dt}dz.$$

Now $X\,dx + Y\,dy + Z\,dz = dR$, if the forces be conservative, and

$$\frac{Du}{Dt}dx + \frac{Dv}{Dt}dy + \frac{Dw}{Dt}dz$$

$$= \frac{D}{Dt}(u\,dx + v\,dy + w\,dz) - u\frac{D\,dx}{Dt} - v\frac{D\,dy}{Dt} - w\frac{D\,dz}{Dt},$$

in which

$$\frac{D\,dx}{Dt} = d\frac{Dx}{Dt} = du, \text{ \&c.}$$

Thus, if $U^2 = u^2 + v^2 + w^2$, we have

$$d\varpi = dR - \frac{D}{Dt}(u\,dx + v\,dy + w\,dz) + \tfrac{1}{2}dU^2 \quad\text{.........(1)},$$

or

$$\frac{D}{Dt}(u\,dx + v\,dy + w\,dz) = d\,(R + \tfrac{1}{2}U^2 - \varpi) \quad\text{......(2)}.$$

Integrating this equation along any finite arc P_1P_2, moving with the fluid, we have

$$\frac{D}{Dt}\int(u\,dx + v\,dy + w\,dz) = (R + \tfrac{1}{2}U^2 - \varpi)_2 - (R + \tfrac{1}{2}U^2 - \varpi)_1 \ldots(3),$$

in which suffixes denote the values of the bracketed function at the points P_2 and P_1 respectively. If the arc be a complete circuit,

$$\frac{D}{Dt}\int(u\,dx + v\,dy + w\,dz) = 0 \quad\text{..............(4)};$$

or, in words,

The line-integral of the tangential component velocity round any closed curve of a moving fluid remains constant throughout all time.

The line-integral in question is appropriately called the *circulation*, and the proposition may be stated :—

The circulation in any closed line moving with the fluid remains constant.

[1] Vortex Motion. *Edinburgh Transactions*, 1869.

In a state of rest the circulation is of course zero, so that, if a fluid be set in motion by pressures transmitted from the outside or by conservative forces, the circulation along any closed line must ever remain zero, which requires that $udx + vdy + wdz$ be a complete differential.

But it does not follow conversely that in irrotational motion there can never be circulation, unless it be known that ϕ is single-valued; for otherwise $\int d\phi$ need not vanish round a closed circuit. In such a case all that can be said is that there is no circulation round any closed curve capable of being contracted to a point without passing out of space occupied by irrotationally moving fluid, or more generally, that the circulation is the same in all mutually reconcilable closed curves. Two curves are said to be reconcilable, when one can be obtained from the other by continuous deformation, without passing out of the irrotationally moving fluid.

Within an oval space, such as that included by an ellipsoid, all circuits are reconcilable, and therefore if a mass of fluid of that form move irrotationally, there can be no circulation along any closed curve drawn within it. Such spaces are called simply-connected. But in an annular space like that bounded by the surface of an anchor ring, a closed curve going round the ring is not continuously reducible to a point, and therefore there may be circulation along it, even although the motion be irrotational throughout the whole volume included. But the circulation is zero for every closed curve which does not pass round the ring, and has the same constant value for all those that do.

[In the above theorems "circulation" is defined without reference to mass. If the fluid be of uniform density, the *momentum* reckoned round a closed circuit is proportional to circulation, but in the case of a compressible fluid a distinction must be drawn. The existence of a velocity-potential does not then imply evanescence of the integral momentum reckoned round a closed circuit.]

241. When $udx + vdy + wdz$ is an exact differential $d\phi$, the velocity in any direction is expressed by the corresponding rate of change of ϕ, which is called the velocity-potential, and

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$$

may be replaced by

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}.$$

If S denote any closed surface, the rate of flow outwards across the element dS is expressed by $dS\,.\,d\phi/dn$, where $d\phi/dn$ is the rate of variation of ϕ in proceeding outwards along the normal. In the case of constant density, the total loss of fluid in time dt is thus

$$\iint \frac{d\phi}{dn}\,dS\,.\,dt,$$

the integration ranging over the whole surface of S. If the space S be full both at the beginning and at the end of the time dt, the loss must vanish; and thus

$$\iint \frac{d\phi}{dn}\,dS=0 \quad\text{.........................(1).}$$

The application of this equation to the element $dxdydz$ gives for the equation of continuity of an incompressible fluid

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=0 \quad\text{.....................(2),}$$

or, as it is generally written,

$$\nabla^2\phi=0 \quad\text{.........................(3);}$$

when it is desired to work with polar co-ordinates, the transformed equation is more readily obtained directly by applying (1) to the corresponding element of volume, than by transforming (2) in accordance with the analytical rules for effecting changes in the independent variables.

Thus, if we take polar co-ordinates in the plane xy, so that

$$x=r\cos\theta,\quad y=r\sin\theta,$$

we find

$$\nabla^2\phi=\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}+\frac{1}{r^2}\frac{d^2\phi}{d\theta^2}+\frac{d^2\phi}{dz^2} \quad\text{.............(4);}$$

or, if we take polar co-ordinates in space,

$$x=r\sin\theta\cos\omega,\quad y=r\sin\theta\sin\omega,\quad z=r\cos\theta,$$

$$\nabla^2\phi=\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}+\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\phi}{d\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{d^2\phi}{d\omega^2}\text{...(5).}$$

Simpler forms are assumed in special cases, such, for example, as that of symmetry round z in (5).

When the fluid is compressible, and the motion such that the squares of small quantities may be neglected, the equation of continuity is by (3), § 238,

$$\frac{ds}{dt}+\nabla^2\phi=0\text{.........................(6)},$$

where any form of $\nabla^2\phi$ may be used that may be most convenient for the problem in hand.

242. The irrotational motion of incompressible fluid within any simply-connected closed space S is completely determined by the normal velocities over the surface of S. If S be a material envelope, it is evident that an arbitrary normal velocity may be impressed upon its surface, which normal velocity must be shared by the fluid immediately in contact, provided that the whole volume inclosed remain unaltered. If the fluid be previously at rest, it can acquire no molecular rotation under the operation of the fluid pressures, which shews that it must be possible to determine a function ϕ, such that $\nabla^2\phi=0$ throughout the space inclosed by S, while over the surface $d\phi/dn$ has a prescribed value, limited only by the condition

$$\iint\frac{d\phi}{dn}\,dS=0\text{.......................(1)}.$$

An analytical proof of this important proposition is indicated in Thomson and Tait's *Natural Philosophy*, § 317.

There is no difficulty in proving that but one solution of the problem is possible. By Green's theorem, if $\nabla^2\phi=0$,

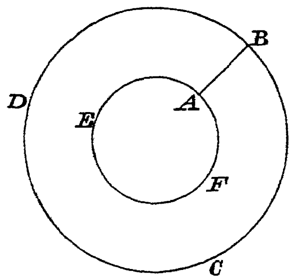
$$\iiint\left(\frac{d\phi^2}{dx^2}+\frac{d\phi^2}{dy^2}+\frac{d\phi^2}{dz^2}\right)dV=\iint\phi\frac{d\phi}{dn}\,dS\text{...........(2)},$$

the integration on the left-hand side ranging over the volume, and on the right over the surface of S. Now if ϕ and $\phi+\Delta\phi$ be two functions, satisfying Laplace's equation, and giving prescribed surface-values of $d\phi/dn$, their difference $\Delta\phi$ is a function also satisfying Laplace's equation, and making $d\Delta\phi/dn$ vanish over the surface of S. Under these circumstances the double integral in (2) vanishes, and we infer that at every point of S $d\Delta\phi/dx$, $d\Delta\phi/dy$, $d\Delta\phi/dz$ must be equal to zero. In other words $\Delta\phi$ must be constant, and the two motions identical. As a particular case, there can be no motion of the irrotational kind within the volume S, independently of a motion of the surface. The restriction to simply-connected spaces is rendered necessary by the failure of Green's theorem, which, as was first pointed out by Helmholtz, is otherwise possible.

When the space S is multiply-connected, the irrotational motion is still determinate, if besides the normal velocity at every point of S there be given the values of the constant circulations in all the possible irreconcilable circuits. For a complete discussion of this question we must refer to Thomson's original memoir, and content ourselves here with the case of a doubly-connected space, which will suffice for illustration.

Let $ABCD$ be an endless tube within which fluid moves irrotationally. For this motion there must exist a velocity-potential, whose differential coefficients, expressing, as they do, the component velocities, are necessarily single-valued, but which need not itself be single-valued. The simplest way of attacking the difficulty presented by the ambiguity of ϕ, is to conceive a barrier AB taken across the ring, so as to close the passage. The space $ABCDBAEF$ is then simply continuous, and Green's theorem applies to it without modification, if allowance be made for a possible finite difference in the value of ϕ on the two sides of the barrier. This difference, if it exist, is necessarily the same at all points of AB, and in the hydrodynamical application expresses the *circulation* round the ring.

Fig. 54.



In applying the equation

$$\iiint\left(\frac{d\phi^2}{dx^2}+\frac{d\phi^2}{dy^2}+\frac{d\phi^2}{dz^2}\right)dV=\iint\phi\frac{d\phi}{dn}dS\text{........ (2)},$$

we have to calculate the double integral over the two faces of the barrier as well as over the original surface of the ring. Now since $\frac{d\phi}{dn}$ has the same value on the two sides,

$$\iint\phi\frac{d\phi}{dn}dS\text{ (over two faces of }AB)=\iint\frac{d\phi}{dn}\kappa dS=\kappa\iint\frac{d\phi}{dn}dS,$$

if κ denote the constant difference of ϕ. Thus, if κ vanish, or there be no circulation round the ring, we infer, just as for a simply-connected space, that ϕ is completely determined by the surface-values of $d\phi/dn$. If there be circulation, ϕ is still determined, if the amount of the circulation be given. For, if ϕ and $\phi + \Delta\phi$ be two functions satisfying Laplace's equation and giving the same amount of circulation and the same normal velocities at S, their difference $\Delta\phi$ also satisfies Laplace's equation and the condition that there shall be neither circulation nor normal velocities over S. But, as we have just seen, under these circumstances $\Delta\phi$ vanishes at every point.

Although in a doubly-connected space irrotational motion is possible independently of surface normal velocities, yet such a motion cannot be generated by conservative forces nor by motions imposed (at any previous time) on the bounding surface, for we have proved that if the fluid be originally at rest, there can never be circulation along any closed curve. Hence, for multiply-connected as well as simply-connected spaces, if a fluid be set in motion by arbitrary deformation of the boundary, the whole mass comes to rest so soon as the motion of the boundary ceases.

If in a fluid moving without circulation all the fluid outside a reentrant tube-like surface of uniform section become instantaneously solid, then also at the same moment all the fluid within the tube comes to rest. This mechanical interpretation, however unpractical, will help the student to understand more clearly what is meant by a fluid having no circulation, and it leads to an extension of Stokes' theorem with respect to molecular rotation. For, if all the fluid (moving subject to a velocity-potential) outside a spherical cavity of any radius become suddenly solid, the fluid inside the cavity can retain no motion. Or, as we may also state it, any spherical portion of an irrotationally moving [incompressible] fluid becoming suddenly solid would possess only a motion of translation, *without rotation*[1].

A similar proposition will apply to a cylinder disc, or cylinder with flat ends, in the case of fluid moving irrotationally in two dimensions only.

[1] Thomson on *Vortex Motion*, *loc. cit.*

The motion of an incompressible fluid which has been once at rest partakes of the remarkable property (§ 79) common to that of all systems which are set in motion with prescribed velocities, namely, that the energy is the least possible. If any other motion be proposed satisfying the equation of continuity and the boundary conditions, its energy is necessarily greater than that of the motion which would be generated from rest[1].

243. The fact that the irrotational motion of incompressible fluid depends upon a velocity-potential satisfying Laplace's equation, is the foundation of a far-reaching analogy between the motion of such a fluid, and that of electricity or heat in a uniform conductor, which it is often of great service to bear in mind. The same may be said of the connection between all the branches of Physics which depend mathematically on a potential, for it often happens that the analogous theorems are far from equally obvious. For example, the analytical theorem that, if $\nabla^2\phi = 0$,

$$\iint \frac{d\phi}{dn}\, dS = 0$$

over a closed surface, is most readily suggested by the fluid interpretation, but once obtained may be interpreted for electric or magnetic forces.

Again, in the theory of the conduction of heat or electricity, it is obvious that there can be no steady motion in the interior of S, without transmission across some part of the bounding surface, but this, when interpreted for incompressible fluids, gives an important and rather recondite law.

244. When a velocity-potential exists, the equation to determine the pressure may be put into a simpler form. We have from (1), § 240,

$$d\varpi = dR - \frac{D}{Dt}\, d\phi + \tfrac{1}{2}\, dU^2 \qquad (1),$$

whence by integration

$$\varpi = \int \frac{dp}{\rho} = R - \frac{D\phi}{Dt} + \tfrac{1}{2}\, U^2.$$

[1] [The reader who wishes to pursue the study of general hydrodynamics is referred to the treatises of Lamb and Basset.]

Now
$$\frac{D\phi}{Dt} = \frac{d\phi}{dt} + u^2 + v^2 + w^2;$$
so that
$$\int \frac{dp}{\rho} = R - \frac{d\phi}{dt} - \tfrac{1}{2} U^2 \text{ .... (2)},$$
which is the form ordinarily given.

If ρ be constant, $\int \frac{dp}{\rho}$ is replaced, of course, by $\frac{p}{\rho}$.

The relation between p and ϕ in the case of impulsive motion from rest may be deduced from (2) by integration. We see that
$$\frac{1}{\rho}\int p\, dt = -\phi \text{ ultimately.}$$
The same conclusion may be arrived at by a direct application of mechanical principles to the circumstances of impulsive motion.

If $p = \kappa\rho$, equation (2) takes the form
$$\kappa \log \rho = R - \frac{d\phi}{dt} - \tfrac{1}{2} U^2 \text{ .... (3)}.$$
If the motion be such that the component velocities are always the same at the same point of space, it is called *steady*, and ϕ becomes independent of the time. The equation of pressure is then
$$\int \frac{dp}{\rho} = R - \tfrac{1}{2} U^2 \text{ .... (4)},$$
or in the case when there are no impressed forces,
$$\int \frac{dp}{\rho} = C - \tfrac{1}{2} U^2 \text{ .... (5)}.$$
In most acoustical applications of (2), the velocities and condensation are small, and then we may neglect the term $\frac{1}{2}U^2$, and substitute $\frac{\delta p}{\rho_0}$ for $\int \frac{dp}{\rho}$, if δp denote the small variable part of p; thus
$$\frac{\delta p}{\rho_0} = R - \frac{d\phi}{dt} \text{ .... (6)},$$
which with
$$\frac{ds}{dt} + \nabla^2 \phi = 0 \text{ .... (7)}$$
are the equations by means of which the small vibrations of an elastic fluid are to be investigated.

If $a^2 = dp/d\rho$, so that $\delta p = a^2\rho_0 s$, (6) becomes

$$a^2 s = R - \frac{d\phi}{dt} \quad\text{.........................(8)},$$

and we get on elimination of s,

$$\frac{d^2\phi}{dt^2} = \frac{dR}{dt} + a^2\nabla^2\phi \quad\text{.......................(9)}.$$

245. The simplest kind of wave-motion is that in which the excursions of every particle are parallel to a fixed line, and are the same in all planes perpendicular to that line. Let us therefore (assuming that $R=0$) suppose that ϕ is a function of x (and t) only. Our equation (9) § 244 becomes

$$\frac{d^2\phi}{dt^2} = a^2\frac{d^2\phi}{dx^2} \quad\text{.........................(1)},$$

the same as that already considered in the chapter on Strings. We there found that the general solution is

$$\phi = f(x - at) + F(x + at) \quad\text{.................(2)},$$

representing the propagation of independent waves in the positive and negative directions with the common velocity a.

Within such limits as allow the application of the approximate equation (1), the velocity of sound is entirely independent of the form of the wave, being, for example, the same for simple waves

$$\phi = A\cos\frac{2\pi}{\lambda}(x - at),$$

whatever the wave-length may be. The condition satisfied by the positive wave, and therefore by the initial disturbance if a positive wave alone be generated, is

$$a\frac{d\phi}{dx} + \frac{d\phi}{dt} = 0,$$

or by (8) § 244

$$u - as = 0 \quad\text{.........................(3)}.$$

Similarly, for a negative wave

$$u + as = 0 \quad\text{.........................(4)}.$$

Whatever the initial disturbance may be (and u and s are both arbitrary), it can always be divided into two parts, satisfying respectively (3) and (4), which are propagated undisturbed. In each component wave the direction of propagation is the same as that of the motion of the *condensed* parts of the fluid.

The rate at which energy is transmitted across unit of area of a plane parallel to the front of a progressive wave may be regarded as the mechanical measure of the intensity of the radiation. In the case of a simple wave, for which

$$\phi = A \cos \frac{2\pi}{\lambda}(x - at) \qquad (5),$$

the velocity $\dot{\xi}$ of the particle at x (equal to $d\phi/dx$) is given by

$$\dot{\xi} = -\frac{2\pi}{\lambda} A \sin \frac{2\pi}{\lambda}(x - at) \qquad (6),$$

and the displacement ξ is given by

$$\xi = -\frac{A}{a} \cos \frac{2\pi}{\lambda}(x - at) \qquad (7).$$

The pressure $p = p_0 + \delta p$, where by (6) § 244

$$\delta p = -\frac{2\pi}{\lambda} \rho_0 a A \sin \frac{2\pi}{\lambda}(x - at) \qquad (8).$$

Hence, if W denote the work transmitted across unit area of the plane x in time t,

$$\frac{dW}{dt} = (p_0 + \delta p)\,\dot{\xi} = \tfrac{1}{2}\rho_0 a \left(\frac{2\pi}{\lambda}\right)^2 A^2 + \text{periodic terms.}$$

If the integration with respect to time extend over any number of complete periods, or practically whenever its range is sufficiently long, the periodic terms may be omitted, and we may take

$$W : t = \tfrac{1}{2}\rho_0 a \left(\frac{2\pi}{\lambda}\right)^2 A^2 \qquad (9);$$

or by (3) and (6), if $\dot{\xi}$ now denote the maximum value of the velocity and s the maximum value of the condensation,

$$W = \tfrac{1}{2}\rho_0 \dot{\xi}^2 a t = \tfrac{1}{2}\rho_0 a^3 s^2 t \qquad (10).$$

Thus the work consumed in generating waves of harmonic type is the same as would be required to give the maximum velocity $\dot{\xi}$ to the whole mass of air through which the waves extend[1].

[1] The earliest statement of the principle embodied in equation (10) that I have met with is in a paper by Sir W. Thomson, "On the possible density of the luminiferous medium, and on the mechanical value of a cubic mile of sun-light." *Phil. Mag.* IX. p. 36. 1855.

In terms of the maximum excursion ξ by (7) and (9)

$$W = 2\pi^2\rho_0 \frac{a^3}{\lambda^2}\xi^2 t = 2\pi^2\rho_0 a t \frac{\xi^2}{\tau^2} \qquad (11)^1,$$

where $\tau\,(=\lambda/a)$ is the periodic time. In a *given medium* the mechanical measure of the intensity is proportional to the square of the amplitude directly, and to the square of the periodic time inversely. The reader, however, must be on his guard against supposing that the mechanical measure of intensity of undulations of different wave lengths is a proper measure of the loudness of the corresponding sounds, as perceived by the ear.

In any plane progressive wave, whether the type be harmonic or not, the whole energy is equally divided between the potential and kinetic forms. Perhaps the simplest road to this result is to consider the formation of positive and negative waves from an initial disturbance, whose energy is wholly potential[2]. The total energies of the two derived progressive waves are evidently equal, and make up together the energy of the original disturbance. Moreover, in each progressive wave the condensation (or rarefaction) is one-half of that which existed at the corresponding point initially, so that the *potential* energy of each progressive wave is *one-quarter* of that of the original disturbance. Since, as we have just seen, the *whole* energy is *one-half* of the same quantity, it follows that in a progressive wave of any type one-half of the energy is potential and one-half is kinetic.

The same conclusion may also be drawn from the general expressions for the potential and kinetic energies and the relations between velocity and condensation expressed in (3) and (4). The potential energy of the element of volume dV is the work that would be gained during the expansion of the corresponding quantity of gas from its actual to its normal volume, the expansion being opposed throughout by the normal pressure p_0. At any stage of the expansion, when the condensation is s', the effective pressure δp is by § 244 $a^2\rho_0 s'$, which pressure has to be multiplied by the corresponding increment of volume $dV.ds'$. The whole work gained during the expansion from dV to $dV(1+s)$ is therefore $a^2\rho_0 dV.\int_0 s'ds'$ or $\frac{1}{2}a^2\rho_0 dV.s^2$. The general expressions for the potential and kinetic energies are accordingly

[1] Bosanquet, *Phil. Mag.* XLV. p. 173. 1873.

[2] *Phil. Mag.* (5) I. p. 260. 1876.

$$\text{potential energy} = \tfrac{1}{2}a^2\rho_0 \iiint s^2\, dV \quad\text{..............(12)},$$

$$\text{kinetic energy} = \tfrac{1}{2}\rho_0 \iiint u^2\, dV \quad\text{..............(13)},$$

and these are equal in the case of plane progressive waves for which

$$u = \pm\, as.$$

If the plane progressive waves be of harmonic type, u and s at any moment of time are circular functions of one of the space co-ordinates (x), and therefore the mean value of their squares is one-half of the maximum value. Hence the total energy of the waves is equal to the kinetic energy of the whole mass of air concerned, moving with the maximum velocity to be found in the waves, or to the potential energy of the same mass of air when condensed to the maximum density of the waves.

[It may be worthy of notice that when terms of the second order are retained, a purely periodic value of u does not correspond to a purely periodic motion. The quantity of fluid which passes unit of area at point x in time dt is $\rho u dt$, or $\rho_0(1+s)\, u dt$. If u be periodic, $\int u dt = 0$, but $\int su dt$ may be finite. Thus in a positive progressive wave

$$\int su dt = a\int s^2 dt,$$

and there is a transference of fluid in the direction of wave propagation.]

246. The first theoretical investigation of the velocity of sound was made by Newton, who assumed that the relation between pressure and density was that formulated in Boyle's law. If we assume $p = \kappa\rho$, we see that the velocity of sound is expressed by $\sqrt{\kappa}$, or $\sqrt{p} \div \sqrt{\rho}$, in which the dimensions of p (= force ÷ area) are $[M][L]^{-1}[T]^{-2}$, and those of ρ (= mass ÷ volume) are $[M][L]^{-3}$. Newton expressed the result in terms of the '*height of the homogeneous atmosphere*,' defined by the equation

$$g\rho h = p \quad\text{..............................(1)},$$

where p and ρ refer to the pressure and the density at the earth's surface. The velocity of sound is thus $\sqrt{(gh)}$, or the velocity which would be acquired by a body falling freely under the action of gravity through half the height of the homogeneous atmosphere.

To obtain a numerical result we require to know a pair of simultaneous values of p and ρ.

[It is found by experiment[1] that at 0° Cent. under the pressure due (at Paris) to 760 mm. of mercury at 0° the density of dry air is ·0012933 gms. per cubic centimetre. If we assume as the density of mercury at 0° 13·5953[2], and $g = 980{\cdot}939$, we have in C.G.S. measure

$$p = 760 \times 13{\cdot}5953 \times 980{\cdot}939, \quad \rho = {\cdot}0012933,$$

whence

$$a = \sqrt{(p/\rho)} = 27994{\cdot}5\,;$$

so that the velocity of sound at 0° would be 279·945 metres per second, falling short of the result of direct observation by about a sixth part.]

Newton's investigation established that the velocity of sound should be independent of the amplitude of the vibration, and also of the pitch, but the discrepancy between his calculated value (published in 1687) and the experimental value was not explained until Laplace pointed out that the use of Boyle's law involved the assumption that in the condensations and rarefactions accompanying sound the temperature remains constant, in contradiction to the known fact that, when air is suddenly compressed, its temperature rises. The laws of Boyle and Charles supply only one relation between the three quantities, pressure, volume, and temperature, of a gas, viz.

$$pv = R\theta \quad\quad (2),$$

where the temperature θ is measured from the zero of the gas thermometer; and therefore without some auxiliary assumption it is impossible to specify the connection between p and v (or ρ). Laplace considered that the condensations and rarefactions concerned in the propagation of sound take place with such rapidity that the heat and cold produced have not time to pass away, and that therefore the relation between volume and pressure is sensibly the same as if the air were confined in an absolutely non-conducting vessel. Under these circumstances the change of pressure corresponding to a given condensation or rarefaction is greater than on the hypothesis of constant temperature, and the velocity of sound is accordingly increased.

[1] On the Densities of the Principal Gases, *Proc. Roy. Soc.* vol. LIII. p. 147, 1893.

[2] Volkmann, *Wied. Ann.* vol. XIII. p. 221, 1881.

In equation (2) let v denote the volume and p the pressure of the unit of mass, and let θ be expressed in centigrade degrees reckoned from the absolute zero[1]. The condition of the gas (if uniform) is defined by any two of the three quantities p, v, θ, and the third may be expressed in terms of them. The relation between the simultaneous variations of the three quantities is

$$\frac{d\theta}{\theta} = \frac{dp}{p} + \frac{dv}{v} \quad \text{.......................(3).}$$

In order to effect the change specified by dp and dv, it is in general necessary to communicate heat to the gas. Calling the necessary quantity of heat dQ, we may write

$$dQ = \left(\frac{dQ}{dv}\right) dv + \left(\frac{dQ}{dp}\right) dp \quad \text{.................(4).}$$

Suppose now (a) that $dp = 0$. Equations (3) and (4) give

$$\frac{dQ}{d\theta}\,(p \text{ const.}) = \left(\frac{dQ}{dv}\right)\frac{v}{\theta},$$

where $\frac{dQ}{d\theta}$ (p const.) expresses the specific heat of the gas under a constant pressure. This being denoted by κ_p, we have

$$\kappa_p = \left(\frac{dQ}{dv}\right)\frac{v}{\theta} \quad \text{..........................(5).}$$

Again, suppose (b) that $dv = 0$. We find in a similar manner that, if κ_v denote the specific heat under a constant volume,

$$\kappa_v = \left(\frac{dQ}{dp}\right)\frac{p}{\theta} \quad \text{..........................(6).}$$

In order to obtain the relation between dp and dv when there is no communication of heat, we have only to put $dQ = 0$. Thus

$$\left(\frac{dQ}{dv}\right) dv + \left(\frac{dQ}{dp}\right) dp = 0,$$

or, on substituting for the differential coefficients of Q their values in terms of κ_v, κ_p,

$$\kappa_p \frac{dv}{v} + \kappa_v \frac{dp}{p} = 0 \quad \text{........................(7).}$$

Since $v = 1/\rho$, $\qquad dv/v = -d\rho/\rho$;

so that

$$a^2 = \frac{dp}{d\rho} = \frac{p}{\rho}\frac{\kappa_p}{\kappa_v} = \frac{p}{\rho}\gamma \quad \text{....................(8),}$$

[1] On the ordinary centigrade scale the absolute zero is about $-273°$.

if, as usual, the ratio of the specific heats be denoted by γ. Laplace's value of the velocity of sound is therefore greater than Newton's in the ratio of $\sqrt{\gamma} : 1$.

By integration of (8), we obtain for the relation between p and ρ, on the supposition of no communication of heat,

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \quad \text{.........................(9)[1]},$$

where p_0, ρ_0 are two simultaneous values. Under the same circumstances the relation between pressure and temperature is by (3)

$$\frac{p}{p_0} = \left(\frac{\theta}{\theta_0}\right)^{\frac{\gamma}{\gamma-1}} \quad \text{......................(10)}.$$

The magnitude of γ cannot be determined with accuracy by direct experiment, but an approximate value may be obtained by a method of which the following is the principle. Air is compressed into a reservoir capable of being put into communication with the external atmosphere by opening a wide valve. At first the temperature of the compressed air is raised, but after a time the superfluous heat passes away and the whole mass assumes the temperature of the atmosphere Θ. Let the pressure (measured by a manometer) be p. The valve is now opened for as short a time as is sufficient to permit the equilibrium of pressure to be completely established, that is, until the internal pressure has become equal to that of the atmosphere P. If the experiment be properly arranged, this operation is so quick that the air in the vessel has not sufficient time to receive heat from the sides, and therefore expands nearly according to the law expressed in (9). Its temperature θ at the moment the operation is complete is therefore determined by

$$\frac{p}{P} = \left(\frac{\Theta}{\theta}\right)^{\frac{\gamma}{\gamma-1}} \quad \text{......................(11)}.$$

The enclosed air is next allowed to absorb heat until it has regained the atmospheric temperature Θ, and its pressure (p') is then observed. During the last change the volume is constant, and therefore the relation between pressure and temperature gives

$$\frac{P}{p'} = \frac{\theta}{\Theta} \quad \text{.........................(12)};$$

[1] It is here assumed that γ is constant. This equation appears to have been given first by Poisson.

so that by elimination of θ/Θ,

$$\frac{p}{P}=\left(\frac{p'}{P}\right)^{\frac{\gamma}{\gamma-1}},$$

whence

$$\gamma=\frac{\log p-\log P}{\log p-\log p'} \quad\text{.....................(13).}$$

By experiments of this nature Clement and Desormes determined $\gamma=1\cdot3492$; but the method is obviously not susceptible of any great accuracy. The value of γ required to reconcile the calculated and observed velocities of sound is 1·408, of the substantial correctness of which there can be little doubt.

We are not, however, dependent on the phenomena of sound for our knowledge of the magnitude of γ. The value of κ_p—the specific heat at constant pressure—has been determined experimentally by Regnault; and although on account of inherent difficulties the experimental method[1] may fail to yield a satisfactory result for κ_v, the information sought for may be obtained indirectly by means of a relation between the two specific heats, brought to light by the modern science of Thermodynamics.

If from the equations

$$\left.\begin{aligned}\frac{dQ}{\theta}&=\kappa_p\frac{dv}{v}+\kappa_v\frac{dp}{p}\\ \frac{d\theta}{\theta}&=\frac{dv}{v}+\frac{dp}{p}\end{aligned}\right\}\quad\text{.....................(14)}$$

we eliminate dp, there results

$$dQ=(\kappa_p-\kappa_v)\frac{pdv}{R}+\kappa_v d\theta \quad\text{...............(15).}$$

Let us suppose that $dQ=0$, or that there is no communication of heat. It is known that the heat developed during the compression of an approximately perfect gas, such as air, is almost exactly the thermal equivalent of the work done in compressing it. This important principle was assumed by Mayer in his celebrated memoir on the dynamical theory of heat, though on grounds which can hardly be considered adequate. However that may be, the principle itself is very nearly true, as has since been proved by the experiments of Joule and Thomson.

If we measure heat in dynamical units, Mayer's principle may be expressed $-\kappa_v d\theta=pdv$ on the understanding that there is

[1] [See, however, Joly, *Phil. Trans.* vol. CLXXXII. A, 1891.]

no communication of heat. Comparing this with (15), we see that

$$\kappa_p - \kappa_v = R \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

and therefore

$$\gamma = \frac{\kappa_p}{\kappa_v} = \frac{\kappa_p}{\kappa_p - R} \quad \ldots\ldots\ldots\ldots\ldots\ldots(17).$$

The value of pv in gravitation measure (gramme, centimetre) is $1033 \div \cdot 001293$, at 0° Cent. so that

$$R = \frac{1033}{\cdot 001293 \times 272\cdot 85}.$$

By Regnault's experiments the specific heat of air is ·2379 of that of water; and in order to raise a gramme of water one degree Cent., 42350 gramme-centimetres of work must be done on it. Hence with the same units as for R,

$$\kappa_p = \cdot 2379 \times 42350.$$

Calculating from these data, we find $\gamma = 1\cdot 410$, agreeing almost exactly with the value deduced from the velocity of sound. This investigation is due to Rankine, who employed in it 1850 to calculate the specific heat of air, taking Joule's equivalent and the observed velocity of sound as data. In this way he anticipated the result of Regnault's experiments, which were not published until 1853.

247. Laplace's theory has often been the subject of misapprehension among students, and a stumblingblock to those remarkable persons, called by De Morgan 'paradoxers.' But there can be no reasonable doubt that, antecedently to all calculation, the hypothesis of no communication of heat is greatly to be preferred to the equally special hypothesis of constant temperature. There would be a real difficulty if the velocity of sound were not decidedly in excess of Newton's value, and the wonder is rather that the cause of the excess remained so long undiscovered.

The only question which can possibly be considered open, is whether a small part of the heat and cold developed may not escape by conduction or radiation before producing its full effect. Everything must depend on the rapidity of the alternations. Below a certain limit of slowness, the heat in excess, or defect, would have time to adjust itself, and the temperature would remain sensibly constant. In this case the relation between pressure and density would be that which leads to Newton's value of the velocity of sound. On the other hand, above a certain limit of quickness, the gas would behave as if confined in a non-conducting vessel, as supposed in Laplace's theory. Now although the circumstances of the actual problem are better represented by the latter than by the former supposition, there may still (it may be said) be a sensible deviation from the law of pressure and density involved in Laplace's theory, entailing a somewhat slower velocity of propagation of sound. This question has been carefully discussed by Stokes in a paper published in 1851[1], of which the following is an outline.

The mechanical equations for the *small* motion of air are

$$\frac{dp}{dx} = -\rho\frac{du}{dt} \text{ \&c.} \quad\text{..........................(1)},$$

with the equation of continuity

$$\frac{ds}{dt} + \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \quad\text{....................(2)}.$$

The temperature is supposed to be uniform except in so far as it is disturbed by the vibrations themselves, so that if θ denote the *excess* of temperature,

$$p = \kappa\rho\,(1 + s + \alpha\theta) \quad\text{....................(3)}.$$

The effect of a small sudden condensation s is to produce an elevation of temperature, which may be denoted by βs. Let dQ be the quantity of heat entering the element of volume in time dt, measured by the rise of temperature that it would produce, if there were no condensation. Then (the distinction between D/Dt and d/dt being neglected)

$$\frac{d\theta}{dt} = \beta\frac{ds}{dt} + \frac{dQ}{dt} \quad\text{......................(4)},$$

dQ/dt being a function of θ and its differential coefficients with respect to space, dependent on the special character of the dissipation. Two extreme cases may be mentioned; the first when the tendency to equalisation of temperature is due to conduction, the second when the operating cause is radiation, and the transparency of the medium such that radiant heat is

[1] *Phil. Mag.* (4) I. 305.

not sensibly absorbed within a distance of several wave-lengths. In the former case $dQ/dt \propto \nabla^2\theta$, and in the latter, which is that selected by Stokes for analytical investigation, $dQ/dt \propto (-\theta)$, Newton's law of radiation being assumed as a sufficient approximation to the truth. We have then

$$\frac{d\theta}{dt} = \beta\frac{ds}{dt} - q\theta \quad \text{......................(5).}$$

In the case of plane waves, to which we shall confine our attention, v and w vanish, while u, p, s, θ are functions of x (and t) only. Eliminating p and u between (1), (2) and (3), we find

$$\frac{d^2s}{dt^2} = \kappa\left(\frac{d^2s}{dx^2} + \alpha\frac{d^2\theta}{dx^2}\right),$$

from which and (5) we get

$$\left(\frac{d}{dt} + q\right)\frac{d^2s}{dt^2} = \kappa\left(\gamma\frac{d}{dt} + q\right)\frac{d^2s}{dx^2} \quad \text{..............(6),}$$

if γ be written (in the same sense as before) for $1 + \alpha\beta$.

If the vibrations be harmonic, we may suppose that s varies as e^{int}, and the equation becomes

$$\frac{d^2s}{dx^2} + \frac{n^2}{\kappa}\cdot\frac{q + in}{q + i\gamma n}\cdot s = 0 \quad \text{.................(7).}$$

Let the coefficient of s in (7) be put into the form $\mu^2e^{-2i\psi}$, where

$$\mu^4 = \frac{n^4}{\kappa^2}\cdot\frac{q^2 + n^2}{q^2 + \gamma^2n^2} \quad \text{....................(8),}$$

and

$$2\psi = \tan^{-1}\frac{\gamma n}{q} - \tan^{-1}\frac{n}{q} = \tan^{-1}\frac{(\gamma - 1)\,nq}{\gamma n^2 + q^2} \quad \text{......(9).}$$

Equation (7) is then satisfied by terms of the form

$$e^{\pm i\mu(\cos\psi - i\sin\psi)x},$$

but (μ being positive, and ψ less than $\frac{1}{2}\pi$) if we wish for the expression of the wave travelling in the positive direction, we must take the lower sign. Discarding the imaginary part, we find as the appropriate solution

$$s = Ae^{-\mu\sin\psi x}\cos(nt - \mu\cos\psi\, x) \quad \text{...........(10).}$$

The first thing to be noticed is that the sound cannot be propagated to a distance unless $\sin\psi$ be insensible.

The velocity of propagation (V) is

$$V = n\mu^{-1}\sec\psi \qquad (11),$$

which, when $\sin\psi$ is insensible, reduces to

$$V = n\mu^{-1} \qquad (12).$$

Now from (9) we see that ψ cannot be insensible, unless q/n is either very great, or very small. On the first supposition from (11), or directly from (7), we have approximately, $V = \sqrt{\kappa}$ (Newton); and on the second, $V = \sqrt{(\kappa\gamma)}$, (Laplace), as ought evidently to be the case, when the meaning of q in (5) is considered. What we now learn is that, if q and n were comparable, the effect would be not merely a deviation of V from either of the limiting values, but a rapid stifling of the sound, which we know does not take place in nature.

Of this theoretical result we may convince ourselves, as Stokes explains, without the use of analysis. Imagine a mass of air to be confined within a closed cylinder, in which a piston is worked with a reciprocating motion. If the period of the motion be very long, the temperature of the air remains nearly constant, the heat developed by compression having time to escape by conduction or radiation. Under these circumstances the pressure is a function of volume, and whatever work has to be expended in producing a given compression is refunded when the piston passes through the same position in the reverse direction; no work is consumed in the long run. Next suppose that the motion is so rapid that there is no time for the heat and cold developed by the condensations and rarefactions to escape. The pressure is still a function of volume, and no work is dissipated. The only difference is that now the variations of pressure are more considerable than before in comparison with the variations of volume. We see how it is that both on Newton's and on Laplace's hypothesis the waves travel without dissipation, though with different velocities.

But in intermediate cases, when the motion of the piston is neither so slow that the temperature remains constant nor so quick that the heat has no time to adjust itself, the result is different. The work expended in producing a small condensation is no longer completely refunded during the corresponding rarefaction on account of the diminished temperature, part of the heat developed by the compression having in the meantime escaped. In fact the passage of heat by conduction or radiation from a warmer to a finitely colder body always involves dissipation, a principle which occupies a fundamental position in the science of Thermodynamics. In order therefore to maintain the motion of the piston, energy must be supplied from without, and if there be only a limited store to be drawn from, the motion must ultimately subside.

Another point to be noticed is that, if q and n were comparable, V would depend upon n, viz. on the pitch of the sound, a state of things which from experiment we have no reason to suspect. On the contrary the evidence of observation goes to prove that there is no such connection.

From (10) we see that the falling off in the intensity, estimated per wave-length, is a maximum with $\tan\psi$, or ψ; and by (9) ψ is a maximum when $q : n = \sqrt{\gamma}$. In this case

$$\mu = n\kappa^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}, \qquad 2\psi = \tan^{-1}\gamma^{\frac{1}{4}} - \tan^{-1}\gamma^{-\frac{1}{4}} \ldots\ldots(13),$$

whence, if we take $\gamma = 1{\cdot}36$, $2\psi = 8^\circ\, 47'$.

Calculating from these data, we find that for each wave-length of advance, the amplitude of the vibration would be diminished in the ratio ·6172.

To take a numerical example, let

$\tau = \frac{1}{300}$ of a second, $\lambda =$ wave-length $= 44$ inches [112 cm.].

In 20 yards [1828 cm.] the intensity would be diminished in the ratio of about 7 millions to one.

Corresponding to this,

$$q = 2198 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

If the value of q were actually that just written, sounds of the pitch in question would be very rapidly stifled. We therefore infer that q is in fact either much greater or else much less. But even so large a value as 2000 is utterly inadmissible, as we may convince ourselves by considering the significance of equation (5).

Suppose that by a rigid envelope transparent to radiant heat, the volume of a small mass of gas were maintained constant, then the equation to determine its thermal condition at any time is

$$\frac{d\theta}{dt} + q\theta = 0,$$

whence $$\theta = Ae^{-qt} \text{..........................(15)},$$

where A denotes the initial excess of temperature, proving that after a time $1/q$ the excess of temperature would fall to less than half its original value. To suppose that this could happen in a two thousandth of a second of time would be in contradiction to the most superficial observation.

We are therefore justified in assuming that q is very small in comparison with n, and our equations then become approximately

$$\mu = \frac{n}{\kappa^{\frac{1}{2}}\gamma^{\frac{1}{2}}}, \qquad 2\psi = \frac{\gamma - 1}{\gamma}\frac{q}{n}, \qquad V = n\mu^{-1} = \kappa^{\frac{1}{2}}\gamma^{\frac{1}{2}},$$

$$s = A\,e^{-(1-\gamma^{-1})\,qx/2V} \cos\frac{2\pi}{\lambda}(Vt - x) \text{ ....(16)}.$$

The effects of a small radiation of heat are to be sought for rather in a damping of the vibration than in an altered velocity of propagation.

Stokes calculates that if $\gamma = 1{\cdot}414$, $V = 1100$, the ratio $(N : 1)$ in which the intensity is diminished in passing over a distance x, is given by $\log_{10} N = {\cdot}0001156\,qx$ in foot-second measure. Although we are not able to make precise measurements of the intensity of sound, yet the fact that audible vibrations can be propagated for many miles excludes any such value of q as could appreciably affect the velocity of transmission.

Neither is it possible to attribute to the air such a conducting power as could materially disturb the application of Laplace's theory. In order to trace the effects of conduction, we have only to replace q in (5) by $-q'd^2/dx^2$. Assuming as a particular solution

$$s = Ae^{i(nt+mx)},$$

we find $$m^2 in\kappa\gamma = in^3 + q'n^2m^2 - \kappa q'm^4,$$

whence, if q' be relatively small,

$$m = \frac{-n}{\sqrt{(\kappa\gamma)}}\left(1 - \frac{\gamma - 1}{\gamma}\frac{q'n}{2\kappa\gamma}i\right) \quad\text{..................(17).}$$

Thus the solution in real quantities is

$$s = A \,.\, Exp\left(-\frac{\gamma - 1}{\gamma}\frac{q'n^2x}{2(\kappa\gamma)^{\frac{3}{2}}}\right).\cos\left(nt - \frac{nx}{\sqrt{(\kappa\gamma)}}\right)\text{......(18),}$$

leaving the velocity of propagation to this order of approximation still equal to $\sqrt{(\kappa\gamma)}$.

From (18) it appears that the first effect of conduction, as of radiation, is on the amplitude rather than on the velocity of propagation. In truth the conducting power of gases is so feeble, and in the case of audible sounds at any rate the time during which conduction can take place is so short, that disturbance from this cause is not to be looked for.

In the preceding discussions the waves are supposed to be propagated in an open space. When the air is confined within a tube, whose diameter is small in comparison with the wave-length, the conditions of the problem are altered, at least in the case of conduction. What we have to say on this head will, however, come more conveniently in another place.

248. From the expression $\sqrt{(p\gamma/\rho)}$, we see that in the same gas the velocity of sound is independent of the density, because if the temperature be constant, p varies as ρ $(p = R\rho\theta)$. On the other hand the velocity of sound is proportional to the square root of the absolute temperature, so that if a_0 be its value at 0° Cent.

$$a = a_0\sqrt{1 + \frac{\theta'}{273}} \quad\text{......................(1),}$$

where the temperature is measured in the ordinary manner from the freezing point of water.

The most conspicuous effect of the dependence of the velocity of sound on temperature is the variability of the pitch of organ pipes. We shall see in the following chapters that the period of the note of a flue organ-pipe is the time occupied by a pulse in running over a distance which is a definite multiple of the length of the pipe, and therefore varies inversely as the velocity of propagation. The inconvenience arising from this alteration of pitch is aggravated by the fact that the reed pipes are not similarly affected; so that a change of temperature puts an organ out of tune with itself.

Prof. Mayer[1] has proposed to make the connection between temperature and wave-length the foundation of a pyrometric method, but I am not aware whether the experiment has ever been carried out.

The correctness of (1) as regards air at the temperatures of 0° and 100° has been verified experimentally by Kundt. See § 260.

In different gases at given temperature and pressure a is inversely proportional to the square roots of the densities, at least if γ be constant[2]. For the non-condensable gases γ does not sensibly vary from its value for air. [Thus in the case of hydrogen the velocity is greater than for air in the ratio

$$\sqrt{(1\cdot2933)} : \sqrt{(\cdot08993)},$$

or

$$3\cdot792 : 1.]$$

The velocity of sound is not entirely independent of the degree of dryness of the air, since at a given pressure moist air is somewhat lighter than dry air. It is calculated that at 50° F. [10° C.], air saturated with moisture would propagate sound between 2 and 3 feet per second faster than if it were perfectly dry. [1 foot = 30·5 cm.]

The formula $a^2 = dp/d\rho$ may be applied to calculate the velocity of sound in liquids, or, if that be known, to infer conversely the coefficient of compressibility. In the case of water it is found by experiment that the compression per atmosphere is ·0000457. Thus, if $dp = 1033 \times 981$ in absolute C.G.S. units,

$$d\rho = \cdot0000457, \quad \text{since } \rho = 1.$$

Hence

$$a = 1489 \text{ metres per second,}$$

which does not differ much from the observed value (1435).

249. In the preceding sections the theory of plane waves has been derived from the general equations of motion. We

[1] On an Acoustic Pyrometer. *Phil. Mag.* XLV. p. 18, 1873.

[2] According to the kinetic theory of gases, the velocity of sound is determined solely by, and is proportional to, the mean velocity of the molecules. Preston, *Phil. Mag.* (5) III. p. 441, 1877. [See also Waterston (1846), *Phil. Trans.* vol. CLXXXIII. A, p. 1, 1892.]

now proceed to an independent investigation in which the motion is expressed in terms of the actual position of the layers of air instead of by means of the velocity-potential, whose aid is no longer necessary inasmuch as in one dimension there can be no question of molecular rotation.

If y, $y + dy/dx \,.\, dx$, define the actual positions at time t of neighbouring layers of air whose equilibrium positions are defined by x and $x + dx$, the density ρ of the included slice is given by

$$\rho : \rho_0 = 1 : \frac{dy}{dx} \quad\text{.....................(1)},$$

whence by (9) § 246,

$$p : p_0 = 1 : \left(\frac{dy}{dx}\right)^\gamma \quad\text{.....................(2)},$$

the expansions and condensations being supposed to take place according to the adiabatic law. The mass of unit of area of the slice is $\rho_0 dx$, and the corresponding moving force is

$$-\,dp/dx \,.\, dx,$$

giving for the equation of motion

$$\rho_0 \frac{d^2y}{dt^2} + \frac{dp}{dx} = 0 \quad\text{.........................(3)}.$$

Between (2) and (3) p is to be eliminated. Thus,

$$\left(\frac{dy}{dx}\right)^{\gamma+1} \frac{d^2y}{dt^2} = \frac{p_0\gamma}{\rho_0} \frac{d^2y}{dx^2} \quad\text{...................(4)}.$$

Equation (4) is an *exact* equation defining the actual abscissa y in terms of the equilibrium abscissa x and the time. If the motion be assumed to be small, we may replace $(dy/dx)^{\gamma+1}$, which occurs as the coefficient of the small quantity d^2y/dt^2, by its approximate value unity; and (4) then becomes

$$\frac{d^2y}{dt^2} = \frac{p_0\gamma}{\rho_0} \frac{d^2y}{dx^2} \quad\text{........................(5)},$$

the ordinary approximate equation.

If the expansion be isothermal, as in Newton's theory, the equations corresponding to (4) and (5) are obtained by merely putting $\gamma = 1$.

Whatever may be the relation between p and ρ, depending on the constitution of the medium, the equation of motion is by (1) and (3)

$$\left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dt^2} = \frac{dp}{d\rho}\frac{d^2y}{dx^2} \quad\text{.......................(6)},$$

from which ρ, occurring in $dp/d\rho$, is to be eliminated by means of the relation between ρ and dy/dx expressed in (1).

250. In the preceding investigations of aerial waves we have supposed that the air is at rest except in so far as it is disturbed by the vibrations of sound, but we are of course at liberty to attribute to the whole mass of air concerned any common motion. If we suppose that the air is moving in the direction contrary to that of the waves and with the same actual velocity, the wave form, if permanent, is stationary in space, and the motion is *steady*. In the present section we will consider the problem under this aspect, as it is important to obtain all possible clearness in our views on the mechanics of wave propagation.

If u_0, p_0, ρ_0 denote respectively the velocity, pressure, and density of the fluid in its undisturbed state, and if u, p, ρ be the corresponding quantities at a point in the wave, we have for the equation of continuity

$$\rho u = \rho_0 u_0 \quad\text{............................(1)},$$

and by (5) § 244 for the equation of energy

$$\int_{p_0}^{p} \frac{dp}{\rho} = \tfrac{1}{2}u_0^2 - \tfrac{1}{2}u^2 \quad\text{.......................(2)}.$$

Eliminating u, we get

$$\int_{p_0}^{p} \frac{dp}{\rho} = \tfrac{1}{2}u_0^2\left(1 - \frac{\rho_0^2}{\rho^2}\right) \quad\text{...................(3)},$$

determining the law of pressure under which alone it is possible for a stationary wave to maintain itself in fluid moving with velocity u_0. From (3)

$$\frac{dp}{d\rho} = u_0^2\frac{\rho_0^2}{\rho^2} \quad\text{.........................(4)},$$

or

$$p = \text{constant} - \frac{u_0^2\rho_0^2}{\rho} \quad\text{....................(5)}.$$

Since the relation between the pressure and the density of actual gases is not that expressed in (5), we conclude that a self-maintaining stationary aerial wave is an impossibility, whatever may be the velocity u_0 of the general current, or in other words that a wave cannot be propagated relatively to the undisturbed parts of the gas without undergoing an alteration of type. Nevertheless, when the changes of density concerned are small, (5) may be satisfied approximately; and we see from (4) that the velocity of stream necessary to keep the wave stationary is given by

$$u_0 = \sqrt{\left(\frac{dp}{d\rho}\right)} \qquad\qquad (6),$$

which is the same as the velocity of the wave estimated relatively to the fluid.

This method of regarding the subject shews, perhaps more clearly than any other, the nature of the relation between velocity and condensation § 245 (3), (4). In a stationary wave-form a loss of velocity accompanies an augmented density according to the principle of energy, and therefore the fluid composing the condensed parts of a wave moves forward more slowly than the undisturbed portions. Relatively to the fluid therefore the motion of the condensed parts is in the same direction as that in which the waves are propagated.

When the relation between pressure and density is other than that expressed in (5), a stationary wave can be maintained only by the aid of an impressed force. By (1) and (2) § 237 we have, on the supposition that the motion is steady,

$$X = u\frac{du}{dx} + \frac{1}{\rho}\frac{dp}{dx} \qquad\qquad (7),$$

while the relation between u and ρ is given by (1). If we suppose that $p = a^2\rho$, (7) becomes

$$X = (u^2 - a^2)\frac{d \log u}{dx} \qquad\qquad (8),$$

shewing that an impressed force is necessary at every place where u is variable and unequal to a.

251. The reason of the change of type which ensues when a wave is left to itself is not difficult to understand. From the ordinary theory we know that an infinitely small disturbance is propagated with a certain velocity a, which velocity is relative to the parts of the medium undisturbed by the wave. Let us consider now the case of a wave so long that the variations of velocity and density are insensible for a considerable distance along it, and at a place where the velocity (u) is finite let us imagine a small secondary wave to be superposed. The velocity with which the secondary wave is propagated through the medium is a, but on account of the local motion of the medium itself the whole velocity of advance is $a+u$, and depends upon the part of the long wave at which the small wave is placed. What has been said of a secondary wave applies also to the parts of the long wave itself, and thus we see that after a time t the place, where a certain velocity u is to be found, is in advance of its original position by a distance equal, not to at, but to $(a+u)t$: or, as we may express it, u is propagated with a velocity $a+u$. In symbolical notation $u = f\{x-(a+u)t\}$, where f is an arbitrary function, an equation first obtained by Poisson[1].

From the argument just employed it might appear at first sight that alteration of type was a necessary incident in the progress of a wave, independently of any particular supposition as to the relation between pressure and density, and yet it was proved in § 250 that in the case of one particular law of pressure there would be no alteration of type. We have, however, tacitly assumed in the present section that a is constant, which is tantamount to a restriction to Boyle's law. Under any other law of pressure $\sqrt{(dp/d\rho)}$ is a function of ρ, and therefore, as we shall see presently, of u. In the case of the law expressed in (5) § 250, the relation between u and ρ for a progressive wave is such that $\sqrt{(dp/d\rho)}+u$ is constant, as much advance being lost by slower propagation due to augmented density as is gained by superposition of the velocity u.

So far as the constitution of the medium itself is concerned there is nothing to prevent our ascribing arbitrary values to both u and ρ, but in a progressive wave a relation between these two quantities must be satisfied. We know already (§ 245) that this is the case when the disturbance is small, and the following argument will not only shew that such a relation is to be expected in cases where the square of the motion must be retained, but will even define the form of the relation.

Whatever may be the law of pressure, the velocity of propagation of small disturbances is by § 245 equal to $\sqrt{(dp/d\rho)}$, and in

[1] Mémoire sur la Théorie du Son. *Journal de l'école polytechnique*, t. VII. p. 319. 1808.

a positive progressive wave the relation between velocity and condensation is

$$u : s = \sqrt{\left(\frac{dp}{d\rho}\right)} \qquad (1).$$

If this relation be violated at any point, a wave will emerge, travelling in the negative direction. Let us now picture to ourselves the case of a positive progressive wave in which the changes of velocity and density are very gradual but become important by accumulation, and let us inquire what conditions must be satisfied in order to prevent the formation of a negative wave. It is clear that the answer to the question whether, or not, a negative wave will be generated at any point will depend upon the state of things in the immediate neighbourhood of the point, and not upon the state of things at a distance from it, and will therefore be determined by the criterion applicable to small disturbances. In applying this criterion we are to consider the velocities and condensations, not absolutely, but relatively to those prevailing in the neighbouring parts of the medium, so that the form of (1) proper for the present purpose is

$$du = \sqrt{\left(\frac{dp}{d\rho}\right)} \cdot \frac{d\rho}{\rho} \qquad (2);$$

whence

$$u = \int \sqrt{\left(\frac{dp}{d\rho}\right)} \cdot \frac{d\rho}{\rho} \qquad (3),$$

which is the relation between u and ρ necessary for a positive progressive wave. Equation (2) was obtained analytically by Earnshaw[1].

In the case of Boyle's law, $\sqrt{(dp/d\rho)}$ is constant, and the relation between velocity and density, given first, I believe, by Helmholtz[2], is

$$u = a \log \frac{\rho}{\rho_0} \qquad (4),$$

if ρ_0 be the density corresponding to $u = 0$.

In this case Poisson's integral allows us to form a definite idea of the change of type accompanying the earlier stages of the progress of the wave, and it finally leads us to a difficulty which has not as yet been surmounted[3]. If we draw a curve to represent

[1] *Phil. Trans.* 1859, p. 146.

[2] *Fortschritte der Physik*, IV. p. 106. 1852.

[3] Stokes, "On a difficulty in the Theory of Sound." *Phil. Mag.* Nov. 1848.

the distribution of velocity, taking x for abscissa and u for ordinate, we may find the corresponding curve after the lapse of time t by the following construction. Through any point on the original curve draw a straight line in the positive direction parallel to x, and of length equal to $(a+u)t$, or, as we are concerned with the shape of the curve only, equal to ut. The locus of the ends of these lines is the velocity curve after a time t.

But this law of derivation cannot hold good indefinitely. The crests of the velocity curve gain continually on the troughs and must at last overtake them. After this the curve would indicate two values of u for one value of x, ceasing to represent anything that could actually take place. In fact we are not at liberty to push the application of the integral beyond the point at which the velocity becomes discontinuous, or the velocity curve has a vertical tangent. In order to find when this happens let us take two neighbouring points on any part of the curve which slopes downwards in the positive direction, and inquire after what time this part of the curve becomes vertical. If the difference of abscissæ be dx, the hinder point will overtake the forward point in the time $dx \div (-du)$. Thus the motion, as determined by Poisson's equation, becomes discontinuous after a time equal to the reciprocal, taken positively, of the greatest negative value of du/dx.

For example, let us suppose that

$$u = U \cos \frac{2\pi}{\lambda} \{x - (a+u)t\},$$

where U is the greatest initial velocity. When $t=0$, the greatest negative value of du/dx is $-2\pi U/\lambda$; so that discontinuity will commence at the time $t = \lambda/2\pi U$.

When discontinuity sets in, a state of things exists to which the usual differential equations are inapplicable; and the subsequent progress of the motion has not been determined. It is probable, as suggested by Stokes, that some sort of reflection would ensue. In regard to this matter we must be careful to keep purely mathematical questions distinct from physical ones. In practice we have to do with spherical waves, whose divergency may of itself be sufficient to hold in check the tendency to discontinuity. In actual gases too it is certain that before discontinuity could enter, the law of pressure would begin to change its form, and the influence of viscosity could no longer be neglected. But these considerations have nothing to do with the mathematical problem of determining what would happen to waves of finite amplitude in a medium, free from viscosity, whose pressure is under all circumstances exactly proportional to its density; and this problem has not been solved.

It is worthy of remark that, although we may of course conceive a wave of finite disturbance to exist at any moment, there is a limit to the duration of its previous independent existence. By drawing lines in the negative instead of in the positive direction we may trace the history of the velocity curve; and we see that as we push our inquiry further and further into past time the forward slopes become easier and the backward slopes steeper. At a time, equal to the greatest positive value of dx/du, antecedent to that at which the curve is first contemplated, the velocity would be discontinuous.

252. The complete integration of the exact equations (4) and (6) § 249 in the case of a progressive wave was first effected by Earnshaw[1]. Finding reason for thinking that in a sound wave the equation

$$\frac{dy}{dt} = F\left(\frac{dy}{dx}\right) \quad\text{.........................}(1)$$

must always be satisfied, he observed that the result of differentiating (1) with respect to t, viz.

$$\frac{d^2y}{dt^2} = \left\{F'\left(\frac{dy}{dx}\right)\right\}^2 \frac{d^2y}{dx^2} \quad\text{.....................}(2),$$

can by means of the arbitrary function F be made to coincide with any dynamical equation in which the ratio of d^2y/dt^2 and d^2y/dx^2 is expressed in terms of dy/dx. The form of the function F being thus determined, the solution may be completed by the usual process applicable to such cases[2].

Writing for brevity α in place of dy/dx, we have

$$dy = \frac{dy}{dx}dx + \frac{dy}{dt}dt = \alpha\, dx + F(\alpha)\, dt,$$

and the integral is to be found by eliminating α between the equations

$$\left.\begin{aligned} y &= \alpha x + F(\alpha)\, t + \phi(\alpha) \\ 0 &= \quad x + F'(\alpha)\, t + \phi'(\alpha) \end{aligned}\right\}\quad\text{.....................}(3),$$

α being equal to ρ_0/ρ, and ϕ being an arbitrary function.

[1] *Proceedings of the Royal Society*, Jan. 6, 1859. *Phil. Trans.* 1860, p. 133.

[2] Boole's *Differential Equations*, Ch. XIV.

If $p = a^2\rho$, the exact equation (6 § 249) is

$$\left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dt^2} = a^2 \frac{d^2y}{dx^2} \quad\text{.....................(4)},$$

by comparison of which with (2) we see that

$$F'(\alpha) = \frac{\pm a}{\alpha} \quad\text{.........................(5)},$$

or on integration

$$F(\alpha) = C \pm a \log \alpha \quad\text{......................(6)},$$

as might also have been inferred from (4) § 251. The constant C vanishes, if $F(\alpha)$, viz. u, vanish when $\alpha = 1$, or $\rho = \rho_0$; otherwise it represents a velocity of the medium as a whole, having nothing to do with the wave as such. For a *positive* progressive wave the lower signs in the ambiguities are to be used. Thus in place of (3), we have

$$\left.\begin{array}{l} y = \alpha x - a \log \alpha\, t + \phi(\alpha) \\ 0 = \alpha x - a t \qquad + \alpha \phi'(\alpha) \end{array}\right\} \quad\text{.................(7)},$$

and

$$u = -a \log \alpha = a \log \frac{\rho}{\rho_0} \quad\text{...................(8)}.$$

If we subtract the second of equations (7) from the first, we get

$$y - at + at \log \alpha = \phi(\alpha) - \alpha \phi'(\alpha),$$

from which by (8) we see that $y - (a + u)t$ is an arbitrary function of α, or of u. Conversely therefore u is an arbitrary function of $y - (a + u)t$, and we may write

$$u = f\{y - (a + u)t\} \quad\text{...................(9)}.$$

Equation (9) is Poisson's integral, considered in the preceding section, where the symbol x has the same meaning as here attaches to y.

253. The problem of plane waves of finite amplitude attracted also the attention of Riemann, whose memoir was communicated to the Royal Society of Göttingen on the 28th of November, 1859[1]. Riemann's investigation is founded on the general hydrodynamical equations investigated in §§ 237, 238, and is not restricted to any particular law of pressure. In order, however, not unduly to

[1] Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. Göttingen, *Abhandlungen*, t. VIII. 1860. See also an excellent abstract in the *Fortschritte der Physik*, XV. p. 123. [Reference may be made also to a paper by C. V. Burton, *Phil. Mag.* XXXV. p. 317, 1893.]

extend the discussion of this part of our subject, already perhaps treated at greater length than its acoustical importance would warrant, we shall here confine ourselves to the case of Boyle's law of pressure.

Applying equations (1), (2) of § 237 and (1) of § 238 to the circumstances of the present problem, we get

$$\frac{du}{dt} + u\frac{du}{dx} = -a^2\frac{d\log\rho}{dx} \quad\ldots\ldots\ldots\ldots(1),$$

$$\frac{d\log\rho}{dt} + u\frac{d\log\rho}{dx} = -\frac{du}{dx} \quad\ldots\ldots\ldots\ldots(2).$$

If we multiply (2) by $\pm a$, and afterwards add it to (1), we obtain

$$\frac{dP}{dt} = -(u+a)\frac{dP}{dx}, \qquad \frac{dQ}{dt} = -(u-a)\frac{dQ}{dx} \quad\ldots\ldots(3),$$

where $$P = a\log\rho + u, \qquad Q = a\log\rho - u \quad\ldots\ldots\ldots(4).$$

Thus $$dP = \frac{dP}{dx}\{dx-(u+a)\,dt\} \quad\ldots\ldots\ldots\ldots(5),$$

$$dQ = \frac{dQ}{dx}\{dx-(u-a)\,dt\} \quad\ldots\ldots\ldots\ldots(6).$$

These equations are more general than Poisson's and Earnshaw's in that they are not limited to the case of a single positive, or negative, progressive wave. From (5) we learn that whatever may be the value of P corresponding to the point x and the time t, the same value of P corresponds to the point $x+(u+a)\,dt$ at the time $t+dt$; and in the same way from (6) we see that Q remains unchanged when x and t acquire the increments $(u-a)\,dt$ and dt respectively. If P and Q be given at a certain instant of time as functions of x, and the representative curves be drawn, we may deduce the corresponding value of u by (4), and thus, as in § 251, construct the curves representing the values of P and Q after the small interval of time dt, from which the new values of u and ρ in their turn become known, and the process can be repeated.

The element of the fluid, to which the values of P and Q at any moment belong, is itself moving with the velocity u, so that the velocities of P and Q relatively to the element are numerically the same, and equal to a, that of P being in the positive direction and that of Q in the negative direction.

We are now in a position to trace the consequences of an initial disturbance which is confined to a finite portion of the medium, e.g. between $x=\alpha$ and $x=\beta$, outside which the medium is at rest and at its normal density, so that the values of P and Q are $a \log \rho_0$. Each value of P propagates itself in turn to the elements of fluid which lie in front of it, and each value of Q to those that lie behind it. The hinder limit of the region in which P is variable, viz. the place where P first attains the constant value $a \log \rho_0$, comes into contact first with the variable values of Q, and moves accordingly with a variable[1] velocity. At a definite time, requiring for its determination a solution of the differential equations, the hinder (left hand) limit of the region through which P varies, meets the hinder (right hand) limit of the region through which Q varies, after which the two regions separate themselves, and include between them a portion of fluid in its equilibrium condition, as appears from the fact that the values of P and Q are both $a \log \rho_0$. In the positive wave Q has the constant value $a \log \rho_0$, so that $u = a \log(\rho/\rho_0)$, as in (4) § 251; in the negative wave P has the same constant value, giving as the relation between u and ρ, $u = -a \log(\rho/\rho_0)$. Since in each progressive wave, when isolated, a law prevails connecting the quantities u and ρ, we see that in the positive wave du vanishes with dP, and in the negative wave du vanishes with dQ. Thus from (5) we learn that in a positive progressive wave du vanishes, if the increments of x and t be such as to satisfy the equation $dx-(u+a)\,dt=0$, from which Poisson's integral immediately follows.

It would lead us too far to follow out the analytical development of Riemann's method, for which the reader must be referred to the original memoir; but it would be improper to pass over in silence an error on the subject of discontinuous motion into which Riemann and other writers have fallen. It has been held that a state of motion is possible in which the fluid is divided into two parts by a surface of discontinuity propagating itself with constant velocity, all the fluid on one side of the surface of discontinuity being in one uniform condition as to density and velocity, and on the other side in a second uniform condition in the same respects. Now, if this motion were possible, a motion of the same kind in which the surface of discontinuity is at rest would also be

[1] At this point an error seems to have crept into Riemann's work, which is corrected in the abstract of the *Fortschritte der Physik*.

possible, as we may see by supposing a velocity equal and opposite to that with which the surface of discontinuity at first moves, to be impressed upon the whole mass of fluid. In order to find the relations that must subsist between the velocity and density on the one side (u_1, ρ_1) and the velocity and density on the other side (u_2, ρ_2), we notice in the first place that by the principle of conservation of matter $\rho_2 u_2 = \rho_1 u_1$. Again, if we consider the momentum of a slice bounded by parallel planes and including the surface of discontinuity, we see that the momentum leaving the slice in the unit of time is for each unit of area $(\rho_2 u_2 = \rho_1 u_1) u_2$, while the momentum entering it is $\rho_1 u_1^2$. The difference of momentum must be balanced by the pressures acting at the boundaries of the slice, so that

$$\rho_1 u_1 (u_2 - u_1) = p_1 - p_2 = a^2 (\rho_1 - \rho_2),$$

whence

$$u_1 = a \sqrt{\left(\frac{\rho_2}{\rho_1}\right)}, \qquad u_2 = a \sqrt{\left(\frac{\rho_1}{\rho_2}\right)} \quad\text{.................(7).}$$

The motion thus determined is, however, not possible; it satisfies indeed the conditions of mass and momentum, but it violates the condition of energy (§ 244) expressed by the equation

$$\tfrac{1}{2} u_2^2 - \tfrac{1}{2} u_1^2 = a^2 \log \rho_1 - a^2 \log \rho_2 \quad\text{...............(8).}$$

This argument has been already given in another form in § 250, which would alone justify us in rejecting the assumed motion, since it appears that no steady motion is possible except under the law of density there determined. From equation (8) of that section we can find what impressed forces would be necessary to maintain the motion defined by (7). It appears that the force X, though confined to the place of discontinuity, is made up of two parts of opposite signs, since by (7) u passes *through* the value a. The whole moving force, viz. $\int X\rho\, dx$, vanishes, and this explains how it is that the condition relating to momentum is satisfied by (7), though the force X be ignored altogether.

253 *a*. Among the phenomena of the second order which admit of a ready explanation, a prominent place must be assigned to the repulsion of resonators discovered independently by Dvořák[1] and Mayer[2]. These observers found that an air resonator of any kind (Ch. XVI.) when exposed to a powerful source

[1] *Pogg. Ann.* CLVII. p. 42, 1876; *Wied. Ann.* III. p. 328, 1878.

[2] *Phil. Mag.* vol. VI. p. 225, 1878.

of sound experiences a force directed inwards from the mouth, somewhat after the manner of a rocket. A combination of four light resonators, mounted anemometer fashion upon a steel point, may be caused to revolve continuously.

If there be no impressed forces, equation (2) § 244 gives

$$\varpi = \int \frac{dp}{\rho} = -\frac{d\phi}{dt} - \tfrac{1}{2}U^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Distinguishing the values of the quantities at two points of space by suffixes, we may write

$$\varpi_1 - \varpi_0 = \frac{d}{dt}(\phi_0 - \phi_1) + \tfrac{1}{2}U_0^2 - \tfrac{1}{2}U_1^2 \ldots\ldots\ldots\ldots(2).$$

This equation holds good at every instant. Integrating it over a long range of time we obtain as applicable to every case of fluid motion in which the flow between the two points does not continually increase

$$\int \varpi_1 dt - \int \varpi_0 dt = \tfrac{1}{2}\int U_0^2 dt - \tfrac{1}{2}\int U_1^2 dt \ldots\ldots\ldots\ldots(3).$$

The first point (with suffix 0) is now to be chosen at such a distance that the variation of pressure and the velocity are there insensible. Accordingly

$$\int \varpi_1 dt = -\tfrac{1}{2}\int U_1^2 dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

This equation is true wherever the second point be taken. If it be in the interior of a resonator, or at a corner where three fixed walls meet, $U_1 = 0$, and therefore

$$\int (\varpi_1 - \varpi_0)\, dt = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

or the mean value of ϖ in the interior is the same as at a distance outside.

By (9) § 246, if the expansions and contractions be adiabatic, $p \propto \rho^\gamma$; and $\varpi = p^{(\gamma-1)/\gamma}$. Thus

$$\surd\left\{\left(\frac{p_1}{p_0}\right)^{(\gamma-1)/\gamma} - 1\right\} dt = 0 \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

If in (6) we suppose that the difference between p_1 and p_0 is comparatively small, we may expand the function there contained by the binomial theorem. The approximate result may be expressed

$$\int \frac{p_1 - p_0}{p_0}\, dt = \frac{1}{2\gamma}\int \left(\frac{p_1 - p_0}{p_0}\right)^2 dt \quad \ldots\ldots\ldots\ldots(7),$$

shewing that the mean value of $(p_1 - p_0)$ is positive, or in other words that the mean pressure in the resonator is *in excess* of the atmospheric pressure[1]. The resonator therefore tends to move as if impelled by a force acting normally over the area of its aperture and directed *inwards*.

The experiment may be made (after Dvorák) with a Helmholtz resonator by connecting the nipple with a horizontal and not too narrow glass tube in which moves a piston of ether. When a fork of suitable pitch, e.g. 256 or 512, is vigorously excited and presented to the mouth of the resonator, the movement of the ether shews an augmentation of pressure, while the similar presentation of the non-vibrating fork is without effect.

If to the first order of small quantities

$$(p - p_0)/p_0 = P \cos nt \quad\text{.....................(8)},$$

its mean value correct to the second order is $P^2/4\gamma$, in which for air and the principal gases $\gamma = 1{\cdot}4$.

If the expansions and contractions be supposed to take place isothermally, the corresponding result is arrived at by putting $\gamma = 1$ in (7).

253 *b*. In § 253 *a* the effect to be explained is intimately connected with the compressibility of the fluid which occupies the interior of the resonator. In the class of phenomena now to be considered the compressibility of the fluid is of secondary importance, and the leading features of the explanation may be given upon the supposition that the fluid retains a constant density throughout.

If ρ be constant, (4) § 253 *a* may be written

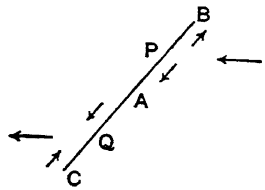
$$\int (p_1 - p_0)\, dt = -\tfrac{1}{2}\rho \int U_1^2 dt \quad\text{..................(1)},$$

shewing that the mean pressure at a place where there is motion is less than in the undisturbed parts of the fluid—a theorem due to Kelvin[2], and applied by him to the explanation of the attractions observed by Guthrie and other experimenters. Thus a vibrating tuning-fork, presented to a delicately suspended rectangle of paper, appears to exercise an attraction, the mean value of U^2 being greater on the face exposed to the fork than upon the back.

[1] *Phil. Mag.* vol. VI. p. 270, 1878.
[2] *Proc. Roy. Soc.* vol. XIX. p. 271, 1887.

In the above experiment the action depends upon the proximity of the source of disturbance. When the flow of fluid, whether steady or alternating, is uniform over a large region, the effect upon an obstacle introduced therein is a question of shape. In the case of a sphere there is manifestly no tendency to turn; and since the flow is symmetrical on the up-stream and down-stream sides, the mean pressures given by (1) balance one another. Accordingly a sphere experiences neither force nor couple. It is otherwise when the form of the body is elongated or flattened. That a flat obstacle tends to turn its flat side to the stream[1] may be inferred from the general character of the lines of flow round it. The pressures at the various points of the surface *BC* (Fig. 54 *a*) depend upon the velocities of the fluid there obtaining. The full pressure due to the complete stoppage of the stream is to be found at two points, where the current divides. It is pretty evident that upon the up-stream side this lies (*P*) on *AB*, and upon the down-stream side upon *AC* at the corresponding point *Q*. The resultant of the pressures thus tends to turn *AB* so as to face the stream.

Fig. 54 *a*.



When the obstacle is in the form of an ellipsoid, the mathematical calculation of the forces can be effected; but it must suffice here to refer to the particular case of a thin circular disc, whose normal makes an angle θ with the direction of the undisturbed stream. It may be proved[2] that the moment M of the couple tending to diminish θ has the value given by

$$M = \tfrac{4}{3}\rho a^3 W^2 \sin 2\theta \quad \text{......................(2)},$$

a being the radius of the disc and W the velocity of the stream. If the stream be alternating instead of steady, we have merely to employ the mean value of W^2, as appears from (1).

The observation that a delicately suspended disc sets itself across the direction of alternating currents of air originated in the attempt to explain certain anomalies in the behaviour of a magnetometer mirror[3]. In illustration, "a small disc of paper, about the size of a sixpence, was hung by a fine silk fibre across

[1] Thomson and Tait's *Natural Philosophy*, § 336, 1867.
[2] W. König, *Wied. Ann.* t. XLIII. p. 51, 1891.
[3] *Proc. Roy. Soc.* vol. XXXII. p. 110, 1881.

the mouth of a resonator of pitch 128. When a sound of this pitch is excited in the neighbourhood, there is a powerful rush of air into and out of the resonator, and the disc sets itself promptly across the passage. A fork of pitch 128 may be held near the resonator, but it is better to use a second resonator at a little distance in order to avoid any possible disturbance due to the neighbourhood of the vibrating prongs. The experiment, though rather less striking, was also successful with forks and resonators of pitch 256."

Upon this principle an instrument may be constructed for measuring the intensities of aerial vibrations of selected pitch[1]. A tube, measuring three quarters of a wave length, is open at one end and at the other is closed air-tight by a plate of glass. At one quarter of a wave length's distance from the closed end is hung by a silk fibre a light mirror with attached magnet, such as is used for reflecting galvanometers. In its undisturbed condition the plane of the mirror makes an angle of 45° with the axis of the tube. At the side is provided a glass window, through which light, entering along the axis and reflected by the mirror, is able to escape from the tube and to form a suitable image upon a divided scale. The tube as a whole acts as a resonator, and the alternating currents at the loop (§ 255) deflect the mirror through an angle which is read in the usual manner.

In an instrument constructed by Boys[2] the sensitiveness is exalted to an extraordinary degree. This is effected partly by the use of a very light mirror with suspension of quartz fibre, and partly by the adoption of double resonance. The large resonator is a heavy brass tube of about 10 cm. diameter, closed at one end, and of such length as to resound to e'. The mirror is hung in a short lateral tube forming a communication between the large resonator and a small glass bulb of suitable capacity. The external vibrations may be regarded as magnified first by the large resonator and then *again* by the small one, so that the mirror is affected by powerful alternating currents of air. The selection of pitch is so definite that there is hardly any response to sounds which are a semi-tone too high or too low.

Perhaps the most striking of all the effects of alternating aerial currents is the rib-like structure assumed by cork filings in

[1] *Phil. Mag.* vol. XIV. p. 186, 1882.

[2] *Nature*, vol. XLII. p. 604, 1890.

Kundt's experiment § 260. Close observation, while the vibrations are in progress, shews that the filings are disposed in thin laminæ transverse to the tube and extending upwards to a certain distance from the bottom. The effect is a maximum at the loops, and disappears in the neighbourhood of the nodes. When the vibrations stop, the laminæ necessarily fall, and in so doing lose much of their sharpness, but they remain visible as transverse streaks.

The explanation of this peculiar behaviour has been given by W. König[1]. We have seen that a single spherical obstacle experiences no force from an alternating current. But this condition of things is disturbed by the presence of a neighbour. Consider for simplicity the case of two spheres at a moderate distance apart, and so situated that the line of centres is either parallel to the stream, Fig. 54 *b*, or perpendicular to it, Fig. 54 *c*. It is easy to recognise that the velocity between the spheres will be less in the first case and greater in the second than on the averted hemispheres. Since the pressure increases as the velocity diminishes, it follows that in the first position the spheres will repel one another, and that in the second position they will attract one another. The result of these forces between neighbours is plainly a tendency to aggregate in laminæ. The case may be contrasted with that of iron filings in a magnetic field, whose direction is parallel to that of the aerial current. There is then attraction in the first position and repulsion in the second, and the result is a tendency to aggregate in *filaments*.

Fig. 54 *b*.

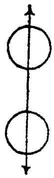
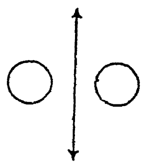
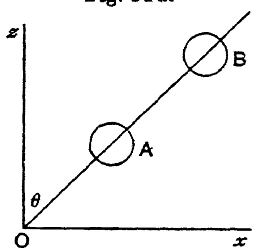


Fig. 54 *c*.



On the foundation of the analysis of Kirchhoff, König has calculated the forces operative in the case of two spheres which are not too close together. If a_1, a_2 be the radii of the spheres, r their distance asunder, θ the angle between the line of centres and the direction of the current taken as axis of z (Fig. 54 *d*), W the velocity of the current, then the components of force upon the sphere B in the direction of z and of x

Fig. 54 *d*.



[1] *Wied. Ann.* t. XLII. pp. 353, 549, 1891.

drawn perpendicular to z in the plane containing z and the line of centres, are given by

$$Z = -\frac{3\pi\rho a_1^3 a_2^3 W^2}{r^4}\cos\theta\,(3 - 5\cos^2\theta)\ldots\ldots\ldots\ldots(3),$$

$$X = -\frac{3\pi\rho a_1^3 a_2^3 W^2}{r^4}\sin\theta\,(1 - 5\cos^2\theta)\ldots\ldots\ldots\ldots(4),$$

the third component Y vanishing by virtue of the symmetry. In the case of Fig. 54 *b* $\theta = 0$, and there is repulsion equal to $6\pi\rho a_1^3 a_2^3 W^2/r^4$; in the case of Fig. 54 *c* $\theta = \frac{1}{2}\pi$, and the force is an attraction $3\pi\rho a_1^3 a_2^3 W^2/r^4$. In oblique positions the direction of the force does not coincide with the line of centres.

If the spheres be rigidly connected, the forces upon the system reduce to a couple (tending to increase θ) of moment given by

$$-Z\sin\theta + X\cos\theta = \frac{3\pi\rho a_1^3 a_2^3 W^2}{r^3}\sin 2\theta\ldots\ldots\ldots(5).$$

When the current is alternating, we are to take the *mean* value of W^2 in (3), (4), (5).

254. The exact experimental determination of the velocity of sound is a matter of greater difficulty than might have been expected. Observations in the open air are liable to errors from the effects of wind, and from uncertainty with respect to the exact condition of the atmosphere as to temperature and dryness. On the other hand when sound is propagated through air contained in pipes, disturbance arises from friction and from transfer of heat; and, although no great errors from these sources are to be feared in the case of tubes of considerable diameter, such as some of those employed by Regnault, it is difficult to feel sure that the ideal plane waves of theory are nearly enough realized.

The following Table[1] contains a list of the principal experimental determinations which have been made hitherto.

Names of Observers.	Velocity of Sound at 0° Cent. in Metres.
Académie des Sciences (1738)	332
Benzenberg (1811)	333·7 332·3
Goldingham (1821)	331·1
Bureau des Longitudes (1822)	330·6
Moll and van Beek	332·2

[1] Bosanquet, *Phil. Mag.* April, 1877.

Names of Observers.	Velocity of Sound at 0° Cent. in Metres.
Stampfer and Myrback	332·4
Bravais and Martins (1844)	332·4
Wertheim	331·6
Stone (1871)	332·4
Le Roux	330·7
Regnault	330·7

In Stone's experiments[1] the course over which the sound was timed commenced at a distance of 640 feet from the source, so that any errors arising from excessive disturbance were to a great extent avoided.

A method has been proposed by Bosscha[2] for determining the velocity of sound without the use of great distances. It depends upon the precision with which the ear is able to decide whether short ticks are simultaneous, or not. In König's[3] form of the experiment, two small electro-magnetic counters are controlled by a fork-interrupter (§ 64), whose period is one-tenth of a second, and give synchronous ticks of the same period. When the counters are close together the audible ticks coincide, but as one counter is gradually removed from the ear, the two series of ticks fall asunder. When the difference of distances is about 34 metres, coincidence again takes place, proving that 34 metres is about the distance traversed by sound in a tenth part of a second.

[On the basis of experiments made in pipes Violle and Vautier[4] give 331·10 as applicable in free air. The result includes a correction, amounting to 0·68, which is of a more or less theoretical character, representing the presumed influence of the pipe ($0·7^{m}$ in diameter).]

[1] *Phil. Trans.* 1872, p. 1.
[2] *Pogg. Ann.* xcii. 486. 1854.
[3] *Pogg. Ann.* cxviii. 610. 1863.
[4] *Ann. de Chim.* t. xix.; 1890.

CHAPTER XII.

VIBRATIONS IN TUBES.

255. WE have already (§ 245) considered the solution of our fundamental equation, when the velocity-potential, in an unlimited fluid, is a function of one space co-ordinate only. In the absence of friction no change would be caused by the introduction of any number of fixed cylindrical surfaces, whose generating lines are parallel to the co-ordinate in question; for even when the surfaces are absent the fluid has no tendency to move across them. If one of the cylindrical surfaces be closed (in respect to its transverse section), we have the important problem of the axial motion of air within a cylindrical pipe, which, when once the mechanical conditions at the ends are given, is independent of anything that may happen outside the pipe.

Considering a simple harmonic vibration, we know (§ 245) that, if ϕ varies as e^{int},

$$\frac{d^2\phi}{dx^2} + k^2\phi = 0 \qquad\qquad (1),$$

where

$$k = \frac{2\pi}{\lambda} = \frac{n}{a} \qquad\qquad (2).$$

The solution may be written in two forms—

$$\left.\begin{aligned} \phi &= (A\cos kx + B\sin kx)\, e^{int} \\ \phi &= (A\, e^{ikx} + B\, e^{-ikx})\, e^{int} \end{aligned}\right\} \qquad\qquad (3),$$

of which finally only the real parts will be retained. The first form will be most convenient when the vibration is stationary, or nearly so, and the second when the motion reduces itself to a positive, or negative, progressive undulation. The constants A and B in the symbolical solution may be complex, and thus the final expression in terms of real quantities will involve *four* arbitrary constants. If we wish to use real quantities throughout, we must take

$$\phi = (A\cos kx + B\sin kx)\cos nt + (C\cos kx + D\sin kx)\sin nt \quad\text{................}(4),$$

but the analytical work would generally be longer. When no ambiguity can arise, we shall sometimes for the sake of brevity drop, or restore, the factor involving the time without express mention. Equations such as (1) are of course equally true whether the factor be understood or not.

Taking the first form in (3), we have

$$\left.\begin{aligned}\phi &= A\cos kx + B\sin kx\\ \frac{d\phi}{dx} &= -kA\sin kx + kB\cos kx\end{aligned}\right\}\text{...............}(5).$$

If there be any point at which either ϕ or $d\phi/dx$ is permanently zero, the ratio $A : B$ must be real, and then the vibration is *stationary*, that is, the same in phase at all points simultaneously.

Let us suppose that there is a node at the origin. Then when $x=0$, $d\phi/dx$ vanishes, the condition of which is $B=0$. Thus

$$\phi = A\cos kx\, e^{int}; \qquad \frac{d\phi}{dx} = -kA\sin kx\, e^{int}\text{.........}(6),$$

from which, if we substitute $Pe^{i\theta}$ for A, and throw away the imaginary part,

$$\left.\begin{aligned}\phi &= P\cos kx\cos(nt+\theta)\\ \frac{d\phi}{dx} &= -kP\sin kx\cos(nt+\theta)\end{aligned}\right\}\text{...............}(7).$$

From these equations we learn that $d\phi/dx$ vanishes wherever $\sin kx = 0$; that is, that besides the origin there are nodes at the points $x=\frac{1}{2}m\lambda$, m being any positive or negative integer. At any of these places infinitely thin rigid plane barriers normal to x might be stretched across the tube without in any way altering the motion. Midway between each pair of consecutive nodes there is a *loop*, or place of no pressure variation, since $\delta p = -\rho\dot\phi$ (6) § 244. At any of these loops a communication with the external atmosphere might be opened, without causing any disturbance of the motion from air passing in or out. The loops are the places of maximum velocity, and the nodes those of maximum pressure variation. At intervals of λ everything is exactly repeated.

If there be a node at $x=l$, as well as at the origin, $\sin kl=0$, or $\lambda=2l/m$, where m is a positive integer. The gravest tone which can be sounded by air contained in a doubly closed pipe of length l is therefore that which has a wave-length equal to $2l$. This statement, it will be observed, holds good whatever be the gas with which the pipe is filled; but the frequency, or the place of the tone in the musical scale, depends also on the nature of the particular gas. The periodic time is given by

$$\tau=\frac{\lambda}{a}=\frac{2l}{a} \quad\text{.........................}\quad (8).$$

The other tones possible for a doubly closed pipe have periods which are submultiples of that of the gravest tone, and the whole system forms a harmonic scale.

Let us now suppose, without stopping for the moment to inquire how such a condition of things can be secured, that there is a loop instead of a node at the point $x=l$. Equation (6) gives $\cos kl=0$, whence $\lambda=4l\div(2m+1)$, where m is zero or a positive integer. In this case the gravest tone has a wave-length equal to four times the length of the pipe reckoned from the node to the loop, and the other tones form with it a harmonic scale, from which, however, all the members of even order are missing.

256. By means of a rigid barrier there is no difficulty in securing a node at any desired point of a tube, but the condition for a loop, i.e. that under no circumstances shall the pressure vary, can only be realized approximately. In most cases the variation of pressure at any point of a pipe may be made small by allowing a free communication with the external air. Thus Euler and Lagrange assumed constancy of pressure as the condition to be satisfied at the end of an open pipe. We shall afterwards return to the problem of the open pipe, and investigate by a rigorous process the conditions to be satisfied at the end. For our immediate purpose it will be sufficient to know, what is indeed tolerably obvious, that the open end of a pipe may be treated as a loop, if the diameter of the pipe be neglected in comparison with the wave-length, provided the external pressure in the neighbourhood of the open end be not itself variable from some cause independent of the motion within the pipe. When there is an independent source of sound, the pressure at the end of the pipe is the same as it would be in the same place, if the pipe were away. The impediment to securing the fulfilment of the condition for a loop at any desired point lies in the inertia of the machinery required to sustain the pressure. For theoretical purposes we may overlook this difficulty, and imagine a massless piston backed by a compressed spring also without mass. The assumption of a loop at an open end of a pipe is tantamount to neglecting the inertia of the outside air.

We have seen that, if a node exist at any point of a pipe, there must be a series, ranged at equal intervals $\frac{1}{2}\lambda$, that midway between each pair of consecutive nodes there must be a loop, and that the whole vibration must be stationary. The same conclusion follows if there be at any point a loop; but it may perfectly well happen that there are neither nodes nor loops, as for example in the case when the motion reduces to a positive or negative progressive wave. In stationary vibration there is no transference of energy along the tube in either direction, for energy cannot pass a node or a loop.

257. The relations between the lengths of an open or closed pipe and the wave-lengths of the included column of air may also be investigated by following the motion of a *pulse*, by which is understood a wave confined within narrow limits and composed of uniformly condensed or rarefied fluid. In looking at the matter from this point of view it is necessary to take into account carefully the circumstances under which the various reflections take place. Let us first suppose that a condensed pulse travels in the positive direction towards a barrier fixed across the tube. Since the energy contained in the wave cannot escape from the tube, there must be a reflected wave, and that this reflected wave is also a wave of condensation appears from the fact that there is no loss of fluid. The same conclusion may be arrived at in another way. The effect of the barrier may be imitated by the introduction of a similar and equidistant wave of condensation moving in the negative direction. Since the two waves are both condensed and are propagated in contrary directions, the velocities of the fluid composing them are equal and opposite, and therefore neutralise one another when the waves are superposed.

If the progress of the negative reflected wave be interrupted by a second barrier, a similar reflection takes place, and the wave, still remaining condensed, regains its positive character. When a distance has been travelled equal to twice the length of the pipe, the original state of things is completely restored, and the same cycle of events repeats itself indefinitely. We learn therefore that the period within a doubly closed pipe is the time occupied by a pulse in travelling twice the length of the pipe.

The case of an open end is somewhat different. The supplementary negative wave necessary to imitate the effect of the open end must evidently be a wave of rarefaction capable of neutralising the positive pressure of the condensed primary wave, and thus in the act of reflection a wave changes its character from condensed to rarefied, or from rarefied to condensed. Another way of considering the matter is to observe that in a positive condensed pulse the momentum of the motion is forwards, and in the absence of the necessary forces cannot be changed by the reflection. But forward motion in the reflected negative wave is indissolubly connected with the rarefied condition.

When both ends of a tube are open, a pulse travelling backwards and forwards within it is completely restored to its original state after traversing twice the length of the tube, suffering in the process two reflections, and thus the relation between length and period is the same as in the case of a tube, whose ends are both closed; but when one end of a tube is open and the other closed, a double passage is not sufficient to close the cycle of changes. The original condensed or rarefied character cannot be recovered until after two reflections from the open end, and accordingly in the case contemplated the period is the time required by the pulse to travel over *four* times the length of the pipe.

258. After the full discussion of the corresponding problems in the chapter on Strings, it will not be necessary to say much on the compound vibrations of columns of air. As a simple example we may take the case of a pipe open at one end and closed at the other, which is suddenly brought to rest at the time $t=0$, after being for some time in motion with a uniform velocity parallel to its length. The initial state of the contained air is then one of uniform velocity u_0 parallel to x, and of freedom from compression and rarefaction. If we suppose that the origin is at the closed end, the general solution is by (7) § 255,

$$\begin{aligned}\phi &= (A_1 \cos n_1 t + B_1 \sin n_1 t) \cos k_1 x \\ &\quad + (A_2 \cos n_2 t + B_2 \sin n_2 t) \cos k_2 x \\ &\quad + \dots \qquad (1),\end{aligned}$$

where $k_r = (r - \frac{1}{2})\pi/l$, $n_r = ak_r$, and $A_1, B_1, A_2, B_2 \dots$ are arbitrary constants.

Since $\dot{\phi}$ is to be zero initially for all values of x, the coefficients B must vanish; the coefficients A are to be determined by the condition that for all values of x between 0 and l,

$$\Sigma\, k_r A_r \sin k_r x = -u_0 \qquad (2),$$

where the summation extends to all integral values of r from 1 to ∞. The determination of the coefficients A from (2) is effected in the usual way. Multiplying by $\sin k_r x\, dx$, and integrating from 0 to l, we get

$$\tfrac{1}{2} l\, k_r A_r = -u_0/k_r,$$

or

$$A_r = -\frac{2u_0}{k_r^2 l} \qquad (3).$$

The complete solution is therefore

$$\phi = -\frac{2u_0}{l} \sum_{r=1}^{r=\infty} \frac{\cos k_r x}{k_r^2} \cos n_r t \qquad (4).$$

259. In the case of a tube stopped at the origin and open at $x = l$, let $\phi = \cos nt$ be the value of the potential at the open end due to an external source of sound. Determining P and θ in equation (7) § 255, we find

$$\phi = \frac{\cos kx}{\cos kl} \cos nt \qquad (1).$$

It appears that the vibration within the tube is a minimum, when $\cos kl = \pm 1$, that is when l is a multiple of $\frac{1}{2}\lambda$, in which case there is a node at $x = l$. When l is an odd multiple of $\frac{1}{4}\lambda$, $\cos kl$ vanishes, and then according to (1) the motion would become infinite. In this case the supposition that the pressure at the open end is independent of what happens within the tube breaks down; and we can only infer that the vibration is very large, in consequence of the isochronism. Since there is a node at $x=0$, there must be a loop when x is an odd multiple of $\frac{1}{4}\lambda$, and we conclude that in the case of isochronism the variation of pressure at the open end of the tube due to the external cause is exactly neutralised by the variation of pressure due to the motion within the tube itself. If there were really at the open end a variation of pressure on the whole, the motion must increase without limit in the absence of dissipative forces.

If we suppose that the origin is a loop instead of a node, the solution is

$$\phi = \frac{\sin kx}{\sin kl} \cos nt \quad\text{......................(2)},$$

where $\phi = \cos nt$ is the given value of ϕ at the open end $x=l$. In this case the expression becomes infinite, when $kl = m\pi$, or $l = \frac{1}{2}m\lambda$.

We will next consider the case of a tube, whose ends are both open and exposed to disturbances of the same period, making ϕ equal to He^{int}, Ke^{int} respectively. Unless the disturbances at the ends are in the same phase, one at least of the coefficients H, K must be complex.

Taking the first form in (3) § 255, we have as the general expression for ϕ

$$\phi = e^{int}(A \cos kx + B \sin kx).$$

If we take the origin in the middle of the tube, and assume that the values He^{int}, Ke^{int} correspond respectively to $x=l$, $x=-l$, we get to determine A and B,

$$H = A \cos kl + B \sin kl,$$
$$K = A \cos kl - B \sin kl,$$

whence

$$A = \frac{H+K}{2\cos kl}, \qquad B = \frac{H-K}{2\sin kl} \quad\text{..............(3)},$$

giving

$$\phi = e^{int}\frac{H \sin k(l+x) + K \sin k(l-x)}{\sin 2kl} \quad\text{.........(4)}.$$

This result might also be deduced from (2), if we consider that the required motion arises from the superposition of the motion, which is due to the disturbance He^{int} calculated on the hypothesis that the other end $x=-l$ is a loop, on the motion, which is due to Ke^{int} on the hypothesis that the end $x=l$ is a loop.

The vibration expressed by (4) cannot be *stationary*, unless the ratio $H : K$ be real, that is unless the disturbances at the ends be in similar, or in opposite, phases. Hence, except in the cases reserved, there is no loop anywhere, and therefore no place at which a branch tube can be connected along which sound will not be propagated[1].

At the middle of the tube, for which $x = 0$,

$$\phi = \frac{H + K}{2 \cos kl} e^{int} \qquad (5),$$

shewing that the variation of pressure (proportional to $\dot{\phi}$) vanishes if $H + K = 0$, that is, if the disturbances at the ends be equal and in *opposite* phases. Unless this condition be satisfied, the expression becomes infinite when $2l = \frac{1}{2}(2m + 1)\lambda$.

At a point distant $\frac{1}{4}\lambda$ from the middle of the tube the expression for ϕ is

$$\phi = \frac{H - K}{2 \sin kl} e^{int} \qquad (6),$$

vanishing when $H = K$, that is, when the disturbances at the ends are equal and in the *same* phase. In general ϕ becomes infinite, when $\sin kl = 0$, or $2l = m\lambda$.

If at one end of an unlimited tube there be a variation of pressure due to an external source, a train of progressive waves will be propagated inwards from that end. Thus, if the length along the tube measured from the open end be y, the velocity-potential is expressed by $\phi = \cos(nt - ny/a)$, corresponding to $\phi = \cos nt$ at $y = 0$; so that, if the cause of the disturbance within the tube be the passage of a train of progressive waves across the open end, the intensity within the tube will be the same as in the space outside. It must not be forgotten that the diameter of the tube is supposed to be infinitely small in comparison with the length of a wave.

[1] An arrangement of this kind has been proposed by Prof. Mayer (*Phil. Mag.* XLV. p. 90, 1873) for comparing the intensities of sources of sound of the same pitch. Each end of the tube is exposed to the action of one of the sources to be compared, and the distances are adjusted until the amplitudes of the vibrations denoted by H and K are equal. The branch tube is led to a manometric capsule (§ 262), and the method assumes that by varying the point of junction the disturbance of the flame can be stopped. From the discussion in the text it appears that this assumption is not theoretically correct.

Let us next suppose that the source of the motion is within the tube itself, due for example to the inexorable motion of a piston at the origin[1]. The constants in (5) § 255 are to be determined by the conditions that when $x=0$, $d\phi/dx = \cos nt$ (say), and that, when $x=l$, $\phi=0$. Thus $kA = -\tan kl$, $kB=1$, and the expression for ϕ is

$$\phi = \frac{\sin k(x-l)}{k\cos kl} \quad\text{.......................(7).}$$

The motion is a minimum, when $\cos kl = \pm 1$, that is, when the length of the tube is a multiple of $\frac{1}{2}\lambda$.

When l is an odd multiple of $\frac{1}{4}\lambda$, the place occupied by the piston would be a node, if the open end were really a loop, but in this case the solution fails. The escape of energy from the tube prevents the energy from accumulating beyond a certain point; but no account can be taken of this so long as the open end is treated rigorously as a loop. We shall resume the question of resonance after we have considered in greater detail the theory of the open end, when we shall be able to deal with it more satisfactorily.

In like manner if the point $x=l$ be a node, instead of a loop, the expression for ϕ is

$$\phi = \frac{\cos k(l-x)}{k\sin kl} \quad\text{.....................(8);}$$

and thus the motion is a minimum when l is an odd multiple of $\frac{1}{4}\lambda$, in which case the origin is a loop. When l is an even multiple of $\frac{1}{4}\lambda$, the origin should be a node, which is forbidden by the conditions of the question. In this case according to (8) the motion becomes infinite, which means that in the absence of dissipative forces the vibration would increase without limit.

260. The experimental investigation of aerial waves within pipes has been effected with considerable success by Kundt[2]. To generate waves is easy enough; but it is not so easy to invent a method by which they can be effectually examined. Kundt discovered that the nodes of stationary waves can be made evident by dust. A little fine sand or lycopodium seed, shaken over the interior of a glass tube containing a vibrating column of air

[1] These problems are considered by Poisson, *Mém. de l'Institut*, t. II. p. 305, 1819.

[2] *Pogg. Ann.* t. CXXXV. p. 337, 1868.

disposes itself in recurring patterns, by means of which it is easy to determine the positions of the nodes and to measure the intervals between them. In Kundt's experiments the origin of the sound was in the longitudinal vibration of a glass tube called the sounding-tube, and the dust-figures were formed in a second and larger tube, called the wave-tube, the latter being provided with a moveable stopper for the purpose of adjusting its length. The other end of the wave-tube was fitted with a cork through which the sounding-tube passed half way. By suitable friction the sounding-tube was caused to vibrate in its gravest mode, so that the central point was nodal, and its interior extremity (closed with a cork) excited aerial vibrations in the wave-tube. By means of the stopper the length of the column of air could be adjusted so as to make the vibrations as vigorous as possible, which happens when the interval between the stopper and the end of the sounding-tube is a multiple of half the wave-length of the sound.

With this apparatus Kundt was able to compare the wave-lengths of the same sound in various gases, from which the relative velocities of propagation are at once deducible, but the results were not entirely satisfactory. It was found that the intervals of recurrence of the dust-patterns were not strictly equal, and, what was worse, that the pitch of the sound was not constant from one experiment to another. These defects were traced to a communication of motion to the wave-tube through the cork, by which the dust-figures were disturbed, and the pitch made irregular in consequence of unavoidable variations in the mounting of the apparatus. To obviate them, Kundt replaced the cork, which formed too stiff a connection between the tubes, by layers of sheet indiarubber tied round with silk, obtaining in this way a flexible and perfectly air-tight joint; and in order to avoid any risk of the comparison of wave-lengths being vitiated by an alteration of pitch, the apparatus was modified so as to make it possible to excite the two systems of dust-figures simultaneously and in response to the same sound. A collateral advantage of the new method consisted in the elimination of temperature-corrections.

In the improved "Double Apparatus" the sounding-tube was caused to vibrate *in its second mode* by friction applied near the middle; and thus the nodes were formed at the points distant from the ends by one-fourth of the length of the tube. At each of these points connection was made with an independent wave-tube, provided with an adjustable stopper, and with branch tubes and stop-cocks suitable for admitting the various gases to be experimented upon. It is evident that dust-figures formed in the two tubes correspond rigorously to the same pitch, and that therefore a comparison of the intervals of recurrence leads to a correct determination of the velocities of propagation, under the circumstances of the experiment, for the two gases with which the tubes are filled.

The results at which Kundt arrived were as follows:—

(*a*) The velocity of sound in a tube diminishes with the diameter. Above a certain diameter, however, the change is not perceptible.

(*b*) The diminution of velocity increases with the wave-length of the tone employed.

(*c*) Powder, scattered in a tube, diminishes the velocity of sound in narrow tubes, but in wide ones is without effect.

(*d*) In narrow tubes the effect of powder increases, when it is very finely divided, and is strongly agitated in consequence.

(*e*) Roughening the interior of a narrow tube, or increasing its surface, diminishes the velocity.

(*f*) In wide tubes these changes of velocity are of no importance, so that the method may be used in spite of them for exact determinations.

(*g*) The influence of the intensity of sound on the velocity cannot be proved.

(*h*) With the exception of the first, the wave-lengths of a tone as shewn by dust are not affected by the mode of excitation.

(*i*) In wide tubes the velocity is independent of pressure, but in small tubes the velocity increases with the pressure.

(*j*) All the observed changes in the velocity were due to friction and especially to exchange of heat between the air and the sides of the tube.

(*k*) The velocity of sound at 100^0 agrees exactly with that given by theory[1].

[1] From some expressions in the memoir already cited, from which the notice in the text is principally derived, Kundt appears to have contemplated a continuation of his investigations; but I am unable to find any later publication on the subject.

We shall return to the question of the propagation of sound in narrow tubes as affected by the causes mentioned above (*j*), and shall then investigate the formulæ given by Helmholtz and Kirchhoff.

[The genesis of the peculiar transverse striation which forms a leading feature of the dust-figures has already been considered § 253 *b*. According to the observations of Dvořák[1] the powerful vibrations which occur in a Kundt's tube are accompanied by certain mean motions of the gas. Thus near the walls there is a flow from the loops to the nodes, and in the interior a return flow from the nodes to the loops. This is a consequence of viscosity acting with peculiar advantage upon the parts of the fluid contiguous to the walls[2]. We may perhaps return to this subject in a later chapter.]

261. In the experiments described in the preceding section the aerial vibrations are *forced*, the pitch being determined by the external source, and not (in any appreciable degree) by the length of the column of air. Indeed, strictly speaking, all sustained vibrations are forced, as it is not in the power of free vibrations to maintain themselves, except in the ideal case when there is absolutely no friction. Nevertheless there is an important practical distinction between the vibrations of a column of air as excited by a longitudinally vibrating rod or by a tuning-fork, and such vibrations as those of the organ-pipe or chemical harmonicon. In the latter cases the pitch of the sound depends principally on the length of the aerial column, the function of the wind or of the flame[3] being merely to restore the energy lost by friction and by communication to the external air. The air in an organ-pipe is to be considered as a column swinging almost freely, the lower end, across which the wind sweeps, being treated roughly as open, and the upper end as closed, or open, as the case may be. Thus the wave-length of the principal tone of a stopped pipe is four times the length of the pipe; and, except at the extremities, there is neither node nor loop. The overtones of the pipe are the *odd*

[1] *Pogg. Ann.* t. CLVII. p. 61, 1876.

[2] On the Circulation of Air observed in Kundt's Tubes, and on some allied Acoustical Problems, *Phil. Trans.* vol. CLXXV. p. 1, 1884.

[3] The subject of sensitive flames with and without pipes is treated in considerable detail by Prof. Tyndall in his work on Sound; but the mechanics of this class of phenomena is still very imperfectly understood. We shall return to it in a subsequent chapter.

harmonics, twelfth, higher third, &c., corresponding to the various subdivisions of the column of air. In the case of the twelfth, for example, there is a node at the point of trisection nearest to the open end, and a loop at the other point of trisection midway between the first and the stopped end of the pipe.

In the case of the open organ-pipe both ends are loops, and there must be at least one internal node. The wave-length of the principal tone is twice the length of the pipe, which is divided into two similar parts by a node in the middle. From this we see the foundation of the ordinary rule that the pitch of an open pipe is the same as that of a stopped pipe of half its length. For reasons to be more fully explained in a subsequent chapter, connected with our present imperfect treatment of the open end, the rule is only approximately correct. The open pipe, differing in this respect from the stopped pipe, is capable of sounding the whole series of tones forming the harmonic scale founded upon its principal tone. In the case of the octave there is a loop at the centre of the pipe and nodes at the points midway between the centre and the extremities.

Since the frequency of the vibration in a pipe is proportional to the velocity of propagation of sound in the gas with which the pipe is filled, the comparison of the pitches of the notes obtained from the same pipe in different gases is an obvious method of determining the velocity of propagation, in cases where the impossibility of obtaining a sufficiently long column of the gas precludes the use of the direct method. In this application Chladni with his usual sagacity led the way. The subject was resumed at a later date by Dulong[1] and by Wertheim[2], who obtained fairly satisfactory results.

262. The condition of the air in the interior of an organ-pipe was investigated experimentally by Savart[3], who lowered into the pipe a small stretched membrane on which a little sand was scattered. In the neighbourhood of a node the sand remained sensibly undisturbed, but, as a loop was approached, it danced with more and more vigour. But by far the most striking form of the

[1] Recherches sur les chaleurs spécifiques des fluides élastiques. *Ann. de Chim.*, t. XLI. p. 113, 1829.

[2] *Ann. de Chim.*, 3ième série, t. XXIII. p. 434, 1848.

[3] *Ann. de Chim.*, t. XXIV. p. 56, 1823.

experiment is that invented by König. In this method the vibration is indicated by a small gas flame, fed through a tube which is in communication with a cavity called a manometric capsule. This cavity is bounded on one side by a membrane on which the vibrating air acts. As the membrane vibrates, rendering the capacity of the capsule variable, the supply of gas becomes unsteady and the flame intermittent. The period is of course too small for the intermittence to manifest itself as such when the flame is looked at steadily. By shaking the head, or with the aid of a moveable mirror, the resolution into more or less detached images may be effected: but even without resolution the altered character of the flame is evident from its general appearance. In the application to organ-pipes, one or more capsules are mounted on a pipe in such a manner that the membranes are in contact with the vibrating column of air; and the difference in the flame is very marked, according as the associated capsule is situated at a node or at a loop.

263. Hitherto we have supposed the pipe to be straight, but it will readily be anticipated that, when the cross section is small and does not vary in area, straightness is not a matter of importance. Conceive a curved axis of x running along the middle of the pipe, and let the constant section perpendicular to this axis be S. When the greatest diameter of S is very small in comparison with the wave-length of the sound, the velocity-potential ϕ becomes nearly invariable over the section; applying Green's theorem to the space bounded by the interior of the pipe and by two cross sections, we get

$$\iiint \nabla^2\phi\, dV = S\,.\,\Delta\left(\frac{d\phi}{dx}\right).$$

Now by the general equation of motion

$$\iiint \nabla^2\phi\, dV = \frac{1}{a^2}\iiint \ddot{\phi}\, dV = \frac{1}{a^2}\frac{d^2}{dt^2}\iiint \phi\, dV = \frac{S}{a^2}\frac{d^2}{dt^2}\int \phi\, dx,$$

and in the limit, when the distance between the sections is made to vanish,

$$\int \phi\, dx = \phi\, dx, \qquad \Delta\left(\frac{d\phi}{dx}\right) = \frac{d^2\phi}{dx^2}\, dx\,;$$

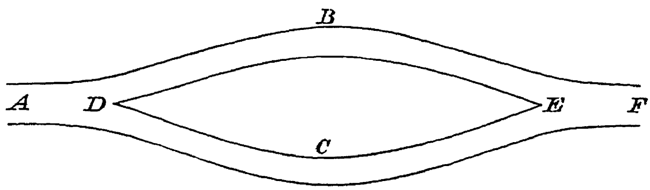
so that

$$\frac{d^2\phi}{dt^2} = a^2\frac{d^2\phi}{dx^2} \quad\text{.........................(1)},$$

shewing that ϕ depends upon x in the same way as if the pipe were straight. By means of equation (1) the vibrations of air in curved pipes of uniform section may be easily investigated, and the results are the rigorous consequences of our fundamental equations (which take no account of friction), when the section is supposed to be infinitely small. In the case of thin tubes such as would be used in experiment, they suffice at any rate to give a very good representation of what actually happens.

264. We now pass on to the consideration of certain cases of connected tubes. In the accompanying figure AD represents a thin pipe, which divides at D into two branches DB, DC. At E the branches reunite and form a single tube EF. The sections of the single tubes and of the branches are assumed to be uniform as well as very small.

Fig. 55.



In the first instance let us suppose that a positive wave of arbitrary type is advancing in A. On its arrival at the fork D, it will give rise to positive waves in B and C, and, unless a certain condition be satisfied, to a negative reflected wave in A. Let the potential of the positive waves be denoted by f_A, f_B, f_C, f being in each case a function of $x - at$: and let the reflected wave be $F(x + at)$. Then the conditions to be satisfied at D are first that the pressures shall be the same for the three pipes, and secondly that the whole velocity of the fluid in A shall be equal to the sum of the whole velocities of the fluid in B and C. Thus, using A, B, C to denote the areas of the sections, we have, § 244,

$$\left.\begin{aligned} f'_A - F' &= f'_B = f'_C \\ A(f'_A + F') &= Bf'_B + Cf'_C \end{aligned}\right\} \quad \text{.........................(1);}$$

whence $$F' = \frac{B + C - A}{B + C + A} f'_A \quad \text{.............................(2),}$$

$$f'_B = f'_C = \frac{2A}{B + C + A} f'_A \quad \text{.......................(3)[1].}$$

[1] These formulæ, as applied to determine the reflected and refracted waves at the junction of two tubes of sections $B + C$, and A respectively, are given by

It appears that f_B and f_C are always the same. There is no reflection, if

$$B + C = A \qquad (4),$$

that is, if the combined sections of the branches be equal to the section of the trunk; and, when this condition is satisfied,

$$f_B = f_C = f_A \qquad (5).$$

The wave then advances in B and C exactly as it would have done in A, had there been no break. If the lengths of the branches between D and E be equal, and the section of F be equal to that of A, the waves on arrival at E combine into a wave propagated along F, and again there is no reflection. The division of the tube has thus been absolutely without effect; and since the same would be true for a negative wave passing from F to A, we may conclude generally that a tube may be divided into two, or more, branches, all of the same length, without in any way influencing the law of aërial vibration, provided that the whole section remain constant. If the lengths of the branches from D to E be unequal, the result is different. Besides the positive wave in F, there will be in general negative reflected waves in B and C. The most interesting case is when the wave is of harmonic type and one of the branches is longer than the other by a multiple of $\frac{1}{2}\lambda$. If the difference be an *even* multiple of $\frac{1}{2}\lambda$, the result will be the same as if the branches were of equal length, and no reflection will ensue. But suppose that, while B and C are equal in section, one of them is longer than the other by an *odd* multiple of $\frac{1}{2}\lambda$. Since the waves arrive at E in opposite phases, it follows from symmetry that the positive wave in F must vanish, and that the pressure at E, which is necessarily the same for all the tubes, must be constant. The waves in B and C are thus reflected as from an open end. That the conditions of the question are thus satisfied may also be seen by supposing a barrier taken across the tube F in the neighbourhood of E in such a way that the tubes B and C communicate without a change of section. The wave in each tube will then pass on into the other without interruption, and the pressure-variation at E, being the resultant of equal and opposite components, will vanish. This being so, the barrier may be removed without altering the conditions, and no wave will be propagated along F, whatever its section may be. The arrange-

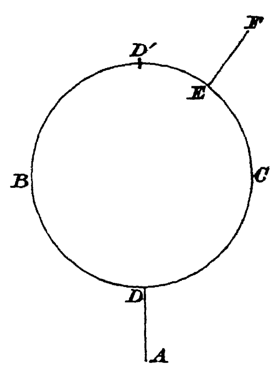
Poisson, *Mém. de l'Institut*, t. II. p. 305, 1819. The reader will not forget that both diameters must be small in comparison with the wave-length.

ment now under consideration was invented by Herschel, and has been employed by Quincke and others for experimental purposes,—an application that we shall afterwards have occasion to describe. The phenomenon itself is often referred to as an example of interference, to which there can be no objection, but the same cannot be said when the reader is led to suppose that the positive waves neutralise each other in *F*, and that there the matter ends. It must never be forgotten that there is no loss of energy in interference, but only a different distribution; when energy is diverted from one place, it reappears in another. In the present case the positive wave in *A* conveys energy with it. If there is no wave along *F*, there are two possible alternatives. Either energy accumulates in the branches, or else it passes back along *A* in the form of a negative wave. In order to see what really happens, let us trace the progress of the waves reflected back at *E*.

These waves are equal in magnitude and start from *E* in opposite phases; in the passage from *E* to *D* one has to travel a greater distance than the other by an odd multiple of $\frac{1}{2}\lambda$; and therefore on arrival at *D* they will be in complete accordance. Under these circumstances they combine into a single wave, which travels negatively along *A*, and there is no reflection. When the negative wave reaches the end of the tube *A*, or is otherwise disturbed in its course, the whole or a part may be reflected, and then the process is repeated. But however often this may happen there will be no wave along *F*, unless by accumulation, in consequence of a coincidence of periods, the vibration in the branches becomes so great that a small fraction of it can no longer be neglected.

Or we may reason thus. Suppose the tube *F* cut off by a barrier as before. The motion in the ring being due to forces acting at *D* is necessarily symmetrical with respect to *D*, and *D'*—the point which divides *DBCD* into equal parts. Hence *D'* is a node, and the vibration is stationary. This being the case, at a point *E* distant $\frac{1}{4}\lambda$ from *D'* on either side, there must be a loop; and if the barrier be removed there will still be no tendency to produce vibration in *F*. If the perimeter of the ring be a multiple of λ, there may be

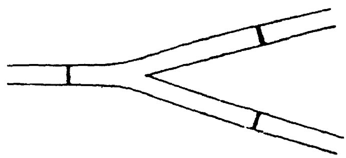
Fig. 56.



vibration within it of the period in question, independently of any lateral openings.

Any combination of connected tubes may be treated in a similar manner. The general principle is that at any junction a space can be taken large enough to include all the region through which the want of uniformity affects the law of the waves, and yet so small that its longest dimension may be neglected in comparison with λ. Under these circumstances the fluid within the space in question may be treated as if the wave-length were infinite, or the fluid itself incompressible, in which case its velocity-potential would satisfy $\nabla^2\phi = 0$, following the same laws as electricity.

Fig. 57.



265. When the section of a pipe is variable, the problem of the vibrations of air within it cannot generally be solved. The case of conical pipes will be treated on a future page. At present we will investigate an approximate expression for the pitch of a nearly cylindrical pipe, taking first the case where both ends are closed. The method that will be employed is similar to that used for a string whose density is not quite constant, §§ 91, 140, depending on the principle that the period of a free vibration fulfils the stationary condition, and may therefore be calculated from the potential and kinetic energies of any hypothetical motion not departing far from the actual type. In accordance with this plan we shall assume that the velocity normal to any section S is constant over the section, as must be very nearly the case when the variation of S is slow. Let X represent the total transfer of fluid at time t across the section at x, reckoned from the equilibrium condition; then $\dot{X}$ represents the total velocity of the current, and $\dot{X} \div S$ represents the actual velocity of the particles of fluid, so that the kinetic energy of the motion within the tube is expressed by

$$T = \tfrac{1}{2}\rho \int \frac{\dot{X}^2}{S}\, dx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The potential energy § 245 (12) is expressed in general by

$$V = \tfrac{1}{2}\, a^2 \rho \iiint s^2\, dV,$$

or, since $dV = Sdx$, by

$$V = \tfrac{1}{2}a^2\rho\int Ss^2 dx \text{ ....(2).}$$

Again, by the condition of continuity,

$$-s = \frac{1}{S}\frac{dX}{dx} \text{ ....(3),}$$

and thus

$$V = \tfrac{1}{2}a^2\rho\int\frac{1}{S}\left(\frac{dX}{dx}\right)^2 dx \text{ ....(4).}$$

If we now assume for X an expression of the same form as would obtain if S were constant, viz.

$$X = \sin\frac{\pi x}{l}\cos nt \text{ ....(5),}$$

we obtain from the values of T and V in (1) and (4),

$$n^2 = \frac{a^2\pi^2}{l^2}\int_0^l \cos^2\frac{\pi x}{l}\frac{dx}{S} \div \int_0^l \sin^2\frac{\pi x}{l}\frac{dx}{S} \text{ ....(6),}$$

or, if we write $S = S_0 + \Delta S$ and neglect the square of ΔS,

$$n^2 = \frac{a^2\pi^2}{l^2}\left\{1 - 2\int_0^l \cos\frac{2\pi x}{l}\frac{\Delta S}{S_0}\frac{dx}{l}\right\} \text{ ....(7).}$$

The result may be expressed conveniently in terms of Δl, the correction that must be made to l in order that the pitch may be calculated from the ordinary formula, as if S were constant. For the value of Δl we have

$$\Delta l = \int_0^l \cos\frac{2\pi x}{l}\frac{\Delta S}{S_0}dx \text{ ....(8).}$$

The effect of a variation of section is greatest near a node or near a loop. An enlargement of section in the first case lowers the pitch, and in the second case raises it. At the points midway between the nodes and loops a slight variation of section is without effect. The pitch is thus decidedly altered by an enlargement or contraction near the middle of the tube, but the influence of a slight conicality would be much less.

The expression for Δl given by (8) is applicable as it stands to the gravest tone only; but we may apply it to the m^{th} tone of the harmonic scale, if we modify it by the substitution of $\cos(2m\pi x/l)$ for $\cos(2\pi x/l)$.

In the case of a tube *open* at both ends (5) is replaced by

$$X = \cos\frac{\pi x}{l}\cos nt \qquad\qquad (9),$$

which leads to

$$\Delta l = -\int_0^l \cos\frac{2\pi x}{l}\frac{\Delta S}{S_0}dx \qquad\qquad (10),$$

instead of (8). The pitch of the sound is now raised by an enlargement at the ends, or by a contraction at the middle, of the tube; and, as before, it is unaffected by a slight general conicality (§ 281).

266. The case of progressive waves moving in a tube of variable section is also interesting. In its general form the problem would be one of great difficulty; but where the change of section is very gradual, so that no considerable alteration occurs within a distance of a great many wave-lengths, the principle of energy will guide us to an approximate solution. It is not difficult to see that in the case supposed there will be no sensible reflection of the wave at any part of its course, and that therefore the energy of the motion must remain unchanged[1]. Now we know, § 245, that for a given area of wave-front, the energy of a train of simple waves is as the square of the amplitude, from which it follows that as the waves advance the amplitude of vibration varies inversely as the square root of the section of the tube. In all other respects the type of vibration remains absolutely unchanged. From these results we may get a general idea of the action of an ear-trumpet. It appears that according to the ordinary approximate equations, there is no limit to the concentration of sound producible in a tube of gradually diminishing section.

The same method is applicable, when the density of the medium varies slowly from point to point. For example, the amplitude of a sound-wave moving upwards in the atmosphere may be determined by the condition that the energy remains unchanged. From § 245 it appears that the amplitude is inversely as the square root of the density[2].

[1] *Phil. Mag.* (5) I. p. 261, 1876.

[2] A delicate question arises as to the ultimate fate of sonorous waves propagated upwards. It should be remarked that in rare air the deadening influence of viscosity is much increased.

CHAPTER XIII.

SPECIAL PROBLEMS. REFLECTION AND REFRACTION OF PLANE WAVES.

267. BEFORE undertaking the discussion of the general equations for aërial vibrations we may conveniently turn our attention to a few special problems, relating principally to motion in two dimensions, which are susceptible of rigorous and yet comparatively simple solution. In this way the reader, to whom the subject is new, will acquire some familiarity with the ideas and methods employed before attacking more formidable difficulties.

In the previous chapter (§ 255) we investigated the vibrations in one dimension, which may take place parallel to the axis of a tube, of which both ends are closed. We will now inquire what vibrations are possible within a closed rectangular box, dispensing with the restriction that the motion is to be in one dimension only. For each simple vibration of which the system is capable, ϕ varies as a circular function of the time, say $\cos kat$, where k is some constant; hence $\ddot{\phi} = -k^2a^2\phi$, and therefore by the general differential equation (9) § 244

$$\nabla^2\phi + k^2\phi = 0 \quad\quad (1).$$

Equation (1) must be satisfied throughout the whole of the included volume. The surface condition to be satisfied over the six sides of the box is simply

$$\frac{d\phi}{dn} = 0 \quad\quad (2),$$

where dn represents an element of the normal to the surface. It is only for special values of k that it is possible to satisfy (1) and (2) simultaneously.

Taking three edges which meet as axes of rectangular co-ordinates, and supposing that the lengths of the edges are respectively α, β, γ, we know (§ 255) that

$$\phi = \cos\left(p\frac{\pi x}{\alpha}\right), \quad \phi = \cos\left(q\frac{\pi y}{\beta}\right), \quad \phi = \cos\left(r\frac{\pi z}{\gamma}\right),$$

where p, q, r are integers, are particular solutions of the problem. By any of these forms equation (2) is satisfied, and provided that k be equal to $p\pi/\alpha$, $q\pi/\beta$, or $r\pi/\gamma$, as the case may be, (1) is also satisfied. It is equally evident that the boundary equation (2) is satisfied over all the surface by the form

$$\phi = \cos\left(p\frac{\pi x}{\alpha}\right)\cos\left(q\frac{\pi y}{\beta}\right)\cos\left(r\frac{\pi z}{\gamma}\right) \quad\text{.........…. (3)},$$

a form which also satisfies (1), if k be taken such that

$$k^2 = \pi^2\left(\frac{p^2}{\alpha^2} + \frac{q^2}{\beta^2} + \frac{r^2}{\gamma^2}\right) \quad\text{.....................… (4)},$$

where as before p, q, r are integers[1].

The general solution, obtained by compounding all particular solutions included under (3), is

$$\phi = \Sigma\Sigma\Sigma\,(A\cos kat + B\sin kat)$$
$$\times\cos\left(p\frac{\pi x}{\alpha}\right)\cos\left(q\frac{\pi y}{\beta}\right)\cos\left(r\frac{\pi z}{\gamma}\right) \text{..............(5)},$$

in which A and B are arbitrary constants, and the summation is extended to all integral values of p, q, r.

This solution is sufficiently general to cover the case of any initial state of things within the box, not involving molecular rotation. The initial distribution of velocities depends upon the initial value of ϕ, or $\int(u_0 dx + v_0 dy + w_0 dz)$, and by Fourier's theorem can be represented by (5), suitable values being ascribed to the coefficients A. In like manner an arbitrary initial distribution of condensation (or rarefaction), depending on the initial value of $\dot{\phi}$, can be represented by ascribing suitable values to the coefficients B.

The investigation might be presented somewhat differently by commencing with assuming in accordance with Fourier's

[1] Duhamel, *Liouville Journ. Math.*, vol. XIV. p. 84, 1849.

theorem that the general value of ϕ at time t can be expressed in the form

$$\phi = \Sigma\Sigma\Sigma\, C \cos\left(p\frac{\pi x}{\alpha}\right)\cos\left(q\frac{\pi y}{\beta}\right)\cos\left(r\frac{\pi z}{\gamma}\right),$$

in which the coefficients C may depend upon t, but not upon x, y, z. The expressions for T and V would then be formed, and shewn to involve only the squares of the coefficients C, and from these expressions would follow the normal equations of motion connecting each normal co-ordinate C with the time.

The gravest mode of vibration is that in which the entire motion is parallel to the longest dimension of the box, and there is no internal node. Thus, if α be the greatest of the three sides α, β, γ, we are to take $p=1$, $q=0$, $r=0$.

In the case of a cubical box, $\alpha=\beta=\gamma$, and then instead of (4) we have

$$k^2 = \frac{\pi^2}{\alpha^2}(p^2+q^2+r^2) \qquad\qquad (6),$$

or, if λ be the wave-length of plane waves of the same period,

$$\lambda = 2\alpha \div \sqrt{(p^2+q^2+r^2)} \qquad\qquad (7).$$

For the gravest mode $p=1$, $q=0$, $r=0$, or $p=0$, $q=1$, $r=0$, &c., and $\lambda=2\alpha$. The next gravest is when $p=1$, $q=1$, $r=0$, &c., and then $\lambda=\sqrt{2}\,\alpha$. When $p=1$, $q=1$, $r=1$, $\lambda=2\alpha/\sqrt{3}$. For the fourth gravest mode $p=2$, $q=0$, $r=0$, &c., and then $\lambda=4\alpha$.

As in the case of the membrane (§ 197), when two or more primitive modes have the same period of vibration, other modes of like period may be derived by composition.

The trebly infinite series of possible simple component vibrations is not necessarily completely represented in particular cases of compound vibrations. If, for example, we suppose the contents of the box in its initial condition to be neither condensed nor rarefied in any part, and to have a uniform velocity, whose components parallel to the axes of co-ordinates are respectively u_0, v_0, w_0, no simple vibrations are generated for which more than one of the three numbers p, q, r is finite. In fact each component initial velocity may be considered separately, and the problem is similar to that solved in § 258.

In future chapters we shall meet with other examples of the vibrations of air within completely closed vessels.

Some of the natural notes of the air contained within a room may generally be detected on singing the scale. Probably it is somewhat in this way that blind people are able to estimate the size of rooms[1].

In long and narrow passages the vibrations parallel to the length are too slow to affect the ear, but notes due to transverse vibrations may often be heard. The relative proportions of the various overtones depend upon the place at which the disturbance is created[2].

In some cases of this kind the pitch of the vibrations, whose direction is principally transverse, is influenced by the occurrence of longitudinal motion. Suppose, for example, in (3) and (4), that $q=1$, $r=0$, and that α is much greater than β. For the principal transverse vibration $p=0$, and $k=\pi/\beta$. But besides this there are other modes of vibration in which the motion is principally transverse, obtained by ascribing to p small integral values. Thus, when $p=1$,

$$k^2=\pi^2\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right),$$

shewing that the pitch is nearly the same as before[3].

268. If we suppose γ to become infinitely great, the box of the preceding section is transformed into an infinite rectangular tube, whose sides are α and β. Whatever may be the motion of the air within this tube, its velocity-potential may be expressed by Fourier's theorem in the series

$$\phi=\Sigma\,\Sigma\,A_{pq}\cos\frac{p\pi x}{\alpha}\cos\frac{q\pi y}{\beta}\quad\ldots\ldots\ldots\ldots\ldots(1),$$

where the coefficients A are independent of x and y. By the use of this form we secure the fulfilment of the boundary condition

[1] A remarkable instance is quoted in Young's *Natural Philosophy*, II. p. 272, from Darwin's *Zoonomia*, II. 487. "The late blind Justice Fielding walked for the first time into my room, when he once visited me, and after speaking a few words said, 'This room is about 22 feet long, 18 wide, and 12 high'; all which he guessed by the ear with great accuracy."

[2] Oppel, *Die harmonischen Obertöne des durch parallele Wände erregten Reflexionstones. Fortschritte der Physik*, XX. p. 130.

[3] There is an underground passage in my house in which it is possible, by singing the right note, to excite free vibrations of many seconds' duration, and it often happens that the resonant note is affected with distinct beats. The breadth of the passage is about 4 feet, and the height about 6½ feet.

that there is to be no velocity across the sides of the tube; the nature of A as a function of z and t depends upon the other conditions of the problem.

Let us consider the case in which the motion at every point is harmonic, and due to a normal motion imposed upon a barrier stretching across the tube at $z=0$. Assuming ϕ to be proportional to e^{ikat} at all points, we have the usual differential equation

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}+k^2\phi=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

which by the conjugate property of the functions must be satisfied separately by each term of (1). Thus to determine A_{pq} as a function of z, we get

$$\frac{d^2A_{pq}}{dz^2}+\left[k^2-\pi^2\left(\frac{p^2}{\alpha^2}+\frac{q^2}{\beta^2}\right)\right]A_{pq}=0 \quad\ldots\ldots\ldots\ldots(3).$$

The solution of this equation differs in form according to the sign of the coefficient of A_{pq}. When p and q are both zero, the coefficient is necessarily positive, but as p and q increase the coefficient changes sign. If the coefficient be positive and be called μ^2, the general value of A_{pq} may be written

$$A_{pq}=B_{pq}\,e^{i(kat+\mu z)}+C_{pq}\,e^{i(kat-\mu z)} \quad\ldots\ldots\ldots\ldots\ldots(4),$$

where, as the factor e^{ikat} is expressed, B_{pq}, C_{pq} are absolute constants. However, the first term in (4) expresses a motion propagated in the negative direction, which is excluded by the conditions of the problem, and thus we are to take simply as the term corresponding to p, q,

$$\phi=C_{pq}\cos\frac{p\pi x}{\alpha}\cos\frac{q\pi y}{\beta}\,e^{i(kat-\mu z)}.$$

In this expression C_{pq} may be complex; passing to real quantities and taking two new real arbitrary constants, we obtain

$$\phi=[D_{pq}\cos(kat-\mu z)+E_{pq}\sin(kat-\mu z)]\cos\frac{p\pi x}{\alpha}\cos\frac{q\pi y}{\beta} \quad\ldots(5).$$

We have now to consider the form of the solution in cases where the coefficient of A_{pq} in (3) is negative. If we call it $-\nu^2$, the solution corresponding to (4) is

$$A_{pq}=e^{ikat}(B_{pq}\,e^{\nu z}+C_{pq}\,e^{-\nu z}) \quad\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

of which the first term is to be rejected as becoming infinite with z. We thus obtain corresponding to (5)

$$\phi = e^{-\nu z}\left[D_{pq}\cos kat + E_{pq}\sin kat\right]\cos\frac{p\pi x}{\alpha}\cos\frac{q\pi y}{\beta} \quad\text{......(7)}.$$

The solution obtained by combining all the particular solutions given by (5) and (7) is the general solution of the problem, and allows of a value of $d\phi/dz$ over the section $z = 0$, arbitrary at every point in both amplitude and phase.

At a great distance from the source the terms given in (7) become insensible, and the motion is represented by the terms of (5) alone. The effect of the terms involving high values of p and q is thus confined to the neighbourhood of the source, and at moderate distances any sudden variations or discontinuities in the motion at $z = 0$ are gradually eased off and obliterated.

If we fix our attention on any particular simple mode of vibration (for which p and q do not both vanish), and conceive the frequency of vibration to increase from zero upwards, we see that the effect, at first confined to the neighbourhood of the source, gradually extends further and further and, after a certain value is passed, propagates itself to an infinite distance, the critical frequency being that of the two dimensional free vibrations of the corresponding mode. Below the critical point no work is required to *maintain* the motion; above it as much work must be done at $z = 0$ as is carried off to infinity in the same time.

268 *a*. If in the general formulæ of § 267 we suppose that $r = 0$, we fall back upon the case of a motion purely two-dimensional. The third dimension (γ) of the chamber is then a matter of indifference; and the problem may be supposed to be that of the vibrations of a rectangular plate of air bounded, for example, by two parallel plates of glass, and confined at the rectangular boundary. In this form it has been treated both theoretically and experimentally by Kundt[1]. The velocity-potential is simply

$$\phi = \cos\left(p\frac{\pi x}{\alpha}\right)\cos\left(q\frac{\pi y}{\beta}\right) \quad\text{.................(1)},$$

where p and q are integers; and the frequency is determined by

$$k^2 = \pi^2(p^2/\alpha^2 + q^2/\beta^2) \quad\text{....................(2)}.$$

[1] *Pogg. Ann.* vol. XL. pp. 177, 337, 1873.

If the plate be *open* at the boundary, an approximate solution may be obtained by supposing that ϕ is there evanescent. In this case the expression for ϕ is derived from (1) by writing sines instead of cosines, while the frequency equation retains the same form (2). This has already been discussed under the head of membranes in § 197. If $\alpha = \beta$, so that the rectangle becomes a square, the various normal modes of the same pitch may be combined, as explained in § 197.

In Kundt's experiments the vibrations were excited through a perforation in one of the glass plates, to which was applied the extremity of a suitably tuned rod vibrating longitudinally, and the division into segments was indicated by the behaviour of cork filings. As regards pitch there was a good agreement with calculation in the case of plates *closed* at the boundary. When the rectangular boundary was *open*, the observed frequencies were too small, a discrepancy to be attributed to the merely approximate character of the assumption that the pressure is there invariable (see § 307).

The theory of the circular plate of air depends upon Bessel's functions, and is considered in § 339.

269. We will now examine the result of the composition of two trains of plane waves of harmonic type, whose amplitudes and wave-lengths are equal, but whose directions of propagation are inclined to one another at an angle 2α. The problem is one of two dimensions only, inasmuch as everything is the same in planes perpendicular to the lines of intersection of the two sets of wave-fronts.

At any moment of time the positions of the planes of maximum condensation for each train of waves may be represented by parallel lines drawn at equal intervals λ on the plane of the paper, and these lines must be supposed to move with a velocity a in a direction perpendicular to their length. If both sets of lines be drawn, the paper will be divided into a system of equal parallelograms, which advance in the direction of one set of diagonals. At each corner of a parallelogram the condensation is doubled by the superposition of the two trains of waves, and in the centre of each parallelogram the rarefaction is a maximum for the same reason. On each diagonal there is therefore a series of maxima and minima condensations, advancing without change of relative position and with velocity $a/\cos\alpha$. Between each adjacent pair of lines of maxima and minima there is a parallel line of zero condensation, on which the two trains of waves neutralize one another. It is especially remarkable that, if the wave-pattern were visible (like the corresponding water wave-pattern to which the whole of the preceding argument is applicable), it would appear to move forwards without change of type in a direction different from that of either component train, and with a velocity different from that with which both component trains move.

In order to express the result analytically, let us suppose that the two directions of propagation are equally inclined at an angle α to the axis of x. The condensations themselves may be denoted by

$$\cos\frac{2\pi}{\lambda}(a\,t - x\cos\alpha - y\sin\alpha)$$

and

$$\cos\frac{2\pi}{\lambda}(a\,t - x\cos\alpha + y\sin\alpha)$$

respectively, and thus the expression for the resultant is

$$s = \cos\frac{2\pi}{\lambda}(a\,t - x\cos\alpha - y\sin\alpha) + \cos\frac{2\pi}{\lambda}(a\,t - x\cos\alpha + y\sin\alpha)$$

$$= 2\cos\frac{2\pi}{\lambda}(a\,t - x\cos\alpha)\,\cos\frac{2\pi}{\lambda}(y\sin\alpha)\,\ldots\ldots\ldots(1).$$

It appears from (1) that the distribution of s on the plane xy advances parallel to the axis of x, unchanged in type, and with a uniform velocity $a/\cos\alpha$. Considered as depending on y, s is a maximum, when $y\sin\alpha$ is equal to 0, λ, 2λ, 3λ, &c., while for the intermediate values, viz. $\frac{1}{2}\lambda$, $\frac{3}{2}\lambda$, &c., s vanishes.

If $\alpha = \frac{1}{2}\pi$, so that the two trains of waves meet one another directly, the velocity of propagation parallel to x becomes infinite, and (1) assumes the form

$$s = 2\cos\left(\frac{2\pi}{\lambda}at\right)\cos\left(\frac{2\pi}{\lambda}y\right)\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

which represents *stationary* waves.

The problem that we have just been considering is in reality the same as that of the reflection of a train of plane waves by an infinite plane wall. Since the expression on the right-hand side of equation (1) is an even function of y, s is symmetrical with respect to the axis of x, and consequently there is no motion across that axis. Under these circumstances it is evident that the motion could in no way be altered by the introduction along the axis of x of an absolutely immovable wall. If α be the angle between the surface and the direction of propagation of the incident waves, the velocity with which the places of maximum condensation (corresponding to the greatest elevation of water-waves) move along the wall is $a/\cos\alpha$. It may be noticed that the aërial pressures have no tendency to move the wall as a whole, except in the case of absolutely perpendicular incidence, since they are at any moment as much negative as positive.

269 *a*. When sound waves proceeding from a distant source are reflected perpendicularly by a solid wall, the superposition of the direct and reflected waves gives rise to a system of nodes and loops, exactly as in the case of a tube considered in § 255. The nodal planes, viz. the surfaces of evanescent motion, occur at distances from the wall which are even multiples of the quarter wave length, and the loops bisect the intervals between the nodes. In exploring experimentally it is usually best to seek the places of minimum effect, but whether these will be nodes or loops depends upon the apparatus employed, a consideration of which the neglect has led to some confusion[1]. Thus a resonator will cease to respond when its mouth coincides with a loop, so that this method of experimenting gives the *loops* whether the resonator be in connection with the ear or with a "manometric capsule" (§ 282). The same conclusion applies also to the use of the unaided ear, except that in this case the head is an obstacle large enough to disturb sensibly the original distribution of the loop and nodes[2]. If on the other hand the indicating apparatus be a small stretched membrane exposed upon both sides, or a sensitive smoke jet or flame, the places of vanishing disturbance are the *nodes*[3].

The complete establishment of stationary vibrations with nodes and loops occupies a certain time during which the sound is to be maintained. When a harmonium reed is sounding steadily in a room free from carpets and curtains, it is easy, listening with a resonator, to find places where the principal tone is almost entirely subdued. But at the first moment of putting down the

[1] N. Savart, *Ann. d. Chim.* LXXI. p. 20, 1839; XL. p. 385, 1845.
[2] *Phil. Mag.* VII. p. 150, 1879.
[3] *Phil. Mag. loc. cit.* p. 153.

key, or immediately after letting it go, the tone in question asserts itself, often with surprising vigour.

The formation of stationary nodes and loops in front of a reflecting wall may be turned to good account when it is desired to determine the wave-lengths of aërial vibrations. The method is especially valuable in the case of very acute sounds and of vibrations of frequency so high as to be inaudible. With the aid of a high pressure sensitive flame vibrations produced by small "bird-calls" may be traced down to a complete wave-length of 6 mm., corresponding to a frequency of about 55,000 per second.

270. So long as the medium which is the vehicle of sound continues of unbroken uniformity, plane waves may be propagated in any direction with constant velocity and with type unchanged; but a disturbance ensues when the waves reach any part where the mechanical properties of the medium undergo a change. The general problem of the vibrations of a variable medium is probably quite beyond the grasp of our present mathematics, but many of the points of physical interest are raised in the case of plane waves. Let us suppose that the medium is uniform above and below a certain infinite plane ($x=0$), but that in crossing that plane there is an abrupt variation in the mechanical properties on which the propagation of sound depends—namely the *compressibility* and the *density*. On the upper side of the plane (which for distinctness of conception we may suppose horizontal) a train of plane waves advances so as to meet it more or less obliquely; the problem is to determine the (refracted) wave which is propagated onwards within the second medium, and also that thrown back into the first medium, or reflected. We have in the first place to form the equations of motion and to express the boundary conditions.

In the upper medium, if ρ be the natural density and s the condensation,

$$\text{density} = \rho\,(1+s),$$

and

$$\text{pressure} = P\,(1+As),$$

where A is a coefficient depending on the compressibility, and P is the undisturbed pressure. In like manner in the lower medium

$$\text{density} = \rho_1\,(1+s_1),$$

$$\text{pressure} = P\,(1+A_1 s_1),$$

the undisturbed pressure being the same on both sides of $x=0$. Taking the axis of z parallel to the line of intersection of the plane of the waves with the surface of separation $x=0$, we have for the upper medium (§ 244),

$$\frac{d^2\phi}{dt^2} = V^2\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}\right) \quad\text{.................(1)},$$

and

$$\frac{d\phi}{dt} + V^2 s = 0 \quad\text{.......................(2)},$$

where

$$V^2 = PA \div \rho \quad\text{.......................(3)}.$$

Similarly, in the lower medium,

$$\frac{d^2\phi_1}{dt^2} = V_1^2\left(\frac{d^2\phi_1}{dx^2}+\frac{d^2\phi_1}{dy^2}\right) \quad\text{.................(4)},$$

and

$$\frac{d\phi_1}{dt} + V_1^2 s_1 = 0 \quad\text{.......................(5)},$$

where

$$V_1^2 = PA_1 \div \rho_1 \quad\text{.......................(6)}.$$

These equations must be satisfied at all points of the fluid. Further the boundary conditions require (i) that at all points of the surface of separation the velocities perpendicular to the surface shall be the same for the two fluids, or

$$d\phi/dx = d\phi_1/dx, \quad \text{when } x=0 \quad\text{..............(7)};$$

(ii) that the pressures shall be the same, whence $A_1 s_1 = As$, or by (2), (3), (5) and (6),

$$\rho\, d\phi/dt = \rho_1\, d\phi_1/dt, \quad \text{when } x=0 \quad\text{..............(8)}.$$

In order to represent a train of waves of harmonic type, we may assume ϕ and ϕ_1 to be proportional to $e^{i(ax+by+ct)}$, where $ax+by=\text{const.}$ gives the direction of the plane of the waves. If we assume for the incident wave,

$$\phi = \phi' e^{i(ax+by+ct)} \quad\text{.....................(9)},$$

the reflected and refracted waves may be represented respectively by

$$\phi = \phi'' e^{i(-ax+by+ct)} \quad\text{.....................(10)},$$

$$\phi_1 = \phi_1 e^{i(a_1x+by+ct)} \quad\text{.....................(11)}.$$

The coefficient of t is necessarily the same in all three waves on account of the periodicity, and the coefficient of y must be the same, since the traces of all the waves on the plane of separation must move together. With regard to the coefficient of x, it appears by substitution in the differential equations that its sign is changed in passing from the incident to the reflected wave; in fact

$$c^2 = V^2 [(\pm a)^2 + b^2] = V_1^2 [a_1^2 + b^2] \text{ ....(12).}$$

Now $b \div \sqrt{(a^2 + b^2)}$ is the sine of the angle included between the axis of x and the normal to the plane of the waves—in optical language, the sine of the angle of incidence, and $b \div \sqrt{(a_1^2 + b^2)}$ is in like manner the sine of the angle of refraction. If these angles be called θ, θ_1, (12) asserts that $\sin\theta : \sin\theta_1$ is equal to the constant ratio $V : V_1$,—the well-known law of sines. The laws of refraction and reflection follow simply from the fact that the velocity of propagation normal to the wave-fronts is constant in each medium, that is to say, independent of the *direction* of the wave-front, taken in connection with the equal velocities of the traces of all the waves on the plane of separation ($V \div \sin\theta = V_1 \div \sin\theta_1$). It remains to satisfy the boundary conditions (7) and (8).

These give

$$\left.\begin{aligned} a(\phi' - \phi'') &= a_1 \phi_1 \\ \rho(\phi' + \phi'') &= \rho_1 \phi_1 \end{aligned}\right\} \text{ ....(13),}$$

whence

$$2\phi' = \left(\frac{\rho_1}{\rho} + \frac{a_1}{a}\right)\phi_1; \qquad 2\phi'' = \left(\frac{\rho_1}{\rho} - \frac{a_1}{a}\right)\phi_1 \text{ ....(14).}$$

This completes the symbolical solution. If a_1 (and θ_1) be real, we see that if the incident wave be

$$\phi = \cos(ax + by + ct),$$

or in terms of V, λ, and θ,

$$\phi = \cos\frac{2\pi}{\lambda}(x\cos\theta + y\sin\theta + Vt) \text{ ....(15),}$$

the reflected wave is

$$\phi = \frac{\dfrac{\rho_1}{\rho} - \dfrac{\cot\theta_1}{\cot\theta}}{\dfrac{\rho_1}{\rho} + \dfrac{\cot\theta_1}{\cot\theta}} \cos\frac{2\pi}{\lambda}(-x\cos\theta + y\sin\theta + Vt) \text{ ... (16),}$$

and the refracted wave is

$$\phi_1 = \frac{2}{\dfrac{\rho_1}{\rho} + \dfrac{\cot\theta_1}{\cot\theta}} \cos\frac{2\pi}{\lambda_1}(x\cos\theta_1 + y\sin\theta_1 + V_1 t) \text{...(17).}$$

The formula for the amplitude of the reflected wave, viz.

$$\frac{\phi''}{\phi'}=\frac{\dfrac{\rho_1}{\rho}-\dfrac{\cot\theta_1}{\cot\theta}}{\dfrac{\rho_1}{\rho}+\dfrac{\cot\theta_1}{\cot\theta}}\quad\text{.......................(18)},$$

is here obtained on the supposition that the waves are of harmonic type; but since it does not involve λ, and there is no change of phase, it may be extended by Fourier's theorem to waves of any type whatever.

If there be no reflected wave, $\cot\theta_1 : \cot\theta = \rho_1 : \rho$, from which and $(1+\cot^2\theta_1) : (1+\cot^2\theta) = V^2 : V_1^2$, we deduce

$$\left(\frac{\rho_1^2}{\rho^2}-\frac{V^2}{V_1^2}\right)\cot^2\theta=\frac{V^2}{V_1^2}-1\quad\text{.................(19)},$$

which shews that, provided the refractive index $V_1 : V$ be intermediate in value between unity and $\rho : \rho_1$, there is always an angle of incidence at which the wave is completely intromitted; but otherwise there is no such angle.

Since (18) is not altered (except as to sign) by an interchange of θ, θ_1; ρ, ρ_1; &c., we infer that a wave incident in the second medium at an angle θ_1 is reflected in the same proportion as a wave incident in the first medium at an angle θ.

As a numerical example let us suppose that the upper medium is air at atmospheric pressure, and the lower medium water. Substituting for $\cot\theta_1$ its value in terms of θ and the refractive index, we get

$$\frac{\cot\theta_1}{\cot\theta}=\frac{V}{V_1}\sqrt{1-\left(\frac{V_1^2}{V^2}-1\right)\tan^2\theta}\quad\text{...........(20)},$$

or, since $V_1 : V = 4{\cdot}3$ approximately,

$$\cot\theta_1/\cot\theta = {\cdot}23\,\sqrt{(1-17{\cdot}5\tan^2\theta)},$$

which shews that the ratio of cotangents diminishes to zero, as θ increases from zero to about 13°, after which it becomes imaginary, indicating total reflection, as we shall see presently. It must be remembered that in applying optical terms to acoustics, it is the *water* that must be conceived to be the 'rare' medium. The ratio of densities is about 770 : 1; so that

$$\frac{\phi''}{\phi'}=\frac{1-{\cdot}0003\sqrt{(1-17{\cdot}5\tan^2\theta)}}{1+{\cdot}0003\sqrt{(1-17{\cdot}5\tan^2\theta)}}$$

$$=1-{\cdot}0006\sqrt{(1-17{\cdot}5\tan^2\theta)}\text{ very nearly.}$$

Even at perpendicular incidence the reflection is sensibly perfect.

If both media be gaseous, $A_1 = A$, if the temperature be constant; and even if the development of heat by compression be taken into account, there will be no sensible difference between A and A_1 in the case of the simple gases. Now, if $A_1 = A$, $\rho_1 : \rho = \sin^2\theta : \sin^2\theta_1$, and the formula for the intensity of the reflected wave becomes

$$\frac{\phi''}{\phi'} = \frac{\sin 2\theta - \sin 2\theta_1}{\sin 2\theta + \sin 2\theta_1} = \frac{\tan(\theta - \theta_1)}{\tan(\theta + \theta_1)} \quad\text{..........} (21),$$

coinciding with that given by Fresnel for light polarized perpendicularly to the plane of incidence. In accordance with Brewster's law the reflection vanishes at the angle of incidence, whose tangent is V/V_1.

But, if on the other hand $\rho_1 = \rho$, the cause of disturbance being the change of compressibility, we have

$$\frac{\phi''}{\phi'} = \frac{\tan\theta_1 - \tan\theta}{\tan\theta_1 + \tan\theta} = \frac{\sin(\theta_1 - \theta)}{\sin(\theta_1 + \theta)} \quad\text{...........} (22),$$

agreeing with Fresnel's formula for light polarized in the plane of incidence. In this case the reflected wave does not vanish at any angle of incidence.

In general, when $\theta = 0$,

$$\phi'' : \phi' = \frac{\rho_1}{\rho} - \frac{V}{V_1} : \frac{\rho_1}{\rho} + \frac{V}{V_1} \quad\text{..............} (23);$$

so that there is no reflection, if $\rho_1 : \rho = V : V_1$. In the case of gases $V^2 : V_1^2 = \rho_1 : \rho$, and then

$$\frac{\phi''}{\phi'} = \frac{\sqrt{\rho_1} - \sqrt{\rho}}{\sqrt{\rho_1} + \sqrt{\rho}} = \frac{V - V_1}{V + V_1} \quad\text{.................} (24).$$

Suppose, for example, that after perpendicular incidence reflection takes place at a surface separating air and hydrogen. We have

$$\rho = \cdot 001276, \quad \rho_1 = \cdot 00008837;$$

whence $\sqrt{\rho} : \sqrt{\rho_1} = 3\cdot 800$, giving

$$\phi'' = -\cdot 5833\, \phi'.$$

The ratio of intensities, which is as the square of the amplitudes, is $\cdot 3402 : 1$, so that about one-third part is reflected.

If the difference between the two media be very small, and we write $V_1 = V + \delta V$, (24) becomes

$$\frac{\phi''}{\phi'} = -\tfrac{1}{2}\frac{\delta V}{V} \quad\text{.......................} (25).$$

If the first medium be air at 0° Cent., and the second medium be air at $t°$ Cent., $V + \delta V = V\sqrt{(1 + \cdot 00366\, t)}$; so that

$$\phi''/\phi' = -\cdot 00091 t.$$

The ratio of the intensities of the reflected and incident sounds is therefore $\cdot 83 \times 10^{-6} \times t^2 : 1$.

As another example of the same kind we may take the case in which the first medium is dry air and the second is air of the same temperature saturated with moisture. At 10° Cent. air saturated with moisture is lighter than dry air by about one part in 220, so that $\delta V = \frac{1}{440} V$ nearly. Hence we conclude from (25) that the reflected sound is only about one 774,000th part of the incident sound.

From these calculations we see that reflections from warm or moist air must generally be very small, though of course the effect may accumulate by repetition. It must also be remembered that in practice the transition from one state of things to the other would be gradual, and not abrupt, as the present theory supposes. If the space occupied by the transition amount to a considerable fraction of the wave-length, the reflection would be materially lessened. On this account we might expect grave sounds to travel through a heterogeneous medium less freely than acute sounds.

The reflection of sound from surfaces separating portions of gas of different densities has engaged the attention of Tyndall, who has devised several striking experiments in illustration of the subject[1]. For example, sound from a high-pitched reed was conducted through a tin tube towards a sensitive flame, which served as an indicator. By the interposition of a coal-gas flame issuing from an ordinary bat's-wing burner between the tube and the sensitive flame, the greater part of the effect could be cut off. Not only so, but by holding the flame at a suitable angle, the sound could be reflected through another tube in sufficient quantity to excite a second sensitive flame, which but for the interposition of the reflecting flame would have remained undisturbed.

[The refraction of Sound has been demonstrated experimentally by Sondhauss[2] with the aid of a collodion balloon charged with carbonic acid.]

[1] *Sound*, 3rd edition, p. 282, 1875.

[2] *Pogg. Ann.* t. 85, p. 378, 1852. *Phil. Mag.* vol. v. p. 73, 1853.

The preceding expressions (16), (17), (18) hold good in every case of reflection from a 'denser' medium; but if the velocity of sound be greater in the lower medium, and the angle of incidence exceed the critical angle, a_1 becomes imaginary, and the formulæ require modification. In the latter case it is impossible that a refracted wave should exist, since, even if the angle of refraction were 90°, its trace on the plane of separation must necessarily outrun the trace of the incident wave.

If $-ia_1'$ be written in place of a_1, the symbolical equations are

Incident wave

$$\phi = e^{i(ax+by+ct)},$$

Reflected wave

$$\phi = \frac{\dfrac{\rho_1}{\rho} + i\dfrac{a_1'}{a}}{\dfrac{\rho_1}{\rho} - i\dfrac{a_1'}{a}} e^{i(-ax+by+ct)},$$

Refracted wave

$$\phi_1 = \frac{2}{\dfrac{\rho_1}{\rho} - i\dfrac{a_1'}{a}} e^{i(-ia_1'x+by+ct)};$$

from which by discarding the imaginary parts, we obtain

Incident wave

$$\phi = \cos(ax + by + ct) \qquad (26),$$

Reflected wave

$$\phi = \cos(-ax + by + ct + 2\epsilon) \qquad (27),$$

Refracted wave

$$\phi = \frac{2}{\left(\dfrac{\rho_1^2}{\rho^2} + \dfrac{a_1'^2}{a^2}\right)^{\frac{1}{2}}} e^{a_1'x} \cos(by + ct + \epsilon) \qquad (28),$$

where

$$\tan\epsilon = \frac{a_1'\rho}{a\rho_1} \qquad (29).$$

These formulæ indicate total reflection. The disturbance in the second medium is not a wave at all in the ordinary sense, and at a short distance from the surface of separation (x negative) becomes insensible. Calculating a_1' from (12) and expressing it in terms of θ and λ, we find

$$a_1' = \frac{2\pi}{\lambda}\sqrt{\sin^2\theta - \frac{V^2}{V_1^2}} \qquad (30),$$

shewing that the disturbance does not penetrate into the second medium more than a few wave-lengths.

The difference of phase between the reflected and the incident waves is 2ϵ, where

$$\tan\epsilon = \frac{\rho}{\rho_1}\sqrt{\tan^2\theta - \frac{V^2}{V_1^2}\sec^2\theta} \text{..............(31)}.$$

If the media have the same compressibilities, $\rho : \rho_1 = V_1^2 : V^2$, and

$$\tan\epsilon = \frac{V_1}{V}\sqrt{\frac{V_1^2}{V^2}\tan^2\theta - \sec^2\theta} \text{..............(32)}.$$

Since there is no loss of energy in reflection and refraction, the work transmitted in any time across any area of the front of the incident wave must be equal to the work transmitted in the same time across corresponding areas of the reflected and refracted waves. These corresponding areas are plainly in the ratio

$$\cos\theta : \cos\theta : \cos\theta_1;$$

and thus by § 245 (τ being the same for all the waves),

$$\cos\theta\,\frac{\rho}{V}(\phi'^2 - \phi''^2) = \cos\theta_1\,\frac{\rho_1}{V_1}\,\phi_1^2,$$

or since $$V : V_1 = \sin\theta : \sin\theta_1,$$

$$\rho\cot\theta\,(\phi'^2 - \phi''^2) = \rho_1\cot\theta_1\,\phi_1^2 \text{..............(33)},$$

which is the energy condition, and agrees with the result of multiplying together the two boundary equations (13).

When the velocity of propagation is greater in the lower than in the upper medium, and the angle of incidence exceeds the critical angle, no energy is transmitted into the second medium; in other words the reflection is total.

The method of the present investigation is substantially the same as that employed by Green in a paper on the Reflection and Refraction of Sound[1]. The case of perpendicular incidence was first investigated by Poisson[2], who obtained formulæ corresponding to (23) and (24), which had however been already given by Young for the reflection of Light. In a subsequent memoir[3] Poisson considered the general case of oblique incidence, limiting himself, however, to gaseous media for which Boyle's law holds good, and by a very complicated analysis arrived at a result equivalent to

[1] *Cambridge Transactions*, vol. VI. p. 403, 1838.

[2] *Mém. de l'Institut*, t. II. p. 305. 1819.

[3] "Mémoire sur le mouvement de deux fluides élastiques superposés." *Mém. de l'Institut*, t. X. p. 317. 1831.

(21). He also verified that the energies of the reflected and refracted waves make up that of the incident wave[1].

271. If the second medium be indefinitely extended downwards with complete uniformity in its mechanical properties, the transmitted wave is propagated onwards continually. But if at $x=-l$ there be a further change in the compressibility, or density, or both, part of the wave will be thrown back, and on arrival at the first surface ($x=0$) will be divided into two parts, one transmitted into the first medium, and one reflected back, to be again divided at $x=-l$, and so on. By following the progress of these waves the solution of the problem may be obtained, the resultant reflected and transmitted waves being compounded of an infinite convergent series of components, all parallel and harmonic. This is the method usually adopted in Optics for the corresponding problem, and is quite rigorous, though perhaps not always sufficiently explained; but it does not appear to have any advantage over a more straightforward analysis. In the following investigation we shall confine ourselves to the case where the third medium is similar in its properties to the first medium.

In the first medium

$$\phi = \phi' e^{i(ax+by+ct)} + \phi'' e^{i(-ax+by+ct)}.$$

In the second medium

$$\psi = \psi' e^{i(a_1x+by+ct)} + \psi'' e^{i(-a_1x+by+ct)}.$$

In the third medium

$$\phi = \phi_1 e^{i(ax+by+ct)},$$

with the conditions

$$c^2 = V^2(a^2+b^2) = V_1^2(a_1^2+b^2) \qquad (1).$$

At the two surfaces of separation we have to secure the equality of normal motions and pressures; for $x=0$,

$$\left.\begin{array}{l} a(\phi'-\phi'') = a_1(\psi'-\psi'') \\ \rho(\phi'+\phi'') = \rho_1(\psi'+\psi'') \end{array}\right\} \qquad (2);$$

for $x=-l$,

$$\left.\begin{array}{l} a_1(\psi' e^{-ia_1l} - \psi'' e^{ia_1l}) = a\phi_1 e^{-ial} \\ \rho_1(\psi' e^{-ia_1l} + \psi'' e^{ia_1l}) = \rho\phi_1 e^{-ial} \end{array}\right\} \qquad (3),$$

[1] [It is interesting and encouraging to note Laplace's remark in a correspondence with T. Young. The great analyst writes (1817) "Je persiste à croire que le problème de la propagation des ondes, lorsqu'elles traversent différens milieux, n'a jamais été résolu, et qu'il surpasse peut-être les forces actuelles de l'analyse" (Young's *Works*, vol. I. p. 374).]

from which ψ' and ψ'' are to be eliminated. We get

$$\left.\begin{aligned}(\phi'-\phi'')\cos a_1 l - i\frac{a_1\rho}{a\rho_1}(\phi'+\phi'')\sin a_1 l = \phi_1 e^{-ial}\\ (\phi'+\phi'')\cos a_1 l - i\frac{a\rho_1}{a_1\rho}(\phi'-\phi'')\sin a_1 l = \phi_1 e^{-ial}\end{aligned}\right\}\dots\dots\dots(4);$$

and from these, if for brevity $a\rho_1/a_1\rho = \alpha$,

$$\frac{\phi''}{\phi'} = \frac{\alpha - \alpha^{-1}}{\alpha + \alpha^{-1} - 2i\cot a_1 l}\dots\dots\dots(5),$$

$$\frac{\phi_1}{\phi'} = \frac{2e^{ial}}{2\cos a_1 l + i\sin a_1 l\,(\alpha + \alpha^{-1})}\dots\dots\dots(6).$$

In order to pass to real quantities, these expressions must be put into the form $Re^{i\theta}$. *If a_1 be real,* we find corresponding to the incident wave

$$\phi = \cos(ax + by + ct),$$

the reflected wave

$$\phi = \frac{(\alpha^{-1} - \alpha)\sin(-ax + by + ct - \epsilon)}{\sqrt{\{4\cot^2 a_1 l + (\alpha + \alpha^{-1})^2\}}}\dots\dots\dots(7),$$

and the transmitted wave

$$\phi = \frac{2\cos(ax + by + ct + al - \epsilon)}{\sqrt{\{4\cos^2 a_1 l + \sin^2 a_1 l\,(\alpha + \alpha^{-1})^2\}}}\dots\dots\dots(8),$$

where

$$\tan\epsilon = \tfrac{1}{2}(\alpha + \alpha^{-1})\tan a_1 l\dots\dots\dots(9).$$

If $\alpha = \rho_1\cot\theta/\rho\cot\theta_1 = 1$, there is no reflected wave, and the transmitted wave is represented by

$$\phi = \cos(ax + by + ct + al - a_1 l),$$

shewing that, except for the alteration of phase, the whole of the medium might as well have been uniform.

If l be small, we have approximately for the reflected wave

$$\phi = \tfrac{1}{2}a_1 l\,(\alpha^{-1} - \alpha)\,\sin(-ax + by + ct),$$

a formula applying when the plate is thin in comparison with the wave-length. Since $a_1 = (2\pi/\lambda_1)\cos\theta_1$, it appears that for a given angle of incidence the amplitude varies inversely as λ_1, or as λ.

In any case the reflection vanishes, if $\cot^2 a_1 l = \infty$, that is, if

$$2l\cos\theta_1 = m\lambda_1,$$

m being an integer. The wave is then wholly transmitted.

At perpendicular incidence, the intensity of the reflection is expressed by

$$\left(\frac{V\rho}{V_1\rho_1}-\frac{V_1\rho_1}{V\rho}\right)\div\sqrt{4\cot^2\frac{2\pi l}{V_1\tau}+\left(\frac{V\rho}{V_1\rho_1}+\frac{V_1\rho_1}{V\rho}\right)^2}\quad\text{.........(10)}.$$

Let us now suppose that the second medium is incompressible, so that $V_1=\infty$; our expression becomes

$$-\frac{\pi\rho_1 l/\rho\lambda}{\sqrt{\{1+\pi^2(\rho_1 l/\rho\lambda)^2\}}}\quad\text{.....................(11)},$$

shewing how the amount of reflection depends upon the relative masses of such quantities of the media as have volumes in the ratio of $l:\lambda$. It is obvious that the second medium behaves like a rigid body and acts only in virtue of its inertia. If this be sufficient, the reflection may become sensibly total.

We have now to consider the case in which a_1 is imaginary. In the symbolical expressions (5) and (6) $\cos a_1 l$ and $i\sin a_1 l$ are real, while α, $\alpha+\alpha^{-1}$, $\alpha-\alpha^{-1}$ are pure imaginaries. Thus, if we suppose that $a_1=ia_1'$, $\alpha=i\alpha'$, and introduce the notation of the hyperbolic sine and cosine (§ 170), we get

$$\frac{\phi''}{\phi'}=\frac{-i(\alpha'+\alpha'^{-1})\sinh a_1'l}{2\cosh a_1'l-i(\alpha'-\alpha'^{-1})\sinh a_1'l},$$

$$\frac{\phi_1}{\phi'}=\frac{2e^{ial}}{2\cosh a_1'l-i(\alpha'-\alpha'^{-1})\sinh a_1'l}.$$

Hence, if the incident wave be

$$\phi=\cos(ax+by+ct),$$

the reflected wave is expressed by

$$\phi=\frac{(\alpha'+\alpha'^{-1})\sinh a_1'l\cos(-ax+by+ct+\epsilon)}{\sqrt{\{4\cosh^2 a_1'l+(\alpha'-\alpha'^{-1})^2\sinh^2 a_1'l\}}}\quad\text{.........(12)},$$

where $\cot\epsilon=\tfrac{1}{2}(\alpha'^{-1}-\alpha')\tanh a_1'l$..............(13),

and the transmitted wave is expressed by

$$\phi=\frac{2\sin(ax+by+ct+al+\epsilon)}{\sqrt{\{4\cosh^2 a_1'l+(\alpha'-\alpha'^{-1})^2\sinh^2 a_1'l\}}}\quad\text{...........(14)}.$$

It is easy to verify that the energies of the reflected and transmitted waves account for the whole energy of the incident wave. Since in the present case the corresponding areas of wave-front are equal for all three waves, it is only necessary to add the squares of the amplitudes given in equations (7), (8), or in equations (12), (14).

272. These calculations of reflection and refraction under various circumstances might be carried further, but their interest would be rather optical than acoustical. It is important to bear in mind that no energy is destroyed by any number of reflections and refractions, whether partial or total, what is lost in one direction always reappearing in another.

On account of the great difference of densities reflection is usually nearly total at the boundary between air and any solid or liquid matter. Sounds produced in air are not easily communicated to water, and *vice versâ* sounds, whose origin is under water, are heard with difficulty in air. A beam of wood, or a metallic wire, acts like a speaking tube, conveying sounds to considerable distances with very little loss.

272 *a*. In preceding sections the surface of separation, at which reflection takes place, is supposed to be absolutely plane. It is of interest, both from an acoustical and from an optical point of view, to inquire what effect would be produced by roughnesses, or corrugations, in the reflecting surface; and the problem thus presented may be solved without difficulty to a certain extent by the method of § 268, especially if we limit ourselves to the case of perpendicular incidence. The equation of the reflecting surface will be supposed to be $z=\zeta$, where ζ is a periodic function of x whose mean value is zero. As a particular case we may take

$$\zeta = c\cos px \qquad\qquad (1);$$

but in general we should have to supplement the first term of the series expressed in (1) by cosines and sines of the multiples of px. The velocity-potential of the incident wave (of amplitude unity) may be written

$$\phi = e^{ik(at+z)} \qquad\qquad (2).$$

For the regularly reflected wave we have $\phi = A_0 e^{-ikz}$, the time factor being dropped for the sake of brevity; but to this must be added terms in $\cos px$, $\cos 2px$, &c. Thus, as the complete value of ϕ in the upper medium,

$$\phi = e^{ikz} + A_0 e^{-ikz} + A_1 e^{-i\mu_1 z}\cos px + A_2 e^{-i\mu_2 z}\cos 2px + \dots \qquad (3),$$

in which

$$\mu_1^2 = k^2 - p^2, \quad \mu_2^2 = k^2 - 4p^2, \qquad\qquad (4).$$

The expression (3), in which for simplicity sines of multiples of px have been omitted from the first, would be sufficiently general even though cosines of multiples of px accompanied $c \cos px$ in (1).

As explained in § 268, much turns upon whether the quantities $\mu_1, \mu_2, \ldots$ are real or imaginary. In the latter case the corresponding terms are sensible only in the neighbourhood of $z=0$. If all the values of μ be imaginary, as happens when $p>k$, the reflected wave soon reduces itself to its first term.

For any real value of μ, say μ_r, the corresponding part of the velocity-potential is

$$\phi = \tfrac{1}{2} A_r \{e^{-i(\mu_r z - rpx)} + e^{-i(\mu_r z + rpx)}\},$$

representing plane waves inclined to z at angles whose sines are $\pm rp/k$. These are known in Optics as the spectra of the rth order. When the wave-length of the corrugation is less than that of the vibration, there are no lateral spectra.

In the lower medium we have

$$\phi_1 = B_0 e^{ik_1 z} + B_1 e^{i\mu_1' z} \cos px + B_2 e^{i\mu_2' z} \cos 2px + \ldots\ldots (5),$$

where $$\mu_1'^2 = k_1^2 - p^2, \qquad \mu_2'^2 = k_1^2 - 4p^2, \ldots\ldots\ldots\ldots\ldots (6).$$

In each exponential the coefficient of z is to be taken positive; if it be imaginary, because the wave is propagated in the negative direction; if it be real, because the disturbance must decrease, and not increase, in penetrating the second medium.

The conditions to be satisfied at the boundary are (§ 270) that

$$\rho\phi = \rho_1 \phi_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

and that $d\phi/dn = d\phi_1/dn$, where dn is perpendicular to the surface $z=\zeta$. Hence

$$\frac{d(\phi-\phi_1)}{dz} - \frac{d(\phi-\phi_1)}{dx}\frac{d\zeta}{dx} = 0 \ldots\ldots\ldots\ldots\ldots (8).$$

Thus far there is no limitation upon either the amplitude (c) or the wave-length ($2\pi/p$) of the corrugation. We will now suppose that the wave-length is very large, so that p^2 may be neglected throughout. Under these conditions, (8) reduces to

$$d(\phi - \phi_1)/dz = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

In the differentiation of (3) and (5) with respect to z, the various terms are multiplied by the coefficients $\mu_1, \mu_2, \ldots \mu_1', \mu_2', \ldots$;

but when p^2 is neglected these quantities may be identified with k, k_1 respectively. Thus at the boundary

$$\frac{d\phi}{dz} = ik\{e^{ik\zeta} - A_0 e^{-ik\zeta} - A_1 e^{-ik\zeta}\cos px - \ldots\ldots\ldots\};$$

and
$$\frac{d\phi_1}{dz} = ik_1\phi_1 = \frac{ik_1\rho\phi}{\rho_1},$$

by (7). Accordingly,

$$k_1\rho\{e^{ik\zeta} + A_0 e^{-ik\zeta} + A_1 e^{-ik\zeta}\cos px + \ldots\ldots\ldots\}$$
$$= k\rho_1\{e^{ik\zeta} - A_0 e^{-ik\zeta} - A_1 e^{-ik\zeta}\cos px - \ldots\ldots\ldots\},$$

or
$$\frac{k_1\rho - k\rho_1}{k_1\rho + k\rho_1} e^{2ik\zeta} + A_0 + A_1\cos px + A_2\cos 2px + \ldots = 0 \quad\ldots\ldots(10).$$

By this equation A_0, A_1, &c. are determined when ζ is known.

If we put $\zeta = 0$, we fall back on previous results (23) § 270 for a truly plane surface. Thus A_1, A_2,... vanish, while

$$A_0 = \frac{k\rho_1 - k_1\rho}{k\rho_1 + k_1\rho} \quad\ldots\ldots\ldots\ldots(11),$$

expressing the amplitude of the wave regularly reflected.

We will now apply (10) to the case of a simple corrugation, as expressed in (1), and for brevity we will denote the right hand member of (11) by R. The determination of A_0, A_1,... requires the expression of $e^{2ik\zeta}$ in Fourier's series. We have (compare § 343)

$$e^{2ikc\cos px} = J_0(2kc) - 2J_2(2kc)\cos 2px + 2J_4(2kc)\cos 4px + \ldots$$
$$+ i\{2J_1(2kc)\cos px - 2J_3(2kc)\cos 3px + 2J_5(2kc)\cos 5px - \ldots\}$$
$$\ldots\ldots\ldots(12),$$

where J_0, J_1,... are the Bessel's functions of the various orders. Thus

$$\left.\begin{aligned} A_0/R &= J_0(2kc), & A_1/R &= 2iJ_1(2kc), \\ A_2/R &= -2J_2(2kc), & A_3/R &= -2iJ_3(2kc), \\ A_4/R &= 2J_4(2kc), & A_5/R &= 2iJ_5(2kc), \\ &\ldots\ldots\ldots\ldots & & \end{aligned}\right\}\ldots\ldots(13),$$

the coefficients of even order being real, and those of odd order pure imaginaries. The complete solution of the problem of reflection, under the restriction that p is small, is then obtained by substitution in (3); and it may be remarked that it is the same as would be furnished by the usual optical methods, which take account only of phase retardations. Thus, as regards the wave reflected parallel to z, the retardation at any point of the surface due to the corrugation is 2ζ, or $2c\cos px$. The influence of the corrugations is therefore to change the amplitude of the reflected vibration in the ratio

$$\int \cos(2kc\cos px)\,dx : \int dx, \quad \text{or} \quad J_0(2kc).$$

In like manner the amplitude of each of the lateral spectra of the first order is $J_1(2kc)$, and so on. The sum of the intensities of all the reflected waves is

$$R^2\{J_0^2 + 2J_1^2 + 2J_2^2 + \ldots\} = R^2 \quad\ldots\ldots\ldots\ldots\ldots(14)$$

by a known theorem; so that, in the case supposed (of p infinitely small), the fraction of the whole energy thrown back is the same as if the surface were smooth.

It should be remarked that in this theory there is no limitation upon the value of $2kc$. If $2kc$ be small, only the earlier terms of the series are sensible, the Bessel's function $J_n(2kc)$ being of order $(2kc)^n$. When on the other hand $2kc$ is large, the early terms are small, while the series is less convergent. The values of J_0 and J_1 are tabulated in § 200. For certain values of $2kc$ individual reflected waves vanish. In the case of the regularly reflected wave, or spectrum of zero order, this first occurs when $2kc = 2{\cdot}404$, § 206, or $c = {\cdot}2\lambda$.

The full solution of the problem of the present section would require the determination of the reflection when k is given for all values of c and for all values of p. We have considered the case of p infinitely small, and we shall presently deal with the case where $p > k$. For intermediate values of p the problem is more difficult, and in considering them we shall limit ourselves to the simpler boundary conditions which obtain when no energy penetrates the second medium. The simplest case of all arises when $\rho_1 = 0$, so that the boundary equation (7) reduces to

$$\phi = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

the condition for an "open end," § 256. We may also refer to the case of a rigid wall, or "closed" end, where the surface condition is

$$d\phi/dn = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16).$$

By (3) and (15) the condition to be satisfied at the surface is

$$e^{2ikz} + A_0 + A_1 e^{i(k-\mu_1)z}\cos px + A_2 e^{i(k-\mu_2)z}\cos 2px + \ldots = 0 \ldots(16).$$

In our problem z is given by (1) as a function of x; and the equations of condition are to be found by equating to zero the coefficients of the various terms involving $\cos px$, $\cos 2px$, &c., when the left hand member of (16) is expanded in Fourier's series. The development of the various exponentials is effected as in (12); and the resulting equations are

$$J_0(2k) + A_0 + iA_1 J_1(k - \mu_1) - A_2 J_2(k - \mu_2) - \ldots = 0 \quad \ldots(17),$$

$$2iJ_1(2k) + A_1\{J_0(k-\mu_1) - J_2(k-\mu_1)\} + A_2\{iJ_1(k-\mu_2) - iJ_3(k-\mu_2)\} + \ldots = 0 \quad \ldots\ldots\ldots (18),$$

$$-2J_2(2k) + A_1\{iJ_1(k-\mu_1) - iJ_3(k-\mu_1)\} + A_2\{J_0(k-\mu_2) + J_4(k-\mu_2)\} + \ldots = 0 \quad \ldots\ldots\ldots(19),$$

and so on, where for the sake of brevity c has been made equal to unity. So far as $(k - \mu)$ may be treated as real, as happens for a large number of terms when p is small relatively to k, the various Bessel's functions are all real, and thus the A's of even order are real and the A's of odd order are pure imaginaries. Accordingly the phase of the perpendicularly reflected wave is the same as if $c = 0$; but it must be remembered that this conclusion is in reality only approximate, because, however small p may be, the μ's end by becoming imaginary.

From the above equations it is easy to obtain the value of A_0 as far as the term in p^4. From (19)

$$A_2 = 2J_2(2k);$$

from (18)

$$iA_1 = 2J_1(2k) + (k - \mu_2) J_2(2k);$$

and finally from (17)

$$-A_0 = J_0(2k) + (k - \mu_1) J_1(2k) + \{\tfrac{1}{2}(k-\mu_1)(k-\mu_2) - \tfrac{1}{4}(k-\mu_2)^2\} J_2(2k) + \ldots \quad \ldots.(20).$$

From (4)

$$k - \mu_1 = \frac{p^2}{2k} + \frac{p^4}{8k^3} + \ldots;$$

so that, as expanded in powers of p with reintroduction of c,

$$-A_0 = J_0(2kc) + \frac{p^2}{k^2} \cdot \tfrac{1}{2}kc \, . \, J_1(2kc) + \frac{p^4}{k^4}\{\tfrac{1}{8}kc \, . \, J_1(2kc) - \tfrac{1}{2}k^2c^2 \, . \, J_2(2kc)\} \quad \ldots\ldots\ldots\ldots(21)^1.$$

[1] Brit. Ass. Rep. 1893, p. 691.

This gives the amplitude of the perpendicularly reflected wave, with omission of p^6 and higher powers of p.

The case of reflection from a fixed wall is a little more complicated. By (8) the boundary condition is

$$d\phi/dz + pc \sin px \, . \, d\phi/dx = 0,$$

which gives

$$e^{2ikz} - A_0 - \frac{\mu_1}{k} A_1 e^{i(k-\mu_1)z} \cos px - \frac{\mu_2}{k} A_2 e^{i(k-\mu_2)z} \cos 2px - \ldots$$

$$- \frac{p^2 c \sin px}{ik} \{A_1 e^{i(k-\mu_1)z} \sin px + 4A_2 e^{i(k-\mu_2)z} \sin 2px + \ldots\} = 0 \qquad \ldots\ldots\ldots(22)$$

as the equation to be satisfied when $z = c \cos px$. The first approximation to A_1 gives

$$A_1 = 2iJ_1(2kc) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23);$$

whence to a second approximation

$$A_0 = J_0(2kc) + \left\{-\tfrac{1}{2}(k-\mu_1) + \frac{p^2 c}{2k}\right\} iA_1$$

$$= J_0(2kc) - \frac{p^2}{2k^2} \, . \, kc \, . \, J_1(2kc) \ldots\ldots\ldots\ldots\ldots\ldots(24).$$

The first approximation to the various coefficients may be found by putting $R = +1$ in (13).

When $p > k$, there are no diffracted spectra, and the whole energy of the wave incident upon an impenetrable medium must be represented in the wave directly reflected. The modulus of A_0 is therefore unity. When $p < k$, the energy is divided between the various spectra, including that of zero order. There is thus a relation between the squares of the moduli of A_0, A_1, A_2, ..., the series being continued as long as μ is real.

A more analytical investigation may be based upon v. Helmholtz's theorem (§ 293), according to which

$$\int \left\{\psi \frac{d\chi}{dn} - \chi \frac{d\psi}{dn}\right\} dS = 0,$$

where S is any closed surface, and ψ and χ satisfy the equation

$$\nabla^2 + k^2 = 0.$$

In order to apply this we take for ψ and χ the real and imaginary parts respectively of ϕ as given by (3). Thus representing each complex coefficient A_n in the form $C_n + iD_n$, we get

$$\psi = \cos kz + C_0 \cos kz + D_0 \sin kz + (C_1 \cos \mu_1 z + D_1 \sin \mu_1 z) \cos px + \ldots \quad (25),$$

$$\chi = \sin kz - C_0 \sin kz + D_0 \cos kz + (-C_1 \sin \mu_1 z + D_1 \cos \mu_1 z) \cos px + \ldots \quad (26).$$

In (25), (26), when the series are carried sufficiently far, the terms change their form on account of μ becoming imaginary; but for the present purpose these terms will not be required, as they disappear when z is very great. The surface of integration S is made up of the reflecting surface and of a plane parallel to it at a great distance. Although this surface is not strictly closed, it may be treated as such, since the part still remaining open laterally at infinity does not contribute sensibly to the result. Now the part of the integral corresponding to the reflecting surface vanishes, either because

$$\psi = \chi = 0,$$

or else because $d\psi/dn = d\chi/dn = 0$;

and we conclude that when z is great

$$\int \left\{ \psi \frac{d\chi}{dz} - \chi \frac{d\psi}{dz} \right\} dx = 0 \quad (27).$$

The application of (27) to the values of ψ and χ in (25), (26) gives

$$C_0^2 + D_0^2 + \frac{\mu_1}{2k}(C_1^2 + D_1^2) + \frac{\mu_2}{2k}(C_2^2 + D_2^2) + \ldots = 1 \quad (28),$$

the series in (28) being continued so far as to include every *real* value of μ.

In (28) $\frac{1}{4}(C_n^2 + D_n^2)$ represents the intensity of each spectrum of the nth order.

The coefficient μ_n/k is equal to $\cos\theta_n$, where θ_n is the obliquity of the diffracted rays. The meaning of this factor will be evident when it is remarked that to each unit of area of the waves incident and directly reflected, there corresponds an area $\cos\theta_n$ of the waves which constitute the spectrum of the nth order.

If all the values of μ are imaginary, as happens when $p > k$, (28) reduces to

$$C_0^2 + D_0^2 = 1 \quad (29),$$

or the intensity of the wave directly reflected is unity. It is of importance to notice the full significance of this result. However deep the corrugations may be, if only they are periodic in a period less than the wave-length of the vibration, the regular reflection is total. An extremely rough wall will thus reflect sound waves of moderate pitch as well as if it were theoretically smooth.

The above investigation is limited to the case where the second medium is impenetrable, so that the whole energy of the incident wave is thrown back in the regularly reflected wave and in the diffracted spectra. It is an interesting question whether the conclusion that corrugations of period less than λ have no effect can be extended so as to apply when there is a wave regularly transmitted. It is evident that the principle of energy does not suffice to decide the question, but it is probable that the answer should be in the negative. If we suppose the corrugations of given period to become very deep and involved, it would seem that the condition of things would at last approach that of a very gradual transition between the media, in which case (§ 148 *b*) the reflection tends to vanish.

Our limits will not allow us to treat at length the problem of oblique incidence upon a corrugated surface; but one or two remarks may be made.

If p^2 may be neglected, the solution corresponding to (13) is

$$A_0 = R\,J_0(2kc\cos\theta) \quad\text{..................(30)},$$

θ being the angle of incidence and reflection, and R the value of A_0, § 270, corresponding to $c=0$. The factor expressing the effect of the corrugations is thus a function of $c\cos\theta$; so that a deep corrugation when θ is large may have the same effect as a shallow one when θ is small.

Whatever be the angle of incidence, there are no reflected spectra (except of zero order) when the wave-length of the corrugation is less than the *half* of that of the vibrations. Hence, if the second medium be impenetrable, the regular reflection under the above condition is total.

The reader who wishes to pursue the study of the theory of gratings is referred to treatises on optics, and to papers by the Author[1], and by Prof. Rowland[2].

[1] The Manufacture and Theory of Diffraction Gratings, *Phil. Mag.* vol. XLVII. pp. 81, 193, 1874; On Copying Diffraction Gratings, and on some Phenomena connected therewith, *Phil. Mag.* vol. XI. p. 196, 1881; *Enc. Brit.* Wave Theory of Light.

[2] Gratings in Theory and Practice, *Phil. Mag.* vol. XXXV. p. 397, 1893.

CHAPTER. XIV.

GENERAL EQUATIONS.

273. IN connection with the general problem of aërial vibrations in three dimensions one of the first questions, which naturally offers itself, is the determination of the motion in an unlimited atmosphere consequent upon arbitrary initial disturbances. It will be assumed that the disturbance is *small*, so that the ordinary approximate equations are applicable, and further that the initial velocities are such as can be derived from a velocity-potential, or (§ 240) that there is no *circulation*. If the latter condition be violated, the problem is one of vortex motion, on which we do not enter. We shall also suppose in the first place that no external forces act upon the fluid, so that the motion to be investigated is due solely to a disturbance actually existing at a time ($t = 0$), previous to which we do not push our inquiries. The method that we shall employ is not very different from that of Poisson[1], by whom the problem was first successfully attacked.

If u_0, v_0, w_0 be the initial velocities at the point x, y, z, and s_0 the initial condensation, we have (§ 244),

$$\phi_0 = \int (u_0 dx + v_0 dy + w_0 dz) \quad\quad (1),$$

$$\dot{\phi}_0 = - a^2 s_0 \quad\quad (2),$$

by which the initial values of the velocity-potential ϕ and of its differential coefficient with respect to time $\dot{\phi}$ are determined. The problem before us is to determine ϕ at time t from the above

[1] Sur l'intégration de quelques équations linéaires aux différences partielles, et particulièrement de l'équation générale du mouvement des fluides élastiques. *Mém. de l'Institut*, t. III. p. 121. 1820.

initial values, and the general equation applicable at all times and places,

$$\left(\frac{d^2}{dt^2} - a^2\nabla^2\right)\phi = 0 \quad\text{.....................} \quad (3).$$

When ϕ is known, its derivatives give the component velocities at any point.

The symbolical solution of (3) may be written

$$\phi = \sin(ia\nabla t).\theta + \cos(ia\nabla t).\chi \quad\text{..............}(4),$$

where θ and χ are two arbitrary functions of x, y, z and $i = \sqrt{(-1)}$. To connect θ and χ with the initial values of ϕ and $\dot{\phi}$, which we shall denote by f and F respectively, it is only necessary to observe that when $t = 0$, (4) gives

$$\phi_0 = \chi, \quad \dot{\phi}_0 = ia\nabla.\theta;$$

so that our result may be expressed

$$\phi = \cos(ia\nabla t).f + \frac{\sin(ia\nabla t)}{ia\nabla}.F \quad\text{..............}(5),$$

in which equation the question of the interpretation of odd powers of ∇ need not be considered, as both the symbolic functions are wholly even.

In the case where ϕ was a function of x only, we saw (§ 245) that its value for any point x at time t depended on the initial values of ϕ and $\dot{\phi}$ at the points whose co-ordinates were $x - at$ and $x + at$, and was wholly independent of the initial circumstances at all other points. In the present case the simplest supposition open to us is that the value of ϕ at a point O depends on the initial values of ϕ and $\dot{\phi}$ at points situated on the surface of the sphere, whose centre is O and radius at; and, as there can be no reason for giving one direction a preference over another, we are thus led to investigate the expression for the mean value of a function over a spherical surface in terms of the successive differential coefficients of the function at the centre.

By the symbolical form of Maclaurin's theorem the value of $F(x, y, z)$ at any point P on the surface of the sphere of radius r may be written

$$F(x, y, z) = e^{x\frac{d}{dx_0} + y\frac{d}{dy_0} + z\frac{d}{dz_0}}.F(x_0, y_0, z_0),$$

the centre of the sphere O being the origin of co-ordinates. In the integration over the surface of the sphere d/dx_0, d/dy_0, d/dz_0 behave as constants; we may denote them temporarily by l, m, n, so that $\nabla^2 = l^2 + m^2 + n^2$.

Thus, r being the radius of the sphere, and dS an element of its surface, since, by the symmetry of the sphere, we may replace any function of $\dfrac{lx+my+nz}{\sqrt{(l^2+m^2+n^2)}}$ by the same function of z without altering the result of the integration,

$$\iint e^{lx+my+nz}\,dS = \iint (e^{\nabla})^{\frac{lx+my+nz}{\sqrt{(l^2+m^2+n^2)}}}\,dS$$

$$= \iint e^{\nabla z}\,dS = 2\pi r\int_{-r}^{+r} e^{\nabla z}\,dz = \frac{2\pi r}{\nabla}\left(e^{\nabla r} - e^{-\nabla r}\right) = 4\pi r^2\,\frac{\sin(i\nabla r)}{i\nabla r}.$$

The mean value of F over the surface of the sphere of radius r is thus expressed by the result of the operation on F of the symbol $\sin(i\nabla r)/i\nabla r$, or, if $\iint d\sigma$ denote integration with respect to angular space,

$$\frac{1}{4\pi}\iint F(r)\,d\sigma = \frac{\sin(i\nabla r)}{i\nabla r}\,.\,F \quad\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

By comparison with (5) we now see that so far as ϕ depends on the initial values of $\dot{\phi}$, it is expressed by

$$\phi = \frac{t}{4\pi}\iint F(at)\,d\sigma \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

or in words, ϕ at any point at time t is the mean of the initial values of $\dot{\phi}$ over the surface of the sphere described round the point in question with radius at, the whole multiplied by t.

By Stokes' rule (§ 95), or by simple inspection of (5), we see that the part of ϕ depending on the initial values of ϕ may be derived from that just written by differentiating with respect to t and changing the arbitrary function. The complete value of ϕ at time t is therefore

$$\phi = \frac{t}{4\pi}\iint F(at)\,d\sigma + \frac{1}{4\pi}\frac{d}{dt}\,t\iint f(at)\,d\sigma\ldots\ldots\ldots(8),$$

which is Poisson's result[1].

On account of the importance of the present problem, it may

[1] Another investigation will be found in Kirchhoff's *Vorlesungen über Mathematische Physik*, p. 317. 1876. [See also Note to § 273 at the end of this volume.]

be well to verify the solution *a posteriori*. We have first to prove that it satisfies the general differential equation (3). Taking for the present the first term only, and bearing in mind the general symbolic equation

$$\frac{d^2}{dt^2}\,t = \frac{1}{t}\frac{d}{dt}\,t^2\frac{d}{dt} \quad \text{.......................} \quad (9),$$

we find from (8)

$$\frac{d^2\phi}{dt^2} = \frac{1}{4\pi t}\frac{d}{dt}\,t^2\iint\frac{d}{dt}F(at)\,d\sigma = \frac{1}{4\pi at}\frac{d}{dt}\iint\frac{d\,F(at)}{d\,(at)}\,dS,$$

dS being the surface element of the sphere $r = at$.

But by Green's theorem

$$\iint\frac{d\,F(r)}{dr}\,dS = \iiint\nabla^2 F\,dV \quad (r < at);$$

and thus

$$\frac{d^2\phi}{dt^2} = \frac{1}{4\pi at}\frac{d}{dt}\cdot\iiint\nabla^2 F\,dV \quad (r < at)$$

$$= \frac{1}{4\pi t}\iint\nabla^2 F\,dS \quad (r = at) = \frac{a^2 t}{4\pi}\iint\nabla^2 F\,d\sigma.$$

Now $\iint\nabla^2 F\,d\sigma$ is the same as $\nabla^2\iint F\,d\sigma$, and thus (3) is in fact satisfied.

Since the second part of ϕ is obtained from the first by differentiation, it also must satisfy the fundamental equation.

With respect to the initial conditions we see that when t is made equal to zero in (8),

$$\phi = \frac{1}{4\pi}\iint f(at)\,d\sigma \quad (t = 0) = f(0);$$

$$\dot\phi = \frac{1}{4\pi}\iint F(at)\,d\sigma \quad (t = 0) + \frac{1}{4\pi}\frac{d^2}{dt^2}\,t\iint f(at)\,d\sigma \quad (t = 0),$$

of which the first term becomes in the limit $F(0)$. When $t = 0$,

$$\frac{d^2}{dt^2}\,t\iint f(at)\,d\sigma = 2\frac{d}{dt}\iint f(at)\,d\sigma \quad (t = 0)$$

$$= 2a\iint f'(at)\,d\sigma \quad (t = 0) = 0,$$

since the oppositely situated elements cancel in the limit, when the radius of the spherical surface is indefinitely diminished. The expression in (8) therefore satisfies the prescribed initial conditions as well as the general differential equation.

274. If the initial disturbance be confined to a space T, the integrals in (8) § 273 are zero, unless some part of the surface of the sphere $r = at$ be included within T. Let O be a point external to T, r_1 and r_2 the radii of the least and greatest spheres described about O which cut it. Then so long as $at < r_1$, ϕ remains equal to zero. When at lies between r_1 and r_2, ϕ may be finite, but for values greater than r_2 ϕ is again zero. The disturbance is thus at any moment confined to those parts of space for which at is intermediate between r_1 and r_2. The limit of the wave is the envelope of spheres with radius at, whose centres are situated on the surface of T. "When t is small, this system of spheres will have an exterior envelope of two sheets, the outer of these sheets being exterior, and the inner interior to the shell formed by the assemblage of the spheres The outer sheet forms the outer limit to the portion of the medium in which the dilatation is different from zero. As t increases, the inner sheet contracts, and at last its opposite sides cross, and it changes its character from being exterior, with reference to the spheres, to interior. It then expands, and forms the inner boundary of the shell in which the wave of condensation is comprised[1]." The successive positions of the boundaries of the wave are thus a series of parallel surfaces, and each boundary is propagated normally with a velocity equal to a.

If at the time $t = 0$ there be no motion, so that the initial disturbance consists merely in a variation of density, the subsequent condition of things is expressed by the first term of (8) § 273. Let us suppose that the original disturbance, still limited to a finite region T, consists of condensation only, without rarefaction. It might be thought that the same peculiarity would attach to the resulting wave throughout the whole of its subsequent course; but, as Prof. Stokes has remarked, such a conclusion would be erroneous. For values of the time less than r_1/a the potential at O is zero; it then becomes negative (s_0 being positive), and continues negative until it vanishes again when $t = r_2/a$, after which it always remains equal to zero. While ϕ is diminishing, the medium at O is in a state of condensation, but as ϕ increases again to zero, the state of the medium at O is one of rarefaction. The wave propagated outwards consists therefore of two parts at least, of which the first is condensed and the last rarefied. Whatever may be the character of the original disturbance within T, the final value of ϕ

[1] Stokes, "Dynamical Theory of Diffraction," *Camb. Trans.* IX p. 15, 1849.

at any external point O is the same as the initial value, and therefore, since $a^2s = -\dot{\phi}$, the mean condensation during the passage of the wave, depending on the integral $\int s\,dt$, is zero. Under the head of spherical waves we shall have occasion to return to this subject (§ 279).

The general solution embodied in (8) § 273 must of course embrace the particular case of plane waves, but a few words on this application may not be superfluous, for it might appear at first sight that the effect at a given point of a disturbance initially confined to a slice of the medium enclosed between two parallel planes would not pass off in any finite time, as we know it ought to do. Let us suppose for simplicity that ϕ_0 is zero throughout, and that within the slice in question the initial value $\dot{\phi}_0$ is constant. From the theory of plane waves we know that at any arbitrary point the disturbance will finally cease after the lapse of a time t, such that at is equal to the distance (d) of the point under consideration from the further boundary of the initially disturbed region; while on the other hand, since the sphere of radius at continues to cut the region, it would appear from the general formula that the disturbance continues. It is true indeed that ϕ remains finite, but this is not inconsistent with rest. It will in fact appear on examination that the mean value of $\dot{\phi}_0$ multiplied by the radius of the sphere is the same whatever may be the position and size of the sphere, provided only that it cut completely through the region of original disturbance. If $at > d$, ϕ is thus constant with respect both to space and time, and accordingly the medium is at rest.

[The same principles may find an application to the phenomena of *thunder*. Along the path of the lightning we may perhaps suppose that the generation of heat is uniform, equivalent to a uniform initial distribution of condensation. It appears that the value of ϕ at O the point of observation can change rapidly only when the sphere $r = at$ meets the path of the discharge at its extremities or very obliquely.]

275. In two dimensions, when ϕ is independent of z, it might be supposed that the corresponding formula would be obtained by simply substituting for the sphere of radius at the circle of equal radius. This, however, is not the case. It may be proved that the mean value of a function $F(x, y)$ over the circumference of a circle of radius r is $J_0(ir\nabla)F_0$, where $i=\sqrt{(-1)}$,

$$\nabla^2 = d^2/dx_0^2 + d^2/dy_0^2,$$

and J_0 is *Bessel's* function of zero order; so that

$$\frac{1}{2\pi r}\int F(x, y)\, ds = \left(1+\frac{r^2\nabla^2}{2^2}+\frac{r^4\nabla^4}{2^2 . 4^2}+\ldots\right)F,$$

differing from what is required to satisfy the fundamental equation.

The correct result applicable to two dimensions may be obtained from the general formula. The element of spherical surface dS may be replaced by $r\,dr\,d\theta/\cos\psi$, where r, θ are plane polar co-ordinates, and ψ is the angle between the tangent plane and that in which the motion takes place. Thus

$$\cos\psi = \frac{\sqrt{(a^2t^2-r^2)}}{at},$$

$F(at)$ is replaced by $F(r, \theta)$, and so

$$\phi = \iint \frac{F(r, \theta)\, r\,dr\,d\theta}{4\pi a\sqrt{(a^2t^2-r^2)}} \qquad\ldots\ldots\ldots\ldots(1),$$

where the integration extends over the area of the circle $r=at$. The other term might be obtained by Stokes' rule.

This solution is applicable to the motion of a layer of gas between two parallel planes, or to that of an unlimited stretched membrane, which depends upon the same fundamental equation.

276. From the solution in terms of initial conditions we may, as usual (§ 66), deduce the effect of a continually renewed disturbance. Let us suppose that throughout the space T (which will ultimately be made to vanish), a uniform disturbance $\dot{\phi}$, equal to $\Phi(t')\,dt'$, is communicated at time t'. The resulting value of ϕ at time t is

$$\frac{S}{4\pi a^2(t-t')}\Phi(t')\,dt',$$

where S denotes the part of the surface of the sphere $r=a(t-t')$ intercepted within T, a quantity which vanishes, unless $a(t-t')$ be comprised between the narrow limits r_1 and r_2. Ultimately $t-t'$ may be replaced by r/a, and $\Phi(t')$ by $\Phi(t-r/a)$; and the result of the integration with respect to dt' is found by writing T (the volume) for $\int a\,S\,dt'$. Hence

$$\phi = \frac{T}{4\pi a^2 r}\Phi\left(t-\frac{r}{a}\right)\qquad\ldots\ldots\ldots\ldots(1),$$

shewing that the disturbance originating at any point spreads itself symmetrically in all directions with velocity a, and with amplitude varying inversely as the distance. Since any number of particular solutions may be superposed, the general solution of the equation

$$\ddot{\phi} = a^2\nabla^2\phi + \Phi \qquad\qquad (2)$$

may be written

$$\phi = \frac{1}{4\pi a^2}\iiint \Phi\left(t - \frac{r}{a}\right)\frac{dV}{r} \qquad\qquad (3),$$

r denoting the distance of the element dV situated at x, y, z from O (at which ϕ is estimated), and $\Phi(t - r/a)$ the value of Φ for the point x, y, z at the time $t - r/a$. Complementary terms, satisfying through all space the equation $\ddot{\phi} = a^2\nabla^2\phi$, may of course occur independently.

In our previous notation (§ 244)

$$\Phi = \frac{d}{dt}\int (X dx + Y dy + Z dz);$$

and it is assumed that $X dx + Y dy + Z dz$ is a complete differential. Forces, under whose action the medium could not adjust itself to equilibrium, are excluded; as for instance, a force uniform in magnitude and direction within a space T, and vanishing outside that space. The nature of the disturbance denoted by Φ is perhaps best seen by considering the extreme case when Φ vanishes except through a small volume, which is supposed to diminish without limit, while the magnitude of Φ increases in such a manner that the whole effect remains finite. If then we integrate equation (2) through a small space including the point at which Φ is ultimately concentrated, we find in the limit

$$0 = a^2\iint \frac{d\phi}{dn} dS + \iiint \Phi\, dV \qquad\qquad (4),$$

shewing that the effect of Φ may be represented by a proportional introduction or abstraction of fluid at the place in question. The simplest source of sound is thus analogous to a focus in the theory of conduction of heat, or to an electrode in the theory of electricity.

277. The preceding expressions are general in respect of the relation to time of the functions concerned; but in almost all the applications that we shall have to make, it will be convenient to analyse the motion by Fourier's theorem and treat separately the simple harmonic motions of various periods, afterwards, if necessary, compounding the results. The values of ϕ and Φ, if simple harmonic at every point of space, may be expressed in the form $R\cos(nt+\epsilon)$,. R and ϵ being independent of time, but variable from point to point. But as in such cases it often conduces to simplicity to add the term $iR\sin(nt+\epsilon)$, making altogether $Re^{i(nt+\epsilon)}$, or $Re^{i\epsilon}\,.\,e^{int}$, we will assume simply that all the functions which enter into a problem are proportional to e^{int}, the coefficients being in general complex. After our operations are completed, the real and imaginary parts of the expressions can be separated, either of them by itself constituting a solution of the question.

Since ϕ is proportional to e^{int}, $\ddot{\phi}=-n^2\phi$; and the differential equation becomes

$$\nabla^2\phi+k^2\phi+a^{-2}\Phi=0\;\dots\dots\dots\dots(1),$$

where, for the sake of brevity, k is written in place of n/a. If λ denote the *wave-length* of the vibration of the period in question,

$$k=n/a=2\pi/\lambda\;\dots\dots\dots\dots(2).$$

To adapt (3) of the preceding section to the present case, it is only necessary to remark that the substitution of $t-r/a$ for t is effected by introducing the factor $e^{-inr/a}$, or e^{-ikr}: thus

$$\Phi(t-r/a)=e^{-ikr}\,\Phi(t),$$

and the solution of (1) is

$$\phi=\frac{1}{4\pi a^2}\iiint\frac{e^{-ikr}}{r}\,\Phi\,dV\;\dots\dots\dots\dots(3),$$

to which may be added any solution of $\nabla^2\phi+k^2\phi=0$.

If the disturbing forces be all in the same phase, and the region through which they act be very small in comparison with the wave-length, e^{-ikr} may be removed from under the integral sign, and at a sufficient distance we may take

$$\phi=\frac{e^{-ikr}}{4\pi a^2 r}\iiint\Phi\,dV,$$

or in real quantities, on restoring the time factor and replacing $\iiint\Phi\,dV$ by Φ_1,

$$\phi=\Phi_1\,\frac{\cos(nt-kr+\epsilon)}{4\pi a^2 r}\;\dots\dots\dots\dots(4).$$

In order to verify that (3) satisfies the differential equation (1), we may proceed as in the theory of the common potential. Considering one element of the integral at a time, we have first to shew that

$$\phi = \frac{e^{-ikr}}{r} \text{ ....(5)}$$

satisfies $\nabla^2\phi + k^2\phi = 0$, at points for which r is finite. The simplest course is to express ∇^2 in polar co-ordinates referred to the element itself as pole, when it appears that

$$\nabla^2 \frac{e^{-ikr}}{r} = \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\frac{e^{-ikr}}{r} = \frac{1}{r}\frac{d^2}{dr^2}r \cdot \frac{e^{-ikr}}{r} = -k^2\frac{e^{-ikr}}{r}.$$

We infer that (3) satisfies $\nabla^2\phi + k^2\phi = 0$, at all points for which Φ vanishes. In the case of a point at which Φ does not vanish, we may put out of account all the elements situated at a finite distance (as contributing only terms satisfying $\nabla^2\phi + k^2\phi = 0$), and for the element at an infinitesimal distance replace e^{-ikr} by unity. Thus on the whole

$$(\nabla^2 + k^2)\phi = \frac{1}{4\pi a^2}\nabla^2\iiint \Phi\frac{dV}{r} = \frac{1}{a^2}\Phi,$$

exactly as in Poisson's theorem for the common potential[1].

278. The effect of a force Φ_1 distributed over a surface S may be obtained as a limiting case from (3) § 277. $\Phi\, dV$ is replaced by $\Phi\, b dS$, b denoting the thickness of the layer; and in the limit we may write $\Phi b = \Phi_1$. Thus

$$\phi = \frac{1}{4\pi a^2}\iint \Phi_1 \frac{e^{-ikr}}{r} dS \text{ ....(1).}$$

The value of ϕ is the same on the two sides of S, but there is discontinuity in its derivatives. If dn be drawn outwards from S normally, (4) § 276 gives

$$\left(\frac{d\phi}{dn}\right)_1 + \left(\frac{d\phi}{dn}\right)_2 = -\frac{1}{a^2}\Phi_1 \text{ ....(2)}^2.$$

If the surface S be plane, the integral in (1) is evidently symmetrical with respect to it, and therefore

$$(d\phi/dn)_1 = (d\phi/dn)_2.$$

[1] See Thomson and Tait's *Natural Philosophy*, § 491.

[2] Helmholtz. *Crelle*, t. 57, p. 21, 1860.

Hence, if $d\phi/dn$ be the given normal velocity of the fluid in contact with the plane, the value of ϕ is determined by

$$\phi = -\frac{1}{2\pi}\iint \frac{d\phi}{dn}\frac{e^{-ikr}}{r}\,dS \quad\text{.....................(3)},$$

which is a result of considerable importance. To exhibit it in terms of real quantities, we may take

$$d\phi/dn = Pe^{i(nt+\epsilon)} \quad\text{.....................(4)},$$

P and ϵ being real functions of the position of dS. The symbolical solution then becomes

$$\phi = -\frac{1}{2\pi}\iint P\, e^{i(nt-kr+\epsilon)}\frac{dS}{r} \quad\text{..................... (5)},$$

from which, if the imaginary part be rejected, we obtain

$$\phi = -\frac{1}{2\pi}\iint P\frac{\cos(nt-kr+\epsilon)}{r}\,dS \quad\text{............... (6)},$$

corresponding to

$$d\phi/dn = P\cos(nt+\epsilon) \quad\text{....................(7)}.$$

The same method is applicable to the general case when the motion is not restricted to be simple harmonic. We have

$$\phi = -\frac{1}{2\pi}\iint V\left(t-\frac{r}{a}\right).\frac{dS}{r} \quad\text{....................(8)},$$

where by $V(t-r/a)$ is denoted the normal velocity at the plane for the element dS at the time $t-r/a$, that is to say, at a time r/a antecedent to that at which ϕ is estimated.

In order to complete the solution of the problem for the unlimited mass of fluid lying on one side of an infinite plane, we have to add the most general value of ϕ, consistent with $V=0$. This part of the question is identical with the general problem of reflection from an infinite rigid plane[1].

It is evident that the effect of the constraint will be represented by the introduction on the other side of the plane of fictitious initial displacements and forces, forming in conjunction with those actually existing on the first side a system perfectly symmetrical with respect to the plane. Whatever the initial values of ϕ and $\dot{\phi}$ may be belonging to any point on the first side, the same must be ascribed to its *image*, and in like manner whatever function of

[1] Poisson, *Journal de l'école polytechnique*, t. VII. 1808.

the time Φ may be at the first point, it must be conceived to be the same function of the time at the other. Under these circumstances it is clear that for all future time ϕ will be symmetrical with respect to the plane, and therefore the normal velocity zero. So far then as the motion on the first side is concerned, there will be no change if the plane be removed, and the fluid continued indefinitely in all directions, provided the circumstances on the second side are the exact reflection of those on the first. This being understood, the general solution of the problem for a fluid bounded by an infinite plane is contained in the formulæ (8) § 273, (3) § 277, and (8) of the present section. They give the result of arbitrary initial conditions (ϕ_0 and $\dot{\phi}_0$), arbitrary applied forces (Φ), and arbitrary motion of the plane (V).

Measured by the resulting potential, a source of given magnitude, i.e. a source at which a given introduction and withdrawal of fluid takes place, is thus twice as effective when close to a rigid plane, as if it were situated in the open; and the result is ultimately the same, whether the source be concentrated in a point close to the plane, or be due to a corresponding normal motion of the surface of the plane itself.

The operation of the plane is to double the effective pressures which oppose the expansion and contraction at the source, and therefore to double the total energy emitted; and since this energy is diffused through only the half of angular space, the intensity of the sound is quadrupled, which corresponds to a doubled amplitude, or potential (§ 245).

We will now suppose that instead of $d\phi/dn = 0$, the prescribed condition at the infinite plane is that $\phi = 0$. In this case the fictitious distribution of ϕ_0, $\dot{\phi}_0$, Φ, on the second side of the plane must be the *opposite* of that on the first side, so that the sum of the values at two corresponding points is always zero. This secures that on the plane of symmetry itself ϕ shall vanish throughout.

Let us next suppose that there are two parallel surfaces S_1, S_2, separated by the infinitely small interval dn, and that the value of Φ_1 on the second surface is equal and opposite to the value of Φ_1 on the first. In crossing S_1, there is by (2) a finite change in the value of $d\phi/dn$ to the amount of Φ_1/a^2, but in crossing S_2 the same finite change occurs in the reverse direction. When dn is reduced without limit, and $\Phi_1 dn$ replaced by Φ_{11}, $d\phi/dn$ will be the same on the two sides of the double sheet, but there will be discontinuity in the value of ϕ to the amount of Φ_{11}/a^2. At the same time (1) becomes

$$\phi = \frac{1}{4\pi a^2}\iint \frac{d}{dn}\left(\frac{e^{-ikr}}{r}\right)\Phi_{11}\, dS \quad\text{.................(9).}$$

If the surface S be plane, the values of ϕ on the two sides of it are numerically equal, and therefore close to the surface itself

$$\phi = \pm \tfrac{1}{2}\, a^{-2}\, \Phi_{11}.$$

Hence (9) may be written

$$\phi = -\frac{1}{2\pi}\iint \frac{d}{dn}\left(\frac{e^{-ikr}}{r}\right)\phi\, dS \quad\text{.................(10),}$$

where ϕ under the integral sign represents the surface-potential, positive on the one side and negative on the other, due to the action of the forces at S. The direction of dn must be understood to be *towards* the side at which ϕ is to be estimated.

279. The problem of spherical waves diverging from a point has already been forced upon us and in some degree considered, but on account of its importance it demands a more detailed treatment. If the centre of symmetry be taken as pole the velocity-potential is a function of r only, and (§ 241) ∇^2 reduces to $\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}$, or to $\frac{1}{r}\frac{d^2}{dr^2}r$. The equation of free motion (3) § 273 thus becomes

$$\frac{d^2(r\phi)}{dt^2} = a^2\frac{d^2(r\phi)}{dr^2} \quad\text{.........................(1),}$$

whence, as in § 245,

$$r\phi = f(at - r) + F(at + r) \quad\text{.................(2).}$$

The values of the velocity and condensation are to be found by differentiation in accordance with the formulæ

$$u = \frac{d\phi}{dr}, \qquad s = -\frac{1}{a^2}\frac{d\phi}{dt} \quad\text{...................(3).}$$

As in the case of one dimension, the first term represents a wave advancing in the direction of r increasing, that is to say, a divergent wave, and the second term represents a wave converging upon the pole. The latter does not in itself possess much interest. If we confine our attention to the divergent wave, we have

$$u = -\frac{f(at-r)}{r^2} - \frac{f'(at-r)}{r}; \qquad as = -\frac{f'(at-r)}{r} \quad\text{.....(4).}$$

When r is very great the term divided by r^2 may be neglected, and then approximately

$$u = as \quad\quad (5),$$

the same relation as obtains in the case of a plane wave, as might have been expected.

If the type be harmonic,

$$r\phi = A\, e^{ik(at-r+\theta)} \quad\quad (6),$$

or, if only the real part be retained,

$$r\phi = A \cos \frac{2\pi}{\lambda}(at + \theta - r) \quad\quad (7).$$

If a divergent disturbance be confined to a spherical shell, within and without which there is neither condensation nor velocity, the character of the wave is limited by a remarkable relation, first pointed out by Stokes[1]. From equations (4) we have

$$(as - u)\, r^2 = f(at - r),$$

shewing that the value of $f(at - r)$ is the same, viz. zero, both inside and outside the shell to which the wave is limited. Hence by (4), if α and β be radii less and greater than the extreme radii of the shell,

$$\int_\alpha^\beta s\, r\, dr = 0 \quad\quad (8),$$

which is the expression of the relation referred to. As in § 274, we see that a condensed or a rarefied wave cannot exist alone. When the radius becomes great in comparison with the thickness, the variation of r in the integral may be neglected, and (8) then expresses that the *mean* condensation is zero.

[Availing himself of Foucault's method for rendering visible minute optical differences, Töpler[2] succeeded in observing spherical sonorous waves originating in small electric sparks, and their reflection from a plane wall. Subsequently photographic records of similar phenomena have been obtained by Mach[3].]

In applying the general solution (2) to deduce the motion resulting from arbitrary initial circumstances, we must remember that in its present form it is too general for the purpose, since it covers the case in which the pole is itself a source, or place where

[1] *Phil. Mag.* XXXIV. p. 52. 1849.

[2] *Pogg. Ann.* vol. CXXXI. pp. 33, 180. 1867.

[3] *Sitzber. der Wiener Akad.*, 1889.

fluid is introduced or withdrawn in violation of the equation of continuity. The total current across the surface of a sphere of radius r is $4\pi r^2 u$, or by (2) and (3)

$$-4\pi\{f(at-r)+F(at+r)\}+4\pi r\{F'(at+r)-f'(at-r)\},$$

so that, if the pole be not a source, $f(at-r)+F(at+r)$, or $r\phi$, must vanish with r. Thus

$$f(at)+F(at)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

an equation which must hold good for all positive values of the argument[1].

By the known initial circumstances the values of u and s are determined for the time $t=0$, and for all (positive) values of r. If these initial values be represented by u_0 and s_0, we obtain from (2) and (3)

$$\left.\begin{array}{l} f(-r)+F(r)=r\int u_0\,dr \\ f(-r)-F(r)=a\int s_0\,r\,dr \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

by which the function f is determined for all negative arguments, and the function F for all positive arguments. The form of f for positive arguments follows by means of (9), and then the whole subsequent motion is determined by (2). The form of F for negative arguments is not required.

The initial disturbance divides itself into two parts, travelling in opposite directions, in each of which $r\phi$ is propagated with constant velocity a, and the inwards travelling wave is continually reflected at the pole. Since the condition to be there satisfied is $r\phi=0$, the case is somewhat similar to that of a parallel tube terminated by an *open* end, and we may thus perhaps better understand why the condensed wave, arising from the liberation of a mass of condensed air round the pole, is followed immediately by a wave of rarefaction.

[The composite character of the wave resulting from an initial condensation may be invoked to explain a phenomenon which has often occasioned surprise. When windows are broken by a violent explosion in their neighbourhood, they are frequently observed to

[1] The solution for spherical vibrations may be obtained without the use of (1) by superposition of trains of plane waves, related similarly to the pole, and travelling outwards in all directions symmetrically.

have fallen *outwards* as if from exposure to a wave of rarefaction. This effect may be attributed to the second part of the compound wave; but it may be asked why should the second part preponderate over the first? If the window were freely suspended, the momentum acquired from the waves of condensation and rarefaction would be equal. But under the actual conditions it may well happen that the force of the condensed wave is spent in overcoming the resistance of the supports, and then the rarefied wave is left free to produce its full effect.]

280. Returning now to the case of a train of harmonic waves travelling outwards continually from the pole as source, let us investigate the connection between the velocity-potential and the quantity of fluid which must be supposed to be introduced and withdrawn alternately. If the velocity-potential be

$$\phi = -\frac{A}{4\pi r}\cos k(at - r) \quad \text{..................(1)},$$

we have, as in the preceding section, for the total current crossing a sphere of radius r,

$$4\pi r^2 \frac{d\phi}{dr} = A\{\cos k(at-r) - kr \sin k(at-r)\} = A \cos kat,$$

where r is small enough. If the maximum rate of introduction of fluid be denoted by A, the corresponding potential is given by (1).

It will be observed that when the source, as measured by A, is finite, the potential and the pressure-variation (proportional to $\dot{\phi}$) are infinite at the pole. But this does not, as might for a moment be supposed, imply an infinite emission of energy. If the pressure be divided into two parts, one of which has the same phase as the velocity, and the other the same phase as the acceleration, it will be found that the former part, on which the work depends, is finite. The infinite part of the pressure does no work on the whole, but merely keeps up the vibration of the air immediately round the source, whose effective inertia is indefinitely great.

We will now investigate the energy emitted from a simple source of given magnitude, supposing for the sake of greater generality that the source is situated at the vertex of a rigid cone of solid angle ω. If the rate of introduction of fluid at the source be $A \cos kat$, we have

$$\omega r^2 \, d\phi/dr = A \cos kat$$

ultimately, corresponding to

$$\phi = -\frac{A}{\omega r}\cos k(at-r) \quad\quad (2);$$

whence

$$\dot{\phi} = \frac{kaA}{\omega r}\sin k(at-r) \quad\quad (3),$$

and

$$\omega r^2 \frac{d\phi}{dr} = A\,\{\cos k(at-r) - kr\sin k(at-r)\} \quad\quad (4).$$

Thus, as in § 245, if dW be the work transmitted in time dt, we get, since $\delta p = -\rho\dot{\phi}$,

$$\frac{dW}{dt} = -\frac{\rho kaA^2}{\omega r}\sin k(at-r)\cos k(at-r) + \rho\frac{k^2aA^2}{\omega}\sin^2 k(at-r).$$

Of the right-hand member the first term is entirely periodic, and in the second the mean value of $\sin^2 k(at-r)$ is $\frac{1}{2}$. Thus in the long run

$$W = \frac{\rho k^2 aA^2}{2\omega}t \quad\quad (5)^1.$$

It will be remarked that when the source is given, the amplitude varies inversely as ω, and therefore the intensity inversely as ω^2. For an acute cone the intensity is greater, not only on account of the diminution in the solid angle through which the sound is distributed, but also because the total energy emitted from the source is itself increased.

When the source is in the open, we have only to put $\omega = 4\pi$, and when it is close to a rigid plane, $\omega = 2\pi$.

The results of this article find an interesting application in the theory of the speaking trumpet, or (by the law of reciprocity §§ 109, 294) hearing trumpet. If the diameter of the large open end be small in comparison with the wave-length, the waves on arrival suffer copious reflection, and the ultimate result, which must depend largely on the precise relative lengths of the tube and of the wave, requires to be determined by a different process. But by sufficiently prolonging the cone, this reflection may be diminished, and it will tend to cease when the diameter of the open end includes a large number of wave-lengths. Apart from friction it would therefore be possible by diminishing ω to obtain from a given source any desired amount of energy, and at the

[1] Cambridge Mathematical Tripos Examination, 1876.

same time by lengthening the cone to secure the unimpeded transference of this energy from the tube to the surrounding air.

From the theory of diffraction it appears that the sound will not fall off to any great extent in a lateral direction, unless the diameter at the large end exceed half a wave-length. The ordinary explanation of the effect of a common trumpet, depending on a supposed concentration of *rays* in the axial direction, is thus untenable.

281. By means of Euler's equation,

$$\frac{d^2(r\phi)}{dt^2}=a^2\frac{d^2(r\phi)}{dr^2}\quad\text{.......................(1)},$$

we may easily establish a theory for conical pipes with open ends, analogous to that of Bernoulli for parallel tubes, subject to the same limitation as to the smallness of the diameter of the tubes in comparison with the wave-length of the sound[1]. Assuming that the vibration is stationary, so that $r\phi$ is everywhere proportional to $\cos kat$, we get from (1)

$$\frac{d^2(r\phi)}{dr^2}+k^2 . r\phi=0\quad\text{.....................(2)},$$

of which the general solution is

$$r\phi=A\cos kr+B\sin kr\quad\text{..................(3)}.$$

The condition to be satisfied at an open end, viz., that there is to be no condensation or rarefaction, gives $r\phi=0$, so that, if the extreme radii of the tube be r_1 and r_2, we have

$$A\cos kr_1+B\sin kr_1=0,\quad A\cos kr_2+B\sin kr_2=0,$$

whence by elimination of $A : B$, $\sin k(r_2-r_1)=0$, or $r_2-r_1=\frac{1}{2}m\lambda$, where m is an integer. In fact since the form of the general solution (3) and the condition for an open end are the same as for a parallel tube, the result that the length of the tube is a multiple of the half wave-length is necessarily also the same.

A cone, which is complete as far as the vertex, may be treated as if the vertex were an open end, since, as we saw in § 279, the condition $r\phi=0$ is there satisfied.

The resemblance to the case of parallel tubes does not extend to the position of the nodes. In the case of the gravest vibration

[1] D. Bernoulli, *Mém. d. l'Acad. d. Sci.* 1762; Duhamel, Liouville *Journ. Math.* vol. XIV. p. 98, 1849.

of a parallel tube open at both ends, the node occupies a central position, and the two halves vibrate synchronously as tubes open at one end and stopped at the other. But if a conical tube were divided by a partition at its centre, the two parts would have different periods, as is evident, because the one part differs from a parallel tube by being contracted at its open end where the effect of a contraction is to depress the pitch, while the other part is contracted at its stopped end, where the effect is to raise the pitch. In order that the two periods may be the same, the partition must approach nearer to the narrower end of the tube. Its actual position may be determined analytically from (3) by equating to zero the value of $d\phi/dr$.

When both ends of a conical pipe are closed, the corresponding notes are determined by eliminating $A : B$ between the equations,

$$A\,(\cos kr_1 + kr_1 \sin kr_1) + B\,(\sin kr_1 - kr_1 \cos kr_1) = 0,$$

$$A\,(\cos kr_2 + kr_2 \sin kr_2) + B\,(\sin kr_2 - kr_2 \cos kr_2) = 0,$$

of which the result may be put into the form

$$kr_2 - \tan^{-1} kr_2 = kr_1 - \tan^{-1} kr_1 \quad\quad (4).$$

If $r_1 = 0$, we have simply

$$\tan kr_2 = kr_2 \quad\quad (5)^1;$$

if r_1 and r_2 be very great, $\tan^{-1} kr_1$ and $\tan^{-1} kr_2$ are both odd multiples of $\frac{1}{2}\pi$, so that $r_2 - r_1$ is a multiple of $\frac{1}{2}\lambda$, as the theory of parallel tubes requires.

[If $r_2 - r_1 = l$, $r_2 + r_1 = r$, (4) may be written

$$\tan kl = \frac{kl}{1 + \frac{1}{4}k^2(r^2 - l^2)} \quad\quad (6).$$

When r is great in comparison with l, the approximate solution of (6) gives

$$\lambda = \frac{2l}{m}\left(1 - \frac{4l^2}{m^2\pi^2r^2}\right) \quad\quad (7),$$

m being an integer. The influence of conicality upon the pitch is thus of the second order.

Experiments upon conical pipes have been made by Boutet[2] and by Blaikley[3].]

1 For the roots of this equation see § 207.

2 *Ann. d. Chim.* vol. XXI. p. 150, 1870.

3 *Phil. Mag.* VI. p. 119, 1878.

282. If there be two distinct sources of sound of the same pitch, situated at O_1 and O_2, the velocity-potential ϕ at a point P whose distances from O_1, O_2 are r_1 and r_2, may be expressed

$$\phi = A\frac{\cos k(at - r_1)}{r_1} + B\frac{\cos k(at - r_2 - \alpha)}{r_2} \quad \text{........(1)},$$

where A and B are coefficients representing the magnitudes of the sources (which without loss of generality may be supposed to have the same sign), and α represents the retardation (considered as a distance) of the second source relatively to the first. The two trains of spherical waves are in agreement at any point P, if $r_2 + \alpha - r_1 = \pm m\lambda$, where m is an integer, that is, if P lie on any one of a system of hyperboloids of revolution having foci at O_1 and O_2. At points lying on the intermediate hyperboloids, represented by $r_2 + \alpha - r_1 = \pm \frac{1}{2}(2m+1)\lambda$, the two sets of waves are opposed in phase, and neutralize one another as far as their actual magnitudes permit. The neutralization is complete, if $r_1 : r_2 = A : B$, and then the density at P continues permanently unchanged. The intersections of this sphere with the system of hyperboloids will thus mark out in most cases several circles of absolute silence. If the distance O_1O_2 between the sources be great in comparison with the length of a wave, and the sources themselves be not very unequal in power, it will be possible to depart from the sphere $r_1 : r_2 = A : B$ for a distance of several wave-lengths, without appreciably disturbing the equality of intensities, and thus to obtain over finite surfaces several alternations of sound and of almost complete silence.

There is some difficulty in actually realising a satisfactory interference of two independent sounds. Unless the unison be extraordinarily perfect, the silences are only momentary and are consequently difficult to appreciate. It is therefore best to employ sources which are mechanically connected in such a way that the relative phases of the sounds issuing from them cannot vary. The simplest plan is to repeat the first sound by reflection from a flat wall (§§ 269, 278), but the experiment then loses something in directness owing to the fictitious character of the second source. Perhaps the most satisfactory form of the experiment is that described in the Philosophical Magazine for June 1877 by myself. "An intermittent electric current, obtained from a fork interrupter making 128 vibrations per second, excited by means of electro-magnets two other forks, whose frequency was 256, (§§ 63, 64).

These latter forks were placed at a distance of about ten yards apart, and were provided with suitably tuned resonators, by which their sounds were reinforced. The pitch of the forks was necessarily identical, since the vibrations were forced by electro-magnetic forces of absolutely the same period. With one ear closed it was found possible to define the places of silence with considerable accuracy, a motion of about an inch being sufficient to produce a marked revival of sound. At a point of silence, from which the line joining the forks subtended an angle of about 60°, the apparent striking up of one fork, when the other was stopped, had a very peculiar effect."

Another method is to duplicate a sound coming along a tube by means of branch tubes, whose open ends act as sources. But the experiment in this form is not a very easy one.

It often happens that considerations of symmetry are sufficient to indicate the existence of places of silence. For example, it is evident that there can be no variation of density in the continuation of the plane of a vibrating plate, nor in the equatorial plane of a symmetrical solid of revolution vibrating in the direction of its axis. More generally, any plane is a plane of silence, with respect to which the sources are symmetrical in such a manner that at any point and at its image in the plane there are sources of equal intensities and of opposite phases, or, as it is often more conveniently expressed, of the same phase and of opposite amplitudes.

If any number of sources in the same phase, whose amplitudes are on the whole as much negative as positive, be placed on the circumference of a circle, they will give rise to no disturbance of pressure at points on the straight line which passes through the centre of the circle and is directed at right angles to its plane. This is the case of the symmetrical bell (§ 232), which emits no sound in the direction of its axis[1].

The accurate experimental investigation of aërial vibrations is beset with considerable difficulties, which have been only partially surmounted hitherto. In order to avoid unwished for reflections it is generally necessary to work in the open air, where delicate apparatus, such as a sensitive flame, is difficult of management. Another impediment arises from the presence of the experimenter himself, whose person is large enough to disturb materially the

[1] *Phil. Mag.* (5), III. p. 460. 1877.

state of things which he wishes to examine. Among indicators of sound may be mentioned membranes stretched over cups, the agitation being made apparent by sand, or by small pendulums resting lightly against them. If a membrane be simply stretched across a hoop, both its faces are acted upon by nearly the same forces, and consequently the motion is much diminished, unless the membrane be large enough to cast a sensible shadow, in which its hinder face may be protected. Probably the best method of examining the intensity of sound at any point in the air is to divert a portion of it by means of a tube ending in a small cone or resonator, the sound so diverted being led to the ear, or to a manometric capsule. In this way it is not difficult to determine places of silence with considerable precision.

By means of the same kind of apparatus it is possible to examine even the *phase* of the vibration at any point in air, and to trace out the surfaces on which the phase does not vary[1]. If the interior of a resonator be connected by flexible tubing with a manometric capsule, which influences a small gas flame, the motion of the flame is related in an invariable manner (depending on the apparatus itself) to the variation of pressure at the mouth of the resonator; and in particular the interval between the lowest drop of the flame and the lowest pressure at the resonator is independent of the absolute time at which these effects occur. In Mayer's experiment two flames were employed, placed close together in one vertical line, and were examined with a revolving mirror. So long as the associated resonators were undisturbed, the serrations of the two flames occupied a fixed relative position, and this relative position was also maintained when one resonator was moved about so as to trace out a surface of invariable phase. For further details the reader must be referred to the original paper.

283. When waves of sound impinge upon an obstacle, a portion of the motion is thrown back as an echo, and under cover of the obstacle there is formed a sort of sound shadow. In order, however, to produce shadows in anything like optical perfection, the dimensions of the intervening body must be considerable. The standard of comparison proper to the subject is the wave-length of the vibration; it requires almost as extreme conditions to produce *rays* in the case of sound, as it requires in optics to avoid producing them. Still, sound shadows thrown by hills, or

[1] Mayer, *Phil. Mag.* (4), XLIV. p. 321. 1872.

buildings, are often tolerably complete, and must be within the experience of all.

For closer examination let us take first the case of plane waves of harmonic type impinging upon an immovable plane screen, of infinitesimal thickness, in which there is an aperture of any form, the plane of the screen ($x = 0$) being parallel to the fronts of the waves. The velocity-potential of the undisturbed train of waves may be taken,

$$\phi = \cos(nt - kx) \qquad \ldots\ldots\ldots\ldots(1).$$

If the value of $d\phi/dx$ over the apertu. be known, formulæ (6) and (7) § 278 allow us to calculate the value of ϕ at any point on the further side. In the ordinary theory of diffraction, as given in works on optics, it is assumed that the disturbance in the plane of the aperture is the same as if the screen were away. This hypothesis, though it can never be rigorously exact, will suffice when the aperture is very large in comparison with the wave-length, as is usually the case in optics.

For the undisturbed wave we have

$$\frac{d\phi}{dx}(x = 0) = k \sin nt \qquad \ldots\ldots\ldots\ldots(2),$$

and therefore on the further side, we get

$$\phi = -\frac{k}{2\pi}\iint \frac{\sin(nt - kr)}{r}\, dS \qquad \ldots\ldots\ldots\ldots(3),$$

the integration extending over the area of the aperture. Since $k = 2\pi/\lambda$, we see by comparison with (1) that in supposing a primary wave broken up, with the view of applying Huygens' principle, dS must be divided by λr, and the phase must be accelerated by a quarter of a period.

When r is large in comparison with the dimensions of the aperture, the composition of the integral is best studied by the aid of Fresnel's[1] zones. With the point O, for which ϕ is to be estimated, as centre describe a series of spheres of radii increasing by the constant difference $\frac{1}{2}\lambda$, the first sphere of the series being of such radius (c) as to touch the plane of the screen. On this plane are thus marked out a series of circles, whose radii ρ are

1 [These zones are usually spoken of as Huygens' zones by optical writers (e.g. Billet, *Traité d'Optique physique*, vol. I. p. 102, Paris, 1858); but, as has been pointed out by Schuster (*Phil. Mag.* vol. XXXI. p. 85, 1891), it is more correct to name them after Fresnel.]

given by $\rho^2+c^2=(c+\frac{1}{2}n\lambda)^2$, or $\rho^2=nc\lambda$, very nearly; so that the rings into which the plane is divided, being of approximately equal area, make contributions to ϕ which are approximately equal in numerical magnitude and alternately opposite in sign. If O lie decidedly within the projection of the area, the first term of the series representing the integral is finite, and the terms which follow are alternately opposite in sign and of numerical magnitude at first nearly constant, but afterwards diminishing gradually to zero, as the parts of the rings intercepted within the aperture become less and less. The case of an aperture, whose boundary is equidistant from O, is excepted.

In a series of this description any term after the first is neutralized almost exactly [that is, so far as first differences are concerned] by half the sum of those which immediately precede and follow it, so that the sum of the whole series is represented approximately by half the first term, which stands over uncompensated. We see that, provided a sufficient number of zones be included within the aperture, the value of ϕ at the point O is independent of the nature of the aperture, and is therefore the same as if there had been no screen at all. Or we may calculate directly the effect of the circle with which the system of zones begins; a course which will have the advantage of bringing out more clearly the significance of the change of phase which we found it necessary to introduce when the primary wave was broken up. Thus, let us conceive the circle in question divided into infinitesimal rings of equal area. The parts of ϕ due to each of these rings are equal in amplitude and of phase ranging uniformly over half a complete period. The phase of the resultant is therefore midway between those of the extreme elements, that is to say, a quarter of a period behind that due to the element at the centre of the circle. The amplitude of the resultant will be less than if all its components had been in the same phase, in the ratio $\int_0^\pi \sin x\,dx : \pi$, or $2:\pi$; and therefore since the area of the circle is $\pi\lambda c$, half the effect of the first zone is

$$\phi = -\tfrac{1}{2}\cdot\frac{2}{\pi}\cdot\frac{\sin(nt-kc-\frac{1}{2}\pi)}{\lambda c}\cdot\pi\lambda c = \cos(nt-kc),$$

the same as if the primary wave were to pass on undisturbed.

When the point O is well away from the projection of the aperture, the result is quite different. The series representing the integral then converges at both ends, and by the same reasoning as before its sum is seen to be approximately zero. We conclude that if the projection of O on the plane $x=0$ fall within the aperture, and be nearer to O by a great many wave-lengths than the nearest point of the boundary of the aperture, then the disturbance at O is nearly the same as if there were no obstacle at all; but, if the projection of O fall outside the aperture and be nearer to O by a great many wave-lengths than the nearest point of the boundary, then the disturbance at O practically vanishes. This is the theory of sound rays in its simplest form.

The argument is not very different if the screen be oblique to the plane of the waves. As before, the motion on the further side of the screen may be regarded as due to the normal motion of the particles in the plane of the aperture, but this normal motion now varies in phase from point to point. If the primary waves proceed from a source at Q, Fresnel's zones for a point P are the series of ellipses represented by $r_1+r_2=PQ+\frac{1}{2}n\lambda$, where r_1 and r_2 are the distances of any point on the screen from Q and P respectively, and n is an integer. On account of the assumed smallness of λ in comparison with r_1 and r_2, the zones are at first of equal area and make equal and opposite contributions to the value of ϕ; and thus by the same reasoning as before we may conclude that at any point decidedly outside the geometrical projection of the aperture the disturbance vanishes, while at any point decidedly within the geometrical projection the disturbance is the same as if the primary wave had passed the screen unimpeded. It may be remarked that the increase of area of the Fresnel's zones due to obliquity is compensated in the calculation of the integral by the correspondingly diminished value of the normal velocity of the fluid. The enfeeblement of the primary wave between the screen and the point P due to divergency is represented by a diminution in the area of the Fresnel's zones below that corresponding to plane incident waves in the ratio $r_1+r_2 : r_1$.

There is a simple relation between the transmission of sound through an aperture in a screen and its reflection from a plane reflector of the same form as the aperture, of which advantage may sometimes be taken in experiment. Let us imagine a source similar to Q and in the same phase to be placed at Q', the *image* of Q in the plane of the screen, and let us suppose that the screen is removed and replaced by a plate whose form and position is exactly that of the aperture; then we know that the effect at P of the two sources is uninfluenced by the presence of the plate, so that the vibration from Q' reflected from the plate and the vibration from Q transmitted round the plate together make up the same vibration as would be received from Q if there were no obstacle at all. Now according to the assumption which we made at the beginning of this section, the unimpeded vibration from Q may be regarded as composed of the vibration that finds its way round the plate and of that which would pass an aperture of the same form in an infinite screen, and thus the vibration from Q as transmitted through the aperture is equal to the vibration from Q' as reflected from the plate.

In order to obtain a nearly complete reflection it is not necessary that the reflecting plate include more than a small number of Fresnel's zones. In the case of direct reflection the radius ρ of the first zone is determined by the equation

$$\rho^2 (1/c_1 + 1/c_2) = \lambda \qquad (4),$$

where c_1 and c_2 are the distances from the reflector of the source and of the point of observation. When the distances concerned are great, the zones become so large that ordinary walls are insufficient to give a complete reflection, but at more moderate distances echos are often nearly perfect. The area necessary for complete reflection depends also upon the wave-length; and thus it happens that a board or plate, which would be quite inadequate to reflect a grave musical note, may reflect very fairly a hiss or the sound of a high whistle. In experiments on reflection by screens of moderate size, the principal difficulty is to get rid sufficiently of the direct sound. The simplest plan is to reflect the sound from an electric bell, or other fairly steady source, round the corner of a large building[1].

284. In the preceding section we have applied Huygens' principle to the case where the primary wave is supposed to be broken up at the surface of an imaginary plane. If we really know what the normal motion at the plane is, we can calculate the disturbance at any point on the further side by a rigorous process. For surfaces other than the plane the problem has not been solved generally; nevertheless, it is not difficult to see that when the radii of curvature of the surface are very great in comparison with the wave-length, the effect of a normal motion of an

[1] *Phil. Mag.* (5), III. p. 458. 1877.

element of the surface must be very nearly the same as if the surface were plane. On this understanding we may employ the same integral as before to calculate the aggregate result. As a matter of convenience it is usually best to suppose the wave to be broken up at what is called in optics a *wave-surface*, that is, a surface at every point of which the *phase* of the disturbance is the same.

Let us consider the application of Huygens' principle to calculate the progress of a given divergent wave. With any point P, at which the disturbance is required, as centre, describe a series of spheres of radii continually increasing by the constant difference $\frac{1}{2}\lambda$, the first of the series being of such radius (c) as to touch the given wave-surface at C. If R be the radius of curvature of the surface in any plane through P and C, the corresponding radius ρ of the outer boundary of the n^{th} zone is given by the equation

$$R+c=\sqrt{\{R^2-\rho^2\}}+\sqrt{\{(c+\tfrac{1}{2}n\lambda)^2-\rho^2\}},$$

from which we get approximately

$$\rho^2=n\lambda\div\left(\frac{1}{R}+\frac{1}{c}\right)\text{......................}(1).$$

If the surface be one of revolution round PC, the area of the first n zones is $\pi\rho^2$, and since ρ^2 is proportional to n, it follows that the zones are of equal area. If the surface be not of revolution, the area of the first n zones is represented $\frac{1}{2}\int\rho^2 d\theta$, where θ is the azimuth of the plane in which ρ is measured, but it still remains true that the zones are of equal area. Since by hypothesis the normal motion does not vary rapidly over the wave-surface, the disturbances at P due to the various zones are nearly equal in magnitude and alternately opposite in sign, and we conclude that, as in the case of plane waves, the aggregate effect is the half of that due to the first zone. The phase at P is accordingly retarded behind that prevailing over the given wave-surface by an amount corresponding to the distance c.

The intensity of the disturbance at P depends upon the area of the first Fresnel's zone, and upon the distance c. In the case of symmetry, we have

$$\frac{\pi\rho^2}{c}=\frac{\pi\lambda R}{R+c},$$

which shews that the disturbance is less than if R were infinite in the ratio $R+c:R$. This diminution is the effect of divergency, and is the same as would be obtained on the supposition that the motion is limited by a conical tube whose vertex is at the centre of curvature (§ 266). When the surface is not of revolution, the value of $\frac{1}{2}\int_0^{2\pi}\rho^2 d\theta \div c$ may be expressed in terms of the principal radii of curvature R_1 and R_2, with which R is connected by the relation

$$1/R = \cos^2\theta/R_1 + \sin^2\theta/R_2.$$

We obtain on effecting the integration

$$\frac{1}{2c}\int_0^{2\pi}\rho^2 d\theta = \frac{\pi\lambda\sqrt{R_1 R_2}}{\sqrt{(R_1+c)(R_2+c)}} \quad\text{..............(2)},$$

so that the amplitude is diminished by divergency in the ratio $\sqrt{(R_1+c)(R_2+c)} : \sqrt{R_1 R_2}$, a result which might be anticipated by supposing the motion limited to a tube formed by normals drawn through a small contour traced on the wave-surface.

Although we have spoken hitherto of diverging waves only, the preceding expressions may also be applied to waves converging in one or in both of the principal planes, if we attach suitable signs to R_1 and R_2. In such a case the area of the first Fresnel's zone is greater than if the wave were plane, and the intensity of the vibration is correspondingly increased. If the point P coincide with one of the principal centres of curvature, the expression (2) becomes infinite. The investigation, on which (2) was founded, is then insufficient; all that we are entitled to affirm is that the disturbance is much greater at P than at other points on the same normal, that the disproportion increases with the frequency, and that it would become infinite for notes of infinitely high pitch, whose wave-length would be negligible in comparison with the distances concerned.

285. Huygens' principle may also be applied to investigate the reflection of sound from curved surfaces. If the material surface of the reflector yielded so completely to the aërial pressures that the normal motion at every point were the same as it would have been in the absence of the reflector, then the sound waves would pass on undisturbed. The reflection which actually ensues when the surface is unyielding may therefore be regarded as due to a normal motion of each element of the reflector, equal and opposite to that of the primary waves at the same point, and may be investigated by the formula proper to plane surfaces in the manner of the preceding section, and subject to a similar limitation as to the relative magnitudes of the wave-length and of the other distances concerned.

The most interesting case of reflection occurs when the surface is so shaped as to cause a concentration of rays upon a particular point (P). If the sound issue originally from a simple source at Q, and the surface be an ellipsoid of revolution having its foci at P and Q, the concentration is complete, the vibration reflected from every element of the surface being in the same phase on arrival at Q. If Q be infinitely distant, so that the incident waves are plane, the surface becomes a paraboloid having its focus at P, and its axis parallel to the incident rays. We must not suppose, however, that a symmetrical wave diverging from Q is converted by reflection at the ellipsoidal surface into a spherical wave converging symmetrically upon P; in fact, it is easy to see that the intensity of the convergent wave must be different in different directions. Nevertheless, when the wave-length is very small in comparison with the radius, the different parts of the convergent wave become approximately independent of one another, and their progress is not materially affected by the failure of perfect symmetry.

The increase of loudness due to curvature depends upon the area of reflecting surface, from which disturbances of uniform phase arrive, as compared with the area of the first Fresnel's zone of a plane reflector in the same position. If the distances of the reflector from the source and from the point of observation be considerable, and the wave-length be not very small, the first Fresnel's zone is already rather large, and therefore in the case of a reflector of moderate dimensions but little is gained by making it concave. On the other hand, in laboratory experiments, when the distances are moderate and the sounds employed are of high pitch, *e.g.* the ticking of a watch or the cracking of electric sparks, concave reflectors are very efficient and give a distinct concentration of sound on particular spots.

286. We have seen that if a ray proceeding from Q passes after reflection at a plane or curved surface through P, the point R at which it meets the surface is determined by the condition that $QR + RP$ is a minimum (or in some cases a maximum). The point R is then the centre of the system of Fresnel's zones; the amplitude of the vibration at P depends upon the area of the first zone, and its phase depends upon the distance $QR+RP$. If there be no point on the surface of the reflector, for which $QR+RP$ is a maximum or a minimum, the system of Fresnel's zones has no centre, and there is no ray proceeding from Q which arrives at P after reflection from the surface. In like manner if sound be reflected more than once, the course of a ray is determined by the condition that its whole length between any two points is a maximum or a minimum.

The same principle may be applied to investigate the *refraction* of sound in a medium, whose mechanical properties vary gradually from point to point. The variation is supposed to be so slow that no sensible reflection occurs, and this is not inconsistent with decided refraction of the rays in travelling distances which include a very great number of wave-lengths. It is evident that what we are now concerned with is not merely the length of the ray, but also the velocity with which the wave travels along it, inasmuch as this velocity is no longer constant. The condition to be satisfied is that the *time* occupied by a wave in travelling along a ray between any two points shall be a maximum or a minimum; so that, if V be the velocity of propagation at any point, and ds an element of the length of the ray, the condition may be expressed, $\delta\int V^{-1}\,ds=0$. This is Fermat's principle of least time.

The further developement of this part of the subject would lead us too far into the domain of geometrical optics. The fundamental assumption of the smallness of the wave-length, on which the doctrine of rays is built, having a far wider application to the phenomena of light than to those of sound, the task of developing its consequences may properly be left to the cultivators of the sister science. In the following sections the methods of optics are applied to one or two isolated questions, whose acoustical interest is sufficient to demand their consideration in the present work.

287. One of the most striking of the phenomena connected with the propagation of sound within closed buildings is that presented by "whispering galleries," of which a good and easily accessible example is to be found in the circular gallery at the base of the dome of St Paul's cathedral. As to the precise mode of action acoustical authorities are not entirely agreed. In the opinion of the Astronomer Royal[1] the effect is to be ascribed to reflection from the surface of the dome overhead, and is to be observed at the point of the gallery diametrically opposite to the source of sound. Every ray proceeding from a radiant point and reflected from the surface of a spherical reflector, will after reflection intersect that diameter of the sphere which contains the radiant point. This diameter is in fact a degraded form of one of the two caustic surfaces touched by systems of rays in general, being the loci of the centres of principal curvature of the surface to which the rays are normal. The concentration of rays on one diameter thus effected, does not require the proximity of the radiant point to the reflecting surface.

Judging from some observations that I have made in St Paul's whispering gallery, I am disposed to think that the principal phenomenon is to be explained somewhat differently. The abnormal loudness with which a whisper is heard is not confined to the position diametrically opposite to that occupied by the whisperer, and therefore, it would appear, does not depend materially upon the symmetry of the dome. The whisper seems to creep round the gallery horizontally, not necessarily along the shorter arc, but rather along that arc towards which the whisperer faces. This is a consequence of the very unequal audibility of a whisper in front of and behind the speaker, a phenomenon which may easily be observed in the open air[2].

Let us consider the course of the rays diverging from a radiant point P, situated near the surface of a reflecting sphere, and let us denote the centre of the sphere by O, and the diameter passing through P by AA', so that A is the point on the surface nearest to P. If we fix our attention on a ray which issues from P at an angle $\pm\theta$ with the tangent plane at A, we see that after any number of reflections it continues to touch a concentric sphere of radius $OP\cos\theta$, so that the whole conical pencil of rays which originally make angles with the tangent plane at A numerically less than θ, is ever afterwards included between the reflecting surface and that of the concentric sphere of radius $OP\cos\theta$. The usual divergence in three dimensions entailing a diminishing intensity varying as r^{-2} is replaced by a divergence in two dimensions, like that of waves issuing from a source situated between

[1] Airy *On Sound*, 2nd edition, 1871, p. 145.

[2] *Phil. Mag.* (5), III. p. 458, 1877.

two parallel reflecting planes, with an intensity varying as r^{-1}. The less rapid enfeeblement of sound by distance than that usually experienced is the leading feature in the phenomena of whispering galleries.

The thickness of the sheet included between the two spheres becomes less and less as A approaches P, and in the limiting case of a radiant point situated on the surface of the reflector is expressed by $OA\,(1-\cos\theta)$, or, if θ be small, $\frac{1}{2}OA\,.\,\theta^2$ approximately. The solid angle of the pencil, which determines the whole amount of radiation in the sheet, is $4\pi\theta$; so that as θ is diminished without limit the intensity becomes infinite, as compared with the intensity at a finite distance from a similar source in the open.

It is evident that this clinging, so to speak, of sound to the surface of a concave wall does not depend upon the exactness of the spherical form. But in the case of a true sphere, or rather of any surface symmetrical with respect to AA', there is in addition the other kind of concentration spoken of at the commencement of the present section which is peculiar to the point A' diametrically opposite to the source. It is probable that in the case of a nearly spherical dome like that of St Paul's a part of the observed effect depends upon the symmetry, though perhaps the greater part is referable simply to the general concavity of the walls.

The propagation of earthquake disturbances is probably affected by the curvature of the surface of the globe acting like a whispering gallery, and perhaps even sonorous vibrations generated at the surface of the land or water do not entirely escape the same kind of influence.

In connection with the acoustics of public buildings there are many points which still remain obscure. It is important to bear in mind that the loss of sound in a single reflection at a smooth wall is very small, whether the wall be plane or curved. In order to prevent reverberation it may often be necessary to introduce carpets or hangings to absorb the sound. In some cases the presence of an audience is found sufficient to produce the desired effect. In the absence of all deadening material the prolongation of sound may be very considerable, of which perhaps the most striking example is that afforded by the Baptistery at Pisa, where the notes of the common chord sung consecutively may be heard ringing on together for many seconds[1]. According to Henry[2] it is important to prevent the repeated reflection of sound backwards and forwards along the *length* of a hall intended for public speaking, which may be accomplished by suitably placed oblique surfaces. In this way the number of reflections in a given time is increased, and the undue prolongation of sound is checked.

288. Almost the only instance of acoustical refraction, which has a practical interest, is the deviation of sonorous rays from a rectilinear course due to heterogeneity of the atmosphere. The variation of pressure at different levels does not of itself give rise to refraction, since the velocity of sound is independent of density; but, as was first pointed out by Prof. Osborne Reynolds[3], the case is different with the variations of temperature which are usually to be met with. The temperature of the atmosphere is determined principally by the condensation or rarefaction, which any portion of air must undergo in its passage from one level to another, and its normal state is one of "convective equilibrium[4]," rather than of uniformity. According to this view the relation between pressure and density is that expressed in (9) § 246, and the velocity of sound is given by

$$V^2 = \frac{dp}{d\rho} = \gamma \frac{p_0}{\rho_0} \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \text{ .... (1).}$$

To connect the pressure and density with the elevation (z), we have the hydrostatical equation

$$dp = -g\rho\, dz \text{ .... (2),}$$

from which and (1) we find

$$V^2 = V_0^2 - (\gamma - 1)\, gz \text{ .... (3),}$$

if V_0 be the velocity at the surface. The corresponding relation between temperature and elevation obtained by means of equation (10) § 246 is

$$\frac{\theta}{\theta_0} = 1 - \frac{\gamma - 1}{V_0^2}\, gz \text{ .... (4),}$$

where θ_0 is the temperature at the surface.

[1] [Some observations of my own, made in 1883, gave the duration as 12 seconds. If a note changes pitch, both sounds are heard together and may give rise to a combination-tone, § 68. See Haberditzl, Ueber die von Dvořák beobachteten Variationston. Wien, *Akad. Sitzber.*, 77, p. 204, 1878.]

[2] *Amer. Assoc. Proc.* 1856, p. 119.

[3] *Proceedings of the Royal Society*, Vol. XXII. p. 531. 1874.

[4] Thomson, *On the convective equilibrium of temperature in the atmosphere. Manchester Memoirs*, 1861—62.

According to (4) the fall of temperature would be about 1° Cent. in 330 feet [100 m.], which does not differ much from the results of Glaisher's balloon observations. When the sky is clear, the fall of temperature during the day is more rapid than when the sky is cloudy, but towards sunset the temperature becomes approximately constant[1]. Probably on clear nights it is often warmer above than below.

The explanation of acoustical refraction as dependent upon a variation of temperature with height is almost exactly the same as that of the optical phenomenon of mirage. The curvature (ρ^{-1}) of a ray, whose course is approximately horizontal, is easily estimated by the method given by Prof. James Thomson[2]. Normal planes drawn at two consecutive points along the ray meet at the centre of curvature and are tangential to the wave-surface in its two consecutive positions. The portions of rays at elevations z and $z+\delta z$ respectively intercepted between the normal planes are to one another in the ratio $\rho : \rho - \delta z$, and also, since they are described in the same time, in the ratio $V : V+\delta V$. Hence in the limit

$$\frac{1}{\rho} = -\frac{d \log V}{dz} \quad\text{.......................}\quad (5).$$

In the normal state of the atmosphere a ray, which starts horizontally, turns gradually upwards, and at a sufficient distance passes over the head of an observer whose station is at the same level as the source. If the source be elevated, the sound is heard at the surface of the earth by means of a ray which starts with a downward inclination; but, if both the observer and the source be on the surface, there is no direct ray, and the sound is heard, if at all, by means of diffraction. The observer may then be said to be situated in a sound shadow, although there may be no obstacle in the direct line between himself and the source. According to (3)

$$2V dV/dz = -(\gamma - 1) g,$$

so that

$$\rho = \frac{2V^2}{(\gamma-1)g} = \frac{4}{\gamma-1} \cdot \frac{V^2}{2g} \quad\text{................}\quad (6);$$

or the radius of curvature of a horizontal ray is about ten times the height through which a body must fall under the action of

[1] *Nature*, Sept. 20, 1877.

[2] See Everett, *On the Optics of Mirage*. *Phil. Mag.* (4) XLV. pp. 161, 248.

gravity in order to acquire a velocity equal to the velocity of sound. If the elevations of the observer and of the source be z_1 and z_2, the greatest distance at which the sound can be heard otherwise than by diffraction is

$$\sqrt{(2z_1\rho)} + \sqrt{(2z_2\rho)} \quad\dots\dots\dots\dots(7).$$

It is not to be supposed that the condition of the atmosphere is always such that the relation between velocity and elevation is that expressed in (3). When the sun is shining, the variation of temperature upwards is more rapid: on the other hand, as Prof. Reynolds has remarked, when rain is falling, a much slower variation is to be expected. In the arctic regions, where the nights are long and still, radiation may have more influence than convection in determining the equilibrium of temperature, and if so the propagation of sound in a horizontal direction would be favoured by the approximately isothermal condition of the atmosphere.

The general differential equation for the path of a ray, when the surfaces of equal velocity are parallel planes, is readily obtained from the law of sines. If θ be the angle of incidence, $V/\sin\theta$ is not altered by a refracting surface, and therefore in the case supposed remains constant along the whole course of a ray. If x be the horizontal co-ordinate, and the constant value of $V/\sin\theta$ be called c, we get $dx/dz = V/\sqrt{(c^2 - V^2)}$,

or
$$x = \int \frac{V dz}{\sqrt{c^2 - V^2}} \quad\dots\dots\dots\dots(8).$$

If the law of velocity be that expressed in (3),

$$dz = -\frac{2V dV}{(\gamma - 1) g},$$

and thus
$$x = -\frac{2}{(\gamma - 1) g}\int \frac{V^2 dV}{\sqrt{c^2 - V^2}},$$

or, on effecting the integration,

$$(\gamma - 1) g\, x = \text{constant} + V\sqrt{(c^2 - V^2)} - c^2 \sin^{-1}(V/c) \quad\dots\dots(9),$$

in which V may be expressed in terms of z by (3).

A simpler result will be obtained by taking an approximate form of (3), which will be accurate enough to represent the cases of practical interest. Neglecting the square and higher powers of z, we may take

$$V^{-1} = V_0^{-1} + \frac{g(\gamma - 1) z}{2V_0^3} \quad\dots\dots\dots\dots(10).$$

Writing for brevity β in place of $\frac{1}{2}g(\gamma-1)/V_0^2$, we have

$$\beta\,dz = dV^{-1}.$$

By substitution in (8)

$$c\beta x = \int \frac{d(c/V)}{\sqrt{(c^2/V^2-1)}} = \log\left[\frac{c}{V} + \sqrt{(c^2/V^2-1)}\right] \quad\text{.....(11)},$$

the origin of x being taken so as to correspond with $V=c$, that is at the place where the ray is horizontal. Expressing V in terms of x, we find

$$2c/V = e^{c\beta x} + e^{-c\beta x},$$

whence

$$\beta z = -V_0^{-1} + \frac{1}{2c}(e^{c\beta x} + e^{-c\beta x}) \quad\text{............ (12)}.$$

The path of each ray is therefore a catenary whose vertex is downwards; the linear parameter is $\dfrac{2V_0^3}{g(\gamma-1)c}$, and varies from ray to ray.

289. Another cause of atmospheric refraction is to be found in the action of wind. It has long been known that sounds are generally better heard to leeward than to windward of the source, but the fact remained unexplained until Stokes[1] pointed out that the increasing velocity of the wind overhead must interfere with the rectilinear propagation of sound rays. From Fermat's law of least time it follows that the course of a ray in a moving, but otherwise homogeneous, medium, is the same as it would be in a medium, of which all the parts are at rest, if the velocity of propagation be increased at every point by the component of the wind-velocity in the direction of the ray. If the wind be horizontal, and do not vary in the same horizontal plane, the course of a ray, whose direction is everywhere but slightly inclined to that of the wind, may be calculated on the same principles as were applied in the preceding section to the case of a variable temperature, the normal velocity of propagation at any point being increased, or diminished, by the local wind-velocity, according as the motion of the sound is to leeward or to windward. Thus, when the wind increases overhead, which may be looked upon as the normal state of things, a horizontal ray travelling to windward is gradually bent upwards, and at a moderate distance passes over the head of an observer; rays travelling with the wind, on the

[1] *Brit. Assoc. Rep.* 1857, p. 22.

other hand, are bent downwards, so that an observer to leeward of the source hears by a direct ray which starts with a slight upward inclination, and has the advantage of being out of the way of obstructions for the greater part of its course.

The law of refraction at a horizontal surface, in crossing which the velocity of the wind changes discontinuously, is easily investigated. It will be sufficient to consider the case in which the direction of the wind and the ray are in the same vertical plane. If θ be the angle of incidence, which is also the angle between the plane of the wave and the surface of separation, U be the velocity of the air in that direction which makes the smaller angle with the ray, and V be the common velocity of propagation, the velocity of the trace of the plane of the wave on the surface of separation is

$$\frac{V}{\sin\theta} + U \qquad\qquad (1),$$

which quantity is unchanged by the refraction. If therefore U' be the velocity of the wind on the second side, and θ' be the angle of refraction,

$$\frac{V}{\sin\theta} + U = \frac{V}{\sin\theta'} + U' \qquad\qquad (2),$$

which differs from the ordinary optical law. If the wind-velocity vary continuously, the course of a ray may be calculated from the condition that the expression (1) remains constant.

If we suppose that $U = 0$, the greatest admissible value of U' is

$$U' = V\{\operatorname{cosec}\theta - 1\} \qquad\qquad (3).$$

At a stratum where U' has this value, the direction of the ray which started at an angle θ has become parallel to the refracting surfaces, and a stratum where U' has a greater value cannot be penetrated at all. Thus a ray travelling upwards in still air at an inclination $(\frac{1}{2}\pi - \theta)$ to the horizon is reflected by a wind overhead of velocity exceeding that given in (3), and this independently of the velocities of intermediate strata. To take a numerical example, all rays whose upward inclination is less than 11°, are totally reflected by a wind of the same azimuth moving at the moderate speed of 15 miles per hour. The effects of such a wind on the propagation of sound cannot fail to be very important. Over the surface of still water sound moving to leeward, being confined between parallel reflecting planes, diverges in two dimensions only, and may therefore be heard at distances far greater than would otherwise be possible. Another possible effect of the reflector overhead is to render sounds audible which in still air would be intercepted by hills or other obstacles intervening. For the production of these phenomena it is not necessary that there be absence of wind at the source of sound, but, as appears at once from the form of (2), merely that the *difference* of velocities $U' - U$ attain a sufficient value.

The differential equation to the path of a ray, when the wind-velocity U is continuously variable, is

$$V\sqrt{1+\left(\frac{dz}{dx}\right)^2} = c \pm U \qquad (4),$$

whence

$$x = \int \frac{V\,dz}{\sqrt{\{(c \pm U)^2 - V^2\}}} \qquad (5).$$

In comparing (5) with (8) of the preceding section, which is the corresponding equation for ordinary refraction, we must remember that V is now constant. If, for the sake of obtaining a definite result, we suppose that the law of variation of wind at different levels is that expressed by

$$U = \alpha + \beta z \qquad (6),$$

we have

$$\beta x = V \int \frac{dU}{\sqrt{\{(c \pm U)^2 - V^2\}}} \qquad (7),$$

which is of the same form as (11) of the preceding section. The course of a ray is accordingly a catenary in the present case also, but there is a most important distinction between the two problems. When the refraction is of the ordinary kind, depending upon a variable velocity of propagation, the direction of a ray may be reversed. In the case of atmospheric refraction, due to a diminution of temperature upwards, the course of a ray is a catenary, whose vertex is downwards, in whichever direction the ray may be propagated. When the refraction is due to wind, whose velocity increases upwards, according to the law expressed in (6) with β positive, the path of a ray, whose direction is upwind, is also along a catenary with vertex downwards, but a ray whose direction is downwind cannot travel along this path. In the latter case the vertex of the catenary along which the ray travels is directed upwards.

290. In the paper by Reynolds already referred to, an account is given of some interesting experiments especially directed to test the theory of refraction by wind. It was found that "In the direction of the wind, when it was strong, the sound (of an electric bell) could be heard as well with the head on the ground as when raised, even when in a hollow with the bell hidden from view by the slope of the ground; and no advantage whatever was gained either by ascending to an elevation or raising the bell. Thus, with the wind over the grass the sound could be heard 140 yards, and over snow 360 yards, either with the head lifted or on the ground; whereas at right angles to the wind on all occasions the range was extended by raising either the observer or the bell."

"Elevation was found to affect the range of sound against the wind in a much more marked manner than at right angles."

"Over the grass no sound could be heard with the head on the ground at 20 yards from the bell, and at 30 yards it was lost with the head 3 feet from the ground, and its full intensity was lost when standing erect at 30 yards. At 70 yards, when standing erect, the sound was lost at long intervals, and was only faintly heard even then; but it became continuous again when the ear was raised 9 feet from the ground, and it reached its full intensity at an elevation of 12 feet."

Prof. Reynolds thus sums up the results of his experiments:—

1. "When there is no wind, sound proceeding over a rough surface is more intense above than below."

2. "As long as the velocity of the wind is greater above than below, sound is lifted up to windward and is not destroyed."

3. "Under the same circumstances it is brought down to leeward, and hence its range extended at the surface of the ground."

Atmospheric refraction has an important bearing on the audibility of fog-signals, a subject which within the last few years has occupied the attention of two eminent physicists, Prof. Henry in America and Prof. Tyndall in this country. Henry[1] attributes almost all the vagaries of distant sounds to refraction, and has shewn how it is possible by various suppositions as to the motion of the air overhead to explain certain abnormal phenomena which have come under the notice of himself and other observers, while

[1] Report of the Lighthouse Board of the United States for the year 1874.

Tyndall[1], whose investigations have been equally extensive, considers the very limited distances to which sounds are sometimes audible to be due to an actual stopping of the sound by a flocculent condition of the atmosphere arising from unequal heating or moisture. That the latter cause is capable of operating in this direction to a certain extent cannot be doubted. Tyndall has proved by laboratory experiments that the sound of an electric bell may be sensibly intercepted by alternate layers of gases of different densities; and, although it must be admitted that the alternations of density were both more considerable and more abrupt than can well be supposed to occur in the open air, except perhaps in the immediate neighbourhood of the solid ground, some of the observations on fog-signals themselves seem to point directly to the explanation in question.

Thus it was found that the blast of a siren placed on the summit of a cliff overlooking the sea was followed by an echo of gradually diminishing intensity, whose duration sometimes amounted to as much as 15 seconds. This phenomenon was observed "when the sea was of glassy smoothness," and cannot apparently be attributed to any other cause than that assigned to it by Tyndall. It is therefore probable that refraction and acoustical opacity are both concerned in the capricious behaviour of fog-signals. *A priori* we should certainly be disposed to attach the greater importance to refraction, and Reynolds has shewn that some of Tyndall's own observations admit of explanation upon this principle. A failure in *reciprocity* can only be explained in accordance with theory by the action of wind (§ 111).

According to the hypothesis of acoustic clouds, a difference might be expected in the behaviour of sounds of long and of short duration, which it may be worth while to point out here, as it does not appear to have been noticed by any previous writer. Since energy is not lost in reflection and refraction, the intensity of radiation at a given distance from a continuous source of sound (or light) is not altered by an enveloping cloud of spherical form and of uniform density, the loss due to the intervening parts of the cloud being compensated by reflection from those which lie beyond the source. When, however, the sound is of short duration, the intensity at a distance may be very much diminished by the cloud on account of the different distances of its reflecting parts and the

[1] *Phil. Trans.* 1874. *Sound*, 3rd edition, Ch. VII.

consequent drawing out of the sound, although the whole intensity, as measured by the time-integral, may be the same as if there had been no cloud at all. This is perhaps the explanation of Tyndall's observation, that different kinds of signals do not always preserve the same order of effectiveness. In some states of the weather a "howitzer firing a 3-lb. charge commanded a larger range than the whistles, trumpets, or syren," while on other days "the inferiority of the gun to the syren was demonstrated in the clearest manner." It should be noticed, however, that in the same series of experiments it was found that the liability of the sound of a gun "to be quenched or deflected by an opposing wind, so as to be practically useless at a very short distance to windward, is very remarkable." The refraction proper must be the same for all kinds of sounds, but for the reason explained above, the diffraction round the edge of an obstacle may be less effective for the report of a gun than for the sustained note of a siren.

Another point examined by Tyndall was the influence of fog on the propagation of sound. In spite of isolated assertions to the contrary[1], it was generally believed on the authority of Derham that the influence of fog was prejudicial. Tyndall's observations prove satisfactorily that this opinion is erroneous, and that the passage of sound is favoured by the homogeneous condition of the atmosphere which is the usual concomitant of foggy weather. When the air is saturated with moisture, the fall of temperature with elevation according to the law of convective equilibrium is much less rapid than in the case of dry air, on account of the condensation of vapour which then accompanies expansion. From a calculation by Thomson[2] it appears that in warm fog the effect of evaporation and condensation would be to diminish the fall of temperature by one-half. The acoustical refraction due to temperature would thus be lessened, and in other respects no doubt the condition of the air would be favourable to the propagation of sound, provided no obstruction were offered by the suspended particles themselves. In a future chapter we shall investigate the disturbance of plane sonorous waves by a small obstacle, and we shall find that the effect depends upon the ratio of the diameter of the obstacle to the wave-length of the sound.

The reader who is desirous of pursuing this subject may

[1] See for example Desor, *Fortschritte der Physik*, XI. p. 217. 1855.

[2] *Manchester Memoirs*, 1861—62.

consult a paper by Reynolds "On the Refraction of Sound by the Atmosphere[1]," as well as the authorities already referred to. It may be mentioned that Reynolds agrees with Henry in considering refraction to be the really important cause of disturbance, but further observations are much needed. See also § 294.

291. On the assumption that the disturbance at an aperture in a screen is the same as it would have been at the same place in the absence of the screen, we may solve various problems respecting the diffraction of sound by the same methods as are employed for the corresponding problems in physical optics. For example, the disturbance at a distance on the further side of an infinite plane wall, pierced with a circular aperture on which plane waves of sound impinge directly, may be calculated as in the analogous problem of the diffraction pattern formed at the focus of a circular object-glass. Thus in the case of a symmetrical speaking trumpet the sound is a maximum along the axis of the instrument, where all the elementary disturbances issuing from the various points of the plane of the mouth are in one phase. In oblique directions the intensity is less; but it does not fall materially short of the maximum value until the obliquity is such that the difference of distances of the nearest and furthest points of the mouth amounts to about half a wave-length. At a somewhat greater obliquity the mouth may be divided into two parts, of which the nearer gives an aggregate effect equal in magnitude, but opposite in phase, to that of the further; so that the intensity in this direction vanishes. In directions still more oblique the sound revives, increases to an intensity equal to ·017 of that along the axis[2], again diminishes to zero, and so on, the alternations corresponding to the bright and dark rings which surround the central patch of light in the image of a star. If R denote the radius of the mouth, the angle, at which the first silence occurs, is $\sin^{-1}(\cdot 610\,\lambda/R)$. When the diameter of the mouth does not exceed $\frac{1}{2}\lambda$, the elementary disturbances combine without any considerable antagonism of phase, and the intensity is nearly uniform in all directions. It appears that concentration of sound along the axis requires that the ratio $R:\lambda$ should be large, a condition not usually satisfied in the ordinary use of speaking trumpets, whose efficiency depends rather upon an increase in the original volume

[1] *Phil. Trans.* Vol. 166, p. 315. 1876.

[2] Verdet, *Leçons d'optique physique*, t. I. p. 306.

of sound (§ 280). When, however, the vibrations are of very short wave-length, a trumpet of moderate size is capable of effecting a considerable concentration along the axis, as I have myself verified in the case of a hiss.

292. Although such calculations as those referred to in the preceding section are useful as giving us a general idea of the phenomena of diffraction, it must not be forgotten that the auxiliary assumption on which they are founded is by no means strictly and generally true. Thus in the case of a wave directly incident upon a screen the normal velocity in the plane of the aperture is not constant, as has been supposed, but increases from the centre towards the edge, becoming infinite at the edge itself. In order to investigate the conditions by which the actual velocity is determined, let us for the moment suppose that the aperture is filled up. The incident wave $\phi = \cos(nt - kx)$ is then perfectly reflected, and the velocity-potential on the negative side of the screen ($x = 0$) is

$$\phi = \cos(nt - kx) + \cos(nt + kx) \qquad \text{..............(1)},$$

giving, when $x = 0$, $\phi = 2\cos nt$. This corresponds to the vanishing of the normal velocity over the area of the aperture; the completion of the problem requires us to determine a variable normal velocity over the aperture such that the potential due to it (§ 278) shall increase by the constant quantity $2\cos nt$ in crossing from the negative to the positive side; or, since the crossing involves simply a change of sign, to determine a value of the normal velocity over the area of the aperture which shall give on the positive side $\phi = \cos nt$ over the same area. The result of superposing the two motions thus defined satisfies all the conditions of the problem, giving the same velocity and pressure on the two sides of the aperture, and a vanishing normal velocity over the remainder of the screen.

If $P\cos(nt + \epsilon)$ denote the value of $d\phi/dx$ at the various points of the area (S) of the aperture, the condition for determining P and ϵ is by (6) § 278,

$$-\frac{1}{2\pi}\iint P\,\frac{\cos(nt - kr + \epsilon)}{r}\,dS = \cos nt \qquad \text{........(2)},$$

where r denotes the distance between the element dS and any fixed point in the aperture. When P and ϵ are known, the complete value of ϕ for any point on the positive side of the screen is given by

$$\phi = -\frac{1}{2\pi}\iint P\frac{\cos(nt - kr + \epsilon)}{r}\,dS \quad\text{.........................(3)},$$

and for any point on the negative side by

$$\phi = +\frac{1}{2\pi}\iint P\frac{\cos(nt - kr + \epsilon)}{r}\,dS + 2\cos nt\cos kx \quad\text{...... (4)}.$$

The expression of P and ϵ for a finite aperture, even if of circular form, is probably beyond the power of known methods; but in the case where the dimensions are very small in comparison with the wave-length the solution of the problem may be effected for the circle and the ellipse. If r be the distance between two points, both of which are situated in the aperture, kr may be neglected, and we then obtain from (2)

$$\epsilon = 0, \qquad 1 = -\frac{1}{2\pi}\iint P\frac{dS}{r} \quad\text{..... .... (5)},$$

shewing that $-P/2\pi$ is the density of the matter which must be distributed over S in order to produce there the constant potential unity. At a distanee from the opening on the positive side we may consider r as constant, and take

$$\phi = M\frac{\cos(nt - kr)}{r} \quad\text{.................... (6)},$$

where $M = -\frac{1}{2\pi}\iint PdS$, denoting the total quantity of matter which must be supposed to be distributed. It will be shewn on a future page (§ 306) that for an ellipse of semimajor axis a, and eccentricity e,

$$M = a \div F(e) \quad\text{.........................(7)},$$

where F is the symbol of the complete elliptic function of the first kind. In the case of a circle, $F(e) = \frac{1}{2}\pi$, and

$$M = \frac{2a}{\pi} \quad\text{................................(8)}.$$

This result is quite different from that which we should obtain on the hypothesis that the normal velocity in the aperture has the value proper to the primary wave. In that case by (3) § 283

$$\phi = -\frac{\pi a^2}{\lambda}\frac{\sin(nt - kr)}{r} \quad\text{.................(9)}.$$

If there be several small apertures, whose distances apart are much greater than their dimensions, the same method gives

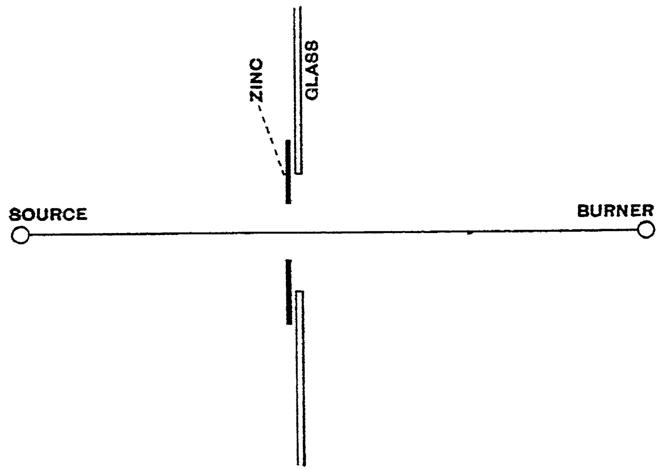
$$\phi = M_1 \frac{\cos(nt - kr_1)}{r_1} + M_2 \frac{\cos(nt - kr_2)}{r_2} + \dots\dots \quad (10).$$

The diffraction of sound is a subject which has attracted but little attention either from mathematicians or experimentalists. Although the general character of the phenomena is well understood, and therefore no very startling discoveries are to be expected, the exact theoretical solution of a few of the simpler problems, which the subject presents, would be interesting; and, even with the present imperfect methods, something probably might be done in the way of experimental examination.

292 *a*. By means of a bird-call giving waves of about 1 cm. wave-length and a high pressure sensitive flame it is possible to imitate many interesting optical experiments. With this apparatus the shadow of an obstacle so small as the hand may be made apparent at a distance of several feet.

An experiment shewing the antagonism between the parts of a wave corresponding to the first and second Fresnel's zones (§ 283)

Fig. 57 *a*.



is very effective. A large glass screen (Fig. 57 *a*) is perforated with a circular hole 20 cm. in diameter, and is so situated between the source of sound and the burner that the aperture corresponds to the first two zones. By means of a zinc plate, held close to the glass, the aperture may be reduced to 14 cm., and then admits only the first zone. If the adjustments are well made, the flame, unaffected by the waves which penetrate the larger aperture, flares violently when the aperture is further restricted by the zinc plate. Or, as an alternative, the perforated plate may be replaced by a disc of 14 cm. diameter, which allows the second zone to be operative while the first is blocked off.

If a, b denote the distances of the screen from the source and from the point of observation, the external radius ρ of the nth zone is given by

$$\sqrt{(a^2+\rho^2)}+\sqrt{(b^2+\rho^2)}-a-b=\tfrac{1}{2}n\lambda,$$

or approximately

$$\rho^2=n\lambda\frac{ab}{a+b} \qquad\qquad (1).$$

When $a=b$,

$$\rho^2=\tfrac{1}{2}n\lambda a \qquad\qquad (2).$$

With the apertures specified above, $\rho^2=49$ for $n=1$; $\rho^2=100$ for $n=2$; so that

$$\lambda a=100,$$

the measurements being in centimetres. This gives the suitable distances when λ is known. In an actual experiment $\lambda=1{\cdot}2$, $a=83$.

The process of augmenting the total effect by blocking out the alternate zones may be carried much further. Thus when a suitable circular grating, cut out of a sheet of zinc, is interposed between the source of sound and the flame, the effect is many times greater than when the screen is removed altogether[1]. As in Soret's corresponding optical experiment, the grating plays the part of a condensing lens.

The focal length of the lens is determined by (1), which may be written in the form

$$\frac{1}{f}=\frac{1}{a}+\frac{1}{b}=\frac{n\lambda}{\rho^2} \qquad\qquad (3);$$

so that

$$f=\rho^2/n\lambda \qquad\qquad (4).$$

In an actual grating constructed upon this plan eight zones—the first, third, fifth &c.—are occupied by metal. The radius of the first zone, or central circle, is 7·6 cm., so that $\rho^2/n=58$. Thus, if $\lambda=1{\cdot}2$ cm., $f=48$ cm. If a and b are equal, each must be 96 cm

[1] "Diffraction of Sound," *Proc. Roy. Inst.* Jan. 20, 1888.

The condition of things at the centre of the shadow of a circular disc is still more easily investigated. If we construct in imagination a system of zones beginning with the circular edge of the disc, we see, as in § 283, that the total effect at a point upon the axis, being represented by the half of that of the first zone, is the same as if no obstacle at all were interposed. This analogue of a famous optical phenomenon is readily exhibited[1]. In one experiment a glass disc 38 cm. in diameter was employed, and its distances from the source and from the flame were respectively 70 cm. and 25 cm. A bird-call giving a pure tone ($\lambda = 1{\cdot}5$ cm.) is suitable, but may be replaced by a toy reed or other source giving short, though not necessarily simple, waves. In private work the ear furnished with a rubber tube may be used instead of a sensitive flame.

The region of no sensible shadow, though not confined to a mathematical point upon the axis, is of small dimensions, and a very moderate movement of the disc in its own plane suffices to reduce the flame to quiet Immediately surrounding the central spot there is a ring of almost complete silence, and beyond that again a moderate revival of effect. The calculation of the intensity of sound at points off the axis of symmetry is too complicated to be entered upon here. The results obtained by Lommel[2] may be readily adapted to the acoustical problem. With the data specified above the diameter of the silent ring immediately surrounding the central region of activity is about 1·7 cm.

293. The value of a function ϕ which satisfies $\nabla^2\phi = 0$ throughout the interior of a simply-connected closed space S can be expressed as the potential of matter distributed over the surface of S. In a certain sense this is also true of the class of functions with which we are now occupied, which satisfy $\nabla^2\phi + k^2\phi = 0$. The following is Helmholtz's proof[3]. By Green's theorem, if ϕ and ψ denote any two functions of x, y, z,

$$\iint \phi \frac{d\psi}{dn} dS - \iiint \phi \nabla^2\psi \, dV = \iint \psi \frac{d\phi}{dn} dS - \iiint \psi \nabla^2\phi \, dV.$$

[1] "Acoustical Observations," *Phil. Mag.* Vol. IX. p. 281, 1880; *Proc. Roy. Inst.* loc. cit.

[2] *Abh. der bayer. Akad. der Wiss.* ii. Cl., XV. Bd., ii. Abth. See also *Encyclopædia Britannica*, Article "Wave Theory."

[3] *Theorie der Luftschwingungen in Röhren mit offenen Enden.* Crelle, Bd. LVII. p. 1. 1860.

To each side add $-\iiint k^2\phi\psi\, dV$; then if

$$a^2(\nabla^2\phi + k^2\phi) + \Phi = 0, \qquad a^2(\nabla^2\psi + k^2\psi) + \Psi = 0,$$

$$a^2\iint \phi \frac{d\psi}{dn}\, dS + \iiint \phi\, \Psi\, dV = a^2 \iint \psi \frac{d\phi}{dn}\, dS + \iiint \psi\, \Phi\, dV \quad \ldots(1).$$

If Φ and Ψ vanish within S, we have simply

$$\iint \phi \frac{d\psi}{dn} dS = \iint \psi \frac{d\phi}{dn}\, dS \quad \ldots\ldots\ldots\ldots(2).$$

Suppose, however, that

$$\phi = \frac{e^{-ikr}}{r} \quad \ldots\ldots\ldots\ldots(3),$$

where r represents the distance of any point from a fixed origin O within S. At all points, except O, Φ vanishes; and the last term in (1) becomes

$$\iiint \psi\, \Phi\, dV = -a^2 \iiint \psi\, \nabla^2\left(\frac{1}{r}\right) dV = 4\pi a^2 \psi,$$

ψ referring to the point O. Thus

$$4\pi\psi = \iint \frac{d\psi}{dn} \frac{e^{-ikr}}{r}\, dS - \iint \psi \frac{d}{dn}\left(\frac{e^{-ikr}}{r}\right) dS + \frac{1}{a^2}\iiint \Psi \frac{e^{-ikr}}{r}\, dV \quad \ldots\ldots\ldots(4),$$

in which, if Ψ vanish, we have an expression for the value of Ψ at any interior point O in terms of the surface values of ψ and of $d\psi/dn$. In the case of the common potential, on which we fall back by putting $k = 0$, ψ would be determined by the surface values of $d\psi/dn$ only. But with k finite, this law ceases to be universally true. For a given space S there is, as in the case investigated in § 267, a series of determinate values of k, corresponding to the periods of the possible modes of simple harmonic vibration which may take place within a closed rigid envelope having the form of S. With any of these values of k, it is obvious that ψ cannot be determined by its normal variation over S, and the fact that it satisfies throughout S the equation $\nabla^2\psi + k^2\psi = 0$. But if the supposed value of k do not coincide with one of the series, then the problem is determinate; for the difference of any two possible solutions, if finite, would satisfy the condition of giving no normal velocity over S, a condition which by hypothesis cannot be satisfied with the assumed value of k.

If the dimensions of the space S be very small in comparison with $\lambda (= 2\pi/k)$, e^{-ikr} may be replaced by unity; and we learn that ψ differs but little from a function which satisfies throughout S the equation $\nabla^2\phi = 0$.

294. On his extension of Green's theorem (1) Helmholtz founds his proof of the important theorem contained in the following statement: *If in a space filled with air which is partly bounded by finitely extended fixed bodies and is partly unbounded, sound waves be excited at any point* A, *the resulting velocity-potential at a second point* B *is the same both in magnitude and phase, as it would have been at* A, *had* B *been the source of the sound.*

If the equation

$$a^2 \iint \left(\phi \frac{d\psi}{dn} - \psi \frac{d\phi}{dn}\right) dS = \iiint (\psi\Phi - \phi\Psi)\, dV \ldots\ldots(1),$$

in which ϕ and ψ are arbitrary functions, and

$$\Phi = -a^2(\nabla^2\phi + k^2\phi), \qquad \Psi = -a^2(\nabla^2\psi + k^2\psi),$$

be applied to a space completely enclosed by a rigid boundary and containing any number of detached rigid fixed bodies, and if ϕ, ψ be velocity-potentials due to sources within S, we get

$$\iiint (\psi\Phi - \phi\Psi)\, dV = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Thus, if ϕ be due to a source concentrated in one point A, $\Phi = 0$ except at that point, and

$$\iiint \psi\Phi\, dV = \psi_A \iiint \Phi\, dV,$$

where $\iiint \Phi\, dV$ represents the intensity of the source. Similarly, if ψ be due to a source situated at B,

$$\iiint \phi\Psi\, dV = \phi_B \iiint \Psi\, dV.$$

Accordingly, if the sources be finite and equal, so that

$$\iiint \Phi\, dV = \iiint \Psi\, dV \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

it follows that

$$\psi_A = \phi_B \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

which is the symbolical statement of Helmholtz's theorem.

If the space S extend to infinity, the surface integral still vanishes, and the result is the same; but it is not necessary to go into detail here, as this theorem is included in the vastly more general principle of reciprocity established in Chapter V. The investigation there given shews that the principle remains true in the presence of dissipative forces, provided that these arise from resistances varying as the first power of the velocity, that the fluid need not be homogeneous, nor the neighbouring bodies rigid or fixed. In the application to infinite space, all obscurity is avoided by supposing the vibrations to be slowly dissipated after having escaped to a distance from A and B, the sources under contemplation.

The reader must carefully remember that in this theorem equal sources of sound are those produced by the periodic introduction and abstraction of equal quantities of fluid, or something whose effect is the same, and that equal sources do not necessarily evolve equal amounts of energy in equal times. For instance, a source close to the surface of a large obstacle emits twice as much energy as an equal source situated in the open.

As an example of the use of this theorem we may take the case of a hearing, or speaking, trumpet consisting of a conical tube, whose efficiency is thus seen to be the same, whether a sound produced at a point outside is observed at the vertex of the cone, or a source of equal strength situated at the vertex is observed at the external point.

It is important also to bear in mind that Helmholtz's form of the reciprocity theorem is applicable only to *simple* sources of sound, which in the absence of obstacles would generate symmetrical waves. As we shall see more clearly in a subsequent chapter, it is possible to have sources of sound, which, though concentrated in an infinitely small region, do not satisfy this condition. It will be sufficient here to consider the case of *double* sources, for which the modified reciprocal theorem has an interest of its own.

Let us suppose that A is a simple source, giving at a point B the potential $-\psi$, and that A' is an equal and opposite source situated at a neighbouring point, whose potential at B is $\psi + \Delta\psi$. If both sources be in operation simultaneously, the potential at B is $\Delta\psi$. Now let us suppose that there is a simple source at B, whose intensity and phase are the same as those of the sources at A and A'; the resulting potential at A is ψ, and at A' $\psi + \Delta\psi$. If the distance AA' be denoted by h, and be supposed to diminish without limit, the velocity of the fluid at A in the direction AA' is the limit of $\Delta\psi/h$. Hence, if we define a unit double source as the limit of two equal and opposite simple sources whose distance is diminished, and whose intensity is increased without limit in such a manner that the product of the intensity and the distance is the same as for two unit simple sources placed at the unit distance apart, we may say that the velocity of the fluid at A in direction AA' due to a unit simple source at B is numerically equal to the potential at B due to a unit *double* source at A, whose axis is in the direction AA'. This theorem, be it observed, is true in spite of any obstacles or reflectors that may exist in the neighbourhood of the sources.

Again, if AA' and BB' represent two unit double sources of the same phase, the velocity at B in direction BB' due to the source AA' is the same as the velocity at A in direction AA' due to the source BB'. These and other results of a like character may also be obtained on an immediate application of the general principle of § 108. These examples will be sufficient to shew that in applying the principle of reciprocity it is necessary to attend to the character of the sources. A double source, situated in an open space, is inaudible from any point in its equatorial plane, but it does not follow that a simple source in the equatorial plane is inaudible from the position of the double source. On this principle, I believe, may be explained a curious experiment by Tyndall[1], in which there was an apparent failure of reciprocity[2]. The source of sound employed was a reed of very high pitch, mounted in a tube, along whose axis the intensity was considerably greater than in oblique directions.

295. The kinetic energy T of the motion within a closed surface S is expressed by

$$T = \tfrac{1}{2}\rho_0 \iiint \Sigma \left(\frac{d\phi}{dx}\right)^2 dV \qquad\qquad (1);$$

[1] *Proceedings of the Royal Institution*, Jan. 1875. Also Tyndall, *On Sound*, 3rd edition, p. 405.

[2] See a note "On the Application of the Principle of Reciprocity to Acoustics." *Royal Society Proceedings*, Vol. xxv. p. 118, 1876, or *Phil. Mag.* (5), III. p. 300.

so that $$\frac{dT}{dt} = \rho_0 \iiint \Sigma \frac{d\phi}{dx} \frac{d\dot{\phi}}{dx} dV$$

$$= \rho_0 \iint \dot{\phi} \frac{d\phi}{dn} dS - \rho_0 \iiint \dot{\phi} \nabla^2 \phi dV \qquad (2),$$

by Green's theorem. For the potential energy V_1 we have by (12) § 245

$$V_1 = \frac{\rho_0}{2a^2} \iiint \dot{\phi}^2 dV \qquad (3),$$

whence $$\frac{dV_1}{dt} = \frac{\rho_0}{a^2} \iiint \dot{\phi} \ddot{\phi} dV = \frac{\rho_0}{a^2} \iiint \left\{ \frac{dR}{dt} + a^2 \nabla^2 \phi \right\} \dot{\phi} dV \ldots (4),$$

by the general equation of motion (9) § 244. Thus, if E denote the whole energy within the space S,

$$\frac{dE}{dt} = \rho_0 \iint \dot{\phi} \frac{d\phi}{dn} dS + \frac{\rho_0}{a^2} \iiint \frac{dR}{dt} \dot{\phi} dV \qquad (5),$$

of which the first term represents the work transmitted across the boundary S, and the second represents the work done by internal sources of sound.

If the boundary S be a fixed rigid envelope, and there be no internal sources, E retains its initial value throughout the motion. This principle has been applied by Kirchhoff[1] to prove the determinateness of the motion resulting from given arbitrary initial conditions. Since every element of E is positive, there can be no motion within S, if E be zero. Now, if there were two motions possible corresponding to the same initial conditions, their difference would be a motion for which the initial value of E was zero; but by what has just been said such a motion cannot exist.

[1] *Vorlesungen über Math. Physik*, p. 311.

CHAPTER XV.

FURTHER APPLICATION OF THE GENERAL EQUATIONS.

296. WHEN a train of plane waves, otherwise unimpeded, impinges upon a space occupied by matter, whose mechanical properties differ from those of the surrounding medium, secondary waves are thrown off, which may be regarded as a disturbance due to the change in the nature of the medium—a point of view more especially appropriate, when the *region of disturbance*, as well as the alteration of mechanical properties, is small. If the medium and the obstacle be fluid, the mechanical properties spoken of are two—the *compressibility* and the *density*: no account is here taken of friction or viscosity. In the chapter on spherical harmonic analysis we shall consider the problem here proposed on the supposition that the obstacle is spherical, without any restriction as to the smallness of the change of mechanical properties; in the present investigation the form of the obstacle is arbitrary, but we assume that the squares and higher powers of the changes of mechanical properties may be omitted.

If ξ, η, ζ denote the displacements parallel to the axes of co-ordinates of the particle, whose equilibrium position is defined by x, y, z, and if σ be the normal density, and m the constant of compressibility so that $\delta p = ms$, the equations of motion are

$$\sigma \frac{d^2\xi}{dt^2} + \frac{d\,(ms)}{dx} = 0 \quad \text{.....................(1)},$$

and two similar equations in η and ζ. On the assumption that the whole motion is proportional to e^{ikat}, where as usual $k = 2\pi/\lambda$, and (§ 244) $a^2 = m/\sigma$, (1) may be written

$$\frac{d\,(ms)}{dx} - \sigma k^2 a^2 \xi = 0 \quad \text{.....................(2)}.$$

The relation between the condensation s, and the displacements ξ, η, ζ, obtained by integrating (3) § 238 with respect to the time, is

$$-s=\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

For the system of primary waves advancing in the direction of $-x$, η and ζ vanish; if ξ_0, s_0 be the values of ξ and s, and m_0, σ_0 be the mechanical constants for the undisturbed medium, we have as in (2)

$$\frac{d(m_0 s_0)}{dx}-\sigma_0 k^2 a^2 \xi_0=0\quad\ldots\ldots\ldots\ldots\ldots\ldots(4);$$

but ξ_0, s_0 do not satisfy (2) at the region of disturbance on account of the variation in m and σ, which occurs there. Let us assume that the complete values are $\xi_0+\xi$, η, ζ, s_0+s[1], and substitute in (2). Then taking account of (4), we get

$$\frac{d(ms)}{dx}-\sigma k^2 a^2\xi+(m-m_0)\frac{ds_0}{dx}+s_0\frac{dm}{dx}-(\sigma-\sigma_0)k^2a^2\xi_0=0,$$

or, as it may also be written,

$$\frac{d}{dx}(ms)-\sigma k^2a^2\xi+\frac{d}{dx}(\Delta m.s_0)-\Delta\sigma.k^2a^2\xi_0=0\;\ldots\ldots\;(5),$$

if Δm, $\Delta\sigma$ stand respectively for $m-m_0$, $\sigma-\sigma_0$. The equations in η and ζ are in like manner

$$\left.\begin{aligned}\frac{d}{dy}(ms)-\sigma k^2a^2\eta+\frac{d}{dy}(\Delta m.s_0)&=0\\ \frac{d}{dz}(ms)-\sigma k^2a^2\zeta+\frac{d}{dz}(\Delta m.s_0)&=0\end{aligned}\right\}\quad\ldots\ldots\ldots(6).$$

It is to be observed that Δm, $\Delta\sigma$ vanish, except through a small space, which is regarded as the region of disturbance; ξ, η, ζ, s, being the result of the disturbance are to be treated as small quantities of the order Δm, $\Delta\sigma$; so that in our approximate analysis the variations of m and σ in the first two terms of (5) and (6) are to be neglected, being there multiplied by small quantities. We thus obtain from (5) and (6) by differentiation and addition, with use of (3), as the differential equation in s,

$$\nabla^2(ms)+k^2ms=k^2a^2\frac{d}{dx}(\Delta\sigma.\xi_0)-\nabla^2(\Delta m.s_0)\;\ldots\ldots(7).$$

[1] [This notation was adopted for brevity. It might be clearer to take $\xi=\xi_0+\Delta\xi$, $s=s_0+\Delta s$, &c.; so that ξ, s, &c. should retain their former meanings.]

As in § 277, the solution of (7) is

$$4\pi m\, s = \iiint \frac{e^{-ikr}}{r} \left\{ \nabla^2 (\Delta m . s_0) - k^2 a^2 \frac{d}{dx} (\Delta\sigma . \xi_0) \right\} dV \ldots\ldots (8),$$

in which the integration extends over a volume completely including the region of disturbance. The integrals in (8) may be transformed with the aid of Green's theorem. Calling the two parts respectively P and Q, we have

$$P = \iiint \frac{e^{-ikr}}{r} \nabla^2 (\Delta m . s_0)\, dV = \iiint \Delta m . s_0 \nabla^2 \left(\frac{e^{-ikr}}{r} \right) dV$$

$$+ \iint \left\{ \frac{e^{-ikr}}{r} \frac{d}{dn} (\Delta m . s_0) - \Delta m . s_0 \frac{d}{dn} \left(\frac{e^{-ikr}}{r} \right) \right\} dS,$$

where S denotes the surface of the space through which the triple integration extends. Now on S, Δm and $\frac{d}{dn}(\Delta m . s_0)$ vanish, so that both the surface integrals disappear. Moreover

$$\nabla^2 \left(\frac{e^{-ikr}}{r} \right) = \frac{1}{r} \frac{d^2}{dr^2} e^{-ikr} = -k^2 \frac{e^{-ikr}}{r};$$

and thus

$$P = -k^2 \iiint \frac{e^{-ikr}}{r} \Delta m . s_0\, dV \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

If the region of disturbance be small in comparison with λ, we may write

$$P = -k^2 s_0 \frac{e^{-ikr}}{r} \iiint \Delta m dV \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

In like manner for the second integral in (8), we find

$$Q = -k^2 a^2 \iiint \frac{e^{-ikr}}{r} \frac{d}{dx} (\Delta\sigma . \xi_0)\, dV$$

$$= k^2 a^2 \iiint \Delta\sigma . \xi_0 \frac{d}{dx} \left(\frac{e^{-ikr}}{r} \right) dV = ik^3 a^2 \xi_0 \mu \frac{e^{-ikr}}{r} \iiint \Delta\sigma dV \ldots (11),$$

where μ denotes the cosine of the angle between x and r. The linear dimension of the region of disturbance is neglected in comparison with λ, and λ is neglected in comparison with r.

If T be the volume of the space through which Δm, $\Delta\sigma$ are sensible, we may write

$$\iiint \Delta m\, dV = T . \Delta m, \qquad \iiint \Delta\sigma\, dV = T . \Delta\sigma,$$

if on the right-hand sides Δm, $\Delta\sigma$ refer to the *mean values* of the variations in question. Thus from (8)

$$s = -\frac{k^2 T e^{-ikr}}{4\pi m\, r}\left\{\Delta m . s_0 - ika^2 \Delta\sigma . \xi_0\, \mu\right\} \;\ldots\ldots(12).$$

To express ξ_0 in terms of s_0, we have from (3), $\xi_0 = -\int s_0\, dx$; and thus, if the condensation for the primary waves be $s_0 = e^{ik(at+x)}$, $ik\xi_0 = -s_0$, and (12) may be put into the form

$$s : s_0 = -\frac{\pi T e^{-ikr}}{\lambda^2 r}\left\{\frac{\Delta m}{m} + \frac{\Delta\sigma}{\sigma}\mu\right\} \;\ldots\ldots\ldots\ldots(13),$$

in which s_0 denotes the condensation of the primary waves at the place of disturbance at time t, and s denotes the condensation of the secondary waves at the same time at a distance r from the disturbance. Since the difference of phase represented by the factor e^{-ikr} corresponds simply to the distance r, we may consider that a simple reversal of phase occurs at the place of disturbance. The amplitude of the secondary waves is inversely proportional to the distance r, and to the *square* of the wave-length λ. Of the two terms expressed in (13) the first is symmetrical in all directions round the place of disturbance, while the second varies as the cosine of the angle between the primary and the secondary rays. Thus a place at which m varies behaves as a *simple* source, and a place at which σ varies behaves as a *double* source (§ 294).

That the secondary disturbance must vary as λ^{-2} may be proved immediately by the method of dimensions. Δm and $\Delta\sigma$ being given, the amplitude is necessarily proportional to T, and in accordance with the principle of energy must also vary inversely as r. Now the only quantities (dependent upon space, time, and mass) of which the ratio of amplitudes can be a function, are T, r, λ, a (the velocity of sound), and σ, of which the last cannot occur in the expression of a simple ratio, as it is the only one of the five which involves a reference to mass. Of the remaining four quantities T, r, λ, and a, the last is the only one which involves a reference to time, and is therefore excluded. We are left with T, r, and λ, of which the only combination varying as Tr^{-1}, and independent of the unit of length, is $Tr^{-1}\lambda^{-2}$.[1]

An interesting application of the results of this section may be made to explain what have been called *harmonic echoes*[2].

[1] "On the Light from the Sky," *Phil. Mag.* Feb. 1871, and "On the scattering of Light by small Particles," *Phil. Mag.* June, 1871.

[2] *Nature*, 1873, VIII. 319.

If the primary sound be a compound musical note, the various component tones are scattered in unlike proportions. The octave, for example, is sixteen times stronger relatively to the fundamental tone in the secondary than it was in the primary sound. There is thus no difficulty in understanding how it may happen that echoes returned from such reflecting bodies as groups of trees may be raised an octave. The phenomenon has also a complementary side. If a number of small bodies lie in the path of waves of sound, the vibrations which issue from them in all directions are at the expense of the energy of the main stream, and where the sound is compound, the exaltation of the higher harmonics in the scattered waves involves a proportional deficiency of them in the direct wave after passing the obstacles. This is perhaps the explanation of certain echoes which are said to return a sound graver than the original; for it is known that the pitch of a pure tone is apt to be estimated too low. But the evidence is conflicting, and the whole subject requires further careful experimental investigation; it may be commended to the attention of those who may have the necessary opportunities. While an alteration in the *character* of a sound is easily intelligible, and must indeed generally happen to a limited extent, a change in the pitch of a simple tone would be a violation of the law of forced vibrations, and hardly to be reconciled with theoretical ideas.

In obtaining (13) we have neglected the effect of the variable nature of the medium *on the disturbance.* When the disturbance on this supposition is thoroughly known, we might approximate again in the same manner. The additional terms so obtained would be necessarily of the second order in Δm, $\Delta\sigma$, so that our expressions are in all cases correct as far as the first powers of those quantities.

Even when the region of disturbance is not small in comparison with λ, the same method is applicable, provided the squares of Δm, $\Delta\sigma$ be really negligible. The total effect of any obstacle may then be calculated by integration from those of its parts. In this way we may trace the transition from a small region of disturbance whose *surface* does not come into consideration, to a thin plate of a few or of a great many square wave-lengths in area, which will ultimately reflect according to the regular optical law. But if the obstacle be at all elongated in the direction of the primary rays, this method of calculation soon ceases to be practically available, because, even although the change of mechanical properties be very small, the interaction of the various parts of the obstacle cannot be left out of account. This caution is more especially needed in dealing with the case of light, where the wave-length is so exceedingly small in comparison with the dimensions of ordinary obstacles.

297. In some degree similar to the effect produced by a change in the mechanical properties of a small region of the fluid, is that which ensues when the square of the motion rises anywhere to such importance that it can be no longer neglected. $\nabla^2\phi + k^2\phi$ then acquires a finite value dependent upon the square of the motion. Such places therefore act like sources of sound; the periods of the sources including the submultiples of the original period. Thus any part of space, at which the intensity accumulates to a sufficient extent, becomes itself a secondary source, emitting the harmonic tones of the primary sound. If there be two primary sounds of sufficient intensity, the secondary vibrations have frequencies which are the sums and differences of the frequencies of the primaries (§ 68)[1].

298. The pitch of a sound is liable to modification when the source and the recipient are in relative motion. It is clear, for instance, that an observer approaching a fixed source will meet the waves with a frequency exceeding that proper to the sound, by the number of wave-lengths passed over in a second of time. Thus if v be the velocity of the observer and a that of sound, the frequency is altered in the ratio $a \pm v : a$, according as the motion is towards or from the source. Since the alteration of pitch is constant, a musical performance would still be heard in tune, although in the second case, when a and v are nearly equal, the fall in pitch would be so great as to destroy all musical character. If we could suppose v to be greater than a, a sound produced after the motion had begun would never reach the observer, but sounds previously excited would be gradually overtaken and heard in the reverse of the natural order. If $v = 2a$, the observer would hear a musical piece in correct time and tune, but *backwards*.

Corresponding results ensue when the source is in motion and the observer at rest; the alteration depending only on the relative motion in the line of hearing. If the source and the observer move with the same velocity there is no alteration of frequency, whether

[1] Helmholtz über Combinationstöne. Pogg. *Ann.* Bd. XCIX. s. 497. 1856.

the medium be in motion, or not. With a relative motion of 40 miles [64 kilometres] per hour the alteration of pitch is very conspicuous, amounting to about a semitone. The whistle of a locomotive is heard too high as it approaches, and too low as it recedes from an observer at a station, changing rather suddenly at the moment of passage.

The principle of the alteration of pitch by relative motion was first enunciated by Doppler[1], and is often called Doppler's principle. Strangely enough its legitimacy was disputed by Petzval[2], whose objection was the result of a confusion between two perfectly distinct cases, that in which there is a relative motion of the source and recipient, and that in which the medium is in motion while the source and the recipient are at rest. In the latter case the circumstances are mechanically the same as if the medium were at rest and the source and the recipient had a common motion, and therefore by Doppler's principle no change of pitch is to be expected.

Doppler's principle has been experimentally verified by Buijs Ballot[3] and Scott Russell, who examined the alterations of pitch of musical instruments carried on locomotives. A laboratory instrument for proving the change of pitch due to motion has been invented by Mach[4]. It consists of a tube six feet [183 cm.] in length, capable of turning about an axis at its centre. At one end is placed a small whistle or reed, which is blown by wind forced along the axis of the tube. An observer situated in the plane of rotation hears a note of fluctuating pitch, but if he places himself in the prolongation of the axis of rotation, the sound becomes steady. Perhaps the simplest experiment is that described by König[5]. Two c'' tuning-forks mounted on resonance cases are prepared to give with each other four beats per second. If the graver of the forks be made to approach the ear while the other remains at rest, one beat is *lost* for each two feet [61 cm.] of approach; if, however, it be the more acute of the two forks which approaches the ear, one beat is *gained* in the same distance.

1 Theorie des farbigen Lichtes der Doppelsterne. Prag, 1842. See Pisko, *Die neueren Apparate der Akustik.* Wien, 1865.

2 *Wien. Ber.* VIII. 134. 1852. *Fortschritte der Physik,* VIII. 167.

3 Pogg. *Ann.* LXVI. p. 321.

4 Pogg. *Ann.* CXII. p. 66, 1861, and CXVI. p. 333, 1862.

5 König's *Catalogue des Appareils d'Acoustique.* Paris, 1865.

A modification of this experiment due to Mayer[1] may also be noticed. In this case one fork excites the vibrations of a second in unison with itself, the excitation being made apparent by a small pendulum, whose bob rests against the extremity of one of the prongs. If the exciting fork be at rest, the effect is apparent up to a distance of 60 feet [1830 cm.], but it ceases when the exciting fork is moved rapidly to or fro in the direction of the line joining the two forks.

There is some difficulty in treating mathematically the problem of a moving source, arising from the fact that any practical source acts also as an obstacle. Thus in the case of a bell carried through the air, we should require to solve a problem difficult enough without including the vibrations at all. But the solution of such a problem, even if it could be obtained, would throw no particular light on Doppler's law, and we may therefore advantageously simplify the question by idealizing the bell into a simple source of sound.

In § 147 we considered the problem of a moving source of disturbance in the case of a stretched string. The theory for aerial waves in one dimension is precisely similar, but for the general case of three dimensions some extension is necessary, in order to take account of the possibility of a motion across the direction of the sound rays. From §§ 273, 276 it appears that the effect at any point O of a source of sound is the same, whether the source be at rest, or whether it move in any manner on the surface of a sphere described about O as centre. If the source move in such a manner as to change its distance (r) from O, its effect is altered in two ways. Not only is the *phase* of the disturbance on arrival at O affected by the variation of distance, but the *amplitude* also undergoes a change. The latter complication however may be put out of account, if we limit ourselves to the case in which the source is sufficiently distant. On this understanding we may assert that the effect at O of a disturbance generated at time t and at distance r is the same as that of a similar disturbance generated at the time $t + \delta t$ and at the distance $r - a\,\delta t$. In the case of a periodic disturbance a velocity of approach (v) is equivalent to an increase of frequency in the ratio $a : a + v$.

299. We will now investigate the forced vibrations of the air contained within a rectangular chamber, due to internal sources

[1] *Phil. Mag.* (4), XLIII. p. 278, 1872.

of sound. By § 267 it appears that the result at time t of an initial condensation confined to the neighbourhood of the point ξ, η, ζ is

$$\dot{\phi} = \Sigma\Sigma\Sigma\, ka\, B_{pqr} \cos kat \cos\left(p\,\frac{\pi x}{\alpha}\right) \cos\left(q\,\frac{\pi y}{\beta}\right) \cos\left(r\,\frac{\pi z}{\gamma}\right) \quad \ldots(1),$$

where

$$ka\, B_{pqr} = \frac{8}{\alpha\beta\gamma} \cos\left(p\,\frac{\pi\xi}{\alpha}\right) \cos\left(q\,\frac{\pi\eta}{\beta}\right) \cos\left(r\,\frac{\pi\zeta}{\gamma}\right) \iiint \dot{\phi}_0\, dx\, dy\, dz \ldots(2),$$

from which the effect of an impressed force may be deduced, as in § 276. The disturbance $\iiint \dot{\phi}_{t'}\, dx\, dy\, dz$ communicated at time t' being denoted by $\iiint \Phi(t')\, dt'\, dx\, dy\, dz$, or $\Phi_1(t')\, dt'$, the resultant disturbance at time t is

$$\dot{\phi} = \frac{8}{\alpha\beta\gamma} \Sigma\Sigma\Sigma \cos\left(p\,\frac{\pi x}{\alpha}\right) \cos\left(q\,\frac{\pi y}{\beta}\right) \cos\left(r\,\frac{\pi z}{\gamma}\right) \times$$

$$\cos\left(p\,\frac{\pi\xi}{\alpha}\right) \cos\left(q\,\frac{\pi\eta}{\beta}\right) \cos\left(r\,\frac{\pi\zeta}{\gamma}\right) \int_{-\infty}^{t} \Phi_1(t') \cos ka(t-t')\, dt' \ldots(3).$$

The symmetry of this expression with respect to x, y, z and ξ, η, ζ is an example of the principle of reciprocity (§ 107).

In the case of a harmonic force, for which $\Phi_1(t') = A \cos mat'$ we have to consider the value of

$$\int_{-\infty}^{t} \cos mat' \cos ka(t-t')\, dt' \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Strictly speaking, this integral has no definite value; but, if we wish for the expression of the forced vibrations only, we must omit the integrated function at the lower limit, as may be seen by supposing the introduction of very small dissipative forces. We thus obtain

$$\int_{-\infty}^{t} \Phi_1(t') \cos ka\,(t-t')\, dt' = A\,\frac{ma \sin mat}{(m^2-k^2)\,a^2} \quad \ldots\ldots(5).$$

As might have been predicted, the expressions become infinite in case of a coincidence between the period of the source and one of the natural periods of the chamber. Any particular normal vibration will not be excited, if the source be situated on one of its loops.

The effect of a multiplicity of sources may readily be inferred by summation or integration.

300. When sound is excited within a cylindrical pipe, the simplest kind of excitation that we can suppose is by the forced vibration of a piston. In this case the waves are plane from the beginning. But it is important also to inquire what happens when the source, instead of being uniformly diffused over the section, is concentrated in one point of it. If we assume (what, however, is not unreservedly true) that at a sufficient distance from the source the waves become plane, the law of reciprocity is sufficient to guide us to the desired information.

Let A be a simple source in an unlimited tube, B, B' two points of the same normal section in the region of plane waves. *Ex hypothesi*, the potentials at B and B' due to the source A are the same, and accordingly by the law of reciprocity equal sources at B and B' would give the same potential at A. From this it follows that the effect of any source is the same at a distance, as if the source were uniformly diffused over the section which passes through it. For example, if B and B' were equal sources in opposite phases, the disturbance at A would be nil.

The energy emitted by a simple source situated within a tube may now be calculated. If the section of the tube be σ, and the source such that in the open the potential due to it would be

$$\phi = -\frac{A}{4\pi} \cdot \frac{\cos k(at - r)}{r} \quad \text{...............(1)},$$

the velocity-potential at a distance within the tube will be the same as if the cause of the disturbance were the motion of a piston at the origin, giving the same total displacement, and the energy emitted will also be the same. Now from (1)

$$2\pi r^2 \frac{d\phi}{dr} = \tfrac{1}{2} A \cos kat \text{ ultimately,}$$

and therefore if ψ be the velocity-potential of the plane waves in the tube (supposed parallel to z), we may take

$$\sigma \frac{d\psi}{dz} = \tfrac{1}{2} A \cos k(at - z) \quad \text{..............(2)},$$

corresponding to which

$$\psi = -\frac{aA}{2\sigma} \cos k(at - z) \quad \text{..................(3)}.$$

Hence, as in § 245, the energy (W) emitted *on each side* of the source is given by

$$\frac{dW}{dt} = \sigma\left(-\rho\,\dot{\psi}\,\frac{d\psi}{dz}\right)_{z=0} = \frac{\rho a A^2}{4\sigma}\cos^2 kat\,;$$

so that in the long run

$$W = \frac{\rho a A^2}{8\sigma}\,t \qquad\qquad (4).$$

If the tube be stopped by an immovable piston placed close to the source, the whole energy is emitted in one direction; but this is not all. In consequence of the doubled pressure, twice as much energy as before is developed, and thus in this case

$$W = \frac{\rho a A^2}{2\sigma}\,t \qquad\qquad (5).$$

The narrower the tube, the greater is the energy issuing from a given source. It is interesting to compare the efficiency of a source at the stopped end of a cylindrical tube with that of an equal source situated at the vertex of a cone. From § 280 we have in the latter case,

$$W' = \rho\,\frac{k^2 a A^2}{2\omega}\,t \qquad\qquad (6),$$

so that

$$W : W' = \omega : k^2\sigma \qquad\qquad (7).$$

The energies emitted in the two cases are the same when $\omega = k^2\sigma$, that is, when the section of the cylinder is equal to the area cut off by the cone from a sphere of radius k^{-1}.

301. We have now to examine how far it is true that vibrations within a cylindrical tube become approximately plane at a sufficient distance from their source. Taking the axis of z parallel to the generating lines of the cylinder, let us investigate the motion, whose potential varies as e^{ikat}, on the positive side of a source, situated at $z = 0$. If ϕ be the potential and ∇^2 stand for $d^2/dx^2 + d^2/dy^2$ the equation of the motion is

$$\left(\frac{d^2}{dz^2} + \nabla^2 + k^2\right)\phi = 0 \qquad\qquad (1).$$

If ϕ be independent of z, it represents vibrations wholly transverse to the axis of the cylinder. If the potential be then proportional to e^{ipat}, it must satisfy

$$(\nabla^2 + p^2)\,\phi = 0 \qquad\qquad (2),$$

as well as the condition that over the boundary of the section

$$\frac{d\phi}{dn} = 0 \quad\text{.............................(3).}$$

In order that these equations may be compatible, p is restricted to certain definite values corresponding to the periods of the natural vibrations. A zero value of p gives $\phi =$ constant, which solution, though it is of no significance in the two dimension problem, we shall presently have to consider. For each admissible value of p, there is a definite normal function u of x and y (§ 92), such that a solution is

$$\phi = A\,u\,e^{ipat} \quad\text{.........................(4).}$$

Two functions u, u', corresponding to different values of p, are conjugate, viz. make

$$\iint u u'\,dx\,dy = 0 \quad\text{.........................(5),}$$

and any function of x and y may be expanded within the contour in the series

$$\phi = A_0 u_0 + A_1 u_1 + A_2 u_2 + \quad\text{.......................(6),}$$

in which u_0, corresponding to $p = 0$, is constant.

In the actual problem ϕ may still be expanded in the same series, provided that A_0, A_1, &c. be regarded as functions of z. By substitution in (1) we get, having regard to (2),

$$u_0\left\{\frac{d^2A_0}{dz^2} + k^2A_0\right\} + u_1\left\{\frac{d^2A_1}{dz^2} + (k^2 - p_1^2)A_1\right\}$$
$$+ u_2\left\{\frac{d^2A_2}{dz^2} + (k^2 - p_2^2)\,A_2\right\} + \ldots = 0 \quad\text{..............(7),}$$

in which, by virtue of the conjugate property of the normal functions, each coefficient of u must vanish separately. Thus

$$\frac{d^2A_0}{dz^2} + k^2A_0 = 0, \qquad \frac{d^2A}{dz^2} + (k^2 - p^2)\,A = 0 \quad\text{...........(8).}$$

The solution of the first of these equations is

$$A_0 = \alpha_0\,e^{ikz} + \beta_0\,e^{-ikz},$$

giving

$$\phi_0 = \alpha_0\,u_0\,e^{ik(at+z)} + \beta_0\,u_0\,e^{ik(at-z)} \quad\text{..................(9).}$$

The solution of the general equation in A assumes a different form, according as $k^2 - p^2$ is positive or negative. If the forced vibration be graver in pitch than the gravest of the purely transverse natural vibrations, every finite value of p^2 is greater than k^2, or $k^2 - p^2$ is always negative. Putting

$$k^2 - p^2 = -\mu^2 \quad\text{.......................(10),}$$

we have $$A = \alpha e^{\mu z} + \beta e^{-\mu z},$$

whence $$\phi = (\alpha e^{\mu z} + \beta e^{-\mu z})\, u e^{ikat} \quad\text{..................(11).}$$

Now under the circumstances supposed, it is evident that the motion does not become infinite with z, so that all the coefficients α vanish. For a somewhat different reason the same is true of α_0, as there can be no wave in the negative direction. We may therefore take

$$\phi = \beta_0 e^{ik(at-z)} + \beta_1 u_1 e^{-\mu_1 z} e^{ikat} + \beta_2 u_2 e^{-\mu_2 z} e^{ikat} + \text{ ....(12),}$$

an expression which reduces to its first term when z is sufficiently great. We conclude that in all cases the waves ultimately become plane, *if the forced vibration be graver than the gravest of the natural transverse vibrations.*

In the case of a circular cylinder, of radius r, the gravest transverse vibration has a wave-length equal to $2\pi r \div 1{\cdot}841 = 3{\cdot}413r$ (§ 339). If then the wave-length of the forced vibration exceed $3{\cdot}413r$, the waves ultimately become plane. It may happen however that the waves ultimately become plane, although the wave-length fall short of the above limit. For example, if the source of vibration be symmetrical with respect to the axis of the tube, *e.g.* a simple source situated on the axis itself, the gravest transverse vibration with which we should have to deal would be more than an octave higher than in the general case, and the wave-length of the forced vibration might have less than half the above value.

From (12), when $z = 0$,

$$\frac{d\phi}{dz} = -ik\beta_0 e^{ikat} - \mu_1 \beta_1 u_1 e^{ikat} - \ldots$$

whence $$\iint \frac{d\phi}{dz}\, d\sigma = -ik\beta_0\, \sigma\, e^{ikat} \quad\text{.................(13),}$$

inasmuch as $\iint u_1\, d\sigma$, $\iint u_2\, d\sigma$, &c., all vanish.

It appears accordingly that the plane waves at a distance are the same as would be produced by a rigid piston at the origin, giving the same mean normal velocity as actually exists. Any normal motion of which the negative and positive parts are equal, produces ultimately no effect.

When there is no restriction on the character of the source, and when some of the transverse natural vibrations are graver than the actual one, some of the values of $k^2 - p^2$ are positive, and then terms enter of the form

$$\phi = \beta u\, e^{ikat}\, e^{-i\sqrt{(k^2-p^2)}z},$$

or in real quantities

$$\phi = \beta u \cos\{k\,at - \sqrt{(k^2 - p^2)}\,z\}\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

indicating that the peculiarities of the source are propagated to an infinite distance.

The problem here considered may be regarded as a generalization of that of § 268. For the case of a circular cylinder it may be worked out completely with the aid of Bessel's functions, but this must be left to the reader.

302. In § 278 we have fully determined the motion of the air due to the normal periodic motion of a bounding plane plate of infinite extent. If $d\phi/dn$ be the given normal velocity at the element dS,

$$\phi = -\frac{1}{2\pi}\iint \frac{d\phi}{dn}\frac{e^{-ikr}}{r}\,dS\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

gives the velocity-potential at any point P distant r from dS. The remainder of this chapter is devoted to the examination of the particular case of this problem which arises when the normal velocity has a given constant value over a circular area of radius R, while over the remainder of the plane it is zero. In particular we shall investigate what forces due to the reaction of the air will act on a rigid circular plate, vibrating with a simple harmonic motion in an equal circular aperture cut out of a rigid plane plate extending to infinity.

For the whole variation of pressure acting on the plate we have (§ 244)

$$\iint \delta p\,dS = -\sigma\iint \dot{\phi}\,dS = -ika\sigma\iint \phi\,dS,$$

where σ is the natural density, and ϕ varies as e^{ikat}. Thus by (1)

$$\iint \delta p dS = \frac{ika\sigma}{\pi}\frac{d\phi}{dn}\Sigma\Sigma\frac{e^{-ikr}}{r}dSdS' \quad\text{..............(2).}$$

In the double sum

$$\Sigma\Sigma\frac{e^{-ikr}}{r}dSdS' \quad\text{.....................(3),}$$

which we have now to evaluate, each pair of elements is to be taken *once* only, and the product is to be summed after multiplication by the factor $r^{-1}e^{-ikr}$, depending on their mutual distance. The best method is that suggested by Prof. Maxwell for the common potential[1]. The quantity (3) is regarded as the work that would be consumed in the complete dissociation of the matter composing the disc, that is to say, in the removal of every element from the influence of every other, on the supposition that the potential of two elements is proportional to $r^{-1}e^{-ikr}$. The amount of work required, which depends only on the initial and final states, may be calculated by supposing the operation performed in any way that may be most convenient. For this purpose we suppose that the disc is divided into elementary rings, and that each ring is carried away to infinity before any of the interior rings are disturbed.

The first step is the calculation of the potential (V) at the edge of a disc of radius c. Taking polar co-ordinates (ρ, θ) with any point of the circumference for pole, we have

$$V = \iint \frac{e^{-ik\rho}}{\rho}\rho\, d\rho\, d\theta = \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^{2c\cos\theta} e^{-ik\rho}d\rho\, d\theta = \frac{2}{ik}\int_0^{\frac{1}{2}\pi}\{1-e^{-2ikc\cos\theta}\}\,d\theta.$$

This quantity must be multiplied by $2\pi c dc$, and afterwards integrated with respect to c between the limits 0 and R. But it will be convenient first to effect a transformation. We have

$$\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} e^{-2ikc\cos\theta}d\theta = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} e^{-2ikc\sin\theta}d\theta$$

$$= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\cos(2kc\sin\theta)\,d\theta - \frac{2i}{\pi}\int_0^{\frac{1}{2}\pi}\sin(2kc\sin\theta)\,d\theta$$

$$= J_0(z) - iK(z) \quad\text{...................................(4),}$$

where z is written for $2kc$. $J_0(z)$ is the Bessel's function of zero

[1] Theory of Resonance. *Phil. Trans.* 1870.

order (§ 200), and $K(z)$ is a function defined by the equation

$$K(z)=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\sin(z\sin\theta)\,d\theta$$

$$=\frac{2}{\pi}\left\{z-\frac{z^3}{1^2.3^2}+\frac{z^5}{1^2.3^2.5^2}-\frac{z^7}{1^2.3^2.5^2.7^2}+\ldots\ldots\right\}\ldots\ldots\ldots(5).$$

Deferring for the moment the further consideration of the function K, we have

$$V=\frac{\pi}{k}[K(z)-i\{1-J_0(z)\}]\ldots\ldots\ldots\ldots\ldots(6),$$

and thus

$$\Sigma\Sigma\frac{e^{-ikr}}{r}dS\,dS'=\frac{\pi^2}{2k^3}\int_0^{2kR}z\,dz\,[K(z)-i\{1-J_0(z)\}].$$

Now by (6) § 200 and (8) § 204

$$\int_0^z z\,dz\,J_0(z)=z\,J_1(z)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7);$$

and thus, if K_1 be defined by

$$K_1(z)=\int_0^z z\,dz\,K(z)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

we may write

$$\Sigma\Sigma\frac{e^{-ikr}}{r}dS\,dS'=\frac{\pi^2}{2k^3}K_1(2kR)-i\frac{\pi^2R^2}{k}\left(1-\frac{J_1(2kR)}{kR}\right)\ldots\ldots(9).$$

From this the total pressure is derived by introduction of the factor $\frac{ika\sigma}{\pi}\frac{d\phi}{dn}$, so that

$$\iint\delta p\,dS=a\sigma\,.\,\pi R^2\,.\,\frac{d\phi}{dn}\left(1-\frac{J_1(2kR)}{kR}\right)+i\frac{a\sigma\pi}{2k^2}\frac{d\phi}{dn}K_1(2kR)\,..(10).$$

The reaction of the air on the disc may thus be divided into two parts, of which the first is proportional to the velocity of the disc, and the second to the acceleration. If ξ denote the displacement of the disc, so that $\dot{\xi}=\frac{d\phi}{dn}$, we have $\ddot{\xi}=ika\,\dot{\xi}=ika\frac{d\phi}{dn}$; and therefore in the equation of motion of the disc, the reaction of the air is represented by a frictional force $a\sigma\,.\,\pi R^2\,.\,\dot{\xi}\left(1-\frac{J_1(2kR)}{kR}\right)$ retarding the motion, and by an accession to the inertia equal to $\frac{\pi\sigma}{2k^3}K_1(2kR)$.

When kR is small, we have from the ascending series for J_1 (5) § 200,

$$1-\frac{J_1(2kR)}{kR}=\frac{k^2R^2}{1.2}-\frac{k^4R^4}{1.2^2.3}+\frac{k^6R^6}{1.2^2.3^2.4}-\frac{k^8R^8}{1.2^2.3^2.4^2.5}+\ldots \quad (11),$$

so that the frictional term is approximately

$$\tfrac{1}{2}a\sigma.\pi R^2.k^2R^2.\dot{\xi}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

From the nature of the case the coefficient of $\dot{\xi}$ must be positive, otherwise the reaction of the air would tend to augment, instead of to diminish, the motion. That $J_1(z)$ is in fact always less than $\frac{1}{2}z$ may be verified as follows. If θ lie between 0 and π, and z be positive, $\sin(z\sin\theta)-z\sin\theta$ is negative, and therefore also

$$\frac{1}{\pi}\int_0^\pi\{\sin(z\sin\theta)-z\sin\theta\}\sin\theta\,d\theta$$

is negative. But this integral is $J_1(z)-\frac{1}{2}z$, which is accordingly negative for all positive values of z.

When kR is great, $J_1(2kR)$ tends to vanish, and then the frictional term becomes simply $a\sigma.\pi R^2.\dot{\xi}$. This result might have been expected; for when kR is very large, the wave motion in the neighbourhood of the disc becomes approximately plane. We have then by (6) and (8) § 245, $dp=a\rho_0\dot{\xi}$, in which ρ_0 is the density (σ); so that the retarding force is $\pi R^2\delta p=a\sigma.\pi R^2.\dot{\xi}$.

We have now to consider the term representing an alteration of inertia, and among other things to prove that this alteration is an increase, or that $K_1(z)$ is positive. By direct integration of the ascending series (5) for K (which is always convergent),

$$K_1(z)=\frac{2}{\pi}\left\{\frac{z^3}{1^2.3}-\frac{z^5}{1^2.3^2.5}+\frac{z^7}{1^2.3^2.5^2.7}\quad\ldots\ldots\right\}\ldots\ldots\ldots(13).$$

When therefore kR is small, we may take as the expression for the increase of inertia

$$\frac{8\sigma R^3}{3}=\sigma.\pi R^2.\frac{8R}{3\pi}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

This part of the reaction of the air is therefore represented by supposing the vibrating plate to carry with it a mass of air equal to that contained in a cylinder whose base is the plate, and whose height is equal to $8R/3\pi$; so that, when the plate is sufficiently small, the mass to be added is independent of the period of vibration.

From the series (5) for $K(z)$, it may be proved immediately that

$$\frac{1}{z}\frac{d}{dz}\left(z\frac{d}{dz}\right)K(z)=\frac{2}{\pi z}-K(z) \quad\text{..............(15)},$$

or

$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+1\right)K(z)=\frac{2}{\pi z} \quad\text{.................(16)}.$$

From the first form (15) it follows that

$$K_1(z)=\int_0^z K(z)\,z\,dz=\frac{2}{\pi}z-z\frac{dK(z)}{dz} \quad\text{..........(17)}.$$

By means of this expression for $K_1(z)$ we may readily prove that the function is always positive. For

$$\frac{dK(z)}{dz}=\frac{d}{dz}\cdot\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\sin(z\sin\theta)\,d\theta=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\cos(z\sin\theta)\sin\theta\,d\theta\text{...(18)};$$

so that

$$K_1(z)=\frac{2z}{\pi}\left\{1-\int_0^{\frac{1}{2}\pi}\cos(z\sin\theta)\sin\theta\,d\theta\right\}$$

$$=\frac{4z}{\pi}\int_0^{\frac{1}{2}\pi}\sin^2(\tfrac{1}{2}z\sin\theta)\sin\theta\,d\theta \quad\text{..............(19)},$$

an integral of which every element is positive. When z is very large, $\cos(z\sin\theta)$ fluctuates with great rapidity, and thus $K_1(z)$ tends to the form

$$K_1(z)=\frac{2}{\pi}.z \quad\text{.........................(20)}.$$

When z is great, the ascending series for K and K_1, though always ultimately convergent, become useless for practical calculation, and it is necessary to resort to other processes. It will be observed that the differential equation (16) satisfied by K is the same as that belonging to the Bessel's function J_0, with the exception of the term on the right-hand side, viz. $2/\pi z$. The function K is therefore included in the form obtained by adding to the general solution of Bessel's equation containing two arbitrary constants any particular solution of (16). Such a particular solution is

$$\tfrac{1}{2}\pi.K(z)=z^{-1}-z^{-3}+1^2.3^2.z^{-5}-1^2.3^2.5^2.z^{-7}+1^2.3^2.5^2.7^2.z^{-9}-\ldots\text{(21)},$$

as may be readily verified on substitution. The series on the right of (21), notwithstanding its ultimate divergency, may be used successfully for computation when z is great. It is in fact the analytical equivalent of $\int_0^\infty e^{-\beta}(z^2+\beta^2)^{-\frac{1}{2}}d\beta$, and we might take

$$K(z)=\frac{2}{\pi}\int_0^\infty \frac{e^{-\beta}d\beta}{\sqrt{(z^2+\beta^2)}}+\text{Complementary Function},$$

determining the two arbitrary constants by an examination of the forms assumed when z is very great. But it is perhaps simpler to follow the method used by Lipschitz[1] for Bessel's functions.

By (4) we have

$$J_0(z)-iK(z)=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} e^{-iz\cos\theta}\,d\theta=\frac{2}{\pi}\int_0^1 \frac{e^{-izv}dv}{\sqrt{(1-v^2)}} \quad\text{.........(22).}$$

Consider the integral $\int\frac{e^{-zw}dw}{\sqrt{(1+w^2)}}$, where w is a complex variable of the form $u+iv$. Representing, as usual, simultaneous pairs of values of u and v by the co-ordinates of a point, we see that the value of the integral will be zero, if the integration with respect to w range round the rectangle, whose angular points are respectively 0, h, $h+i$, i, where h is any real positive quantity. Thus

$$\int_0^h \frac{e^{-zu}du}{\sqrt{1+u^2}}+\int_0^i \frac{e^{-z(h+iv)}d(iv)}{\sqrt{1+(h+iv)^2}}+\int_h^0 \frac{e^{-z(u+i)}du}{\sqrt{1+(u+i)^2}}+\int_i^0\frac{e^{-izv}d(iv)}{\sqrt{1-v^2}}=0,$$

from which, if we suppose that $h=\infty$,

$$\int_0^1\frac{e^{-izv}dv}{\sqrt{1-v^2}}=-i\int_0^\infty\frac{e^{-zu}du}{\sqrt{1+u^2}}+i\int_0^\infty\frac{e^{-z(u+i)}du}{\sqrt{1+(u+i)^2}}\text{...........(23).}$$

Replacing uz by β, we may write (23) in the form

$$\int_0^1\frac{e^{-izv}dv}{\sqrt{(1-v^2)}}=-i\int_0^\infty\frac{e^{-\beta}d\beta}{z\sqrt{(1+\beta^2/z^2)}}+\frac{e^{-i(z-\frac{1}{4}\pi)}}{\sqrt{(2z)}}\int_0^\infty\frac{e^{-\beta}\beta^{-\frac{1}{2}}d\beta}{\sqrt{(1-i\beta/2z)}}\text{...(24).}$$

The first term on the right in (24) is entirely imaginary; it therefore follows by (22) that $\frac{1}{2}\pi J_0(z)$ is the real part of the second term. By expanding the binomial under the integral sign, and afterwards integrating by the formula

$$\int_0^\infty e^{-\beta}\beta^{q-\frac{1}{2}}d\beta=\Gamma(q+\tfrac{1}{2}),$$

we obtain as the expansion for $J_0(z)$ in negative powers of z,

$$J_0(z)=\sqrt{\left(\frac{2}{\pi z}\right)}\left\{1-\frac{1^2.3^2}{1.2.(8z)^2}+\dots\right\}\cos(z-\tfrac{1}{4}\pi)$$
$$+\sqrt{\left(\frac{2}{\pi z}\right)}\left\{\frac{1^2}{1.8z}-\frac{1^2.3^2.5^2}{1.2.3.(8z)^3}+\dots\right\}\sin(z-\tfrac{1}{4}\pi)\text{..... (25).}$$

[1] Crelle, Bd. LVI. 1859. Lömmel, *Studien über die Bessel'schen Functionen*, p. 59.

By stopping the expansion after any desired number of terms, and forming the expression for the remainder, it may be proved that the error committed by neglecting the remainder cannot exceed the last term retained (§ 200).

In like manner the imaginary part of the right-hand member of (24) is the equivalent of $-\frac{1}{2}i\pi K(z)$, so that

$$K(z)=\frac{2}{\pi}\left\{z^{-1}-z^{-3}+1^2.3^2.z^{-5}-1^2.3^2.5^2.z^{-7}+\ldots\ldots\right\}$$
$$+\sqrt{\left(\frac{2}{\pi z}\right)}\left\{1-\frac{1^2.3^2}{1.2.(8z)^2}+\frac{1^2.3^2.5^2.7^2}{1.2.3.4.(8z)^4}-\ldots\ldots\right\}\sin\left(z-\tfrac{1}{4}\pi\right)$$
$$-\sqrt{\left(\frac{2}{\pi z}\right)}\left\{\frac{1^2}{1.8z}-\frac{1^2.3^2.5^2}{1.2.3.(8z)^3}+\ldots\ldots\right\}\cos\left(z-\tfrac{1}{4}\pi\right)\ldots\ldots\ldots(26).$$

The value of $K_1(z)$ may now be determined by means of (17). We find

$$\frac{dK}{dz}=-\frac{2}{\pi}\left\{z^{-2}-3.z^{-4}+1^2.3^2.5.z^{-6}-1^2.3^2.5^2.7.z^{-8}+\ldots\ldots\right\}$$
$$+\sqrt{\left(\frac{2}{\pi z}\right)}\cos\left(z-\tfrac{1}{4}\pi\right)\left\{1+\frac{3.5.1}{1.2.(8z)^2}-\frac{3.5.7.9.1.3.5}{1.2.3.4.(8z)^4}+\ldots\ldots\right\}$$
$$\sqrt{\left(\frac{2}{\pi z}\right)}\sin\left(z-\tfrac{1}{4}\pi\right)\left\{\frac{3}{1.(8z)}-\frac{3.5.7.1.3}{1.2.3.(8z)^3}\right.$$
$$\left.+\frac{3.5.7.9.11.1.3.5.7}{1.2.3.4.5.(8z)^5}-\ldots\ldots\right\}\ldots\ldots(27).$$

The final expression for $K_1(z)$ may be put into the form

$$K_1(z)=\frac{2}{\pi}\left\{z+z^{-1}-3.z^{-3}+1^2.3^2.5.z^{-5}-1^2.3^2.5^2.7.z^{-7}+\ldots\ldots\right\}$$
$$-\sqrt{\frac{2z}{\pi}}.\cos\left(z-\tfrac{1}{4}\pi\right)\left\{1-\frac{(1^2-4)(3^2-4)}{1.2.(8z)^2}\right.$$
$$\left.+\frac{(1^2-4)(3^2-4)(5^2-4)(7^2-4)}{1.2.3.4.(8z)^4}-\ldots\ldots\right\}$$
$$-\sqrt{\frac{2z}{\pi}}.\sin\left(z-\tfrac{1}{4}\pi\right)\left\{\frac{1^2-4}{1.8z}-\frac{(1^2-4)(3^2-4)(5^2-4)}{1.2.3.(8z)^3}+\ldots\ldots\right\}^{1}\ldots(28).$$

It appears then that K_1 does not vanish when z is great, but approximates to $2z/\pi$. But although the accession to the inertia,

[1] As was to be expected, the series within brackets are the same as those that occur in the expression of the function $J_1(z)$.

which is proportional to K_1, becomes infinite with R, it vanishes ultimately when compared with the area of the disc, and with the other term which represents the dissipation. And this agrees with what we should anticipate from the theory of plane waves.

If, independently of the reaction of the air, the mass of the plate be M, and the force of restitution be $\mu\xi$, the equation of motion of the plate when acted on by an impressed force F, proportional to e^{ikat}, will be

$$\left\{M+\frac{\pi\sigma}{2k^3}K_1(2kR)\right\}\ddot{\xi}+a\sigma\pi R^2\left(1-\frac{J_1(2kR)}{kR}\right)\dot{\xi}+\mu\xi=F\text{...(29)};$$

or by (13), if, as will be usual in practical applications, kR be small,

$$\left(M+\frac{8\sigma R^3}{3}\right)\ddot{\xi}+\frac{a\sigma\pi k^2R^4}{2}\dot{\xi}+\mu\xi=F\text{...........(30)}.$$

Two particular cases of this problem deserve notice. First let M and μ vanish, so that the plate, itself devoid of mass, is subject to no other forces than F and those arising from aerial pressures. Since $\ddot{\xi}=ika\dot{\xi}$, the frictional term is relatively negligible, and we get when kR is very small,

$$a\sigma\pi R^2.\frac{8kR}{3\pi}\dot{\xi}=-iF\text{....................(31)}.$$

Next let M and μ be such that the natural period of the plate, when subject to the reaction of the air, is the same as that imposed upon it. Under these circumstances

$$\left(M+\frac{8\sigma R^3}{3}\right)\ddot{\xi}+\mu\xi=0,$$

and therefore

$$a\sigma\pi R^2.\frac{k^2R^2}{2}.\dot{\xi}=F\text{..................(32)}.$$

Comparing with (31), we see that the amplitude of vibration is greater in the case when the inertia of the air is balanced, in the ratio of $16:3\pi kR$, shewing a large increase when kR is small. In the first case the phase of the motion is such that comparatively very little work is done by the force F; while in the second, the inertia of the air is compensated by the spring, and then F, being of the same phase as the velocity, does the maximum amount of work.

CHAPTER XVI.

THEORY OF RESONATORS.

303. IN the pipe closed at one end and open at the other we had an example of a mass of air endowed with the property of vibrating in certain definite periods peculiar to itself in more or less complete independence of the external atmosphere. If the air beyond the open end were entirely without mass, the motion within the pipe would have no tendency to escape, and the contained column of air would behave like any other complex system not subject to dissipation. In actual experiment the inertia of the external air cannot, of course, be got rid of, but when the diameter of the pipe is small, the effect produced in the course of a few periods may be insignificant, and then vibrations once excited in the pipe have a certain degree of persistence. The narrower the channel of communication between the interior of a vessel and the external medium, the greater does the independence become. Such cavities constitute resonators; in the presence of an external source of sound, the contained air vibrates in unison, and with an amplitude dependent upon the relative magnitudes of the natural and forced periods, rising to great intensity in the case of approximate isochronism. When the original cause of sound ceases, the resonator yields back the vibrations stored up as it were within it, thus becoming itself for a short time a secondary source. The theory of resonators constitutes an important branch of our subject.

As an introduction to it we may take the simple case of a stopped cylinder, in which a piston moves without friction. On the further side of the piston the air is supposed to be devoid of inertia, so that the pressure is absolutely constant. If now the piston be set into vibration of very long period, it is clear that the contained air will be at any time very nearly in the equilibrium condition (of uniform density) corresponding to the momentary position of the piston. If the mass of the piston be very considerable in comparison with that of the included air, the natural vibrations resulting from a displacement will occur nearly as if the air had no inertia; and in deriving the period from the kinetic and potential energies, the former may be calculated without allowance for the inertia of the air, and the latter as if the rarefaction and condensation were uniform. Under the circumstances contemplated the air acts merely as a spring in virtue of its resistance to compression or dilatation; the form of the containing vessel is therefore immaterial, and the period of vibration remains the same, provided the capacity be not varied.

When a gas is compressed or rarefied, the mechanical value of the resulting displacement is found by multiplying each infinitesimal increment of volume by the corresponding pressure and integrating over the range required. In the present case it is of course only the difference of pressure on the two sides of the piston which is really operative, and this for a small change is proportional to the alteration of volume. The whole mechanical value of the small change is the same as if the expansion were opposed throughout by the *mean*, that is half the final, pressure; thus corresponding to a change of volume from S to $S+\delta S$, since $p=a^2\rho$,

$$V=p\cdot\frac{\delta S}{2S}\cdot\delta S=\tfrac{1}{2}\rho a^2\frac{(\delta S)^2}{S}\quad\ldots\ldots\ldots\ldots\ldots\ldots(1)^1.$$

If A denote the area of the piston, M its mass, and x its linear displacement, $\delta S=Ax$, and the equation of motion is

$$M\ddot{x}+\frac{\rho a^2A^2}{S}x=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

indicating vibrations, whose periodic time is

$$\tau=2\pi\div aA\sqrt{\frac{\rho}{MS}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Let us now imagine a vessel containing air, whose interior communicates with the external atmosphere by a narrow aperture or neck. It is not difficult to see that this system is capable of vibrations similar to those just considered, the air in the neighbourhood of the aperture supplying the place of the piston. By sufficiently increasing S, the period of the vibration may be made as long as we please, and we obtain finally a state of things in

[1] Compare (12) § 245.

which the kinetic energy of the motion may be neglected except in the neighbourhood of the aperture, and the potential energy may be calculated as if the density in the interior of the vessel were uniform. In flowing through the aperture under the operation of a difference of pressure on the two sides, or in virtue of its own inertia after such pressure has ceased, the air moves approximately as an incompressible fluid would do under like circumstances, provided that the space through which the kinetic energy is sensible be very small in comparison with the length of the wave. The suppositions on which we are about to proceed are not of course strictly correct as applied to actual resonators such as are used in experiment, but they are near enough to the mark to afford an instructive view of the subject and in many cases a foundation for a sufficiently accurate calculation of the pitch. They become rigorous only in the limit when the wave-length is indefinitely great in comparison with the dimensions of the vessel.

[On the above principles we may at once calculate the pitch of a resonator of volume S, whose cavity communicates with the external air by a long cylindrical neck of length L and area A. The mass of the aerial piston is ρAL; so that (3) gives as the period of vibration

$$\tau = \frac{2\pi}{a}\sqrt{\left(\frac{LS}{A}\right)} \qquad\qquad (4);$$

or, if λ be the length of plane waves of the same pitch,

$$\lambda = a\tau = 2\pi\sqrt{\left(\frac{LS}{A}\right)} \qquad\qquad (5).$$

If the cross-section of the neck be a circle of radius R, $A = \pi R^2$, and we obtain the formula (8) of § 307.]

304. The kinetic energy of the motion of an incompressible fluid through a given channel may be expressed in terms of the density ρ, and the rate of transfer, or current, $\dot{X}$, for under the circumstances contemplated the character of the motion is always the same. Since T necessarily varies as ρ and as $\dot{X}^2$, we may put

$$T = \tfrac{1}{2}\rho\frac{\dot{X}^2}{c} \qquad\qquad (1),$$

where the constant c, which depends only on the nature of the channel, is a linear quantity, as may be inferred from the fact that the dimensions of $\dot{X}$ are 3 in space and -1 in time. In fact, if ϕ be the velocity-potential,

$$T = \tfrac{1}{2}\rho \iiint \left[\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right] dx\, dy\, dz = \tfrac{1}{2}\rho \iint \phi \frac{d\phi}{dn} dS,$$

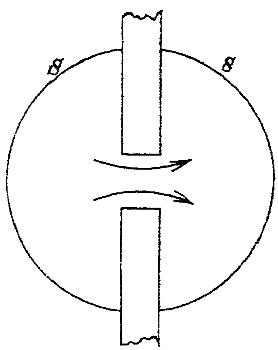
by Green's theorem, where the integration is to be extended over a surface including the whole region through which the motion is sensible. At a sufficient distance on either side of the aperture, ϕ becomes constant, and if the constant values be denoted by ϕ_1 and ϕ_2, and the integration be now limited to that half of S towards which the fluid flows, we have

$$T = \tfrac{1}{2}\rho\,(\phi_1 - \phi_2) \iint \frac{d\phi}{dn} dS = \tfrac{1}{2}\rho\,(\phi_1 - \phi_2)\,\dot{X}.$$

Now, since within S ϕ is determined linearly by its surface values, $\iint \frac{d\phi}{dn} dS$, or $\dot{X}$, is proportional to $(\phi_1 - \phi_2)$. If we put $\dot{X} = c\,(\phi_1 - \phi_2)$, we get as before $T = \tfrac{1}{2}\rho \dot{X}^2/c$.

The nature of the constant c will be better understood by considering the electrical problem, whose conditions are mathematically identical with those of that under discussion. Let us suppose that the fluid is replaced by uniformly conducting material, and that the boundary of the channel or aperture is replaced by insulators. We know that if by battery power or otherwise, a difference of electric potential be maintained on the two sides, a steady current through the aperture of proportional magnitude will be generated. The ratio of the total current to the electromotive force is called the *conductivity* of the channel, and thus we see that our constant c represents simply this conductivity, on the supposition that the specific conducting power of the hypothetical substance is unity. The same thing may be otherwise expressed by saying that c is the side of the cube, whose resistance between opposite faces is the same as that of the channel. In the sequel we shall often avail ourselves of the electrical analogy.

Fig. 58.



When c is known, the proper tone of the resonator can be easily deduced. Since

$$V = \tfrac{1}{2}\rho a^2 \frac{X^2}{S}, \quad T = \tfrac{1}{2}\rho \frac{\dot{X}^2}{c} \text{ ....} \quad (2),$$

the equation of motion is

$$\ddot{X} + \frac{a^2 c}{S} X = 0 \text{ ....} (3),$$

indicating simple oscillations performed in a time

$$\tau = 2\pi \div \sqrt{\frac{a^2 c}{S}} \text{ ....} (4).$$

If N be the frequency, or number of complete vibrations executed in the unit time,

$$N = \frac{a}{2\pi}\sqrt{\frac{c}{S}} \text{ ....} (5).$$

The wave-length λ, which is the quantity most closely connected with the dimensions of the cavity, is given by

$$\lambda = \frac{a}{N} = 2\pi\sqrt{\frac{S}{c}} \text{ ....} \quad (6),$$

and varies directly as the linear dimension. The wave-length, it will be observed, is a function of the size and shape of the resonator only, while the frequency depends also upon the nature of the gas; and it is important to remark that it is on the nature of the gas in and near the channel that the pitch depends and not on that occupying the interior of the vessel, for the inertia of the air in the latter situation does not come into play, while the compressibility of all gases is very approximately the same. Thus in the case of a pipe, the substitution of hydrogen for air in the neighbourhood of a node would make but little difference, but its effect in the neighbourhood of a loop would be considerable.

Hitherto we have spoken of the channel of communication as single, but if there be more than one channel, the problem is not essentially altered. The same formula for the frequency is still applicable, if as before we understand by c the whole conductivity between the interior and exterior of the vessel. When the channels are situated sufficiently far apart to act independently one of another, the resultant conductivity is the simple sum of those belonging to the separate channels; otherwise the resultant is less than that calculated by mere addition.

If there be two precisely similar channels, which do not interfere, and whose conductivity taken separately is c, we have

$$N = \sqrt{2} \times \frac{a}{2\pi}\sqrt{\frac{c}{S}} \quad\text{.....................}\quad (7),$$

shewing that the note is higher than if there were only one channel in the ratio $\sqrt{2} : 1$, or by rather less than a fifth—a law observed by Sondhauss and proved theoretically by Helmholtz in the case, where the channels of communication consist of simple holes in the infinitely thin sides of the reservoir.

305. The investigation of the conductivity for various kinds of channels is an important part of the theory of resonators; but in all except a very few cases the accurate solution of the problem is beyond the power of existing mathematics. Some general principles throwing light on the question may however be laid down, and in many cases of interest an approximate solution, sufficient for practical purposes, may be obtained.

We know (§§ 79, 242) that the energy of a fluid flowing through a channel cannot be greater than that of any fictitious motion giving the same total current. Hence, if the channel be narrowed in any way, or any obstruction be introduced, the conductivity is thereby diminished, because the alteration is of the nature of an additional constraint. Before the change the fluid was free to adopt the distribution of flow finally assumed. In cases where a rigorous solution cannot be obtained we may use the minimum property to estimate an inferior limit to the conductivity; the energy calculated from a hypothetical law of flow can never be less than the truth, and must exceed it unless the hypothetical and the actual motion coincide.

Another general principle, which is of frequent use, may be more conveniently stated in electrical language. The quantity with which we are concerned is the conductivity of a certain conductor composed of matter of unit specific conductivity. The principle is that if the conductivity of any part of the conductor be increased that of the whole is increased, and if the conductivity of any part be diminished that of the whole is diminished, exception being made of certain very particular cases, where no alteration ensues. In its passage through a conductor electricity distributes itself, so that the energy dissipated is for a given total current the least possible (§ 82). If now the specific resistance of any part be diminished, the total dissipation would be less than before, even if the distribution of currents remained unchanged. *A fortiori* will this be the case, when the currents redistribute themselves so as to make the dissipation a minimum. If an infinitely thin lamina of matter stretching across the channel be made perfectly conducting, the resistance of the whole will be diminished, unless the lamina coincide with one of the undisturbed equipotential surfaces. In the excepted case no effect will be produced.

306. Among different kinds of channels an important place must be assigned to those consisting of simple apertures in unlimited plane walls of infinitesimal thickness. In practical applications it is sufficient that a wall be very thin in proportion to the dimensions of the aperture, and approximately plane within a distance from the aperture large in proportion to the same quantity.

On account of the symmetry on the two sides of the wall, the motion of the fluid in the plane of the aperture must be normal, and therefore the velocity-potential must be constant; over the remainder of the plane the motion must be exclusively tangential, so that to determine ϕ on one side of the plane we have the conditions (i) $\phi =$ constant over the aperture, (ii) $d\phi/dn = 0$ over the rest of the plane of the wall, (iii) $\phi =$ constant at infinity.

Since we are concerned only with the differences of ϕ we may suppose that at infinity ϕ vanishes. It will be seen that conditions (ii) and (iii) are satisfied by supposing ϕ to be the potential of attracting matter distributed over the aperture; the remainder of the problem consists in determining the distribution of matter so that its potential may be constant over the same area. The problem is mathematically the same as that of determining the distribution of electricity on a charged conducting plate situated in an open space, whose form is that of the aperture under consideration, and the conductivity of the aperture may be expressed in terms of the *capacity* of the plate of the statical problem. If ϕ denote the constant potential in the aperture, the electrical resistance (for one side only) will be

$$\phi_1 \div \iint \frac{d\phi}{dn}\, d\sigma,$$

the integration extending over the area of the opening.

Now $\iint \frac{d\phi}{dn}\, d\sigma = 2\pi \times$ (whole quantity of matter distributed), and thus, if M be the capacity, or charge corresponding to unit-potential, the total resistance is $(\pi M)^{-1}$. Accordingly for the conductivity, which is the reciprocal of the resistance,

$$c = \pi M \quad\text{.............................}(1).$$

So far as I am aware, the ellipse is the only form of aperture for which c or M can be determined theoretically[1], in which case the result is included in the known solution of the problem of determining the distribution of charge on an ellipsoidal conductor. From the fact that a shell bounded by two concentric, similar and similarly situated ellipsoids exerts no force on an internal particle, it is easy to see that the superficial density at any point of an ellipsoid necessary to give a constant potential is proportional to the perpendicular (p) let fall from the centre upon the tangent plane at the point in question. Thus if ρ be the density, $\rho = \kappa p$; the whole quantity of matter Q is given by

$$Q = \iint \rho\, dS = \kappa \iint p\, dS = 4\pi\kappa abc \quad\text{...........}(2),[2]$$

so that

$$\rho = \frac{Qp}{4\pi\, abc} \quad\text{............................}(3).$$

In the usual notation

$$p = 1 \div \sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}},$$

or, since

$$z^2/c^2 = 1 - x^2/a^2 - y^2/b^2,$$

$$p = c \div \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{c^2x^2}{a^4} + \frac{c^2y^2}{b^4}}\,.$$

If we now suppose that c is infinitely small, we obtain the particular case of an elliptic plate, and if we no longer distinguish between the two surfaces, we get

$$\rho = \frac{Q}{2\pi ab} \div \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \quad\text{..................}(4).$$

[1] The case of a resonator with an elliptic aperture was considered by Helmholtz (Crelle, Bd. 57, 1860), whose result is equivalent to (8).

[2] $2c$ being for the moment the third principal axis of the ellipsoid.

We have next to find the value of the constant potential (P). By considering the value of P at the centre of the plate, we see that

$$P = \iint \frac{\rho\, dS}{r} = \iint \rho\, dr\, d\theta.$$

Integrating first with respect to r, we have

$$\int_0^r \rho\, dr = Q \div 4a \sqrt{(1 - e^2 \cos^2\theta)},$$

e being the eccentricity; and thus

$$P = \frac{Q}{a} \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{(1 - e^2 \cos^2\theta)}} = \frac{Q}{a} F(e),$$

where F is the symbol of the complete elliptic function of the first order. Putting $P = 1$, we find

$$\frac{c}{\pi} = M = \frac{a}{F(e)} \qquad \text{.........................(5)},$$

as the final expression for the capacity of an ellipse, whose semi-major axis is a and eccentricity is e. In the particular case of the circle, $e = 0$, $F(e) = \frac{1}{2}\pi$, and thus for a circle of radius R,

$$c = 2R \qquad \text{.............................(6)}.$$

If the capacity of the resonator be S, we find from (6) § 304

$$\lambda = \pi \sqrt{\left(\frac{2S}{R}\right)} \qquad \text{............................ (7)}.$$

The area of the ellipse (σ) is given by

$$\sigma = \pi a^2 \sqrt{(1 - e^2)},$$

and hence in terms of σ

$$\frac{1}{c} = \frac{1}{2}\sqrt{\left(\frac{\pi}{\sigma}\right)} \cdot \frac{2F(e)\,(1 - e^2)^{\frac{1}{4}}}{\pi} \qquad \text{..................(8)}.$$

When e is small, we obtain by expanding in powers of e previous to integration,

$$F(e) = \frac{1}{2}\pi \left\{1 + \frac{1^2}{2^2} e^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} e^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} e^6 + \dots\right\} \qquad \text{.....(9)},$$

whence

$$\frac{2F(e)\,(1 - e^2)^{\frac{1}{4}}}{\pi} = 1 - \frac{e^4}{64} - \frac{e^6}{64} + \dots$$

Neglecting e^8 and higher powers, we have therefore

$$c = 2\sqrt{\left(\frac{\sigma}{\pi}\right)}\cdot\left(1+\frac{e^4}{64}+\frac{e^6}{64}+\ldots\right)\qquad\qquad(10).$$

From this result we see that, if its eccentricity be small, the conductivity of an elliptic aperture is very nearly the same as that of a circular aperture *of equal area*. Among various forms of aperture of given area there must be one which has a minimum conductivity, and, though a formal proof might be difficult, it is easy to recognise that this can be no other than the circle. An inferior limit to the value of c is thus always afforded by the conductivity of the circle of equal area, that is $2\sqrt{(\sigma/\pi)}$, and when the true form is nearly circular, this limit may be taken as a close approximation to the real value.

The value of λ is then given by

$$\lambda = 2^{\frac{1}{2}}\pi^{\frac{5}{4}}\sigma^{-\frac{1}{4}}S^{\frac{1}{2}}\qquad\qquad(11).$$

In order to shew how slightly a moderate eccentricity affects the value of c, I have calculated the following short table with the aid of Legendre's values of $F(e)$. Putting $e=\sin\psi$, we have $\cos\psi$ as the ratio of axes, and for the conductivity

$$c=2\sqrt{\left(\frac{\sigma}{\pi}\right)}\cdot\frac{\pi}{2\sqrt{(\cos\psi)}\,.\,F(\sin\psi)}.$$

ψ.	$e=\sin\psi$.	$b : a=\cos\psi$.	$\pi\div 2F(e)(1-e^2)^{\frac{1}{4}}$.
0°	·00000	1·00000	1·0000
20°	·34204	·93969	1·0002
30°	·50000	·86603	1·0013
40°	·64279	·76604	1·0044
50°	·76604	·64279	1·0122
60°	·86603	·50000	1·0301
70°	·93969	·34202	1 0724
80°	·98481	·17365	1·1954
90°	1·00000	·00000	∞

The value of the last factor given in the fourth column is the ratio of the conductivity of the ellipse to *that of a circle of equal area*. It appears that even when the ellipse is so eccentric that the ratio of the axes is 2 : 1, the conductivity is increased by only about 3 per cent., which would correspond to an alteration of little more than a comma (§ 18) in the pitch of a resonator. There seems no reason to suppose that this approximate independence of shape is a property peculiar to the ellipse, and we may conclude with some confidence that in the case of any moderately elongated oval aperture, the conductivity may be calculated from the area alone with a considerable degree of accuracy.

If the area be given, there is no superior limit to c. For suppose the area σ to be distributed over n equal circles sufficiently far apart to act independently. The area of each circle is σ/n, and its conductivity is $2(n\pi)^{-\frac{1}{2}}\sigma^{\frac{1}{2}}$. The whole conductivity is n times as great, and therefore increases indefinitely with n. As a general rule, the more the opening is elongated or broken up, the greater will be the conductivity for a given area.

To find a superior limit to the conductivity of a given aperture we may avail ourselves of the principle that any addition to the aperture must be attended by an increase in the value of c. Thus in the case of a square, we may be sure that c is less than for the circumscribed circle, and we have already seen that it is greater than for the circle of equal area. If b be the side of the square

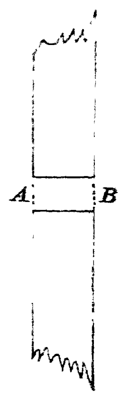
$$\frac{2b}{\sqrt{\pi}} < c < \sqrt{2}\, b.$$

The tones of a resonator with a square aperture calculated from these two limits would differ by about a whole tone; the graver of them would doubtless be much the nearer to the truth. This example shews that even when analysis fails to give a solution in the mathematical sense, we need not be altogether in the dark as to the magnitudes of the quantities with which we are dealing.

In the case of similar orifices, or systems of orifices, c varies as the linear dimension.

307. Most resonators used in practice have necks of greater or less length, and even when there is nothing that would be called a neck, the thickness of the side of the reservoir cannot always be neglected. We shall therefore examine the conductivity of a channel formed by a cylindrical boring through an obstructing plate bounded by parallel planes, and, though we fail to solve the problem rigorously, we shall obtain information sufficient for most practical purposes. The thickness of the plate we shall call L, and the radius of the cylindrical channel R.

Fig. 59.



Whatever the resistance of the channel may be, it will be lessened by the introduction of infinitely thin discs of perfect conductivity at A and B fig. 59. The effect of the discs is to produce constant potential over their areas, and the problem thus modified is susceptible of rigorous solution. Outside A and B the motion is the same as that previously investigated, when the obstructing plate is infinitely thin; between A and B the flow is uniform. The resistance is therefore on the whole

$$\frac{1}{2R}+\frac{L}{\pi R^2},$$

whence
$$c=\frac{\pi R^2}{L+\frac{1}{2}\pi R}\ \dots\dots\dots\dots\dots\dots\dots\dots(1).$$

If α denote the correction, which must be added to L on account of an open end,

$$\alpha=\tfrac{1}{4}\pi R\ \dots\dots\dots\dots\dots\dots\dots\dots(2).$$

This correction is in general under the mark, but, when L is very small in comparison with R, the assumed motion coincides more and more nearly with the actual motion, and thus the value of α in (2) tends to become correct.

A superior limit to the resistance may be calculated from a hypothetical motion of the fluid. For this purpose we will suppose infinitely thin pistons introduced at A and B, the effect of which will be to make the normal velocity constant at those places. Within the tube the flow will be uniform as before, but for the external space we have a new problem to consider:—To determine the motion of a fluid bounded by an infinite plane, the normal velocity over a circular area of the plane having a given constant value, and over the remainder of the plane being zero.

The potential may still be regarded as due to matter distributed over the disc, but it is no longer constant over the area; the *density* of the matter, however, being proportional to $d\phi/dn$ is constant.

The kinetic energy of the motion

$$=\tfrac{1}{2}\iint\phi\frac{d\phi}{dn}d\sigma=\tfrac{1}{2}\frac{d\phi}{dn}\iint\phi\, d\sigma,$$

the integration going over the area of the circle.

The total current through the plane

$$=\iint \frac{d\phi}{dn}\,d\sigma = \pi R^2 \frac{d\phi}{dn}.$$

Hence
$$\frac{2 \text{ kinetic energy}}{(\text{current})^2} = \frac{\iint \phi\, d\sigma}{\pi^2 R^4 \dfrac{d\phi}{dn}}.$$

If the density of the matter be taken as unity, $d\phi/dn = 2\pi$, and the required ratio is expressed by $P/\pi^3 R^4$, where P denotes the potential on itself of a circular layer of matter of unit density and of radius R.

The simplest method of calculating P depends upon the consideration that it represents the work required to break up the disc into infinitesimal elements and to remove them from each other's influence[1]. If we take polar co-ordinates (ρ, θ), the pole being at the edge of the disc whose radius is a, we have for the potential at the pole, $V = \iint d\theta\, d\rho$, the limits of ρ being 0 and $2a\cos\theta$, and those of θ being $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

Thus
$$V = 4a \qquad (3).$$

Now let us cut off a strip of breadth da from the edge of the disc. The work required to remove this to an infinite distance is $2\pi a\, da \,.\, 4a$. If we gradually pare the disc down to nothing and carry all the parings to infinity[2], we find for the total work by integrating with respect to a from 0 to R,

$$P = \frac{8\pi R^3}{3}.$$

The limit to the resistance (for one side) is thus $8/3\pi^2 R$; we conclude that the resistance of the whole channel is less than

$$\frac{L}{\pi R^2} + \frac{16}{3\pi^2 R} \qquad (4).$$

Collecting our results, we see that

$$\frac{\pi}{4} R < \alpha < \frac{8}{3\pi} R \qquad (5),$$

[1] A part of § 302 is repeated here for the sake of those who may wish to avoid the difficulties of the more complete investigation.

[2] This method of calculating P was suggested to the author by Professor Clerk Maxwell.

or in decimals,

$$\left.\begin{array}{l}\alpha > \cdot 785\, R \\ \alpha < \cdot 849\, R\end{array}\right\} \text{..........................(6).}$$

It must be observed that α here denotes the correction for one end. The whole resistance corresponds to a length $L + 2\alpha$ of tube having the section πR^2.

When L is very great in relation to R, we may take simply

$$c = \frac{\pi R^2}{L} \text{ .... (7).}$$

In this case we have from (6) § 304

$$\lambda = \frac{2\sqrt{\pi} \, . \, \sqrt{(SL)}}{R} \text{ ....(8).}$$

The correction for an open end (α) is a function of L, coinciding with the lower limit, viz. $\frac{1}{4}\pi R$, when L vanishes. As L increases, α increases with it; but does not, even when L is infinite, attain the superior limit $8R/3\pi$. For consider the motion going on in any middle piece of the tube. The kinetic energy is greater than corresponds merely to the length of the piece. If therefore the piece be removed, and the free ends brought together, the motion otherwise continuing as before, the kinetic energy will be diminished more than corresponds to the length of the piece subtracted. *A fortiori* will this be true of the real motion which would exist in the shortened tube. That, when $L = \infty$, α does not become $8R/3\pi$ is evident, because the normal velocity at the end, far from being constant, as was assumed in the calculation of this result, must increase from the centre outwards and become infinite at the edge.

A further approximation to the value of α may be obtained by assuming a variable velocity at the plane of the mouth. The calculation will be found in Appendix A. It appears that in the case of an infinitely long tube α cannot be so great as $\cdot 82422\, R$. The real value of α is probably not far from $\cdot 82\, R$.

308. Besides the cylinder there are very few forms of channel whose conductivity can be determined mathematically. When however the form is approximately cylindrical we may obtain limits, which are useful as allowing us to estimate the effect of such departures from mathematical accuracy as must occur in practice.

An inferior limit to the resistance of any elongated and approximately straight conductor may be obtained immediately by the imaginary introduction of an infinite number of plane perfectly conducting layers perpendicular to the axis. If σ denote the area of the section at any point x, the resistance between two layers distant dx will be $\sigma^{-1}dx$, and therefore the whole actual resistance is certainly greater than

$$\int \sigma^{-1}dx \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

unless indeed the conductor be truly cylindrical.

In order to find a superior limit we may calculate the kinetic energy of the current on the hypothesis that the velocity parallel to the axis is uniform over each section. The hypothetical motion is that which would follow from the introduction of an infinite number of rigid pistons moving freely, and the calculated result is necessarily in excess of the truth, unless the section be absolutely constant. We shall suppose for the sake of simplicity that the channel is symmetrical about an axis, in which case of course the motion of the fluid is symmetrical also.

If U denote the total current, we have *ex hypothesi* for the axial velocity at any point x

$$u = \sigma^{-1} U \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

from which the radial velocity v is determined by the equation of continuity (6 § 238),

$$\frac{d(ru)}{dx} + \frac{d(rv)}{dr} = 0.$$

Thus
$$rv = \text{const.} - \tfrac{1}{2} U r^2 \frac{d\sigma^{-1}}{dx},$$

or, since there is no source of fluid on the axis,

$$v = -\tfrac{1}{2} U r \frac{d\sigma^{-1}}{dx} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The kinetic energy may now be calculated by simple integration:—

$$\int u^2\sigma\, dx = U^2 \int \sigma^{-1} dx,$$

$$\iint v^2 2\pi r\, dr\, dx = \frac{\pi U^2}{8} \int y^4 \left(\frac{d\sigma^{-1}}{dx}\right)^2 dx,$$

if y be the radius of the channel at the point x, so that $\sigma = \pi y^2$.

Thus $$\frac{2 \text{ kinetic energy}}{(\text{current})^2} = \int \frac{1}{\pi y^2} \left\{1 + \tfrac{1}{2}\left(\frac{dy}{dx}\right)^2\right\} dx \quad\text{.........(4).}$$

This is the quantity which gives a superior limit to the resistance. The first term, which corresponds to the component velocity u, is the same as that previously obtained for the lower limit, as might have been foreseen. The difference between the two, which gives the utmost error involved in taking either of them as the true value, is

$$\frac{1}{2\pi}\int \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 dx \quad\text{.......................(5).}$$

In a nearly cylindrical channel dy/dx is a small quantity and so the result found in this manner is closely approximate. It is not necessary that the section should be nearly constant, but only that it should vary slowly. The success of the approximation in this and similar cases depends upon the fact that the quantity to be estimated is at a minimum. Any reasonable approximation to the real motion will give a result very near the truth according to the principles of the differential calculus.

By means of the properties of the potential and stream functions (§ 238) the present problem admits of actual approximate solution. If ϕ and ψ denote the values of these functions at any point x, r; u, v denote the axial and transverse velocities,

$$u = \frac{d\phi}{dx} = \frac{1}{r}\frac{d\psi}{dr}, \qquad v = \frac{d\phi}{dr} = -\frac{1}{r}\frac{d\psi}{dx} \quad\text{...........(6),}$$

whence by elimination

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \frac{d^2\phi}{dx^2} = 0 \quad\text{.................. (7),}$$

$$\frac{d^2\psi}{dr^2} - \frac{1}{r}\frac{d\psi}{dr} + \frac{d^2\psi}{dx^2} = 0 \quad\text{.................. (8).}$$

If F denote the value of ϕ as a function of x when $r=0$, the general values of ϕ and ψ may be expressed in terms of F by means of (7) and (8) in the series

$$\left.\begin{aligned}\phi &= F - \frac{r^2 F''}{2^2} + \frac{r^4 F^{\text{iv}}}{2^2 \cdot 4^2} - \frac{r^6 F^{\text{vi}}}{2^2 \cdot 4^2 \cdot 6^2} + \dots \\ \psi &= \frac{r^2 F'}{2} - \frac{r^4 F'''}{2^2 \cdot 4} + \frac{r^6 F^{\text{v}}}{2^2 \cdot 4^2 \cdot 6} - \frac{r^8 F^{\text{vii}}}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \dots\end{aligned}\right\} \dots\dots\dots (9),$$

where accents denote differentiation with respect to x. At the boundary of the channel where $r=y$, ψ is constant, say ψ_1. Then

$$\psi_1 = \frac{y^2 F'}{2} - \frac{y^4 F'''}{2^2 \cdot 4} + \frac{y^6 F^{\text{v}}}{2^2 \cdot 4^2 \cdot 6} - \dots\dots\dots\dots (10)$$

is the equation connecting y and F. In the present problem y is given, and we have to express F by means of it. By successive approximation we obtain from (10)

$$F' = \frac{2\psi_1}{y^2} + \frac{y^2}{8}\left\{\frac{d^2}{dx^2}\left(\frac{2\psi_1}{y^2}\right) + \frac{1}{8}\frac{d^2}{dx^2}\, y^2 \frac{d^2}{dx^2}\left(\frac{2\psi_1}{y^2}\right)\right\}$$
$$- \frac{y^4}{12 \cdot 4^2}\frac{d^4}{dx^4}\left(\frac{2\psi_1}{y^2}\right) \dots\dots\dots\dots (11).$$

The total stream is given by the integral

$$\int_0^y \frac{d\phi}{dx}\, 2\pi r\, dr = \int_0^y \frac{1}{r}\frac{d\psi}{dr}\, 2\pi r\, dr = 2\pi\psi_1;$$

and therefore the resistance between any two equipotential surfaces is represented by

$$\frac{1}{2\pi\psi_1}\int F' dx.$$

The expression for the resistance admits of considerable simplification by integration by parts in the case when the channel is truly cylindrical in the neighbourhood of the limits of integration. In this way we find for the final result,

$$\text{resistance} = \int \frac{dx}{\pi y^2}\left\{1 + \tfrac{1}{2}y'^2 - \frac{(3y'^2 - yy'')^2}{48}\right\} \dots\dots\dots (12)[1],$$

y', y'' denoting the differential coefficients of y with respect to x.

It thus appears that the superior limit of the preceding investigation is in fact the correct result to the second order of

[1] *Proceedings of the London Mathematical Society*, Vol. VII. p. 70, 1876.

approximation. If we regard y as a function of ωx, where ω is a small quantity, (12) is correct as far as terms containing ω^4.

309. Our knowledge of the laws on which the pitch of resonators depends, is due to the labours of several experimenters and mathematicians.

The observation that for a given mouthpiece the pitch of a resonator depends mainly upon the volume S is due to Liscovius, who found that the pitch of a flask partly filled with water was not altered when the flask was inclined. This result was confirmed by Sondhauss[1]. The latter observer found further, that in the case of resonators without necks, the influence of the aperture depended mainly upon its area, although when the shape was very elongated, a certain rise of pitch ensued. He gave the formula

$$N = 52400 \frac{\sigma^{\frac{1}{4}}}{S^{\frac{1}{2}}} \qquad \text{(1)},$$

the unit of length being the millimetre.

The *theory* of this kind of resonator we owe to Helmholtz[2], whose formula is

$$N = \frac{a\sigma^{\frac{1}{4}}}{2^{\frac{1}{2}} \pi^{\frac{5}{4}} S^{\frac{1}{2}}} \qquad \text{(2)},$$

applicable to circular apertures.

For flasks with long necks, Sondhauss[3] found

$$N = 46705 \frac{\sigma^{\frac{1}{2}}}{L^{\frac{1}{2}} S^{\frac{1}{2}}} \qquad \text{(3)},$$

corresponding to the theoretical

$$N = \frac{a}{2\pi} \frac{\sigma^{\frac{1}{2}}}{L^{\frac{1}{2}} S^{\frac{1}{2}}} \qquad \text{(4)}.$$

In practice it does not often happen either that the neck is so long that the correction for the open ends can be neglected, as (4) supposes, or, on the other hand, so short that it can itself be neglected, as supposed in (2). Wertheim[4] was the first

[1] Ueber den Brummkreisel und das Schwingungsgesetz der cubischen Pfeifen. *Pogg. Ann.* LXXXI. pp. 235, 347. 1850.

[2] Crelle, Bd. LVII. 1—72. 1860.

[3] Ueber die Schallschwingungen der Luft in erhitzten Glasröhren und in gedeckten Pfeifen von ungleicher Weite. *Pogg. Ann.* LXXIX. p. 1. 1850.

[4] Mémoire sur les vibrations sonores de l'air. *Ann. d. Chim.* (3) XXXI. p. 385. 1851.

to shew that the effect of an open end could be represented by an addition (α) to the length, independent, or nearly so, of L and λ.

The approximate theoretical determination of α is due to Helmholtz, who gave $\frac{1}{4}\pi R$ as the correction for an open end fitted with an infinite flange. His method consisted in inventing forms of tube for which the problem was soluble, and selecting that one which agreed most nearly with a cylinder. The correction $\frac{1}{4}\pi R$ is rigorously applicable to a tube whose radius at the open end and at a great distance from it is R, but which in the neighbourhood of the open end bulges slightly.

From the fact that the true cylinder may be derived by introducing an obstruction, we may infer that the result thus obtained is too small.

It is curious that the process followed in this work, which was first given in the memoir on resonance, leads to exactly the same result, though it would be difficult to conceive two methods more unlike each other.

The correction to the length will depend to some extent upon whether the flow of air from the open end is obstructed, or not. When the neck projects into open space, there will be less obstruction than when a backward flow is prevented by a flange as supposed in our approximate calculations. However, the uncertainty introduced in this way is not very important, and we may generally take $\alpha = \frac{1}{4}\pi R$ as a sufficient approximation. In practice, when the necks are short, the hypothesis of the flange agrees pretty well with fact, and when the necks are long, the correction is itself of subordinate importance.

The general formula will then run

$$N = \frac{a}{2\pi}\sqrt{\frac{\sigma}{S\{L + \frac{1}{2}\sqrt{(\pi\sigma)}\}}} \quad\text{..................(5)},$$

where σ is the area of the section of the neck, or in numbers

$$N = \frac{a}{6{\cdot}2832}\,\frac{\sigma^{\frac{1}{4}}}{S^{\frac{1}{2}}\sqrt{(L + {\cdot}8863\sqrt{\sigma})}} \quad\text{..................(6)}.$$

A formula not differing much from this was given, as the embodiment of the results of his measurements, by Sondhauss[1] who

[1] *Pogg. Ann.* CXL. pp. 53, 219. 1870.

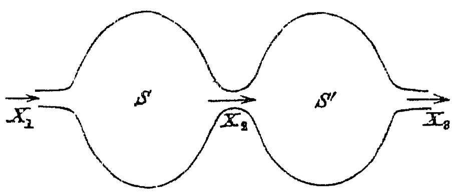
at the same time expressed a conviction that it was no mere empirical formula of interpolation, but the expression of a natural law. The theory of resonators with necks was given about the same time[1] in a memoir 'on Resonance' published in the *Philosophical Transactions* for 1871, from which most of the last few pages is derived.

310 The simple method of calculating the pitch of resonators with which we have been occupied is applicable to the gravest mode of vibration only, the character of which is quite distinct. The overtones of resonators with contracted necks are relatively very high, and the corresponding modes of vibration are by no means independent of the inertia of the air in the interior of the reservoir. The character of these modes will be more evident, when we come to consider the vibrations of air within a completely closed vessel, such as a sphere, but it will rarely happen that the pitch can be calculated theoretically.

There are, however, cases of multiple resonance to which our theory is applicable. These occur when two or more vessels communicate by channels with each other and with the external air; and are readily treated by Lagrange's method, provided of course that the wave-length of the vibration is sufficiently large in comparison with the dimensions of the vessels.

Suppose that there are two reservoirs, S, S', communicating with each other and with the external air by narrow passages or

Fig. 60.



necks. If we were to consider SS' as a single reservoir and apply our previous formula, we should be led to an erroneous result; for that formula is founded on the assumption that within the reservoir the inertia of the air may be left out of account, whereas it is evident that the energy of the motion through the connecting passage may be as great as through the two others. However, an

[1] *Proceedings of the Royal Society*, Nov. 24, 1870.

investigation on the same general plan as before meets the case perfectly. Denoting by X_1, X_2, X_3 the total transfers of fluid through the three passages, we have as in (2) § 304 for the kinetic energy the expression

$$T = \tfrac{1}{2}\rho \left\{ \frac{\dot{X}_1^2}{c_1} + \frac{\dot{X}_2^2}{c_2} + \frac{\dot{X}_3^2}{c_3} \right\} \text{..................(1)},$$

and for the potential energy,

$$V = \tfrac{1}{2}\rho a^2 \left\{ \frac{(X_2 - X_1)^2}{S} + \frac{(X_3 - X_2)^2}{S'} \right\} \text{..............(2)}.$$

An application of Lagrange's method gives as the differential equations of motion,

$$\left.\begin{aligned} \frac{\ddot{X}_1}{c_1} + a^2 \frac{X_1 - X_2}{S} &= 0 \\ \frac{\ddot{X}_2}{c_2} + a^2 \left\{ \frac{X_2 - X_1}{S} + \frac{X_2 - X_3}{S'} \right\} &= 0 \\ \frac{\ddot{X}_3}{c_3} + a^2 \frac{X_3 - X_1}{S'} &= 0 \end{aligned}\right\} \text{................(3)}.$$

By addition and integration,

$$\frac{X_1}{c_1} + \frac{X_2}{c_2} + \frac{X_3}{c_3} = 0 \text{......................(4)}.$$

Hence on elimination of X_2,

$$\left.\begin{aligned} \ddot{X}_1 + \frac{a^2}{S} \left\{ (c_1 + c_2) X_1 + \frac{c_1 c_2}{c_3} X_3 \right\} &= 0 \\ \ddot{X}_3 + \frac{a^2}{S'} \left\{ (c_3 + c_2) X_3 + \frac{c_3 c_2}{c_1} X_1 \right\} &= 0 \end{aligned}\right\} \text{.........(5)}.$$

Assuming $X_1 = Ae^{pt}$, $X_3 = Be^{pt}$, we obtain on substitution and determination of $A : B$,

$$p^4 + p^2 a^2 \left\{ \frac{c_1 + c_2}{S} + \frac{c_3 + c_2}{S'} \right\} + \frac{a^4}{SS'} \left\{ c_1 c_3 + c_2 (c_1 + c_3) \right\} = 0 \text{...(6)},$$

as the equation to determine the natural tones. If N be the frequency of vibration, $N^2 = -p^2/4\pi^2$, the two values of p^2 being of course real and negative. The formula simplifies considerably if $c_3 = c_1$, $S' = S$; but it will be more instructive to work out this case from the beginning. Let $c_1 = c_3 = mc_2 = mc$.

The differential equations take the form

$$\left.\begin{aligned}\ddot{X}_1+\frac{a^2c}{S}\{(1+m)X_1+X_3\}=0\\ \ddot{X}_3+\frac{a^2c}{S}\{(1+m)X_3+X_1\}=0\end{aligned}\right\}\text{............ (7)},$$

while from (4) $X_2=-\dfrac{X_1+X_3}{m}$.

Hence

$$\left.\begin{aligned}(\ddot{X}_1+\ddot{X}_3)+\frac{a^2c}{S}(m+2)(X_1+X_3)=0\\ (\ddot{X}_1-\ddot{X}_3)+\frac{a^2c}{S}m(X_1-X_3)=0\end{aligned}\right\}\text{......... (8)}.$$

The whole motion may be divided into two parts. For the first of these

$$X_1+X_3=0\text{.........................(9)},$$

which requires that $X_2=0$. The motion is therefore the same as might take place were the communication between S and S' cut off, and has its frequency given by

$$N^2=\frac{a^2c_1}{4\pi^2S}=\frac{a^2mc}{4\pi^2S}\text{....................(10)}.$$

The density of the air is the same in both reservoirs.

For the other component part, $X_1-X_3=0$, so that

$$X_2=-\frac{2X_1}{m};\qquad N'^2=\frac{a^2(m+2)c}{4\pi^2S}\text{..........(11)}.$$

The vibrations are thus opposed in phase. The ratio of frequencies is given by $N'^2:N^2=m+2:m$, shewing that the second mode has the shorter period. In this mode of vibration the connecting passage acts in some measure as a second opening to both vessels, and thus raises the pitch. If the passage be contracted, the interval of pitch between the two notes is small.

A particular case of the general formula worthy of notice is obtained by putting $c_3=0$, which amounts to suppressing one of the communications with the external air. We thus obtain

$$p^4+a^2p^2\left(\frac{c_1+c_2}{S}+\frac{c_2}{S'}\right)+\frac{a^4c_1c_2}{SS'}=0\text{...........(12)},$$

or, if $$S = S', \quad c_1 = mc_2 = mc,$$

$$p^4 + a^2p^2 \frac{(m+2)c}{S} + \frac{a^4 m c^2}{S^2} = 0 \text{..............}(13),$$

whence $$N^2 = \frac{a^2 c}{8\pi^2 S} \{m + 2 \pm \sqrt{(m^2+4)}\} \text{ .... } (14).$$

If we further suppose $m = 1$, or $c_2 = c_1$,

$$N^2 = \frac{a^2 c}{8\pi^2 S} (3 \pm \sqrt{5}).$$

If N' be the frequency for a simple resonator (S, c),

$$N'^2 = \frac{a^2 c}{4\pi^2 S},$$

and thus $$N_1^2 : N'^2 = \frac{3 + \sqrt{5}}{2} = 2{\cdot}618,$$

$$N'^2 : N_2^2 = \frac{2}{3 - \sqrt{5}} = 2{\cdot}618.$$

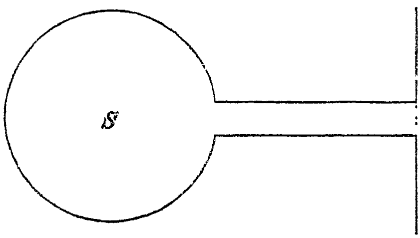
It appears that the interval from N_1 to N' is the same as from N' to N_2, namely, $\sqrt{(2{\cdot}618)} = 1{\cdot}618$, or rather more than a fifth. It will be found that whatever the value of m may be, the interval between the two tones cannot be less than 2·414, which is about an octave and a minor third. The corresponding value of m is 2.

A similar method is applicable to any combination, however complicated, of reservoirs and connecting passages under the single restriction as to the comparative magnitudes of the reservoirs and wave-lengths; but the example just given is sufficient to illustrate the theory of multiple resonance. A few measurements of the pitch of double resonators are detailed in my memoir on resonance, already referred to.

311. The equations which we have employed hitherto take no account of the escape of energy from a resonator. If there were really no transfer of energy between a resonator and the external atmosphere, the motion would be isolated and of little practical interest: nevertheless the characteristic of a resonator consists in its vibrations being in great measure independent. Vibrations, once excited, will continue for a considerable number of periods without much loss of energy, and their frequency will be almost entirely independent of the rate of dissipation. The rate of dissipation is, however, an important feature in the character of a resonator, on which its behaviour under certain circumstances materially depends. It will be understood that the dissipation here spoken of means only the escape of energy from the vessel and its neighbourhood, and its diffusion in the surrounding medium, and not the transformation of ordinary energy into heat. Of such transformation our equations take no account, unless special terms be introduced for the purpose of representing the effects of viscosity, and of the conduction and radiation of heat.

[The influence of the conduction of heat has been considered by Koláček[1].]

Fig. 61.



In a previous chapter (§ 278) we saw how to express the motion on the right of the infinite flange (Fig. 61), in terms of the normal velocity of the fluid over the disc A. We found, § 278 (3),

$$\phi = -\frac{1}{2\pi}\iint \frac{d\phi}{dn}\frac{e^{-ikr}}{r}\,d\sigma,$$

where ϕ is proportional to e^{int}.

If r be the distance between any two points of the disc, kr is a small quantity, and $e^{-ikr} = 1 - ikr$ approximately.

Thus $$\phi_A = -\frac{1}{2\pi}\left(\iint \frac{d\phi}{dn}\frac{d\sigma}{r} - ik\iint \frac{d\phi}{dn}\,d\sigma\right) \text{...........(1).}$$

The first term depends upon the distribution of the current. If we suppose that $d\phi/dn$ is constant, we obtain ultimately a term representing an increase of inertia, or a correction to the length, equal to $8R/3\pi$. This we have already considered, under the supposition of a piston at A. The second term, on which the dissipation depends, is independent of the distribution of current,

[1] *Wied. Ann.* t. 12, p. 353, 1881.

being a function of the total current ($\dot{X}$) only. Confining our attention to this term, we have

$$\phi_A = \frac{ik\dot{X}}{2\pi} \quad \text{.........................} \quad (2).$$

Assuming now that $\phi \propto e^{int}$, we have for the part of the variation of pressure at A, on which dissipation depends,

$$\delta p = -\rho\dot{\phi}_A = -i\rho n\phi_A = \frac{\rho nk\dot{X}}{2\pi} = \frac{\rho n^2\dot{X}}{2\pi a} \quad \text{.........} \quad (3).$$

The corresponding work done during a transfer of fluid δX is $\frac{\rho n^2\dot{X}}{2\pi a}\delta X$; and since, as in § 304, the expressions for the potential and kinetic energies are

$$V = \tfrac{1}{2}\rho a^2\frac{X^2}{S}, \qquad T = \tfrac{1}{2}\rho\frac{\dot{X}^2}{c} \quad \text{...............} \quad (4),$$

the equation of motion (§ 80) is

$$\ddot{X} + \frac{n^2c}{2\pi a}\dot{X} + \frac{a^2c}{S}X = 0 \quad \text{................} \quad (5)^1,$$

in place of (3) § 304. In the valuation of c an allowance must be included for the inertia of the fluid on the right-hand side of A, corresponding to the term omitted in the expression for δp.

Equation (5) is of the standard form for the free vibrations of dissipative systems of one degree of freedom (§ 45). The amplitude varies as $e^{-n^2ct/4\pi a}$, being diminished in the ratio $e : 1$ after a time equal to $4\pi a/n^2c$. If the pitch (determined by n) be given, the vibrations have the greatest persistence when c is smallest, that is, when the neck is most contracted.

If S be given, we have on substituting for c its value in terms of S and n,

$$\frac{4\pi a}{n^2c} = \frac{4\pi a^3}{n^4S} \quad \text{.........................} \quad (6),$$

shewing that under these circumstances the duration of the motion increases rapidly as n diminishes.

In the case of similar resonators $c \propto n^{-1}$, and then

$$\frac{4\pi a}{n^2c} \propto \frac{1}{n},$$

[1] Equation (5) is only approximate, inasmuch as the dissipative force is calculated on the supposition that the vibration is permanent; but this will lead to no material error when the dissipation is small.

which shews that in this case the same proportional loss of amplitude always occurs after the lapse of the same number of periods. This result may be obtained by the method of dimensions, as a consequence of the principle of dynamical similarity.

As an example of (5), I may refer to the case of a globe with a neck, intended for burning phosphorus in oxygen gas, whose capacity is ·251 cubic feet [7100 c.c.]. It was found by experiment that the note of maximum resonance made 120 vibrations per second, so that $n = 120 \times 2\pi$. Taking the velocity of sound (a) at 1120 feet [34200 cent.] per second, we find from these data

$$\frac{4\pi a^3}{n^4 S} = \frac{1}{5} \text{ of a second nearly.}$$

Judging from the sound produced when the globe is struck, I think that this estimate must be too low; but it should be observed that the absence of the infinite flange assumed in the theory must influence very materially the rate of dissipation.

We will now examine the forced vibrations due to a source of sound external to the resonator. If the pressure δp at the mouth of the resonator due to the source, i.e. calculated on the supposition that the mouth is closed, be $F e^{ikat}$, the equation of motion corresponding to (5), but applicable to the forced vibration only, is

$$\frac{\rho}{c}\ddot{X} + \frac{\rho k^2 a}{2\pi}\dot{X} + \frac{\rho a^2}{S}X = F e^{ikat} \quad \text{..............(7).}$$

If $X = X_0 e^{i(kat+\epsilon)}$, where X_0 is real,

$$\frac{F^2}{\rho^2 a^4 X_0^2} = \left(\frac{1}{S} - \frac{k^2}{c}\right)^2 + \left(\frac{k^3}{2\pi}\right)^2.$$

The maximum variation of pressure (G) inside the resonator is connected with X_0 by the equation

$$G = \frac{a^2 \rho X_0}{S} \quad \text{..........................(8),}$$

since $X_0 \div S$ is the maximum condensation. Thus

$$\frac{F^2}{G^2} = \left(1 - \frac{k^2 S}{c}\right)^2 + \left(\frac{k^3 S}{2\pi}\right)^2 \quad \text{.................(9),}$$

which agrees with the equation obtained by Helmholtz for the case where the communication with the external air is by a simple aperture (§ 306). The present problem is nearly, but not quite, a case of that treated in § 46, the difference depending upon the fact that the coefficient of dissipation in (7) is itself a function of the period, and not an absolutely constant quantity. If the period, determined by k, and S be given, (9) shews that the internal variation of pressure (G) is a maximum when $c = k^2 S$, that is, when the natural note of the resonator (calculated without allowance for dissipation) is the same as that of the generating sound. The maximum vibration, when the coincidence of periods is perfect, varies inversely as S; but, if S be small, a very slight inequality in the periods is sufficient to cause a marked falling off in the intensity of the resonance (§ 49). In the practical use of resonators it is not advantageous to carry the reduction of S and c very far, probably because the arrangements necessary for connecting the interior with the ear or other sensitive apparatus involve a departure from the suppositions on which the calculations are founded, which becomes more and more important as the dimensions are reduced. When the sensitive apparatus is not in connection with the interior, as in the experiment of reinforcing the sound of a tuning-fork by means of a resonator, other elements enter into the question, and a distinct investigation is necessary (§ 319).

In virtue of the principle of reciprocity the investigation of the preceding paragraph may be applied to calculate the effect of a source of sound situated in the interior of a resonator.

312. We now pass on to the further discussion of the problem of the open pipe. We shall suppose that the open end of the pipe is provided with an infinite flange, and that its diameter is small in comparison with the wave-length of the vibration under consideration.

As an introduction to the question, we will further suppose that the mouth of the pipe is fitted with a freely moving piston without thickness and mass. The preceding problems, from which the present differs in reality but little, have already given us reason to think that the presence of the piston will cause no important modification. Within the tube we suppose (§ 255) that the velocity-potential is

$$\phi = (A \cos kx + B \sin kx)\, e^{int} \text{.................(1)},$$

where, as usual, $k = 2\pi/\lambda = n/a$. At the mouth, where $x = 0$,

$$\phi_0 = A e^{int}; \qquad \left(\frac{d\phi}{dx}\right)_0 = kB\, e^{int} \text{..............(2)}.$$

On the right of the piston the relation between ϕ_0 and $\left(\frac{d\phi}{dx}\right)_0$ is by § 302

$$\iint \phi_0 d\sigma \; : \; \left(\frac{d\phi}{dx}\right)_0 = i\,\frac{\pi R^2}{k}\left\{1 - \frac{J_1(2kR)}{kR}\right\} - \frac{\pi}{2k^2} K_1(2kR)\ldots\ldots(3),$$

R being the radius of the pipe. From this the solution of the problem may be obtained without any restriction as to the smallness of kR: since, however, it is only when kR is small that the presence of the piston would not materially modify the question, we may as well have the benefit of the simplification at once by taking as in (1) § 311

$$\iint \phi_0 \, d\sigma \; : \; \left(\frac{d\phi}{dx}\right)_0 = \frac{i\pi k R^4}{2} - \frac{8R^3}{3}\ldots\ldots\ldots\ldots\ldots(4).$$

Now, since the piston occupies no space, the values of $(d\phi/dx)_0$ must be the same on both sides of it; and since there is no mass, the like must be true of the values of $\iint \phi_0 d\sigma$. Thus

$$A\sigma = kB\left\{-\frac{8R^3}{3} + i\,\frac{k\pi R^4}{2}\right\}$$

or

$$A = B\left\{-\frac{8kR}{3\pi} + i\,\frac{k^2R^2}{2}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Substituting in (1), we find on rejecting the imaginary part, and putting for brevity $B = 1$,

$$\phi = \left\{\sin kx - \frac{8kR}{3\pi}\cos kx\right\}\cos nt - \tfrac{1}{2}k^2R^2\cos kx\,\sin nt\ldots\ldots(6).$$

In this expression the term containing $\sin nt$ depends upon the dissipation, and is the same as if there were no piston, while that involving $8kR/3\pi$ represents the effect of the inertia of the external air in the neighbourhood of the mouth. In order to compare with previous results, let α be such that

$$\sin kx - \frac{8kR}{3\pi}\cos kx = \sin k(x - \alpha);$$

then, the squares of small quantities being neglected,

$$\alpha = \frac{8R}{3\pi}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

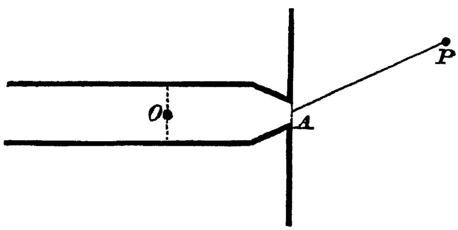
and

$$\phi = \sin k(x - \alpha)\;\cos nt - \tfrac{1}{2}k^2R^2\cos kx\,\sin nt\ldots\ldots\ldots\ldots(8).$$

These formulæ shew that, if the dissipation be left out of account, the velocity-potential is the same as if the tube were lengthened by $8/3\pi$ of the radius, and the open end then behaved as a loop. The amount of the correction agrees with what previous investigations would have led us to expect as the result of the introduction of the piston. We have seen reason to know that the true value of α lies between $\frac{1}{4}\pi R$ and $8R/3\pi$, and that the presence of the piston does not affect the term representing the dissipation. But, before discussing our results, it will be advantageous to investigate them afresh by a rather different method, which besides being of somewhat greater generality, will help to throw light on the mechanics of the question.

313. For this purpose it will be convenient to shift the origin in the negative direction to such a distance from the mouth that the waves are there approximately plane, a displacement which according to our suppositions need not amount to more than a small fraction of the wave-length. The difficulty of the question consists in finding the connection between the waves in the pipe, which at a sufficient distance from the mouth are plane, and the diverging waves outside, which at a moderate distance may be treated as spherical. If the transition take place within a space small compared with the wave-length, which it must evidently do, if the diameter be small enough, the problem admits of solution, whatever may be the form of the pipe in the neighbourhood of the mouth.

Fig. 62.



At a point P, whose distance from A is moderate, the velocity-potential is (§ 279)

$$\psi = \frac{A'}{r} e^{-ikr} e^{int} \quad\text{.......................(1)},$$

whence

$$\frac{d\psi}{dr} = -\frac{A' e^{i(nt-kr)}}{r^2}(1 + ikr) \quad\text{.................(2)}.$$

Let us consider the behaviour of the mass of air included between the plane section at O and a hemispherical surface whose centre is A, and radius r, r being large in comparison with the diameter of the pipe, but small in comparison with the wave-length. Within this space the air must move approximately as an incompressible fluid would do. Now the current across the hemispherical surface

$$= 2\pi r^2 \frac{d\psi}{dr} = -2\pi A'(1 + ikr)e^{i(nt-kr)} = -2\pi A' e^{int} \quad\ldots\ldots\ldots(3),$$

if the square of kr be neglected.

If, as before, we take for the velocity-potential within the pipe

$$\phi = (A \cos kx + B \sin kx)\, e^{int} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

we have for the current across the section at O,

$$\sigma \left(\frac{d\phi}{dx}\right)_0 = \sigma k B\, e^{int} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5);$$

and thus

$$\sigma k B = -2\pi A' \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

This is the first condition; the second is to be found from the consideration that the total current (whose two values have just been equated) is proportional to the difference of potential at the terminals. Thus, if c denote the conductivity of the passage between the terminal surfaces,

$$\sigma \left(\frac{d\phi}{dx}\right)_0 = c\,(\psi_r - \phi_0),$$

or

$$\frac{\sigma k B}{c} = \frac{A'}{r} e^{-ikr} - A \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

On substituting for A' its value from (6), we have

$$-A = \sigma k B \left(\frac{1}{c} + \frac{e^{-ikr}}{2\pi r}\right) = \sigma k B \left\{\frac{1}{c} + \frac{1}{2\pi r} - \frac{ik}{2\pi}\right\}.$$

In this expression the second term is negligible in comparison with the first, for c is at most a quantity of the same order as the radius of the tube, and when the mouth is much contracted it is smaller still. Thus we may take

$$A = \sigma k B \left(-\frac{1}{c} + \frac{ik}{2\pi}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Substituting this in (4), we have for the imaginary expression of the velocity-potential within the tube, if B be put equal to unity,

$$\phi = \left\{\sin kx + \sigma k \left(-\frac{1}{c} + \frac{ik}{2\pi}\right) \cos kx\right\} e^{int},$$

or, if only the real part be retained,

$$\phi = \left\{\sin kx - \frac{\sigma k}{c}\cos kx\right\}\cos nt - \frac{k^2\sigma}{2\pi}\cos kx \sin nt \ldots\ldots(9).$$

Following Helmholtz, we may simplify our results by introducing a quantity α defined by the equation

$$\tan k\alpha = \frac{k\sigma}{c} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Thus

$$\phi = \frac{\sin k(x-\alpha)}{\cos k\alpha}\cos nt - \frac{k^2\sigma}{2\pi}\cos kx \sin nt \ldots\ldots\ldots\ldots(11),$$

and the corresponding potential outside the mouth is

$$\psi = -\frac{\sigma k}{2\pi r}\cos(nt - kr) \ldots\ldots\ldots\ldots\ldots\ldots(12).$$

If R be the radius of the tube, we may replace σ by πR^2.

When the tube is a simple cylinder, and the origin lies at a distance ΔL from the mouth, we know that $\sigma c^{-1} = \Delta L + \mu R$, where μ is a number rather greater than $\frac{1}{4}\pi$. In such a case (the origin being taken sufficiently near the mouth) $k\alpha$ is a small quantity, and therefore from (10)

$$\alpha = \frac{\sigma}{c} = \Delta L + \mu R \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

At the same time $\cos k\alpha$ may be identified with unity. The principal term in ϕ, involving $\cos nt$, may then be calculated, as if the tube were prolonged, and there were a loop at a point situated at a distance μR beyond the actual position of the mouth, in accordance with what we found before. These results, approximate for ordinary tubes, become rigorous when the diameter is reduced without limit, friction being neglected.

If there be no flange at A, the value of c is slightly modified by the removal of what acts as an obstruction, but the principal effect is on the term representing the dissipation. If we suppose as an approximation that the waves diverging from A are spherical, we must take for the current $4\pi r^2\, d\psi/dr$ instead of $2\pi r^2\, d\psi/dr$. The ultimate effect of the alteration will be to halve the expression for the velocity-potential outside the mouth, as well as the corresponding second term in ϕ (involving $\sin nt$). The amount of dissipation is thus seen to depend materially on the degree in which the waves are free to diverge, and our analytical expressions must not be regarded as more than rough estimates.

The correct theory of the open organ-pipe, including equations (11) and (12), was discovered by Helmholtz[1], whose method, however, differs considerably from that here adopted. The earliest solutions of the problem by Lagrange, D. Bernoulli, and Euler, were founded on the assumption that at an open end the pressure could not vary from that of the surrounding atmosphere, a principle which may perhaps even now be considered applicable to an end whose openness is ideally perfect. The fact that in all ordinary cases energy escapes is a proof that there is not anywhere in the pipe an absolute loop, and it might have been expected that the inertia of the air just outside the mouth would have the effect of an increase in the length. The positions of the nodes in a sounding pipe were investigated experimentally by Savart[2] and Hopkins[3], with the result that the interval between the mouth and the nearest node is always less than the half of that separating consecutive nodes.

[The correction necessary for an open end is the origin of a departure from the simple law of octaves, which according to elementary theory would connect the notes of closed and open pipes of the same length. Thus in the application to an organ-pipe let αR denote the correction for the upper end when open, and l the length of the pipe including the correction for the mouth at the lower end. The whole effective length of the open pipe is then $l + \alpha R$, while the effective length of the pipe if closed at the upper end is l simply. The open pipe is practically the longer, and the interval between the notes is less than the octave of the simple theory[4].

It may be worthy of remark that the correction, assumed to be independent of wave-length, does not disturb the harmonic relations between the partial tones, whether a pipe be open or closed.]

314. Experimental determinations of the correction for an open end have generally been made without the use of a flange, and it therefore becomes important to form at any rate a rough estimate of its effect. No theoretical solution of the problem of an unflanged open end has hitherto been given, but it is easy to

[1] Crelle, Bd. 57, p. 1. 1860.

[2] Recherches sur les vibrations de l'air. *Ann. d. Chim.* t. XXIV. 1823.

[3] Aerial vibrations in cylindrical tubes. *Cambridge Transactions*, Vol. V. p. 231. 1833.

[4] Bosanquet, *Phil. Mag.* VI. p. 63, 1878.

see (§§ 79, 307) that the removal of the flange will reduce the correction materially below the value ·82 R (Appendix A). In the absence of theory I have attempted to determine the influence of a flange experimentally[1]. Two organ-pipes nearly enough in unison with one another to give countable beats were blown from an organ bellows; the effect of the flange was deduced from the difference in the frequencies of the beats according as one of the pipes was flanged or not. The correction due to the flange was about ·2R. A (probably more trustworthy) repetition of this experiment by Mr Bosanquet gave ·25R. If we subtract ·22R from ·82R, we obtain ·6R, which may be regarded as about the probable value of the correction for an unflanged open end, on the supposition that the wave-length is great in comparison with the diameter of the pipe.

Attempts to determine the correction entirely from experiment have not led hitherto to very precise results. Measurements by Wertheim[2] on doubly open pipes gave as a mean (for each end) ·663 R, while for pipes open at one end only the mean result was ·746 R. In two careful experiments by Bosanquet[3] on doubly open pipes the correction for one end was ·635 R, when $\lambda = 12\,R$, and ·543 R, when $\lambda = 30\,R$. Bosanquet lays it down as a general rule that the correction (expressed as a fraction of R) increases with the ratio of diameter to wave-length; part of this increase may however be due to the mutual reaction of the ends, which causes the plane of symmetry to behave like a rigid wall. When the pipe is only moderately long in proportion to its diameter, a state of things is approached which may be more nearly represented by the presence than by the absence of a flange. The comparison of theory and observation on this subject is a matter of some difficulty, because when the correction is small, its value, as calculated from observation, is affected by uncertainties as to absolute pitch and the velocity of sound, while for the case, when the correction is relatively larger, which experiment is more competent to deal with, there is at present no theory. Probably a more accurate value of the correction could be obtained from a resonator of the kind considered in § 306, where the communication with

[1] *Phil. Mag.* (5) III. 456. 1877. [The earliest experiments of the kind are those of Gripon (*Ann. d. Chim.* III. p. 384, 1874) who shewed that the effect of a large flange is proportional to the diameter of the pipe.]

[2] *Ann. d. Chim.* (3) t. XXXI. p. 394, 1851.

[3] *Phil. Mag.* (5) IV. p. 219. 1877.

the outside air is by a simple aperture; the "length" is in that case zero, and the "correction" is everything. Some measurements of this kind, in which, however, no great accuracy was attempted, will be found in my memoir on resonance[1].

[Careful experimental determinations of the correction for an unflanged open end have been made by Blaikley[2], who employed a vertical tube of thin brass 2·08 inches (5·3 cm.) in diameter. The lower part of the tube was immersed in water, the surface of which defined the "closed end," and the experiment consisted in varying the degree of immersion until the resonance to a fork of known pitch was a maximum. If the two shortest distances of the water surface from the open end thus found be l_1 and l_2, $(l_2 - l_1)$ represents the half wave-length, and the "correction for the open end" is $\frac{1}{2}(l_2 - l_1) - l_1$. The following are the results obtained by Blaikley, expressed as a fraction of the radius. They relate to the same tube resounding to forks of various pitch.

c'	253·68	·565
e'	317·46	·595
g'	380·81	·564
$b\flat'$	444·72	·587
c''	507·45	·568

The mean correction is thus ·576 R.]

Various methods have been used to determine the pitch of resonators experimentally. Most frequently, perhaps, the resonators have been made to *speak* after the manner of organ-pipes by a stream of air blown obliquely across their mouths. Although good results have been obtained in this way, our ignorance as to the mode of action of the wind renders the method unsatisfactory. In Bosanquet's experiments the pipes were not actually made to speak, but short discontinuous jets of air were blown across the open end, the pitch being estimated from the free vibrations as the sound died away. A method, similar in principle, that I have sometimes employed with advantage consists in exciting free vibrations by means of a blow. In order to obtain as well defined a note as possible, it is of importance to accommodate the hardness of the substance with which the resonator comes into contact to the pitch,

[1] *Phil. Trans.* 1871. See also Sondhauss, *Pogg. Ann.* t. 140, 53, 219 (1870), and some remarks thereupon by myself (*Phil. Mag.*, Sept. 1870).

[2] *Phil. Mag.* vol. 7, p. 339, 1879.

a low pitch requiring a soft blow. Thus the pitch of a test-tube may be determined in a moment by striking it against the bent knee.

In using this method we ought not entirely to overlook the fact that the natural pitch of a vibrating body is altered by a term depending upon the square of the dissipation. With the notation of § 45, the frequency is diminished from n to $n(1-\frac{1}{8}k^2n^{-2})$, or if x be the number of vibrations executed while the amplitude falls in the ratio $e:1$, from n to

$$n\left(1-\frac{1}{8\pi^2x^2}\right).$$

The correction, however, would rarely be worth taking into account.

The measurements given in my memoir on resonance were conducted upon a different principle by estimating the note of maximum resonance. The ear was placed in communication with the interior of the cavity, while the chromatic scale was sounded. In this way it was found possible with a little practice to estimate the pitch of a good resonator to about a quarter of a semitone. In the case of small flasks with long necks, to which the above method would not be applicable, it was found sufficient merely to hold the flask near the vibrating wires of a pianoforte. The resonant note announced itself by a quivering of the body of the flask, easily perceptible by the fingers. In using this method it is important that the mind should be free from bias in subdividing the interval between two consecutive semitones. When the theoretical result is known, it is almost impossible to arrive at an independent opinion by experiment.

315. We will now, following Helmholtz, examine more closely the nature of the motion within the pipe, represented by the formula (11) § 313. We have

$$\phi = L\cos(nt-\theta) \qquad\qquad (1),$$

where

$$L^2 = \frac{\sin^2 k(x-\alpha)}{\cos^2 k\alpha} + \frac{k^4\sigma^2}{4\pi^2}\cos^2 kx \qquad\qquad (2),$$

$$\tan\theta = -\frac{k^2\sigma\cos k\alpha\cos kx}{2\pi\sin k(x-\alpha)} \qquad\qquad (3).$$

In the expression for L^2 the second term is very small, and therefore the maximum values of ϕ occur very nearly when

$$k(x-\alpha)=(-m+\tfrac{1}{2})\pi,$$

or

$$-x=\tfrac{1}{2}m\lambda-\tfrac{1}{4}\lambda-\alpha \quad\text{.................... (4)},$$

where m is a positive integer.

The distance between consecutive maxima is thus $\frac{1}{2}\lambda$, and the value of the maximum is $\sec^2 k\alpha$. The minimum values of L^2 occur approximately when $k(x-\alpha)=-m\pi$,

or

$$-x=\tfrac{1}{2}m\lambda-\alpha \quad\text{.........................(5)},$$

and their magnitude is given by

$$L^2=\frac{k^4\sigma^2}{4\pi^2}\cos^2 kx=\frac{k^4\sigma^2}{4\pi^2}\cos^2 k\alpha \quad\text{..............(6)}.$$

In like manner,

$$\frac{d\phi}{dx}=J\cos(nt-\chi) \quad\text{.......................(7)},$$

where

$$J^2=k^2\frac{\cos^2 k(x-\alpha)}{\cos^2 k\alpha}+\frac{k^6\sigma^2}{4\pi^2}\sin^2 kx \quad\text{..............(8)},$$

$$\tan\chi=\frac{k^2\sigma\cos k\alpha\sin kx}{2\pi\cos k(x-\alpha)} \quad\text{....................(9)}.$$

The maximum values of J^2 occur when

$$-x=\tfrac{1}{2}m\lambda-\alpha \quad\text{...................... (10)},$$

and the minimum values, when

$$-x=\tfrac{1}{2}m\lambda-\tfrac{1}{4}\lambda-\alpha \quad\text{....................(11)}.$$

The approximate magnitude of the maximum is $k^2\sec^2 k\alpha$, and that of the minimum $k^6\sigma^2\cos^2 k\alpha \div 4\pi^2$. It appears that the maxima of velocity occur in the same parts of the tube as the minima of condensation (and rarefaction), and the minima of velocity in the same places as the maxima of condensation. The series of loops and nodes are arranged as if the first loop were at a distance α beyond the mouth.

With regard to the phases, we see that both θ and χ are in general small; and therefore with the exception of the places where L^2 and J^2 are near their minima the whole motion is synchronous, as if there were no dissipation.

Hitherto we have considered the problem of the passage of plane waves along the pipe and their gradual diffusion from the mouth, without regard to the origin of the plane waves themselves. All that we have assumed is that the origin of the motion is somewhere within the pipe. We will now suppose that the motion is due to the known vibration of a piston, situated at $x=-l$, the origin of co-ordinates being at the mouth. Thus, when $x=-l$,

$$\frac{d\phi}{dx}=G\cos nt \quad\text{.......................}(12),$$

and this must be made to correspond with the expression for the plane waves, generalized by the introduction of arbitrary amplitude and phase.

We may take

$$\frac{d\phi}{dx}=BJ\cos(nt-\epsilon-\chi)\quad\text{..............}(13),$$

where J and χ have the values given in (8), (9), while B and ϵ are arbitrary. Comparing (12) and (13) we conclude that

$$\tan\epsilon=\frac{k^2\sigma\cos k\alpha\sin kl}{2\pi\cos k(l+\alpha)}\quad\text{.................}(14),$$

$$G^2=B^2k^2\left\{\frac{\cos^2 k(l+\alpha)}{\cos^2 k\alpha}+\frac{k^4\sigma^2}{4\pi^2}\sin^2 kl\right\}\quad\text{.........}(15),$$

by which B and ϵ are determined.

In accordance with (12) § 313, the corresponding divergent wave is represented by

$$\psi=-\frac{\sigma kB}{2\pi r}\cos(nt-\epsilon-kr)\quad\text{..............}(16).$$

If G be given, B is greatest, when $\cos k(l+\alpha)=0$, that is when the piston is situated at an approximate node. In that case

$$B=\frac{2\pi}{k^3\sigma\cos k\alpha}G\quad\text{....................}(17),$$

shewing that the magnitude of the resulting vibration is very great, though not infinite, since $\cos k\alpha$ cannot vanish. When the mouth is much contracted, $\cos k\alpha$ may become small, but in this case it is necessary that the adjustment of periods be very exact in order that the first term of (15) may be negligible in comparison with the second. In ordinary pipes $\cos k\alpha$ is nearly equal to unity.

The minimum of vibration occurs when l is such that

$\cos k(l+\alpha) = \pm 1$, that is, when the piston is situated at a loop. In that case

$$B = \frac{G \cos k\alpha}{k} \quad\text{.....................(18).}$$

The vibration outside the tube is then, according to the value of α, equal to or smaller than the vibration which there would be if there were no tube and the vibrating plate were made part of the yz plane.

316. Our equations may also be applied to the investigation of the motion excited in a tube by external sources of sound. Let us suppose in the first place that the mouth of the tube is closed by a fixed plate forming part of the yz plane, and that the potential due to the external sources (approximately constant over the plate) is under these circumstances

$$\psi = H \cos nt \quad\text{.........................(1),}$$

where ψ is composed of the potential due to each source and its image in the yz plane, as explained in § 278. Inside the tube let the potential be

$$\phi = H \cos kx \cos nt \quad\text{.................... (2),}$$

so that ϕ and its differential coefficient are continuous across the barrier. The physical meaning of this is simple. We imagine within the tube such a motion as is determined by the conditions that the velocity at the mouth is zero, and that the condensation at the mouth is the same as that due to the sources of sound when the mouth is closed. It is obvious that under these circumstances the closing plate may be removed without any alteration in the motion. Now, however, there is in general a finite velocity at $x = -l$, and therefore we cannot suppose the pipe to be there stopped. But when there happens to be a node at $x = -l$, that is to say when l is such that $[\sin kl] = 0$, all the conditions are satisfied, and the actual motion within the pipe is that expressed by (2)[1]. This motion is evidently the same as might obtain if the pipe were closed at both ends; and in external space the potential is the same as if the mouth of the pipe were closed with the rigid plate.

In the general case in order to reduce the air at $x = -l$ to rest, we must superpose on the motion represented by (2) another of

[1] [An error, pointed out by Dr Burton, is here corrected.]

the kind investigated in § 313, so determined as to give at $x = -l$ a velocity equal and opposite to that of the first. Thus, if the second motion be given by

$$d\phi/dx = BJ\cos(nt - \epsilon - \chi),$$

we have $\epsilon + \chi = 0$, and

$$B^2\left\{\frac{\cos^2 k(l+\alpha)}{\cos^2 k\alpha} + \frac{k^4\sigma^2}{4\pi^2}\sin^2 kl\right\} = H^2\sin^2 kl \quad\text{.........(3).}$$

When $\sin kl = 0$, we have, as above explained, $B = 0$. The maximum value of B occurs when $\cos k(l+\alpha) = 0$, and then

$$B = \frac{2\pi H}{k^2\sigma} \quad\text{.........................(4)[1].}$$

It appears, as might have been expected, that the resonance is greatest when the reduced length is an odd multiple of $\frac{1}{4}\lambda$.

317. From the principle that in the neighbourhood of a node the inertia of the air does not come much into play, we see that in such places the form of a tube is of little consequence, and that only the capacity need be attended to. This consideration allows us to calculate the pitch of a pipe which is cylindrical through most of its length (l), but near the closed end expands into a bulb of small capacity (S). The reduced length is then evidently

$$l + \alpha + S\sigma^{-1} \quad\text{...........................(1),}$$

where α is the correction for the open end, and σ is the area of the transverse section of the cylindrical part. This formula is often useful, and may be applied also when the deviation from the cylindrical form does not take the shape of an enlargement.

When the enlargement represented by S is too large to allow of the above treatment, we may proceed as follows. The dissipation being neglected, the velocity-potential in the tube may be taken to be

$$\phi = \sin k(x - \alpha)\cos nt,$$

the origin being at the mouth, while $\alpha = \frac{1}{4}\pi R$ approximately. At $x = -l$, we have

$$\dot\phi = n\sin k(l+\alpha)\sin nt,$$

and
$$\frac{d\phi}{dx} = k\cos k(l+\alpha)\cos nt.$$

[1] Helmholtz, *Crelle*, Bd. 57, 1860.

Now the condensation is given by $s = -a^{-2}\dot{\phi}$, and the condition to be satisfied at $x = -l$ is

$$S\frac{ds}{dt} = -\sigma\frac{d\phi}{dx} \quad\text{.........................(2),}$$

if it be assumed that the condensation within S is sensibly uniform. Thus

$$Sn^2 a^{-2} \sin k(l+\alpha) = \sigma k \cos k(l+\alpha),$$

or, since $n = ak$,

$$\tan k(l+\alpha) = \frac{\sigma}{kS} \quad\text{.........................(3)}$$

is the equation determining the pitch. Numerical examples of the application of (3) are given in my memoir on resonance (*Phil. Trans.* 1871, p. 117).

Similar reasoning proves that in any case of stationary vibrations, for which the wave-length is several times as great as the diameter of the bulb, the end of the tube adjoining the bulb behaves approximately as an open end if kS be much greater than σ, and as a stopped end if kS be much less than σ.

318. The action of a resonator when under the influence of a source of sound in unison with itself is a point of considerable delicacy and importance, and one on which there has been a good deal of confusion among acoustical writers, the author not excepted.

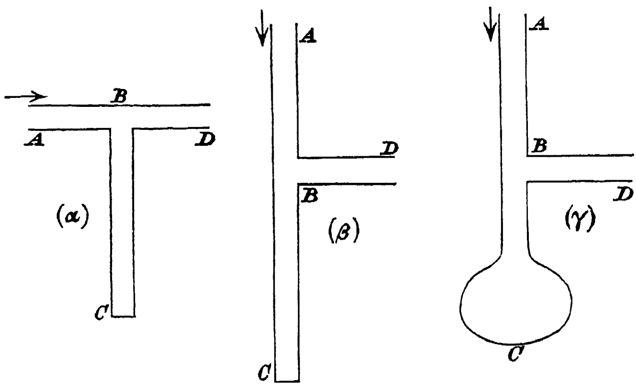
There are cases where a resonator absorbs sound, as it were attracting the vibrations to itself and so diverting them from regions where otherwise they would be felt. For example, suppose that there is a simple source of sound B situated in a narrow tube at a distance $\frac{1}{4}\lambda$ (or any odd multiple thereof) from a closed end, and not too near the mouth: then at any distant external point A, its effect is nil. This is an immediate consequence of the principle of reciprocity, because if A were the source, there could be no variation of potential at B. The restriction, precluding too great a proximity to the mouth, may be dispensed with, if we suppose the source B to be diffused uniformly over the cross section, instead of concentrated in one point. Then, whatever may be the size and shape of the section, there is absolutely no disturbance on the further side. This is clear from the theory of vibrations in one dimension; the reciprocal form of the proposition—that whatever sources of disturbance may exist beyond the section, $\iint \psi d\sigma = 0$—may be proved from Helmholtz's formula (2) § 293, by taking for ϕ the velocity potential of the purely axial vibration of the same period.

It is scarcely necessary to say that, whenever no energy is emitted, the source does no work; and this requires, not that there shall be no variation of pressure at the source, for that in the case of a simple source is impossible, but that the variable part of the pressure shall have exactly the phase of the acceleration, and no component with the phase of the velocity.

Other examples of the absorption of sound by resonators are afforded by certain modifications of Herschel's[1] interference tube used by Quincke[2] to stop tones of definite pitch from reaching the ear.

In the combinations of pipes represented in Fig. 63, the sound enters freely at A; at B it finds itself at the mouth of a resonator of pitch identical with its own. Under these circumstances it is absorbed, and there is no vibration propagated along BD. It is clear that the cylindrical tube BC may be replaced by any other resonator of the same pitch (γ), without prejudice to the action of the apparatus. The ordinary explanation by interference (so called) of direct and reflected waves is then less applicable.

Fig. 63.



These cases where the source is at the mouth of a resonator must not be confused with others where the source is in the interior. If B be a source at the bottom of a stopped tube whose

[1] *Phil Mag.* 1833, Vol. III. p. 405. [2] Pogg. *Ann.* CXXVIII. 177, 1866.

reduced length is $\frac{1}{4}\lambda$, the intensity at an external point A may be vastly greater than if there had been no tube. In fact the potential at A due to the source at B is the same as it would be at B were the source at A.

319. For a closer examination of the mechanics of resonance, we shall obtain the problem in a form disembarrassed of unnecessary difficulties by supposing the resonator to consist of a small circular plate, backed by a spring, and imbedded in an indefinite rigid plane. It was proved in a previous chapter, (30) § 302, that if M be the mass of the plate, ξ its displacement, $\mu\xi$ the force of restitution, R the radius, and σ the density of the air, the equation of vibration is

$$\left(M+\frac{8\sigma R^3}{3}\right)\ddot{\xi}+\frac{a\sigma\pi k^2R^4}{2}\dot{\xi}+\mu\xi=F \qquad (1),$$

where F and ξ are proportional to e^{ikat}.

If the natural period of vibration (the reaction of external air included) coincide with that imposed, the equation reduces to

$$\tfrac{1}{2}a\sigma\pi k^2R^4\dot{\xi}=F \qquad (2).$$

Let us now suppose that F is due to an external source of sound, giving when the plate is at rest a potential ψ_0, which will be nearly constant over the area of the plate. Thus

$$F=-\delta p\,.\,\pi R^2=ika\sigma\,.\,\pi R^2\,.\,\psi_0 \qquad (3);$$

so that

$$\pi R^2\dot{\xi}=\dot{X}=2i\pi k^{-1}\psi_0=i\lambda\psi_0 \qquad (4),$$

and the potential ϕ due to the motion of the plate at a distance r will be

$$\phi=-\frac{\dot{X}}{2\pi}\frac{e^{-ikr}}{r}=-\frac{i\psi_0}{k}\frac{e^{-ikr}}{r}=\psi_0\frac{e^{-ikr}}{ikr} \qquad (5),$$

independent, it should be observed, of the area of the plate.

Leaving for the present the case of perfect isochronism, let us suppose that

$$-\left(M+\frac{8\sigma R^3}{3}\right)k'^2a^2+\mu=0 \qquad (6),$$

so that $2\pi/k'$ is the wave-length of the natural note of the resonator. If M' be written for $M+\frac{8}{3}\sigma R^3$, the equation corresponding to (5) takes the form

$$\phi=\psi_0\frac{e^{-ikr}}{ikr}\div\left[1-2iM'\frac{k'^2-k^2}{\pi\sigma k^3R^4}\right] \qquad (7),$$

from which we may infer as before that if $k' = k$ the efficiency of the resonator as a source is independent of R. When the adjustment is imperfect, the law of falling off depends upon $M'R^{-4}$. Thus if M' be great and R small, although the maximum efficiency of the resonator is no less, a greater accuracy of adjustment is required in order to approach the maximum (§ 49). In the case of resonators with simple apertures $M' = \frac{16}{3}\sigma R^3$, so that $M'R^{-4}$ varies as R^{-1}. Accordingly resonators with small apertures require the greatest precision of tuning, but the difference is not important. From a comparison of the present investigation with that of § 311 it appears that the conditions of efficiency are different according as internal or external effects are considered.

We will now return to the case of isochronism and suppose further that the external source of sound to which the resonator A responds, is the motion of a similar plate B, whose distance c from A is a quantity large in comparison with the dimensions of the plates. The intensity of B may be supposed to be such that its potential is

$$\psi = \frac{e^{-ikr}}{r} \qquad\qquad (8).$$

Accordingly $\psi_0 = c^{-1} e^{-ikc}$, and therefore by (5)

$$\phi = \psi_0 \frac{e^{-ikr}}{ikr} = \frac{e^{-ikc}}{ikc} \cdot \frac{e^{-ikr}}{r} \qquad\qquad (9),$$

shewing that at equal distances from their sources

$$\phi : \psi = e^{-ikc} : ikc \qquad\qquad (10).$$

The relation of phases may be represented by regarding the induced vibration ϕ as proceeding from B by way of A, and as being subject to an additional retardation of $\frac{1}{4}\lambda$, so that the whole retardation between B and A is $c + \frac{1}{4}\lambda$. In respect of amplitude ϕ is greater than ψ in the ratio of $1 : kc$.

Thus when kc is small, the induced vibration is much the greater, and the total sound is much louder than if A were not permitted to operate. In this case the phase is retarded by a quarter of a period.

It is important to have a clear idea of the cause of this augmentation of sound. In a previous chapter (§ 280) we saw that, when A is fixed, B gives out much less sound than might at first have been expected from the pressure developed. The explanation was that the *phase* of the pressure was unfavourable;

the larger part of it is concerned only in overcoming the inertia of the surrounding air, and is ineffective towards the performance of work. Now the pressure which sets A in motion is the whole pressure, and not merely the insignificant part that would of itself do work. The motion of A is determined by the condition that that component of the whole pressure upon it, which has the phase of the velocity, shall vanish. But of the pressure that is due to the motion of A, the larger part has the phase of the acceleration; and therefore the prescribed condition requires an equality between the small component of the pressure due to A's motion, and a pressure comparable with the large component of the pressure due to B's motion. The result is that A becomes a much more powerful source than B. Of course no work is done by the piston A; its effect is to augment the work done at B, by modifying the otherwise unfavourable relation between the phases of the pressure and of the velocity.

The infinite plane in the preceding discussion is only required in order that we may find room behind it for our machinery of springs. If we are content with still more highly idealized sources and resonators, we may dispense with it. To each piston must be added a duplicate, vibrating in a similar manner, but in the opposite direction, the effect of which will be to make the normal velocity of the fluid vanish over the plane AB. Under these circumstances the plane is without influence and may be removed. If the size of the plates be reduced without limit they become ultimately equivalent to simple sources of fluid; and we conclude that a simple source B will become more efficient than before in the ratio of $1 : kc$, when at a small distance c from it there is allowed to operate a simple resonator (as we may call it) of like pitch, that is, a source in which the inertia of the immediately surrounding fluid is compensated by some adequate machinery, and which is set in motion by external causes only.

In the present state of our knowledge of the mechanics of vibrating fluids, while the difficulties of deduction are for the most part still to be overcome, any simplification of conditions which allows progress to be made, without wholly destroying the practical character of the question, may be a step of great importance. Such, for example, was the introduction by Helmholtz of the idea of a source concentrated in one point, represented analytically by the violation at that point of the equation of continuity. Perhaps in like manner the idea of a simple resonator may be useful, although the thing would be still more impossible to construct than a simple source.

320. We have seen that there is a great augmentation of sound, when a suitably tuned resonator is close to a simple source. Much more is this the case, when the source of sound is compound. The potential due to a double source is (§§ 294, 324)

$$r\psi = \mu e^{-ikr}\left(1 + \frac{1}{ikr}\right) \text{.....................(1).}$$

If the resonator be at a small distance c,

$$\psi_0 = \mu_0 \frac{e^{-ikc}}{ikc^2},$$

and therefore the potential due to the resonator at a distance r' is

$$\phi = \mu_0 \frac{e^{-ikc}}{ikc^2} \cdot \frac{e^{-ikr'}}{ikr'} = \mu_0 \frac{e^{-ikc}}{i^2k^2c^2} \cdot \frac{e^{-ikr'}}{r'} \text{..............(2).}$$

If μ_0 vanish, the resonator is without effect; but when $\mu_0 = \pm 1$, that is, when the resonator lies on the axis of the double source, we have

$$\phi = \mp \frac{e^{-ikc}}{k^2c^2} \cdot \frac{e^{-ikr'}}{r'} \text{......................(3).}$$

At a distance from the double source its potential is

$$\psi = \mu \frac{e^{-ikr}}{r} \text{.........................(4).}$$

Thus we may consider that the potential due to the resonator is greater than that due to the double source in the ratio $k^2c^2 : 1$, the angular variation being disregarded.

A vibrating rigid sphere gives the same kind of motion to the surrounding air as a double source situated at its centre; but the substitution suggested by this fact is only permissible when the radius of the sphere is small in comparison with c: otherwise the presence of the sphere modifies the action of the resonator. Nevertheless the preceding investigation shews how powerful in general the action of a resonator is when placed in a suitable position close to a compound source of sound, whose character is such that it would of itself produce but little effect at a distance.

One of the best examples of this use of a resonator is afforded by a vibrating bar of glass, or metal, held at the nodes. A strip of plate glass about a foot [30 cm.] long and an inch [2·5 cm.] broad, of medium thickness (say $\frac{1}{8}$ inch [·32 cm.]), supported at about 3 inches [7·6 cm.] from the ends by means of string twisted round it, answers the purpose very well. When struck by a hammer it gives but little sound except overtones; and even these may almost be got rid of by choosing a hammer of suitable softness. This deficiency of sound is a consequence of the small dimensions of the bar in comparison with the wave-length, which allows of the easy transference of air from one side to the other. If now the mouth of a resonator of the right pitch[1] be held over one of the free ends, a sound of considerable force and purity may be obtained by a well-managed blow. In this way an improved harmonicon may be constructed, with tones much lower than would be practicable without resonators. In the ordinary instrument the wave-lengths are sufficiently short to permit the bar to communicate vibrations to the air independently.

The reinforcement of the sound of a bell in a well-known experiment due to Savart[2] is an example of the same mode of action; but perhaps the most striking instance is in the arrangement adopted by Helmholtz in his experiments requiring pure tones, which are obtained by holding tuning-forks over the mouths of resonators.

321. When two simple resonators A_1, A_2, separately in tune with the source, are close together, the effect is less than if there were only one. If the potentials due respectively to A_1, A_2 be ϕ_1, ϕ_2, we may take

$$\phi_1 = A_1 \frac{e^{-ikr_1}}{r_1}, \qquad \phi_2 = A_2 \frac{e^{-ikr_2}}{r_2}.$$

Let R represent the distance A_1A_2, and ψ_1, ψ_2, the potentials that would exist at A_1, A_2, if there were no resonators; then the conditions to determine A_1, A_2 are by (5) § 319

$$\left.\begin{aligned} \psi_1 + A_2/R = +ikA_1 \\ \psi_2 + A_1/R = +ikA_2 \end{aligned}\right\} \quad\text{.....................}\quad (1).$$

[1] To get the best effect, the mouth of the resonator ought to be pretty close to the bar; and then the pitch is decidedly lower than it would be in the open. The final adjustment may be made by varying the amount of obstruction. This use of resonators is of great antiquity.

[2] *Ann. d. Chim.* t. xxiv. 1823.

By hypothesis ψ_1 and ψ_2 are nearly equal, and therefore

$$A_1 = A_2 = \frac{R}{-1 + ikR}\psi \qquad\qquad (2).$$

Since ikR is small, the effect is much less than if there were only one resonator. It must be observed however that the diminished effectiveness is due to the resonators putting one another out of tune, and if this tendency be compensated by an alteration in the spring, any number of resonators near together have just the effect of one. This point is illustrated by § 302, where it will be seen (32) that though the resonance does not depend upon the size of the plate, still the inertia of the air, which has to be compensated by a spring, does depend upon it.

322. It will be proper to say a few words in this place on an objection, which has been brought forward by Bosanquet[1] as possibly invalidating the usual calculations of the pitch of resonators and of the correction to the length of organ-pipes. When fluid flows in a steady stream through a hole in a thin plate, the motion on the low pressure side is by no means of the character investigated in § 306. Instead of diverging after passing the hole so as to follow the surface of the plate, the fluid shapes itself into an approximately cylindrical jet, whose form for the case of two dimensions can be calculated[2] from formulæ given by Kirchhoff. On the high pressure side the motion does not deviate so widely from that determined by the electrical law. In like manner fluid passing outwards from a pipe continues to move in a cylindrical stream. If the external pressure be the greater, the character of the motion is different. In this case the stream lines converge from all directions to the mouth of the pipe, afterwards gathering themselves into a parallel bundle, whose section is considerably less than that of the pipe. It is clear that, if the formation of jets took place to any considerable extent during the passage of air through the mouths of resonators, our calculations of pitch would have to be seriously modified.

The precise conditions under which jets are formed is a subject of great delicacy. It may even be doubted whether they would occur at all in frictionless fluid moving with velocities so small that the corresponding pressures, which are proportional to the squares of

[1] *Phil. Mag.* Vol. IV. p. 125, 1877.
[2] *Phil. Mag.* Vol. II. p. 441, 1876.

the velocities, are inconsiderable. But with air, as we actually have it, moving under the action of the pressures to be found in resonators, it must be admitted that jets may sometimes occur. While experimenting about two years ago with one of König's brass resonators of pitch c', I noticed that when the corresponding fork, strongly excited, was held to the mouth, a wind of considerable force issued from the nipple at the opposite side. This effect may rise to such intensity as to blow out a candle upon whose wick the stream is directed. It does not depend upon any peculiar motion of the air near the ends of the fork, as is proved by mounting the fork upon its resonance-box and presenting the open end of the box, instead of the fork itself, to the mouth of the resonator, when the effect is obtained with but slightly diminished intensity. A similar result was obtained with a fork and resonator, of pitch an octave lower (c). Closer examination revealed the fact that at the sides of the nipple the outward flowing stream was replaced by one in the opposite direction, so that a tongue of flame from a suitably placed candle appeared to enter the nipple at the same time that another candle situated immediately in front was blown away. The two effects are of course in reality alternating, and only appear to be simultaneous in consequence of the inability of the eye to follow such rapid changes. The formation of jets must make a serious draft on the energy of the motion, and this is no doubt the reason why it is necessary to close the nipple in order to obtain a powerful sound from a resonator of this form, when a suitably tuned fork is presented to it.

At the same time it does not appear probable that jet formation occurs to any appreciable extent at the mouths of resonators as ordinarily used. The near agreement between the observed and the calculated pitch is almost a sufficient proof of this. Another argument tending to the same conclusion may be drawn from the persistence of the free vibrations of resonators (§ 311), whose duration seems to exclude any important cause of dissipation beyond the communication of motion to the surrounding air.

In the case of organ-pipes, where the vibrations are very powerful, these arguments are less cogent, but I see no reason for thinking that the motion at the upper open end differs greatly from that supposed in Helmholtz's calculation. No conclusion to the contrary can, I think, safely be drawn from the phenomena of steady motion. In the opposite extreme case of impulsive motion jets certainly cannot be formed, as follows from Thomson's principle of least energy (§ 79), and it is doubtful to which extreme the case of periodic motion may with greatest plausibility be assimilated. Observation by the method of intermittent illumination (§ 42) might lead to further information upon this subject.

322*a*. As has already been mentioned, the free vibrations of the body of air contained in a resonator may be excited by a suitable blow delivered to the latter. The gas does not at first partake of the sudden movement imposed upon the walls, and the relative motion thus initiated is the origin of free vibrations of the kind considered in preceding sections. When corks are drawn from partially empty bottles, or when the lids are suddenly removed from tubular pasteboard pencil-cases, free vibrations of the resonating air columns are initiated in like manner.

If the vibrations are to be maintained with a view to the emission of a continued sound, the vibrating body must be in communication with a source of energy (§ 68 *a*), and the reaction between the two must be rightly accommodated with respect to phase. The question whether the source of energy or the resonator is to be regarded as the origin of the sound is of no particular significance and will be variously answered according to the point of view of the moment. In the organ the pipe, rather than the compressed air within the bellows or even the escaping wind, is regarded as the parent of the sound, but when a similar pipe is maintained in action by a flame the credit of the joint performance is usually given to the latter.

Up to this point the explanation of maintained vibrations is simple enough; but the complete theory in any particular case demands such an investigation of the reaction as will determine the phase relation. On this depends the whole question whether the reaction is favourable or unfavourable to the continuance of the vibrations, and the determination is often a matter of difficulty.

Before proceeding to discuss the action of the blast it will be desirable to say something further upon the organ-pipe considered simply as a resonator. We have seen (§ 314) how to take account of an upper open end, but according to the rule of Cavaillé-Coll the whole addition which must be made to the measured length of an open pipe in order to bring about agreement with the simple formula (8) § 255 amounts to as much as $3\frac{1}{3}R$, very much greater than the correction ($1{\cdot}2R$) necessary for a simple tube of circular section open at both ends. This discrepancy is sometimes attributed to the blast. But it must be remembered that the lower end is very much less open than the upper end, and that if a sensible correction on account of deficient openness is required for the latter, a much more important correction will probably be necessary for the former. Observations by the author[1] have shewn that this is the case. A pipe fitted with a sliding prolongation was tuned to maximum resonance with a given (256) fork as in Blaikley's experiment (§ 314). It was then blown from a well-regulated bellows with measured pressures of wind, and the pitch of the sounds so obtained was referred to that of the fork by the method of beats (§ 30). The results shewed that at practical pressures the pitch of the pipe as sounded by wind was *higher* than its natural note of maximum resonance; so that the considerable correction to the length found by Cavaillé-Coll is not attributable to the blast, but to the contracted character of the lower end treated as open in the elementary theory. In order to estimate the natural note an even larger "correction to the length" would be required.

The rise of pitch due to the wind increases with pressure. Thus in the case referred to above the pipe under a pressure of 1·06 inches (2·7 cm.) of water gave a note about 2 vibrations per second sharper than that of the fork, but when the wind pressure was raised to 4·2 inches (10·7 cm.) the excess was as much as 11 vibrations per second. When the pressure was raised much further, the pipe was "over blown" and gave the octave of its proper pitch. This, of course, corresponds to another mode of vibration of the aerial column.

It remains to consider the maintaining action of the blast. The vibrations of a column of air may be encouraged either by the introduction of fluid at a place where the density varies and at a moment of condensation (and by the similar abstraction of fluid at a moment of rarefaction), or by a suitable acceleration of the parts of the column situated near a loop. Since the blast of an organ acts at an open end of the pipe, it is clear that here we

[1] *Phil. Mag.* III. p. 462, 1877; XIII. p. 340, 1882.

have to do with the latter alternative. The sheet of wind directed across the lip of the pipe is easily deflected. When during the vibration the external air tends to enter the pipe, it carries the jet with it more or less completely. Half a period later when the natural flow is outwards, the jet is deflected in the corresponding direction. In either case the jet encourages the prevailing motion, and thus renders possible the maintenance of the vibration.

For ready speech it is necessary that the sheet of wind be accurately adjusted. But Schneebeli[1] has shewn that when the vibration is once started there is more latitude. In an experimental arrangement the jet was so adjusted as to pass entirely outside the pipe. Under these circumstances there was failure to speak until by a temporary strong blast directed upon it from outside the jet was bent inwards to the proper position. The pipe then spoke and continued in action until by a pressure in the reverse direction the jet was bent back. The motion of the jet may be made apparent with the aid of smoke or by means of a piece of tissue paper held so as to vibrate with it. Both Schneebeli and H. Smith[2] insist upon a comparison between the jet and the tongue of a reed organ-pipe, but the modes of action appear to be essentially different.

The above view of the matter, which is that adopted by v. Helmholtz in the fourth edition of his great work, appears to be satisfactory as a general explanation of the maintenance of a continued vibration, but it cannot be regarded as complete. In matters of this kind practice is usually in advance of theory; and many generations of practical men have brought the organ-pipe to a high degree of excellence.

Another view that has been favourably entertained by many good authorities regards the pipe as merely reinforcing by its resonance a sound primarily due to the friction of the jet playing against the lip, and there seems to be no doubt that sounds may thus originate[3]. Perhaps after all there is less difference than might at first appear between the two views, and the latter may be especially appropriate when the initiation of the sound rather than its maintenance is under consideration. A detailed discussion

[1] *Pogg. Ann.* Bd. 153, p. 301, 1874.

[2] *Nature*, 1873, 1874, 1875.

[3] See for example Melde's *Akustik*, p. 252; Sondhauss *Pogg. Ann.* XCI. p. 126, 1854.

of the question will be found in an essay by Van Schaik[1]. For a fuller explanation we must probably await a better knowledge of the mechanics of jets.

322 *b.* The character of the sound emitted from a pipe depends upon the presence or absence of the various overtones, a matter which requires further consideration. When a system vibrates freely, the overtones may be harmonic or inharmonic according to the nature of the system, and the composition of the sound depends upon the initial circumstances. But in the case of a maintained vibration like that now before us the motion is strictly periodic, and the overtones must be harmonic if present at all. The frequency of the whole vibration will correspond approximately with that natural to the pipe in its gravest mode[2], but the agreement between the pitch of an audible overtone and that of any free vibration may be much less close. The strength of any overtone thus depends upon two things: first upon the extent to which the maintaining forces possess a component of the right kind, and secondly upon the degree of approximation between the overtone and some natural tone of the vibrating body. In organ-pipes the sharpness of the upper lip and the comparative thinness of the sheet of wind are favourable to the production of overtones; so that in narrow open pipes v. Helmholtz was able to hear plainly the first six partial tones. In wider open pipes, on the other hand, the agreement between the overtones and the natural tones is less close. In consequence, pipes of this class, especially if of wood, give a softer quality of sound, in which besides the fundamental only the octave and twelfth are to be detected[3].

When a bottle (§ 26), or a spherical resonator, is blown by wind after the manner of an organ-pipe, there are no natural tones in the neighbourhood of the harmonics, and the resulting sound is almost free from overtones.

322 *c.* When two organ-pipes of the same pitch stand side by side, complications ensue which not unfrequently give trouble in practice. In extreme cases the pipes may almost reduce one another to silence. Even when the mutual influence is more moderate, it may still go so far as to cause the pipes to speak

[1] *Ueber die Tonerregung in Labialpfeifen.* Rotterdam, 1891.

[2] We are not now speaking of "over blowing."

[3] *Tonempfindungen.* Fourth edition, p. 155, 1877.

in absolute unison, in spite of inevitable small natural differences. The simplest case that can be considered is that of a pipe, along the median plane of which a thin resisting wall is supposed to be introduced. If this wall occupy the whole plane, the original pipe is divided into two, independent of, and perfectly similar to one another. And the pitch of these segments is the same as that of the original pipe, fluid friction being neglected, since during the vibrations of the latter there is no motion across the median plane of symmetry. But the case is altered if the wall be limited to the part of the plane included within the pipe, for then the two vibrating columns are free to react upon one another. The system as a whole has two degrees of freedom—we are not now regarding overtones—and free vibrations are performed in two distinct periods. The first of these is characterised by synchronism of phase between the vibrations of the component columns, and the pitch is accordingly the same as before the separation into two parts. But in the second mode the phases of vibration of the component columns are opposed, so that the air which escapes from one open end is absorbed by the contiguous open end of the other part. In consequence the "correction for the open ends" is much diminished in amount, and the pitch in this mode is correspondingly raised. So long as the motion is free, temporary vibrations in both modes may co-exist, and would give rise to beats; but it does not follow that both can be maintained by the blast. This would indeed seem improbable beforehand, and experiment shews that after the first moment the vibrations are confined to the second mode. The contiguous open ends act as opposed sources, and but little sound escapes, although within the pipes, and indeed outside in the immediate neighbourhood of their mouths, the vigour of the vibrations is unimpaired. Effects of the same kind are produced when two distinct but similar pipes are mounted side by side, and under the influence of the blast the compound system may vibrate in one mode only, in spite of small differences of pitch between the notes of the pipes when sounding separately[1].

322 *d.* Direct observation of the state of things within a vibrating air column is of course a matter of great difficulty, but

[1] *Proceedings of the Musical Association*, Dec. 1878.

interesting results have been obtained by Töpler and Boltzmann[1], calling to their aid the method of optical interference to meet the difficulty arising out of the invisibility of air and the method of stroboscopic vision to meet that arising out of the rapidity of the changes. The upper end of an organ-pipe, closed by a thin plate of metal, was provided with sides of worked glass projecting above beyond the metal plate, and by suitable optical arrangements interference was produced between light which passed above and below. The space above being occupied by air at normal density and that below by air in a state of increased or diminished density according to the phase at the moment, the interference bands undergo displacements synchronous with the aerial vibration. Observed directly these displacements would escape the eye; but by the aid of a fork electrically maintained and provided with suitable slits (§ 42) the light may be rendered intermittent in a period nearly coincident with that of the vibration, and then the sequence of changes becomes apparent. From the observed movement of the bands it is possible to infer not merely the total change of density from maximum to minimum, but the law of the variation of density as a function of time.

When a pipe of large section was but moderately blown, the change of density at the node amounted to ·009 of an atmosphere, and the law was very nearly simple harmonic. Under a greater pressure of wind the simple harmonic law was widely departed from, the bands shifting themselves almost suddenly from one extreme position to the other. In this case the amplitude of the first overtone (the twelfth) was about one quarter of that of the fundamental tone. The whole variation of density was ·019 atmosphere.

322 *e*. In some experimental investigations a form of pipe more completely symmetrical with respect to the axis has been employed[2]. The lip is constituted by the entire circular edge of the pipe as defined by a plane perpendicular to the axis, and upon this an annular sheet of wind is brought to bear. A similar arrangement is adopted in the ordinary steam whistle.

Another way of applying wind to evoke the speech of small pipes has been experimented upon by Sondhauss[3], and the rationale

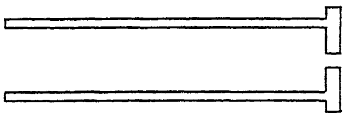
[1] *Pogg. Ann.* CXLI. p. 321, 1870.

[2] Gripon, *Ann. d. Chemie*, III. p. 384, 1874.

[3] *Pogg. Ann.* XCI. p. 126, 1854.

is even less understood. A tube entirely open at one end is partially closed at the other by a plate of wood or metal 2 or 3 mm. thick and pierced by a cylindrical aperture with sharp edges (Fig. 63 *a*).

Fig. 63 *a*.



To set the pipe into action it is only necessary to insert the open end into a reservoir of wind. For rough purposes when it is not required to register the pressure nor to preserve a constant temperature, the mouth suffices, and the sound may be evoked either by pressure or by suction. The cylindrical aperture may be replaced by one of conical form, but in that case the wind must flow from the narrower towards the wider end. The sounds tabulated by Sondhauss vary from a' to f^6, corresponding in all cases to proper tones of the tube.

The whistling sounds of the unaided mouth are evidently of this class, the adjustment of pitch (from about c'' to c^5) being effected mainly by varying the internal capacity (§ 304). The formation of sound in whistling is sometimes said to be connected with a vibration of the lips, but this appears to be a mistake. I have found it possible to whistle through a suitable conical aperture in a piece of box-wood held tightly between the lips.

The occurrence of vibration may be taken as evidence that the steady flow of air through the passages in question is unstable. Contrary to what occurs in the organ-pipe and in sensitive flames, the deformations of the jet would seem here to be of the symmetrical sort. There is perhaps a tendency alternately to follow and to depart from the course marked out by the walls.

322 *f*. An important part of our present subject relates to the maintenance of vibrations by means of heat, and it will be possible to give at least a general account of the manner in which the effect takes place. In almost all cases where heat is communicated to a body expansion ensues, and this expansion may be made to do mechanical work. If the phases of the forces thus operative be favourable, a vibration may be maintained.

An instructive example is afforded by Trevelyan's rocker, consisting of a mass of iron or copper, so shaped that during vibration the weight is alternately carried on one or other of two adjacent and parallel ridges. When the instrument is heated and placed upon a block of cold lead, the vibrations persist so long as the heat remains sufficient. "Sir John Leslie first suggested that the cause of these vibrations is to be found in the expansion of the cold block by the heat which flows into it from the hot metal at the points of contact. Faraday[1], Seebeck[2], and Tyndall[3] have adopted this explanation; and they have shewn that most of the facts that they and others have ascertained respecting these vibrations are easily explained upon this view of their cause, supposing only that the expansion is sufficiently great to produce any sensible effect. Forbes[4], on the other hand, after an extensive series of experiments, was led to reject Sir John Leslie's explanation, one of his principal reasons for doing so being the impossibility, as it appeared to him, that the expansion occasioned by so slow a process as the conduction of heat could produce any sensible mechanical effect."

Davis, from whom[5] the above sentences are quoted, has examined the question mathematically, and has shewn that the explanation is adequate. It is evidently important that the lower body should possess a high rate of expansibility with temperature. In this respect lead stands high among the metals, and rock salt, which Tyndall found to answer well, is even more expansible.

The objection taken by Forbes may be met by the reply that the conduction of heat is not a slow process when small distances and masses are in question; and the special repulsion invoked by him as the basis of an alternative explanation would be of unsuitable character in respect of phase. It is essential that the phase of the force should be in arrear of the phase of the negative displacement.

In an experiment due to Page[6] the vibrations are made independent of an initial difference of temperature, the local heating at the points of contact being obtained with the aid of an

[1] *Proc. of Roy. Inst.* vol. II. p. 119, 1831.

[2] *Pogg. Ann.* vol. LI. p. 1, 1840.

[3] *Phil. Mag.* vol. VIII. p. 1, 1854.

[4] *Phil. Mag.* vol. IV. pp. 15, 182, 1834.

[5] *Phil. Mag.* vol. XLV. p. 296, 1873.

[6] Silliman's *Journal*, vol. IX. p. 105, 1850.

electric current caused to pass from one body to the other. In this arrangement there is no contraction in the upper body to be deducted from the expansion in the lower. On a similar principle Gore[1] has contrived a continuous motion of a copper ball which travels upon circular rails themselves connected with a powerful battery.

322 *g*. But the most interesting examples of vibrations maintained by heat are those which occur when the resonating body is gaseous. "If heat be periodically communicated to, and abstracted from, a mass of air vibrating (for example) in a cylinder bounded by a piston, the effect produced will depend upon the phase of the vibration at which the transfer of heat takes place. If heat be given to the air at the moment of greatest condensation, or be taken from it at the moment of greatest rarefaction, the vibration is encouraged. On the other hand, if heat be given at the moment of greatest rarefaction, or abstracted at the moment of greatest condensation, the vibration is discouraged. The latter effect takes place of itself (§ 247) when the rapidity of alternation is neither very great nor very small in consequence of radiation; for when air is condensed it becomes hotter, and communicates heat to surrounding bodies. The two extreme cases are exceptional, though for different reasons. In the first, which corresponds to the suppositions of Laplace's theory of the propagation of sound, there is not sufficient time for a sensible transfer to be effected. In the second, the temperature remains nearly constant, and the loss of heat occurs during the *process* of condensation, and not when the condensation is effected. This case corresponds to Newton's theory of the velocity of sound. When the transfer of heat takes place at the moment of greatest condensation or of greatest rarefaction, the pitch is not affected.

If the air be at its normal density at the moment when the transfer of heat takes place, the vibration is neither encouraged nor discouraged, but the pitch is altered. Thus the pitch is *raised* if heat be communicated to the air a quarter period *before* the phase of greatest condensation; and the pitch is *lowered* if the heat be communicated a quarter period *after* the phase of greatest condensation.

[1] *Phil. Mag.* vol. xv. p. 519, 1858; vol. xviii. p. 94, 1859.

In general both kinds of effects are produced by a periodic transfer of heat. The pitch is altered, and the vibrations are either encouraged or discouraged. But there is no effect of the second kind if the air concerned be at a loop, i.e. a place where the density does not vary, nor if the communication of heat be the same at any stage of rarefaction as at the corresponding stage of condensation[1]."

Thus in any problem which may present itself of the maintenance of a vibration by heat, the principal question to be considered is the *phase* of the communication of heat relatively to that of the vibration.

322 *h*. The sounds emitted by a jet of hydrogen burning in a pipe open at both ends, were noticed soon after the discovery of the gas, and have been the subject of several elaborate inquiries. The fact that the notes are substantially the same as those which may be elicited in other ways, e.g. by blowing, was announced by Chladni. Faraday[2] proved that other gases were competent to take the place of hydrogen, though not without disadvantage. But it is to Sondhauss[3] that we owe the most detailed examination of the circumstances under which the sound is produced. His experiments prove the importance of the part taken by the column of gas in the tube which supplies the jet. For example, sound cannot be got with a supply tube which is plugged with cotton in the neighbourhood of the jet, although no difference can be detected by the eye between the flame thus obtained and others which are competent to excite sound. When the supply tube is unobstructed, the sounds obtainable by varying the resonator are limited as to pitch, often dividing themselves into distinct groups. In the intervals between the groups no coaxing will induce a maintained sound; and it may be added that, for a part of the interval at any rate, the influence of the flame is inimical, so that a vibration started by a blow is damped more rapidly than if the jet were not ignited.

Forms of resonator other than the open pipe may be employed, and sometimes with advantage. Very low notes can be got from spherical resonators, such as the large globes employed for demon-

[1] *Proc. Roy. Inst.* vol. VIII. p. 536, 1878; *Nature*, vol. XVIII. p. 319, 1878.

[2] *Quart. Journ. Sci.* vol. V. p. 274, 1818.

[3] *Pogg. Ann.* vol. CIX. pp. 1, 426, 1860.

strating the combustion of phosphorus in oxygen gas. A globe of this kind gave in its natural condition a deep and pure tone of 64 vibrations per second. When it was fitted with a longer and narrower neck formed from a pasteboard tube, the calculated frequency fell to 25, and the vibrations, though vigorous enough to extinguish the flame, were hardly audible. When it is desired to excite very deep sounds, the supply tube should be made of considerable length, and the orifice must not be much contracted.

Singing flames may sometimes replace electrically maintained tuning-forks for the production of pure tones, when absolute constancy of pitch is not insisted upon. In order to avoid progressive deterioration of the air, it is advisable to use a resonator open above as well as below. A bulbous chimney, such as are often used with paraffin lamps, meets this requirement, and at the same time emits a pure tone. Or an otherwise cylindrical pipe may be blocked in the middle by a loosely fitting plug[1].

As Wheatstone shewed, the intermittence of a singing flame is easily made manifest by an oscillating, or a revolving, mirror. A more minute examination is best effected by the stroboscopic method, § 42. Drawings of the transformations thus observed have been given by Töpler[2], from which it appears that at one phase the flame may withdraw itself entirely within the supply tube.

Vibrations capable of being maintained are not always self-starting. The initial impulse may be given by a blow administered to the resonator, or by a gentle blast directed across the mouth. In the striking experiments of Schaffgotsch and Tyndall[3] a flame, previously silent, responds to a sound in unison with its own. In some cases the vibrations thus initiated rise to such intensity as to extinguish the flame.

The experiments of Sondhauss shew that a relationship of proportionality subsists between the lengths of the supply tubes and of the sounding columns. When the nature of the gas is varied, the same supply tube being retained, the mean lengths of

1 *Phil. Mag.* vol. VII. p. 149, 1879.
2 *Pogg. Ann.* vol. CXXVIII. p. 126, 1866.
3 *Sound*, 3rd edition, p. 224, 1875.

the speaking columns are approximately as the square roots of the density of the gas. A connection is thus established between the natural note of a supply tube and the notes which can be sounded with its aid.

Partly in consequence of the peculiar and ill understood behaviour of flames, and partly for other reasons, the thorough explanation of the phenomena now under consideration is a matter of some difficulty; but there can be no doubt that they fall under the head of vibrations maintained by heat, the heat being communicated periodically to the mass of air confined in the sounding tube at a place where, in the course of a vibration, the pressure varies. Although some authors have shewn a tendency to lay stress upon the effects of the draught of air through the pipe, the sounds, as we have seen, can be readily produced, not only when there is no through draught, but even when the flame is so situated that there is no sensible periodic motion of the air in its neighbourhood.

In consequence of the variable pressure within the resonator, the issue of gas, and therefore the development of heat, varies during the vibration. The question is under what circumstances the variation is of the kind necessary for the maintenance of the vibration. If we were to suppose, as we might at first be inclined to do, that the issue of gas is greatest when the pressure in the resonator is least, and that the phase of greatest development of heat coincides with that of the greatest issue of gas, we should have the condition of things the most unfavourable of all to the persistence of the vibration. It is not difficult, however, to see that both suppositions are incorrect. In the supply tube (supposed to be unplugged, and of not too small bore) stationary, or approximately stationary, vibrations are excited, whose phase is either the same or the opposite of that of the vibration in the resonator. If the length of the supply tube from the burner to the open end in the gas-generating flask be less than a quarter of the wave-length in hydrogen of the actual vibration, the greatest issue of gas *precedes* by a quarter period the phase of greatest condensation; so that, if the development of heat is *retarded* somewhat in comparison with the issue of gas, a state of things exists *favourable* to the maintenance of the sound. Some such retardation is inevitable, because a jet of inflammable gas can burn only at the outside; but in many cases a still more potent cause may be found in the fact that during the retreat of the gas in the supply tube small quantities of air may enter from the interior of the resonator, whose expulsion must be effected before the inflammable gas can again begin to escape.

If the length of the supply tube amounts to exactly one quarter of the wave-length, the stationary vibration within it will be of such a character that a node is formed at the burner, the variable part of the pressure just inside the burner being the same as in the interior of the resonator. Under these circumstances there is nothing to make the flow of gas, or the development of heat, variable, and therefore the vibration cannot be maintained. This particular case is free from some of the difficulties which attach themselves to the general problem, and the conclusion is in accordance with Sondhauss' observations.

When the supply tube is somewhat longer than a quarter of the wave, the motion of the gas is materially different from that first described. Instead of preceding, the greatest outward flow of gas *follows* at a quarter period interval the phase of greatest condensation, and therefore if the development of heat be somewhat retarded, the whole effect is unfavourable. This state of things continues to prevail, as the supply tube is lengthened, until the length of half a wave is reached, after which the motion again changes sign, so as to restore the possibility of maintenance. Although the size of the flame and its position in the tube (or neck of resonator) are not without influence, this sketch of the theory is sufficient to explain the fact, formulated by Sondhauss, that the principal element in the question is the length of the supply tube.

322 *i*. Another phenomenon of the class now under consideration occasionally obtrudes itself upon the notice of glass-blowers. When a bulb about 2 cm. in diameter is blown at the end of a somewhat narrow tube, 12 or 15 cm. long, a sound is sometimes heard proceeding from the heated glass. For experimental purposes it is well to use hard glass, which can afterwards be reheated at pleasure without losing its shape. As was found by De la Rive, the production of sound is facilitated by the presence of vapour, especially of alcohol and ether.

It was proved by Sondhauss[1] that a vibration of the glass

[1] *Pogg. Ann.* vol. LXXIX. p. 1, 1850.

itself is no essential part of the phenomenon, and the same indefatigable observer was very successful in discovering the connection (§§ 303, 309) between the pitch of the note and the dimensions of the apparatus. But no adequate explanation of the production of sound was given.

For the sake of simplicity, a simple tube, hot at the closed end and getting gradually cooler towards the open end, may be considered. At a quarter of a period *before* the phase of greatest condensation (which occurs almost simultaneously at all parts of the column) the air is moving inwards, i.e. towards the closed end, and therefore is passing from colder to hotter parts of the tube; but the heat received at this moment (of normal density) has no effect either in encouraging or discouraging the vibration. The same would be true of the entire operation of the heat, if the adjustment of temperature were instantaneous, so that there was never any sensible difference between the temperatures of the air and of the neighbouring parts of the tube. But in fact the adjustment of temperature takes *time*, and thus the temperature of the air deviates from that of the neighbouring parts of the tube, inclining towards the temperature of that part of the tube *from* which the air has just come. From this it follows that at the phase of greatest condensation heat is received by the air, and at the phase of greatest rarefaction heat is given up from it, and thus there is a tendency to maintain the vibrations. It must not be forgotten, however, that apart from transfer of heat altogether, the condensed air is hotter than the rarefied air, and that in order that the whole effect of heat may be on the side of encouragement, it is necessary that previous to condensation the air should pass not merely towards a hotter part of the tube, but towards a part of the tube which is hotter than the air will be when it arrives there. On this account a great range of temperature is necessary for the maintenance of vibration, and even with a great range the influence of the transfer of heat is necessarily unfavourable at the closed end, where the motion is very small. This is probably the reason of the advantage of a bulb. It is obvious that if the *open* end of the tube were heated, the effect of the transfer of heat would be even more unfavourable than in the case of a temperature uniform throughout.

322*j*. The last example of the production of sound by heat which we shall here consider is a very striking phenomenon discovered by Rijke[1]. When a piece of fine metallic gauze, stretching across the lower part of a tube open at both ends and held vertically, is heated by a gas flame placed under it, a sound of considerable power and lasting for several seconds is observed almost immediately *after* the removal of the flame. Differing in this respect from the case of sonorous flames (§ 322), the generation of sound was found by Rijke to be closely connected with the formation of a through draught impinging upon the heated gauze. In this form of the experiment the heat is soon abstracted, and then the sound ceases; but by keeping the gauze hot by the current from a powerful galvanic battery Rijke was able to obtain the prolongation of the sound for an indefinite period.

These notes may be obtained upon a large scale and form a very effective lecture experiment. For this purpose a cast iron pipe 5 feet (152 cm.) long and $4\frac{3}{4}$ inches (12 cm.) in diameter may be employed. The gauze (iron wire) is of about 32 meshes to the linear inch (2·54 cm.), and may advantageously be used in two thicknesses. It may be moulded with a hammer on a circular wooden block of somewhat smaller diameter than that of the pipe, and will then retain its position in the pipe by friction. When it is desired to produce the sound, the gauze caps are pushed up the pipe to a distance of about a foot (30·5 cm.), and a gas flame from a large rose burner is adjusted underneath, at such a level as to heat the gauze to bright redness. For this purpose the vertical tube of the lamp should be prolonged, if necessary, by an additional length of brass tubing. When a good red heat is attained, the flame is suddenly removed, either by withdrawing the lamp or by stopping the supply of gas. In about a second the sound begins, and presently rises to such intensity as to shake the room, after which it gradually dies away. The whole duration of the sound may be about 10 seconds[2].

In discussing the question of maintenance in accordance with the views already explained, we have to examine the character of the variable communication of heat from the gauze to the air. So far as the communication is affected directly by variations of pressure or density, the influence is unfavourable, inasmuch as the air will receive less heat from the gauze when its own temperature is raised by condensation. The maintenance depends

[1] *Pogg. Ann.* vol. CVII. p. 339, 1859; *Phil. Mag.* vol. XVII. p. 419, 1859.
[2] *Phil. Mag.* vol. VII. p. 155, 1879.

upon the variable transfer of heat due to the varying *motions* of the air through the gauze, this motion being compounded of a uniform motion upwards with a motion, alternately upwards and downwards, due to the vibration. In the lower half of the tube these motions conspire a quarter period *before* the phase of greatest condensation, and oppose one another a quarter period after that phase. The rate of transfer of heat will depend mainly upon the temperature of the air in contact with the gauze, being greatest when that temperature is lowest. Perhaps the easiest way to trace the mode of action is to begin with the case of a simple vibration without a steady current. Under these circumstances the whole of the air which comes in contact with the metal, in the course of a complete period, becomes heated; and after this state of things is established, there is comparatively little further transfer of heat. The effect of superposing a small steady upwards current is now easily recognized. At the limit of the inwards motion, i.e. at the phase of greatest condensation, a small quantity of air comes into contact with the metal, which has not done so before, and is accordingly cool; and the heat communicated to this quantity of air acts in the most favourable manner for the maintenance of the vibration.

A quite different result ensues if the gauze be placed in the *upper* half of the tube. In this case the fresh air will come into the field at the moment of greatest rarefaction, when the communication of heat has an unfavourable instead of a favourable effect. The principal note of the tube therefore cannot be sounded.

A complementary phenomenon discovered by Bosscha[1] and Riess[2] may be explained upon the same principles. If a current of *hot* air impinge upon *cold* gauze, sound is produced; but in order to obtain the principal note of the tube the gauze must be in the upper, and not as before in the lower, half of the tube. In an experiment due to Riess the sound is maintained indefinitely. The upper part of a brass tube is kept cool by water contained in a surrounding vessel, through the bottom of which the tube passes. In this way the gauze remains comparatively cool, although exposed to the heat of a gas flame situated an inch or two below it. The experiment sometimes succeeds better when the draught

[1] *Pogg. Ann.* vol. CVII. p. 342, 1859.

[2] *Pogg. Ann.* vol. CVIII. p. 653, 1859; CIX. p. 145, 1860.

is checked by a plate of wood placed somewhat closely over the top of the tube.

Both in Rijke's and Riess' experiments the variable transfer of heat depends upon the motion of vibration, while the effect of the transfer depends upon the variation of pressure. The gauze must therefore be placed where both effects are sensible, i.e. neither near a node nor near a loop. About a quarter of the length of the tube, from the lower or upper end, as the case may be, appears to be the most favourable position[1].

322 *k*. It remains to consider briefly another class of maintained aerial vibrations where the maintenance is provided for by the actual mechanical introduction of fluid, taking effect at a node and near the phase of maximum condensation. Well-known examples are afforded by such reed instruments as the clarinette, and by the various wind instruments actuated directly by the lips. The notes obtained are determined in the main by the aerial columns, modified, it may be, to some extent by the maintaining appliances. The reeds of the harmonium and organ come under a different head. The pitch is there determined almost entirely by the tongues themselves vibrating under their own elasticity, resonating air columns being either altogether absent or playing but a subordinate part.

In the instruments now under discussion the air column and the wind-pipe are connected by a narrow aperture, which is opened and closed periodically by a vibrating tongue. Tongues are distinguished by v. Helmholtz as in-beating and out-beating. In the first case the passage is opened when the tongue moves inwards, i.e. against the wind, as happens in the clarinette. Lip instruments, such as the trombone, belong to the second class, the passage being open when the lips are moved outwards or with the wind.

Let us consider the case of a cylindrical pipe, open at the further end, in which the air vibrates at such a pitch as to make the quarter wave-length equal to the length of the pipe. The end of the column where the tongue is situated thus coincides with an approximate node, and the aerial vibration can be maintained if the passage is open at the moment of greatest condensation, so

[1] *Proc. Roy. Inst.* vol. VIII. p. 536, 1879; *Nature*, vol. XVIII. p. 319, 1878.

that air from the wind-pipe is then forcibly injected. The tongue is maintained in motion by the variable pressure within the pipe, and the phase of its motion will depend upon whether it is in-beating or out-beating. In the latter case its phase is nearly the opposite to that of the forces operative upon it, being open when the pressure tending to close it is greatest. This is the state of things in lip instruments, the lips being heavy in relation to the rapidity of the vibrations actually performed, § 46. When the tongue is light and stiff, it must be made in-beating, as in the clarinette, and its phase is then in approximate agreement with the phase of the forces. A slight departure in the proper direction from precise opposition or precise agreement of phase, as the case may be, will allow of the communication of sufficient energy to maintain the motion in spite of dissipative influences. A more complete analytical statement of the circumstances has been given by v. Helmholtz[1], to whom the whole theory is due.

The character of the sounds from the various wind instruments used in music differs greatly. Strongly contrasted qualities are obtained from the trombone and the euphonion; the former brilliant and piercing, and the latter mellow. Blaikley[2] has analysed the sounds from a number of instruments, and has called attention to various circumstances, such as the size of the bell-mouth, and the shape of the cup applied to the lips, upon which the differences probably depend. The pressures used in practice, rising to 40 inches (102 cm.) of water in the case of the euphonion, have been measured by Stone[3].

[1] *Tonempfindungen*, 4th edition, appendix VII.

[2] *Phil. Mag.* vol. VI. p. 119, 1878.

[3] *Phil. Mag.* vol. XLVIII. p. 113, 1874.

CHAPTER XVII.

APPLICATIONS OF LAPLACE'S FUNCTIONS.

323. THE general equation of a velocity potential, when referred to polar co-ordinates, takes the form (§ 241)

$$r^2 \frac{d^2\psi}{dr^2} + 2r\frac{d\psi}{dr} + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2\psi}{d\omega^2} + k^2r^2\psi = 0 \ldots (1).$$

If k vanish, we have the equation of the ordinary potential, which, as we know, is satisfied, if $\psi = r^n S_n$, where S_n denotes the spherical surface harmonic[1] of order n. On substitution it appears that the equation satisfied by S_n is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS_n}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2S_n}{d\omega^2} + n(n+1)S_n = 0 \ldots\ldots (2).$$

Now, whatever the form of ψ may be, it can be expanded in a series of spherical harmonics

$$\psi = \psi_0 + \psi_1 + \psi_2 + \ldots\ldots + \psi_n + \ldots\ldots\ldots\ldots (3),$$

where ψ_n will satisfy an equation such as (2).

Comparing (1) and (2) we see that to determine ψ_n as a function of r, we have

$$r^2\frac{d^2\psi_n}{dr^2} + 2r\frac{d\psi_n}{dr} - n(n+1)\psi_n + k^2r^2\psi_n = 0;$$

or, as it may also be written,

$$\frac{d^2(r\psi_n)}{d(kr)^2} - \frac{n(n+1)}{(kr)^2}(r\psi_n) + r\psi_n = 0 \ldots\ldots\ldots\ldots (4).$$

[1] On the theory of these functions the latest English works are Todhunter's *The Functions of Laplace, Lamé, and Bessel*, Ferrers' *Spherical Harmonics* and Gray and Mathews' *Bessel's Functions*, Macmillan, 1895.

In order to solve this equation, we may observe that when r is very great, the middle term is relatively negligible, and that then the solution is

$$r\psi_n = Ae^{ikr} + Be^{-ikr} \quad\text{.....................} (5).$$

The same form may be assumed to hold good for the complete equation (4), if we look upon A and B no longer as constants, but as functions of r, whose nature is to be determined. Substituting in (4), we find for B,

$$-\frac{d^2B}{d(ikr)^2} + 2\frac{dB}{d(ikr)} + \frac{n(n+1)}{(ikr)^2}B = 0 \quad\text{...........}(6).$$

Let us assume

$$B = B_0 + B_1(ikr)^{-1} + B_2(ikr)^{-2} + \ldots + B_s(ikr)^{-s} + \ldots \quad (7),$$

and substitute in (6). Equating to zero the coefficient of $(ikr)^{-s-2}$, we obtain

$$B_{s+1} = B_s\frac{n(n+1) - s(s+1)}{2(s+1)} = B_s\frac{(n-s)(n+s+1)}{2(s+1)} \quad\text{......}(8).$$

Thus $\qquad B_1 = \tfrac{1}{2}n(n+1)B_0,$

$$B_2 = B_1\frac{(n-1)(n+2)}{2\,.\,2} = \frac{(n-1)n(n+1)(n+2)}{2\,.\,4}B_0, \quad \text{\&c.};$$

so that

$$B = B_0\left\{1 + \frac{n(n+1)}{2\,.\,ikr} + \frac{(n-1)\ldots(n+2)}{2\,.\,4\,.\,(ikr)^2} + \frac{(n-2)\ldots(n+3)}{2\,.\,4\,.\,6\,.\,(ikr)^3} + \ldots + \frac{1\,.\,2\,.\,3\ldots 2n}{2\,.\,4\,.\,6\ldots 2n\,.\,(ikr)^n}\right\} \quad\text{..............}(9).$$

Denoting with Prof. Stokes[1] the series within brackets by $f_n(ikr)$, we have

$$B = B_0 f_n(ikr) \quad\text{.........................}(10).$$

In like manner by changing the sign of i, we get

$$A = A_0 f_n(-ikr) \quad\text{.......................}(11).$$

The symbols A_0 and B_0, though independent of r, are functions of the angular co-ordinates: in the most general case, they are any two spherical surface harmonics of order n. Equation (5) may therefore be written

$$r\psi_n = S_n e^{-ikr} f_n(ikr) + S_n' e^{+ikr} f_n(-ikr) \quad\text{........}(12).$$

[1] On the Communication of Vibrations from a Vibrating Body to a surrounding Gas. *Phil. Trans.* 1868.

By differentiation of (12)

$$\frac{d\psi_n}{dr} = -\frac{S_n}{r^2} e^{-ikr} F_n(ikr) - \frac{S_n'}{r^2} e^{+ikr} F_n(-ikr) \text{....(13)},$$

where

$$F_n(ikr) = (1 + ikr) f_n(ikr) - ikr f_n'(ikr) \text{ ...... (14)}.$$

The forms of the functions F, as far as $n = 7$, are exhibited in the accompanying table:

$F_0(y) = y + 1$
$F_1(y) = y + 2 + 2y^{-1}$
$F_2(y) = y + 4 + 9y^{-1} + 9y^{-2}$
$F_3(y) = y + 7 + 27y^{-1} + 60y^{-2} + 60y^{-3}$
$F_4(y) = y + 11 + 65y^{-1} + 240y^{-2} + 525y^{-3} + 525y^{-4}$
$F_5(y) = y + 16 + 135y^{-1} + 735y^{-2} + 2625y^{-3} + 5670y^{-4} + 5670y^{-5}$
$F_6(y) = y + 22 + 252y^{-1} + 1890y^{-2} + 9765y^{-3} + 34020y^{-4} + 72765y^{-5} + 72765y^{-6}$
$F_7(y) = y + 29 + 434y^{-1} + 4284y^{-2} + 29925y^{-3} + 148995y^{-4} + 509355y^{-5} + 1081080y^{-6}$
$+ 1081080y^{-7}$

In order to find the leading terms in $F_n(ikr)$ when ikr is small, we have on reversing the series in (9)

$$f_n(ikr) = 1.3.5\ldots(2n-1)(ikr)^{-n}\left\{1 + ikr + \frac{n-1}{2n-1}(ikr)^2 + \ldots\right\} \text{......(15)},$$

whence by (14) we find

$$F_n(ikr) = 1.3.5\ldots(2n-1)(n+1)(ikr)^{-n} \times \left\{1 + ikr + \frac{n^2(ikr)^2}{(n+1)(2n-1)} + \ldots\right\} \text{..........(16)}.$$

324. An important case of our general formulæ occurs when ψ represents a disturbance which is propagated wholly *outwards*. At a great distance from the origin, $f_n(ikr) = f_n(-ikr) = 1$, and thus, if we restore the time factor (e^{ikat}), we have

$$r\psi_n = S_n e^{ik(at-r)} + S_n' e^{ik(at+r)} \text{ ....(1)},$$

of which the second part represents a disturbance travelling inwards. Under the circumstances contemplated we are therefore to take $S_n' = 0$, and thus

$$r\psi_n = S_n f_n(ikr) e^{ik(at-r)} \text{.....................(2)},$$

which represents in the most general manner the n^{th} harmonic component of a disturbance of the given period diffusing itself outwards into infinite space.

The origin of the disturbance may be in a prescribed normal motion of the surface of a sphere of radius c. Let us suppose that at any point on the sphere the outward velocity is represented by $U e^{ikat}$, U being in general a function of the position of the point considered.

If U be expanded in the spherical harmonic series

$$U = U_0 + U_1 + U_2 + \dots + U_n + \dots\dots\dots\dots\dots(3),$$

we must have by (13) § 323

$$U_n = -\frac{S_n}{c^2} e^{-ikc} F_n(ikc)\dots\dots\dots\dots\dots\dots\dots(4).$$

The complete value of ψ is thus

$$\psi = -\frac{c^2}{r} e^{ik(at-r+c)} \Sigma \frac{U_n}{F_n(ikc)} f_n(ikr)\dots\dots\dots\dots(5),$$

where the summation is to be extended to all (integral) values of n. The real part of this equation will give the velocity potential due to the normal velocity $U \cos kat$[1] at the surface of the sphere $r = c$.

Prof. Stokes has applied this solution to the explanation of a remarkable experiment by Leslie, according to which it appeared that the sound of a bell vibrating in a partially exhausted receiver is diminished by the introduction of hydrogen. This paradoxical phenomenon has its origin in the augmented wave-length due to the addition of hydrogen, in consequence of which the bell loses its hold (so to speak) on the surrounding gas. The general explanation cannot be better given than in the words of Prof. Stokes:

"Suppose a person to move his hand to and fro through a small space. The motion which is occasioned in the air is almost exactly the same as it would have been if the air had been an incompressible fluid. There is a mere local reciprocating motion, in which the air immediately in front is pushed forward, and that immediately behind impelled after the moving body, while in the anterior space generally the air recedes from the encroachment of the moving body, and in the posterior space generally flows in from all sides to supply the vacuum which tends to be created; so that in lateral directions the flow of the fluid is backwards, a

[1] The assumption of a real value for U is equivalent to limiting the normal velocity to be in the same phase all over the sphere $r=c$. To include the most general aerial motion U would have to be treated as complex.

portion of the excess of fluid in front going to supply the deficiency behind. Now conceive the periodic time of the motion to be continually diminished. Gradually the alternation of movement becomes too rapid to permit of the full establishment of the merely local reciprocating flow; the air is sensibly compressed and rarefied, and a sensible sound wave (or wave of the same nature, in case the periodic time be beyond the limits suitable to hearing) is propagated to a distance. The same takes place in any gas; and the more rapid be the propagation of condensations and rarefactions in the gas, the more nearly will it approach, in relation to the motions we have under consideration, to the condition of an incompressible fluid; the more nearly will the conditions of the displacement of the gas at the surface of the solid be satisfied by a merely local reciprocating flow."

In discussing the solution (5), Prof. Stokes goes on to say,

"At a great distance from the sphere the function $f_n(ikr)$[1] becomes ultimately equal to 1, and we have

$$\psi = -\frac{c^2}{r} e^{ik(at-r+c)} \Sigma \frac{U_n}{F_n(ikc)} \quad \text{..................... (6).}$$

"It appears (from the value of $d\psi/dr$) that the component of the velocity along the radius vector is of the order r^{-1}, and that in any direction perpendicular to the radius vector of the order r^{-2}, so that the lateral motion may be disregarded except in the neighbourhood of the sphere.

"In order to examine the influence of the lateral motion in the neighbourhood of the sphere, let us compare the actual disturbance at a great distance with what it would have been if all lateral motion had been prevented, suppose by infinitely thin conical partitions dividing the fluid into elementary canals, each bounded by a conical surface having its vertex at the centre.

"On this supposition the motion in any canal would evidently be the same as it would be in all directions if the sphere vibrated by contraction and expansion of the surface, the same all round, and such that the normal velocity of the surface was the same as it is at the particular point at which the canal in question abuts on the surface. Now if U were constant the expansion of U would

[1] I have made some slight changes in Prof. Stokes' notation.

be reduced to its first term U_0, and seeing that $f_0(ikr) = 1$, we should have from (5),

$$\psi = -\frac{c^2}{r} e^{ik(at-r+c)} \frac{U_0}{F_0(ikc)}.$$

This expression will apply to any particular canal if we take U_0 to denote the normal velocity at the sphere's surface for that particular canal; and therefore to obtain an expression applicable at once to all the canals, we have merely to write U for U_0. To facilitate a comparison with (5) and (6), I shall, however, write ΣU_n for U. We have then,

$$\psi = -\frac{c^2}{r} e^{ik(at-r+c)} \frac{\Sigma U_n}{F_0(ikc)} \qquad (7).$$

It must be remembered that this is merely an expression applicable at once to all the canals, the motion in each of which takes place wholly along the radius vector, and accordingly the expression is not to be differentiated with respect to θ or ω with the view of finding the transverse velocities.

"On comparing (7) with the expression for the function ψ in the actual motion at a great distance from the sphere (6), we see that the two are identical with the exception that U_n is divided by two different constants, namely $F_0(ikc)$ in the former case and $F_n(ikc)$ in the latter. The same will be true of the leading terms (or those of the order r^{-1}) in the expressions for the condensation and velocity. Hence if the mode of vibration of the sphere be such that the normal velocity of its surface is expressed by a Laplace's function of any one order, the disturbance at a great distance from the sphere will vary from one direction to another according to the same law as if lateral motions had been prevented, the amplitude of excursion at a given distance from the centre varying in both cases as the amplitude of excursion, in a normal direction, of the surface of the sphere itself. The only difference is that expressed by the symbolic ratio $F_n(ikc) : F_0(ikc)$. If we suppose $F_n(ikc)$ reduced to the form $\mu_n(\cos\alpha_n + i\sin\alpha_n)$, the amplitude of vibration in the actual case will be to that in the supposed case as μ_0 to μ_n, and the phases in the two cases will differ by $\alpha_0 - \alpha_n$.

"If the normal velocity of the surface of the sphere be not expressible by a single Laplace's Function, but only by a series, finite or infinite, of such functions, the disturbance at a given great distance from the centre will no longer vary from one direction to another according to the same law as the normal velocity of the surface of the sphere, since the modulus μ_n and likewise the amplitude α_n of the imaginary quantity $F_n(ikc)$ vary with the order of the function.

"Let us now suppose the disturbance expressed by a Laplace's function of some one order, and seek the numerical value of the alteration of intensity at a distance, produced by the lateral motion which actually exists.

"The intensity will be measured by the *vis viva* produced in a given time, and consequently will vary as the density multiplied by the velocity of propagation multiplied by the square of the amplitude of vibration. It is the last factor alone that is different from what it would have been if there had been no lateral motion. The amplitude is altered in the proportion of μ_0 to μ_n, so that if $\mu_n^2 : \mu_0^2 = I_n$, I_n is the quantity by which the intensity that would have existed if the fluid had been hindered from lateral motion has to be divided.

"If λ be the length of the sound-wave corresponding to the period of the vibration, $k = 2\pi/\lambda$, so that kc is the ratio of the circumference of the sphere to the length of a wave. If we suppose the gas to be air and λ to be 2 feet, which would correspond to about 550 vibrations in a second, and the circumference $2\pi c$ to be 1 foot (a size and pitch which would correspond with the case of a common house-bell), we shall have $kc = \frac{1}{2}$. The following table gives the values of the squares of the modulus and of the

kc	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	
4	17	16·25	14·879	13·848	20·177	Values of μ_n^2.
2	5	5	9·3125	80	1495·8	
1	2	5	89	3965	300137	
0·5	1·25	16·25	1330·2	236191	72086371	
0·25	1·0625	64·062	20878	14837899	18160×10^6	
4	1	0·95588	0·87523	0·81459	1·1869	Values of I_n.
2	1	1	1·8625	16	299·16	
1	1	2·5	44·5	1982·5	150068	
0·5	1	13	1064·2	188953	57669097	
0·25	1	60·294	19650	13965×10^3	17092×10^6	

ratio I_n for the functions $F_n(ikc)$ of the first five orders, for each of the values 4, 2, 1, $\frac{1}{2}$, and $\frac{1}{4}$ of kc. It will presently appear why the table has been extended further in the direction of values greater than $\frac{1}{2}$ than it has in the opposite direction. Five significant figures at least are retained.

"When $kc = \infty$ we get from the analytical expressions $I_n = 1$. We see from the table that when kc is somewhat large I_n is liable to be a *little* less than 1, and consequently the sound to be *a little* more intense than if lateral motion had been prevented. The possibility of that is explained by considering that the waves of condensation spreading from those compartments of the sphere which at a given moment are vibrating positively, *i.e.* outwards, after the lapse of a half period may have spread over the neighbouring compartments, which are now in their turn vibrating positively, so that these latter compartments in their outward motion work against a somewhat greater pressure than if such compartment had opposite to it only the vibration of the gas which it had itself occasioned; and the same explanation applies *mutatis mutandis* to the waves of rarefaction. However, the increase of sound thus occasioned by the existence of lateral motion is but small in any case, whereas when kc is somewhat small I_n increases enormously, and the sound becomes a mere nothing compared with what it would have been had lateral motion been prevented.

"The higher be the order of the function, the greater will be the number of compartments, alternately positive and negative as to their mode of vibration at a given moment, into which the surface of the sphere will be divided. We see from the table that for a given periodic time as well as radius the value of I_n becomes considerable when n is somewhat high. However practically vibrations of this kind are produced when the elastic sphere executes, not its principal, but one of its subordinate vibrations, the pitch corresponding to which rises with the order of vibration, so that k increases with that order. It was for this reason that the table was extended from $kc = 0{\cdot}5$ further in the direction of high pitch than low pitch, namely, to three octaves higher and only one octave lower.

"When the sphere vibrates symmetrically about the centre, *i.e.* so that any two opposite points of the surface are at a given moment moving with equal velocities in opposite directions, or more generally when the mode of vibration is such that there is no change of position of the centre of gravity of the volume, there is no term of order 1. For a sphere vibrating in the manner of a bell the principal vibration is that expressed by a term of the order 2, to which I shall now more particularly attend.

"Putting, for shortness, $k^2c^2 = q$, we have

$$\mu_0^2 = q + 1, \quad \mu_2^2 = (q^{\frac{1}{2}} + 9q^{-\frac{1}{2}})^2 + (4 - 9q^{-1})^2 = q - 2 + 9q^{-1} + 81q^{-2},$$

$$I_2 = \frac{q^3 - 2q^2 + 9q + 81}{q^2(q+1)}.$$

"The minimum value of I_2 is determined by

$$q^3 - 6q^2 - 84q - 54 = 0,$$

giving approximately,

$$q = 12{\cdot}859, \qquad kc = 3{\cdot}586, \qquad \mu_0^2 = 13{\cdot}859, \qquad \mu_2^2 = 12{\cdot}049,$$

$$I_2 = {\cdot}86941;$$

so that the utmost increase of sound produced by lateral motion amounts to about 15 per cent.

"I now come more particularly to Leslie's experiments. Nothing is stated as to the form, size, or pitch of his bell; and even if these had been accurately described, there would have been a good deal of guess-work in fixing on the size of the sphere which should be considered the best representative of the bell. Hence all we can do is to choose such values for k and c as are comparable with the probable conditions of the experiment.

"I possess a bell, belonging to an old bell-in-air apparatus, which may probably be somewhat similar to that used by Leslie. It is nearly hemispherical, the diameter is 1·96 inch, and the pitch an octave above the middle c of a piano. Taking the number of vibrations 1056 per second, and the velocity of sound in air 1100 feet per second, we have $\lambda = 12{\cdot}5$ inches. To represent the bell by a sphere of the same radius would be very greatly to underrate the influence of local circulation, since near the mouth the gas has but a little way to get round from the outside to the inside or the reverse. To represent it by a sphere of half the radius would still apparently be to underrate the effect. Nevertheless for the sake of rather under-estimating than exaggerating the influence of the cause here investigated, I will make these two suppositions successively, giving respectively $c = {\cdot}98$ and $c = {\cdot}49$, $kc = {\cdot}4926$, and $kc = {\cdot}2463$ for air.

"If it were not for lateral motion the intensity would vary from gas to gas in the proportion of the density into the velocity of propagation, and therefore as the pressure into the square root of the density under a standard pressure, if we take the factor depending on the development of heat as sensibly the same for the gases and gaseous mixtures with which we have to deal. In the following Table the first column gives the gas, the second the

Gas.	p	D	Q_r	$c = {\cdot}98$			$c = {\cdot}49$		
				q	I_2	Q	q	I_2	Q
Air	1	1	1	·2427	1136	1	·06067	20890	1
Hydrogen	1	·0690	·2627	·01674	284700	·001048	·004186	4604000	·001191
Air rarefied	·01	·01	·01	·2427	1136	·01	·06067	20890	·01
The same filled with H...	1	·0783	·2798	·01900	220600	·001440	·004751	3572000	·001637
Air of same density	·0783	·0783	·0783	·2427	1136	·0783	·06067	20890	·0783
Air rarefied ½	·5	·5	·5	·2427	1136	·5	·06067	20890	·5
The same filled with H ..	1	·5345	·7311	·1297	4322	·1921	·0324	74890	·2039

pressure p, in atmospheres, the third the density D under the pressure p, referred to the density of the air at the atmospheric pressure as unity, the fourth, Q_r, what would have been the intensity had the motion been wholly radial, referred to the intensity in air at atmospheric pressure as unity, or, in other words, a quantity varying as $p \times$ (the density at pressure 1)$^{\frac{1}{2}}$. Then follow the values of q, I_2, and Q, the last being the actual intensity referred to air as before.

"An inspection of the numbers contained in the columns headed Q will shew that the cause here investigated is amply sufficient to account for the facts mentioned by Leslie."

The importance of the subject, and the masterly manner in which it has been treated by Prof. Stokes, will probably be thought sufficient to justify this long quotation. The simplicity of the true explanation contrasts remarkably with conjectures that had previously been advanced. Sir J. Herschel, for example, thought that the mixture of two gases tending to propagate sound with different velocities might produce a confusion resulting in a rapid stifling of the sound.

[The subject now under consideration may be still more simply illustrated by the problems of §§ 268, 301. The former, for instance, may be regarded as the extreme case of the present, in which the spherical surface is reduced to a plane vibrating in rectangular segments. If we suppose the size of these segments, determined by p and q, to be given, and trace the effect of gradually increasing frequency, we see that it is only when the frequency attains a certain value that sensible vibrations are propagated to infinity, the law of diminution with distance being exponential in its form. On the other hand vibrations whose frequency exceeds the critical value are propagated without loss, escaping the attenuation to which spherical waves must of necessity submit.]

325. The term of zero order

$$\psi_0 = \frac{S_0}{r} e^{ik(at-r)} \quad \text{.......................} \quad (1).$$

where S_0 is a complex constant, corresponds to the potential of a *simple source* of arbitrary intensity and phase, situated at the centre of the sphere (§ 279). If, as often happens in practice, the source of sound be a solid body vibrating without much change of volume, this term is relatively deficient. In the case of a rigid sphere vibrating about a position of equilibrium, the deficiency is absolute[1], inasmuch as the whole motion will then be represented by a term of order 1; and whenever the body is very small in comparison with the wave-length, the term of zero order must be insignificant. For if we integrate the equation of motion, $\nabla^2\psi + k^2\psi = 0$, over the small volume included between the body and a sphere closely surrounding it, we see that the whole quantity of fluid which enters and leaves this space is small, and that therefore there is but little total flow across the surface of the sphere.

Putting $n=1$, we get for the term of the first order

$$r\psi_1 = S_1 e^{ik(at-r)}\left\{1+\frac{1}{ikr}\right\} \qquad (2),$$

and S_1 is proportional to the cosine of the angle between the direction considered and some fixed axis. This expression is of the same form as the potential of a *double* source (§ 294), situated at the centre, and composed of two equal and opposite simple sources lying on the axis in question, whose distance apart is infinitely small, and intensities such that the product of the intensities and distance is finite. For, if x be the axis, and the cosine of the angle between x and r be μ, it is evident that the potential of the double source is proportional to

$$\frac{d}{dx}\left(\frac{e^{-ikr}}{r}\right) = \mu\frac{d}{dr}\left(\frac{e^{-ikr}}{r}\right) = -ik\frac{\mu e^{-ikr}}{r}\left\{1+\frac{1}{ikr}\right\}.$$

It appears then that the disturbance due to the vibration of a sphere as a rigid body is the same as that corresponding to a double source at the centre whose axis coincides with the line of the sphere's vibration.

The reaction of the air on a small sphere vibrating as a rigid body with a harmonic motion, may be readily calculated from preceding formulæ. If $\dot{\xi}$ denote the velocity of the sphere at time t,

$$U_1 e^{ikat} = \dot{\xi}\mu \qquad (3),$$

and therefore for the value of ψ at the surface of the sphere, we have from (5) § 324,

$$\dot{\psi} = -ikac\dot{\xi}\mu\frac{f_1(ikc)}{F_1(ikc)} \qquad (4).$$

[1] The centre of the sphere being the origin of coordinates.

The force Ξ due to aerial pressures accelerating the motion is given by

$$\Xi = -\iint \mu \delta p \, dS = \rho \iint \mu \dot{\psi} \, dS$$

$$= - ika\rho c^3 \dot{\xi} \frac{f_1(ikc)}{F_1(ikc)} \int 2\pi \mu^2 d\mu = - ika \frac{4\pi c^3}{3} \rho \dot{\xi} \frac{f_1(ikc)}{F_1(ikc)}.$$

If we write

$$\frac{f_1(ikc)}{F_1(ikc)} = p - iq \quad \text{.....................(5)},$$

then
$$\Xi = -p \cdot \tfrac{4}{3}\pi\rho c^3 \cdot \ddot{\xi} - qka \cdot \tfrac{4}{3}\pi\rho c^3 \cdot \dot{\xi} \quad \text{..............(6)},$$

inasmuch as
$$\ddot{\xi} = ika\,\dot{\xi}.$$

The operation of the air is therefore to increase the effective inertia of the sphere by p times the inertia of the air displaced, and to retard the motion by a force proportional to the velocity, and equal to $\frac{4}{3}\pi\rho c^3 \cdot qka\dot{\xi}$, these effects being in general functions of the frequency of vibration. By introduction of the values of f_1 and F_1 we find

$$\frac{f_1(ikc)}{F_1(ikc)} = \frac{2 + k^2c^2 - ik^3c^3}{4 + k^4c^4} \quad \text{.................(7)};$$

so that,
$$p = \frac{2 + k^2c^2}{4 + k^4c^4}, \qquad q = \frac{k^3c^3}{4 + k^4c^4} \quad \text{................(8)}.$$

When kc is small, we have approximately $p = \frac{1}{2}$, $q = \frac{1}{4}k^3c^3$. Hence the effective inertia of a small sphere is increased by one-half of that of the air displaced—a quantity independent of the frequency and the same as if the fluid were incompressible. The dissipative term, which corresponds to the energy emitted, is of high order in kc, and therefore (the effects of viscosity being disregarded) the vibrations of a small sphere are but slowly damped.

The motion of an ellipsoid through an incompressible fluid has been investigated by Green[1], and his result is applicable to the calculation of the increase of effective inertia due to a compressible fluid, provided the dimensions of the body be small in comparison with the wave-length of the vibration. For a small circular disc vibrating at right angles to its plane, the increase of effective inertia is to the mass of a sphere of fluid, whose radius is equal to

[1] *Edinburgh Transactions*, Dec. 16, 1833. Also Green's *Mathematical Papers*, edited by Ferrers. Macmillan & Co., 1871.

that of the disc, as 2 to π. The result for the case of a sphere given above was obtained by Poisson[1], a short time before the publication of Green's paper.

It has been proved by Maxwell[2] that the various terms of the harmonic expansion of the common potential may be regarded as due to *multiple points* of corresponding degrees of complexity. Thus V_i is proportional to $\frac{d^i}{dh_1 dh_2 \dots dh_i}\left(\frac{1}{r}\right)$, where there are i differentiations of r^{-1} with respect to the axes h_1, h_2, &c., any number of which may in particular cases coincide. It might perhaps have been expected that a similar law would hold for the velocity potential with the substitution of $r^{-1}e^{-ikr}$ for r^{-1}. This however is not the case; it may be shewn that the potential of a quadruple source, denoted by $\frac{d^2}{dh_1 dh_2}\cdot\frac{e^{-ikr}}{r}$, corresponds in general not to the term of the second order simply, viz., $S_2\frac{e^{-ikr}}{r}f_2(ikr)$, but to a combination of this with a term of zero order. The analogy therefore holds only in the single instance of the *double* point or source, though of course the function $r^{-1}e^{-ikr}$ after any number of differentiations continues to satisfy the fundamental equation

$$(\nabla^2 + k^2)\psi = 0.$$

It is perhaps worth notice that the disturbance outside any imaginary sphere which completely encloses the origin of sound may be represented as due to the normal motion of the surface of any smaller concentric sphere, or, as a particular case when the radius of the sphere is infinitely small, as due to a source concentrated in one point at the centre. This source will in general be composed of a combination of multiple sources of all orders of complexity.

326. When the origin of the disturbance is the vibration of a rigid body parallel to its axis of revolution, the various spherical harmonics S_n reduce to simple multiples of the zonal harmonic $P_n(\mu)$, which may be defined as the coefficient of e^n in the expansion of $\{1 - 2e\mu + e^2\}^{-\frac{1}{2}}$ in rising powers of e. [For the forms of these functions see § 334.] And whenever the solid, besides being

[1] *Mémoires de l'Académie des Sciences*, Tom. XI. p. 521.

[2] Maxwell's *Electricity and Magnetism*, Ch. IX.

symmetrical about an axis, is also symmetrical with respect to an equatorial plane (whose intersection with the axis is taken as origin of co-ordinates), the expansion of the resulting disturbance in spherical harmonics will contain terms of odd order only. For example, if the vibrating body were a circular disc moving perpendicularly to its plane, the expansion of ψ would contain terms proportional to $P_1(\mu), P_3(\mu), P_5(\mu)$, &c. In the case of the sphere, as we have seen, the series reduces absolutely to its first term, and this term will generally be preponderant.

On the other hand we may have a vibrating system symmetrical about an axis and with respect to an equatorial plane, but in such a manner that the motions of the parts on the two sides of the plane are opposed. Under this head comes the ideal tuning-fork, composed of equal spheres or parallel circular discs, whose distance apart varies periodically. Symmetry shews that the velocity-potential, being the same at any point and at its image in the plane of symmetry, must be an even function of μ, and therefore expressible by a series containing only the even functions $P_0(\mu)$, $P_2(\mu)$, &c. The second function $P_2(\mu)$ would usually preponderate, though in particular cases, as for example if the body were composed of two discs very close together in comparison with their diameter, the symmetrical term of zero order might become important. A comparison with the known solution for the sphere whose surface vibrates according to any law, will in most cases furnish material for an estimate as to the relative importance of the various terms.

[The accompanying table, p. 251, giving P_n as a function of θ, or $\cos^{-1}\mu$, is abbreviated from that of Perry[1].]

327. The total emission of energy by a vibrating sphere is found by multiplying the variable part of the pressure (proportional to $\dot{\psi}$) by the normal velocity and integrating over the surface (§ 245). In virtue of the conjugate property the various spherical harmonic terms may be taken separately without loss of generality. We have (§ 323)

$$\left.\begin{aligned} \dot{\psi}_n &= ika\,\frac{S_n\,e^{ik(at-r)}}{r}\,f_n(ikr) \\ \frac{d\psi_n}{dr} &= -\frac{S_n\,e^{ik(at-r)}}{r^2}\,F_n(ikr) \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(1),$$

[1] *Phil. Mag.* vol. XXXII., p. 516, 1891.

Table of Zonal Spherical Harmonics.

θ	P_1	P_2	P_3	P_4	P_5	P_6	P_7
0°	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
2	·9994	·9982	·9963	·9939	·9909	·9872	·9829
4	·9976	·9927	·9854	·9758	·9638	·9495	·9329
6	·9945	·9836	·9674	·9459	·9194	·8881	·8522
8	·9903	·9709	·9423	·9048	·8589	·8053	·7448
10	·9848	·9548	·9106	·8532	·7840	·7045	·6164
12	·9781	·9352	·8724	·7920	·6966	·5892	·4732
14	·9703	·9122	·8283	·7224	·5990	·4635	·3219
16	·9613	·8860	·7787	·6454	·4937	·3322	·1699
18	·9511	·8568	·7240	·5624	·3836	·2002	·0289
20	·9397	·8245	·6649	·4750	·2715	·0719	– ·1072
22	·9272	·7895	·6019	·3845	·1602	– ·0481	– ·2201
24	·9135	·7518	·5357	·2926	·0525	– ·1559	– ·3095
26	·8988	·7117	·4670	·2007	– ·0489	– ·2478	– ·3717
28	·8829	·6694	·3964	·1105	– ·1415	– ·3211	– ·4052
30	·8660	·6250	·3248	·0234	– ·2233	– ·3740	– ·4101
32	·8480	·5788	·2527	– ·0591	– ·2923	– ·4052	– ·3876
34	·8290	·5310	·1809	– ·1357	– ·3473	– ·4148	– ·3409
36	·8090	·4818	·1102	– ·2052	– ·3871	– ·4031	– ·2738
38	·7880	·4314	·0413	– ·2666	– ·4112	– ·3719	– ·1918
40	·7660	·3802	– ·0252	– ·3190	– ·4197	– ·3234	– ·1003
42	·7431	·3284	– ·0887	– ·3616	– ·4128	– ·2611	– ·0065
44	·7193	·2762	– ·1485	– ·3940	– ·3914	– ·1878	·0846
46	·6947	·2238	– ·2040	– ·4158	– ·3568	– ·1079	·1666
48	·6691	·1716	– ·2547	– ·4270	– ·3105	– ·0251	·2349
50	·6428	·1198	– ·3002	– ·4275	– ·2545	+ ·0563	·2854
52	·6157	·0686	– ·3401	– ·4178	– ·1910	+ ·1326	·3153
54	·5878	·0182	– ·3740	– ·3984	– ·1223	+ ·2002	·3234
56	·5592	– ·0310	– ·4016	– ·3698	– ·0510	+ ·2559	·3095
58	·5299	– ·0788	– ·4229	– ·3331	·0206	+ ·2976	·2752
60	·5000	– ·1250	– ·4375	– ·2891	·0898	+ ·3232	·2231
62	·4695	– ·1694	– ·4455	– ·2390	·1545	+ ·3321	·1571
64	·4384	– ·2117	– ·4470	– ·1841	·2123	+ ·3240	·0818
66	·4067	– ·2518	– ·4419	– ·1256	·2615	+ ·2996	·0021
68	·3746	– ·2896	– ·4305	– ·0650	·3005	+ ·2605	– ·0763
70	·3420	– ·3245	– ·4130	– ·0038	·3281	+ ·2089	– ·1485
72	·3090	– ·3568	– ·3898	·0568	·3434	+ ·1472	– ·2099
74	·2756	– ·3860	– ·3611	·1153	·3461	+ ·0795	– ·2559
76	·2419	– ·4132	– ·3275	·1705	·3362	+ ·0076	– ·2848
78	·2079	– ·4352	– ·2894	·2211	·3143	– ·0644	– ·2943
80	·1736	– ·4548	– ·2474	·2659	·2810	– ·1321	– ·2835
82	·1392	– ·4709	– ·2020	·3040	·2378	– ·1926	– ·2536
84	·1045	– ·4836	– ·1539	·3345	·1861	– ·2431	– ·2067
86	·0698	– ·4927	– ·1038	·3569	·1278	– ·2811	– ·1460
88	·0349	– ·4982	– ·0522	·3704	·0651	– ·3045	– ·0735
90	·0000	– ·5000	– ·0000	·3750	·0000	– ·3125	·0000

or on rejecting the imaginary part

$$\left.\begin{aligned}\dot{\psi}_n &= -\frac{kaS_n}{r}\{\beta' \cos k(at-r) + \alpha' \sin k(at-r)\} \\ \frac{d\psi_n}{dr} &= -\ \frac{S_n}{r^2}\{\ \alpha \cos k(at-r) - \beta \sin k(at-r)\}\end{aligned}\right\} \quad \ldots\ldots(2),$$

where $$F = \alpha + i\beta, \qquad f = \alpha' + i\beta' \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Thus $$\iint \dot{\psi}_n \frac{d\psi_n}{dr}\, dS = \iint \dot{\psi}_n \frac{d\psi_n}{dr} \,.\, r^2 d\sigma$$

$$= \frac{ka}{r}\iint S_n^2 d\sigma\,\{\alpha\beta' \cos^2 k(at-r) - \alpha'\beta \sin^2 k(at-r)$$
$$+ (\alpha\alpha' - \beta\beta') \sin k(at-r) \cos k(at-r)\}.$$

When this is integrated over a long range of time, the periodic terms may be omitted, and thus

$$\int . \iint \dot{\psi}_n \frac{d\psi_n}{dr}\, dS \,.\, dt = \frac{kat}{2r}(\alpha\beta' - \alpha'\beta)\iint S_n^2 d\sigma \quad \ldots\ldots\ (4).$$

Now, since there can be on the whole no accumulation of energy in the space included between two concentric spherical surfaces, the rates of transmission of energy across these surfaces must be the same, that is to say $r^{-1}(\alpha'\beta - \beta'\alpha)$ must be independent of r. In order to determine the constant value, we may take the particular case of r indefinitely great, when

$$F_n(ikr) = ikr, \qquad \alpha = 0, \qquad \beta = kr,$$
$$f_n(ikr) = 1, \qquad \alpha' = 1, \qquad \beta' = 0.$$

Thus $$\alpha'\beta - \beta'\alpha = kr, \text{ identically} \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

It may be observed that the left-hand member of (5) when multiplied by i is the imaginary part of $(\alpha + i\beta)(\alpha' - i\beta')$ or of $F_n(ikr) f_n(-ikr)$, so that our result may be expressed by saying that the imaginary part of $F_n(ikr) f_n(-ikr)$ is ikr, or

$$F_n(ikr) f_n(-ikr) - F_n(-ikr) f_n(ikr) = 2ikr \quad \ldots\ldots\ (6).$$

In this form we shall have occasion presently to make use of it.

The same conclusion may be arrived at somewhat more directly by an application of Helmholtz's theorem (§ 294), *i.e.* that if two functions u and v satisfy through a closed space S the equation $(\nabla^2 + k^2)u = 0$, then

$$\iint \left(u\frac{dv}{dn} - v\frac{du}{dn}\right) dS = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

If we take for S the space between two concentric spheres, making

$$u=\frac{S_n e^{-ikr} f_n(ikr)}{r}, \qquad v=\frac{S_n e^{+ikr} f_n(-ikr)}{r},$$

we find that $r^{-1}\{F_n'(ikr) f_n(-ikr) - F_n(-ikr) f_n(ikr)\}$ must be independent of r.

We have therefore

$$\int . \iint \dot{\psi}_n \frac{d\psi_n}{dr} dS . dt = -\tfrac{1}{2}k^2 at \iint S_n^2 d\sigma;$$

so that the expression for the energy emitted in time t is (since $\delta p = -\rho\dot{\psi}$)

$$W = \tfrac{1}{2}k^2 \rho\, at \iint S_n^2 d\sigma \quad\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

It will be more instructive to exhibit W as a function of the normal motion at the surface of a sphere of radius c. From (2)

$$\frac{d\psi_n}{dr} = -\frac{S_n}{c^2}[\cos kat(\alpha\cos kc + \beta\sin kc) + \sin kat(\alpha\sin kc - \beta\cos kc)],$$

so that, if the amplitude of $d\psi_n/dr$ be U_n, we have as the relation between S_n and U_n

$$c^4 U_n^2 = (\alpha^2+\beta^2) S_n^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Thus

$$W = \frac{k^2 c^4 \rho a t}{2(\alpha^2+\beta^2)} \iint U_n^2 d\sigma \quad\ldots\ldots\ldots\ldots\ldots(10).$$

This formula may be verified for the particular cases $n=0$ and $n=1$, treated in §§ 280, 325 respectively.

328. If the source of disturbance be a normal motion of a small part of the surface of the sphere ($r=c$) in the immediate neighbourhood of the point $\mu=1$, we must take in the general solution applicable to divergent waves, viz.

$$\psi = -\frac{c^2}{r} e^{ik(at-r+c)} \Sigma \frac{U_n}{F_n(ikc)} f_n(ikr) \quad\ldots\ldots\ldots\ldots(1),$$

$$U_n = \tfrac{1}{2}(2n+1) P_n(\mu) . \int_{-1}^{+1} U P_n(\mu)\, d\mu$$

$$= \tfrac{1}{2}(2n+1) P_n(\mu) \int_{-1}^{+1} U d\mu = \frac{2n+1}{4\pi c^2} P_n(\mu) \iint U dS \quad\ldots\ldots\ldots(2);$$

for where U is sensible, $P_n(\mu)=1$. Thus

$$\psi = -\frac{e^{ik(at-r+c)}}{4\pi r}\cdot\iint U\,dS\,.\,\Sigma\,(2n+1)P_n(\mu)\frac{f_n(ikr)}{F_n(ikc)} \quad\text{......(3).}$$

In this formula $\iint U\,dS$ measures the intensity of the source.

If ikc be very small,

$$\frac{f_0(ikr)}{F_0(ikc)} = 1 - ikc + \ldots, \qquad \frac{f_1(ikr)}{F_1(ikc)} = \tfrac{1}{2}ikc\left(1+\frac{1}{ikr}\right)+\ldots \text{ \&c.};$$

so that ultimately

$$\psi = -\frac{e^{ik(at-r)}}{4\pi r}\iint U\,dS \quad\text{..................(4),}$$

and the waves diverge as from a simple source of equal magnitude.

We will now examine the problem when kc is not very small, taking for simplicity the case where ψ is required at a great distance only, so that $f_n(ikr)=1$. The factor on which the relative intensities in various directions depend is

$$\Sigma\,\frac{(2n+1)}{2}\,\frac{P_n(\mu)}{F_n(ikc)} \quad\text{.................... (5),}$$

and a complete solution of the question would involve a discussion of this series as a function of μ and kc.

Thus, if

$$\Sigma\,\frac{(2n+1)}{2}\,\frac{P_n(\mu)}{F_n(ikc)} = F + iG \quad\text{.............. (6),}$$

$$\psi = -\frac{1}{2\pi r}\iint U\,dS\,.\,\{F^2+G^2\}^{\frac{1}{2}}\,.\,e^{ik(at-r+c)+i\theta} \quad\text{.........(7),}$$

where

$$\tan\theta = G : F \quad\text{....... .... (8).}$$

The intensity of the vibrations in the various directions is thus measured by F^2+G^2. If, as before, $F_n=\alpha+i\beta$,

$$\left.\begin{aligned} F &= \Sigma\,\frac{2n+1}{2}\,\frac{\alpha P_n(\mu)}{\alpha^2+\beta^2} \\ -G &= \Sigma\,\frac{2n+1}{2}\,\frac{\beta P_n(\mu)}{\alpha^2+\beta^2} \end{aligned}\right\} \quad\text{................. (9).}$$

The following table gives the means of calculating F and G for any value of μ, when $kc=\frac{1}{2}$, 1, or 2. In the last case it is necessary to go as far as $n=7$ to get a tolerably accurate result, and for larger values of kc the calculation would soon become very laborious. In all problems of this sort the harmonic analysis seems to lose its power when the waves are very small in comparison with the dimensions of bodies.

$kc = \frac{1}{2}$.

n	2α	2β	$(n+\frac{1}{2})\alpha \div (\alpha^2+\beta^2)$	$(n+\frac{1}{2})\beta \div (\alpha^2+\beta^2)$
0	+ 2	+ 1	+ ·4	+ ·2
1	+ 4	− 7	+ ·1846153	− ·3230768
2	− 64	− 35	− ·0601391	− ·0328885
3	− 466	+ 853	− ·0034527	+ ·0063201
4	+ 14902	+ 8141	+ ·0004653	+ ·0002542
5	+ 175592	− 321419	+ ·0000144	− ·0000264

$kc = 1$.

n	α	β	$(n+\frac{1}{2})\alpha \div (\alpha^2+\beta^2)$	$(n+\frac{1}{2})\beta \div (\alpha^2+\beta^2)$
0	+ 1	+ 1	+ ·25	+ ·25
1	+ 2	− 1	+ ·6	− ·3
2	− 5	− 8	− ·140449	− ·224719
3	− 53	+ 34	− ·046784	+ ·030013
4	+ 296	+ 461	+ ·004438	+ ·006912
5	+ 4951	− 3179	+ ·000787	− ·000505
6	− 40613	− 63251	− ·000047	− ·000073
7	− 936340	+ 601217	− ·000006	+ ·000004

$kc = 2$.

n	α	β	$(n+\frac{1}{2})\alpha \div (\alpha^2+\beta^2)$	$(n+\frac{1}{2})\beta \div (\alpha^2+\beta^2)$
0	+ 1	+ 2	+ ·1	+ ·2
1	+ 2	+ 1	+ ·6	+ ·3
2	+ 1·75	− 2·5	+ ·46980	− ·67114
3	− 8	− 4	− ·35	− ·175
4	− 16·1875	+ 35·125	− ·04870	+ ·10567
5	+ 186·625	+ 85·4375	+ ·02436	+ ·01115
6	+ 538·80	− 1177·3	+ ·00209	− ·00456
7	− 8621·7	− 3945·8	− ·00072	− ·00033

The most interesting question on which this analysis informs us is the influence which a rigid sphere, situated close to the source, has on the intensity of sound in different directions. By the principle of reciprocity (§ 294) the source and the place of observation may be interchanged. When therefore we know the relative intensities at two distant points B, B', due to a source A on the surface of the sphere, we have also the relative intensities (measured by potential) at the point A, due to distant sources at B and B'. On this account the problem has a double interest.

As a numerical example I have calculated the values of $F+iG$ and F^2+G^2 for the above values of kc, when $\mu=1$, $\mu=-1$, $\mu=0$, that is, looking from the centre of the sphere, in the direction of the source, in the opposite direction, and laterally.

When kc is zero, the value of F^2+G^2 is ·25, which therefore represents on the same scale as in the table the intensity due to an unobstructed source of equal magnitude. We may interpret kc as the ratio of the circumference of the sphere to the wave-length of the sound.

kc	μ	$F+iG$	F^2+G^2
$\frac{1}{2}$	1	·521503 + ·149417i	·294291
	−1	·159149 − ·484149i	·259729
	0	·430244 − ·216539i	·231999
1	1	·667938 + ·238369i	·502961
	−1	− ·440055 − ·302609i	·285220
	0	+ ·321903 − ·364974i	·236828
2	1	·79683 + ·23421i	·6898
	−1	·24954 + ·50586i	·3182
	0	− ·15381 − ·57662i	·3562

In looking at these figures the first point which attracts attention is the comparatively slight deviation from uniformity in the intensities in different directions. Even when the circumference of the sphere amounts to twice the wave-length, there is scarcely anything to be called a sound shadow. But what is perhaps still more unexpected is that in the first two cases the intensity behind the sphere exceeds that in a transverse direction. This result depends mainly on the preponderance of the term of the first order, which vanishes with μ. The order of the more important terms increases with kc; when kc is 2, the principal term is that of the second order.

Up to a certain point the augmentation of the sphere will increase the total energy emitted, because a simple source emits twice as much energy when close to a rigid plane as when entirely in the open. Within the limits of the table this effect masks the obstruction due to an increasing sphere, so that when $\mu = -1$, the intensity is greater when the circumference is twice the wave-length than when it is half the wave-length, the source itself remaining constant.

If the source be not simple harmonic with respect to time, the relative proportions of the various constituents will vary to some extent both with the size of the sphere and with the direction of the point of observation, illustrating the fundamental character of the analysis into simple harmonics.

When kc is decidedly less than one-half, the calculation may be conducted with sufficient approximation algebraically. The result is

$$F^2 + G^2 = \tfrac{1}{4} + \tfrac{1}{12} k^2c^2 (\tfrac{7}{4}\mu^2 - \tfrac{4}{3})$$
$$+ \tfrac{1}{4} k^4c^4 \left(1 + \tfrac{3}{2}\mu + \tfrac{50}{81} P_2 + \tfrac{25}{81} P_2^2 - \tfrac{7}{20}\mu P_3 + \tfrac{6}{175} P_4\right)$$
$$+ \text{terms in } k^6c^6 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10).$$

It appears that so far as the term in k^2c^2, the intensity is an even function of μ, viz. the same at any two points diametrically opposed. For the principal directions $\mu = \pm 1$, or 0, the numerical calculation of the coefficient of k^4c^4 is easy on account of the simple values then assumed by the functions P. Thus

$$(\mu = 1), \quad F^2 + G^2 = \tfrac{1}{4} + \tfrac{5}{144} k^2c^2 + {\cdot}77755\, k^4c^4 + \ldots\ldots$$
$$(\mu = -1), \quad F^2 + G^2 = \tfrac{1}{4} + \tfrac{5}{144} k^2c^2 + {\cdot}02755\, k^4c^4 + \ldots\ldots$$
$$(\mu = 0), \quad F^2 + G^2 = \tfrac{1}{4} - \tfrac{1}{9} k^2c^2 + {\cdot}19534\, k^4c^4 + \ldots\ldots$$

When k^4c^4 can be neglected. the intensity is less in a lateral direction than immediately in front of or behind the sphere. Or, by the reciprocal property, a source at a distance will give a greater intensity on the surface of a small sphere at the point furthest from the source than in a lateral position.

If we apply these formulæ to the case of $kc = \tfrac{1}{2}$, we get

$$(\mu = 1), \quad F^2 + G^2 = {\cdot}3073,$$
$$(\mu = -1), \quad F^2 + G^2 = {\cdot}2604,$$
$$(\mu = 0), \quad F^2 + G^2 = {\cdot}2344,$$

which agree pretty closely with the results of the more complete calculation.

For other values of μ, the coefficient of k^4c^4 in (10) might be calculated with the aid of tables of Legendre's functions, or from the following algebraic expression in terms of μ[1],

$$1 + \tfrac{3}{2}\mu + \tfrac{50}{81}P_2 + \tfrac{25}{81}P_2{}^2 - \tfrac{7}{20}\mu P_3 + \tfrac{6}{175}P_4$$
$$= \cdot 78138 + 1\cdot 5\,\mu + \cdot 85938\,\mu^2 - \cdot 03056\,\mu^4.$$

The *difference* of intensities in the directions $\mu = +1$ and $\mu = -1$ may be very simply expressed. Thus

$$(F^2 + G^2)_{\mu=1} - (F^2 + G^2)_{\mu=-1} = \tfrac{3}{4}k^4c^4.$$

If $kc = \tfrac{4}{8}$, $\tfrac{3}{4}k^4c^4 = \cdot 0148$.

If $kc = \tfrac{3}{8}$, $\tfrac{3}{4}k^4c^4 = \cdot 0029$.

If $kc = \tfrac{1}{8}$, $\tfrac{3}{4}k^4c^4 = \cdot 0002$.

At the same time the total value of $F^2 + G^2$ approximates to $\cdot 25$, when kc is small.

These numbers have an interesting bearing on the explanation of the part played by the two ears in the perception of the quarter from which a sound proceeds.

It should be observed that the variations of intensity in different directions about which we have been speaking are due to the presence of the sphere as an obstacle, and not to the fact that the source is on the circumference of the sphere instead of at the centre. At a great distance a small displacement of a source of sound will affect the *phase* but not the *intensity* in any direction.

In order to find the alteration of phase we have for a small sphere

$$F = \tfrac{1}{2} + k^2c^2(-\tfrac{1}{2} + \tfrac{3}{4}\mu - \tfrac{5}{18}P_2), \quad G = kc\,(-\tfrac{1}{2} + \tfrac{3}{4}\mu),$$

$$\tan\theta = G : F = kc\,(-1 + \tfrac{3}{2}\mu), \text{ or } \theta = kc\,(-1 + \tfrac{3}{2}\mu) \text{ nearly.}$$

Thus in (7) $e^{ik\,(at-r+c)+i\theta} = e^{ik\,(at-r+\frac{3}{2}\mu c)}$,

from which we may infer that the phase at a distance is the same as if the source had been situated at the point $\mu = 1$, $r = \tfrac{3}{2}c$ (instead of $r = c$), and there had been no obstacle.

329. The functional symbols f and F may be expressed in terms of P. It is known[2] that

$$P_n(\mu) = 1 - \frac{n}{1}.\frac{n+1}{1}.\frac{1-\mu}{2} + \frac{n(n-1)}{1.2}\frac{(n+1)(n+2)}{1.2}\frac{(1-\mu)^2}{2^2} - \ldots$$

[1] For the forms of the functions P, see § 334.

[2] Thomson and Tait's *Nat. Phil.* § 782 (quoted from Murphy).

or, on changing μ into $1-\mu$,

$$P_n(1-\mu)=1-\frac{n}{1}\cdot\frac{n+1}{1}\cdot\frac{\mu}{2}+\frac{n(n-1)}{1\,.\,2}\cdot\frac{(n+1)(n+2)}{1\,.\,2}\cdot\frac{\mu^2}{2^2}-\ldots(1).$$

Consider now the symbolic operator $P_n\left(1-\frac{d}{dy}\right)$, and let it operate on y^{-1}.

Since $$\left(\frac{d}{dy}\right)^s\cdot\frac{1}{y}=(-1)(-2)\ldots\ldots(-s)\,y^{-s-1},$$

$$P_n\left(1-\frac{d}{dy}\right)\cdot\frac{1}{y}=y^{-1}+\frac{n(n+1)}{1\,.\,2}y^{-2}+\frac{(n-1)\ldots\ldots(n+2)}{2\,.\,4}y^{-3}+\ldots\ldots$$

A comparison with (9) § 323 now shews that

$$f_n(y)=y\,P_n\left(1-\frac{d}{dy}\right)\cdot\frac{1}{y}\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

from which we deduce by a known formula,

$$\frac{e^{-y}}{y}f_n(y)=e^{-y}P_n\left(1-\frac{d}{dy}\right)\frac{1}{y}=(-1)^n P_n\left(\frac{d}{dy}\right)\cdot\frac{e^{-y}}{y}\ldots\ldots(3).$$

In like manner,

$$\frac{e^{+y}}{y}f_n(-y)=P_n\left(\frac{d}{dy}\right)\cdot\frac{e^{+y}}{y}.$$

If we now identify y with ikr, we see that the general solution, (12) § 323, may be written

$$\psi_n=(-1)^n ikS_n P_n\left(\frac{d}{d\,.\,ikr}\right)\cdot\frac{e^{-ikr}}{ikr}+ikS_n' P_n\left(\frac{d}{d\,.\,ikr}\right)\cdot\frac{e^{+ikr}}{ikr}\ldots(4),$$

from which the second term is to be omitted, if no part of the disturbance be propagated inwards.

Again from (14) § 323 we see that

$$\frac{F_n(y)}{y^2}=\left(1-\frac{d}{dy}\right)\cdot\frac{f_n(y)}{y},$$

whence $$F_n(y)=y^2P_n\left(1-\frac{d}{dy}\right)\left(1-\frac{d}{dy}\right)\cdot\frac{1}{y}\ldots\ldots\ldots\ldots(5),$$

and $$\frac{F_n(y)\,e^{-y}}{y^2}=-(-)^n P_n\left(\frac{d}{dy}\right)\frac{d}{dy}\cdot\frac{e^{-y}}{y}\ldots\ldots\ldots\ldots\ldots(6).$$

Similarly, $$\frac{F_n(-y)\,e^{+y}}{y^2}=-P_n\left(\frac{d}{dy}\right)\frac{d}{dy}\cdot\frac{e^{+y}}{y}\ldots\ldots\ldots\ldots\ldots(7).$$

Using these expressions in (13) § 323, we get

$$\frac{d\psi_n}{dr} = (-)^{n+1} k^2 S_n P_n \left(\frac{d}{d.ikr}\right) \frac{d}{d.ikr} \cdot \frac{e^{-ikr}}{ikr}$$
$$- k^2 S_n' P_n \left(\frac{d}{d.ikr}\right) \frac{d}{d.ikr} \cdot \frac{e^{+ikr}}{ikr} \quad \text{.........} (8).$$

330. We have already considered in some detail the form assumed by our general expressions when there is no source at infinity. An equally important class of cases is defined by the condition that there be no source at the origin. We shall now investigate what restriction is thereby imposed on our general expressions.

Reversing the series for f_n, we have

$$r\psi_n = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{(ikr)^n} \{S_n e^{-ikr} (1 + ikr + \ldots)$$
$$+ (-1)^n S_n' e^{+ikr} (1 - ikr + \ldots)\},$$

shewing that, as r diminishes without limit, $r\psi_n$ approximates to

$$r\psi_n = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{(ikr)^n} \{S_n + (-1)^n S_n'\}.$$

In order therefore that ψ_n may be finite at the origin,

$$S_n + (-1)^n S_n' = 0 \quad \text{.........................} (1)$$

is a necessary condition; that it is sufficient we shall see later.

Accordingly (12) § 323 becomes

$$r\psi_n = S_n \{e^{-ikr} f_n (ikr) - (-1)^n e^{+ikr} f_n (-ikr)\} \quad \text{......} (2).$$

If, separating the real and imaginary parts of f_n, we write (as before)

$$f_n = \alpha' + i\beta' \quad \text{.........................} (3),$$

(2) may be put into the form

$$r\psi_n = -2i^{n+1} S_n \{\alpha' \sin (kr + \tfrac{1}{2} n\pi) - \beta' \cos (kr + \tfrac{1}{2} n\pi)\} \quad \text{......} (4).$$

Another form may be derived from (4) § 329. We have

$$\psi_n = -2ik (-1)^n S_n P_n \left(\frac{d}{d \,.\, ikr}\right) \cdot \frac{e^{+ikr} - e^{-ikr}}{2ikr}$$
$$= -2ik (-1)^n S_n P_n \left(\frac{d}{d \,.\, ikr}\right) \cdot \frac{\sin kr}{kr} \quad \text{...........} (5).$$

Since the function P_n is either wholly odd or wholly even, the expression for ψ_n is wholly real or wholly imaginary.

In order to prove that the value of ψ_n in (5) remains finite when r vanishes, we begin by observing that

$$\frac{2 \sin kr}{kr} = \int_{-1}^{+1} e^{-ikr\mu}\, d\mu \quad \text{.....................(6)},$$

so that
$$2P_n\left(\frac{d}{d\,.\,ikr}\right)\frac{\sin kr}{kr} = \int_{-1}^{+1} P_n\left(\frac{d}{d\,.\,ikr}\right) e^{ikr\mu}\, d\mu$$
$$= \int_{-1}^{+1} P_n(\mu)\, e^{ikr\mu}\, d\mu \quad \text{..............(7)},$$

as is obvious when it is considered that the effect of differentiating $e^{ikr\mu}$ any number of times with respect to ikr is to multiply it by the corresponding power of μ. It remains to expand the expression on the right in ascending powers of r. We have

$$\int_{-1}^{+1} P_n(\mu)\, e^{ikr\mu}\, d\mu = \int_{-1}^{+1} d\mu\, P_n(\mu)\left\{1 + ikr\,.\,\mu + \frac{(ikr)^2}{1\,.\,2}\,.\,\mu^2 + \ldots + \frac{(ikr)^n}{1\,.\,2\ldots n}\,.\,\mu^n + \ldots\right\}.$$

Now any positive integral power of μ, such as μ^p, can be expanded in a terminating series of the functions P, the function of highest order being P_p. It follows that, if $p < n$,

$$\int_{-1}^{+1} \mu^p P_n(\mu)\, d\mu = 0,$$

by known properties of these functions; so that the lowest power of ikr in $\int_{-1}^{+1} P_n(\mu)\, e^{ikr\mu}\, d\mu$ is $(ikr)^n$. Retaining only the leading term, we may write

$$\int_{-1}^{+1} P_n(\mu)\, e^{ikr\mu}\, d\mu = \frac{(ikr)^n}{1\,.\,2\ldots n}\int_{-1}^{+1} \mu^n P_n(\mu)\, d\mu.$$

From the expression for $P_n(\mu)$ in terms of μ, viz.

$$P_n(\mu) = \frac{1\,.\,3\,.\,5\ldots(2n-1)}{1\,.\,2\,.\,3\ldots n}\left\{\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\,.\,4\,.\,(2n-1)(2n-3)}\mu^{n-4} - \ldots\right\} \quad \text{...........(8)},$$

we see that

$$\mu^n = \frac{1\,.\,2\,.\,3\ldots n}{1\,.\,3\,.\,5\ldots(2n-1)} P_n(\mu) + \text{terms in } \mu \text{ of lower order than } \mu^n;$$

and therefore

$$\int_{-1}^{+1} \mu^n P_n(\mu)\, d\mu = \frac{1.2.3 \dots n}{1.3.5 \dots (2n-1)} \int_{-1}^{+1} [P_n(\mu)]^2\, d\mu$$

$$= \frac{1.2.3 \dots n}{1.3.5 \dots (2n-1)} \cdot \frac{2}{2n+1} \dots\dots\dots\dots (9).$$

Accordingly, by (5) and (7)

$$\psi_n = -2ik(-1)^n S_n \frac{(ikr)^n}{1.3.5 \dots (2n+1)} + \dots \dots\dots\dots (10),$$

which shews that ψ_n vanishes with r, except when $n = 0$.

The complete series for ψ_n, when there is no source at the pole, is more conveniently obtained by the aid of the theory of Bessel's functions. The differential equations (4) § 200, satisfied by these functions, viz.

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{m^2}{z^2}\right) y = 0 \dots\dots\dots\dots (11),$$

may also be written in the form

$$\frac{d^2(yz^{\frac{1}{2}})}{dz^2} + \left(1 - \frac{4m^2-1}{4z^2}\right) yz^{\frac{1}{2}} = 0 \dots\dots\dots\dots (12).$$

It is known (§ 200) that the solution of (11) subject to the condition of finiteness when $z = 0$, is $y = A J_m(z)$, where

$$J_m(z) = \frac{z^m}{2^m \Gamma(m+1)} \left\{1 - \frac{z^2}{2.(2m+2)} + \frac{z^4}{2.4.(2m+2)(2m+4)} - \dots\right\} \dots\dots\dots (13),$$

is the Bessel's function of order m.

When m is integral, $\Gamma(m+1) = 1.2.3 \dots m$; but here we have to do with m fractional and of the form $n + \frac{1}{2}$, n being integral. In this case

$$\Gamma(m+1) = \frac{1.3.5 \dots (2n+1)}{2^{n+1}} \cdot \sqrt{\pi} \dots\dots\dots\dots (14).$$

Referring now to (12), we see that the solution of

$$\frac{d^2\theta}{dz^2} + \left(1 - \frac{4m^2-1}{4z^2}\right) \theta = 0 \dots\dots\dots\dots (15),$$

under the same condition of finiteness when $z = 0$, is

$$\theta = A z^{\frac{1}{2}} J_m(z) \dots\dots\dots\dots (16).$$

Now the function ψ_n, with which we are at present concerned, satisfies (4) § 323, viz.

$$\frac{d^2(r\psi_n)}{d(kr)^2}+\left(1-\frac{n(n+1)}{(kr)^2}\right)r\psi_n=0 \quad \text{...........} (17),$$

which is of the same form as (15), if $m=n+\frac{1}{2}$; so that the solution is

$$\psi_n=A(kr)^{-\frac{1}{2}}J_{n+\frac{1}{2}}(kr)$$

$$=A\frac{(kr)^n\sqrt{2}}{1.3\ldots(2n+1)\sqrt{\pi}}\left\{1-\frac{(kr)^2}{2.(2n+3)}+\frac{(kr)^4}{2.4.(2n+3)(2n+5)}-\ldots\right\} \quad \text{.........} (18).$$

Determining the constant by a comparison with (10), we find

$$\psi_n=-2(-1)^n i^{n+1}kS_n\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}J_{n+\frac{1}{2}}(kr)$$

$$=-2ik(-1)^nS_n\frac{(ikr)^n}{1.3.5\ldots(2n+1)}\left\{1-\frac{k^2r^2}{2(2n+3)}+\frac{k^4r^4}{2.4.(2n+3)(2n+5)}-\frac{k^6r^6}{2.4.6.(2n+3)(2n+5)(2n+7)}+\ldots\right\}$$

$$\text{..............}(19),$$

as the complete expression for ψ_n in rising powers of r.

Comparing the different expressions (5) and (19) for ψ_n, we obtain

$$P_n\left(\frac{d}{d.ikr}\right).\frac{\sin kr}{kr}=i^n\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}J_{n+\frac{1}{2}}(kr) \quad \text{.............}(20).$$

If $F=\alpha+i\beta$, the corresponding expressions for $d\psi_n/dr$, are

$$\frac{d\psi_n}{dr}=-\frac{S_n}{r^2}\{e^{-ikr}F_n(ikr)-(-1)^n e^{+ikr}F_n(-ikr)\}$$

$$=\frac{2i^{n+1}S_n}{r^2}\{\alpha\sin(kr+\tfrac{1}{2}n\pi)-\beta\cos(kr+\tfrac{1}{2}n\pi)\}$$

$$=-2ik^2(-1)^nS_nP_n\left(\frac{d}{d.ikr}\right)\frac{d}{d.kr}.\frac{\sin kr}{kr}$$

$$=\frac{2n(-1)^nk^2S_n(ikr)^{n-1}}{1.3.5\ldots(2n+1)}\left\{1-\frac{n+2}{2n(2n+3)}k^2r^2+\ldots\right\} \quad \text{......}(21).$$

It will be convenient to write down for reference the forms of ψ and $d\psi/dr$ for the first three orders.

$$n=0 \quad \begin{cases} \psi_0 = -2ikS_0 \dfrac{\sin kr}{kr}, \\ \dfrac{d\psi_0}{dr} = \dfrac{2ikS_0}{r}\left\{\dfrac{\sin kr}{kr} - \cos kr\right\}. \end{cases}$$

$$n=1 \quad \begin{cases} \psi_1 = \dfrac{2S_1}{r}\left\{\cos kr - \dfrac{\sin kr}{kr}\right\}, \\ \dfrac{d\psi_1}{dr} = -\dfrac{2S_1}{r^2}\left\{2\cos kr + \left(kr - \dfrac{2}{kr}\right)\sin kr\right\}. \end{cases}$$

$$n=2 \quad \begin{cases} \psi_2 = -\dfrac{2iS_2}{r}\left\{\left(1 - \dfrac{3}{k^2r^2}\right)\sin kr + \dfrac{3}{kr}\cos kr\right\}, \\ \dfrac{d\psi_2}{dr} = \dfrac{2iS_2}{r^2}\left\{\left(4 - \dfrac{9}{k^2r^2}\right)\sin kr - \left(kr - \dfrac{9}{kr}\right)\cos kr\right\}. \end{cases}$$

331. One of the most interesting applications of these results is to the investigation of the motion of a gas within a rigid spherical envelope. To determine the free periods we have only to suppose that $d\psi/dr$ vanishes, when r is equal to the radius of the envelope. Thus in the case of the symmetrical vibrations, we have to determine k,

$$\tan kr = kr \quad\text{.........................(1)},$$

an equation which we have already considered in the chapter on membranes, § 207. The first finite root ($kr = 1{\cdot}4303\,\pi$) corresponds to the symmetrical vibration of lowest pitch. In the case of a higher root, the vibration in question has spherical nodes, whose radii correspond to the inferior roots.

Any cone, whose vertex is at the origin, may be made rigid without affecting the conditions of the question.

The loops, or places of no pressure variation, are given by $(kr)^{-1}\sin kr = 0$, or $kr = m\pi$, where m is any integer, except zero.

The case of $n=1$, when the vibrations may be called diametral, is perhaps the most interesting. S_1, being a harmonic of order 1, is proportional to $\cos\theta$ where θ is the angle between r and some fixed direction of reference. Since $d\psi_1/d\theta$ vanishes only at the poles, there are no conical nodes[1] with vertex at the centre. Any meridianal plane, however, is nodal, and may be supposed rigid. Along any specified radius vector, ψ_1 and $d\psi_1/d\theta$ vanish, and change sign, with $\cos kr - (kr)^{-1}\sin kr$, viz. when $\tan kr = kr$.

To find the spherical nodes, we have

$$\tan kr = \frac{2kr}{2 - k^2r^2} \qquad\text{.........................(2).}$$

The first root is $kr = 0$. Calculating from Trigonometrical Tables by trial and error, I find for the next root, which corresponds to the vibration of most importance within a sphere, $kr = 119{\cdot}26 \times \pi/180$; so that $r : \lambda = {\cdot}3313$.

The air sways from side to side in much the same manner as in a doubly closed pipe. Without analysis we might anticipate that the pitch would be higher for the sphere than for a closed pipe of equal length, because the sphere may be derived from the cylinder with closed ends, by filling up part of the latter with obstructing material, the effect of which must be to sharpen the *spring*, while the mass to be moved remains but little changed. In fact, for a closed pipe of length $2r$,

$$r : \lambda = {\cdot}25.$$

The sphere is thus higher in pitch than the cylinder by about a Fourth.

The vibration now under consideration is the gravest of which the sphere is capable; it is more than an octave graver than the gravest radial vibration. The next vibration of this type is such that $kr = 340{\cdot}35\ \pi/180$, or

$$r : \lambda = {\cdot}9454,$$

and is therefore higher than the first radial.

When kr is great, the roots of (2) may be conveniently calculated by means of a series. If $kr = \sigma\pi - y$, [where σ is an integer,] then

$$\tan y = \frac{2(\sigma\pi - y)}{(\sigma\pi - y)^2 - 2},$$

from which we find

$$kr = \sigma\pi - \frac{2}{\sigma\pi} - \frac{16}{3\sigma^3\pi^3} + \text{.....................} \qquad (3).$$

[1] A node is a surface which might be supposed rigid, viz. one across which there is no motion.

When $n=2$, the general expression for S_n is

$$S_2 = A_0(\cos^2\theta - \tfrac{1}{3}) + (A_1 \cos\omega + B_1 \sin\omega)\sin\theta\cos\theta + (A_2\cos 2\omega + B_2 \sin 2\omega)\sin^2\theta \quad \text{....(4)},$$

from which we may select for special consideration the following notable cases:

(α) the zonal harmonic,

$$S_2 = A_0(\cos^2\theta - \tfrac{1}{3}) \quad \text{.....................(4a)}.$$

Here $d\psi_2/d\theta$ is proportional to $\sin 2\theta$, and therefore vanishes when $\theta = \frac{1}{2}\pi$. This shews that the equatorial plane is a nodal surface, so that the same motion might take place within a closed hemisphere. Also since S_2 does not involve ω, any meridianal plane may be regarded as rigid.

(β) the sectorial harmonic

$$S_2 = A_2 \cos 2\omega \sin^2\theta \quad \text{...................... (5)}.$$

Here again $d\psi_2/d\theta$ varies as $\sin 2\theta$, and the equatorial plane is nodal. But $d\psi_2/d\omega$ varies as $\sin 2\omega$, and therefore does not vanish independently of θ, except when $\sin 2\omega = 0$. It appears accordingly that two, and but two, meridianal planes are nodal, and that these are at right angles to one another.

(γ) the tesseral harmonic,

$$S_2 = A_1 \cos\omega \sin\theta \cos\theta \quad \text{...................(6)}.$$

In this case $d\psi_2/d\theta$ vanishes independently of ω with $\cos 2\theta$, that is, when $\theta = \frac{1}{4}\pi$, or $\frac{3}{4}\pi$, which gives a nodal cone of revolution whose vertical angle is a right angle. $d\psi_2/d\omega$ varies as $\sin\omega$, and thus there is one meridianal nodal plane, and but one[1].

The spherical nodes are given by

$$\tan kr = \frac{k^3r^3 - 9kr}{4k^2r^2 - 9} \quad \text{......................(7)},$$

of which the first finite solution is

$$kr = 3{\cdot}3422,$$

giving a tone graver than any of the radial group.

In the case of the general harmonic, the equation giving the

[1] [I owe to Prof. Lamb the remark that the difference between (β) and (γ) is only in relation to the axes of reference.]

tones possible within a sphere of radius r may be written (21) § 330

$$\tan(kr + \tfrac{1}{2}n\pi) = \beta : \alpha \text{ ....(8)},$$

or

$$P_n\left(\frac{d}{d\,.\,ikr}\right)\frac{d}{d.kr}\cdot\frac{\sin kr}{kr} = 0 \text{ ....(9)},$$

or again,

$$2\,kr\,J'_{n+\frac{1}{2}}(kr) = J_{n+\frac{1}{2}}(kr) \text{..................(10)}.$$

[For the roots of

$$\frac{d}{dz}\{z^{-\frac{1}{2}}J_\nu(z)\} = 0 \text{ ....(11)},$$

equivalent to (10), Prof. McMahon gives[1]

$$z_\nu^{(s)} = \beta' - \frac{m+7}{8\beta'} - \frac{4(7m^2+154m+95)}{3(8\beta')^3} - \frac{32(83m^3+3535m^2+3561m+6133)}{15(8\beta')^5} - \text{...........(12)},$$

where $m = 4\nu^2$, and

$$\beta' = \tfrac{1}{4}(2\nu + 4s + 1) \text{......................(13)}.$$

If $n = 1$, so that $\nu = \frac{3}{2}$,

$$m = 9, \qquad \beta' = s + 1,$$

and (12) gives a result in harmony with (3).]

Table A shews the values of λ for a sphere of radius unity, corresponding to the more important modes of vibration. In B is exhibited the frequency of the various vibrations referred to the gravest of the whole system. The Table is extended far enough to include two octaves.

TABLE A,

Giving the values of λ for a sphere of unit radius.

Order of Harmonic.

Number of internal spherical nodes.		0	1	2	3	4	5	6
	0	1·3983	3·0186	1·8800	1·392	1·113	·9300	·8002
	1	·81334	1·0577	·86195	·7320	·6385		
	2	·57622	·68251	·59208	·5248			
	3	·44670	·50653	·45380				
	4	·36485	·40330					
	5	·30833	·33523					

[1] *Annals of Mathematics*, vol. IX. no. 1.

TABLE B.

Pitch of each tone, referred to gravest.	Order of Harmonic.	Number of internal spherical nodes.	Pitch of each tone, referred to gravest.	Order of Harmonic.	Number of internal spherical nodes.
1·0000	1	0	2 8540	1	1
1·6056	2	0	3·2458	5	0
2·1588	0	0	3·5021	2	1
2·169	3	0	3·7114	0	1
2·712	4	0	3·772	6	0

332. If we drop unnecessary constants, the particular solution for the vibrations of gas within a spherical case of radius unity is represented by

$$\psi_n = S_n (kr)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(kr) \cos(kat - \theta) \text{ ....(1)},$$

where k is a root of

$$2k J'_{n+\frac{1}{2}}(k) = J_{n+\frac{1}{2}}(k) \text{......................(2)}.$$

In generalising this, we must remember that S_n may be composed of several terms, corresponding to each of which there may exist a vibration of arbitrary amplitude and phase. Further, each term in S_n may be associated with any, or all, of the values of k, determined by (2). For example, under the head of $n = 2$, we might have

$$\psi_2 = A(\cos^2\theta - \tfrac{1}{3})(k_1 r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(k_1 r) \cos(k_1 at + \theta_1)$$
$$+ B \cos 2\omega \sin^2\theta (k_2 r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(k_2 r) \cos(k_2 at + \theta_2),$$

k_1 and k_2 being different roots of

$$2k J'_{\frac{5}{2}}(k) = J_{\frac{5}{2}}(k).$$

Any two of the constituents of ψ are conjugate, *i.e.* will vanish when multiplied together and integrated over the volume of the sphere. This follows from the property of the spherical harmonics, wherever the two terms considered correspond to different values of n, or to two different constituents of S_n[1]. The only case remaining for consideration requires us to shew that

$$\int_0^1 r^2 dr \,.\, (k_1 r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(k_1 r) \,.\, (k_2 r)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(k_2 r) = 0 \text{ ......(3)},$$

[1] Thomson and Tait's *Nat. Phil.* p. 151.

where k_1 and k_2 are *different* roots of

$$2k\,J'_{n+\frac{1}{2}}(k) = J_{n+\frac{1}{2}}(k) \quad\text{.................... (4)},$$

and this is an immediate consequence of a fundamental property of these functions (§ 203). There is therefore no difficulty in adapting the general solution to prescribed initial circumstances.

In order to illustrate this subject we will take the case where initially the gas is in its position of equilibrium but is moving with constant velocity parallel to x. This condition of things would be approximately realised, if the case, having been previously in uniform motion, were suddenly stopped.

Since there is no initial condensation or rarefaction, all the quantities θ_n vanish. If $d\psi/dx$ be initially unity, we have $\psi = x = r\mu$, which shews that the solution contains only terms of the first order in spherical harmonics. The solution is therefore of the form

$$\psi = A_1 (k_1 r)^{-\frac{1}{2}} J_{\frac{3}{2}}(k_1 r)\,\mu \cos k_1 at + A_2 (k_2 r)^{-\frac{1}{2}} J_{\frac{3}{2}}(k_2 r)\,\mu \cos k_2 at + \text{..................} \quad (5),$$

where k_1, k_2, &c. are roots of

$$2kJ_{\frac{3}{2}}'(k) = J_{\frac{3}{2}}(k) \quad\text{.......................... (6)}.$$

To determine the coefficients, we have initially for values of r from 0 to 1,

$$r = A_1 (k_1 r)^{-\frac{1}{2}} J_{\frac{3}{2}}(k_1 r) + A_2 (k_2 r)^{-\frac{1}{2}} J_{\frac{3}{2}}(k_2 r) + \text{.........} \quad (7).$$

Multiplying by $r^{\frac{3}{2}} J_{\frac{3}{2}}(kr)$ and integrating with respect to r from 0 to 1, we find

$$\int_0^1 r^{\frac{5}{2}} J_{\frac{3}{2}}(kr)\,dr = Ak^{-\frac{1}{2}} \int_0^1 [J_{\frac{3}{2}}(kr)]^2\, r\,dr \quad\text{.............. (8)},$$

the other terms on the right vanishing in virtue of the conjugate property. Now by (16), § 203,

$$2\int_0^1 [J_{\frac{3}{2}}(kr)]^2\, r\,dr = [J_{\frac{3}{2}}'(k)]^2 + \left(1 - \frac{9}{4k^2}\right)[J_{\frac{3}{2}}(k)]^2$$

$$= \left(1 - \frac{2}{k^2}\right)[J_{\frac{3}{2}}(k)]^2 \quad\text{..................... (9)},$$

by (6).

The evaluation of $\int_0^1 r^{\frac{5}{2}} J_{\frac{3}{2}}(kr)\,dr$ may be effected by the aid of a general theorem relating to these functions. By the fundamental differential equation

$$\int_0^r r^{n+1}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dJ_n(kr)}{dr}\right)+\left(k^2-\frac{n^2}{r^2}\right)J_n(kr)\right]dr=0,$$

whence by integration by parts we obtain,

$$k^2\int_0^r r^{n+1}J_n(kr)\,dr=nr^n J_n(kr)-r^{n+1}\frac{dJ_n(kr)}{dr} \quad\text{..........(10)},$$

or, if we make $r=1$,

$$k^2\int_0^1 r^{n+1}J_n(kr)\,dr=nJ_n(k)-kJ_n'(k)\quad\text{..............(11)}.$$

Thus in the case, with which we are here concerned,

$$k^2\int_0^1 r^{\frac{5}{2}}J_{\frac{3}{2}}(kr)\,dr=\tfrac{3}{2}J_{\frac{3}{2}}(k)-kJ_{\frac{3}{2}}'(k)=J_{\frac{3}{2}}(k) \quad\text{by (6)}.$$

Equation (8) therefore takes the form

$$A=\frac{2k^{\frac{1}{2}}}{(k^2-2)J_{\frac{3}{2}}(k)}\quad\text{..........................(12)},$$

and the final solution is

$$\psi=\Sigma\frac{2r^{-\frac{1}{2}}\mu}{k^2-2}\,\frac{J_{\frac{3}{2}}(kr)}{J_{\frac{3}{2}}(k)}\cos kat\quad\text{.....................(13)},$$

where the summation is to be extended to all the admissible values of k.

When $t=0$, and $r=1$, we must have $\psi=\mu$, and accordingly

$$\Sigma\frac{2}{k^2-2}=1\quad\text{..............................(14)}.$$

It will be remembered that the higher values of k are approximately, (3) § 331,

$$k=\sigma\pi-\frac{2}{\sigma\pi}\quad\text{..............................(15)}.$$

The first value of k is 2·0815, and the second 5·9402, whence

$$\frac{2}{k_1^2-2}=\cdot 85742,\qquad \frac{2}{k_2^2-2}=\cdot 06009,$$

shewing that the first term in the series for ψ is by far the most important.

It may be well to recall here that

$$J_{\frac{3}{2}}(z)=\sqrt{\frac{2}{\pi z}}\left(\frac{\sin z}{z}-\cos z\right) \text{ ....(16).}$$

Equation (14) may be verified thus: the quantities k are the roots of

$$\frac{d}{dz}\left\{z^{-\frac{1}{2}}J_{\frac{3}{2}}(z)\right\}=0,$$

or, if $\phi=z^{-\frac{1}{2}}J_{\frac{3}{2}}(z)$, the roots of $\phi'=0$, where ϕ satisfies

$$\phi''+\frac{2}{z}\phi'+\left(1-\frac{2}{z^2}\right)\phi=0 \text{ ....(17).}$$

Now, since the leading term in the expansion of ϕ' in ascending powers of z is independent of z, we may write

$$\phi'=\text{const.}\left\{1-\frac{z^2}{k_1^2}\right\}\left\{1-\frac{z^2}{k_2^2}\right\}\text{ ......}$$

whence, by taking the logarithms and differentiating,

$$-\frac{\phi''}{\phi'}=\frac{2z}{k_1^2-z^2}+\frac{2z}{k_2^2-z^2}+\dots$$

If we now put $z^2=2$, we get by (17),

$$\Sigma\frac{2}{k^2-2}=-\frac{\phi''}{z\phi'},\ (z^2=2)=1.$$

333. In a similar manner we may treat the problem of the vibrations of air included between rigid concentric spherical surfaces, whose radii are r_1 and r_2. For by (13) § 323, if $d\psi_n/dr$ vanish for these values of r,

$$e^{2ikr_1}\frac{F_n(-ikr_1)}{F_n(+ikr_1)}=e^{2ikr_2}\frac{F_n(-ikr_2)}{F_n(+ikr_2)},$$

whence

$$\tan k(r_1-r_2)=\frac{(\beta/\alpha)_1-(\beta/\alpha)_2}{1+(\beta/\alpha)_1(\beta/\alpha)_2}\text{ ....(1),}$$

where as before

$$F_n(+ikr)=\alpha+i\beta \text{ ....(2).}$$

When the difference between r_1 and r_2 is very small compared with either, the problem identifies itself with that of the vibration of a spherical sheet of air, and is best solved independently. In (1)

§ 323, if ψ be independent of r, as it is evident that it must approximately be in the case supposed, we have

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi}{d\theta}\right)+\frac{1}{\sin^2\theta}\frac{d^2\psi}{d\omega^2}+k^2r^2\psi=0 \quad\text{......}\ (3),$$

whose solution is simply

$$\psi_n=S_n\text{..................................}(4),$$

while the admissible values of k^2 are given by

$$k^2r^2=n(n+1)\text{........................}(5).$$

The interval between the gravest tone ($n=1$) and the next is such that two of them would make a twelfth (octave + fifth). The problem of the spherical sheet of gas will be further considered in the following chapter. [For a derivation of (5) from the fundamental determinant, equivalent to (1), the reader may be referred to a short paper[1] by Mr Chree.]

334. The next application that we shall make of the spherical harmonic analysis is to investigate the disturbance which ensues when plane waves of sound impinge on an obstructing sphere. Taking the centre of the sphere as origin of polar co-ordinates, and the direction from which the waves come as the axis of μ, let ϕ be the potential of the unobstructed plane waves. Then, leaving out an unnecessary complex coefficient, we have

$$\phi=e^{ik(at+x)}=e^{ikat}\,.\,e^{ikr\mu}\text{.....................}(1),$$

and the solution of the problem requires the expansion of $e^{ikr\mu}$ in spherical harmonics. On account of the symmetry the harmonics reduce themselves to Legendre's functions $P_n(\mu)$, so that we may take

$$e^{ikr\mu}=A_0+A_1P_1+\ldots+A_nP_n+\text{...........}(2),$$

where $A_0\ldots$ are functions of r, but not of μ. From what has been already proved we may anticipate that A_n, considered as a function of r, must vary as

$$P_n\left(\frac{d}{d\,.\,ikr}\right)\frac{\sin kr}{kr},\quad\text{or as}\quad r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(kr),$$

but the same result may easily be obtained directly. Multiplying

[1] *Messenger of Mathematics*, vol. XV. p. 20, 1886.

(2) by $P_n(\mu)$, and integrating with respect to μ from $\mu=-1$ to $\mu=+1$, we find

$$\int_{-1}^{+1} P_n(\mu)\, e^{ikr\mu}\, d\mu = A_n \int_{-1}^{+1} (P_n)^2\, d\mu = \frac{2A_n}{2n+1} \quad \text{.........(3);}$$

and, as in § 330,

$$\int_{-1}^{+1} P_n(\mu)\, e^{ikr\mu}\, d\mu = 2P_n\left(\frac{d}{d\,.\,ikr}\right).\frac{\sin kr}{kr},$$

so that finally

$$\frac{A_n}{2n+1} = P_n\left(\frac{d}{d\,.\,ikr}\right).\frac{\sin kr}{kr} = i^n\sqrt{\frac{\pi}{2kr}}\,.\,J_{n+\frac{1}{2}}(kr) \quad \text{......(4).}$$

In the problem in hand the whole motion outside the sphere may be divided into two parts; the first, that represented by ϕ and corresponding to undisturbed plane waves, and the second a disturbance due to the presence of the sphere, and radiating outwards from it. If the potential of the latter part be ψ, we have (2) § 324 on replacing the general harmonic S_n by $a_n P_n(\mu)$,

$$\left.\begin{aligned} r\psi_n &= a_n P_n(\mu)\,.\,e^{-ikr} f_n(ikr) \\ r^2\frac{d\psi_n}{dr} &= -\,a_n P_n(\mu)\,.\,e^{-ikr} F_n(ikr) \end{aligned}\right\} \quad \text{............(5).}$$

The velocity-potential of the whole motion is found by addition of ϕ and ψ, the constants a_n being determined by the boundary conditions, whose form depends upon the character of the obstruction presented by the sphere. The simplest case is that of a rigid and fixed sphere, and then the condition to be satisfied when $r=c$ is that

$$\frac{d\phi}{dr} + \frac{d\psi}{dr} = 0 \quad \text{..........................(6),}$$

a relation which must of course hold good for each harmonic element separately. For the element of order n, we get

$$a_n = (2n+1)\frac{kc^2 e^{ikc}}{F_n(ikc)} P_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}.\frac{\sin kc}{kc} \quad \text{........(7).}$$

Corresponding to the plane waves $\phi = e^{ik(at+x)}$, the disturbance due to the presence of the sphere is expressed by

$$\psi = \frac{kc^2}{r} e^{ik(at-r+c)}$$
$$\times \sum_{n=0}^{n=\infty} \frac{2n+1}{F_n(ikc)} P_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}.\frac{\sin kc}{kc}\,.\,P_n(\mu)\,.\,f_n(ikr) \text{...(8).}$$

At a sufficient distance from the source of disturbance we may take $f_n(ikr)=1$. In order to pass to the solution of a real problem, we may separate the real and imaginary parts, and throw away the latter. On this supposition the plane waves are represented by

$$[\phi] = \cos k\,(at+x) \quad\quad\quad (9).$$

Confining ourselves for simplicity's sake to parts of space at a great distance from the sphere, where $f_n(ikr)=1$, we proceed to extract the real part of (8). Since the functions P are wholly even or wholly odd,

$$P_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\,.\,\frac{\sin kc}{kc}$$

is wholly real or wholly imaginary, so that this factor presents no difficulty. $\{F_n(ikc)\}^{-1}$, however, is complex, and since $F_n(ikc)=\alpha+i\beta$,

$$\{F_n(ikc)\}^{-1} = \frac{\alpha - i\beta}{\alpha^2+\beta^2} = \frac{e^{i\gamma}}{\sqrt{(\alpha^2+\beta^2)}},$$

where $\tan\gamma = -\beta/\alpha$. [If the positive value of $\sqrt{(\alpha^2+\beta^2)}$ be taken in all cases, γ must be so chosen that $\cos\gamma$ has the same sign as α.]

Thus

$$\psi = \Sigma\,(2n+1)\,\frac{kc^2}{r}\,e^{i[k(at-r+c)+\gamma]}$$

$$\times\,\{\alpha^2+\beta^2\}^{-\frac{1}{2}}\,P_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\,.\,\frac{\sin kc}{kc}\,.\,P_n(\mu) \quad\quad (10).$$

When therefore n is even,

$$[\psi] = (2n+1)\,\frac{kc^2}{r}\cos\{k\,(at-r+c)+\gamma\}$$

$$\times\,\{\alpha^2+\beta^2\}^{-\frac{1}{2}}\,P_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\,.\,\frac{\sin kc}{kc}\,.\,P_n(\mu) \quad\quad (11),$$

while, if n be odd,

$$[\psi] = (2n+1)\,\frac{kc^2}{r}\,i\sin\{k\,(at-r+c)+\gamma\}$$

$$\times\,\{\alpha^2+\beta^2\}^{-\frac{1}{2}}\,P_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\,.\,\frac{\sin kc}{kc}\,.\,P_n(\mu) \quad\quad (12).$$

As examples we may write down the terms in $[\psi]$, involving harmonics of orders 0, 1, 2. The following table of the functions $P_n(\mu)$ will be useful.

$P_0 = 1,$ $\qquad P_1 = \mu,$

$P_2 = \frac{3}{2}(\mu^2 - \frac{1}{3}),$ $\qquad P_3 = \frac{5}{2}(\mu^3 - \frac{3}{5}\mu),$

$P_4 = \frac{35}{8}(\mu^4 - \frac{6}{7}\mu^2 + \frac{3}{35}),$ $\qquad P_5 = \frac{63}{8}(\mu^5 - \frac{10}{9}\mu^3 + \frac{5}{21}\mu).$

We have,

$$n = 0, \quad \alpha^2 + \beta^2 = 1 + k^2c^2, \quad \tan\gamma_0 = -kc,$$

$$[\psi_0] = \frac{kc^2}{r}\{1 + k^2c^2\}^{-\frac{1}{2}} \frac{d}{d\,.\,kc} \frac{\sin kc}{kc} \,.\, \cos\{k(at - r + c) + \gamma_0\} \ldots (13);$$

$$n = 1, \quad \alpha^2 + \beta^2 = k^2c^2 + \frac{4}{k^2c^2}, \quad \tan\gamma_1 = -\frac{k^2c^2 - 2}{2kc},$$

$$[\psi_1] = \frac{3kc^2}{r}\left\{k^2c^2 + \frac{4}{k^2c^2}\right\}^{-\frac{1}{2}} \frac{d^2}{d(kc)^2} \cdot \frac{\sin kc}{kc} \,.\, \mu \,.\, \sin\{k(at - r + c) + \gamma_1\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14);$$

$$n = 2, \quad \alpha^2 + \beta^2 = k^2c^2 - 2 + \frac{9}{k^2c^2} + \frac{81}{k^4c^4}, \quad \tan\gamma_2 = -\frac{kc(k^2c^2 - 9)}{4k^2c^2 - 9},$$

$$[\psi_2] = -\frac{45\,kc^2}{4r}\left\{k^2c^2 - 2 + \frac{9}{k^2c^2} + \frac{81}{k^4c^4}\right\}^{-\frac{1}{2}}$$
$$\times\left\{\frac{d^3}{d(kc)^3} + \frac{1}{3}\frac{d}{d(kc)}\right\}\frac{\sin kc}{kc} \,.\, (\mu^2 - \tfrac{1}{3})\cos\{k(at - r + c) + \gamma_2\} \ldots (15).$$

The solution of the problem here obtained, though analytically quite general, is hardly of practical use except when kc is a small quantity. In this case we may advantageously expand our results in rising powers of kc.

$$[\psi_0] = -\frac{k^2c^3}{3r}(1 - \tfrac{3}{5}k^2c^2 + \tfrac{3}{7}k^4c^4 - \tfrac{19}{54}k^6c^6 + \ldots)$$
$$\times\cos\{k(at - r + c) + \gamma_0\} \ldots\ldots\ldots\ldots (16).$$

$$[\psi_1] = -\frac{k^2c^3}{2r}(1 - \tfrac{3}{10}k^2c^2 - \tfrac{3}{28}k^4c^4 + \tfrac{1}{27}k^6c^6 + \ldots)$$
$$\times\mu\,.\,\sin\{k(at - r + c) + \gamma_1\} \ldots\ldots\ldots\ldots (17),$$

$$[\psi_2] = -\frac{k^4c^5}{9r}(1 - \tfrac{25}{126}k^2c^2 + \tfrac{13}{567}k^4c^4 + \ldots)$$
$$\times(\mu^2 - \tfrac{1}{3})\cos\{k(at - r + c) + \gamma_2\} \ldots\ldots (18).$$

It appears that while $[\psi_0]$ and $[\psi_1]$ are of the same order in the small quantity kc, $[\psi_2]$ is two orders higher. We shall find presently that the higher harmonic components in $[\psi]$ depend upon still more elevated powers of kc. For a first approximation, then, we may confine ourselves to the elements of order 0 and 1.

Although $[\psi_0]$ contains a cosine, and $[\psi_1]$ a sine, they nevertheless differ in phase by a small quantity only. Comparing two of the values of $d\psi_n/dr$ in (21) § 330 we see that

$$\alpha \sin (kc + \tfrac{1}{2} n\pi) - \beta \cos (kc + \tfrac{1}{2} n\pi)$$
$$= -(-1)^n \frac{n (kc)^{n+1}}{1.3.5 \ldots (2n+1)} + \text{higher powers of } kc$$

identically. Dividing by $\alpha \cos (kc + \tfrac{1}{2} n\pi)$, we get ultimately

$$\tan (kc + \tfrac{1}{2} n\pi) - \frac{\beta}{\alpha} = - \frac{(-1)^n}{\alpha \cos (kc + \tfrac{1}{2} n\pi)} \cdot \frac{n (kc)^{n+1}}{1.3.5 \ldots (2n+1)}.$$

When n is *even*, this equation becomes on substitution for α of its leading term from (16) § 323,

$$\tan kc - \frac{\beta}{\alpha} = - \frac{n}{(n+1)(2n+1)} \frac{(kc)^{2n+1}}{\{1.3.5 \ldots (2n-1)\}^2} \ldots (19).$$

For example, if $n = 2$,

$$\tan kc - \left(\frac{\beta}{\alpha}\right)_2 = - \frac{2 (kc)^5}{3^3 . 5} + \ldots\ldots$$

When n is at all high, the expressions $\tan kc$ and β/α become very nearly identical for moderate values of kc.

When n is *odd*, we get in a nearly similar manner,

$$\cot kc + \frac{\beta}{\alpha} = \frac{n (kc)^{2n-1}}{(n+1)(2n+1)\{1.3.5 \ldots (2n-1)\}^2} + \ldots\ldots (20).$$

[From (19) we see that when n is even $\tan \gamma$, or $-\beta/\alpha$, is approximately equal to $-\tan kc$, and from (20) when n is odd that $\cot \gamma = \tan kc$. In the first case, by (16) § 323, α has the sign of i^{-n} or of $(-1)^{\frac{1}{2}n}$; and in the second case α has the sign of i^{-n+1} or of $(-1)^{\frac{1}{2}(n-1)}$. In both cases the approximate solution may be expressed

$$\gamma = -kc + \tfrac{1}{2} n\pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20').$$[1]]

The velocity-potential of the disturbance due to a small rigid and fixed sphere is therefore approximately,

$$[\psi_0] + [\psi_1] = - \frac{k^2 c^3}{3r} (1 + \tfrac{3}{2} \mu) \cos k (at - r)$$
$$= - \frac{\pi T}{r \lambda^2} (1 + \tfrac{3}{2} \mu) \cos k (at - r) \ldots\ldots (21),$$

[1] This emendation and others consequential to it are due to Dr Burton.

if T denote the volume of the obstacle, the corresponding direct wave being

$$[\phi] = \cos k\,(at + x) \quad\text{.....................(22).}$$

For a given obstacle and a given distance the ratio of the amplitudes of the scattered and the direct waves is in general proportional to the inverse square of the wave-length, and the ratio of intensities is proportional to the inverse fourth power (§ 296).

In order to compare the intensities of the primary and scattered sounds, we may suppose the former to originate in a simple source, provided it be sufficiently distant (R) from T. Thus, if

$$[\phi] = \frac{\cos k\,(at - R)}{R} \quad\text{.....................(23),}$$

$$[\psi] = -\frac{\pi T}{rR\lambda^2}(1 + \tfrac{3}{2}\mu)\cos k\,(at - r) \quad\text{...........(24);}$$

so that at equal distances from their sources the secondary and the primary waves are in the ratio

$$-\frac{\pi T}{R\lambda^2}(1 + \tfrac{3}{2}\mu) \quad\text{.........................(25).}$$

The intensities are therefore in the ratio

$$\frac{\pi^2 T^2}{R^2\lambda^4}(1 + \tfrac{3}{2}\mu)^2 \quad\text{......................(26),}$$

which, in the case of $\mu = +1$, gives approximately

$$\frac{61{\cdot}72\,T^2}{R^2\lambda^4} \quad\text{...........................(27).}$$

It must be well understood that in order that this result may apply, λ must be great compared with the linear dimension of T, and R must be great compared with λ.

To find the leading term in the expression for ψ_n, when kc is small, we have in the first place,

$$(2n+1)\,P_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\cdot\frac{\sin kc}{kc}$$

$$= \frac{n\,i^n\,(kc)^{n-1}}{1\,.\,3\,.\,5\ldots(2n-1)}\left\{1 - \frac{(n+2)\,k^2c^2}{2\,.\,n\,.\,(2n+3)} + \ldots\right\} \quad\text{......(28).}$$

Again,

$$\alpha^2+\beta^2=F_n(ikc)\times F_n(-ikc)$$
$$=\{1.3.5\ldots(2n-1)(n+1)(kc)^{-n}\}^2\left\{1+\frac{(n-1)k^2c^2}{(n+1)(2n-1)}+\ldots\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29);$$

so that

$$\{\alpha^2+\beta^2\}^{-\frac{1}{2}}=\frac{(kc)^n}{1.3\ldots(2n-1)(n+1)}\left\{1-\frac{(n-1)k^2c^2}{2.(n+1)(2n-1)}+\ldots\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30).$$

Hence, from (10),

$$\psi_n=\frac{c(kc)^{2n}\,n\,i^n P_n(\mu)}{r\{1.3.5\ldots(2n-1)\}^2(n+1)}e^{i[k(at-r+c)+\gamma_n]}$$
$$\times\left\{1-k^2c^2\left(\frac{n-1}{(2n+2)(2n-1)}+\frac{n+2}{2n(2n+3)}\right)+\ldots\right\}\ldots(31).$$

When n is even, [since $\gamma=-kc+\frac{1}{2}n\pi$ approximately,]

$$[\psi_n]=\frac{c(kc)^{2n}n\,i^nP_n(\mu)}{r\{1.3\ldots\ldots(2n-1)\}^2(n+1)}\cos\{k(at-r)+\tfrac{1}{2}n\pi\}$$
$$\times\left\{1-k^2c^2\left(\frac{n-1}{(2n+2)(2n-1)}+\frac{n+2}{2n(2n+3)}\right)+\ldots\ldots\right\}\ldots\ldots(32);$$

while if n be odd, we have merely to replace i^n by i^{n+1} [and cos by sin], the result being then still real.

By means of (31) we may verify the first two terms in the expressions for $[\psi_1]$, $[\psi_2]$, in (17), (18). To the case of $n=0$, (31) does not apply.

Again, by (31),

$$[\psi_3]=\frac{k^6c^7}{120\,r}\{1-\tfrac{77}{540}k^2c^2\}\{\mu^3-\tfrac{3}{5}\mu\}\sin\{k(at-r+c)+\gamma_3\}\ldots(33),$$

$$[\psi_4]=\frac{k^8c^9}{3150\,r}\{\mu^4-\tfrac{6}{7}\mu^2+\tfrac{3}{35}\}\cos\{k(at-r+c)+\gamma_4\}\ldots\ldots\ldots\ldots(34).$$

Combining (17), (18), (33), (34), we have the value of $[\psi]$ complete as far as the terms which are of the order k^6c^6 compared with the two leading terms given in (21). In compounding the partial expressions, it is as necessary to be exact with respect to the phases of the components as with respect to their amplitudes; but for purposes requiring only one harmonic element at a time, the phase is often of subordinate importance. In such cases we may take

$$\gamma = \mp kc + \tfrac{1}{2} n\pi.$$

From (31) or (32) it appears that the leading term in ψ_n rises two orders in kc with each step in the order of the harmonic; and that ψ_n is itself expressed by a series containing only even, or only odd, powers of kc. But besides being of higher order in kc, the leading term becomes rapidly smaller as n increases, on account of the other factors which it contains. This is evident, because for all values of n and μ, $P_n(\mu) < 1$; the same is true of $n/(n+1)$; while i^n only affects the phase.

In particular cases any one of the harmonic elements of $[\psi]$ may vanish. From (11), (12), since $(\alpha^2+\beta^2)^{-\frac{1}{2}}$ cannot vanish, we have in such a case

$$P_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\,\frac{\sin kc}{kc} = 0,$$

the same equation as that which gives the periods of the vibrations of order n in a closed sphere of radius c. A little consideration will shew that this result might have been expected. The table of § 331 is applicable to this question and shews, among other things, that when kc is small, no harmonic element in $[\psi]$ can vanish.

In consequence of the aerial pressures the sphere is acted on by a force parallel to the axis of μ, whose tendency is to set the sphere into vibration. The magnitude of this force, if σ be the density of the fluid, is given by

$$2\pi c^2 \sigma \int_{-1}^{+1} (\dot{\phi} + \dot{\psi})\, \mu\, d\mu,$$

in which, by the conjugate property of Legendre's functions, only the term of the first order affects the result of the integration. Now, when $r = c$,

$$\phi_1 = 3\, e^{ikat}\,\frac{d}{d\,.\,ikc}\,\frac{kc}{kc}\,.\,\mu,$$

$$\psi_1 = 3kc\, e^{ikat}\,\frac{f_1(ikc)}{F_1(ikc)}\,\frac{d}{d\,.\,ikc}\,.\,\frac{d}{d\,.\,kc}\,.\,\frac{\sin kc}{kc}\,.\,\mu,$$

where

$$f_1(ikc) = 1 + \frac{1}{ikc}, \qquad F_1(ikc) = ikc + 2 + \frac{2}{ikc}.$$

In order that the force may vanish, it would be necessary that

$$\frac{d}{d\,.\,kc}\cdot\frac{\sin kc}{kc}+kc\frac{f_1(ikc)}{F_1(ikc)}\frac{d^2}{(d\,.\,kc)^2}\frac{\sin kc}{kc}=0,$$

which cannot be satisfied by any real value of kc. We conclude that, if the sphere be free to move, it will always be set into vibration.

If instead of being absolutely plane, the primary waves have their origin in a unit source at a great, though finite, distance R from the centre of the sphere, we have

$$\phi=-\frac{1}{4\pi R}e^{ik(at-R)}\,\Sigma\,(2n+1)\,P_n(\mu)\times P_n\left(\frac{d}{d\,.\,ikr}\right)\frac{\sin kr}{kr}\ldots(35),$$

$$\psi=-\frac{kc^2}{4\pi rR}e^{ik(at-R-r+c)}\,\Sigma\,(2n+1)\frac{P_n(\mu)\,f_n(ikr)}{F_n(ikc)}$$

$$\times P_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\frac{\sin kc}{kc}\;\ldots\ldots\ldots\ldots\ldots\ldots(36).$$

On the sphere itself $r=c$, so that the value of the total potential at any point at the surface is

$$\phi+\psi=-\frac{e^{ik(at-R)}}{4\pi R}\,\Sigma\,(2n+1)\,P_n(\mu)$$

$$\times\left[P_n\left(\frac{d}{d\,.\,ikc}\right)\frac{\sin kc}{kc}+kc\frac{f_n(ikc)}{F_n(ikc)}P_n\left(\frac{d}{d\,.\,ikc}\right)\frac{d}{d\,.\,kc}\frac{\sin kc}{kc}\right].$$

This expression may be simplified. We have

$$P_n\left(\frac{d}{d\,.\,ikc}\right)\frac{\sin kc}{kc}=\frac{1}{2ikc}\{-(-1)^n e^{-ikc}f_n(ikc)+e^{+ikc}f_n(-ikc)\},$$

$$\frac{d}{d\,.\,kc}\cdot P_n\left(\frac{d}{d\,.\,ikc}\right)\frac{\sin kc}{kc}=\frac{1}{2ik^2c^2}\{(-1)^n e^{-ikc}F_n(ikc)-e^{+ikc}F_n(-ikc)\},$$

and thus the quantity within square brackets may be written

$$\frac{e^{ikc}}{2ikc}\,\frac{F_n(ikc)\,f_n(-ikc)-F_n(-ikc)\,f_n(ikc)}{F_n(ikc)},$$

which by (6) § 327 is identical with $e^{ikc}\,[F_n(ikc)]^{-1}$. Thus

$$\phi+\psi=-\frac{e^{ik(at-R+c)}}{4\pi R}\,\Sigma\,(2n+1)\frac{P_n(\mu)}{F_n(ikc)}\;\ldots\ldots\ldots(37),$$

which is the same as if the source had been on the sphere, and the point at which the potential is required at a great distance (§ 328), and is an example of the general Principle of Reciprocity.

By assuming the principle, and making use of the result (3) of § 328, we see that if the source of the primary waves be at a finite distance R, the value of the total potential at any point on the sphere is

$$\phi + \psi = -\frac{1}{4\pi R} e^{ik(at-R+c)} \Sigma (2n+1) P_n(\mu) \frac{f_n(ikR)}{F_n(ikc)} \ldots\ldots (38).$$

If A and B be any two points external to the sphere, a unit source at A will give the same total potential at B, as a unit source at B would give at A. In either case the total potential is made up of two parts, of which the first is the same as if there were no obstacle to the free propagation of the waves, and the second represents the disturbance due to the obstacle. Of these two parts the first is obviously the same, whichever of the two points be regarded as source, and therefore the other parts must also be equal, that is the value of ψ at B when A is a source is equal to the value of ψ at A when B is an equal source. Now when the source A is at a great distance R, the value of ψ at a point B whose angular distance from A is $\cos^{-1}\mu$, and linear distance from the centre is r, is (36)

$$\psi = -\frac{kc^2}{4\pi rR} e^{ik(at-R-r+c)} \Sigma (2n+1) \frac{P_n(\mu) f_n(ikr)}{F_n(ikc)}$$

$$\times P_n\left(\frac{d}{d.ikc}\right) \frac{d}{d.kc} \cdot \frac{\sin kc}{kc},$$

and accordingly this is also the value of ψ at a great distance R, when the source is at B. But since ψ is a disturbance radiating outwards from the sphere, its value at any finite distance R may be inferred from that at an infinite distance by introducing into each harmonic term the factor $f_n(ikR)$. We thus obtain the following symmetrical expression

$$\psi = -\frac{kc^2}{4\pi rR} e^{ik(at-R-r+c)} \Sigma (2n+1) \frac{P_n(\mu)}{F_n(ikc)}$$

$$\times f_n(ikR) \, . \, f_n(ikr) \, P_n\left(\frac{d}{d.ikc}\right) \frac{d}{d.kc} \frac{\sin kc}{kc} \ldots\ldots\ldots (39),$$

which gives this part of the potential at either point, when the other is a unit source.

It should be observed that the general part of the argument does not depend upon the obstacle being either spherical or rigid.

From the expansion of $e^{ikr\mu}$ in spherical harmonics, we may deduce that of the potential of waves issuing from a unit simple source A finitely distant (r) from the origin of co-ordinates. The potential at a point B at an infinite distance R from the origin, and in a direction making an angle $\cos^{-1}\mu$ with r, will be

$$\phi = \frac{e^{-ik(R-\mu r)}}{4\pi R},$$

the time factor being omitted.

Hence by the expansion of $e^{ikr\mu}$

$$\phi = \frac{e^{-ikR}}{4\pi R}\,\Sigma\,(2n+1)\,P_n\left(\frac{d}{d\,.\,ikr}\right)\frac{\sin kr}{kr}\,.\,P_n(\mu);$$

from which we pass to the case of a finite R by the simple introduction of the factor $f_n(ikR)$.

Thus the potential at a finitely distant point B of a unit source at A is

$$\phi = \frac{e^{ik(at-R)}}{4\pi R}\,\Sigma\,(2n+1)\,P_n\left(\frac{d}{d\,.\,ikr}\right)\frac{\sin kr}{kr}\,f_n(ikR)\,.\,P_n(\mu)\ldots(40).$$

335. Having considered at some length the case of a rigid spherical obstacle, we will now sketch briefly the course of the investigation when the obstacle is gaseous. Although in all natural gases the compressibility is nearly the same, we will suppose for the sake of generality that the matter occupying the sphere differs in compressibility, as well as in density, from the medium in which the plane waves advance.

Exterior to the sphere, ϕ is the same exactly, and ψ is of the same form as before. For the motion inside the sphere, if $k' = 2\pi/\lambda'$ be the internal wave-length, (2) § 330,

$$\psi_n = \frac{a_n' P_n}{r}\{e^{-ik'r} f_n(ik'r) - (-1)^n e^{+ik'r} f_n(-ik'r)\},$$

$$\frac{d\psi_n}{dr} = \frac{2a_n' P_n}{r^2}\,.\,i^{n+1}\{\alpha \sin(k'r + \tfrac{1}{2}n\pi) - \beta\cos(k'r + \tfrac{1}{2}n\pi)\},$$

satisfying the condition of continuity through the centre.

If σ, σ' be the natural densities, m, m' the compressibilities,

$$k'^2/k^2 = \sigma'/\sigma\,.\,m/m' \ldots\ldots\ldots\ldots\ldots\ldots(1);$$

and the conditions, to be satisfied by each harmonic element separately, are

$$d\phi/dr + d\psi/dr \text{ (outside)} = d\psi/dr \text{ (inside)} \quad \ldots\ldots\ldots (2),$$

$$\sigma\{\phi + \psi \text{ (outside)}\} = \sigma'\psi \text{ (inside)} \quad \ldots\ldots\ldots (3),$$

expressing respectively the equalities of the normal motions and of the pressures on the two sides of the bounding surface. From these equations the complete solution may be worked out; but we will here confine ourselves to finding the value of the leading terms, when kc, $k'c$ are very small.

In this case, when $r = c$,

$$\left.\begin{aligned} \psi_0 \text{ (inside)} &= -2ik'a_0' \\ d\psi_0/dr \text{ (inside)} &= \tfrac{2}{3}\, ik'^3 ca_0' \end{aligned}\right\} \quad \ldots\ldots\ldots (4),$$

$$\left.\begin{aligned} \phi_0 &= 1 \\ d\phi_0/dr &= -\tfrac{1}{3}k^2 c \end{aligned}\right\} \quad \ldots\ldots\ldots (5),$$

$$\left.\begin{aligned} \psi_0 \text{ (outside)} &= a_0/c \\ d\psi_0/dr \text{ (outside)} &= -a_0/c^2 \end{aligned}\right\} \quad \ldots\ldots\ldots (6).$$

Using these in (2), (3), and eliminating a_0', retaining only the principal term, we find

$$a_0 = -\frac{k^2c^3}{3} \cdot \frac{m' - m}{m'} \quad \ldots\ldots\ldots (7).$$

In like manner for the term of first order,

$$\left.\begin{aligned} \psi_1 \text{ (inside)} &= -\tfrac{2}{3}a_1'k'^2 c\mu \\ d\psi_1/dr \text{ (inside)} &= -\tfrac{2}{3}a_1'k'^2\mu \end{aligned}\right\} \quad \ldots\ldots\ldots (8),$$

$$\left.\begin{aligned} \phi_1 &= ikc\mu \\ d\phi_1/dr &= ik\mu \end{aligned}\right\} \quad \ldots\ldots\ldots (9),$$

$$\left.\begin{aligned} \psi_1 \text{ (outside)} &= a_1/ikc^2 \,.\, \mu \\ d\psi_1/dr \text{ (outside)} &= -2a_1/ikc^3 \,.\, \mu \end{aligned}\right\} \quad \ldots\ldots\ldots (10),$$

which give

$$a_1 = \frac{k^2c^3(\sigma - \sigma')}{\sigma + 2\sigma'} \quad \ldots\ldots\ldots (11).$$

At a distance from the sphere the disturbance due to it is expressed by

$$\psi = \frac{1}{r}\, e^{ik(at-r)}\{a_0 + a_1\mu\}$$

$$= -\frac{k^2c^3}{3r}\, e^{ik(at-r)}\left\{\frac{m' - m}{m'} + 3\frac{\sigma' - \sigma}{\sigma + 2\sigma'}\,\mu\right\} \quad \ldots\ldots\ldots (12).$$

If we introduce the relations

$$T = \tfrac{4}{3}\pi c^3, \quad k = 2\pi/\lambda,$$

and throw away the imaginary part, we obtain

$$\psi = -\frac{\pi T}{\lambda^2 r}\left\{\frac{m' - m}{m'} + 3\,\frac{\sigma' - \sigma}{\sigma + 2\sigma'}\mu\right\}\cos k\,(at - r) \ldots\ldots (13),$$

as the expression for the most important part of the disturbance, corresponding to (21) § 334 for a fixed rigid sphere. It appears, as might have been expected, that the term of zero order is due to the variation of compressibility, and that of order one to the variation of density.

From (13) we may fall back on the case of a rigid fixed sphere, by making both σ' and m' infinite. It is not sufficient to make σ' by itself infinite, apparently because, if m' at the same time remained finite, $k'c$ would not be small, as the investigation has assumed.

When $m' - m$, $\sigma' - \sigma$ are small, (13) becomes equivalent to

$$\psi = -\frac{\pi T}{\lambda^2 r}\left\{\frac{m' - m}{m} + \frac{\sigma' - \sigma}{\sigma}\mu\right\}\cos k\,(at - r),$$

corresponding to $\phi = \cos kat$ at the centre of the sphere. This agrees with the result (13) of § 296, in which the obstacle may be of any form.

In actual gases $m' = m$, and the term of zero order disappears. If the gas occupying the spherical space be incomparably lighter than the other gas, $\sigma' = 0$, and

$$\psi = 3\,\frac{\pi T}{\lambda^2 r}\,\mu \cos k\,(at - r) \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

so that in the term of order one, the effect is twice that of a rigid body, and has the reverse sign.

The greater part of this chapter is taken from two papers by the author "On the vibrations of a gas contained within a rigid spherical envelope," and an "Investigation of the disturbance produced by a spherical obstacle on the waves of sound[1]," and from the paper by Professor Stokes already referred to.

[1] *Math. Society's Proceedings*, March 14, 1872; Nov. 14, 1872.

335 *a.* An interesting function, which has been considered by Prof. Lamb,[1] relates to the maximum disturbance that can be produced by an infinitesimal resonator exposed to plane waves.

The value of ϕ for the expansion of the primary waves is given by (1), (2), (4), § 334. By (5) § 334 and (3) § 329 the value of ψ for the secondary waves may be taken to be

$$\psi = ika_n(-1)^n P_n(\mu) \,.\, P_n\left(\frac{d}{d\,.\,ikr}\right)\left(\frac{e^{-ikr}}{ikr}\right),$$

in which

$$P_n\left(\frac{d}{d\,.\,ikr}\right)\left(\frac{e^{-ikr}}{ikr}\right) = \frac{1}{i} P_n\left(\frac{d}{d\,.\,ikr}\right)\left(\frac{\cos kr}{kr} - i\frac{\sin kr}{kr}\right).$$

If we omit the common factor $P_n(\mu)$, we have

$$\phi + \psi = \{2n+1-(-1)^n ika_n\} Pn\left(\frac{d}{d\,.\,ikr}\right)\frac{\sin kr}{kr}$$

$$+(-1)^n ka_n P_n\left(\frac{d}{d\,.\,ikr}\right)\frac{\cos kr}{kr} \quad \text{..............(1).}$$

Now the only condition imposed upon the appliances introduced at r is that they shall do no work. This requires that $\phi+\psi$ be in the same phase as $d\phi/dr+d\psi/dr$, viz. that the ratio of (1) and of its derivative with respect to r shall be *real*. Since P_n is a wholly odd or wholly even function, this requires that

$$\frac{2n+1-(-1)^n ika_n}{(-1)^n ka_n} \text{ be real.}$$

If a_n, which may be complex, be written $Ae^{i\alpha}$, we get

$$kA = -(-1)^n(2n+1)\sin\alpha \quad \text{..................(2).}$$

Thus A is a maximum when

$$\sin\alpha = -(-1)^n \text{........................(3),}$$

and the maximum value is

$$A = \frac{2n+1}{k} \quad \text{..........................(4).}$$

[1] *London Math. Soc. Proc.* Vol. XXXII. p. 11, 1900.

By (3) and (4), $$a_n = -(-1)^n i \frac{2n+1}{k} \quad\quad (5),$$

so that in (1), $$2n+1-(-1)^n ika_n = 0 \quad\quad (6),$$
but $\phi+\psi$ does not itself vanish.

If the incident plane waves are regarded as due to a source at a great distance R, we have, in order to secure the value unity at the resonator as supposed,

$$\phi = \frac{Re^{-ikR}}{R} \quad\quad (7),$$

with which we may compare

$$\psi = \frac{a_n e^{-ikr}}{r} P_n(\mu) \quad\quad (8).$$

The work emitted by the *primary* source being represented by

$$R^2 \int_{-1}^{+1} d\mu,$$

that emitted, or rather diverted, by the resonator will be

$$\text{Mod}^2 a_n \int_{-1}^{+1} P_n^2(\mu)\, d\mu.$$

Now $$\int_{-1}^{+1} P_n^2(\mu)\, d\mu = \frac{2}{2n+1}$$

and $$\int_{-1}^{+1} d\mu = 2.$$

Also $$\text{Mod}^2 a_n = \frac{(2n+1)^2}{k^2} \quad\quad (9);$$

so that the ratio of works is

$$\frac{2n+1}{k^2R^2} \quad\quad (10).$$

This agrees with the result of § 319 for a symmetrical resonator

$(n=0)$.

Prof. Lamb expresses his conclusion in terms of the energy transmitted in the primary waves across unit of area. This being taken as unity, the work emitted by the resonator is given by multiplying (10) by the area of the sphere of radius R, viz. $4\pi R^2$. We get accordingly

$$\frac{(2n+1)\lambda^2}{\pi} \qquad\qquad (11),$$

$2\pi/\lambda$ being substituted for k. This formula, given by Prof. Lamb, expresses the whole energy emitted by the resonator in terms of the energy of the primary waves per unit of area.

It is worthy of remark that we have nowhere assumed that r at the surface of the resonator is small. The results therefore apply to resonators of finite size, provided that the symmetrical constitution implied in the harmonic analysis is maintained. And the maximum energy emitted is the same whatever be the size of the resonator.

The case of $n=1$ is in some respects the simplest, inasmuch as the resonator may then consist of a rigid sphere held to a fixed point by elastic attachments. As a particular case of (1) we have

$$i(\phi_1+\psi_1)=\mu(3+ika_1)\frac{d}{dkr}\frac{\sin kr}{kr}-ka_1\mu\frac{d}{dkr}\frac{\cos kr}{kr} \quad (12).$$

This may be considered to represent the force acting upon the sphere due to the pressures. Its derivative with respect to r will represent in like manner the acceleration of the sphere, and by suitable choice of mass and spring all the conditions may be satisfied, provided that the ratio of these quantities is real. The maximum a_1 is, as in (5),

$$a_1=3i/k \qquad\qquad (13);$$

and, as in (11), the energy emitted by the resonator is, on the scale there adopted, $3\lambda^2/\pi$.

It may occasion surprise that the energy emissible in the present case is 3 times that emissible from a symmetrical resonator; but the 3 may be got rid of by another presentation of the matter. We have supposed hitherto that the sphere is capable of vibration in the line of symmetry defined by the direction of propagation of the primary waves. If the sphere, considered as infinitesimal, be capable of vibration along one line only, its efficiency as a resonator is proportional to the cosine squared of the angle between this direction and that of the primary waves. This limitation to a single direction of vibration is really the standard case, while that previously considered involves three degrees of freedom. If now we inquire what is the *average* efficiency of the resonator for primary waves reaching it in one direction, we see that the above specially favoured efficiencies must be reduced, in the ratio

$$\int_0^1 \mu^2 d\mu : \int_0^1 d\mu ;$$

that is, in the ratio of 3 : 1. The *average* efficiency of the resonator when $n=1$ is then the same as when $n=0$, and a like result applies whatever n may be. The increased efficiency represented by the factors $2n+1$ must be regarded as due to the cooperation of $2n+1$ degrees of freedom.

CHAPTER XVIII.

SPHERICAL SHEETS OF AIR. MOTION IN TWO DIMENSIONS.

336. IN a former chapter (§ 135), we saw that a proof of Fourier's theorem might be obtained by considering the mechanics of a vibrating string. A similar treatment of the problem of a spherical sheet of air will lead us to a proof of Laplace's expansion for a function which is arbitrary at every point of a spherical surface.

As in § 333, if ψ is the velocity-potential, the equation of continuity, referred to the ordinary polar co-ordinates θ, ω, takes the form,

$$c^2 \frac{d^2\psi}{dt^2} = a^2 \left\{ \frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\psi}{d\theta} \right) + \frac{1}{\sin^2\theta} \frac{d^2\psi}{d\omega^2} \right\}.$$

Whatever may be the character of the free motion, it can be analysed into a series of simple harmonic vibrations, the nature of which is determined by the corresponding functions ψ, considered as dependent on space. Thus, if $\psi \propto e^{ikat}$, the equation to determine ψ as a function of θ and ω is

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\psi}{d\theta} \right) + \frac{1}{\sin^2\theta} \frac{d^2\psi}{d\omega^2} + k^2c^2\psi = 0 \quad \text{.........(1).}$$

Again, whatever function ψ may be, it can be expanded by Fourier's theorem[1] in a series of sines and cosines of the multiples of ω. Thus

$$\psi = \psi_0 + \psi_1 \cos\omega + \psi_1' \sin\omega + \psi_2 \cos 2\omega + \psi_2' \sin 2\omega$$
$$+ \ldots\ldots + \psi_s \cos s\omega + \psi_s' \sin s\omega + \ldots\ldots\ldots \text{(2),}$$

[1] We here introduce the condition that ψ recurs after one revolution round the sphere.

where the coefficients $\psi_0, \psi_1 \ldots \psi_1', \psi_2' \ldots$ are functions of θ only; and by the conjugate property of the circular functions, each term of the series must satisfy the equation independently. Accordingly,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi_s}{d\theta}\right) - \frac{s^2\psi_s}{\sin^2\theta} + k^2c^2\psi_s = 0 \quad \ldots\ldots\ldots (3)$$

is the equation from which the character of ψ_s or ψ_s' is to be determined. This equation may be written in various ways.

In terms of $\mu\ (=\cos\theta)$,

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\psi_s}{d\mu}\right\} + h^2\psi_s - \frac{s^2}{1-\mu^2}\psi_s = 0 \quad \ldots\ldots\ldots (4);$$

or, if $\nu = \sin\theta$,

$$\nu^2(1-\nu^2)\frac{d^2\psi_s}{d\nu^2} + \nu(1-2\nu^2)\frac{d\psi_s}{d\nu} + \nu^2h^2\psi_s - s^2\psi_s = 0 \ldots (5),$$

where h^2 is written for k^2c^2.

When the original function ψ is symmetrical with respect to the pole, that is, depends upon latitude only, s vanishes, and the equations simplify. This case we may conveniently take first. In terms of μ,

$$(1-\mu^2)\frac{d^2\psi_0}{d\mu^2} - 2\mu\frac{d\psi_0}{d\mu} + h^2\psi_0 = 0 \ldots\ldots\ldots\ldots (6).$$

The solution of this equation involves two arbitrary constants, multiplying two definite functions of μ, and may be obtained in the ordinary way by assuming an ascending series and determining the exponents and coefficients by substitution. Thus

$$\psi_0 = A\left\{1 - \frac{h^2}{1.2}\mu^2 + \frac{h^2(h^2-2.3)}{1.2.3.4}\mu^4 - \frac{h^2(h^2-2.3)(h^2-4.5)}{1.2.3.4.5.6}\mu^6 + \&c.\right\}$$

$$+ B\left\{\mu - \frac{h^2-1.2}{1.2.3}\mu^3 + \frac{(h^2-1.2)(h^2-3.4)}{1.2.3.4.5}\mu^5 - \&c.\right\} \quad \ldots\ldots (7),$$

in which A and B are arbitrary constants.

Let us now further suppose that ψ besides being symmetrical round the pole is also symmetrical with respect to the equator (which is accordingly *nodal*), or in other words that ψ is an *even* function of the sine of the latitude (μ). Under these circumstances it is clear that B must vanish, and the value of ψ be expressed simply by the first series, multiplied by the arbitrary constant A. This value of the velocity-potential is the logical consequence of the original differential equation and of the two restrictions as to symmetry. The value of h^2 might appear to be arbitrary, but from what we know of the mechanics of the problem, it is certain beforehand that h^2 is really limited to a series of particular values. The condition, which yet remains to be introduced and by which h is determined, is that the original equation is satisfied at the pole itself, or in other words that the pole is not a source; and this requires us to consider the value of the series when $\mu=1$. Since the series is an even function of μ, if the pole $\mu=+1$ be not a source, neither will be the pole $\mu=-1$. It is evident at once that if h^2 be of the form $n(n+1)$, where n is an even integer, the series terminates, and therefore remains finite when $\mu=1$; but what we now want to prove is that, if the series remain finite for $\mu=1$, h^2 is necessarily of the above-mentioned form. By the ordinary rule it appears at once that, whatever be the value of h^2, the ratio of successive terms tends to the limit μ^2, and therefore the series is convergent for all values of μ less than unity. But for the extreme value $\mu=1$, a higher method of discrimination is necessary.

It is known[1] that the infinite hypergeometrical series

$$1+\frac{ab}{cd}+\frac{a(a+1)b(b+1)}{c(c+1)d(d+1)}+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)\,d(d+1)(d+2)}+\ldots(8)$$

is convergent, if $c+d-a-b$ be greater than 1, and divergent if $c+d-a-b$ be equal to, or less than 1. In the latter case the value of $c+d-a-b$ affords a criterion of the degree of divergency. Of two divergent series of the above form, for which the values of $c+d-a-b$ are different, that one is *relatively* infinite for which the value of $c+d-a-b$ is the smaller.

Our present series (7) may be reduced to the standard form by taking $h^2=n(n+1)$, where n is not assumed to be integral. Thus

[1] Boole's *Finite Differences*, p. 79.

$$1-\frac{h^2}{1.2}\mu^2+\frac{h^2(h^2-2.3)}{1.2.3.4}\mu^4-\dots$$

$$=1-\frac{n(n+1)}{1.2}\mu^2+\frac{n(n+1)(n-2)(n+3)}{1.2.3.4}\mu^4-\dots$$

$$=1+\frac{(-\frac{1}{2}n)(\frac{1}{2}n+\frac{1}{2})}{1.\frac{1}{2}}\mu^2+\frac{(-\frac{1}{2}n)(-\frac{1}{2}n+1)(\frac{1}{2}n+\frac{1}{2})(\frac{1}{2}n+\frac{1}{2}+1)}{1.2.\frac{1}{2}.\frac{3}{2}}\mu^4$$
$$+\dots\dots\dots\dots(9),$$

which is of the standard form, if

$$a=-\tfrac{1}{2}n, \qquad b=\tfrac{1}{2}n+\tfrac{1}{2}, \qquad c=\tfrac{1}{2}, \qquad d=1.$$

Accordingly, since $c+d-a-b=1$, the series is divergent for $\mu=1$, *unless it terminate;* and it terminates only when n is an even integer. We are thus led to the conclusion that when the pole is not a source, and ψ_0 is an even function of μ, h^2 must be of the form $n(n+1)$, where n is an even integer.

In like manner, we may prove that when ψ_0 is an odd function of μ, and the poles are not sources, $A=0$, and h^2 must be of the form $n(n+1)$, n being an *odd* integer.

If n be fractional, both series are divergent for $\mu=\pm 1$, and although a combination of them may be found which remains finite at one or other pole, there can be no combination which remains finite at *both* poles. If therefore it be a condition that no point on the surface of the sphere is a source, we have no alternative but to make n integral, and even then we do not secure finiteness at the poles unless we further suppose $A=0$, when n is odd, and $B=0$, when n is even. We conclude that for a complete spherical layer, the only admissible values of ψ, which are functions of latitude only, and proportional to harmonic functions of the time, are included under

$$\psi=C\,P_n(\mu),$$

where $P_n(\mu)$ is Legendre's function, and n is any odd or even integer. The possibility of expanding an arbitrary function of latitude in a series of Legendre's functions is a necessary consequence of what has now been proved. Any possible motion of the layer of gas is represented by the series

$$\psi=A_0+P_1(\mu)\left(A_1\cos\frac{\sqrt{(1.2)}\,.\,at}{c}+B_1\sin\frac{\sqrt{(1.2)}\,.\,at}{c}\right)+\dots$$
$$+P_n(\mu)\left(A_n\cos\frac{\sqrt{\{n(n+1)\}}\,.\,at}{c}+B_n\sin\frac{\sqrt{\{n(n+1)\}}\,.\,at}{c}\right)+\dots(10).$$

When $t = 0$,

$$\psi = A_0 + A_1 P_1(\mu) + \ldots + A_n P_n(\mu) + \ldots\ldots\ldots(11),$$

and the value of ψ when $t = 0$ is an *arbitrary* function of latitude.

The method that we have here followed has also the advantage of proving the conjugate property,

$$\int_{-1}^{+1} P_n(\mu)\, P_m(\mu)\, d\mu = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

where n and m are different integers. For the functions $P(\mu)$ are the *normal* functions (§ 94) for the vibrating system under consideration, and accordingly the expression for the kinetic energy can only involve the *squares* of the generalized velocities. If (12) do not hold good, the *products* also of the velocities must enter.

The value of ψ appropriate to a *plane* layer of vibrating gas can of course be deduced as a particular case of the general solution applicable to a spherical layer. Confining ourselves to the case where there is no source at the pole ($\mu = 1$), we have to investigate the limiting form of $\psi = CP_n(\mu)$, where $n(n+1) = k^2c^2$, when c^2 and n^2 are infinite. At the same time $\mu - 1$ and ν are infinitesimal, and $c\nu$ passes into the plane polar radius (r), so that $n\nu = kr$. For this purpose the most convenient form of $P_n(\mu)$ is that of Murphy[1]:

$$P_n(\cos\theta) = 1 - \frac{n(n+1)}{1^2}\sin^2\frac{\theta}{2} + \frac{(n-1)n(n+1)(n+2)}{1^2 \cdot 2^2}\sin^4\frac{\theta}{2} - \ldots\ldots\ldots\ldots\ldots(13).$$

The limit is evidently

$$\psi = C\left\{1 - \frac{k^2r^2}{2^2} + \frac{k^4r^4}{2^2 \cdot 4^2} - \frac{k^6r^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots\right\} = CJ_0(kr)\ldots\ldots(14),$$

shewing that the Bessel's function of zero order is an extreme case of Legendre's functions.

When the spherical layer is not complete, the problem requires a different treatment. Thus, if the gas be bounded by walls stretching along two parallels of latitude, the complete integral involving two arbitrary constants will in general be necessary.

[1] Thomson and Tait's *Nat. Phil.* § 782. [$t = \sin^2 \frac{1}{2}\theta$, not $4 \sin^2 \frac{1}{2}\theta$.] Todhunter's *Laplace's Functions*, § 19.

The ratio of the constants and the admissible values of h^2 are to be determined by the two boundary conditions expressing that at the parallels in question the motion is wholly in longitude. The value of μ being throughout numerically less than unity, the series are always convergent.

If the portion of the surface occupied by gas be that included between two parallels of latitude at equal distances from the equator, the question becomes simpler, since then one or other of the constants A and B in (7) vanishes in the case of each normal function.

337. When the spherical area contemplated includes a pole, we have, as in the case of the complete sphere, to introduce the condition that the pole is not a source. For this purpose the solution in terms of ν, i.e. $\sin\theta$, will be more convenient.

If we restrict ourselves for the present to the case of symmetry, we have, putting $s = 0$ in (5) § 336,

$$\nu(1-\nu^2)\frac{d^2\psi_0}{d\nu^2} + (1-2\nu^2)\frac{d\psi_0}{d\nu} + h^2\nu\psi_0 = 0 \quad \text{.........(1).}$$

One solution of this equation is readily obtained in the ordinary way by assuming an ascending series and substituting in the differential equation to determine the exponents and coefficients. We get[1]

$$\psi_0 = A\left\{1 + \frac{0.1-h^2}{2^2}\nu^2 + \frac{(0.1-h^2)(2.3-h^2)}{2^2.4^2}\nu^4 + \frac{(0.1-h^2)(2.3-h^2)(4.5-h^2)}{2^2.4^2.6^2}\nu^6 + \ldots\right\} \quad \text{......(2).}$$

This value of ψ_0 is the most general solution of (1), subject to the condition of finiteness when $\nu = 0$. The complete solution involving two arbitrary constants provides for a source of arbitrary intensity at the pole, in which case the value of ψ_0 is infinite when $\nu = 0$. Any solution which remains finite when $\nu = 0$ and involves one arbitrary constant, is therefore the most general possible under the restriction that the pole be not a source. Accordingly it is unnecessary for our purpose to complete the solution. The nature of the second function (involving a logarithm of ν) will be illustrated in the particular case of a plane layer to be considered presently.

[1] Heine's *Kugelfunctionen*, § 28.

By writing $n(n+1)$ for h^2 the series within brackets becomes

$$1-\frac{n(n+1)}{2^2}\nu^2+\frac{(n-2)n(n+1)(n+3)}{2^2.4^2}\nu^2-\ldots\ldots \quad (3),$$

or, when reduced to the standard hypergeometrical form,

$$1+\frac{(-\frac{1}{2}n)(\frac{1}{2}n+\frac{1}{2})}{1.1}\nu^2+\frac{(-\frac{1}{2}n)(-\frac{1}{2}n+1)(\frac{1}{2}n+\frac{1}{2})(\frac{1}{2}n+\frac{1}{2}+1)}{1.2.1.2}\nu^4+\ldots,$$

corresponding to

$$a=-\tfrac{1}{2}n,\quad b=\tfrac{1}{2}n+\tfrac{1}{2},\quad c=1,\quad d=1.$$

Since $c+d-a-b=\frac{3}{2}$, the series converges for all values of ν from 0 to 1 inclusive. To values of $\theta\,(=\sin^{-1}\nu)$ greater than $\frac{1}{2}\pi$ the solution is inapplicable.

When n is an integer, the series becomes identical with Legendre's function $P_n(\mu)$. If the integer be even, the series terminates, but otherwise remains infinite. Thus, when $n=1$, the series is identical with the expansion of μ, viz. $\sqrt{(1-\nu^2)}$, in powers of ν.

The expression for ψ in terms of ν may be conveniently applied to the investigation of the free symmetrical vibrations of a spherical layer of air, bounded by a small circle, whose radius is less than the quadrant. The condition to be satisfied is simply $d\psi/d\nu=0$, an equation by which the possible values of h^2, or k^2c^2, are connected with the given boundary value of ν.

Certain particular cases of this problem may be treated by means of Legendre's functions. Suppose, for example, that $n=6$, so that $h^2=k^2c^2=42$. The corresponding solution is $\psi=AP_6(\mu)$. The greatest value of μ for which $d\psi/d\mu=0$ is $\mu=\cdot 8302$, corresponding to $\theta=33^\circ\,53'=\cdot 59137$ radians[1].

If we take $c\theta=r$, so that r is the radius of the small circle measured along the sphere, we get

$$kr=\sqrt{(42)}\times\cdot 59137=3\cdot 8325,$$

which is the equation connecting the value of $k\,(=2\pi/\lambda)$ with the curved radius r, in the case of a small circle, whose angular radius is $33^\circ\,53'$. If the layer were plane (§ 339), the value of kr would be $3\cdot 8317$; so that it makes no perceptible difference in the pitch of the gravest tone whether the radius (r) of given length be

[1] The radian is the unit of circular measure.

straight, or be curved to an arc of 33°. The result of the comparison would, however, be materially different, if we were to take the length of the circumference as the same in the two cases, that is, replace $c\theta = r$ by $c\nu = r$.

In order to deduce the symmetrical solution for a plane layer, it is only necessary to make c infinite, while $c\nu$ remains finite. On account of the infinite value of h^2, the solution assumes the simple form

$$\psi = A\left\{1 - \frac{h^2\nu^2}{2^2} + \frac{h^4\nu^4}{2^2 . 4^2} - \frac{h^6\nu^6}{2^2 . 4^2 . 6^2} + \ldots\right\} \ldots\ldots\ldots(4),$$

or, if we write $c\nu = r$, where r is the polar radius in two dimensions,

$$\psi = A\left\{1 - \frac{k^2r^2}{2^2} + \frac{k^4r^4}{2^2 . 4^2} - \ldots\ldots\ldots\right\} = A\, J_0(kr) \ldots\ldots\ldots(5),$$

as in (14) § 336.

The differential equation for ψ in terms of ν, when c is infinite and $c\nu = r$, becomes

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + k^2\psi = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

An independent investigation and solution for the plane problem will be given presently.

338. When s is different from zero, the differential equation satisfied by the coefficients of $\sin s\omega$, $\cos s\omega$, is

$$\nu^2(1-\nu^2)\frac{d^2\psi_s}{d\nu^2} + \nu(1-2\nu^2)\frac{d\psi_s}{d\nu} + \nu^2h^2\psi_s - s^2\psi_s = 0 \ldots\ldots(1),$$

and the solution, subject to the condition of finiteness when $\nu = 0$[1], is easily found to be

$$\psi_s = A\nu^s\left\{1 + \frac{s(s+1)-h^2}{2(2s+2)}\nu^2 + \frac{s(s+1)-h^2}{2(2s+2)} . \frac{(s+2)(s+3)-h^2}{4(2s+4)}\nu^4 + \ldots\right\};$$

or, if we put $h^2 = n(n+1)$,

$$\psi_s = A\nu^s\left\{1 + \frac{(s-n)(s+n+1)}{2.(2s+2)}\nu^2 + \frac{(s-n)(s-n+2)(s+n+1)(s+n+3)}{2.4.(2s+2)(2s+4)}\nu^4 + \ldots\right\} \ldots\ldots(2).$$

[1] The solution may be completed by the addition of a second function derived from (2) by changing the sign of s, which occurs in (1) only as s^2, but a modification is necessary, when s is a positive integer. The method of procedure will be exemplified presently in the case of the plane layer.

We have here the complete solution of the problem of the vibrations of a spherical layer of gas bounded by a small circle whose radius is less than the quadrant. For each value of s, there are a series of possible values of n, determined by the condition $d\psi_s/d\nu = 0$; with any of these values of n the function on the right-hand side of (2), when multiplied by $\cos s\omega$ or $\sin s\omega$, is a normal function of the system. The aggregate of all the normal functions corresponding to every admissible value of s and n, with an arbitrary coefficient prefixed to each, gives an expression capable of being identified with the initial value of ψ, i.e. with a function given arbitrarily over the area of the small circle.

When the radius of the sphere c is infinitely great, h^2 is infinite. If $c\nu = r$, $h^2\nu^2 = k^2r^2$, and (2) becomes

$$\psi_s = A'r^s\left\{1 - \frac{k^2r^2}{2.(2s+2)} + \frac{k^4r^4}{2.4.(2s+2)(2s+4)} - \ldots\right\} \ldots\ldots(3),$$

a function of r proportional to $J_s(kr)$.

In terms of μ, the differential equation satisfied by the coefficient of $\cos s\omega$, or $\sin s\omega$, is

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\psi_s}{d\mu}\right\} + h^2\psi_s - \frac{s^2}{1-\mu^2}\psi_s = 0 \ \ldots\ldots\ldots\ldots(4).$$

Assuming $\psi_s = (1-\mu^2)^{\frac{1}{2}s}\phi_s$, we find as the equation for ϕ_s

$$(1-\mu^2)\frac{d^2\phi_s}{d\mu^2} - 2(s+1)\mu\frac{d\phi_s}{d\mu} + \{h^2 - s(s+1)\}\phi_s = 0 \ \ldots.(5),$$

which will be more easily dealt with.

To solve it, let

$$\phi_s = \mu^\alpha + a_2\mu^{\alpha+2} + a_4\mu^{\alpha+4} + \ldots + a_{2m}\mu^{\alpha+2m} + \ldots,$$

and substitute in (5). The coefficient of the lowest power of μ is $\alpha(\alpha-1)$; so that $\alpha = 0$, or $\alpha = 1$. The relation between a_{2m+2}, and a_{2m}, found by equating to zero the coefficient of $\mu^{\alpha+2m}$, is

$$a_{2m+2} = a_{2m}\frac{(\alpha+2m+s-n)(\alpha+2m+s+n+1)}{(\alpha+2m+1)(\alpha+2m+2)},$$

where $n(n+1) = h^2$.

The complete value of ϕ_s is accordingly given by

$$\phi_s = A\left\{1 + \frac{(s-n)(s+n+1)}{1.2}\mu^2 + \frac{(s-n)(s-n+2)(s+n+1)(s+n+3)}{1.2.3.4}\mu^4 + \frac{(s-n)(s-n+2)(s-n+4)(s+n+1)(s+n+3)(s+n+5)}{1.2.3.4.5.6}\mu^6 + \dots\right\}$$

$$+ B\left\{\mu + \frac{(s-n+1)(s+n+2)}{2.3}\mu^3 + \frac{(s-n+1)(s-n+3)(s+n+2)(s+n+4)}{2.3.4.5}\mu^5 + \dots\right\} \dots(6),$$

where A and B are arbitrary constants;

and $$\psi_s = (1-\mu^2)^{\frac{1}{2}s}\phi_s \dots\dots\dots\dots(7).$$

We have now to prove that the condition that neither pole is a source requires that $n-s$ be a positive integer, in which case one or other of the series in the expression for ϕ_s terminates. For this purpose it will not be enough to shew that the series (unless terminating) are infinite when $\mu = \pm 1$; it will be necessary to prove that they remain divergent after multiplication by $(1-\mu^2)^{\frac{1}{2}s}$, or as we may put it more conveniently, that they are infinite when $\mu = \pm 1$ *in comparison with* $(1-\mu^2)^{-\frac{1}{2}s}$. It will be sufficient to consider in detail the case of the first series.

We have

$$1 + \frac{(s-n)(s+n+1)}{1.2} + \frac{(s-n)(s-n+2)(s+n+1)(s+n+3)}{1.2.3.4} + \dots\dots$$

$$= 1 + \frac{(\frac{1}{2}s - \frac{1}{2}n)(\frac{1}{2}s + \frac{1}{2}n + \frac{1}{2})}{1.\frac{1}{2}} + \frac{(\frac{1}{2}s - \frac{1}{2}n)(\frac{1}{2}s - \frac{1}{2}n + 1)(\frac{1}{2}s + \frac{1}{2}n + \frac{1}{2})(\frac{1}{2}s + \frac{1}{2}n + \frac{1}{2} + 1)}{1.2.\frac{1}{2}.\frac{3}{2}} + \dots;$$

which is of the standard form (8) § 336

$$1 + \frac{ab}{cd} + \frac{a(a+1)b(b+1)}{c(c+1)d(d+1)} + \dots,$$

if $$a = \tfrac{1}{2}s - \tfrac{1}{2}n, \quad b = \tfrac{1}{2}s + \tfrac{1}{2}n + \tfrac{1}{2}, \quad c = 1, \quad d = \tfrac{1}{2}.$$

The degree of divergency is determined by the value of $a+b-c-d$, which is here equal to $s-1$.

On the other hand, the binomial theorem gives for the expansion of $(1-\mu^2)^{-\frac{1}{2}s}$

$$1+\frac{\frac{1}{2}s}{1}\mu^2+\frac{\frac{1}{2}s\,(\frac{1}{2}s+1)}{1\,.\,2}\mu^4+\ldots\ldots,$$

which is of the standard form, if

$$a=\tfrac{1}{2}s,\quad c=1,\quad b=d,\quad \text{and makes}\quad a+b-c-d=\tfrac{1}{2}s-1.$$

Since $s-1>\frac{1}{2}s-1$, it appears that the series in the expression for ϕ_s are infinities of a higher order than $(1-\mu^2)^{-\frac{1}{2}s}$, and therefore remain infinite after multiplication by $(1-\mu^2)^{\frac{1}{2}s}$. Accordingly ψ_s cannot be finite at both poles unless one or other of the series terminate, which can only happen when $n-s$ is zero, or a positive integer. If the integer be even, we have still to suppose $B=0$; and if the integer be odd, $A=0$, in order to secure finiteness at the poles.

In either case the value of ϕ_s for the complete sphere may be put into the form

$$\phi_s=\frac{d^{n+s}}{d\mu^{n+s}}(1-\mu^2)^n=\frac{d^sP_n(\mu)}{d\mu^s}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (8),$$

where the constant multiplier is omitted. The complete expression for that part of ψ which contains $\cos s\omega$ or $\sin s\omega$ as a factor is therefore

$$\psi=\frac{\cos s\omega}{\sin s\omega}\sum_{n=s}^{n=\infty}A_n\nu^s\frac{d^s}{d\mu^s}P_n(\mu)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

where A_n is constant with respect to μ and ω, but as a function of the time will vary as

$$\cos\left(\frac{\sqrt{(n\,.\,n+1)}\,at}{c}+\epsilon\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

For most purposes, however, it is more convenient to group the terms for which n is the same, rather than those for which s is the same. Thus for any value of n

$$\psi=\sum_{s=0}^{s=n}\nu^s\frac{d^sP_n(\mu)}{d\mu^s}(A_s\cos s\omega+B_s\sin s\omega)\ \ldots\ldots\ldots\ (11),$$

where every coefficient A_s, B_s may be regarded as containing a time factor of the form (10).

Initially ψ is an arbitrary function of μ and ω, and therefore any such function is capable of being represented in the form

$$\psi = \sum_{n=0}^{n=\infty} \sum_{s=0}^{s=n} \nu^s \frac{d^s P_n(\mu)}{d\mu^s} (A_s^n \cos s\omega + B_s^n \sin s\omega) \ldots (12),$$

which is Laplace's expansion in spherical surface harmonics.

From the differential equation (5), or from its general solution (6), it is easy to prove that ϕ_s is of the same form as $d\phi_{s-1}/d\mu$, so that we may write

$$\phi_s = \left(\frac{d}{d\mu}\right)^s \phi_0 \ldots\ldots\ldots\ldots\ldots\ldots (13),$$

(in which no connection between the arbitrary constants is asserted), or in terms of ψ by (7),

$$\psi_s = (1-\mu^2)^{\frac{1}{2}s} \left(\frac{d}{d\mu}\right)^s \psi_0 \ldots\ldots\ldots\ldots (14).$$

Equation (13) is a generalization of the property of Laplace's functions used in (8).

The corresponding relations for the plane problem may be deduced, as before, by attaching an infinite value to n, which in (13), (14) is arbitrary, and writing $n\nu = kr$. Since $\mu^2 + \nu^2 = 1$,

$$\psi_s = \nu^s \left(-\frac{\mu}{\nu}\frac{d}{d\nu}\right)^s \psi_0,$$

ψ_0 being regarded as a function of ν. In the limit μ (even though subject to differentiation) may be identified with unity, and thus we may take

$$\psi_s = (-2kr)^s \left(\frac{d}{d\,.\,(kr)^2}\right)^s \psi_0 \ldots\ldots\ldots\ldots (15).$$

When the pole is not a source, ψ_s is proportional to $J_s(kr)$. The constant coefficient, left undetermined by (15), may be readily found by a comparison of the leading terms. It thus appears that

$$J_s(kr) = (-2kr)^s \left(\frac{d}{d\,.\,(kr)^2}\right)^s J_0(kr) \ldots\ldots\ldots (16),$$

a well-known property of Bessel's functions[1].

The vibrations of a plane layer of gas are of course more easily dealt with, than those of a layer of finite curvature, but I have preferred to exhibit the indirect as well as the direct method of investigation, both for the sake of the spherical problem

[1] Todhunter's *Laplace's Functions*, § 390.

itself with the corresponding Laplace's expansion[1], and because the connection between Bessel's and Laplace's functions appears not to be generally understood. We may now, however, proceed to the independent treatment of the plane problem.

339. If in the general equation of simple aerial vibrations

$$\nabla^2\psi + k^2\psi = 0,$$

we assume that ψ is independent of z, and introduce plane polar coordinates, we get (§ 241)

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} + k^2\psi = 0 \quad\text{..............(1)};$$

or, if ψ be expanded in Fourier's series

$$\psi = \psi_0 + \psi_1 + \ldots + \psi_n + \quad\text{....................(2)},$$

where ψ_n is of the form $A_n \cos n\theta + B_n \sin n\theta$,

$$\frac{d^2\psi_n}{dr^2} + \frac{1}{r}\frac{d\psi_n}{dr} + \left(k^2 - \frac{n^2}{r^2}\right)\psi_n = 0 \quad\text{..............(3)}[2].$$

This equation is of the same form as that with which we had to deal in treating of circular membranes (§ 200); the principal mathematical difference between the two questions lies in the fact that while in the case of membranes the condition to be satisfied at the boundary is $\psi = 0$, in the present case interest attaches itself rather to the boundary condition $d\psi/dr = 0$, corresponding to the confinement of the gas by a rigid cylindrical envelope[3].

The pole not being a source, the solution of (3) is

$$\psi_n = A\, J_n(kr) \quad\text{.......................(4)},$$

and the equation giving the possible periods of vibration within a cylinder of radius r, is

$$J_n'(kr) = 0 \quad\text{.........................(5)}.$$

The lower values of kr satisfying (5) are given in the following table[4], which was calculated from Hansen's tables of the functions

[1] I have been much assisted by Heine's *Handbuch der Kugelfunctionen*, Berlin, 1861, and by Sir W. Thomson's papers on Laplace's Theory of the Tides, *Phil. Mag.* Vol. L. 1875.

[2] I here recur to the usual notation, but the reader will understand that n corresponds to the s of preceding sections. The n of Laplace's functions is now infinite.

[3] [The symmetrical vibrations within a cylindrical boundary, corresponding to $n=0$, were considered by Duhamel (*Liouville Journ. Math.* Vol. 14, p. 69, 1849).]

[4] Notes on Bessel's Functions. *Phil. Mag.* Nov. 1872.

J by means of the relations allowing J_n to be expressed in terms of J_0 and J_1.

Number of internal circular nodes.	$n=0$	$n=1$	$n=2$	$n=3$
0	3·832	1·841	3·054	4·201
1	7·015	5·332	6·705	8·015
2	10·174	8·536	9·965	11·344
3	13·324	11·706		
4	16·471	14·864		
5	19·616	18·016		

[For the roots of the equation $J_n'(z)=0$, Prof. McMahon[1] finds

$$z_n^{(s)}=\beta'-\frac{m+3}{8\beta'}-\frac{4(7m^2+82m-9)}{3(8\beta')^3}$$

$$-\frac{32(83m^3+2075m^2-3039m+3527)}{15(8\beta')^5}\ldots\ldots(6\,a),$$

where $m=4n^2$, and $\beta'=\frac{1}{4}\pi(2n+4s+1)$. It will be found that $n=0$ in (6 a) gives the same result as $n=1$ in (4) § 206, in accordance with the identity $J_0'(z)=-J_1(z)$.]

The particular solution may be written

$$\psi_n=(A\cos n\theta+B\sin n\theta)J_n(kr)\cos kat$$

$$+(C\cos n\theta+D\sin n\theta)J_n(kr)\sin kat\ldots\ldots\ldots\ldots(6),$$

where A, B, C, D are arbitrary for every admissible value of n and k. As in the corresponding problems for the sphere and circular membrane, the sum of all the particular solutions must be general enough to represent, when $t=0$, arbitrary values of ψ and $\dot{\psi}$.

As an example of compound vibrations we may suppose, as in § 332, that the initial condition of the gas is that defined by

$$\dot{\psi}=0,\quad \psi=x=r\cos\theta.$$

Under these circumstances (6) reduces to

$$\psi=A_1\cos\theta\,J_1(k_1r)\cos k_1at+A_2\cos\theta\,J_1(k_2r)\cos k_2at+\ldots(7),$$

and, if we suppose the radius of the cylinder to be unity, the admissible values of k are the roots of

$$J_1'(k)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

[1] *Annals of Mathematics*, Vol. IX. No. 1.

The condition to determine the coefficients A is that for all values of r from $r=0$ to $r=1$,

$$r = A_1 J_1(k_1 r) + A_2 J_1(k_2 r) + \ldots \ldots \ldots \ldots (9),$$

whence, as in § 332,

$$A = \frac{2}{(k^2-1) J_1(k)} \ldots \ldots \ldots \ldots (10).$$

The complete solution is therefore

$$\psi = \Sigma \frac{2 \cos\theta\, J_1(kr)}{(k^2-1) J_1(k)} \cos kat \ldots \ldots \ldots \ldots (11),$$

where the summation extends to all the values of k determined by (8).

If we put $t=0$ and $r=1$, we get from (9) and (10)

$$\Sigma \frac{2}{k^2-1} = 1 \ldots \ldots \ldots \ldots (12),$$

an equation which may be verified numerically, or by an analytical process similar to that applied in the case of (14) § 332. We may prove that

$$\log J_1'(z) = \text{constant} + \Sigma \log\left(1 - \frac{z^2}{k^2}\right),$$

whence by differentiation

$$\frac{J_1''(z)}{J_1'(z)} = -\Sigma \frac{2z}{k^2 - z^2}.$$

From this (12) is derived by putting $z=1$, and having regard to the fundamental differential equation satisfied by J_1, which shews that

$$J_1''(1) : J_1'(1) = -1.$$

[More generally, if $J_n'(k) = 0$,

$$\Sigma \frac{2n}{k^2 - n^2} = 1.\ ]$$

Hitherto we have supposed the cylinder complete, so that ψ recurs after each revolution, which requires that n be integral; but if instead of the complete cylinder we take the sector included between $\theta = 0$ and $\theta = \beta$, fractional values of n will in general present themselves. Since $d\psi/d\theta$ vanishes at both limits of θ, ψ must be of the form

$$\psi = A \cos(kat + \epsilon) \cos n\theta\, J_n(kr) \ldots \ldots \ldots \ldots (13),$$

where $n = \nu\pi/\beta$, ν being integral. If β be an aliquot part of π (or π itself), the complete solution involves only integral values of n, as might have been foreseen; but, in general, functions of fractional order must be introduced.

An interesting example occurs when $\beta = 2\pi$, which corresponds to the case of a cylinder, traversed by a rigid wall stretching from the centre to the circumference (compare § 207). The effect of the wall is to render possible a difference of pressure on its two sides; but when no such difference occurs, the wall may be removed, and the vibrations are included under the theory of a complete cylinder. This state of things occurs when ν is even. But when ν is odd, n is of the form (integer $+\frac{1}{2}$), and the pressures on the two sides of the wall are different. In the latter case J_n is expressible in finite terms. The gravest tone is obtained by taking $\nu = 1$, or $n = \frac{1}{2}$, when

$$\psi = A \cos (kat + \epsilon) . \cos \tfrac{1}{2}\theta . \frac{\sin kr}{\sqrt{(kr)}} \text{ ....(14)},$$

and the admissible values of k are the roots of $\tan k = 2k$. The first root (after $k = 0$) is $k = 1{\cdot}1655$, corresponding to a tone decidedly graver than any of which the complete cylinder is capable.

The preceding analysis has an interesting application to the mathematically analogous problem of the vibrations of water in a cylindrical vessel of uniform depth. The reader may consult a paper on waves by the author in the *Philosophical Magazine* for April, 1876, and papers by Prof. Guthrie to which reference is there made. The observation of the periodic time is very easy, and in this way may be obtained an experimental solution of problems, whose theoretical treatment is far beyond the power of known methods.

340. Returning to the complete cylinder, let us suppose it closed by rigid transverse walls at $z = 0$, and $z = l$, and remove the restriction that the motion is to be the same in all transverse sections. The general differential equation (§ 241) is

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} + \frac{d^2\psi}{dz^2} + k^2\psi = 0 \text{ ....(1)}.$$

Let ψ be expanded by Fourier's theorem in the series

$$\psi = H_0 + H_1 \cos\frac{\pi z}{l} + H_2 \cos\frac{2\pi z}{l} + \dots + H_p \cos\left(p\,\frac{\pi z}{l}\right) + \dots (2),$$

where the coefficients H_p may be functions of r and θ. This form secures the fulfilment of the boundary conditions, when $z=0$, $z=l$, and each term must satisfy the differential equation separately. Thus

$$\frac{d^2H_p}{dr^2}+\frac{1}{r}\frac{dH_p}{dr}+\frac{1}{r^2}\frac{d^2H_p}{d\theta^2}+\left(k^2-p^2\frac{\pi^2}{l^2}\right)H_p=0 \text{ ...... (3)},$$

which is of the same form as when the motion is independent of z, k^2 being replaced by $k^2-p^2\pi^2l^{-2}$. The particular solution may therefore be written

$$\psi=(A_n\cos n\theta+B_n\sin n\theta)\,.\cos p\frac{\pi z}{l}\,.\,J_n(\sqrt{k^2-p^2\pi^2l^{-2}}\,.\,r)\cos kat$$

$$+(C_n\cos n\theta+D_n\sin n\theta)\cos p\frac{\pi z}{l}\,.\,J_n(\sqrt{k^2-p^2\pi^2l^{-2}}\,.\,r)\sin kat\text{...(4)},$$

which must be generalized by a triple summation, with respect to all integral values of p and n, and also with respect to all the values of k, determined by the equation,

$$J_n'(\sqrt{k^2-p^2\pi^2l^{-2}}\,.\,r)=0 \text{ .... (5)}.$$

If $r=1$, and K denote the values of k given in the table (§ 339), corresponding to purely transverse vibrations, we have

$$k^2=K^2+p^2\pi^2/l^2 \text{ .... (6)}.$$

The purely axial vibrations correspond to a zero value of K, not included in the table.

341. The complete integral of the equation

$$\frac{d^2\psi_n}{dr^2}+\frac{1}{r}\frac{d\psi_n}{dr}+\left(k^2-\frac{n^2}{r^2}\right)\psi_n=0 \text{ .... (1)},$$

when there is no limitation as to the absence of a source at the pole, involves a second function of r, which may be denoted by $J_{-n}(kr)$. Thus, omitting unnecessary constant multipliers, we may take (§ 200)

$$\psi_n=Ar^{+n}\left\{1-\frac{k^2r^2}{2\,.\,2+2n}+\frac{k^4r^4}{2\,.\,4\,.\,2+2n\,.\,4+2n}-\dots\right\}$$

$$+Br^{-n}\left\{1-\frac{k^2r^2}{2\,.\,2-2n}+\frac{k^4r^4}{2\,.\,4\,.\,2-2n\,.\,4-2n}-\dots\right\}\text{ .... (2)},$$

but the second series requires modification, if n be integral. When $n=0$, the two series become identical, and thus the immediate result of supposing $n=0$ in (2) lacks the necessary generality. The required solution may, however, be obtained by the ordinary rule applicable to such cases. Denoting the coefficients of A and B in (2) by $f(n)$, $f(-n)$, we have

$$\begin{aligned}\psi &= Af(n) + Bf(-n)\\ &= (A+B)f(0) + (A-B)f'(0)\,n + (A+B)f''(0)\frac{n^2}{1.2} + \dots\end{aligned}$$

by Maclaurin's theorem. Hence, taking new arbitrary constants, we may write as the limiting form of (2),

$$\psi_0 = A\,f(0) + B\,f'(0).$$

In this equation $f(0)$ is $J_0(kr)$; to find $f'(0)$ we have

$$\begin{aligned}f'(n) &= r^n \log r\left\{1 - \frac{k^2r^2}{2\,.\,2+2n} + \frac{k^4r^4}{2\,.\,4\,.\,2+2n\,.\,4+2n} - \dots\right\}\\ &+ r^n\frac{d}{dn}\left\{1 - \frac{k^2r^2}{2\,.\,2+2n} + \frac{k^4r^4}{2\,.\,4\,.\,2+2n\,.\,4+2n} - \dots\right\}.\end{aligned}$$

If u denote the general term (involving r^{2m}) of the series within brackets, taken without regard to sign,

$$\frac{1}{u}\frac{du}{dn} = \frac{d\log u}{dn} = -\frac{2}{2+2n} - \frac{2}{4+2n} - \dots - \frac{2}{2m+2n},$$

so that $$\left(\frac{du}{dn}\right)_{n=0} = -u_{n=0}\,S_m,$$

if $$S_m = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m} \qquad\qquad (3).$$

Thus $$\begin{aligned}f'(0) &= \log r\left\{1 - \frac{k^2r^2}{2^2} + \frac{k^4r^4}{2^2\,.\,4^2} - \frac{k^6r^6}{2^2\,.\,4^2\,.\,6^2} + \dots\right\}\\ &+ \left\{\frac{k^2r^2}{2^2} - \frac{k^4r^4}{2^2\,.\,4^2}S_2 + \frac{k^6r^6}{2^2\,.\,4^2\,.\,6^2}S_3 - \dots\dots\right\},\end{aligned}$$

and the complete integral for the case $n = 0$ is

$$\begin{aligned}\psi_0 &= (A + B\log r)\left\{1 - \frac{k^2r^2}{2^2} + \frac{k^4r^4}{2^2\,.\,4^2} - \dots\right\}\\ &+ B\left\{\frac{k^2r^2}{2^2} - \frac{k^4r^4}{2^2\,.\,4^2}S_2 + \frac{k^6r^6}{2^2\,.\,4^2\,.\,6^2}S_3 - \dots\right\} \qquad (4).\end{aligned}$$

For the general integral value of n the corresponding expression may be derived by means of (15) § 338

$$\psi_n = (-2kr)^n\left(\frac{d}{d\,(kr)^2}\right)^n\psi_0 \qquad\qquad (5).$$

The formula of derivation (5) may be obtained directly from the differential equation (1). Writing z for kr and putting

$$\psi_n = z^n \phi_n \qquad\qquad (6),$$

we find in place of (1)

$$\frac{d^2\phi_n}{dz^2} + \frac{2n+1}{z}\frac{d\phi_n}{dz} + \phi_n = 0 \qquad\qquad (7).$$

Again (7) may be put into the form

$$z^2 \frac{d^2\phi_n}{d(z^2)^2} + (n+1)\frac{d\phi_n}{d\,.\,z^2} + \tfrac{1}{4}\phi_n = 0 \qquad\qquad (8),$$

from which it follows at once that

$$\phi_n = \frac{d}{d\,.\,z^2}\phi_{n-1} \qquad\qquad (9);$$

so that

$$\phi_n = \left(\frac{d}{d\,.\,z^2}\right)^n \phi_0 \qquad\qquad (10),$$

or by (6)

$$\psi_n = z^n \left(\frac{d}{d\,.\,z^2}\right)^n \psi_0 \qquad\qquad (11),$$

which is equivalent to (5), since the constants in ψ_0 are arbitrary in both equations.

The serial expressions for ψ_n thus obtained are convergent for all values of the argument, but are practically useless when the argument is great. In such cases we must have recourse to semi-convergent series corresponding to that of (10) § 200.

Equation (1) may be put into the form

$$\frac{d^2(z^{\frac{1}{2}}\psi_n)}{dz^2} - \frac{(n-\frac{1}{2})(n+\frac{1}{2})}{z^2}(z^{\frac{1}{2}}\psi_n) + z^{\frac{1}{2}}\psi_n = 0 \qquad\qquad (12),$$

whence by § 323 (4), (12), we find as the general solution of (1)

$$\psi_n = C(ikr)^{-\frac{1}{2}}e^{-ikr}\left\{1 - \frac{1^2-4n^2}{1\,.\,8ikr} + \frac{(1^2-4n^2)(3^2-4n^2)}{1\,.\,2\,.\,(8ikr)^2} - \frac{(1^2-4n^2)(3^2-4n^2)(5^2-4n^2)}{1\,.\,2\,.\,3\,.\,(8ikr)^3} + \ldots\right\}$$
$$+ D(ikr)^{-\frac{1}{2}}e^{+ikr}\left\{1 + \frac{1^2-4n^2}{1\,.\,8ikr} + \frac{(1^2-4n^2)(3^2-4n^2)}{1\,.\,2\,.\,(8ikr)^2} + \frac{(1^2-4n^2)(3^2-4n^2)(5^2-4n^2)}{1\,.\,2\,.\,3\,.\,(8ikr)^3} + \ldots\right\} \qquad\qquad (13).$$

When n is integral, these series are infinite and ultimately divergent, but (§§ 200, 302) this circumstance does not interfere with their practical utility.

The most important application of the complete integral of (1) is to represent a disturbance diverging from the pole, a problem which has been treated by Stokes in his memoir on the communication of vibrations to a gas. The condition that the disturbance represented by (13) shall be exclusively divergent is simply $D=0$, as appears immediately on introduction of the time factor e^{ikat} by supposing r to be very great; the principal difficulty of the question consists in discovering what relation between the coefficients of the ascending series corresponds to this condition, for which purpose Stokes employs the solution of (1) in the form of a definite integral. We shall attain the same object, perhaps more simply, by using the results of § 302.

By (22), (24) § 302

$$-\left(\frac{\pi}{2iz}\right)^{\frac{1}{2}} e^{-iz}\left\{1-\frac{1^2}{1\,.\,8iz}+\frac{1^2\,.\,3^2}{1\,.\,2\,.\,(8iz)^2}+\ldots\right\}$$

$$=\tfrac{1}{2}\pi\{K(z)+i\,J_0(z)\}-\int_0^\infty \frac{e^{-\beta}\,d\beta}{\sqrt{(\beta^2+z^2)}}\quad\ldots\ldots\ldots(14),$$

and thus the question reduces itself to the determination of the form of the right-hand member of (14) when z is small. By (5) § 302 and (5) § 200 we have

$$\tfrac{1}{2}\pi\{K(z)+i\,J_0(z)\}=z+\tfrac{1}{2}i\pi+\text{higher terms in } z\ldots\ldots(15),$$

so that all that remains is to find the form of the definite integral in (14), when z is small. Putting $\sqrt{(\beta^2+z^2)}=y-\beta$, we have

$$\int_0^\infty \frac{e^{-\beta}\,d\beta}{\sqrt{(\beta^2+z^2)}}=\int_z^\infty e^{-\frac{y^2-z^2}{2y}}\frac{dy}{y}=\int_z^\infty e^{\frac{1}{2}z^2/y}e^{-\frac{1}{2}y}\frac{dy}{y}.$$

When z is small, $z^2/2y$ is also small throughout the range of integration, and thus we may write

$$\int_0^\infty \frac{e^{-\beta}\,d\beta}{\sqrt{(\beta^2+z^2)}}=\int_z^\infty\left\{1+\frac{z^2}{2y}+\frac{1}{2}\frac{z^4}{4y^2}+\ldots\right\}\frac{e^{-\frac{1}{2}y}}{y}\,dy.$$

The first integral on the right is

$$\int_z^\infty \frac{e^{-\frac{1}{2}y}}{y}\,dy=\int_{\frac{1}{2}z}^\infty \frac{e^{-v}\,dv}{v}=-\gamma-\log\left(\tfrac{1}{2}z\right)+\tfrac{1}{2}z+\ldots\ldots(16)^1,$$

[1] De Morgan's *Differential and Integral Calculus*, p. 653.

where γ is Euler's constant (·5772...); and, as we may easily satisfy ourselves by integration by parts, the other integrals do not contribute anything to the leading terms. Thus, when z is very small,

$$-\left(\frac{\pi}{2iz}\right)^{\frac{1}{2}} e^{-iz}\left\{1-\frac{1^2}{1\,.\,8iz}+\frac{1^2\,.\,3^2}{1\,.\,2\,.\,(8iz)}-\frac{1^2\,.\,3^2\,.\,5^2}{1\,.\,2\,.\,3\,.\,(8iz)^3}+\dots\right\}$$

$$=\gamma+\log\left(\tfrac{1}{2}z\right)+\tfrac{1}{2}i\pi+\dots\dots\dots(17).$$

Replacing z by kr, and comparing with the form assumed by (4) when r is small, we see that in order to make the series identical we must take

$$A=\gamma+\log\tfrac{1}{2}+\log k+\tfrac{1}{2}i\pi,\qquad B=1;$$

so that a series of waves diverging from the pole, whose expression in descending series is

$$\psi_0=-\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}\left\{1-\frac{1^2}{1\,.\,8ikr}+\frac{1^2\,.\,3^2}{1\,.\,2\,.\,(8ikr)^2}-\dots\right\}\dots\dots(18),$$

is represented also by the ascending series

$$\psi_0=\left(\gamma+\log\frac{ikr}{2}\right)\left\{1-\frac{k^2r^2}{2^2}+\frac{k^4r^4}{2^2\,.\,4^2}-\dots\right\}$$

$$+\frac{k^2r^2}{2^2}S_1-\frac{k^4r^4}{2^2\,.\,4^2}S_2+\frac{k^6r^6}{2^2\,.\,4^2\,.\,6^2}S_3-\dots\dots\dots(19).$$

In applying the formula of derivation (11) to the descending series, the parts containing e^{-ikr} and e^{+ikr} as factors will evidently remain distinct, and the complete integral for the general value of n, subject to the condition that the part containing e^{+ikr} shall not appear, will be got by differentiation from the complete integral for $n=0$ subject to the same condition. Thus, since by (5) $\psi_1=d\psi_0/dr$,

$$\psi_1=\left(\frac{\pi i}{2kr}\right)^{\frac{1}{2}} e^{-ikr}\left\{1-\frac{-1\,.\,3}{1\,.\,8ikr}+\frac{-1\,.\,1\,.\,3\,.\,5}{1\,.\,2\,.\,(8ikr)^2}\right.$$

$$\left.-\frac{-1\,.\,1\,.\,3^2\,.\,5\,.\,7}{1\,.\,2\,.\,3\,.\,(8ikr)^3}+\dots\right\}\dots\dots\dots(20),$$

or, in terms of the ascending series,

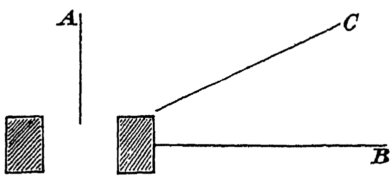
$$\psi_1 = \frac{1}{kr}\left\{1 - \frac{k^2r^2}{2^2} + \frac{k^4r^4}{2^2\,4^2} - \dots\right\}$$

$$- \left(\gamma + \log\frac{ikr}{2}\right)\left\{\frac{kr}{2} - \frac{k^3r^3}{2^2.4} + \frac{k^5r^5}{2^2.4^2.6} - \dots\right\}$$

$$+ \frac{kr}{2}S_1 - \frac{k^3r^3}{2^2.4}S_2 + \frac{k^5r^5}{2^2.4^2.6}S_3 - \dots\dots\dots\dots(21).$$

These expressions are applied by Prof. Stokes to shew how feebly the vibrations of a string, (corresponding to the term of order one), are communicated to the surrounding gas. For this purpose he makes a comparison between the actual sound, and what would have been emitted in the same direction, were the lateral motion of the gas in the neighbourhood of the string prevented. For a piano string corresponding to the middle C, the radius of the wire may be about ·02 inch, and λ is about 25 inches; and it appears that the sound is nearly 40,000 times weaker than it would have been if the motion of the particles of air had taken place in planes passing through the axis of the string. "This shews the vital importance of sounding-boards in stringed instruments. Although the amplitude of vibration of the particles of the sounding-board is extremely small compared with that of the particles of the string, yet as it presents a broad surface to the air it is able to excite loud sonorous vibrations, whereas were the string supported in an absolutely rigid manner, the vibrations which it could excite directly in the air would be so small as to be almost or altogether inaudible."

Fig. 64.



"The increase of sound produced by the stoppage of lateral motion may be prettily exhibited by a very simple experiment. Take a tuning-fork, and holding it in the fingers after it has been made to vibrate, place a sheet of paper, or the blade of a broad knife, with its edge parallel to the axis of the fork, and as near to the fork as conveniently may be without touching. If the plane of the obstacle coincide with either of the planes of symmetry of the fork, as represented in section at A or B, no effect is produced; but if it be placed in an intermediate position, such as C, the sound becomes much stronger[1]."

342. The real expression for the velocity-potential of symmetrical waves diverging in two dimensions is obtained from (18) § 341 after introduction of the time factor e^{ikat} by rejecting the imaginary part; it is

$$\psi_0 = -\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} \cos k\left(at - r - \tfrac{1}{8}\lambda\right)\left\{1 - \frac{1^2 . 3^2}{1 . 2 . (8kr)^2} + \ldots\right\}$$

$$+\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} \sin k\left(at - r - \tfrac{1}{8}\lambda\right)\left\{\frac{1^2}{1 . 8kr} - \frac{1^2 . 3^2 . 5^2}{1 . 2 . 3 . (8kr)^3} + \ldots\right\} \ldots\ldots(1),$$

in which, as usual, two arbitrary constants may be inserted, one as a multiplier of the whole expression and the other as an addition to the time.

The problem of a linear source of uniform intensity may also be treated by the general method applicable in three dimensions. Thus by (3) § 277, if ρ be the distance of any element dx from O, the point at which the potential is to be estimated, and r be the smallest value of ρ, so that $\rho^2 = r^2 + x^2$, we may take

$$\phi = 2\int_0^\infty \frac{e^{-ik\rho}\,dx}{\rho} = 2\int_r^\infty \frac{e^{-ik\rho}\,d\rho}{\sqrt{(\rho^2 - r^2)}} \ldots\ldots\ldots\ldots\ldots(2),$$

which must be of the same form as (1). Taking $y = \rho - r$, we may write in place of (2)

$$\phi = 2\int_0^\infty \frac{e^{-ikr}\,e^{-iky}\,dy}{\sqrt{y} . \sqrt{(2r + y)}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

from which the various expressions follow as in (14) § 341. When kr is great, an approximate value of the integral may be obtained by neglecting the variation of $\sqrt{(2r + y)}$, since on account of the rapid fluctuation of sign caused by the factor e^{-iky} we need attend

[1] *Phil. Trans.* vol. 158, p. 447, 1868.

only to small values of y. Now

$$\int_0^\infty \frac{\cos x\,dx}{\sqrt{x}} = \int_0^\infty \frac{\sin x\,dx}{\sqrt{x}} = \sqrt{\left(\frac{\pi}{2}\right)} \text{ ....(4)},$$

so that $$\phi = \sqrt{\left(\frac{\pi}{kr}\right)}\, e^{-ikr}(1-i) = \sqrt{\left(\frac{2\pi}{kr}\right)}\, e^{-ik(r+\frac{1}{8}\lambda)} \text{ ....(5)}.$$

Introducing the factor e^{ikat}, and rejecting the imaginary part of the expression, we have finally

$$\phi = \sqrt{\left(\frac{2\pi}{kr}\right)} \cos k\,(at - r - \tfrac{1}{8}\lambda) \text{ ....(6)},$$

as the value of the velocity-potential at a great distance. A similar argument is applicable to shew that (1) is also the expression for the velocity-potential on one side of an infinite plane (§ 278) due to the uniform normal motion of an infinitesimal strip bounded by parallel lines.

In like manner we may regard the term of the first order (20) § 341 as the expression of the velocity-potential due to double sources uniformly distributed along an infinite straight line.

From the point of view of the present section we see the significance of the retardation of $\frac{1}{8}\lambda$, which appears in (1) and in the results of the following section (16), (17). In the ordinary integration for surface distributions by Fresnel's zones (§ 283) the whole effect is the half of that of the first zone, and the phase of the effect of the first zone is midway between the phases due to its extreme parts, i.e. $\frac{1}{4}\lambda$ behind the phase due to the central point. In the present case the retardation of the resultant relatively to the central element is less, on account of the preponderance of the central parts.

[From the formulæ of the present section for the velocity-potential of a linear source we may obtain by integration a corresponding expression for a source which is uniformly distributed over a *plane*. The waves issuing from this latter are necessarily plane waves, of which the velocity-potential can at once be written down, and the comparison of results leads to the evaluation of certain definite integrals relating to Bessel's and allied functions[1].]

[1] On Point-, Line-, and Plane-Sources of Sound. *Proc. London Math. Soc.* Vol. XIX. p. 504, 1888.

343. In illustration of the formulæ of § 341 we may take the problem of the disturbance of plane waves of sound by a cylindrical obstacle, whose radius is small in comparison with the length of the waves, and whose axis is parallel to their plane. (Compare § 335.)

Let the plane waves be represented by

$$\phi = e^{ik(at+x)} = e^{ikat} \cdot e^{ikr\cos\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The general expansion of ϕ in Fourier's series may be readily effected, the coefficients of the various terms being, as might be anticipated, simply the Bessel's functions of corresponding orders. [Thus, as in (12) § 272 a,

$$e^{ikr\cos\theta} = J_0(kr) + 2iJ_1(kr)\cos\theta + \ldots + 2i^n J_n(kr)\cos n\theta + \ldots.]$$

But, as we confine ourselves here to the case where c the radius of the cylinder is small, we will at once expand in powers of r.

Thus, when $r = c$, if e^{ikat} be omitted,

$$\phi = 1 - \tfrac{1}{4}k^2c^2 + ikc \cdot \cos\theta + \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\frac{d\phi}{dr} = -\tfrac{1}{2}k^2c + ik \cdot \cos\theta + \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The amount and even the law of the disturbance depends upon the character of the obstacle. We will begin by supposing the material of the cylinder to be a gas of density σ' and compressibility m'; the solution of the problem for a rigid obstacle may finally be derived by suitable suppositions with respect to σ', m'. If k' be the internal value of k, we have inside the cylinder by the condition that the axis is not a source (§ 339),

$$\psi = A_0\left\{1 - \frac{k'^2r^2}{2^2} + \frac{k'^4r^4}{2^2 \cdot 4^2} - \ldots\right\} + A_1 r\left\{1 - \frac{k'^2r^2}{2 \cdot 4} + \frac{k'^4r^4}{2 \cdot 4^2 \cdot 6} - \ldots\right\}\cos\theta;$$

so that, when $r = c$,

$$\psi\,(\text{inside}) = A_0(1 - \tfrac{1}{4}k'^2c^2) + A_1 c\,(1 - \tfrac{1}{8}k'^2c^2) \cdot \cos\theta \ldots(4),$$

$$\frac{d\psi}{dr}\,(\text{inside}) = -\tfrac{1}{2}A_0 k'^2 c + A_1(1 - \tfrac{3}{8}k'^2c^2)\cos\theta \quad \ldots\ldots\ldots(5).$$

Outside the cylinder, when $r = c$, we have by (19), (21) § 341,

$$\psi = B_0\left(\gamma + \log\frac{ikc}{2}\right) + \frac{B_1\cos\theta}{kc} \quad \ldots\ldots\ldots\ldots\ldots(6),$$

$$\frac{d\psi}{dr} = \frac{B_0}{c} - \frac{B_1\cos\theta}{kc^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

The conditions to be satisfied at the surface of separation are thus

$$-A_0 k'^2 c^2 = -k^2 c^2 + 2B_0 \qquad \text{..................(8)},$$

$$\frac{\sigma'}{\sigma} A_0 (1 - \tfrac{1}{4}k'^2 c^2) = 1 - \tfrac{1}{4}k^2 c^2 + B_0\left(\gamma + \log\frac{ikc}{2}\right) \qquad \text{......(9)},$$

$$A_1\left(1 - \frac{3k'^2 c^2}{8}\right) = ik - \frac{B_1}{kc^2} \qquad \text{..................(10)},$$

$$\frac{\sigma'}{\sigma} A_1 c\left(1 - \frac{k'^2 c^2}{8}\right) = ikc + \frac{B_1}{kc} \qquad \text{..............(11)},$$

from which by eliminating A_0, A_1 we get approximately

$$B_0 = \tfrac{1}{2}k^2 c^2\left(1 - \frac{k'^2}{k^2}\cdot\frac{\sigma}{\sigma'}\right) = \tfrac{1}{2}k^2 c^2 \frac{m' - m}{m'} \qquad \text{..........(12)},$$

$$B_1 = ik^2 c^2 \frac{\sigma' - \sigma}{\sigma' + \sigma} \qquad \text{......................(13)}.$$

Thus at a distance from the cylinder we have by (18) and (20) § 341,

$$\psi = -B_0\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr} + B_1\left(\frac{\pi i}{2kr}\right)^{\frac{1}{2}} e^{-ikr} \cdot \cos\theta$$

$$= -k^2 c^2 e^{-ikr}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}\left\{\frac{m' - m}{2m'} + \frac{\sigma' - \sigma}{\sigma' + \sigma}\cos\theta\right\}$$

$$= -\frac{2\pi \cdot \pi c^2}{r^{\frac{1}{2}}\lambda^{\frac{3}{2}}} e^{-i(kr + \frac{1}{4}\pi)}\left\{\frac{m' - m}{2m'} + \frac{\sigma' - \sigma}{\sigma' + \sigma}\cos\theta\right\} \qquad \text{...(14)}.$$

Hence, corresponding to the primary wave

$$\phi = \cos\frac{2\pi}{\lambda}(at + x) \qquad \text{....................(15)},$$

the scattered wave is approximately

$$\psi = -\frac{2\pi \cdot \pi c^2}{r^{\frac{1}{2}}\lambda^{\frac{3}{2}}}\left\{\frac{m' - m}{2m'} + \frac{\sigma' - \sigma}{\sigma' + \sigma}\cos\theta\right\}\cos\frac{2\pi}{\lambda}(at - r - \tfrac{1}{8}\lambda) \qquad \text{....(16)}.$$

The fact that ψ varies inversely as $\lambda^{\frac{3}{2}}$ might have been anticipated by the method of dimensions, as in the corresponding problem for the sphere (§ 296). As in that case, the symmetrical part of the divergent wave depends upon the variation of compressibility, and would disappear in the application to an actual gas; and the term of the first order depends upon the variation of density.

By supposing σ' and m' to become infinite, in such a manner that their ratio remains finite, we obtain the solution corresponding to a rigid and immoveable obstacle,

$$\psi = -\frac{2\pi \,.\, \pi c^2}{r^{\frac{1}{2}}\lambda^{\frac{3}{2}}}(\tfrac{1}{2} + \cos\theta)\cos\frac{2\pi}{\lambda}(at - r - \tfrac{1}{8}\lambda) \quad\ldots\ldots(17).$$

The exceeding smallness of the obstruction offered by fine wires or fibres to the passage of sound is strikingly illustrated in some of Tyndall's experiments. A piece of stiff felt half an inch in thickness allows much more sound to pass than a *wetted* pocket-handkerchief, which in consequence of the closing of its pores behaves rather as a thin lamina. For the same reason fogs, and even rain and snow, interfere but little with the free propagation of sounds of moderate wave-length. In the case of a hiss, or other very acute sound, the effect would perhaps be apparent.

[The partial reflections from sheets of muslin may be utilized to illustrate an important principle. If a pure tone of high (inaudible) pitch be reflected from a single sheet so as to impinge upon a sensitive flame, the intensity will probably be insufficient to produce a visible effect. If, however, a moderate number of such sheets be placed parallel to one another and at such equal distances apart that the partial reflections agree in phase, then the flame may be powerfully affected. The parallelism and equidistance of the sheets may be maintained mechanically by a lazy-tongs arrangement, which nevertheless allows the common distance to be varied. It is then easy to trace the dependence of the action upon the accommodation of the interval to the wave length of the sound. Thus, if the incidence were perpendicular, the flame would be most powerfully influenced when the interval between adjacent sheets was equal to the *half* wave length; and although the exigencies of experiment make it necessary to introduce obliquity, allowance for this is readily made[1].]

[1] Iridescent Crystals, *Proc. Roy. Inst.* April 1889. See also *Phil. Mag.* vol. XXIV. p. 145, 1887; vol. XXVI. p. 256, 1888.

CHAPTER XIX.

FRICTION AND HEAT CONDUCTION.

344. THE equations of Chapter XI. and the consequences that we have deduced from them are based upon the assumption (§ 236), that the mutual action between any two portions of fluid separated by an imaginary surface is normal to that surface. Actual fluids however do not come up to this ideal; in many phenomena the defect of fluidity, usually called viscosity or fluid friction, plays an important and even a preponderating part. It will therefore be proper to inquire whether the laws of aerial vibrations are sensibly influenced by the viscosity of air, and if so in what manner.

In order to understand clearly the nature of viscosity, let us conceive a fluid divided into parallel strata in such a manner that while each stratum moves in its own plane with uniform velocity, a change of velocity occurs in passing from one stratum to another. The simplest supposition which we can make is that the velocities of all the strata are in the same direction, but increase uniformly in magnitude as we pass along a line perpendicular to the planes of stratification. Under these circumstances a tangential force between contiguous strata is called into play, in the direction of the relative motion, and of magnitude proportional to the rate at which the velocity changes, and to a coefficient of viscosity, commonly denoted by the letter μ. Thus, if the strata be parallel to xy and the direction of their motion be parallel to y, the tangential force, reckoned (like a pressure) per unit of area, is

$$\mu \frac{dv}{dz} \qquad \text{(1)}.$$

The dimensions of μ are $[ML^{-1}T^{-1}]$.

The examination of the origin of the tangential force belongs to molecular science. It has been explained by Maxwell in accordance with the kinetic theory of gases as resulting from interchange of molecules between the strata, giving rise to diffusion of momentum. Both by theory and experiment the remarkable conclusion has been established that within wide limits the force is independent of the density of the gas. For air at θ^0 Centigrade Maxwell[1] found

$$\mu = \cdot 0001878\,(1 + \cdot 00366\,\theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

the centimetre, gramme, and second being units.

345. The investigation of the equations of fluid motion in which regard is paid to viscous forces can scarcely be considered to belong to the subject of this work, but it may be of service to some readers to point out its close connection with the more generally known theory of solid elasticity.

The potential energy of unit of volume of uniformly strained isotropic matter may be expressed[2]

$$V = \tfrac{1}{2} m\delta^2 + \tfrac{1}{2} n\,(e^2 + f^2 + g^2 - 2fg - 2ge - 2ef + a^2 + b^2 + c^2)$$
$$= \tfrac{1}{2}\kappa\delta^2 + \tfrac{1}{2} n\,(2e^2 + 2f^2 + 2g^2 - \tfrac{2}{3}\delta^2 + a^2 + b^2 + c^2) \quad \ldots\ldots\ldots (1),$$

in which $\delta\,(= e + f + g)$ is the dilatation, e, f, g, a, b, c are the six components of strain, connected with the actual displacements α, β, γ by the equations

$$e = \frac{d\alpha}{dx}, \qquad f = \frac{d\beta}{dy}, \qquad g = \frac{d\gamma}{dz} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$a = \frac{d\beta}{dz} + \frac{d\gamma}{dy}, \quad b = \frac{d\gamma}{dx} + \frac{d\alpha}{dz}, \quad c = \frac{d\alpha}{dy} + \frac{d\beta}{dx} \quad \ldots\ldots\ldots\ldots (3),$$

and m, n, κ are constants of elasticity, connected by the equation

$$\kappa = m - \tfrac{1}{3} n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

of which n measures the *rigidity*, or resistance to *shearing*, and κ measures the resistance to change of *volume*. The components of stress P, Q, R, S, T, U, corresponding respectively to e, f, g, a, b, c, are found from V by simple differentiation with respect to those quantities; thus

$$P = \kappa\delta + 2n\,(e - \tfrac{1}{3}\delta), \text{ \&c.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$S = na, \text{ \&c.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

[1] On the Viscosity or Internal Friction of Air and other Gases. *Phil. Trans.* vol. 156, p. 249, 1866.

[2] Thomson and Tait's *Natural Philosophy*. Appendix C.

If X, Y, Z be the components of the applied force reckoned per unit of volume, the equations of equilibrium are of the form

$$\frac{dP}{dx}+\frac{dU}{dy}+\frac{dT}{dz}+X=0, \text{ \&c.} \quad\quad (7),$$

from which the equations of motion are immediately obtainable by means of D'Alembert's principle. In terms of the displacements α, β, γ, these equations become

$$\kappa\frac{d\delta}{dx}+\tfrac{1}{3}n\frac{d\delta}{dx}+n\nabla^2\alpha+X=0, \text{ \&c.} \quad\quad (8),$$

where

$$\delta=\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz} \quad\quad (9).$$

In the ordinary theory of fluid friction no forces of restitution are included, but on the other hand we have to consider viscous forces whose relation to the velocities (u, v, w) of the fluid elements is of precisely the same character as that of the forces of restitution to the displacements (α, β, γ) of an isotropic solid. Thus if δ' be the velocity of dilatation, so that

$$\delta'=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} \quad\quad (10),$$

the force parallel to x due to viscosity is, as in (8),

$$\kappa\frac{d\delta'}{dx}+\tfrac{1}{3}n\frac{d\delta'}{dx}+n\nabla^2 u \quad\quad (11).$$

So far κ and n are arbitrary constants; but it has been argued with great force by Prof. Stokes, that there is no reason why a motion of dilatation uniform in all directions should give rise to viscous force, or cause the pressure to differ from the statical pressure corresponding to the actual density. In accordance with this argument we are to put $\kappa=0$; and, as appears from (6), n coincides with the quantity previously denoted by μ. The frictional terms are therefore

$$\mu\left\{\nabla^2 u+\tfrac{1}{3}\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right\}, \text{ \&c.};$$

and (§ 237) the equations of motion take the form

$$\rho\left(\frac{Du}{Dt}-X\right)+\frac{dp}{dx}-\mu\nabla^2 u-\tfrac{1}{3}\mu\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0 \quad\quad (12);$$

or, if there be no applied forces and the square of the motion be neglected,

$$\rho_0 \frac{du}{dt} + \frac{dp}{dx} - \mu\nabla^2 u - \tfrac{1}{3}\mu \frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0 \quad\text{.........(13).}$$

We may observe that the dissipative forces here considered correspond to a dissipation function, whose form is the same with respect to u, v, w as that of V with respect to α, β, γ, in the theory of isotropic solids. Thus putting $\kappa = 0$, we have from (1)

$$F = \tfrac{1}{2}\mu \iiint \left[2\left(\frac{du}{dx}\right)^2 + 2\left(\frac{dv}{dy}\right)^2 + 2\left(\frac{dw}{dz}\right)^2 - \tfrac{2}{3}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)^2 + \left(\frac{dv}{dz} + \frac{dw}{dy}\right)^2 + \left(\frac{dw}{dx} + \frac{du}{dz}\right)^2 + \left(\frac{du}{dy} + \frac{dv}{dx}\right)^2 \right] dx\,dy\,dz \quad\text{......(14),}$$

in agreement with Prof. Stokes' calculation[1]. The theory of friction for the case of a compressible fluid was first given by Poisson[2].

346. We will now apply the differential equations to the investigation of plane waves of sound. Supposing that v and w are zero and that u, p, &c. are functions of x only, we obtain from (13) § 345

$$\rho_0 \frac{du}{dt} + \frac{dp}{dx} - \frac{4\mu}{3}\frac{d^2u}{dx^2} = 0 \quad\text{....................(1).}$$

The equation of continuity (3) § 238 is in this case

$$\frac{ds}{dt} + \frac{du}{dx} = 0 \quad\text{........................(2),}$$

and the relation between the variable part of the pressure δp and the condensation s is as usual (§ 244)

$$\delta p = a^2 \rho_0 s \quad\text{.........................(3).}$$

Thus, eliminating δp and s between (1), (2), (3), we obtain

$$\frac{d^2u}{dt^2} - a^2\frac{d^2u}{dx^2} - \frac{4\mu}{3\rho_0}\frac{d^3u}{dx^2\,dt} = 0 \quad\text{................(4),}$$

which is the equation given by Stokes[3].

Let us now inquire how a train of harmonic waves of wave-length λ, which are maintained at the origin ($x = 0$), fade away

[1] *Cambridge Transactions*, vol. IX. § 49, 1851.
[2] *Journal de l'École Polytechnique*, t. XIII. cah. 20, p. 139.
[3] *Cambridge Transactions*, vol. VIII. p. 287, 1845.

as x increases. Assuming that u varies as e^{int}, we find as in § 148,

$$u = Ae^{-\alpha x}\cos(nt - \beta x) \qquad (5),$$

where
$$\beta^2 - \alpha^2 = \frac{n^2 a^2}{a^4 + \dfrac{16\mu^2 n^2}{9\rho_0^2}}, \quad 2\alpha\beta = \frac{4\mu n^3/3\rho_0}{a^4 + \dfrac{16\mu^2 n^2}{9\rho_0^2}} \qquad (6).$$

In the application to air at ordinary pressures μ may be considered to be a very small quantity and its square may be neglected. Thus

$$\beta = \frac{n}{a}, \qquad \alpha = \frac{2\mu n^2}{3\rho_0 a^3} \qquad (7).$$

It appears that to this order of approximation the velocity of sound is unaffected by fluid friction. If we replace n by $2\pi a\lambda^{-1}$, the expression for the coefficient of decay becomes

$$\alpha = \frac{8\pi^2\mu}{3\lambda^2\rho_0 a} \qquad (8),$$

shewing that the influence of viscosity is greatest on the waves of short wave-length. The amplitude is diminished in the ratio $e : 1$, when $x = \alpha^{-1}$. In C. G. S. measure we may take

$$\rho_0 = \cdot0013, \quad \mu = \cdot00019, \quad a = 33200;$$

whence
$$x = 8800\lambda^2 \qquad (9).$$

Thus the amplitude of waves of one centimetre wave-length is diminished in the ratio $e : 1$ after travelling a distance of 88 metres. A wave-length of 10 centimetres would correspond nearly to g^{iv}; for this case $x = 8800$ metres. It appears therefore that at atmospheric pressures the influence of friction is not likely to be sensible to ordinary observation, except near the upper limit of the musical scale. The mellowing of sounds by distance, as observed in mountainous countries, is perhaps to be attributed to friction, by the operation of which the higher and harsher components are gradually eliminated. It must often have been noticed that the sound s is scarcely, if at all, returned by echos, and I have found[1] that at a distance of 200 metres a powerful hiss loses its character, even when there is no reflection. Probably this effect also is due to viscosity.

[1] Acoustical Observations, *Phil. Mag.* vol. III. p. 456, 1877.

In highly rarefied air the value of α as given in (8) is much increased, μ being constant. Sounds even of grave pitch may then be affected within moderate distances.

From the observations of Colladon in the lake of Geneva it would appear that in water grave sounds are more rapidly damped than acute sounds. At a moderate distance from a bell, struck under water, he found the sound short and sharp, without musical character.

347. The effect of viscosity in modifying the motion of air in contact with vibrating solids will be best understood from the solution of the problem for a very simple case given by Stokes. Let us suppose that an infinite plane (yz) executes harmonic vibrations in a direction (y) parallel to itself. The motion being in parallel strata, u and w vanish, and the variable quantities are functions of x only. The first of equations (13) § 345 shews that the pressure is constant; the corresponding equation in v takes the form

$$\frac{dv}{dt} = \frac{\mu}{\rho}\frac{d^2v}{dx^2} \quad\text{..........(1)},$$

similar to the equation for the linear conduction of heat. If we now suppose that v is proportional to e^{int}, the resulting equation in x is

$$\frac{d^2v}{dx^2} = i\frac{n\rho}{\mu}v \quad\text{..........(2)},$$

and its general solution

$$v = Ae^{-mx} + Be^{+mx} \quad\text{..........(3)},$$

where

$$m = \sqrt{\left(\frac{n\rho}{2\mu}\right)}(1+i) \quad\text{..........(4)},$$

If the gas be on the positive side of the vibrating plane the motion is to vanish when $x = +\infty$. Hence $B = 0$, and the value of v becomes on rejection of the imaginary part

$$v = Ae^{-\sqrt{\left(\frac{n\rho}{2\mu}\right)}x}\cos\left\{nt - \sqrt{\left(\frac{n\rho}{2\mu}\right)}x\right\} \quad\text{..........(5)},$$

corresponding to the motion

$$V = A\cos nt \quad\text{..........(6)}$$

at $x = 0$. The velocity of the fluid in contact with the plane is usually assumed to be the same as that of the plane itself on the apparently sufficient ground that the contrary would imply an infinitely greater smoothness of the fluid with respect to the solid than with respect to itself. On this supposition (5) expresses the motion of the fluid on the positive side due to a motion of the plane given by (6).

The tangential force per unit area acting on the plane is

$$\mu\, dv/dx_{(x=0)},$$

or $$\mu \sqrt{\left(\frac{n\rho}{2\mu}\right)}\{-\cos nt + \sin nt\} = -\sqrt{(\tfrac{1}{2}n\rho\mu)}\left(V + \frac{1}{n}\frac{dV}{dt}\right) \dots (7),$$

if $A = 1$. The first term represents a dissipative force tending to stop the motion; the second represents a force equivalent to an increase in the inertia of the vibrating body. The magnitude of both forces depends upon the frequency of the vibration.

We will apply this result to calculate approximately the velocity of sound in tubes so narrow that the viscosity of air exercises a sensible influence. As in § 265, let X denote the total transfer of fluid across the section of the tube at the point x. The force, due to hydrostatic pressure, acting on the slice between x and $x + \delta x$ is, as usual,

$$-S\frac{dp}{dx}\delta x = a^2\rho\,\delta x\,\frac{d^2X}{dx^2} \dots\dots\dots\dots (8).$$

The force due to viscosity may be inferred from the investigation for a vibrating plane, provided that the thickness of the layer of air adhering to the walls of the tube be small in comparison with the diameter. Thus, if P be the perimeter of the tube, and V be the velocity of the current at a distance from the walls of the tube, the tangential force on the slice, whose volume is $S\delta x$, is by (7)

$$-P\delta x\sqrt{(\tfrac{1}{2}n\rho\mu)}\left(V + \frac{1}{n}\frac{dV}{dt}\right),$$

or on replacing V by $\dfrac{dX}{dt} \div S$

$$-P\delta x\sqrt{(\tfrac{1}{2}n\rho\mu)}\left(\frac{dX}{dt} + \frac{1}{n}\frac{d^2X}{dt^2}\right) \div S \dots\dots\dots\dots (9).$$

The equation of motion *for this period* is therefore

$$\rho\delta x\frac{d^2X}{dt^2} + \sqrt{(\tfrac{1}{2}n\rho\mu)}\frac{P\delta x}{S}\left(\frac{dX}{dt} + \frac{1}{n}\frac{d^2X}{dt^2}\right) = a^2\rho\,\delta x\,\frac{d^2X}{dx^2},$$

or $$\frac{d^2X}{dt^2}\left\{1+\frac{P}{S}\sqrt{\left(\frac{\mu}{2n\rho}\right)}\right\}+\frac{P}{S}\sqrt{\left(\frac{n\mu}{\rho}\right)}\frac{dX}{dt}=a^2\frac{d^2X}{dx^2}\quad\text{....(10).}$$

The velocity of sound is approximately

$$a\left\{1-\tfrac{1}{2}\frac{P}{S}\sqrt{\left(\frac{\mu}{2n\rho}\right)}\right\}\quad\text{........................(11),}$$

or in the case of a circular tube of radius r,

$$a\left\{1-\frac{1}{r}\sqrt{\left(\frac{\mu}{2n\rho}\right)}\right\}\quad\text{........................(12).}$$

The result expressed in (12) was first obtained by Helmholtz.

348[1]. In the investigation of Kirchhoff[2], to which we now proceed, account is taken not only of viscosity but of the equally important effects arising from the generation of heat and its communication by conduction to and from the solid walls of a narrow tube.

The square of the motion being neglected, the "equation of continuity" (3) § 237 is

$$\frac{ds}{dt}+\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0\quad\text{.................. (1);}$$

so that the dynamical equations (13) § 345 may be written in the form

$$\frac{du}{dt}+\frac{1}{\rho_0}\frac{dp}{dx}=\frac{\mu}{\rho_0}\nabla^2u+\frac{\mu}{3\rho_0}\frac{d^2s}{dx\,dt}\quad\text{.............. (2).}$$

The thermal questions involved have already been considered in § 247. By equation (4)

$$\frac{d\theta}{dt}=\beta\frac{ds}{dt}+\nu\nabla^2\theta\quad\text{.................... (3),}$$

where ν is a constant representing the thermometric conductivity.

By (3) § 247

$$p/\rho_0=b^2(1+s+\alpha\theta)\quad\text{.....................(4),}$$

in which b denotes Newton's value of the velocity of sound, viz. $\sqrt{(p_0/\rho_0)}$. If we denote Laplace's value for the velocity by a,

$$a^2/b^2=\gamma=1+\alpha\beta\quad\text{.......................(5),}$$

so that $$\beta=(a^2-b^2)/b^2\alpha\quad\text{........................ (6).}$$

[1] This and the following §§ appear for the first time in the second edition. The first edition closed with § 348, there devoted to the question of dynamical similarity.

[2] *Pogg. Ann.* vol. cxxxiv., p. 177, 1868.

It will simplify the equations if we introduce a new symbol θ' in place of θ, connected with it by the relation $\theta' = \theta/\beta$. Thus (3) becomes

$$\frac{d\theta'}{dt} - \frac{ds}{dt} = \nu \nabla^2 \theta \qquad (7),$$

and the typical equation (2) may be written

$$\frac{du}{dt} + b^2 \frac{ds}{dx} + (a^2 - b^2)\frac{d\theta'}{dx} = \mu' \nabla^2 u - \mu'' \frac{d^2 s}{dx\,dt} \qquad (8),$$

where μ' is equal to μ/ρ_0. μ'' represents a second constant, whose value according to Stokes' theory is $\frac{1}{3}\mu'$. This relation is in accordance with Maxwell's kinetic theory, which on the introduction of more special suppositions further gives

$$\nu = \tfrac{5}{2}\mu' \qquad (9).$$

In any case μ', μ'', ν may be regarded as being of the same order of magnitude.

We will now, following Kirchhoff closely, introduce the supposition that the variables u, v, w, s, θ' are functions of the time on account only of the factor e^{ht}, where h is a constant to be afterwards taken as imaginary. Differentiations with respect to t are then represented by the insertion of the factor h, and the equations become

$$du/dx + dv/dy + dw/dz + hs = 0 \qquad (10),$$

$$\left.\begin{aligned} hu - \mu'\nabla^2 u &= -\,dP/dx \\ hv - \mu'\nabla^2 v &= -\,dP/dy \\ hw - \mu'\nabla^2 w &= -\,dP/dz \end{aligned}\right\} \qquad (11),$$

$$P = (b^2 + h\mu'')\, s + (a^2 - b^2)\,\theta' \qquad (12),$$

$$s = \theta' - (\nu/h)\,\nabla^2\theta' \qquad (13).$$

By (13), if s be eliminated, (12) and (10) become

$$P = (a^2 + h\mu'')\,\theta' - \frac{\nu}{h}(b^2 + h\mu'')\,\nabla^2\theta' \qquad (14),$$

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} + h\theta' - \nu\nabla^2\theta' = 0 \qquad (15).$$

By differentiation of equations (11) with respect to x, y, z, with subsequent addition and use of (14), (15), we find as the equation in θ'

$$h^2\theta' - \{a^2 + h(\mu' + \mu'' + \nu)\}\,\nabla^2\theta' + \frac{\nu}{h}\{b^2 + h(\mu' + \mu'')\}\,\nabla^4\theta' = 0 \qquad (16).$$

A solution of (16) may be obtained in the form

$$\theta' = A_1 Q_1 + A_2 Q_2 \quad\text{.................... (17)},$$

where Q_1, Q_2 are functions satisfying respectively

$$\nabla^2 Q_1 = \lambda_1 Q_1, \qquad \nabla^2 Q_2 = \lambda_2 Q_2 \quad\text{............ (18)},$$

λ_1, λ_2 being the roots of

$$h^2 - \{a^2 + h(\mu' + \mu'' + \nu)\}\lambda + \frac{\nu}{h}\{b^2 + h(\mu' + \mu'')\}\lambda^2 = 0 \text{...(19)},$$

while A_1, A_2 denote arbitrary constants.

In correspondence with this value of θ', particular solutions of equations (11) are obtained by equating u, v, w to the differential coefficients of

$$B_1 Q_1 + B_2 Q_2,$$

taken with respect to x, y, z. The relation of the constants B_1, B_2 to A_1, A_2 appears at once from (15), which gives

$$\nabla^2(B_1 Q_1 + B_2 Q_2) + (h - \nu\nabla^2)(A_1 Q_1 + A_2 Q_2) = 0,$$

so that by (18)

$$B_1 = A_1\left(\nu - \frac{h}{\lambda_1}\right), \qquad B_2 = A_2\left(\nu - \frac{h}{\lambda_2}\right) \text{.........(20)}.$$

More general solutions may be obtained by addition to u, v, w respectively of u', v', w', where u', v', w' satisfy

$$\nabla^2 u' = \frac{h}{\mu'}u', \qquad \nabla^2 v' = \frac{h}{\mu'}v', \qquad \nabla^2 w' = \frac{h}{\mu'}w' \text{ ... (21)}.$$

Thus

$$\left.\begin{aligned} u &= u' + B_1 dQ_1/dx + B_2\, dQ_2/dx \\ v &= v' + B_1 dQ_1/dy + B_2\, dQ_2/dy \\ w &= w' + B_1 dQ_1/dz + B_2\, dQ_2/dz \end{aligned}\right\} \text{............ (22)},$$

where B_1, B_2 have the values above given.

By substitution in (15) of the values of u, v, w specified in (22) it appears that

$$\frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz} = 0 \text{ .... (23)}.$$

349. These results are first applied by Kirchhoff to the case of plane waves, supposed to be propagated in infinite space in the direction of $+x$. Thus v' and w' vanish, while u', Q_1, Q_2 are independent of y and z. It follows from (23) § 348 that u' also vanishes. The equations for Q_1 and Q_2 are

$$d^2 Q_1/dx^2 = \lambda_1 Q_1, \qquad d^2 Q_2/dx^2 = \lambda_2 Q_2 \text{ .... (1)};$$

so that we may take

$$Q_1 = e^{-x\sqrt{\lambda_1}}, \qquad Q_2 = e^{-x\sqrt{\lambda_2}} \quad\text{...............}(2),$$

where the signs of the square roots are to be so chosen that the real parts are positive. Accordingly

$$u = A_1\lambda_1^{\frac{1}{2}}\left(\frac{h}{\lambda_1} - \nu\right)e^{-x\sqrt{\lambda_1}} + A_2\lambda_2^{\frac{1}{2}}\left(\frac{h}{\lambda_2} - \nu\right)e^{-x\sqrt{\lambda_2}} \quad\text{......}(3),$$

$$\theta' = A_1e^{-x\sqrt{\lambda_1}} + A_2e^{-x\sqrt{\lambda_2}} \quad\text{..................}(4),$$

in which the constants A_1, A_2 may be regarded as determined by the values of u and θ' when $x = 0$.

The solution, as expressed by (3), (4), is too general for our present purpose, providing as it does for arbitrary communication of heat at $x = 0$. From the quadratic in λ, (19) § 348, we see that if μ', μ'', ν be regarded as small quantities, one of the values of λ, say λ_1, is approximately equal to h^2/a^2, while the other λ_2 is very great. The solution which we require is that corresponding to λ_1 simply. The second approximation to λ_1 is by (19) § 348

$$\lambda_1 = \frac{h^2}{a^2 + h(\mu' + \mu'' + \nu)} + \frac{\nu b^2\lambda_1^2}{ha^2} = \frac{h^2}{a^2}\left\{1 - \frac{h(\mu' + \mu'' + \nu)}{a^2}\right\} + \frac{\nu b^2 h^3}{a^6};$$

so that $$\pm\sqrt{\lambda_1} = \frac{h}{a} - \frac{h^2}{2a^3}\{\mu' + \mu'' + \nu(1 - b^2/a^2)\} \quad\text{.........}(5).$$

If we now write in for h, we see that the typical solution is

$$u = e^{-m'x}\,e^{in(t - x/a)} \quad\text{........................}(6),$$

where $$m' = \frac{n^2}{2a^3}\left\{\mu' + \mu'' + \nu\left(1 - \frac{b^2}{a^2}\right)\right\} \quad\text{..............}(7).$$

In (6) an arbitrary multiplier and an arbitrary addition to t may, as usual, be introduced; and, if desired, the solution may be realized by omitting the imaginary part.

These results are in harmony with those already obtained for particular cases. Thus, if $\nu = 0$, (7) gives

$$m' = \frac{n^2}{2a^3}(\mu' + \mu''),$$

in agreement with (7) § 346, where

$$\mu'' = \tfrac{1}{3}\mu' = \tfrac{1}{3}\mu/\rho.$$

On the other hand if viscosity be left out of account, so that $\mu' = \mu'' = 0$, we fall back upon (18) § 247. It is unnecessary to add anything to the discussions already given.

In the case of spherical waves, propagated in the direction of $+r$, Kirchhoff finds in like manner as the expression for the radial velocity

$$\frac{d}{dr}\frac{e^{-m'r}}{r}\cdot e^{in(t-r/a)} \qquad\text{(8)},$$

where m' has the same value (7) as before.

350. We will now pass on to the more important problem and suppose that the air is contained in a cylindrical tube of circular section, and that the motion is symmetrical with respect to the axis of x. If $y^2+z^2=r^2$, and

$$v=q\cdot y/r, \qquad w=q\cdot z/r,$$
$$v'=q'\cdot y/r, \qquad w'=q'\cdot z/r,$$

then u, u', q, q', Q_1, Q_2 are to be regarded as functions of x and r. We suppose further that as functions of x these quantities are proportional to e^{mx}, where m is a complex constant to be determined. The equations (18) § 348 for Q_1, Q_2 become

$$\frac{d^2Q_1}{dr^2}+\frac{1}{r}\frac{dQ_1}{dr}=(\lambda_1-m^2)\,Q_1 \qquad\text{(1)},$$

$$\frac{d^2Q_2}{dr^2}+\frac{1}{r}\frac{dQ_2}{dr}=(\lambda_2-m^2)\,Q_2 \qquad\text{(2)}.$$

For u', q' equations (21), (23) give

$$\frac{d^2u'}{dr^2}+\frac{1}{r}\frac{du'}{dr}=\left(\frac{h}{\mu'}-m^2\right)u' \qquad\text{(3)},$$

$$\frac{d^2q'}{dr^2}+\frac{1}{r}\frac{dq'}{dr}-\frac{q'}{r^2}=\left(\frac{h}{\mu'}-m^2\right)q' \qquad\text{(4)},$$

$$mu'+\frac{dq'}{dr}+\frac{q'}{r}=0 \qquad\text{(5)}.$$

These three equations are satisfied if u' be determined by means of the first, and q' is chosen so that

$$q'=-\frac{m}{h/\mu'-m^2}\frac{du'}{dr} \qquad\text{(6)},$$

a relation obtained by subtracting from (4) the result of differentiating (5) with respect to r. The solution of (3) may be written $u'=AQ$, in which A is a constant, and Q a function of r satisfying

$$\frac{d^2Q}{dr^2}+\frac{1}{r}\frac{dQ}{dr}=\left(\frac{h}{\mu'}-m^2\right)Q \qquad\text{(7)}.$$

Thus, by (20), (22) § 348,

$$u = AQ - A_1 m\left(\frac{h}{\lambda_1} - \nu\right) Q_1 - A_2 m\left(\frac{h}{\lambda_2} - \nu\right) Q_2 \text{ ...... } (8),$$

$$q = -A\frac{m}{h/\mu' - m^2}\frac{dQ}{dr} - A_1\left(\frac{h}{\lambda_1} - \nu\right)\frac{dQ_1}{dr} - A_2\left(\frac{h}{\lambda_2} - \nu\right)\frac{dQ_2}{dr} \text{...} (9),$$

$$\theta' = A_1 Q_1 + A_2 Q_2 \text{ .... } (10).$$

On the walls of the tube u, q, θ' must satisfy certain conditions. It will here be supposed that there is neither motion of the gas nor change of temperature; so that when r has a value equal to the radius of the tube, u, q, θ' vanish. The condition of which we are in search is thus expressed by the evanescence of the determinant of (8), (9), (10), viz.:

$$\frac{m^2 h}{h/\mu' - m^2}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\frac{d \log Q}{dr} + \left(\frac{h}{\lambda_1} - \nu\right)\frac{d \log Q_1}{dr} - \left(\frac{h}{\lambda_2} - \nu\right)\frac{d \log Q_2}{dr} = 0 \text{ ...... } (11).$$

The three functions Q, Q_1, Q_2, which are required to remain finite when $r = 0$, are Bessel's functions of order zero (§ 200), so that we may write in the usual notation

$$\left.\begin{aligned} Q &= J_0\{r\sqrt{(m^2 - h/\mu')}\} \\ Q_1 &= J_0\{r\sqrt{(m^2 - \lambda_1)}\} \\ Q_2 &= J_0\{r\sqrt{(m^2 - \lambda_2)}\} \end{aligned}\right\} \text{ .... } (12).$$

In equation (11) the values of λ_1, λ_2 are independent of r, being determined by (19) § 348. In the application to air under normal conditions μ', μ'', ν may be regarded as small, and we have approximately

$$\lambda_1 = h^2/a^2, \qquad \lambda_2 = ha^2/\nu b^2 \text{ .... } (13).$$

A second approximation to the value of λ_1 has already been given in (5) § 349. It is here assumed that the velocity of propagation of viscous effects of the pitch in question, viz. $\sqrt{(2\mu' n)}$, § 347, is small compared with that of sound, so that $in\mu'/a^2$, or $h\mu'/a^2$, is a small quantity.

In interpreting the solution there are two extreme cases worthy of special notice. The first of these, which is that considered by Kirchhoff, arises when μ', μ'', ν are treated as very small, so small that the layer of gas immediately affected by the walls of the tube is but an insignificant fraction of the whole contents. When μ' &c. vanish, we have

$$\lambda_1 = h^2/a^2, \qquad m^2 = h^2/a^2,$$

so that $r\sqrt{(m^2-\lambda_1)}$ is here to be regarded as small. On the other hand $r\sqrt{(m^2-h/\mu')}$, $r\sqrt{(m^2-\lambda_2)}$ are large.

The value of $J_0(z)$, when z is small, is given by the ascending series (5) § 200; from which it follows at once that

$$d\log J_0(z)/dz = -\tfrac{1}{2}z.$$

When z is very large and such that its imaginary part is positive, (10) § 200 gives

$$d\log J_0(z)/dz = -\tan(z-\tfrac{1}{4}\pi) = -i.$$

Thus, if we retain only the terms of highest order,

$$\left.\begin{aligned} d\log Q/dr &= \sqrt{(h/\mu')} \\ d\log Q_1/dr &= \tfrac{1}{2}r(\lambda_1-m^2) \\ d\log Q_2/dr &= \sqrt{(ha^2/\nu b^2)} \end{aligned}\right\} \quad\text{..............} (14).$$

Using these in (11) with the approximate values of λ_1, λ_2 from (13), we find

$$m^2 = \frac{h^2}{a^2}\left(1+\frac{2\gamma'}{r\sqrt{h}}\right) \quad\text{....................} (15),$$

where $$\gamma' = \sqrt{\mu'} + (a/b - b/a)\sqrt{\nu} \quad\text{.................}(16),$$

and the sign of $\sqrt{h}$ is to be so chosen that the real part is positive.

We now write

$$h = ni \quad\text{...........................} (17),$$

so that the frequency is $n/2\pi$. Thus

$$\sqrt{h} = \sqrt{(\tfrac{1}{2}n)}\,.\,(1+i) \quad\text{......................} (18);$$

and $$m = \pm(m'+im'') \quad\text{.....................}(19),$$

where by (15)

$$m' = \frac{\sqrt{n}\,.\,\gamma'}{\sqrt{2}\,.\,ar}, \qquad m'' = \frac{n}{a} + \frac{\sqrt{n}\,.\,\gamma'}{\sqrt{2}\,.\,ar} \quad\text{...........}(20).$$

If we restore the hitherto suppressed factors dependent upon x and t, we have

$$u = BRe^{ht+mx}, \qquad q = BR'e^{ht+mx}, \qquad \theta' = BR''e^{ht+mx},$$

where B is an arbitrary constant, and R, R', R'' are certain functions of r, which vanish when r is equated to the radius of the tube, and which for points lying at a finite distance from the walls assume the values

$$R = 1, \qquad R' = 0, \qquad R'' = -1/a.$$

The realized solution for u, applicable at points which lie at a finite distance from the walls, may be written

$$u = C_1 e^{m'x} \sin(nt + m''x + \delta_1) + C_2 e^{-m'x} \sin(nt - m''x + \delta_2) \ldots (21),$$

where C_1, C_2, δ_1, δ_2 denote four real arbitrary constants. Accordingly m' determines the attenuation which the waves suffer in their progress, and m'' determines the velocity of propagation. This velocity is

$$n/m'' = a\left\{1 - \frac{\gamma'}{\sqrt{(2n)} \cdot r}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (22),$$

in harmony with (12) § 347.

The diminution of the velocity of sound in narrow tubes, as indicated by the wave-length of stationary vibrations, was observed by Kundt (§ 260), and has been specially investigated by Schneebeli[1] and A. Seebeck[2]. From their experiments it appears that the diminution of velocity varies as r^{-1}, in accordance with (22), but that, when n varies, it is proportional rather to $n^{-\frac{3}{4}}$ than to $n^{-\frac{1}{2}}$. Since μ is independent of the density (ρ), the effect would be increased in rarefied air.

We will now turn to the consideration of another extreme case of equation (11). This arises when the tube is such that the layer immediately affected by the friction, instead of merely forming a thin coating to the walls, extends itself over the whole section, as must inevitably happen if the diameter be sufficiently reduced. Under these circumstances hr^2/μ' is a small, and not, as in the case treated by Kirchhoff, a large quantity, and the arguments of *all* the three functions in (12) are to be regarded as small.

One result of the investigation may be foreseen. When the diameter of the tube is very much reduced, the conduction of heat from the centre to the circumference of the column of air becomes more and more free. In the limit the temperature of the solid walls controls that of the included gas, and the expansions and rarefactions take place isothermally. Under these circumstances there is no dissipation due to conduction, and everything is the same as if no heat were developed at all. Consequently the coefficient of heat-conduction will not appear in the result, which

[1] *Pogg. Ann.* vol. CXXXVI. p. 296, 1869.
[2] *Pogg. Ann.* vol. CXXXIX. p. 104, 1870.

will involve, moreover, the Newtonian value (b) of the velocity of sound, and not that of Laplace (a).

When z is small,

$$J_0(z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 \cdot 4^2},$$

so that approximately

$$d \log J_0(z)/dz = -\tfrac{1}{2}z\,(1 + \tfrac{1}{8}z^2) \;\ldots\ldots\ldots\ldots\ldots (23).$$

When the results of the application of (23) to Q, Q_1, Q_2 are introduced into (11), the equation may be divided by $\frac{1}{2}r$, and the left-hand member will then consist of two parts, of which the first is independent of r and the second is proportional to r^2. The first part reduces itself without further approximations to $\nu\,(\lambda_2 - \lambda_1)$. For the second part the leading terms only need be retained. Thus with use of (13)

$$\nu\,(\lambda_2 - \lambda_1) - \frac{a^2 r^2}{8}\left\{\frac{m^2}{\mu'} + \frac{h^2(a^2 - b^2)}{\nu b^4}\right\} = 0,$$

whence

$$m^2 = \frac{8\,\mu' h}{b^2 r^2} - \frac{h^2 \mu'(a^2 - b^2)}{\nu b^4}.$$

The ratio of the second term to the first is of the order hr^2/ν, by supposition a small quantity, so that we are to take simply

$$m^2 = \frac{8\mu' h}{b^2 r^2} = \frac{8i\mu' n}{b^2 r^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24),$$

as the solution applicable under the supposed conditions.

Before leaving this question it may be worth while to consider briefly the corresponding problem in two dimensions, although it is of less importance than that of the circular tube treated by Kirchhoff. The analysis is a little simpler; but, as it follows practically the same course, we may content ourselves with a mere indication of the necessary changes. The motion is supposed to be independent of z and to take place between parallel walls at $y = \pm y_1$.

The equations (1) to (11) of the preceding investigation may be regarded as still applicable in the present problem, if we write v for q and y for r, with omission of the terms where r occurs in the denominator. The general solution of the equations corresponding to (1), (2), (7) contains two functions whose form is that of sines and cosines of multiples of y. But from (8), (9), (10) it is evident that the conditions of the problem at $y = 0$ require the absence of the sine function, so that in (12) we are simply to replace the function J_0 by the cosine.

In the case where μ' &c. are regarded as infinitely small we have as in (14), when $y = y_1$,

$$\left.\begin{array}{l} d \log Q/dy = \sqrt{(h/\mu')} \\ d \log Q_2/dy = \sqrt{(ha^2/\nu b^2)} \end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (25),$$

but in place of the second of equations (14)

$$d \log Q_1/dy = y_1(\lambda_1 - m^2) \quad\ldots\ldots\ldots\ldots\ldots\ldots (26).$$

When these values are substituted in (11), the resulting equation is unchanged, except that r is replaced by $2y_1$. The same substitution is to be made in (15), (20), (22). The latter gives for the velocity of sound

$$a\left\{1 - \frac{1}{2y_1}\frac{\gamma'}{\sqrt{(2n)}}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (27).$$

It is worth notice that (27) is what (11) § 347 becomes for this case when we replace $\sqrt{\mu'}$ by γ'; and we may perhaps infer that the same change is sufficient to render that equation applicable to a section of any form when thermal effects are to be taken into account.

In the second extreme case where the distance between the walls $(2y_1)$ is so small that hy_1^2/ν is to be neglected, we have in place of (23)

$$d \log \cos z/dz = -z(1 + \tfrac{1}{3}z^2) \quad\ldots\ldots\ldots\ldots\ldots\ldots (28).$$

The equations following are thus adapted to our present purpose if we replace $\frac{1}{8}r^2$ by $\frac{1}{3}y_1^2$. The analogue of (24) is accordingly

$$m^2 = \frac{3\mu' h}{b^2 y_1^2} = \frac{3i\mu' n}{b^2 y_1^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots (29).$$

351. The results of § 350 have an important bearing upon the explanation of the behaviour of porous bodies in relation to sound. Tyndall has shewn that in many cases sound penetrates such bodies more freely than would have been expected, although it is reflected from thin layers of continuous solid matter. On the other hand a hay-stack seems to form a very perfect obstacle. It is probable that porous walls give a diminished reflection, so that within a building so bounded resonance is less prolonged than if the walls were formed of continuous matter.

When we inquire into the mechanical question, it is evident that sound is not destroyed by obstacles as such. In the absence of dissipative forces, what is not transmitted must be reflected. Destruction depends upon viscosity and upon conduction of heat; but the influence of these agencies is enormously augmented by the contact of solid matter exposing a large surface. At such a surface the tangential as well as the normal motion is hindered, and a passage of heat to and fro takes place, as the neighbouring air is heated and cooled during its condensations and rarefactions. With such rapidity of alternation as we are concerned with in the case of audible sounds, these influences extend to only a very thin layer of the air and of the solid, and are thus greatly favoured by a fine state of division.

Let us conceive an otherwise continuous wall, presenting a flat face, to be perforated by a great number of similar narrow channels, uniformly distributed, and bounded by surfaces everywhere perpendicular to the face of the wall. If the channels be sufficiently numerous, the transition, when sound impinges, from simple plane waves on the outside to the state on the inside of aerial vibration corresponding to the interior of a channel of unlimited length, occupies a space which is small relatively to the wave-length of the vibration, and then the connection between the condition of things inside and outside admits of simple expression.

Considering first the interior of one of the channels, and taking the axis of x parallel to the axis of the channel, we suppose that as functions of x the velocity components u, v, w and the condensation s are proportional to e^{ikx}, while as functions of t everything is proportional to e^{int}, n being real. The relationship between k and n depends upon the nature of the gas and upon the size and form of the channel, and has been determined for certain important cases in § 350, ik being there denoted by m. Supposing it to be known, we will go on to shew how the problem of reflection is to be dealt with.

For this purpose consider the equation of continuity as integrated over the cross-section σ of the channel. Since the walls of the channel are impenetrable,

$$\frac{d}{dt}\iint s\,d\sigma+\frac{d}{dx}\iint u\,d\sigma=0,$$

so that
$$n\iint s\,d\sigma+k\iint u\,d\sigma=0\;\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This equation is applicable at points distant from the open end more than several diameters of the channel.

Taking now the origin of x at the face of the wall, we have to form corresponding expressions for the waves outside; and we may there neglect the effects of viscosity of conduction of heat. If a be the velocity of sound in the open, and $k_0 = n/a$, we may write for waves incident and reflected perpendicularly

$$s = (e^{ik_0x} + Be^{-ik_0x})\, e^{int} \quad\text{.....................}\ (2),$$

$$u = a\,(-e^{ik_0x} + Be^{-ik_0x})\, e^{int} \quad\text{.................}\ (3);$$

so that the incident wave is

$$s = e^{i(nt+k_0x)} \quad\text{..........................}\ (4),$$

or, on throwing away the imaginary part,

$$s = \cos(nt + k_0x) \quad\text{......................}\ (5).$$

These expressions are applicable when x exceeds a moderate multiple of the distance between the channels. Close up to the face the motion will be more complicated; but we have no need to investigate it in detail. The ratio of u and s at a place near the wall is given with sufficient accuracy by putting $x = 0$ in (2) and (3),

$$\frac{u}{as} = \frac{B-1}{B+1} \quad\text{..........................}\ (6).$$

We now assume that a space, defined by parallel planes one on either side of $x = 0$, may be taken so thin relatively to the wave-length that the mean pressures are sensibly the same at the two boundaries, and that the flow into the space at one boundary is sensibly equal to the flow out of the space at the other boundary, and yet broad enough relatively to the transverse dimensions of the channels to allow the application of (6) at one bounding plane and of (1) at the other bounding plane. The equality of flow does not imply an equality of mean velocities, since the areas concerned are different. The mean velocities will be inversely proportional to the corresponding areas—that is in the ratio $\sigma : \sigma + \sigma'$, if σ' denote the area of the unperforated part of the wall corresponding to each channel. By (1) and (6) the connection between the inside and outside motion is expressed by

$$-\frac{n}{k}\sigma = \frac{a\,(B-1)}{B+1}\,(\sigma + \sigma').$$

We will denote the ratio of the unperforated to the perforated parts of the wall by g, so that $g = \sigma'/\sigma$. Thus

$$\frac{1-B}{1+B} = \frac{k_0}{k(1+g)} \quad\text{.......................(7).}$$

If $g=0$, $k=k_0$, that is, if the wall be abolished, or if it be reduced to infinitely thin partitions between the channels while at the same time the dissipative effects are neglected, there is no reflection. If there are no perforations ($g=\infty$), then $B=1$, signifying total reflection. Generally in place of (7) we may write

$$B = \frac{k(1+g)-k_0}{k(1+g)+k_0} \quad\text{.......................(8),}$$

which is the solution of the problem proposed. It is understood that waves which have once entered the wall do not return. When dissipative forces act, this condition may always be satisfied by supposing the channels to be long enough. The necessary length of channel, or thickness of wall, will depend upon the properties of the gas and upon the size and shape of the channels. Even in the absence of dissipative forces there must be reflection, except in the extreme case $g=0$. Putting $k=k_0$ in (8), we have

$$B = \frac{g}{2+g} \quad\text{.......................(9).}$$

If $g=1$, that is if half the wall be cut away, $B=\frac{1}{3}$, $B^2=\frac{1}{9}$, so that the reflection is but small. If the channels be circular and arranged in square order as close as possible to one another, $g=(4-\pi)/\pi$, whence $B=\cdot 121$, $B^2=\cdot 015$, nearly all the motion being transmitted.

If the channels be circular in section and so small that nr^2/ν may be neglected, we have, (24) § 350,

$$-k^2 = m^2 = \frac{8i\mu' n}{b^2 r^2} \quad\text{.......................(10);}$$

so that (21) the wave propagated into a channel is proportional to

$$e^{m'x}\sin(nt+m''x+\delta_1) \quad\text{.......................(11),}$$

where
$$m' = m'' = \frac{2\sqrt{(\mu' n)}}{br} = \frac{2\sqrt{(\mu'\gamma n)}}{ar} \quad\text{.............(12),}$$

γ being the ratio of specific heats § 246.

To take a numerical example, suppose that the pitch is 256, so that $n = 2\pi \times 256$. The value of μ' for air is ·16 C.G.S., and that of ν is ·256. If we take $r = \frac{1}{1000}$ cm., we find $nr^2/8\nu$ equal to about $\frac{1}{1000}$. If r were ten times as great, the approximation in (10) would perhaps still be sufficient.

From (12), if $n = 2\pi \times 256$,

$$m' = m'' = {\cdot}00115/r \qquad (13);$$

so that, if $r = \frac{1}{1000}$ cm., $m' = 1{\cdot}15$. In this case the amplitude is reduced in the ratio $e : 1$ in passing over the distance $1/m'$, that is about one centimetre. The distance penetrated is proportional to the radius of the channel.

The amplitude of the reflected wave is by (8)

$$B = \frac{m'(1+g)(1-i) - k_0}{m'(1+g)(1-i) + k_0},$$

or, as we may write it,

$$B = \frac{M - 1 - iM}{M + 1 - iM} \qquad (14),$$

where

$$M = (1+g)\, m'/k_0 \qquad (15).$$

If I be the intensity of the reflected sound, that of the incident sound being unity,

$$I = \frac{2M^2 - 2M + 1}{2M^2 + 2M + 1} \qquad (16).$$

The intensity of the intromitted sound is given by

$$1 - I = \frac{4M}{2M^2 + 2M + 1} \qquad (17).$$

By (12), (15)

$$M = \frac{2(1+g)\sqrt{(\mu'\gamma)}}{r\sqrt{n}} \qquad (18).$$

If we suppose $r = \frac{1}{1000}$ cm., and $g = 1$, we shall have a wall of pretty close texture. In this case by (18), $M = 47{\cdot}4$ and $1 - I = {\cdot}0412$. A loss of 4 per cent. may not appear to be important; but we must remember that in prolonged resonance we are concerned with the accumulated effect of a large number of reflections, so that a comparatively small loss in a single reflection may well be material. The thickness of the porous layer necessary to produce this effect is less than one centimetre.

Again, suppose $r = \frac{1}{100}$ cm., $g = 1$. We find $M = 4{\cdot}74$, $1 - I = {\cdot}342$; and the necessary thickness would be less than 10 centimetres.

If r be much greater than $\frac{1}{100}$ cm., the exchange of heat between the air and the sides of the channel is no longer sufficiently free to allow of the use of (24) § 350. When the diameter is so great that the thermal and viscous effects extend through only a small fraction of it, we have the case discussed by Kirchhoff (15) § 350. Here

$$k = \frac{n}{a}\left\{1 + \frac{\gamma'(1-i)}{r\sqrt{(2n)}}\right\} \quad\text{.................. (19),}$$

which value is to be substituted in (8). If for simplicity we put $g = 0$, we find

$$B = \frac{\gamma'(1-i)}{2r\sqrt{(2n)}} \quad\text{...................... (20),}$$

$$I = \gamma'^2/4r^2 n \quad\text{...................... (21).}$$

The supposition that $g = 0$ is, however, inconsistent with the circular section; and it is therefore preferable to use the solution corresponding to (27) § 350, applicable when the channels assume the form of narrow crevasses[1]. We have merely to replace r in (19), (20), (21) by $2y_1$, $2y_1$ being the width of a crevasse. The incident sound is absorbed more and more completely as the width of the channels increases; but at the same time a greater length of channel, or thickness of wall, becomes necessary in order to prevent a return from the further side. If $g = 0$, there is no theoretical limit to the absorption; and, as we have seen, a moderate value of g does not of itself entail more than a comparatively small reflection. A loosely compacted hay-stack would seem to be as effective an absorbent of sound as anything likely to be met with.

In large spaces bounded by non-porous walls, roof, and floor, and with few windows, a prolonged resonance seems inevitable. The mitigating influence of thick carpets in such cases is well known. The application of similar material to the walls and to the roof appears to offer the best chance of further improvement.

352. One of the most curious consequences of viscosity is the generation in certain cases of regular vortices. Of this an example, discovered by Dvořák, has already been mentioned in § 260. In

[1] It may be remarked that even in the two-dimensional problem the supposition $g = 0$ involves an infinite capacity for heat in the material composing the partitions.

a theoretically inviscid fluid no such effect could occur, § 240; and, even when viscosity enters, the phenomenon is one of the second order, dependent, that is, upon the *square* of the motion. Three problems of this kind have been treated by the author[1] on a former occasion, but here we must limit ourselves to Dvořák's phenomenon, further simplifying the question by taking the case of two dimensions and by neglecting the terms dependent upon the development and conduction of heat.

If we suppose that $p = a^2\rho$, and write s for $\log(\rho/\rho_0)$, the fundamental equations (12) § 345 are

$$a^2\frac{ds}{dx} = -\frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} + \mu'\nabla^2 u + \mu''\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}\right)\ldots(1),$$

with a corresponding equation for v, and the equation of continuity § 238

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{ds}{dt}+u\frac{ds}{dx}+v\frac{ds}{dy}=0\ldots\ldots\ldots\ldots\ldots(2).$$

Whatever may be the actual values of u and v, we may write

$$u = \frac{d\phi}{dx}+\frac{d\psi}{dy}, \qquad v = \frac{d\phi}{dy}-\frac{d\psi}{dx} \;\ldots\ldots\ldots\ldots (3),$$

in which

$$\nabla^2\phi = \frac{du}{dx}+\frac{dv}{dy}, \qquad \nabla^2\psi = \frac{du}{dy}-\frac{dv}{dx} \;\ldots\ldots\ldots (4).$$

From (1), (2)

$$\left(a^2+\mu''\frac{d}{dt}\right)\frac{ds}{dx} = -\frac{du}{dt}+\mu'\nabla^2 u$$
$$- u\frac{du}{dx} - v\frac{du}{dy} - \mu''\frac{d}{dx}\left(u\frac{ds}{dx}+v\frac{ds}{dy}\right)\ldots\ldots(5),$$

$$\left(a^2+\mu''\frac{d}{dt}\right)\frac{ds}{dy} = -\frac{dv}{dt}+\mu'\nabla^2 v$$
$$- u\frac{dv}{dx} - v\frac{dv}{dy} - \mu''\frac{d}{dy}\left(u\frac{ds}{dx}+v\frac{ds}{dy}\right)\ldots\ldots(6).$$

Again, from (5), (6),

$$\left(a^2+\mu'\frac{d}{dt}+\mu''\frac{d}{dt}\right)\nabla^2 s - \frac{d^2 s}{dt^2} = \frac{d}{dt}\left(u\frac{ds}{dx}+v\frac{ds}{dy}\right)$$
$$-(\mu'+\mu'')\nabla^2\left(u\frac{ds}{dx}+v\frac{ds}{dy}\right) - \frac{d}{dx}\left(u\frac{du}{dx}+v\frac{du}{dy}\right) - \frac{d}{dy}\left(u\frac{dv}{dx}+v\frac{dv}{dy}\right)$$
$$\ldots\ldots\ldots(7).$$

[1] On the Circulation of Air observed in Kundt's Tubes, and on some allied Acoustical Problems. *Phil. Trans.* vol. 175, p. 1, 1883.

For the first approximation the terms of the second order in u, v, s are to be omitted. If we assume that as functions of t all the periodic quantities are proportional to e^{int}, and write q for $a^2 + in\mu' + in\mu''$, (7) becomes

$$q\nabla^2 s + n^2 s = 0 \qquad (8).$$

Now by (2), (4) $\quad \nabla^2\phi = -ins = i(q/n)\nabla^2 s,$

so that

$$\phi = iqs/n \qquad (9)[1],$$

and

$$u = \frac{iq}{n}\frac{ds}{dx} + \frac{d\psi}{dy}, \qquad v = \frac{iq}{n}\frac{ds}{dy} - \frac{d\psi}{dx} \qquad (10).$$

Substituting in (5), (6), with omission of the terms of the second order, we get in view of (8),

$$(\mu'\nabla^2 - in)\frac{d\psi}{dy} = 0, \qquad (\mu'\nabla^2 - in)\frac{d\psi}{dx} = 0,$$

whence

$$(\mu'\nabla^2 - in)\,\psi = 0 \qquad (11).$$

If we eliminate s directly from equations (1), we get

$$\left(\mu'\nabla^4 - \frac{d}{dt}\nabla^2\right)\psi = \frac{d}{dy}\left(u\frac{du}{dx} + v\frac{du}{dy}\right) - \frac{d}{dx}\left(u\frac{dv}{dx} + v\frac{dv}{dy}\right)$$

$$= \frac{d}{dy}(v\nabla^2\psi) + \frac{d}{dx}(u\nabla^2\psi)$$

$$= \left(\frac{du}{dx} + \frac{dv}{dy}\right)\nabla^2\psi + u\frac{d\nabla^2\psi}{dx} + v\frac{d\nabla^2\psi}{dy} \qquad (12).$$

If we now assume that as functions of x the quantities s, ψ, &c. are proportional to e^{ikx}, equations (8), (11) may be written

$$(d^2/dy^2 - k''^2)\,s = 0 \qquad (13),$$

where $k''^2 = k^2 - n^2/q$,

$$(d^2/dy^2 - k'^2)\,\psi = 0 \qquad (14),$$

where $k'^2 = k^2 + in/\mu'$.

If the origin for y be in the middle between the two parallel bounding planes, s must be an even function of y, and ψ must be an odd function. Thus we may write

$$s = A\cosh k''y \,.\, e^{int} \,.\, e^{ikx}, \qquad \psi = B\sinh k'y \,.\, e^{int} \,.\, e^{ikx} \qquad (15),$$

$$\left.\begin{aligned} u &= (-kq/n \,.\, A\cosh k''y + k'B\cosh k'y)\, e^{int} \,.\, e^{ikx} \\ v &= (iqk''/n \,.\, A\sinh k''y - ikB\sinh k'y)\, e^{int} \,.\, e^{ikx} \end{aligned}\right\} \qquad (16).$$

[1] It is unnecessary to add a complementary function ϕ' satisfying $\nabla^2\phi' = 0$, for the motion corresponding thereto may be regarded as covered by ψ.

If the fixed walls are situated at $y = \pm y_1$, u and v must vanish for these values of y. Eliminating from (16) the ratio of A to B, we get as the equation for determining k,

$$k^2 \tanh k'y_1 = k'k'' \tanh k''y_1 \quad \text{..............} (17),$$

where k', k'' are the functions of k above defined. Equation (17) may be regarded as a modified and simplified form of (11) § 350, modified on account of the change from symmetry about an axis to two dimensions, and simplified by the omission of the thermal terms represented by ν. The comparison is readily made. Since $\lambda_2 = \infty$, the third term in (11), involving Q_2, disappears altogether, and then λ_1^{-1} divides out. In (11), (12) r is to be replaced by y, and J_0 by cosine, as has already been explained. Further,

$$m^2 = -k^2, \quad h = in.$$

We now introduce further approximations dependent upon the assumption that the direct influence of viscosity extends through a layer whose thickness is a small fraction only of y_1. In this case $k^2 = n^2/a^2$ nearly, so that $k''y_1$ is a small quantity and $k'y_1$ is a large quantity, and we may take

$$\tanh k'y_1 = \pm 1, \qquad \tanh k''y_1 = \pm k''y_1.$$

Equation (17) then becomes

$$k^2 = k'k''^2 y_1 \quad \text{........................} (18),$$

or, if we introduce the values of k', k'' from (13), (14),

$$k^2 = y_1 (k^2 - n^2/q)(k^2 + in/\mu')^{\frac{1}{2}}.$$

Thus approximately

$$k = \pm \frac{n}{a} \left\{ 1 + \frac{1 - i}{2y_1 \sqrt{(2n/\mu')}} \right\} \quad \text{..............} \mathbf{(19)},$$

in agreement with the result already indicated in § 350.

In taking approximate forms for (16) we must specify which half of the symmetrical motion we contemplate. If we choose that for which y is *negative*, we replace $\cosh k'y$ and $\sinh k'y$ by $\frac{1}{2} e^{-k'y}$. For $\cosh k''y$ we may write unity, and for $\sinh k''y$ simply $k''y$. If we change the arbitrary multiplier so that the maximum value of u is u_0 and for the present take u_0 equal to unity, we have

$$\left.\begin{aligned} u &= (-1 + e^{-k'(y+y_1)})\, e^{ikx}\, e^{int} \\ v &= ik/k' \,.\, (y/y_1 + e^{-k'(y+y_1)})\, e^{ikx}\, e^{int} \end{aligned}\right\} \quad \text{...........} (20),$$

in which, of course, u and v vanish when $y = -y_1$.

If in (20) we change k into $-k$ and then take the mean, we obtain

$$\left.\begin{aligned} u &= (-1 + e^{-k'(y+y_1)}) \cos kx \, e^{int} \\ v &= -k/k' \,.\, (y/y_1 + e^{-k'(y+y_1)}) \sin kx \, e^{int} \end{aligned}\right\} \quad \text{.........(21).}$$

Although k is not absolutely a real quantity, we may consider it to be so with sufficient approximation for our purpose. We may also take in (14)

$$k' = \sqrt{(in/\mu')} = \beta(1+i) \quad \text{..................(22),}$$

if $\beta = \sqrt{(n/2\mu')}$. Using this approximation in (21), we get in terms of real quantities,

$$\left.\begin{aligned} u &= \cos kx \left[-\cos nt + e^{-\beta(y+y_1)} \cos\{nt - \beta(y+y_1)\}\right] \\ v &= -\frac{k \sin kx}{\beta\sqrt{2}} \left[\frac{y}{y_1} \cos(nt - \tfrac{1}{4}\pi) \right. \\ &\qquad \left. + e^{-\beta(y+y_1)} \cos\{nt - \tfrac{1}{4}\pi - \beta(y+y_1)\}\right] \end{aligned}\right\} \quad \text{...(23).}$$

It will shorten the expressions with which we have to deal if we measure y from the wall (on the negative side) instead of, as hitherto, from the plane of symmetry, for which purpose we must write y for $y + y_1$. Thus

$$\left.\begin{aligned} u_1 &= \cos kx \left[-\cos nt + e^{-\beta y} \cos(nt - \beta y)\right] \\ v_1 &= \frac{k \sin kx}{\beta\sqrt{2}} \left[\frac{y_1 - y}{y_1} \cos(nt - \tfrac{1}{4}\pi) - e^{-\beta y} \cos(nt - \tfrac{1}{4}\pi - \beta y)\right] \end{aligned}\right\} \quad \text{...(24),}$$

the subscripts indicating the order of the terms.

These are the values of the velocities when the square of the motion is neglected. In proceeding to a second approximation we require to form expressions for the right-hand members of (7) and (12), which for the purposes of the first approximation were neglected altogether. The additional terms dependent upon the square of the motion are partly independent of the time and partly of double frequency involving $2nt$. The latter are not of much interest, so that we shall confine ourselves to the non-periodic part. Further simplifications are admissible in virtue of the small thickness of the retarded layer in proportion to the width of the channel ($2y_1$) and still more in proportion to the wave-length (λ). Thus k/β is a small quantity and may usually be neglected.

From (24)

$$\nabla^2\psi_1 = \beta\sqrt{2}\,.\cos kx\, e^{-\beta y} \sin(nt - \tfrac{1}{4}\pi - \beta y) \text{......(25)},$$

$$du_1/dx + dv_1/dy = k \sin kx \cos nt \text{ .... (26)},$$

$$u_1 \frac{d\nabla^2\psi_1}{dx} + v_1 \frac{d\nabla^2\psi_1}{dy} = \tfrac{1}{2}k\beta \sin 2kx\, e^{-\beta y}(-\cos\beta y + e^{-\beta y}) + \text{terms in } 2nt \text{ ....(27)},$$

$$\left(\frac{du_1}{dx} + \frac{dv_1}{dy}\right)\nabla^2\psi_1 = -\tfrac{1}{4}k\beta \sin 2kx\, e^{-\beta y}(\sin\beta y + \cos\beta y) + \text{terms in } 2nt \text{ ....(28)}.$$

Thus for the non-periodic part of ψ of the second order, we have from (12)

$$\nabla^4\psi_2 = -\frac{k\beta}{4\mu'} \sin 2kx\, e^{-\beta y}\{\sin\beta y + 3\cos\beta y - 2e^{-\beta y}\} \text{...(29)}.$$

In this we identify ∇^4 with $(d/dy)^4$, so that

$$\psi_2 = \frac{k \sin 2kx\, e^{-\beta y}}{16\mu'\beta^3}\{\sin\beta y + 3\cos\beta y + \tfrac{1}{2}e^{-\beta y}\} \text{ ....(30)},$$

to which may be added a complementary function, satisfying $\nabla^4\psi_2 = 0$, of the form

$$\psi_2 = \frac{\sin 2kx}{16\mu'\beta^3}\{A \sinh 2k(y_1 - y) + B(y_1 - y)\cosh 2k(y_1 - y)\} \text{...(31)},$$

or, as we may take it approximately, if y_1 be small compared with λ,

$$\psi_2 = \frac{k \sin 2kx}{16\mu'\beta^3}\{A'(y_1 - y) + B'(y_1 - y)^3\} \text{ ....(32)}.$$

Equations (30), (32) give the non-periodic part of ψ of the second order.

The value of s to a second approximation would have to be investigated by means of (7). It will be composed of two parts, the first independent of t, the second a harmonic function of $2nt$. In calculating the part of $d\phi/dx$ independent of t from

$$\nabla^2\phi = -ds/dt - u\,ds/dx - v\,ds/dy,$$

we shall obtain nothing from ds/dt. In the remaining terms on the right-hand side it will be sufficient to employ the values u, v, s of the first approximation. From

$$ds/dt = -du/dx - dv/dy,$$

in conjunction with (26), we get

$$s = -u_0/a \,.\, \sin kx \, \sin nt,$$

whence $$d^2\phi_2/d(\beta y)^2 = ku_0^2/2a\beta^2 \,.\, \cos^2 kx \, e^{-\beta y} \sin \beta y.$$

From this it is easily seen that the part of u_2 resulting from $d\phi/dx$ in (3) is of order k^2/β^2 in comparison with the part (33) resulting from ψ_2, and may be omitted. Accordingly by (30), with introduction of the value of β and (in order to restore homogeneity) of u_0^2,

$$u_2 = -\frac{u_0^2 \sin 2kx \, e^{-\beta y}}{8a}\{4 \sin \beta y + 2 \cos \beta y + e^{-\beta y}\} \;\ldots (33),$$

$$v_2 = -\frac{2ku_0^2 \cos 2kx \, e^{-\beta y}}{8\beta a}\{\sin \beta y + 3 \cos \beta y + \tfrac{1}{2} e^{-\beta y}\} \ldots (34);$$

and from (32)

$$u_2 = -\frac{u_0^2 \sin 2kx}{8\beta a}\{A' + 3B'(y_1 - y)^2\} \;\ldots\ldots\ldots\ldots\ldots (35),$$

$$v_2 = -\frac{2ku_0^2 \cos 2kx}{8\beta a}\{A'(y_1 - y) + B'(y_1 - y)^3\} \ldots\ldots (36).$$

The complete value of the terms of the second order in u, v are given by addition of (33), (35) and of (34), (36). The constants A', B' are to be determined by the condition that these values vanish when $y = 0$. We thus obtain as the complete expression of the terms of the second order

$$u_2 = -\frac{u_0^2 \sin 2kx}{8a}\left\{e^{-\beta y}(4 \sin \beta y + 2 \cos \beta y + e^{-\beta y}) + \tfrac{3}{2} - \tfrac{9}{2}\frac{(y_1 - y)^2}{y_1^2}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots (37),$$

$$v_2 = -\frac{2ku_0^2 \cos 2kx}{8\beta a}\left\{e^{-\beta y}(\sin \beta y + 3 \cos \beta y + \tfrac{1}{2} e^{-\beta y})\right.$$
$$\left. + \tfrac{3}{2}\beta(y_1 - y) - \tfrac{3}{2}\beta\frac{(y_1 - y)^3}{y_1^2}\right\} \ldots (38).$$

Outside the thin film of air immediately influenced by the friction we may put $e^{-\beta y} = 0$, and then

$$u_2 = -\frac{3u_0^2 \sin 2kx}{16a}\left\{1 - \frac{3(y_1 - y)^2}{y_1^2}\right\} \ldots\ldots\ldots\ldots\ldots (39),$$

$$v_2 = -\frac{3u_0^2 \,.\, 2k \cos 2kx}{16a}\left\{y_1 - y - \frac{(y_1 - y)^3}{y_1^2}\right\} \ldots\ldots (40).$$

From (39) we see that u_2 changes sign as we pass from the boundary $y=0$ to the plane of symmetry $y=y_1$, the critical value of y being $y_1(1-\sqrt{\frac{1}{3}})$, or $\cdot 423 y_1$.

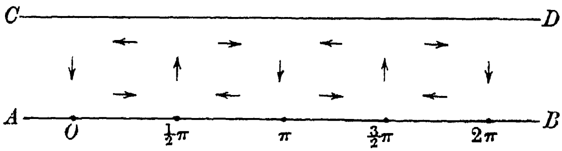
The value of u_1 from (24) corresponding to (39) is

$$u_1 = -u_0 \cos kx \cos nt \quad\text{.....................(41)},$$

so that the loops correspond to $kx=0, \pi, 2\pi, \ldots$, and the nodes correspond to $kx=\frac{1}{2}\pi, \frac{3}{2}\pi, \ldots$.

The steady motion represented by (39), (40) is of a very simple character. It consists of a series of vortices periodic with respect to x in the distance $\frac{1}{2}\lambda$. From (40) it appears that v is positive at the nodes and negative at the loops, vanishing of course in each case both at the wall $y=0$ and at the plane of symmetry $y=y_1$.

Fig. 65.



In the figure AB represents the wall, CD the plane of symmetry, and the directions of motion in the vortices are indicated by arrows. It is especially to be remarked that the velocity of the vortical motion is independent of μ', so that this effect is not to be obviated by taking the viscosity infinitely small. In that way the tendency to generate the vortices may indeed be diminished, but in the same proportion the maintenance of the vortices is facilitated, so that when the motion has reached a final state the vortices are as important with a small as with a large viscosity. The fact that when viscosity is neglected from the first no such vortices make their appearance in the solution shews what extreme care is required in dealing with problems relating to the behaviour of slightly viscous fluid in contact with solid bodies.

In estimating the mean motion to the second order there is another point to be considered which has not yet been touched upon. The values of u_1 and v_1 in (24) are, it is true, strictly periodic, but the same property does not attach to the motions thereby defined of the particles of the fluid. In our notation u is not the velocity of any individual particle of the fluid, but of the particle, whichever it may be, that at the moment under consideration occupies the position x, y, (§ 237). If $x+\xi$, $y+\eta$ define the actual position at time t of the particle whose mean position during several vibrations is (x, y), then the actual velocities of the particle at time t are, not u_1, v_1, but

$$u_1+\frac{du_1}{dx}\xi+\frac{du_1}{dy}\eta, \qquad v_1+\frac{dv_1}{dx}\xi+\frac{dv_1}{dy}\eta:$$

and thus the mean velocity parallel to x is not necessarily zero, but is equal to the mean value of

$$\xi\, du_1/dx+\eta\, du_1/dy \qquad (42),$$

in which again

$$\xi=\int u_1 dt, \qquad \eta=\int v_1 dt \qquad (43).$$

In the present case the mean value of (42) is

$$-u_0^2/4a\,.\sin 2kx\, e^{-\beta y}(e^{-\beta y}-\cos\beta y) \qquad (44),$$

which is to be regarded as an addition to (37). However, at a short distance from the wall (44) may be neglected, so that (39) remains adequate.

We have seen that the width of the direct current along the wall $y=0$ is $\cdot 423\, y_1$, and that of the return current, measured up to the plane of symmetry, is $\cdot 577\, y_1$. The ratio of these widths is not altered by the inclusion of the second half of the channel lying beyond the plane of symmetry; so that the direct current is distinctly narrower than the return current. This disproportion will be increased in the case of a tube of circular section. The point under consideration depends in fact only upon a complementary function analogous to (32), and is so simple that it may be worth while briefly to indicate the steps of the calculation.

The equation for ψ_2 is[1]

$$\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}-4k^2\right)^2\psi_2=0 \qquad (45);$$

but, if we suppose that the radius of the tube is small in comparison with λ, k^2 may be omitted. The general solution is

$$\psi_2=\{A+Br^2+B'r^2\log r+Cr^4\}\sin 2kx \qquad (46),$$

so that

$$u_2=d\psi_2/r\,dr=\{2B+B'(2\log r+1)+4Cr^2\}\sin 2kx \qquad (47),$$

[1] Stokes, *Trans. Camb. Phil. Soc.*, vol. IX. 1856; Basset's *Hydrodynamics*, § 485.

whence $B' = 0$, by the condition at $r = 0$. Again,

$$v_2 = -d\psi_2/rdx = -2k\{Ar^{-1} + Br + Cr^3\}\cos 2kx \quad \text{...(48)},$$

whence $A = 0$.

We may therefore take

$$\left.\begin{aligned} u_2 &= \{2B + 4Cr^2\}\sin 2kx \\ v_2 &= -2k\{Br + Cr^3\}\cos 2kx \end{aligned}\right\} \quad \text{..............(49)}.$$

If, as in (40), $v_2 = 0$, when $r = R$, $B + CR^2 = 0$, and

$$u_2 = 2C(2r^2 - R^2)\sin 2kx \quad \text{..................(50)}.$$

Thus u_2 vanishes, when

$$r = R/\sqrt{2} = \cdot 707 R, \qquad R - r = \cdot 293 R.$$

The direct current is thus limited to an annulus of thickness $\cdot 293 R$, the return current occupying the whole interior and having therefore a diameter of $2 \times \cdot 707 R$, or $1\cdot 414 R$.

CHAPTER XX.

CAPILLARITY.

353. THE subject of the present chapter is the behaviour of inviscid incompressible fluid vibrating under the action of gravity and capillary force, more especially the latter. In virtue of the first condition we may assume the existence of a velocity-potential (ϕ), which by the second condition must satisfy (§ 241) the equation

$$\nabla^2\phi = 0 \qquad\qquad (1),$$

throughout the interior of the fluid. In terms of ϕ the equation for the pressure is (§ 244)

$$\delta p/\rho = R - d\phi/dt \qquad\qquad (2),$$

if we assume that the motion is so small that its square may be neglected. The only impressed force, acting upon the interior of the fluid, which we have occasion to consider is that due to gravity; so that, if z be measured vertically downwards, $R = gz$, and (2) becomes

$$\delta p/\rho = gz - d\phi/dt \qquad\qquad (3).$$

Let us now consider the propagation of waves upon the horizontal surface ($z = 0$) of water, or other liquid, of uniform depth l, limiting our attention to the case of two dimensions, where the motion is confined to the plane zx. The general solution of (1) under this condition, and that the motion is proportional to e^{ikx}, is

$$\phi = e^{ikx}(Ae^{kz} + Be^{-kz});$$

or, with regard to the condition that the vertical velocity must vanish at the bottom where $z = l$,

$$\phi = C\cosh k(z - l) \,.\, e^{ikx} \qquad\qquad (4).$$

If the motion be proportional also to e^{int}, and we throw away the imaginary part in (4), we get as the expression for waves propagated in the negative direction

$$\phi = C \cosh k(z-l) \cos(nt+kx) \quad\text{..............(5)},$$

in which it remains to find the connection between n and k.

If h denote the elevation of the water surface at the point x, and T the constant tension, the pressure at the surface due to capillarity is $-Td^2h/dx^2$, and (3) becomes

$$\frac{T}{\rho}\frac{d^2h}{dx^2} = gh + \frac{d\phi}{dt}:$$

or, if we differentiate with respect to t and remember that $dh/dt = -d\phi/dz$,

$$\frac{T}{\rho}\frac{d^3\phi}{dx^2dz} = g\frac{d\phi}{dz} - \frac{d^2\phi}{dt^2} \quad\text{....................(6)}.$$

Applying this equation to (5) where $z=0$, we get for the velocity of propagation

$$V^2 = n^2/k^2 = (g/k + Tk/\rho)\tanh kl \quad\text{..............(7)[1]},$$

where, as usual,

$$k = 2\pi/\lambda \quad\text{.......................... (8)}.$$

In many cases the depth of liquid is sufficient to allow us to take $\tanh kl = 1$; and then

$$V^2 = \frac{g\lambda}{2\pi} + \frac{2\pi T}{\rho\lambda} \quad\text{.................... (9)[2]},$$

gives the relation between V and λ. When λ is great, the waves move mainly under gravity and with velocity approximately equal to $\sqrt{(g\lambda/2\pi)}$. On the other hand, when λ is small, the influence of capillarity becomes predominant and the expression for the velocity assumes the form

$$V = \sqrt{(2\pi T/\rho\lambda)} \quad\text{......................(10)}.$$

Since $\lambda = V\tau$, the relation between wave-length and periodic time corresponding to (10) is

$$\lambda^3/\tau^2 = 2\pi T/\rho \quad\text{......................(11)}.$$

Except as regards the numerical factor, the relations (10), (11) can be deduced by considerations of dimensions from the fact that the dimensions of T are those of a force divided by a line.

[1] A more general formula for the velocity of propagation (n/k) at the interface between two liquids is given in (7) § 365.

[2] Kelvin, *Phil. Mag.* vol. XLII. p. 375, 1871.

If we inquire what values of λ correspond to a given value of V, we obtain from the quadratic (9)

$$\lambda = \pi V^2/g \pm \pi/g \,.\, \sqrt{(V^4 - 4Tg/\rho)} \quad\text{..........}\quad (12),$$

which shews that for no wave-length can V be less than V_0, where

$$V_0 = (4Tg/\rho)^{\frac{1}{4}} \quad\text{..........}\quad (13).$$

The values of λ and of τ corresponding to the minimum velocity are given by

$$\lambda_0 = 2\pi\,(T/g\rho)^{\frac{1}{2}}, \qquad \tau_0 = 2\pi\,(T/4g^3\rho)^{\frac{1}{4}} \quad\text{.........}\quad (14).$$

If we take in C.G.S. measure $g = 981$, and for water $\rho = 1$, $T = 76$, we have $V_0 = 23{\cdot}1$, $\lambda_0 = 1{\cdot}71$, $1/\tau = 13{\cdot}6$.

The accompanying table gives a few corresponding values of wave-length, velocity, and frequency in the neighbourhood of the critical point:—

Wave-length	·5	1·0	1·7	2·5	3·0	5·0
Velocity	31·5	24·7	23·1	23·9	24·9	29·5
Frequency	63·0	24·7	13·6	9·6	8·3	5·9

A comparison of Kelvin's formula (9) with observation has been effected by Matthiessen[1], the ripples being generated by touching the surface of the various liquids with dippers attached to vibrating forks of known pitch. Among the liquids tried were water, mercury, alcohol, ether, bisulphide of carbon; and the agreement was found to be satisfactory. The observations include frequencies as high as 1832, and wave-lengths as small as ·04 cm.

Somewhat similar experiments have been carried out by the author[2] with the view of determining T by a method independent of any assumption respecting angles of contact between fluid and solid, and admitting of application to surfaces purified to the utmost from grease. In order to see the waves well, the light was made intermittent in a period equal to that of the waves (§ 42), and Foucault's optical method was employed for rendering visible small departures from truth in plane or spherical reflecting

[1] *Wied. Ann.* vol. XXXVIII. p. 118, 1889.

[2] On the Tension of Water Surfaces, clean and contaminated. *Phil. Mag.* vol. XXX. p. 386, 1890.

surfaces. From the measured values of τ and λ, T may be determined by (11), corrected, if necessary, for any small effect of gravity. The values thus found were for clean water 74·0 C.G.S., for a surface greasy to the point where camphor motions nearly cease 53·0, for a surface saturated with olive-oil 41·0, and for one saturated with oleate of soda 25·0. It should be remembered that the tension of contaminated surfaces is liable to variations dependent upon the extension which has taken place, or is taking place; but it is not necessary for the purposes of this work to enter further upon the question of "superficial viscosity."

354. Another way of generating capillary waves, or crispations as they were termed by Faraday, depends upon the principle discussed in § 68 *b*. If a glass plate, held horizontally and made to vibrate as for the production of Chladni's figures, be covered with a thin layer of water or other mobile liquid, the phenomena in question may be readily observed[1]. Over those parts of the plate which vibrate sensibly the surface is ruffled by minute waves, the degree of fineness increasing with the frequency of vibration. The same crispations are observed upon the surface of liquid in a large wine-glass or finger-glass which is caused to vibrate in the usual manner by carrying the moistened finger round the circumference (§ 234). All that is essential to the production of crispations is that a body of liquid with a free surface be constrained to execute a vertical vibration. It is indifferent whether the origin of the motion be at the bottom, as in the first case, or, as in the second, be due to the alternate advance and retreat of a lateral boundary, to accommodate itself to which the neighbouring surface must rise and fall.

More than sixty years ago the nature of these vibrations was examined by Faraday[2] with great ingenuity and success. The conditions are simplest when the motion of the vibrating horizontal plate on which the liquid is spread is a simple up and down motion without rotation. To secure this Faraday attached the plate to the centre of a strip of glass or lath of deal, supported at the nodes, and caused to vibrate by friction. Still more convenient is a large iron bar, maintained in vibration electrically, to which the plate may be attached by cement.

[1] On the Crispations of Fluid resting upon a Vibrating Support. *Phil. Mag.* vol. XVI. p. 50, 1883.

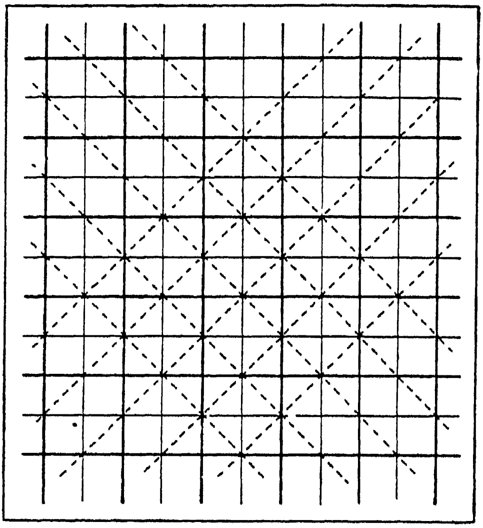
[2] *Phil. Trans.* 1831, p. 299.

The vibrating liquid standing upon the plate presents appearances which at first are rather difficult to interpret, and which vary a good deal with the nature of the liquid in respect of transparency and opacity, and with the incidence of the light. The vibrations are too quick to be followed by the eye; and thus the effect observed is an average, due to the superposition of an indefinite number of elementary impressions corresponding to the various phases.

If the plate be rectangular, the motion of the liquid consists of two sets of stationary vibrations superposed, the ridges and furrows of the two sets being perpendicular to one another and usually parallel to the edges of the plate. Confining our attention for the moment to one set of stationary waves, let us consider what appearance it might be expected to present. At one moment the ridges form a set of parallel and equidistant lines, the interval being λ. Midway between these are the lines which represent at that moment the position of the furrows. After the lapse of a $\frac{1}{4}$ period the surface is flat; after another $\frac{1}{4}$ period the ridges and furrows are again at their maximum developement, but the positions are exchanged. Now, since only an average effect can be perceived, it is clear that no distinction is recognizable between the ridges and the furrows, and that the observed effect must be periodic within a distance equal to $\frac{1}{2}\lambda$. If the liquid on the plate be rendered moderately opaque by addition of aniline blue, and be seen by diffused transmitted light, the lines of ridge and furrow will appear bright in comparison with the intermediate nodal lines where the normal depth is preserved throughout the vibration. The gain of light when the thickness is small will, in accordance with the law of absorption, outweigh the loss of light which occurs half a period later when the furrow is replaced by a ridge.

The actual phenomenon is more complicated in consequence of the coexistence of the two sets of ridges and furrows in perpendicular directions (x, y). In the adjoining figure (Fig. 66) the thick lines represent the ridges, and the thin lines the furrows, of the two systems at a moment of maximum excursion. One quarter period later the surface is flat, and one half period later the ridges and furrows are interchanged. The places of maximum elevation and depression are the intersections of the thick lines with one another and of the thin lines with one another, places not distinguishable by ordinary vision. They appear like holes in the sheet of colour. The nodal lines where the normal depth of colour is preserved throughout the vibration are shewn dotted; they are inclined at 45°, and pass through the intersections of the thin lines with the thick lines. The pattern is recurrent in the

Fig. 66.



directions both of x and y, and in each case with an interval equal to the real wave-length (λ). The distance between the bright spots measured parallel to x or y is thus λ; but the shortest distance between these spots is in directions inclined at 45°, and is equal to $\lambda/\sqrt{2}$.

As in all similar cases, these stationary waves may be resolved into their progressive components by a suitable motion of the eye. Consider, for example, the simple set of waves represented by

$$2 \cos kx \cos nt = \cos (nt + kx) + \cos (nt - kx).$$

This is with reference to an origin fixed in space. But let us refer the phenomenon to an origin moving forward with the velocity (n/k) of the waves, so as to obtain the impression that would be produced upon the eye, or in a photographic camera, carried forward in this manner. Writing $kx' + nt$ for kx, we get

$$\cos (kx' + 2nt) + \cos kx'.$$

Now the average effect of the first term is independent of x', so that what is seen is simply that set of progressive waves which moves with the eye.

In order to see the progressive waves it is not necessary to move the head as a whole, but only to turn the eye as when we follow the motion of a real object. To do this without assistance is not very easy at first, especially if the area of the plate be somewhat small. By moving a pointer at various speeds until the right one is found, the eye may be guided to do what is required of it; and after a few successes repetition becomes easy.

Faraday's assertion that the waves have a period double that of the support has been disputed, but it may be verified in various ways. Observation by stroboscopic methods is perhaps the most satisfactory. The violence of the vibrations and the small depth of the liquid interfere with an accurate calculation of frequency on the basis of the observed wave-length. The theory of vibrations in the sub-octave has already been considered (§ 68 *b*).

355. Typical stationary waves are formed by the superposition of equal positive and negative progressive waves of like frequency. If the one set be derived from the other by reflection, the equality of frequencies is secured automatically; but if the two sets of waves originate in different sources, the unison is a matter of adjustment, and a question arises as to the effect of a slight error. We may take as the expression for the two sets of progressive waves of equal amplitude and of approximately equal frequency

$$\cos(kx - nt) + \cos(k'x + n't),$$

or, which is the same,

$$2\cos\{\tfrac{1}{2}(k+k')x + \tfrac{1}{2}(n'-n)t\} \times \cos\{\tfrac{1}{2}(k'-k)x + \tfrac{1}{2}(n'+n)t\} \quad \text{.........(1)}.$$

If $n' = n$, $k' = k$, the waves are absolutely stationary; but we have now to interpret (1) when $(n'-n)$, $(k'-k)$ are merely small.

The position at any time t of the crests and hollows of the nearly stationary waves represented by (1) is given by

$$\tfrac{1}{2}(k+k')x + \tfrac{1}{2}(n'-n)t = m\pi \quad \text{..............(2)},$$

where m is an integer. The velocity of displacement U is accordingly

$$U = (n-n')/(k+k'),$$

or approximately

$$U = (n - n')/2k \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$[1],

from which it appears that in every case the shifting takes place in the direction of waves of higher pitch, or towards the source of graver pitch. If V be the velocity (n/k) of propagation of the progressive waves, (3) may be written

$$U/V = (n - n')/2n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The slow travel under these circumstances of the places where the maximum displacements occur is a general phenomenon, not dependent upon the peculiarities of any particular kind of waves; but the most striking example is that afforded by capillary waves and described by Lissajous[2]. In his experiment two nearly unisonant forks touch the surface of water so as to form approximately stationary waves in the region between the points of contact. Since the crests and troughs cannot be distinguished, the pattern seen has an apparent wave-length half that of the real waves, and it travels slowly towards the graver fork. A frequency of about 50 will be found suitable for convenient observation.

If the waves be aerial, there is no difference of velocity; but (4) still holds good, and gives the rate at which the ear must travel in order to remain continually in a loop or in a node.

356. One of the best opportunities for the examination of capillary waves occurs when they are reduced to rest by a contrary movement of the water. Waves of this kind are sometimes described as standing waves, and they may usually be observed when the uniform motion of a stream is disturbed by obstacles. Thus when the surface is touched by a small rod, or by a fishing-line, or is displaced by the impact of a gentle stream of air from a small nozzle, a beautiful pattern is often displayed, stationary with respect to the obstacle. This was described and figured by Scott Russell[3], who remarked that the purity of the water had much to do with the extent and range of the phenomenon. On the up-stream side of the obstacle the wave-length is short, and, as was first clearly shewn by Kelvin, the force governing the vibra-

[1] *Phil. Mag.* vol. XVI. p. 57, 1883.
[2] *Compt. Rend.* vol. LXVII. p. 1187, 1868.
[3] *Brit. Ass. Rep.* 1844, p. 375, Plate 57. See also Poncelet, *Ann. d. Chim.* vol. XLVI. p. 5, 1831.

tions is principally cohesion. On the down-stream side the waves are longer and are governed principally by gravity. Both sets of waves move with the same velocity relatively to the water (§ 353); namely, that required in order that they may maintain a fixed position relatively to the obstacle. The same condition governs the velocity and therefore the wave-lengths of those parts of the pattern where the fronts are oblique to the direction of motion. If the angle between this direction and the normal to the wave-front be called θ, the velocity of propagation must be equal to $v_0 \cos\theta$, where v_0 represents the velocity of the water.

If v_0 be less than 23 cm. per sec., no wave-pattern is possible, for no waves can then move over the surface so slowly as to maintain a stationary position with respect to the obstacle. When v_0 exceeds 23 cm. per sec., a pattern is formed; but the angle θ has a limit defined by $v_0 \cos\theta = 23$, and the curved wave-front has a corresponding asymptote.

It would lead us too far to go further into the matter here, but it may be mentioned that the problem in two dimensions admits of analytical treatment[1], and that the solution explains satisfactorily one of the peculiar features of the case, namely, the limitation of the smaller capillary waves to the up-stream side, and of the larger (gravity) waves to the down-stream side of the obstacle.

357. A large class of phenomena, interesting not only in themselves but also as throwing light upon others yet more obscure, depend for their explanation upon the transformations undergone by a cylindrical body of liquid when slightly displaced from its equilibrium configuration and then left to itself. Such a cylinder is formed when liquid issues under pressure through a circular orifice, at least when gravity may be neglected; and the behaviour of the jet, as studied experimentally by Savart, Magnus, Plateau and others, is substantially independent of the forward motion common to all its parts. It will save repetition and be more in accordance with the general character of this work if we commence our investigation with the theory of an infinite cylinder of liquid, considered as a system in equilibrium under the action

[1] On the form of Standing Waves on the Surface of Running Water. *Proc. Lond. Math. Soc.* vol. xv. p. 69, 1883.

of the capillary force. With a solution of this mechanical problem most of the experimental results will easily be connected.

Taking cylindrical coordinates z, r, ϕ, the equation of the slightly disturbed surface may be written

$$r = a_0 + f(\phi, z) \quad\text{.......................}\quad (1),$$

in which $f(\phi, z)$ is always a small quantity. By Fourier's theorem the arbitrary function f may be expanded in a series of terms of the type $\alpha_n \cos n\phi \cos kz$; and, as we shall see in the course of the investigation, each of these terms may be considered independently of the others. Either cosine may be replaced by a sine; and the summation extends to all positive values of k and to all positive integral values of n, zero included.

During the motion the quantity a_0 does not remain absolutely constant; its value must be determined by the condition that the enclosed *volume* is invariable. Now for the surface

$$r = a_0 + \alpha_n \cos n\phi \cos kz \quad\text{..................}\quad (2),$$

we find

$$\text{Volume} = \tfrac{1}{2}\iint r^2 \, d\phi \, dz = z\,(\pi a_0^2 + \tfrac{1}{4}\pi\alpha_n^2);$$

so that, if a denote the radius of the section of the undisturbed cylinder,

$$a^2 = a_0^2 + \tfrac{1}{4}\alpha_n^2,$$

whence approximately

$$a_0 = a\,(1 - \tfrac{1}{8}\alpha_n^2/a^2) \quad\text{....................}\quad (3).$$

This holds good when $n = 1, 2, 3 \ldots$. If $n = 0$, (2) gives in place of (3)

$$a_0 = a\,(1 - \tfrac{1}{4}\alpha_0^2/a^2) \quad\text{....................}\quad (4).$$

The potential energy of the system in any configuration, due to the capillary force, is proportional simply to the surface. Now in (2)

$$\text{Surface} = \iint \left\{1 + \left(\frac{dr}{dz}\right)^2 + \left(\frac{dr}{r d\phi}\right)^2\right\}^{\frac{1}{2}} r \, d\phi \, dz$$

$$= z\,\{2\pi a_0 + \tfrac{1}{4}\pi k^2 \alpha_n^2 a + \tfrac{1}{4}\pi n^2 \alpha_n^2/a\};$$

so that by (3), if σ denote the surface corresponding upon the average to unit of length,

$$\sigma = 2\pi a + \tfrac{1}{4}\pi\,(k^2a^2 + n^2 - 1)\,\alpha_n^2/a \quad\text{............}\quad (5).$$

The potential energy due to capillarity, estimated per unit length and from the configuration of equilibrium, is accordingly

$$P = \tfrac{1}{4}\pi T (k^2 a^2 + n^2 - 1)\, \alpha_n^2/a \text{ .... (6)},$$

T denoting, as usual, the superficial tension.

In (6) it is supposed that k and n are not zero. If k be zero, (6) requires to be doubled in order to give the potential energy corresponding to

$$r = a_0 + \alpha_n \cos n\phi \text{ .... (7)}:$$

and again, if n be zero, we are to take

$$P = \tfrac{1}{2}\pi T (k^2 a^2 - 1)\, \alpha_0^2/a \text{ .... (8)},$$

corresponding to

$$r = a_0 + \alpha_0 \cos kz \text{ .... (9)}.$$

From (6) it appears that when n is unity or any greater integer, the value of P is positive, shewing that for all displacements of these kinds the original equilibrium is stable. For the case of displacements symmetrical about the axis ($n = 0$), we see from (8) that the equilibrium is stable or unstable according as ka is greater or less than unity, i.e. according as the wave-length ($2\pi/k$) of the symmetrical deformation is less or greater than the circumference of the cylinder, a proposition first established by Plateau.

If the expression for r in (2) involve a number of terms with various values of n and k, and with arbitrary substitution of sines for cosines, the corresponding expression for P is found by simple addition of the expressions relating to the component terms, and it contains the squares only (and not the products) of the quantities α.

We have now to consider the kinetic energy of the motion. Since the fluid is supposed to be inviscid, there is a velocity-potential ψ, and this in virtue of the incompressibility satisfies Laplace's equation. Thus, (4) § 241,

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \frac{1}{r^2}\frac{d^2\psi}{d\phi^2} + \frac{d^2\psi}{dz^2} = 0,$$

or, if in order to correspond with (2) we assume that the variable part is proportional to $\cos n\phi \cos kz$,

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} - \left(\frac{n^2}{r^2} + k^2\right)\psi = 0 \text{ .... (10)}.$$

The solution of (10) under the condition that there is no introduction or abstraction of fluid along the axis of symmetry is § 200

$$\psi = \beta_n J_n(ikr) \cos n\phi \cos kz \quad \text{.............. (11).}$$

The constant β_n is to be found from the condition that the radial velocity when $r=a$ coincides with that implied in (2). Thus

$$ik\beta_n J_n'(ika) = d\alpha_n/dt \quad \text{.................. (12).}$$

If ρ be the density, the kinetic energy of the motion is by Green's theorem (2) § 242

$$\tfrac{1}{2}\rho \iint [\psi\, d\psi/dr]_{r=a}\, a\, d\phi\, dz = \tfrac{1}{4}\pi\rho z . ika . J_n(ika)\, J_n'(ika) . \beta_n^2 ;$$

so that by (12), if K denote the kinetic energy per unit of length,

$$K = \tfrac{1}{4}\pi\rho a^2 \frac{J_n(ika)}{ika . J_n'(ika)} \left(\frac{d\alpha_n}{dt}\right)^2 \quad \text{............ (13).}$$

When $n=0$, we must take in place of (13)

$$K = \tfrac{1}{2}\pi\rho a^2 \frac{J_0(ika)}{ika . J_0'(ika)} \left(\frac{d\alpha_0}{dt}\right)^2 \quad \text{............ (14).}$$

The most general value of K is to be found by simple summation from the particular values expressed in (13), (14). Since the expressions for P and K involve the squares only, and not the products, of the quantities α, $d\alpha/dt$, and the corresponding quantities in which cosines are replaced by sines, it follows that the motions represented by (2) take place in perfect independence of one another, so long as the whole displacement is small.

For the free motion we get by Lagrange's method from (6), (13)

$$\frac{d^2\alpha_n}{dt^2} + \frac{T}{\rho a^3} \frac{ika . J_n'(ika)}{J_n(ika)} (n^2 + k^2a^2 - 1)\, \alpha_n = 0 \quad \text{......(15),}$$

which applies without change to the case $n=0$. Thus, if α_n varies as $\cos(pt - \epsilon)$,

$$p^2 = \frac{T}{\rho a^3} \frac{ika . J_n'(ika)}{J_n(ika)} (n^2 + k^2a^2 - 1) \quad \text{........ (16)[1],}$$

giving the frequency of vibration in the cases of stability.

If $n=0$, and $ka<1$, the solution changes its form. If we suppose that α_0 varies as $e^{\pm qt}$,

$$q^2 = \frac{T}{\rho a^3} \frac{ika . J_0'(ika)}{J_0(ika)} (1 - k^2a^2) \quad \text{..............(17).}$$

[1] *Proc. Roy. Soc.* vol. xxix. p. 94, 1879.

When n is greater than unity, the circumstances are usually such that the motion is approximately in two dimensions only. We may then advantageously introduce into (16) the supposition that ka is small. In this way we get, (5) § 200,

$$p^2 = n(n^2 - 1 + k^2a^2)\frac{T}{\rho a^3}\left[1 + \frac{k^2a^2}{n(2n+2)}\right] \qquad (18),$$

or, if ka be neglected altogether,

$$p^2 = (n^3 - n)\frac{T}{\rho a^3} \qquad (19),$$

the two-dimensional formula. When $n = 1$, there is no force of restitution for a displacement purely in two dimensions. If λ denote the wave-length measured round the circumference, $\lambda = 2\pi a/n$. Thus in (19), if n and a are infinite,

$$p^2 = \frac{T}{\rho}\left(\frac{2\pi}{\lambda}\right)^3 \qquad (20),$$

in agreement with the theory of capillary waves upon a plane surface. Compare (7) § 353. A similar conclusion may be reached by the consideration of waves whose length is measured axially. Thus, if $\lambda = 2\pi/k$, and $a = \infty$, $n = 0$, (16) reduces to (20) in virtue of the relation, §§ 302, 350,

$$\text{Limit}_{z=\infty}\, iJ_0'(iz)/J_0(iz) = 1.$$

358. Many years ago Bidone investigated by experiment the behaviour of jets of water issuing horizontally under considerable pressure from orifices in thin plates. If the orifice be circular, the section of the jet, though diminished in area, retains the circular form. But if the orifice be not circular, curious transformations ensue. The peculiarities of the orifice are exaggerated in the jet, but in an inverted manner. Thus in the case of an elliptical aperture, with major axis horizontal, the sections of the jet taken at increasing distances gradually lose their ellipticity until at a certain distance the section is circular. Further out the section again assumes ellipticity, but now with major axis vertical, and (in the circumstances of Bidone's experiments) the ellipticity increases until the jet is reduced to a flat sheet in the vertical plane, very broad and thin. This sheet preserves its continuity to a considerable distance (e.g. six feet) from the orifice, where finally it is penetrated by air. If the orifice be in the form of an equilateral triangle, the jet resolves itself into three sheets disposed symmetrically round the axis, the planes of the sheets being perpendicular to the sides of the orifice; and in like manner if the aperture be a regular polygon of any number of sides, there are developed a corresponding number of sheets perpendicular to the sides of the polygon.

Bidone explains the formation of these sheets by reference to simpler cases of meeting streams. Thus equal jets, moving in the same straight line with equal and opposite velocities, flatten themselves into a disc situated in the perpendicular plane. If the axes of the jets intersect obliquely, a sheet is formed symmetrically in the plane perpendicular to that of the impinging jets. Those portions of a jet which proceed from the outlying parts of a single unsymmetrical orifice are regarded as behaving in some degree like independent meeting streams.

In many cases, especially when the orifices are small and the pressures low, the extension of the sheets reaches a limit. Sections taken at still greater distances from the orifice shew a gradual gathering together of the sheets, until a compact form is regained similar to that at the first contraction. Beyond this point, if the jet retains its coherence, sheets are gradually thrown out again, but in directions bisecting the angles between the directions of the former sheets. These sheets may in their turn reach a limit of developement, again contract, and so on. The forms assumed in the case of orifices of various shapes including the rectangle, the equilateral triangle, and the square, have been carefully investigated and figured by Magnus. Phenomena of this kind are of every day occurrence, and may generally be observed whenever liquid falls from the lip of a moderately elevated vessel.

As was first suggested by Magnus[1] and Buff[2], the cause of the contraction of the sheets after their first developement is to be found in the capillary force, in virtue of which the fluid behaves as if enclosed in an envelope of constant tension; and the recurrent form of the jet is due to *vibrations* of the fluid column about the circular figure of equilibrium, superposed upon the general progressive motion. Since the phase of the vibration, initiated during passage through the aperture, depends upon the

[1] Hydraulische Untersuchungen, *Pogg. Ann.* vol. xcv, p. 1, 1855.

[2] *Pogg. Ann.* vol. c, p. 168, 1857.

time elapsed, it is always the same at the same point in space, and thus the motion is *steady* in the hydrodynamical sense, and the boundary of the jet is a fixed surface. Relatively to the water the waves here concerned are progressive, such as may be compounded of two stationary systems, and they move up stream with a velocity equal to that of the water so as to maintain a fixed position relatively to external objects, § 356.

If the departure from the circular form be small, the vibrations are those considered in § 357, of which the frequency is determined by equations (16), (18), (19). The distance between consecutive corresponding points of the recurrent figure, or, as it may be called, the wave-length of the figure, is the space travelled over by the stream during one vibration. Thence results a relation between wave-length and period. If the circumference of the jet be small in comparison with the wave-length, so that (19) § 357 is applicable, the periodic time is independent of the wave-length; and then the wave-length is directly proportional to the velocity of the jet, or to the square root of the pressure. The elongation of wave-length with increasing pressure was remarked by Bidone and by Magnus, but no definite law was arrived at.

In the experiments of the author[1] upon elliptical, triangular, and square apertures, the jets were caused to issue horizontally in order to avoid the complications due to gravity; and, if the pressure were not too high, the law above stated was found to be verified. At higher pressures the observed wave-lengths had a marked tendency to increase more rapidly than the velocity of the jet. This result points to a departure from the law of isochronous vibration. Strict isochronism is only to be expected when vibrations are infinitely small, that is when the section of the jet never deviates more than infinitesimally from the circular form. Under the high pressures in question the departures from circularity were very considerable, and there is no reason for expecting that such vibrations will be executed in precisely the same time as vibrations of infinitely small amplitude.

The increase of amplitude under high pressure is easily explained, inasmuch as the lateral velocities to which the vibrations are mainly due vary in direct proportion to the longitudinal velocity of the jet. Consequently the amplitude varies approxi-

[1] *Proc. Roy. Soc.* vol. XXIX, p. 71, 1879.

mately as the square root of the pressure, or as the wave-length. In general, the periodic time of a vibration is an even function of amplitude (§ 67); and thus, if h represent the head of liquid, the wave-length may be expected to be a function of h of the form $(M+Nh)\sqrt{h}$, where M and N are constants for a given aperture. It appears from experiment, and might perhaps have been expected, that N is here positive.

For a comparison with theory it is necessary to keep within the range of the law of isochronism; and it is convenient to employ in the calculations the area of the section of the jet in place of the mean radius. Thus, if $A = \pi a^2$, (19) § 357 may be written

$$p = \pi^{\frac{3}{4}} T^{\frac{1}{2}} \rho^{-\frac{1}{2}} A^{-\frac{3}{4}} \sqrt{(n^3 - n)} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

in which A is to be determined by experiments upon the rate of total discharge. For the case of water (§ 353) we may take in C.G.S. measure $T = 74$, $\rho = 1$; so that for the frequency of the gravest vibration ($n = 2$) we get from (1)

$$p/2\pi = 7{\cdot}91 A^{-\frac{3}{4}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

For a sectional area of one square centimetre there are thus about 8 vibrations per second. A pitch of 256 would correspond to a diameter of about one millimetre.

For the general value of n, we have

$$p/2\pi = 3{\cdot}23 A^{-\frac{3}{4}} \sqrt{(n^3 - n)} \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If h be the head of water to which the velocity of the jet is due and λ the wave-length,

$$\lambda = \frac{\sqrt{(2gh)}\,.\,A^{\frac{3}{4}}}{3{\cdot}23\sqrt{(n^3 - n)}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

In one experiment with an elliptical aperture ($n = 2$) the observed value of λ was 3·95 while the value calculated from (4) is 3·93. In the case of a triangular aperture ($n = 3$) the observed value of λ was 2·3 and the calculated was 2·1. Again, the observed value for a square aperture ($n = 4$) was 1·85 and the calculated 1·78. The excess of the observed over the calculated values in the last two cases may perhaps have been due to excessive departure from the circular figure.

The general theory, unrestricted to small amplitudes, would doubtless involve great complications; but some information respecting it may be obtained with facility by the method of dimensions. If the *shape* of the orifice be given, λ may be regarded as a function of T, ρ, A, and H the pressure under which the jet escapes. Of these T is a force divided by a line, so that its dimensions are 1 in mass, 0 in length, and -2 in time; ρ is of dimensions 1 in mass, -3 in length, 0 in time; A is of dimensions 0 in mass, 2 in length, 0 in time; and finally H is of dimensions 1 in mass, -1 in length, and -2 in time. If we assume

$$\lambda \propto T^x \rho^y A^z H^u,$$

then $$x + y + u = 0, \quad -3y + 2z - u = 1, \quad -2x - 2u = 0,$$

whence $$u = -x, \quad y = 0, \quad z = \tfrac{1}{2}(1 - x);$$

so that $$\lambda \propto A^{\frac{1}{2}} (TA^{-\frac{1}{2}} H^{-1})^x.$$

The exponent x is here undetermined; and, since any number of terms with different values of x may occur simultaneously, all that we can infer is that λ is of the form

$$\lambda = A^{\frac{1}{2}} . f(TA^{-\frac{1}{2}} H^{-1}),$$

or, if we prefer it,

$$\lambda = T^{-\frac{1}{2}} H^{\frac{1}{2}} A^{\frac{3}{4}} . F(HA^{\frac{1}{2}} T^{-1}) \quad\text{.................(5)},$$

where f and F are arbitrary functional symbols. Thus for a given liquid and shape of orifice there is complete dynamical similarity if the pressure be taken inversely proportional to the linear dimension. The simple case previously considered where the departures from circularity are small, and the vibrations take place approximately in two dimensions, corresponds to F = constant.

The method of determining T by observations upon λ is scarcely delicate enough to compete with others that may be employed for the same purpose when the tension is constant. But the possibility of thus experimenting upon surfaces which have been formed but a fraction of a second earlier is of considerable interest. In this way it may be proved with great ease that the tension of a soapy solution immediately after the formation of a free surface differs comparatively little from that of pure water, whereas when a few seconds have elapsed the difference becomes very great[1].

[1] On the Tension of Recently Formed Liquid Surfaces, *Proc. Roy. Soc.* vol. XLVII, p. 281, 1890.

Hitherto it has been supposed for the sake of simplicity that the jet after its issue from the nozzle is withdrawn from the action of gravity. If the direction of projection be vertically downwards, as is often convenient, the velocity of flow (v) continually increases, while at the same time the area of the section diminishes, the relation being vA = constant. But, so far as regards λ, the disturbance which thus ensues is less than might have been expected, for the changes in v and A compensate one another to a considerable extent. By (1)

$$\lambda \propto v/p \propto v^{\frac{3}{2}} \propto h^{\frac{3}{4}},$$

if h denote the whole difference of level between the surface of liquid in the reservoir and the place where λ is measured.

359. In § 358 the motion of the liquid is regarded as steady, every portion as in turn it passes the orifice being similarly affected. Under these circumstances no term corresponding to $n=0$ can appear in the mathematical expressions; but it must not be forgotten that for certain disturbances of this type the cylindrical form is unstable and that therefore the jet cannot long preserve its integrity. The minute disturbances required to bring the instability into play are such as act differently at different moments of time, and have their origin in eddying motions of the fluid due to friction, and especially in vibration communicated to the nozzle and of such a character as to render the rate of discharge subject to a slight periodic variation. If v be the velocity of the jet and τ the period of the vibration, the cylindrical column issuing from a circular orifice is launched subject to a disturbance of wave-length (λ) equal to $v\tau$. If this wave-length exceed the circumference of the jet ($2\pi a$), the disturbance grows exponentially, until finally the column of liquid is divided into detached masses separated by the common interval λ, and passing a fixed point with velocity v and frequency $1/\tau$. Even though no regular vibration has access to the nozzle, the instability cannot fail to assert itself, and casual disturbances of a complex character will bring about disintegration. It will be convenient to discuss in the first place somewhat in detail the theory of the case of $n=0$ in (16), (17) § 357, and then to consider its application to the beautiful phenomena described by Savart and to a large extent explained by Plateau.

If $ka = z$, and we introduce the notation of § 221 *a*, (17) § 357 becomes

$$q^2 = \frac{T}{\rho a^3} \frac{z I_1(z)}{I_0(z)} (1 - z^2) \qquad \text{(1)}.$$

In this equation $I_1(z)$ and $I_0(z)$ are both positive, so that as z decreases (or as λ increases) q first becomes real when $z = 1$. At this point instability commences, and at first the degree of instability is infinitely small. Also when z is very small, or λ is very great,

$$q^2 = \frac{T}{\rho a^3} \frac{z^2}{2}$$

ultimately, so that q is again small. For some value of z between 0 and 1, q is a maximum, and the investigation of this value is a matter of importance, because, as has already been shewn § 87, the unstable equilibrium will give way by preference in the mode so characterized.

The function to be made a maximum is

$$z(1 - z^2) I_1(z)/I_0(z) \qquad \text{(2)},$$

or, expanded in powers of z,

$$\frac{1}{2}\left(z^2 - \frac{9}{8} z^4 + \frac{7}{2^4 . 3} z^6 - \frac{25}{2^{10}} z^8 + \frac{91}{2^{11} . 3 . 5} z^{10} + \ldots\right).$$

Hence, to find the maximum, we obtain on differentiation

$$1 - \frac{9}{4} z^2 + \frac{7}{2^4} z^4 - \frac{100}{2^{10}} z^6 + \frac{91}{2^{11} . 3} z^8 + \ldots = 0.$$

If the last terms be neglected, the quadratic gives $z^2 = \cdot 4914$. If this value be substituted in the small terms, the equation becomes

$$\cdot 98928 - \tfrac{9}{4} z^2 + \tfrac{7}{16} z^4 = 0,$$

whence $\qquad z^2 = \cdot 486, \qquad z = \cdot 679$[1].

The values of expression (2), or of its square root, to which q is proportional, may be calculated from tables of I_0 and I_1, § 221 *a*. We have

z	$\{(2)\}^{\frac{1}{2}}$	z	$\{(2)\}^{\frac{1}{2}}$
0·0	·0000	0·6	·3321
0·1	·0703	0·7	·3433
0·2	·1382	0·8	·3269
0·3	·2012	0·9	·2647
0·4	·2567	1·0	·0000
0·5	·3015		

[1] On the Instability of Jets, *Proc. Lond. Math. Soc.* vol. x, p. 7, 1878.

From these values we find for the maximum by Lagrange's interpolation formula $z = \cdot 696$, corresponding to

$$\lambda = 2\pi a/z = 4\cdot 51 \times 2a \qquad (3).$$

Hence the maximum instability occurs when the wave-length of disturbance is about half as great again as that at which instability first commences.

Taking for water in C.G.S. units $T = 73$, $\rho = 1$, we get for the case of maximum instability

$$q^{-1} = \frac{(2a)^{\frac{3}{2}}}{73^{\frac{1}{2}} . 2^{\frac{1}{2}} \times \cdot 343} = \cdot 120\, (2a)^{\frac{3}{2}} \qquad (4).$$

This is the time in which the disturbance is multiplied in the ratio $e : 1$. Thus in the case of a diameter of one centimetre the disturbance is multiplied 2·7 times in about $\frac{1}{8}$ second. If the disturbance be multiplied 1000 fold in time t, $qt = 3 \log_e 10 = 6\cdot 9$, so that $t = \cdot 828\, (2a)^{\frac{3}{2}}$. For example, if the diameter be one millimetre, the disturbance is multiplied 1000 fold in about $\frac{1}{40}$ second. In view of these estimates the rapid disintegration of a jet of water will not cause surprise.

The above theory of the instability of a cylindrical surface separating liquid from gas may be extended to meet the case where the liquid is outside and the gas, whose inertia is neglected, is inside the surface. This represents a jet of gas discharged under liquid; and it appears that the degree of maximum instability is even higher than before, and that it occurs when $\lambda = 6\cdot 48 \times 2a$[1]. But it is scarcely necessary for our purpose to pursue this part of the subject further.

360. The application of our mathematical results to actual jets presents no great difficulty. The disturbances, by which equilibrium is upset, are impressed upon the fluid as it leaves the aperture, and the continuous portion of the jet represents the distance travelled over during the time necessary to produce disintegration. Thus the length of the continuous portion necessarily depends upon the character of the disturbances in respect of amplitude and wave-length. It may be increased considerably, as Savart shewed[2], by a suitable isolation of the reservoir from

[1] On the Instability of Cylindrical Fluid Surfaces, *Phil. Mag.* vol. XXXIV, p. 177, 1892.

[2] *Ann. de Chimie*, LIII, p. 337, 1833.

tremors, whether due to external sources or to the impact of the jet itself in the vessel placed to receive it. Nevertheless it does not appear possible to carry the prolongation very far. Whether the residual disturbances are of external origin or are due to friction, or to some peculiarity of the fluid motion within the reservoir, has not been satisfactorily determined. On this point Plateau's explanations are not very clear, and he sometimes expresses himself as if the time of disintegration depended only upon the capillary tension without reference to initial disturbances at all.

Two laws were formulated by Savart with respect to the length of the continuous portion of a jet, and have been to a certain extent explained by Plateau[1]. For a given fluid and a given orifice the length is approximately proportional to the square root of the head. This follows at once from theory, if it can be assumed that the disturbances remain always of the same character, so that the *time* of disintegration is constant. When the head is given, Savart found the length to be proportional to the diameter of the orifice. From (4) § 359 it appears that the time in which a small disturbance is multiplied in a given ratio varies not as a, but as $a^{\frac{3}{2}}$. Again, when the fluid is changed, the time varies as $\rho^{\frac{1}{2}}T^{-\frac{1}{2}}$. But it may well be doubted whether the length of the continuous portion obeys any very simple laws, even when external disturbances are avoided as far as possible.

When a jet falls vertically downwards, the circumstances upon which its stability or instability depend are continually changing, more especially if the initial velocity be very small. The kind of disturbance to which the jet is most sensitive as it leaves the nozzle is one which impresses upon it undulations of length equal to about $4\frac{1}{2}$ times the initial diameter. But as the jet falls, its velocity increases, with consequent lengthening of the undulations, and its diameter diminishes, so that the degree of instability soon becomes much reduced. On the other hand, the kind of disturbance which will be effective in a later stage is altogether ineffective in the earlier stages. The change of conditions during fall has thus a protective influence, and the continuous part tends to become longer than would be the case were the velocity constant, the initial disturbances being unaltered.

[1] Statique experimentale et théorique des Liquides soumis aux seules forces moléculaires, Paris, 1873.

When the circumstances are such that the reservoir is influenced by the shocks due to the impact of the jet, the disintegration often assumes a complete regularity and is attended by a musical note (Savart). The impact of the regular series of drops, which at any moment strike the receiving vessel, determines the rupture into similar drops of the portion of the jet at the same moment passing the orifice. The pitch of the note, though not definite, cannot differ greatly from that which corresponds to the division of the column into wave-lengths of maximum instability; and in fact Savart found that the frequency was directly as the square root of the head, inversely as the diameter of the orifice, and independent of the nature of the fluid—laws which follow immediately from Plateau's theory.

From the observed pitch of the note due to a jet of given diameter, and issuing under a given head, the wave-length of the nascent divisions can be at once deduced. Reasoning from some observations of Savart, Plateau found in this way 4·38 as the ratio of the length of a division to the diameter of the jet. Now that the length of a division can be estimated *a priori*, it is preferable to reverse Plateau's calculation and to exhibit the frequency of vibration in terms of the other data of the problem. Thus

$$\text{frequency} = \frac{v}{4{\cdot}51 \times 2a} \qquad (1),$$

and in many cases, where the jet is not too fine, v may be replaced by $\sqrt{(2gh)}$ with sufficient accuracy.

But the most certain method of attaining complete regularity of resolution is to bring the reservoir under the influence of an external vibrator, whose pitch is approximately the same as that proper to the jet. Magnus[1] employed a Neef's hammer, attached to the frame which supported the reservoir. Perhaps an electrically maintained tuning-fork is still better. Magnus shewed that the most important part of the effect is due to the forced vibration of that side of the vessel which contains the orifice, and that but little of it is propagated through the air. With respect to the limits of pitch, Savart found that the note might be a fifth above, and more than an octave below, that proper to the jet. According to theory there is no well defined lower limit; while, on the other side the external vibration cannot be efficient if it tends to produce divisions

[1] *Pogg. Ann.* vol. CVI, p. 1, 1859.

whose length is less than the circumference of the jet. This gives for the interval defining the upper limit $\pi : 4{\cdot}51$, or about a fifth. In the case of Plateau's numbers ($\pi : 4{\cdot}38$) the discrepancy is a little greater.

361. The question of the influence of vibrations of low frequency is difficult to treat experimentally in consequence of the complications which arise from the almost universal presence of harmonic overtones. It is evident that the octave, for example, of the principal tone, though present in a very subordinate degree, may nevertheless be the more important agent of the two in determining the behaviour of the jet, if its pitch happen to lie in the neighbourhood of that of maximum instability. In my own experiments[1] tuning-forks were employed as sources of vibration, and in every case the behaviour of the jet on its horizontal course was examined not only by direct inspection, but also by the method of intermittent illumination (§ 42) so arranged that there was one view for each complete period of the phenomenon to be observed. Except when it was important to eliminate the octave as far as possible, the vibration was communicated to the reservoir through the table on which it stood. The forks were either screwed to the table and vibrated by a bow, or maintained electrically, the former method being adequate when only one fork was required at a time. The circumstances of the jet were such that the pitch of maximum sensitiveness, as determined by calculation, was 259, and that forming the transition between stability and instability 372.

With pitches varying downwards from 370 to about 180, the observed phenomena agreed perfectly with the unambiguous predictions of theory. From the point—decidedly below 370—at which a regular effect was first observed, there was always one drop for each complete vibration of the fork, and a single stream, each drop breaking away under precisely the same conditions as its predecessor. After passing 180 it becomes a question whether the octave of the fork's note may not produce an effect as well as the prime. If this effect be sufficient, the number of drops is doubled, and when the prime is very subordinate indeed, there is a double stream, alternate drops breaking away under different conditions and (under the action of gravity) taking sensibly

[1] *Proc. Roy. Soc.* vol. xxxiv, p. 133, 1882.

different courses. In these experiments the influence of the prime was usually sufficient to determine the number of drops, even in the neighbourhood of pitch 128. Sometimes, however, the octave became predominant and doubled the number of drops. When the octave is not strong enough actually to double the drops, it often produces an effect which is very apparent to an observer examining the transformation through the revolving holes. On one occasion a vigorous bowing of the fork, which favours the octave, gave at first a double stream, but this after a few seconds passed into a single one. Near the point of resolution those consecutive drops which ultimately coalesce as the fork dies down are connected by a ligament. If the octave is strong enough, this ligament subsequently breaks, and the drops are separated; otherwise the ligament draws the half-formed drops together, and the stream becomes single. The transition from the one state of things to the other could be watched with facility.

In order to get rid entirely of the influence of the octave a different arrangement was necessary. It was found that the desired result could be arrived at by holding a 128 fork in the hand over a resonator of the same pitch resting upon the table. The transformation was now quite similar in appearance to that effected by a fork of frequency 256, the only differences being that the drops were bigger and twice as widely spaced, and that the *spherule*, which results from the gathering together of the ligament, was much larger. We may conclude that the cause of the doubling of a jet by the sub-octave of the note natural to it is to be found in the presence of the second component from which hardly any musical notes are free.

When two forks of pitches 128 and 256 were sounded together, the single or double stream could be obtained at pleasure by varying the relative intensities. Any imperfection in the tuning is rendered very evident by the behaviour of the jet, which performs evolutions synchronous with the audible beats. This observation, which does not require the aid of the stroboscopic disc, suggests that the effect depends in some degree upon the relative phases of the two tones, as might be expected *a priori*. In some cases the influence of the sub-octave is shewn more in making the alternate drops unequal in magnitude than in projecting them into very different paths.

Returning now to the case of a single fork screwed to the table, it was found that as the pitch was lowered below 128, the double stream was regularly established. The action of the twelfth ($85\frac{1}{3}$) below the principal note demands special attention. At this pitch we might expect the first three components of a compound note to influence the result. If the third component were pretty strong, it would determine the number of drops, and the result would be a three-fold stream. In the case of a fork screwed to the table the third component of the note must be extremely weak if not altogether missing; but the second (octave) component is fairly strong, and in fact determined the number of drops ($190\frac{2}{3}$). At the same time the influence of the prime ($85\frac{1}{3}$) is sufficient to cause the alternate drops to pursue different paths, so that a double stream is observed.

By the addition of a 256 fork there was no difficulty in obtaining a triple stream; but it was of more interest to examine whether it were possible to reduce the double stream to a single one with only $85\frac{1}{3}$ drops per second. In order to secure as strong and as pure a fundamental tone as possible and to cause it to act upon the jet in the most favourable manner, the air space in the reservoir (an aspirator bottle) above the water was tuned to the note of the fork by sliding a plate of glass over the neck so as partially to cover it (§ 305). When the fork was held over the resonator thus formed, the pressure which expels the jet was rendered variable with a frequency of $85\frac{1}{3}$, and overtones were excluded as far as possible. To the unaided eye, however, the jet still appeared double, though on more attentive examination one set of drops was seen to be decidedly smaller than the other. With the revolving disc, giving about 85 views per second, the real state of the case was made clear. The smaller drops were the *spherules*, and the stream was single in the same sense as the streams given by pure tones of frequencies 128 and 256. The increased size of the spherule is of course to be attributed to the greater length of the ligament, the principal drops being now three times as widely spaced as when the jet is under the influence of the 256 fork.

With still graver forks screwed to the table the number of drops continued to correspond to the second component of the note. The double octave of the principal note (64) gave 128 drops per second, and the influence of the prime was so feeble that the duplicity of the stream was only just recognisable. Below 64 the observations were not carried, and even at this pitch attempts to attain a single stream of drops were unsuccessful.

362. Savart's experiments upon this subject have been further developed by Mr C. A. Bell, who shewed that a jet may be made to play the part of a telephonic receiver[1]. The external vibrations may be conveyed to the nozzle through a string telephone (§ 156 *a*). An india rubber membrane, stretched over the upper end of a metal tube, receives the jet and communicates the vibration due to the varying impact to the cavity behind, with which the ear may be connected. The diameter and velocity of the jet require to be accommodated to the general character as to pitch of the sounds to be dealt with. "When the membrane is held close under the jet orifice, no sound will be audible in the ear-piece; but as the receiving tube is gradually withdrawn along the jet path, a sound will be heard corresponding in pitch and quality to the disturbing sound—provided, of course, that the jet is at such pressure as to be capable of responding to all the higher tones to which the disturbing sound may owe its timbre. The intensity of this sound grows as the distance between jet orifice and membrane is increased. Finally, while the jet is still continuous above the membrane, a point of maximum intensity and purity of tone will be reached; and if the membrane be carried beyond this point the sound heard will at first increase in loudness, becoming harsh in character at the same time, and at a still lower point will degenerate into an unmusical roar. In the latter case the jet will be seen to break above the membrane."

From the fact that small jets travelling at high speeds respond equally to sounds whose pitch varies over a wide range Mr Bell argues that Plateau's theory is inadequate, and he looks rather to vortex motion, dependent upon unequal velocity at the centre and at the exterior of the column, as the real cause of the phenomena presented by these jets.

As an example of a jet self-excited, the interrupter of § 235 *r* may be referred to. In this case the machinery by which the effect is carried back to the nozzle is electric. But ordinary mechanical devices answer the purpose equally well. The introduction of a resonator, such as the fork of § 235 *r*, or the telephone

[1] *Phil. Trans.* vol. 177, p. 383, 1886.

plate which may be made to take its place, if the telephone be brought in contact with the nozzle, gives greater regularity to the process, and usually allows also of a greater latitude in respect of pitch. It should not be forgotten that in all these cases of self-excitation a certain condition as to phase needs to be satisfied. If for instance in the interrupter of § 235 r, supposed to be working well, the platinum points be displaced through half the interval between consecutive drops, it is evident that the action will cease until some fresh accommodation is brought about.

363. When a small jet is projected upwards in a nearly vertical direction, there are complications dependent upon the collisions of the drops with one another. Such collisions are inevitable in consequence of the different velocities acquired by the drops as they break away irregularly from the continuous portion of the column. Even when the resolution is regularized by the action of external vibrations of suitable frequency, the drops must still come into contact before they reach the summit of their parabolic path. In the case of a continuous jet the "equation of continuity" shews that as the jet loses velocity in ascending, it must increase in section. When the stream consists of drops following the same path in single file, no such increase of section is possible; and then the constancy of the total stream demands a gradual approximation of the drops, which in the case of a nearly vertical direction of motion cannot stop short of actual contact. Regular vibration has, however, the effect of postponing the collisions and consequent scattering of the drops, and in the case of a direction of motion less nearly vertical may prevent them altogether.

The behaviour of a nearly vertical fountain is influenced in an extraordinary manner by the neighbourhood of an electrified body. The experiment may be tried with a jet from a nozzle of 1 mm. diameter rising about 50 centims. In its normal state the jet resolves itself into drops, which even before passing the summit, and still more after passing it, are scattered through a considerable width. When a feebly electrified body is presented to it, the jet undergoes a remarkable transformation, and appears to become coherent; but under more powerful electrical action the scattering becomes even greater than at first. The second effect is readily attributed to the mutual repulsion of the electrified drops, but the action of feeble electricity in producing apparent coherence depends upon a different principle.

It has been shewn by Beetz[1] that the coherence is apparent only, and that the place where the jet breaks into drops is not perceptibly shifted by the electricity. By screening various parts with metallic plates connected to earth, Beetz further proved that, contrary to the opinion of earlier observers, the seat of sensitiveness is not at the root of the jet where it leaves the orifice, but at the place of resolution into drops. As in Lord Kelvin's water-dropping apparatus for atmospheric electricity, the drops carry away with them an electric charge, which may be collected by receiving them in an insulated vessel.

It may be proved by instantaneous illumination that the normal scattering is due to the rebound of the drops when they come into collision. Under moderate electrical influence there is no material change in the resolution into drops nor in the subsequent motion of the drops up to the moment of collision. The difference begins here. Instead of rebounding after collision, as the unelectrified drops of clean water generally do, the electrified drops *coalesce*, and thus the jet is no longer scattered about[2]. An elaborate discussion of this subject would be out of place here. It must suffice to say that the effect depends upon a *difference* of potential between the drops at the moment of collision, and that when this difference is too small to cause coalescence there is complete electrical insulation between the contiguous masses.

When the jet is projected upwards at a moderate obliquity, the scattering is confined to the vertical plane. Under these circumstances there are few or no collisions, as the drops have room to clear one another, and moderate electrical influence is without effect. At a higher obliquity the drops begin to be scattered out of the vertical plane, which is a sign that collisions are taking place. Moderate electrical influence will reduce the scattering to the vertical plane by causing coalescence of drops which come into contact.

If, as in Savart's beautiful experiments, the resolution into drops is regularized by external vibrations of suitable frequency,

[1] *Pogg. Ann.* vol. CXLIV. p. 443, 1872.

[2] The influence of Electricity on Colliding Water Drops, *Proc. Roy. Soc.* vol. XXVIII. p. 406, 1879.

the principal drops follow the same course, and unless the projection is nearly vertical there are no collisions between them. But it sometimes happens that the spherules are thrown out laterally in a distinct stream, making a considerable angle with the main stream. This is the result of collisions between the spherules and the principal drops. It may even happen that the former are reflected backwards and forwards several times until at last they escape laterally. In all cases the behaviour under feeble electrical influence is a criterion of the occurrence of collisions.

In an experiment, due to Magnus[1], the spherules are diverted from the main stream without collisions by electrical attraction. Advantage may be taken of this to obtain a regular procession of drops finer than would otherwise be possible.

364. The detached masses of liquid into which a jet is resolved do not at once assume and retain a spherical figure, but execute a series of vibrations, being alternately compressed and elongated in the direction of the axis of symmetry. When the resolution is effected in a perfectly periodic manner, each drop is in the same phase of its vibration as it passes through a given point of space; and thence arises the remarkable appearance of alternate swellings and contractions described by Savart. The interval from one swelling to the next is the space described by the drop during one complete vibration about its figure of equilibrium, and is therefore, as Plateau shewed, proportional *cæteris paribus* to the square root of the head.

The time of vibration is of course itself a function of the nature of the fluid (T, ρ) and of the size of the drop, to the calculation of which we now proceed. It may be remarked that the argument from dimensions is sufficient to shew that the time (τ) of an infinitely small vibration of any type is proportional to $\sqrt{(\rho V/T)}$, where V is the volume of the drop.

In the mathematical investigation of the small vibrations of a liquid mass about its spherical figure of equilibrium, we will confine ourselves to modes of vibration symmetrical about an axis, which suffice for the problem in hand. These modes require for their expression only Legendre's functions P_n; the more general

[1] *Pogg. Ann.* vol. CVI. p. 27, 1859.

problem, involving Laplace's functions, may be treated in the same way and leads to the same results.

The radius r of the surface bounding the liquid may be expanded at any time t in the series (§ 336)

$$r = a_0 + a_1 P_1(\mu) + \dots + a_n P_n(\mu) + \dots\dots\dots\dots (1),$$

where $a_1, a_2 \dots$ are small quantities relatively to a_0, and μ represents, as usual, the cosine of the colatitude (θ).

For the volume (V) included within the surface (1) we have

$$V = \tfrac{2}{3}\pi \int_{-1}^{+1} r^3 d\mu = \tfrac{4}{3}\pi a_0^3 [1 + 3\Sigma (2n+1)^{-1} a_n^2/a_0^2] \dots (2),$$

the summation commencing at $n = 1$. Thus, if a be the radius of the sphere of equilibrium,

$$a = a_0 [1 + \Sigma (2n+1)^{-1} a_n^2/a_0^2] \dots\dots\dots\dots (3).$$

The potential energy of capillarity is the product of the tension T and of the surface S. To calculate S we have

$$S = 2\pi \int r \sin\theta \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}} d\theta = 2\pi \int \left\{ r^2 + \tfrac{1}{2}\left(\frac{dr}{d\theta}\right)^2 \right\} \sin\theta \, d\theta.$$

For the first part

$$\int_{-1}^{+1} r^2 d\mu = 2a_0^2 + 2\Sigma (2n+1)^{-1} a_n^2.$$

For the second part

$$\tfrac{1}{2} \int \left(\frac{dr}{d\theta}\right)^2 \sin\theta \, d\theta = \tfrac{1}{2} \int_{-1}^{+1} (1-\mu^2) \left[\Sigma a_n \frac{dP_n}{d\mu} \right]^2 d\mu.$$

The value of the quantity on the right may be found with the aid of the formula

$$\int_{-1}^{+1} (1-\mu^2) \frac{dP_m}{d\mu} \frac{dP_n}{d\mu} d\mu = n(n+1) \int_{-1}^{+1} P_m P_n \, d\mu,$$

in which m is an integer equal to or different from n. Thus

$$\tfrac{1}{2} \int_{-1}^{+1} \left(\frac{dr}{d\theta}\right)^2 \sin\theta \, d\theta = \tfrac{1}{2} \int_{-1}^{+1} (1-\mu^2) \Sigma a_n^2 \left(\frac{dP_n}{d\mu}\right)^2 d\mu$$

$$= \tfrac{1}{2} \Sigma n(n+1) a_n^2 \int_{-1}^{+1} P_n^2 d\mu = \Sigma n(n+1)(2n+1)^{-1} a_n^2.$$

Accordingly

$$S = 4\pi a_0^2 + 2\pi \Sigma (2n+1)^{-1} (n^2+n+2) a_n^2$$

$$= 4\pi a^2 + 2\pi \Sigma (n-1)(n+2)(2n+1)^{-1} a_n^2 \dots\dots (4)$$

by (3).

Thus, if T be the cohesive tension, the potential energy (P) corresponding thereto may be taken to be

$$P = 2\pi T . \Sigma (n-1)(n+2)(2n+1)^{-1} a_n^2 \quad \ldots\ldots\ldots (5).$$

We have now to calculate the kinetic energy of the motion. The velocity-potential ψ may be expanded in the series

$$\psi = \beta_0 + \beta_1 r P_1(\mu) + \ldots + \beta_n r^n P_n(\mu) + \quad \ldots\ldots\ldots (6);$$

and thus for the kinetic energy we get

$$K = \tfrac{1}{2}\rho \iint \psi \, d\psi/dr . a^2 \, d\phi \, d\mu$$

$$= 2\pi\rho a^2 . \Sigma (2n+1)^{-1} n a^{2n-1} \beta_n^2.$$

But by comparison of the value of $d\psi/dr$ from (6) with (1) we find

$$n a^{n-1} \beta_n = da_n/dt;$$

and thus

$$K = 2\pi\rho a^3 . \Sigma (2n^2 + n)^{-1} (da_n/dt)^2 \quad \ldots\ldots\ldots\ldots (7).$$

Since the products of the quantities a_n and da_n/dt do not occur in the expressions for P and K, the motions represented by the various terms take place independently of one another. The equation for a_n is by Lagrange's method (§ 87)

$$\frac{d^2 a_n}{dt^2} + n(n-1)(n+2)\frac{T}{\rho a^3} a_n = 0 \quad \ldots\ldots\ldots\ldots (8);$$

so that, if $a_n \propto \cos(pt + \epsilon)$,

$$p^2 = n(n-1)(n+2)\frac{T}{\rho a^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)^1.$$

The periodic time is equal to $2\pi/p$, so that in terms of V (equal to $\frac{4}{3}\pi a^3$)

$$\tau = \frac{\sqrt{\{3\pi\rho V/T\}}}{\sqrt{\{n(n-1)(n+2)\}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

or in the particular case of n equal to 2

$$\tau = \sqrt{\{3\pi\rho V/8T\}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11).$$

To find the radius of the sphere of water which vibrates seconds, we put in (9) $p = 2\pi$, $T = 74$, $\rho = 1$, $n = 2$. Thus $a = 2{\cdot}47$ centims., or a little less than one inch.

An attempt to compare (11) with the phenomena observed in a jet did not bring out a good agreement. A stream of 19·7 cub.

[1] *Proc. Roy. Soc.* vol. XXIX. p. 97, 1879; Webb, *Mess. of Math.* vol. IX. p. 177, 1880.

cent. per second was broken up under the action of a fork making 128 vibrations per second. Neglecting the mass of the small spherules, we may take for the volume of each principal drop 19·7/128, or ·154 cub. cent. Thence by (11), putting $\rho = 1$, $T = 74$, we have $\tau = {\cdot}0494$ second. This is the calculated value. By observation of the vibrating jet the distance between the first and second swellings, corresponding to the maximum oblateness of the drops, was 16·5 centims. The level of the contraction midway between the two swellings was 36·8 centims. below the surface of the liquid in the reservoir, corresponding to a velocity of 269 centims. per second. These data give for the time of vibration

$$\tau = 16{\cdot}5/269 = {\cdot}0612 \text{ second.}$$

The discrepancy between the two values of τ is probably attributable to excessive amplitude, entailing a departure from the law of isochronism. Observations upon the vibrations of drops delivered singly from pipettes have been made by Lenard[1].

The tendency of the capillary force is always towards the restoration of the spherical figure of equilibrium. By electrifying the drop we may introduce a force operative in the opposite direction. It may be proved[2] that if Q be the charge of electricity in electrostatic measure, the formula corresponding to (9) is

$$p^2 = \frac{n(n-1)}{\rho a^3}\left\{(n+2)T - \frac{Q^2}{4\pi a^3}\right\} \quad \ldots\ldots\ldots\ldots (12).$$

If $T > Q^2/16\pi a^3$, the spherical form is stable for all displacements. When Q is great, the spherical form becomes unstable for all values of n below a certain limit, the maximum instability corresponding to a great, but still finite, value of n. Under these circumstances the liquid is thrown out in fine jets, whose fineness, however, has a limit.

Observations upon the swellings and contractions of a regularly resolved jet may be made stroboscopically, one view corresponding to each complete period of the vibrator; or photographs may be taken by the instantaneous illumination furnished by a powerful electric spark[3].

1 *Wied. Ann.* vol. xxx. p. 209, 1887.

2 *Phil. Mag.* vol. xiv. p. 184, 1882.

3 Some Applications of Photography, *Proc. Roy. Soc. Inst.* vol. xiii. p. 261, 1891; *Nature*, vol. xliv. p. 249, 1891.

In the mathematical investigations of this chapter no account has been taken of viscosity. Plateau held the opinion that the difference between the wave-length of spontaneous division of a jet ($4{\cdot}5 \times 2a$) and the critical wave-length ($\pi \times 2a$) was an effect of viscosity; but we have seen that it is sufficiently accounted for by *inertia*. The inclusion of viscosity considerably complicates the mathematical problem[1], and it will not here be attempted. The result is to shew that, when viscosity is paramount, long threads do not tend to divide themselves into drops at mutual distances comparable with the diameter of the thread, but rather to give way by attenuation at few and distant places. This appears to be in agreement with the observed behaviour of highly viscous threads of glass, or treacle, when supported only at the terminals. A separation into numerous drops, or a varicosity pointing to such a resolution, may thus be taken as evidence that the fluidity has been sufficient to bring inertia into play.

A still more general investigation, in which the influence of electrification is considered, has been given by Basset[2].

[1] *Phil. Mag.* vol. XXXIV. p. 145, 1892.

[2] *Amer. Journ. of Math.* vol. XVI. No. 1.

CHAPTER XXI.

VORTEX MOTION AND SENSITIVE JETS.

365. A LARGE and important group of acoustical phenomena have their origin in the instability of certain fluid motions of the kind classified in hydrodynamics as steady. A motion, the same at all times, satisfies the dynamical conditions, and is thus in a sense possible; but the smallest departure from the ideal so defined tends spontaneously to increase, and usually with great rapidity according to the law of compound interest. Examples of such instability are afforded by sensitive jets and flames, æolian tones, and by the flute pipes of the organ. These phenomena are still very imperfectly understood; but their importance is such as to demand all the consideration that we can give them.

So long as we regard the fluid as absolutely inviscid there is nothing to forbid a finite slip at the surface where two masses come into contact. At such a surface the vorticity (§ 239) is infinite, and the surface may be called a vortex sheet. The existence of a vortex sheet is compatible with the dynamical conditions for steady motion; but, as was remarked at an early date by v. Helmholtz[1], the steady motion is unstable. The simplest case occurs when a plane vortex sheet separates two masses of fluid which move with different velocities, but without internal relative motion—a problem considered by Lord Kelvin in his investigation of the influence of wind upon waves[2]. In the following discussion the method of Lord Kelvin is applied to determine the law of falling away from steady motion in some of the simpler cases of a plane surface of separation.

[1] *Phil. Mag.* vol. XXXVI. p. 337, 1868.

[2] *Phil. Mag.* vol. XLII. p. 368, 1871. See also *Proc. Math. Soc.* vol. X. p. 4, 1878; Basset's *Hydrodynamics*, § 391, 1888; Lamb's *Hydrodynamics*, § 224, 1895.

Let us suppose that below the plane $z=0$ the fluid is of constant density ρ and moves parallel to x with velocity V, and that above that plane the density is ρ' and the velocity V'. As in § 353, let z be measured downwards, and let there be rigid walls bounding the lower fluid at $z=l$ and the upper fluid at $z=-l'$. The disturbance is supposed to involve x and t only through the factors e^{ikx}, e^{int}. The velocity potential $(Vx+\phi)$ in the lower fluid satisfies Laplace's equation, and thus ϕ by the condition at $z=l$ takes the form

$$\phi = C \cosh k(z-l) \,.\, e^{i(nt+kx)} \quad\text{.................(1)};$$

and a similar expression,

$$\phi' = C' \cosh k(z+l') \,.\, e^{i(nt+kx)} \quad\text{.............. (2)},$$

applies to the lower fluid, if the whole velocity-potential be there $(V'x+\phi')$. The connection between ϕ and the elevation (h) at the common surface is

$$-\frac{d\phi}{dz}(z=0) = \frac{dh}{dt} + V\frac{dh}{dx};$$

so that, if

$$h = He^{i(nt+kx)} \quad\text{...................... (3)},$$

$$kC \sinh kl = i(n+kV)H \quad\text{.............. (4)}.$$

In like manner,

$$-kC' \sinh kl' = i(n+kV')H \quad\text{.............. (5)}.$$

We have now to express the condition relating to pressures at $z=0$. The general equation (2), § 244, gives for the lower fluid

$$\begin{aligned}\frac{\delta p}{\rho} &= -gh - \frac{d\phi}{dt} - \tfrac{1}{2}\left(V+\frac{d\phi}{dx}\right)^2 - \tfrac{1}{2}\left(\frac{d\phi}{dz}\right)^2 \\ &= -gh - in\phi - ikV\phi,\end{aligned}$$

squares of small quantities being neglected. In like manner for the upper fluid at $z=0$

$$\frac{\delta p'}{\rho'} = -gh - in\phi' - ikV'\phi'.$$

If there be no capillary tension, δp and $\delta p'$ are equal. If the capillary tension be T, the difference is

$$\delta p - \delta p' = -T\, d^2h/dx^2 = k^2Th,$$

so that

$$g(\rho-\rho')h + k^2Th = i\rho'(n+kV')\phi' - i\rho(n+kV)\phi \quad\text{........(6)}.$$

When the values of ϕ, ϕ' at $z=0$ are introduced from (1), (2), (4), (5), the condition becomes

$$g(\rho-\rho')+k^2T=k\rho(V+n/k)^2\coth kl+k\rho'(V'+n/k)^2\coth kl' \qquad \text{..............(7).}$$

This is the equation which determines the values of n/k. If the roots of the quadratic are real, waves are propagated with the corresponding real velocities; if on the other hand the roots are imaginary, exponential functions of the time enter into the solution, indicating that the steady motion is unstable. The criterion of stability is accordingly

$$(\rho\coth kl+\rho'\coth kl')\{g(\rho-\rho')+Tk^2\} - k\rho\rho'\coth kl\coth kl'(V-V')^2>0 \text{......(8).}$$

If g and T both vanish, the motion is unstable for all disturbances, that is, whatever may be the value of k. If T vanish, the operation of gravity may be to secure stability for certain values of k, but it cannot render the steady motion stable on the whole. For when k is infinitely great, that is, when the corrugations are infinitely fine, $\coth kl=\coth kl'=1$, and the term in g disappears from the criterion. In spite of the impressed forces tending to stability the motion is necessarily unstable for waves of infinitesimal length; and this conclusion may be extended to vortex sheets of any form and to impressed forces of any kind.

If T be finite, then on the contrary there is of necessity stability for waves of infinitesimal length, although there may be instability for waves of finite length.

For further examination we may take the simpler conditions which arise when l and l' are infinite. The criterion of stability then becomes

$$(\rho+\rho')\{g(\rho-\rho')+Tk^2\}-k\rho\rho'(V-V')^2>0 \text{ ...... (9),}$$

and the critical case is determined by equating the left-hand member to zero. This gives a quadratic in k. If the roots of the quadratic are imaginary, the criterion (9) is satisfied for all intermediate values of k, as well as for the infinitely small and infinitely large values by which it is satisfied in all cases, provided that $\rho>\rho'$. The condition of complete stability is thus

$$4g(\rho-\rho')T>\frac{\rho^2\rho'^2(V-V')^4}{(\rho+\rho')^2} \text{ .... (10).}$$

Let W denote the minimum velocity (§ 353) of waves when $V=0$, $V'=0$. Then by (7)

$$(\rho+\rho')^2 W^4 = 4g(\rho-\rho')T \qquad (11),$$

and (10) may be written

$$W^4 > \frac{\rho\rho'(V-V')^2}{(\rho+\rho')^2} \qquad (12).$$

If $(V-V')$ do not exceed the value thus determined, the steady motion is stable for all disturbances; otherwise there will be some finite wave-lengths for which disturbances increase exponentially.

If we now omit the terms in (7) dependent upon gravity and upon capillarity, the equation becomes

$$\rho(n+kV)^2 \coth kl + \rho'(n+kV')^2 \coth kl' = 0 \qquad (13).$$

When $l=l'$, or when both these quantities are infinite, we have simply

$$\rho(n+kV)^2 + \rho'(n+kV')^2 = 0 \qquad (14),$$

or

$$\frac{n}{k} = -\frac{\rho V + \rho' V' \pm i\sqrt{(\rho\rho')}\,.\,(V-V')}{\rho+\rho'} \qquad (15).$$

We see from (15) that, as was to be expected, a motion common to both parts of the liquid has no dynamical significance. An equal addition to V and V' is equivalent to a deduction of like amount from n/k. If $\rho=\rho'$, (15) becomes

$$n/k = -\tfrac{1}{2}(V+V') \pm \tfrac{1}{2}i(V-V') \qquad (16).$$

The essential features of the case are brought out by the simple case where $V'=-V$, so that the steady motions of the two masses of fluid are equal and opposite. We have then

$$n/k = \pm iV \qquad (17);$$

and for the elevation,

$$h = He^{\pm kVt}\cos(kx+\epsilon) \qquad (18),$$

corresponding to

$$h = H\cos(nx+\epsilon) \qquad (19),$$

initially.

If when $t=0$, $dh/dt=0$,

$$h = H\cosh kVt\cos(nx+\epsilon) \qquad (20),$$

indicating that the waves upon the surface of separation are stationary, and increase in amplitude with the time according to the law of the hyperbolic cosine. The rate of increase of the term with the positive exponent is extremely rapid. Since $k = 2\pi/\lambda$, the amplitude is multiplied by e^{π}, or about 23, in the time occupied by either stream in passing over a distance λ.

If $V' = V$, the roots (16) are equal, but the general solution may be obtained by the usual method. Thus, if we put

$$V' = V(1+\alpha),$$

where α is ultimately to vanish,

$$n/k = -V \pm \tfrac{1}{2}i\alpha V;$$

and
$$h = e^{ik(x-Vt)}\{Ae^{\frac{1}{2}ikVt.\alpha} + Be^{-\frac{1}{2}ikVt.\alpha}\},$$

where A, B are arbitrary constants. Passing now to the limit where $\alpha = 0$, and taking new arbitrary constants, we get

$$h = e^{ik(x-Vt)}\{C + Dt\},$$

or in real quantities,

$$h = \{C + Dt\}\cos k(x - Vt + \epsilon).$$

If initially
$$h = \cos kx, \quad dh/dt = 0,$$

$$h = \cos k(Vt - x) + kVt \sin k(Vt - x) \quad \text{.........} \quad (21).$$

The peculiarity of this case is that previous to the displacement there is no real surface of separation at all.

The general solution involving l and l' may be adapted to represent certain cases of disturbance of a two-dimensional jet of width $2l$ playing into stationary fluid. For if the disturbance be *symmetrical*, so that the median plane is a plane of symmetry, the conditions are the same as if a fixed wall were there introduced. If the surrounding fluid be unlimited, $l' = \infty$, $\coth kl' = 1$; and the equation determining n becomes, if $V' = 0$, $\rho' = \rho$,

$$(n + kV)^2 \coth kl + n^2 = 0 \quad \text{..............} \quad (22),$$

of which the solution is

$$\frac{n}{kV} = \frac{-1 \pm i\sqrt{(\tanh kl)}}{1 + \tanh kl} \quad \text{.................} \quad (23).$$

Thus
$$h = He^{\pm\mu kVt}\cos k\left\{x - \frac{Vt}{1 + \tanh kl}\right\} \quad \text{...........} \quad (24),$$

where
$$\mu = \frac{\sqrt{(\tanh kl)}}{1 + \tanh kl} \quad \text{......................} \quad (25).$$

This represents the progression of symmetrical disturbances in a jet of width $2l$ playing into a stationary environment of the same density.

If kl be very small, so that the wave-length is large in comparison with the thickness of the jet,

$$h = He^{\pm\sqrt{(kl)}\,.\,kVt} \cos k\{x - Vt\} \quad\text{..............(26).}$$

The investigation of the asymmetrical disturbance of a jet requires the solution of the problem of a single vortex sheet when the condition to be satisfied at $z = l$ is $\phi = 0$, instead of as hitherto $d\phi/dz = 0$. The value of ϕ is

$$\phi = -i(n + kV)H \frac{\sinh k(z - l)}{k \cosh kl} e^{int} e^{ikx} \quad\text{....................(27);}$$

from which, if as before $d\phi'/dz = 0$ when $z = -l'$,

$$\rho(n + kV)^2 \tanh kl + \rho'(n + kV')^2 \coth kl' = 0 \quad\text{... (28).}$$

If $l' = \infty$, $\rho' = \rho$, $V' = 0$,

$$(n + kV)^2 \tanh kl + n^2 = 0 \quad\text{.............. (29).}$$

This is applicable to a jet of width $2l$, moving with velocity V in still fluid and displaced in such a manner that the sinuosities of its two surfaces are parallel.

When kl is small, we have approximately

$$h = He^{\pm\sqrt{(kl)}\,.\,kVt} \cos k(x - kl\,.\,Vt) \quad\text{........... (30).}$$

By a combination of the solutions represented by (26), (30), we may determine the consequences of any displacements in two dimensions of the two surfaces of a thin jet moving with velocity V in still fluid of its own density.

366. The investigations of § 365 may be considered to afford an adequate general explanation of the sensitiveness of jets. In the ideal case of abrupt transitions of velocity, constituting vortex sheets, in frictionless fluid, the motion is always unstable, and the degree of instability increases as the wave-length of the disturbance diminishes.

The direct application of this result to actual jets would lead us to the conclusion that their sensitiveness increases indefinitely with pitch. It is true that, in the case of certain flames, the pitch of the most efficient sounds is very high, not far from the upper limit of human hearing; but there are other kinds of sensitive jets on which these high sounds are without effect, and which require for their excitation a moderate or even a grave pitch.

A probable explanation of the discrepancy readily suggests itself. The calculations are founded upon the supposition that the changes of velocity are discontinuous—a supposition that cannot possibly agree with reality. In consequence of fluid friction a surface of discontinuity, even if it could ever be formed, would instantaneously disappear, the transition from the one velocity to the other becoming more and more gradual, until the layer of transition attained a sensible width. When this width is comparable with the wave-length of a sinuous disturbance, the solution for an abrupt transition ceases to be applicable, and we have no reason for supposing that the instability would increase for much shorter wave-lengths.

A general idea of the influence of viscosity in broadening a jet may be obtained from Fourier's solution of the problem where the initial width is supposed to be infinitesimal. Thus, if in the general equations v and w vanish, while u is a function of y only, the equation satisfied by u is (as in § 347)

$$\frac{du}{dt} = \frac{\mu}{\rho}\frac{d^2u}{dy^2} \qquad (1).$$

The solution of this equation for the case where u is initially sensible only at $y = 0$ is

$$u = U_1 \frac{e^{-y^2/4\nu t}}{2\sqrt{(\pi\nu t)}} \qquad (2),$$

where $\nu = \mu/\rho$, and U_1 denotes the initial value of $\int u\,dy$. When $y^2 = 4\nu t$, the value of u is less than that to be found at the same time at $y = 0$ in the ratio $e : 1$. For air $\nu = \cdot 16$ C.G.S., and thus after a time t the thickness ($2y$) of the jet is comparable in magnitude with $1\cdot 6\sqrt{t}$; for example, after one second it may be considered to be about $1\frac{1}{2}$ cm.

There is therefore ample foundation for the suspicion that the phenomena of sensitive jets may be greatly influenced by fluid friction, and deviate materially from the results of calculations based upon the supposition of discontinuous changes of velocity. Under these circumstances it becomes important to investigate the character of the equilibrium of stratified motion in cases more nearly approaching what is met with in practice. A complete investigation which should take account of all the effects of viscosity would encounter many formidable difficulties. For the present purpose we shall treat the fluid as frictionless and be content to obtain solutions for laws of stratification which are free from discontinuity. For the undisturbed motion the component velocities v, w are zero, and u is a function of y only, which we will denote by U. A curve in which U is ordinate and y is abscissa represents the law of stratification, and may be called for brevity the velocity curve. The vorticity Z (§ 239) of the steady motion is equal to $\frac{1}{2}dU/dy$.

If in the disturbed motion, assumed to be in two dimensions, the velocities be denoted by $U+u$, v, and the vorticity by $Z+\zeta$, the general equation (4), § 239, takes the form

$$\frac{d(Z+\zeta)}{dt}+(U+u)\frac{d(Z+\zeta)}{dx}+v\frac{d(Z+\zeta)}{dy}=0,$$

in which $$dZ/dt=0, \quad dZ/dx=0.$$

Thus, if the square of the disturbances be neglected, the equation may be written

$$\frac{d\zeta}{dt}+U\frac{d\zeta}{dx}+v\frac{dZ}{dy}=0 \qquad (3);$$

and the equation of continuity for an incompressible fluid gives

$$\frac{du}{dx}+\frac{dv}{dy}=0 \qquad (4).$$

If the values of Z and ζ in terms of the velocities be substituted in (3),

$$\left(\frac{d}{dt}-U\frac{d}{dx}\right)\left(\frac{du}{dy}-\frac{dv}{dx}\right)+v\frac{d^2U}{dy^2}=0 \qquad (5).$$

We now introduce the supposition that as functions of x and t, u and v are proportional to $e^{int}.e^{ikx}$. From (4)

$$iku+dv/dy=0 \qquad (6);$$

and if this value of u be substituted in (5), we obtain

$$\left(\frac{n}{k}+U\right)\left(\frac{d^2v}{dy^2}-k^2v\right)-\frac{d^2U}{dy^2}v=0 \qquad (7)^1.$$

[1] *Proc. Math. Soc.* vol. XI. p. 68, 1880.

In (7) k may be regarded as real, and in any particular problem that may be proposed the principal object is to determine the corresponding value of n, and especially whether it is real or imaginary. One general proposition of importance relates to the case where d^2U/dy^2 is of one sign, so that the velocity curve is wholly convex, or wholly concave, throughout the entire space between two fixed walls at which the condition $v=0$ is satisfied. Let $n/k = p + iq$, $v = \alpha + i\beta$, where p, q, α, β are real. Substituting in (7) we get

$$\frac{d^2\alpha}{dy^2} + i\frac{d^2\beta}{dy^2} = \left[k^2 + \frac{d^2U}{dy^2}\frac{p+U-iq}{(p+U)^2+q^2}\right](\alpha+i\beta)=0;$$

or, on equating separately to zero the real and imaginary parts,

$$\frac{d^2\alpha}{dy^2} = k^2\alpha + \frac{d^2U}{dy^2}\frac{(p+U)\,\alpha + q\beta}{(p+U)^2+q^2} \quad\ldots\ldots\ldots\ldots\ldots(8),$$

$$\frac{d^2\beta}{dy^2} = k^2\beta + \frac{d^2U}{dy^2}\frac{-q\alpha + (p+U)\,\beta}{(p+U)^2+q^2} \quad\ldots\ldots\ldots\ldots(9).$$

Multiplying (8) by β, (9) by α, and subtracting, we get

$$\beta\frac{d^2\alpha}{dy^2} - \alpha\frac{d^2\beta}{dy^2} = \frac{d}{dy}\left(\beta\frac{d\alpha}{dy} - \alpha\frac{d\beta}{dy}\right) = \frac{d^2U}{dy^2}\,\frac{q\,(\alpha^2+\beta^2)}{(p+U)^2+q^2}\ldots(10).$$

At the limits v, and therefore both α and β, are by hypothesis zero. Hence integrating (10) between the limits, we see that q must be zero, if d^2U/dy^2 is of one sign throughout the range of integration. Accordingly n is real, and the motion, if not absolutely stable, is at any rate not exponentially unstable.

Another general conclusion worthy of notice can be deduced from (7). Writing it in the form

$$\frac{d^2v}{dy^2} = \left\{k^2 + \frac{d^2U/dy^2}{U+n/k}\right\}v,$$

we see that, if n be real, v cannot pass from one zero value to another zero value, unless d^2U/dy^2 and $(n+kU)$ be somewhere of contrary signs. Thus if we suppose that U is positive and d^2U/dy^2 negative throughout, and that V is the greatest value of U, we find that $n+kV$ must be positive.

367. A class of problems admitting of fairly simple solution is obtained by supposing the vorticity Z to be constant throughout layers of finite thickness and to change its value only in passing a limited number of planes, for each of which y is constant. In such cases the velocity curve is composed of portions of straight lines which meet one another at finite angles. This state of things is supposed to be disturbed by bending the surfaces of transition.

Throughout any layer of constant vorticity $d^2U/dy^2 = 0$, and thus by (7), § 366, wherever $n + kU$ is not equal to zero,

$$\frac{d^2v}{dy^2} - k^2v = 0 \qquad (1),$$

of which the solution is

$$v = Ae^{ky} + Be^{-ky} \qquad (2).$$

If there are several layers in each of which Z is constant, the various solutions of the form (2) are to be fitted together, the arbitrary constants being so chosen as to satisfy certain boundary conditions. The first of these conditions is evidently the continuity of v, or as it may be expressed,

$$\Delta v = 0 \qquad (3).$$

The other necessary condition may be obtained by integrating (7), § 366, across the surface of transition. Thus

$$\left(\frac{n}{k} + U\right) . \Delta \left(\frac{dv}{dy}\right) - \Delta \left(\frac{dU}{dy}\right) . v = 0 \qquad (4).$$

These are the conditions that the velocity shall be continuous at the places where dU/dy changes its value.

In the problems which we shall consider the fluid is either bounded by a fixed plane at which y is constant, or else extends to infinity. For the former the condition is simply $v = 0$. If there be a layer extending to infinity in the positive direction, A must vanish in the expression (2) applicable to this layer; if a layer extend to infinity in the negative direction, the corresponding B must vanish.

Under the first head we will consider a problem of some generality, where the stratified steady motion takes place between fixed walls at $y = 0$ and at $y = b_1 + b' + b_2$.

The vorticity is constant throughout each of the three layers bounded by $y = 0$, $y = b_1$; $y = b_1$, $y = b_1 + b'$; $y = b_1 + b'$, $y = b_1 + b' + b_2$ (Fig. 67). There are thus two internal surfaces where the vorticity changes. The values of U at these surfaces may be denoted by U_1, U_2.

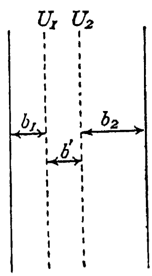


Fig. 67.

In conformity with (3) and with the condition that $v=0$ when $y=0$, we may take in the first layer

$$v=v_1=\sinh ky \qquad (5);$$

in the second layer

$$v=v_2=v_1+M_1\sinh k(y-b_1) \qquad (6);$$

in the third layer

$$v=v_3=v_2+M_2\sinh k(y-b_1-b') \qquad (7).$$

The condition that $v=0$, when $y=b_1+b'+b_2$, now gives

$$0=M_2\sinh kb_2+M_1\sinh k(b_2+b')+\sinh k(b_2+b'+b_1) \qquad (8).$$

We have still to express the other two conditions (4) at the surfaces of transition. At the first surface

$$v=\sinh kb_1, \qquad \Delta(dv/dy)=kM_1;$$

at the second surface

$$v=M_1\sinh kb'+\sinh k(b_1+b'), \quad \Delta(dv/dy)=kM_2.$$

If we denote the values of $\Delta(dU/dy)$ at the two surfaces respectively by Δ_1, Δ_2, our conditions become

$$(n+kU_1)M_1-\Delta_1\sinh kb_1=0 \qquad (9),$$

$$(n+kU_2)M_2-\Delta_2\{M_1\sinh kb'+\sinh k(b_1+b')\}=0 \qquad (10).$$

By (8), (9), (10) the values of M_1, M_2, n are determined.

The equation for n is found by equating to zero the determinant of the three equations. It may be written

$$An^2+Bn+C=0 \qquad (11),$$

where

$$A=\sinh k(b_2+b'+b_1) \qquad (12),$$

$$B=k(U_1+U_2)\sinh k(b_2+b'+b_1)$$
$$+\Delta_2\sinh kb_2\ \sinh k(b_1+b')+\Delta_1\sinh kb_1\ \sinh k(b_2+b') \qquad (13),$$

$$C=k^2U_1U_2\sinh k(b_2+b'+b_1)$$
$$+kU_1\Delta_2\sinh kb_2\ \sinh k(b_1+b')+kU_2\Delta_1\sinh kb_1\ \sinh k(b_2+b')$$
$$+\Delta_1\Delta_2\sinh kb_1\ \sinh kb_2\ \sinh kb' \qquad (14).$$

To find the character of the roots we have to form the expression for B^2-4AC. On reduction we get

$$B^2-4AC=\{k(U_1-U_2)\sinh k(b_2+b'+b_1)$$
$$+\Delta_1\sinh kb_1\ \sinh k(b_2+b')-\Delta_2\sinh kb_2\ \sinh k(b_1+b')\}^2$$
$$+4\Delta_1\Delta_2\sinh^2 kb_1\ \sinh^2 kb_2 \qquad (15).$$

Hence if Δ_1, Δ_2 have the same sign, that is, if the velocity curve (§ 366) be of one curvature throughout, $B^2 - 4AC$ is positive, and the two values of n are real. Under these circumstances the disturbed motion is stable.

We will now suppose that the surfaces at which the vorticity changes are symmetrically situated, so that $b_1 = b_2 = b$.

In this case we find

$$A = \sinh k(2b + b') \qquad (16),$$

$$B = k(U_1 + U_2)\sinh k(2b + b') + (\Delta_1 + \Delta_2)\sinh kb \, \sinh k(b + b') \qquad (17),$$

$$C = k^2 U_1 U_2 \sinh k(2b + b') + k(U_1\Delta_2 + U_2\Delta_1)\sinh kb \, \sinh k(b + b') + \Delta_1\Delta_2 \sinh^2 kb \, \sinh kb' \qquad (18),$$

$$B^2 - 4AC = 4\Delta_1\Delta_2 \sinh^4 kb + \{k(U_1 - U_2)\sinh k(2b + b') + (\Delta_1 - \Delta_2)\sinh kb \, \sinh k(b + b')\}^2 \qquad (19).$$

Under this head there are two sub-cases which may be especially noted. The first is that in which the values of U are the same on both sides of the median plane, so that the middle layer is a region of constant velocity without vorticity, and the velocity curve is that shewn in Fig. 68. We may suppose that $U = V$ in the middle layer, and that $U = 0$ at the walls, without loss of generality, since any constant velocity (U_0) superposed upon this system merely alters n by the corresponding quantity $-kU_0$, as is evident from (7), § 366.

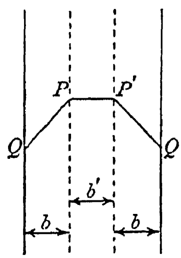


Fig. 68.

Thus $$U_1 = U_2 = V, \quad \Delta_2 = \Delta_1 = \Delta = -V/b;$$

and $$B^2 - 4AC = 4\Delta^2 \sinh^4 kb.$$

Hence $$n + kV = \frac{V}{b}\,\frac{\sinh kb \, \sinh k(b + b') \pm \sinh^2 kb}{\sinh k(2b + b')} \qquad (20).$$

As was to be expected, since the curvature of the velocity curve is of one sign, the values of n in (20) are real. It is easy from the symmetry to see that the two normal disturbances are such that the values of v at the surfaces of separation are either equal or opposite for a given value of x. In the first case the surfaces are bent towards the same side, and (as may be found from the equations or inferred from the particular case presently to be mentioned) the corresponding value of n in (20) has the upper sign. In the second case the motion is symmetrical with respect to the median plane which behaves as a fixed wall.

If the middle layer be absent ($b'=0$), one value of n, that corresponding to the symmetrical motion, vanishes. The remaining value is given by

$$n + kV = \frac{2\sinh^2 kb}{\sinh 2kb} = \frac{V \tanh kb}{b} \quad\ldots\ldots\ldots\ldots (21).$$

The other case which we shall consider is that in which the velocities U on the two sides of the median plane are opposite to one another; so that

$$U_1 = -U_2 = V, \quad \Delta_2 = -\Delta_1 = -\mu V \quad\ldots\ldots\ldots (22).$$

Here $B = 0$, and

$$C = -k^2V^2 \sinh k(2b+b') - 2k\mu V^2 \sinh kb \ \sinh k(b+b') \\ - \mu^2 V^2 \sinh^2 kb \ \sinh kb'.$$

For the sake of brevity we will write $kb = \beta$, $kb' = \beta'$; so that the equation for n becomes

$$\frac{n^2}{k^2V^2} = \frac{k^2\sinh(2\beta+\beta') + 2k\mu\sinh\beta\,\sinh(\beta+\beta') + \mu^2\sinh^2\beta\,\sinh^2\beta'}{k^2\sinh(2\beta+\beta')}.$$

$$= \frac{\{\mu\sinh\beta\,\sinh\beta' + k\sinh(\beta+\beta')\}^2 - k^2\sinh^2\beta}{k^2\sinh\beta'\,\sinh(2\beta+\beta')} \quad\ldots\ldots (23).$$

Here the two values of n are equal and opposite; and, since Δ_1, Δ_2 are of opposite signs, the question is open as to whether n is real or imaginary.

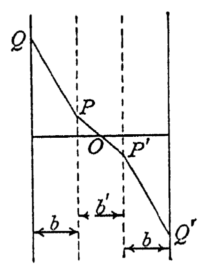


Fig. 69.

It is at once evident that n is real if μ be positive, that is, if Δ_1 and V are of the same sign as in Fig. 69.

Even when μ is negative, n^2 is necessarily positive for great values of k, that is, for small wave-lengths. For we have ultimately from (23)

$$n = \pm kV.$$

We may now inquire for what values of μ n^2 may be negative when k is very small, that is, when the wave-length is very great. Equating the numerator of (23) to zero, and expanding the hyperbolic sines, we get as a quadratic in μ,

$$\mu^2 b^2 b' + 2\mu b(b+b') + 2b + b' = 0,$$

whence $\quad \mu = 1/b, \quad$ or $\quad \mu = -1/b - 2/b' \quad\ldots\ldots\ldots\ldots\ldots (24).$

When μ lies between these limits (and then only), n^2 is negative, and the disturbance (of great wave-length) increases exponentially with the time.

We may express these results by means of the velocity V_0 at the wall where $y=0$. We have

$$V_0 = V\frac{b+\frac{1}{2}b'}{\frac{1}{2}b'} + \Delta_1 b = V\left(\frac{b+\frac{1}{2}b'}{\frac{1}{2}b'} + \mu b\right).$$

The limiting values of V_0 are therefore $bV/\frac{1}{2}b'$ and 0. The velocity curve corresponding to the first limit is shewn in Fig. 70 by the line $QPOP'Q'$, the point Q being found by drawing a line AQ parallel to OP to meet the wall in Q. If $b'=2b$, QP is parallel to OA, or the velocity is constant in each of the extreme layers.

At the second limit $V_0=0$, and the velocity curve is that shewn in Fig. 71.

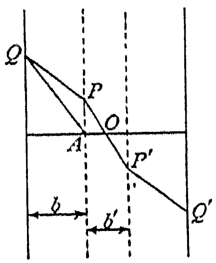


Fig. 70.

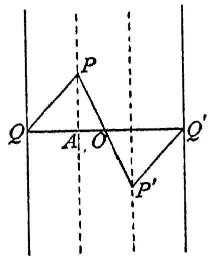


Fig. 71.

It is important to notice that motions represented by velocity curves intermediate between these limits are unstable in a manner not possible to motions in which the velocity curve, as in Fig. 68, is of one curvature throughout.

According to the first approximation, the motion of Fig. 71 is on the border-line between stability and instability for disturbances of great wave-length; but, if we pursue the calculation, we find that it is really unstable. Taking in (23)

$$\mu = -1/b - 2/b',$$

we get, after reduction,

$$\frac{n^2}{k^2V^2} = -\frac{k^2b^2}{3} \qquad (25),$$

indicating instability.

From the second form of (23) we see that, whatever may be the value of k, it is possible so to determine μ that the disturbance shall be unstable. The condition is simply that μ must be between the limits

$$-k\frac{\sinh k(b+b') \pm \sinh kb}{\sinh kb \ \sinh kb'},$$

or $$-k\{\coth kb + \coth \tfrac{1}{2}kb'\}, \quad -k\{\coth kb + \tanh \tfrac{1}{2}kb'\} \ldots (26),$$

of which the first corresponds to the superior limit to the *numerical value* of μ.

When k is very large, the limits are very great and very close. When k is small, they become

$$-1/b - 2/b' \quad \text{and} \quad -1/b,$$

as has already been proved. As k increases from 0 to ∞, the numerical value of the upper limit increases continuously from $1/b + 2/b'$ to ∞, and in like manner that of the inferior limit from $1/b$ to ∞. The motion therefore cannot be stable for *all values of k*, if μ (being negative) exceed numerically $1/b$. The final condition of complete stability is therefore that algebraically

$$\mu > -1/b \ldots\ldots\ldots\ldots (27).$$

In the transition case

$$V_0 = \left(\mu - \frac{1}{b} + \frac{2}{b'}\right) Vb = \frac{2Vb}{b'} \ldots\ldots\ldots\ldots (28);$$

it is that represented in Fig. 70. If PQ be bent more downwards than is there shewn, as for example in Fig. 71, the steady motion is certainly unstable.

Reverting to the general equations (11), (12), (13), (14), (15), let us suppose that $\Delta_2 = 0$, amounting to the abolition of the corresponding surface of discontinuity. We get

$$B = k(U_1 + U_2)\sinh k(b_2 + b' + b_1) + \Delta_1 \sinh kb_1 \ \sinh k(b_2 + b'),$$
$$B^2 - 4AC = \{k(U_1 - U_2)\sinh k(b_2 + b' + b_1) + \Delta_1 \sinh kb_1 \ \sinh k(b_2 + b')\}^2;$$

so that $$n = -kU_2 \ldots\ldots\ldots\ldots (29),$$

or $$n = -kU_1 - \frac{\Delta_1 \sinh kb_1 \ \sinh k(b_2 + b')}{\sinh k(b_1 + b' + b_2)} \ldots\ldots\ldots (30).$$

The latter is the general solution for two layers of constant vorticity of breadths b_1 and $b' + b_2$. An equivalent result may be obtained by supposing in (11) &c. that $b' = 0$, or that $b_1 = 0$.

The occurrence of (29) suggests that any value of $-kU$ is admissible as a value of n, and the meaning of this is apparent from the fundamental equation (7), § 366. For, at the place where $n+kU=0$, (1) need not be satisfied, that is, the arbitrary constants in (2) may change their values. It is evident that, with the prescribed values of n and k, a solution may be found satisfying the required conditions at the walls and at the surfaces where dU/dy changes value, as well as equation (3) at the plane where $n+kU=0$. In this motion an additional vorticity is supposed to be communicated to the fluid at the plane in question, and it moves with the fluid at velocity U.

We may inquire what occurs at a second place in the fluid where the velocity happens to be the same as at the first place of added vorticity. The second place may be either within a layer of originally uniform vorticity, or upon a surface of transition. In the first case nothing very special presents itself. If there be no new vorticity at the second place, the value of v is definite as usual, save as to one arbitrary multiplyer. But, consistently with the given value of n, there may be new vorticity at the second as well as at the first place, and then the complete value of v for the given n may be regarded as composed of two parts, each proportional to one of the new vorticities and each affected by an arbitrary multiplyer.

If the second place lie upon a surface of transition, it follows from (4) that $v=0$, since $\Delta(dU/dy)$ is finite. From this fact we might be tempted to infer that the surface in question behaves like a fixed wall, but a closer examination shews that the inference would be unwarranted. In order to understand this, it may be well to investigate the relation between v and the displacement of the surface, supposed also to be proportional to $e^{int} . e^{ikx}$. Thus, if the equation of the surface be

$$F = y - he^{int+ikx} = 0 \text{..................} (31),$$

the condition to be satisfied is[1]

$$\frac{dF}{dt} + U\frac{dF}{dx} + v\frac{dF}{dy} = 0 \text{..................} (32),$$

so that

$$-ih(n+kU) + v = 0 \text{..................} (33)$$

[1] Lamb's *Hydrodynamics*, § 10.

is the required relation. A finite h is thus consistent with an evanescent v.

368. In the problems of § 367 the fluid is bounded by fixed walls; in those to which we now proceed, it will be considered to be unlimited. As a first example, let us suppose that on the upper side of a layer of thickness b the undisturbed velocity U is equal to $+V$, and on the lower side to $-V$, while inside the layer

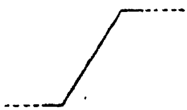


Fig. 72.


Fig. 73.


Fig. 74.

it changes uniformly, Fig. 72. The vorticity within the layer is V/b, and outside the layer it is zero.

The most straightforward method of attacking this problem is perhaps on the lines of § 367. From $y=-\infty$ to $y=0$, we should assume an expression of the form $v_1=e^{ky}$, satisfying the necessary condition when $y=-\infty$. Then from $y=0$ to $y=b$,

$$v_2=v_1+M_1 \sinh ky\,;$$

and from $y=b$ to $y=+\infty$,

$$v_3=v_2+M_2 \sinh k(y-b).$$

But by the conditions at $+\infty$, v_3 must be of the form e^{-ky}, so that

$$1+M_1+M_2e^{-kb}=0.$$

The two other conditions may then be formed as in § 367, and the two constants M_1, M_2 eliminated, giving finally an equation for n. But it will be more appropriate and instructive to follow a different course, suggested by vortex theory.

If we write the fundamental equation

$$\left(\frac{n}{k}+U\right)\left(\frac{d^2v}{dy^2}-k^2v\right)-\frac{d^2U}{dy^2}v=0 \text{ .... (1)},$$

in the form

$$d^2v/dy^2-k^2v=Y \text{......................(2)},$$

we see that, if $Y=0$ from $y=-\infty$ to $y=+\infty$, then $v=0$. Any value that v may have may thus be regarded as dependent upon Y, and further, in virtue of the linearity, as compounded by simple addition of the values corresponding to the partial values of Y.

In the applications which we have in view Y vanishes, except at certain definite places—the surfaces of discontinuity—where alone d^2U/dy^2 differs from zero. The complete value of v may thus be found by summation of partial values, each corresponding to a single surface of discontinuity.

To find the partial value corresponding to a surface of discontinuity situate at $y=y_1$, we have to suppose in (2) that Y vanishes at all other places, while v vanishes at $\pm\infty$. Thus, when $y>y_1$, v must be proportional to $e^{-k(y-y_1)}$, and when $y<y_1$, v must be proportional to $e^{+k(y-y_1)}$. Moreover, since v itself must be continuous at $y=y_1$, the coefficients of the exponentials must be equal, so that the value may be written

$$v=Ce^{\pm k(y-y_1)} \quad\text{.........................(3)},$$

when C is some constant.

In the particular problem above proposed there are two surfaces of discontinuity, at $y=0$ and at $y=b$; and accordingly the complete value of v may be written in the form

$$v=Ae^{\pm ky}+Be^{\pm k(y-b)} \quad\text{....................(4)}.$$

We have now to satisfy at each surface the equation of condition (4), § 367. When $y=0$, we have from (4)

$$v_0=A+Be^{-kb}, \qquad \Delta(dv/dy)_0=-2kA,$$

while $U=-V$, $\qquad \Delta(dU/dy)=+2V/b$;

and when $y=b$,

$$v_b=Ae^{-kb}+B, \qquad \Delta(dv/dy)_b=-2kB,$$

while $U=+V$, $\qquad \Delta(dU/dy)=-2V/b$.

The conditions to be satisfied by $B:A$ and n are thus

$$A\{n-kV+V/b\}+B\{Ve^{-kb}/b\}=0 \quad\text{...........(5)},$$

$$A\{Ve^{-kb}/b\}-B\{n+kV-V/b\}=0 \quad\text{...........(6)};$$

from which by elimination of $B:A$,

$$n^2=\frac{V^2}{b^2}\{(kb-1)^2-e^{-2kb}\} \quad\text{................(7)}.$$

When kb is small, that is, when the wave-length is great in comparison with b, the case approximates to that of a sudden transition from the velocity $-V$ to the velocity $+V$. Then from (7)

$$n^2=-k^2V^2 \quad\text{.........................(8)},$$

in agreement with the value already found (17), § 365. In this case the steady motion is unstable. On the other hand, when kb is great, we find from (7)

$$n^2 = k^2V^2 \quad\text{.............................(9)};$$

and, since the two values of n are real, the motion is stable. It appears, therefore, that so far from the instability increasing indefinitely with vanishing wave-length, as happens when the transition from $-V$ to $+V$ is sudden, a diminution of wave-length below a certain value is accompanied by an instability which gradually decreases, and is finally exchanged for actual stability. The following table exhibits more in detail the progress of b^2n^2/V^2 as a function of kb:—

kb	b^2n^2/V^2	kb	b^2n^2/V^2
·2	−·03032	1·0	−·13534
·4	−·08933	1·2	−·05072
·6	−·14120	1·3	+·01573
·8	−·16190	2·0	+·98168

We see that the instability is greatest when $kb = ·8$ nearly, that is, when $\lambda = 8b$; and that the passage from instability to stability takes place when $kb = 1·3$ nearly, or $\lambda = 5b$.

Corresponding with the two values of n, there are two ratios of $B : A$ determined by (5) or (6), each of which gives a normal mode of disturbance, and by means of these normal modes arbitrary initial circumstances may be represented. It will be seen that for the stable disturbances the ratio $B : A$ is real, indicating that the sinuosities of the two surfaces are at every moment in the same phase.

We may next take an example from a jet of thickness $2b$ moving in still fluid, supposing that the velocity in the middle of the jet is V, and that it falls uniformly to zero on either side, (Fig. 73). Taking the origin of y in the middle line, we may write

$$U = V(1 \mp y/b) \quad\text{.................... (10)},$$

in which the $-$ sign applies to the upper, and the $+$ sign to the lower half of the jet (Fig. 73). There are now three surfaces $y = -b$, $y = 0$, $y = +b$, at which the form of v suffers discontinuity. As in (4) we may take

$$v = Ae^{\pm k(y+b)} + Be^{\pm ky} + Ce^{\pm k(y-b)} \quad\text{........... (11)};$$

so that, when

$$y=-b, \quad U=0, \quad \Delta(dU/dy)=V/b,$$
$$v=A+Be^{-kb}+Ce^{-2kb}, \quad \Delta(dv/dy)=-2kA;$$

when
$$y=0, \quad U=V, \quad \Delta(dU/dy)=-2V/b,$$
$$v=Ae^{-kb}+B+Ce^{-kb}, \quad \Delta(dv/dy)=-2kB;$$

when
$$y=b, \quad U=0, \quad \Delta(dU/dy)=V/b,$$
$$v=Ae^{-2kb}+Be^{-kb}+C, \quad \Delta(dv/dy)=-2kC.$$

The introduction of these values into the equations of condition (4), § 367 gives

$$mA+\gamma B+\gamma^2 C=0 \quad \text{.........................} \quad (12),$$

$$\gamma A+(\tfrac{3}{2}-\tfrac{1}{2}m-kb)B+\gamma C=0 \quad \text{...........} \quad (13),$$

$$\gamma^2 A+\gamma B+mC=0 \quad \text{.........................} \quad (14),$$

which are the equations determining $A : B : C$ and n.

By the symmetries of the case, or by inspection of (12), (13), (14), we see that one of the normal disturbances is defined by

$$B=0, \quad A+C=0 \quad \text{...................} \quad (15),$$

and that the corresponding value of m is γ^2. Thus for the symmetrical disturbance

$$n=-\frac{V}{2b}(1-e^{-2kb}) \quad \text{...................} \quad (16),$$

indicating stability, so far as this mode is concerned.

The general determinant of the system of three equations may be put into the form

$$(m-\gamma^2)\{m^2+(\gamma^2+2kb-3)m+\gamma^2(1+2kb)\}=0 \quad ..(17),$$

in which the first factor corresponds to the symmetrical disturbance already considered. The two remaining values of n are real, if

$$(\gamma^2+2kb-3)^2-4\gamma^2(1+2kb)>0 \quad \text{..........} \quad (18),$$

but not otherwise. When kb is infinite, $\gamma=0$, and (18) is satisfied; so that the motion is stable when the wave-length of disturbance is small in comparison with the thickness ($2b$) of the jet. On the other hand, as may be proved without difficulty by expanding γ, or e^{-kb}, in (18), the motion is unstable, when the wave-length is great in comparison with the thickness of the jet.

The values of the left-hand member of (18) can be more easily computed when it is thrown into the form

$$(5+2kb-e^{-2kb})^2-16\,(1+2kb) \quad\ldots\ldots\ldots\ldots (19).$$

Some corresponding values of (19) and $2kb$ are tabulated below:—

$2kb$	(19)	$2kb$	(19)
·5	− ·054	2·5	− ·975
1·0	− ·279	3·0	− ·794
1·5	− ·599	3·5	− ·263
2·0	− ·876	4·0	+ ·671

The imaginary part of n, when such exists, is proportional to the square root of (19). The wave-length of maximum instability is thus determined approximately by $2kb = 2{\cdot}5$, or $\lambda = 2{\cdot}5 \times 2b$. The critical wave-length is given by $2kb = 3{\cdot}5$ nearly, or $\lambda = 1{\cdot}8 \times 2b$, smaller wave-lengths than this leading to stability, and greater wave-lengths to instability. In these respects there is a fairly close analogy with cylindrical columns of liquid under capillary force (§ 357), although the nature of the equilibrium itself and the manner in which it is departed from are so entirely different.

One more step in the direction of generality may be taken by supposing the maximum velocity V to extend through a layer of finite thickness b' in the middle of the jet (Fig. 74). In this layer accordingly there is no vorticity, while in the adjacent layers of thickness b the vorticity and velocity remain as before.

Taking, as in (11), four constants A, B, C, D to represent the discontinuities at the four surfaces considered in order, and writing $\gamma = e^{-kb}$, $\gamma' = e^{-kb'}$, we have at the first surface

$$U=0, \qquad \Delta\,(dU/dy) = +\,V/b,$$
$$v = A+\gamma B+\gamma\gamma' C+\gamma^2\gamma' D, \qquad \Delta\,(dv/dy) = -\,2kA\,;$$

at the second surface

$$U=V, \qquad \Delta\,(dU/dy) = -\,V/b,$$
$$v = \gamma A+B+\gamma' C+\gamma\gamma' D, \qquad \Delta\,(dv/dy) = -\,2kB\,;$$

at the third surface

$$U=V, \qquad \Delta\,(dU/dy) = -\,V/b,$$
$$v = \gamma\gamma' A+\gamma' B+C+\gamma D, \qquad \Delta\,(dv/dy) = -\,2kC\,;$$

at the fourth surface

$$U=0, \qquad \Delta\,(dU/dy) = +\,V/b,$$
$$v = \gamma^2\gamma' A + \gamma\gamma' B + \gamma C + D, \qquad \Delta\,(dv/dy) = -\,2kD.$$

Using these values in (4) § 367, we get

$$A\,\{1 + 2bn/V\} + \gamma B + \gamma\gamma' C + \gamma^2\gamma' D = 0 \ldots\ldots (20),$$

$$\gamma A + B\,\{1 - 2b\,(k + n/V)\} + \gamma' C + \gamma\gamma' D = 0 \ldots\ldots (21),$$

$$\gamma\gamma' A + \gamma' B + C\,\{1 - 2b\,(k + n/V)\} + \gamma D = 0 \ldots\ldots (22),$$

$$\gamma^2\gamma' A + \gamma\gamma' B + \gamma C + D\,\{1 + 2bn/V\} = 0 \ldots\ldots (23).$$

The elimination of the ratios $A : B : C : D$ would give a biquadratic in n, which, however, may be split into two quadratics, one relating to symmetrical disturbances for which $A + D = 0$, $B + C = 0$; and the other to disturbances for which $A - D = 0$, $B - C = 0$. The resulting equation in n may be written

$$\left(\frac{2bn}{V}\right)^2 + (\pm\,\gamma' \mp \gamma'\gamma^2 + 2kb)\,\frac{2bn}{V}$$
$$\pm\,\gamma' - 1 + 2kb + \gamma^2\,(1 \mp \gamma' \mp 2kb\gamma') = 0 \ldots\ldots (24).$$

In (24) the upper signs of the ambiguities correspond to the symmetrical disturbances. The roots are real, and the corresponding disturbances are stable, if

$$(\pm\,\gamma' \mp \gamma'\gamma^2 + 2kb)^2 - 4\,[\pm\,\gamma' - 1 + 2kb + \gamma^2\,(1 \mp \gamma' \mp 2kb\gamma')] \ldots (25),$$

be positive.

In what follows we will limit our attention to the symmetrical disturbances, that is, to the upper signs in (25), and to terms of orders not higher than the first in b'. The expression (25) may then be reduced to

$$(1 - \gamma^2 - 2kb)^2 + 2kb'\,(1 + \gamma^2)\,(1 - \gamma^2 - 2kb) \ldots\ldots (26).$$

If kb be very small, this becomes

$$4k^4b^4 - 8kb' \,.\, k^2b^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (27).$$

If b' is zero (27) is positive, and the disturbance is stable, as we found before; but, if b and b' be of the same order of magnitude and both small compared with λ, it follows from (27) that the disturbance is unstable, although it be symmetrical.

If in (24) we suppose that $b' = 0$, we fall back upon the suppositions of the previous problem. For the symmetrical disturbances, putting $\gamma' = 1$ in (24), we get

$$\left(\frac{2bn}{V}\right)^2 + (1 - \gamma^2 + 2kb)\frac{2bn}{V} + 2kb(1 - \gamma^2) = 0,$$

shewing that the values of $2bn/V$ are $\gamma^2 - 1$ and $-2kb$. The former agrees with (16), and the latter gives $n + kV = 0$. We have already seen that any value of $-kU$ is a possible solution for n.

If on the other hand we suppose that $b = 0$, we fall back upon the case of a jet of uniform velocity V and thickness b' moving in still fluid. The equation for n becomes, after division by b^2,

$$n^2 + (1 \pm \gamma')kV \,.\, n + \tfrac{1}{2}(1 \pm \gamma')k^2V^2 = 0,$$

or

$$(n + kV)^2 \frac{1 \pm \gamma'}{1 \mp \gamma'} + n^2 = 0 \quad\text{.................}(28).$$

In (28) $\dfrac{1+\gamma'}{1-\gamma'} = \coth \tfrac{1}{2}kb'$, $\dfrac{1-\gamma'}{1+\gamma'} = \tanh \tfrac{1}{2}kb'$;

so that the result is in harmony with (22), (29), § 365, where l corresponds with $\tfrac{1}{2}b'$.

Another particular case of (24), comparable with previous results, is obtained by supposing b' to be infinite.

369. When d^2U/dy^2 is finite, we must fall back upon the general equation § 366

$$\left(\frac{n}{k} + U\right)\left(\frac{d^2v}{dy^2} - k^2v\right) - \frac{d^2U}{dy^2}v = 0 \quad\text{..............}(1),$$

from which the curve representing v as a function of y can theoretically be constructed when n (being real) is known. In fact we may regard (1) as determining the curvature with which we are to proceed in tracing the curve through any point. At a place when $n + kU$ vanishes, that is, where the stream-velocity is equal to the wave-velocity, the curvature becomes infinite, unless v vanishes. The character of the infinity at such a place (suppose $y = 0$) would be most satisfactorily investigated by means of the complete solution of some particular case. It is, however, sufficient to examine the form of solution in the neighbourhood of $y = 0$, and for this purpose the differential equation may be simplified. Thus, when y is small, $n + kU$ may be treated as proportional to y, and

d^2U/dy^2 as approximately constant. In comparison with the large term, k^2v^2 may be neglected, and it suffices to consider

$$d^2v/dy^2+y^{-1}v=0 \quad\text{.......................(2)},$$

a known constant multiplying y being omitted for the sake of brevity. This falls under the head of Ricati's equation

$$d^2v/dy^2+y^{\mu}v=0 \quad\text{.......................(3)},$$

of which the solution is in general (m fractional)[1]

$$v=\sqrt{y}\,.\,\{AJ_m(\xi)+BJ_{-m}(\xi)\} \quad\text{................(4)},$$

where $\quad m=1/(\mu+2), \qquad \xi=2my^{1/2m}$..................(5).

When, as in the present case, m is integral, $J_{-m}(\xi)$ is to be replaced (§ 341) by the function of the second kind $Y_m(\xi)$. The general solution of (2) is accordingly

$$v=\sqrt{y}\,.\,\{AJ_1(2\sqrt{y})+BY_1(2\sqrt{y})\} \quad\text{..............(6)}.$$

In passing through zero y changes sign and with it the character of the functions. If we regard (6) as applicable on the positive side, then on the negative side we may write

$$v=\sqrt{y}\,.\,\{CJ_1(2\sqrt{y})+DY_1(2\sqrt{y})\} \quad\text{.............. (7)},$$

the argument of the functions in (7) being pure imaginaries.

From the known forms of the functions (§ 341) we may deduce, as applicable when y is small,

$$v=A\,\{y-\tfrac{1}{2}y^2\}$$
$$+B\,\{\tfrac{1}{2}(1-y+\tfrac{1}{4}y^2)-\log(2\sqrt{y})\,.\,(y-\tfrac{1}{2}y^2)+y-\tfrac{5}{4}y^2\} \quad\text{......(8)};$$

so that ultimately

$$v=\tfrac{1}{2}B, \quad \frac{dv}{dy}=A-\tfrac{1}{2}B\log y, \quad \frac{d^2v}{dy^2}=-A-\tfrac{1}{2}By^{-1} \quad\text{......(9)},$$

v remaining finite in any case.

We will now shew that any value of $-kU$ is an admissible value of n in (1). The place where $n+kU=0$ is taken as origin of y; and in the first instance we will suppose that $n+kU$ vanishes nowhere else. In the immediate neighbourhood of $y=0$ the solutions applicable upon the two sides are (6), (7), and they are subject to the condition that v shall be continuous. Hence by

[1] Lommel, *Studien über die Bessel'schen Functionen* § 31, Leipzig, 1868; Gray and Matthews' *Bessel Functions*, p. 233, 1895.

(9), $B=D$, leaving three constants arbitrary. The manner in which the functions start from $y=0$ being thus ascertained, their further progress is subject to the original equation (1), which completely defines them when the three arbitraries are known. In the present case two relations are given by the conditions to be satisfied at the fixed walls or other boundaries of the fluid, and thus is determined the entire form of v, save as to a constant multiplyer. If B and D are finite, there is infinite vorticity at the origin.

Any other places at which $n+kU=0$ may be treated in a similar manner, and the most general solution will contain as many arbitrary constants as there are places of infinite vorticity. But the vorticity need not be infinite merely because $n+kU=0$; and in fact a particular solution may be obtained with only one infinite vorticity. At any other of the critical places, such for example as we may now suppose the origin to be, B and D may vanish, so that $v=0$, $d^2v/dy^2=A$, or C.

From this discussion it would seem that the infinities which present themselves when $n+kU=0$ do not seriously interfere with the application of the general theory, so long as the square of the disturbance from steady motion is neglected.

A large part of the preceding paragraphs is taken from certain papers by the author[1]. The reader should also consult Lord Kelvin's writings[2] in which the effects of viscosity are dealt with.

370. It remains to describe the phenomena of sensitive flames and to indicate, so far as can be done, the application of theoretical principles. In a sense the combination of flame and resonator described in § 322 *h* may be called sensitive, but in this case it is rather the resonator to which the name attaches, the office of the flame being to maintain by a periodic supply of heat the vibration of the resonator when once started. Following Tyndall, we may conveniently limit the term to naked flames and jets, where the origin of the sensitiveness is undoubtedly to be found in the instability which accompanies vortex motion.

The earliest observation upon this subject was that of Prof.

1 *Proc. Math. Soc.*, vol. XI. p. 57, 1880; vol. XIX. p. 67, 1887. It is hoped shortly to communicate a supplement.

2 *Phil. Mag.* vol. XXIV. pp. 188, 272, 1887.

Leconte[1], who noticed the jumping of the flame from an ordinary fishtail burner in response to certain notes of a violoncello. The sensitive condition demanded that in the absence of sound the flame should be on the point of flaring. When the pressure of gas was reduced, the sensitiveness was lost.

An independent observation of the same nature drew the attention of Prof. Barrett to sensitive flames; and he investigated the kind of burner best suited to work with the ordinary pressure of the gas mains[2]. "It is formed of glass tubing about $\frac{3}{8}$ of an inch (1 cm.) in diameter, contracted to an orifice $\frac{1}{16}$ of an inch (·16 cm.) in diameter. It is very essential that this orifice should be slightly V-shaped....Nothing is easier than to form such a burner; it is only necessary to draw out a piece of glass tubing in a gas flame, and with a pair of scissors snip the contraction into the shape indicated."

But the most striking by far is the high-pressure flame employed by Tyndall. The gas is supplied from a special holder under a pressure of say 25 cm. of water to a pinhole steatite burner, and the flame rises to a height of about 40 cm. Under the influence of a sound of suitable (very high) pitch the flame roars, and drops down to perhaps half its original height[3]. Tyndall shewed that the seat of sensitiveness is at the root of the flame. Sound coming along a tube is ineffective when presented to the flame a little higher up, and also when caused to impinge upon the burner below the place of issue.

It is to Tyndall that we owe also the demonstration that it is not to the flame as such that these extraordinary effects are to be ascribed. Phenomena substantially the same are obtained when a jet of unignited gas, of carbonic acid, hydrogen, or even air itself, issues from an orifice under proper pressure. They may be rendered visible in two ways. By association with smoke the whole course of the jet may be made apparent; and it is found that suitable smoke jets can surpass even flames in delicacy. "The notes here effective are of much lower pitch than those which are most efficient in the case of flames." Another way of making the sensitiveness of an air-jet visible to the eye is to cause

[1] On the Influence of Musical Sounds on the Flame of a Jet of Coal-gas. *Phil. Mag.* vol. xv. p. 235, 1858.

[2] *Phil. Mag.* vol. XXXIII. p. 216, 1867.

[3] *Phil. Mag.* vol. XXXIII. pp. 92, 375, 1867; *Sound*, 3rd ed. ch. VI.

it to impinge upon a flame, such as a candle flame, which plays merely the part of an indicator.

In the sensitive flame of Prof. Govi[1] and of Mr Barry[2] the gas is unignited at the burner, but catches fire on the further side of wire-gauze held at a suitable distance. On the same principle is an arrangement employed by the author[3]. A jet of coal gas from a pinhole burner rises vertically in the interior of a cavity from which air is excluded. It then passes into a brass tube a few inches long, and on reaching the top burns in the open. The front wall of the cavity is formed of a flexible membrane of tissue-paper, through which external sounds can reach the burner. In these cases the sensitive agent is the unignited part of the jet. Used in this way a given burner requires a much less pressure of gas than is necessary when the flame is allowed to reach it, and the sounds which have the most influence are graver.

Struck by the analogy between these phenomena and those of water-jets investigated by Savart and Plateau, the earlier observers seem to have leaped to the conclusion that the manner of disintegration was also similar—symmetrical, that is, about the axis; and Prof. Leconte went so far as to deduce the existence of a cohesive force in gases. A surface tension, however, requires a very abrupt transition between the properties of the matter on the two sides, such as could have only a momentary existence when there is a tendency to mix, so that it appears extremely unlikely that capillarity plays here any sensible part.

The question of the manner of disintegration, whether it be by gradually increasing *varicosity* or by gradually increasing *sinuosity*, is of the greatest importance, and the answer is still, perhaps, in some cases open to doubt. But that the latter is predominant in general follows from a variety of arguments. The necessity, as remarked by Barrett, for an unsymmetrical orifice points strongly in this direction. The same conclusion is drawn by Ridout[4] from the results of some ingenious experiments. The latter observer found further that fishtail flames, formed by the union at a small angle of jets from two perfectly similar glass nozzles, shewed a

[1] Torino, *Atti Acad. Sci.* vol. v. p. 396, 1869.

[2] Tyndall's *Sound*, 3rd edition, p. 240.

[3] *Camb. Phil. Soc. Proc.* vol. IV. p. 17, 1883.

[4] *Nature*, vol. XVIII. p. 604, 1878.

sensitiveness dependent upon the direction of the sound. If this direction lie in the plane of symmetry containing the flame (that perpendicular to the plane of the nozzles), there is no response.

Even in the case of the tall high-pressure flame from a pin-hole burner, where to all appearance both the nozzle and the flame (when undisturbed) are perfectly symmetrical, there is reason to believe that the manner of disintegration is sinuous, or unsymmetrical. Perhaps the easiest road to this conclusion is by examining the behaviour of the flame when exposed to stationary sonorous waves, such as may be derived by superposing upon direct waves from a source giving a pure tone the waves reflected perpendicularly from a flat obstacle, e.g. a sheet of glass. According to the analogy with capillary jets, an analogy pushed further than it will bear by most writers upon this subject, the flame should be excited when the nozzle is situated at a node, where the pressure varies most, and remain unaffected at a loop where the pressure does not vary at all. There was no difficulty in proving experimentally[1] that the facts are precisely the opposite. The source of sound was a bird-call (§ 371), and the observations were made by moving the burner to and fro in front of the reflector until the positions were found in which the flame was least disturbed. These positions were very well defined, and the measurements shewed distances from the reflector proportional to the series of numbers 1, 2, 3, &c., and therefore corresponding to nodes. If the positions had coincided with loops, the distances would have formed a series proportional to the *odd* numbers 1, 3, 5, &c. The wave-length of the sound, determined by the doubled interval between consecutive minima, was 31·2 mm., corresponding to pitch $f\sharp^{v\prime}$.

A few observations were made at the same time on the positions of the silences as estimated by the ear listening through a tube. As was to be expected, they coincided with the loops, bisecting the intervals given by the flame. When the flame was in a position of minimum effect, and the free end of the tube was held close to the burner at an equal distance from the reflector, the sound heard was a maximum, and diminished when the end of the tube was displaced a little in either direction. It was thus established that the flame is affected where the ear would not be affected, and *vice versâ*.

[1] *Phil. Mag.* vol. VII. p. 153, 1879.

Flames from pinhole burners, which perform well in other respects, seem always to shew a marked difference according to the direction in which the sound arrives. If, while a bird-call is in operation, the burner be turned steadily round its axis, two positions differing by 180° are found, in which there is little or no response. This peculiarity may sometimes be turned to account in experiment[1]. Thus after such an adjustment has been made that the direct sound has no effect, vigorous flaring may yet result from the impact of sound from the same source after reflection from a small pane of glass, the pane being held so that the direction of arrival is at 90° to that of the direct sound, and this although the distance travelled by the reflected sound is the greater.

Tyndall[2] lays it down as an essential condition of complete success in the more delicate experiments with these flames, "that a free way should be open for the transmission of the vibrations from the flame, *backwards*, through the gaspipe which feeds it. The orifices of the stopcocks near the flame ought to be as wide as possible." The recommendation is probably better justified than the reason given for it. Prof. Barrett[3] attributes the evil effect of a partially opened stopcock to the irregular flow and consequent ricochetting of the current of gas from side to side of the pipe. In some experiments of my own[4] the introduction of a glass nozzle into the supply pipe, making the flow of gas in the highest degree irregular, did not interfere, nor did other obstructions unless attended by hissing sounds. The prejudicial action of a partially opened stopcock was thus naturally attributed to the production of internal sounds of the kind to which the flame is sensitive, and this view of the matter was confirmed by some observations of the pressure of the gas in the neighbourhood of the burner. "In the path of the gas there were inserted two stopcocks, one only a little way behind the manometer junction, the other separated from it by a long length of india-rubber tubing. When the first cock was fully open, and the flame was brought near the flaring-point by adjustment of the distant cock, the sensitiveness to external sounds was great,

[1] *Proc. Roy. Inst.* vol. XII. p. 192, 1888; *Nature*, vol. XXXVIII. p. 208, 1888.
[2] *Phil. Mag.* vol. XXXIII. p. 99, 1867.
[3] *Phil. Mag.* vol. XXXIII. p. 288, 1867.
[4] *Phil. Mag.* vol. XIII. p. 345, 1882.

and the manometer indicated a pressure of 10 inches (25·4 cm.) of water. But when the distant cock stood fully open and the adjustment was effected at the other, high sensitiveness could not be obtained; and the reason was obvious, because the flame flared without external excitation while the pressure was still an inch (2·54 cm.) short of that which had been borne without flinching in the former arrangement. On opening again the neighbouring cock to its full extent, and adjusting the distant one until the pressure at the manometer measured 9 inches (22·9 cm.), the flame was found comparatively insensitive."

The most direct and satisfactory evidence as to the manner of disintegration is of course that of actual observation. Using a jet of phosphorus smoke from a glass nozzle and a stroboscopic disc, I was able (in 1879) to see the sinuosities when the jet was disturbed by a fork of pitch 256 vibrating in its neighbourhood[1]. Moreover by placing the nozzle exactly in the plane of symmetry between the prongs of the tuning-fork, it could be verified that the disturbance required is motion transverse to the jet. In this position there was but little effect; but the slightest displacement led to an early rupture of the jet.

"In order to exalt the sensitiveness of jets to notes of moderate pitch, I found the use of resonators advantageous. These may be of Helmholtz's pattern; but suitably selected wide-mouth bottles answer the purpose. What is essential is that the jet should issue from the nozzle in the region of rapid reciprocating motion at the mouth of the resonator, and in a transverse direction.

"Good results were obtained at a pitch of 256. When two forks of about this pitch, and slightly out of tune with one another, were allowed to sound simultaneously, the evolutions of the smoke-jet in correspondence with the audible beats were very remarkable. By gradually raising the pressure at which the smoke is supplied, in the manner usual in these experiments, a high degree of sensitiveness may be attained, either with a drawn-out glass nozzle or with the steatite pinhole burner used by Tyndall. In some cases (even at pitch 256) the combination of jet and resonator proved almost as sensitive to sound as the ear itself.

"The behaviour of the sensitive jet does not depend upon the smoke-particles, whose office is merely to render the effects more

[1] *Phil. Mag.* vol. XVII. p. 188, 1884.

easily visible. I have repeated these observations without smoke by simply causing air-jets from the same nozzles to impinge upon the flame of a candle placed at a suitable distance. In such cases, as has been pointed out by Tyndall, the flame acts merely as an indicator of the condition of the otherwise invisible jet. Even without a resonator the sensitiveness of such jets to hissing sounds may be taken advantage of to form a pretty experiment.

"The combination of jet, resonator, and flame shows sometimes a tendency to *speak* on its own account; but I did not succeed in getting a well-sustained sound. Such as it is, the effect probably corresponds to one observed by Savart and Plateau with water-jets breaking up under the operation of the capillary tension and, when resolved into drops, impinging upon a solid obstacle, such as the bottom of a sink, in mechanical connection with the nozzle from which the jet originally issues. In virtue of the connexion, any regular cycle in the mode of disintegration is able, as it were, to propagate itself."

"In the hope of being able to make better observations upon the transformations of unstable jets, I next had recourse to coloured water issuing under water. In this form the experiment is more manageable than in the case of smoke-jets, which are difficult to light, and liable to be disturbed by the slightest draught. Permanganate of potash was preferred as a colouring agent, and the colour may be discharged by mixing with the general mass of liquid a little acid ferrous sulphate. The jets were usually projected downwards into a large beaker or tank of glass, and were lighted from behind through a piece of ground glass.

"The notes of maximum sensitiveness of these liquid jets were found to be far graver than for smoke-jets or for flames. Forks vibrating from 20 to 50 times per second appeared to produce the maximum effect, to observe which it is only necessary to bring the stalk of the fork into contact with the table supporting the apparatus. The general behaviour of the jet could be observed without stroboscopic appliances by causing the liquid in the beaker to vibrate from side to side under the action of gravity. The line of colour proceeding from the nozzle is seen to become gradually more and more sinuous, and a little further down presents the appearance of a rope bent backwards and forwards upon itself. I have followed the process of disintegration with gradually increasing frequencies of vibrational disturbance from 1 or 2 per second up to about 24 per second, using electro-magnetic interruptors to send intermittent currents through an electro-magnet which acted upon a soft-iron armature attached to the nozzle. At each stage the pressure at which the jet is supplied should be adjusted so as to give the right degree of sensitiveness. If the pressure be too great, the jet flares independently of the imposed vibration, and the transformations become irregular: in the contrary case the phenomena, though usually observable, are not so well marked as when a suitable adjustment is made. After a little practice it is possible to interpret pretty well what is seen directly; but in order to have before the eye an image of what is really going on, we must have recourse to intermittent vision. The best results are obtained with two forks slightly out of tune, one of which is used to effect the disintegration of the jet, and the other (by means of perforated plates attached to its prongs) to give an intermittent view. The difference of frequencies should be about one per second. When the means of obtaining uniform rotation are at hand, a stroboscopic disk may be substituted for the second fork[1].

"The carrying out of these observations, especially when it is desired to make a drawing, is difficult unless we can control the plane of the bendings. In order to see the phases properly it is necessary that the plane of bendings should be perpendicular to the line of vision; but with a symmetrical nozzle this would occur only by accident. The difficulty may be got over by slightly nicking the end of the drawn-out glass nozzle at two opposite points (Barrett). In this way the plane of bending is usually rendered determinate, being that which includes the nicks, so that by turning the nozzle round its axis the sinuosities of the jet may be properly presented to the eye.

"Occasionally the jet appears to divide itself into two parts imperfectly connected by a sort of sheet. This seems to correspond to the duplication of flames and smoke-jets under powerful sonorous action, and to be due to what we may regard as the broken waves taking alternately different courses."

"It has already been noticed that the notes appropriate to water-jets are far graver than for air-jets from the same nozzles.

[1] In the original paper (*Phil. Mag.* vol. XVII. p. 188, 1884) drawings by Mrs Sidgwick are given. See also *Proc. Roy. Inst.* vol. XIII. p. 261, 1891, for reproductions of instantaneous photographs.

Moreover, the velocities suitable in the former case are much less than in the latter. This difference relates not, as might perhaps be at first supposed, to the greater density, but to the smaller viscosity of the water, measured of course kinematically. It is not difficult to see that the density, presumed to be the same for the jet and surrounding fluid, is immaterial, except of course in so far as a denser fluid requires a greater pressure to give it an assigned velocity. The influence of fluid viscosity upon these phenomena is explained in a former paper on the Stability or Instability of certain Fluid Motions[1]; and the laws of dynamical similarity with regard to fluid friction, laid down by Prof. Stokes[2], allow us to compare the behaviour of one fluid with another. The dimensions of the kinematic coefficient of viscosity are those of an area divided by a time. If we use the same nozzle in both cases, we must keep the same standard of length; and thus the times must be taken inversely, and the velocities directly, as the coefficients of viscosity. In passing from air to water the pitch and velocity are to be reduced some ten times. But, in spite of the smaller velocity, the water-jet will require the greater pressure behind it, inasmuch as the densities differ in a ratio exceeding 100 : 1."

Guided by these considerations, I made experiments to try whether the jets would behave differently in warm (less viscous) water, and as to the effect of substituting for water a mixture of alcohol and water in equal parts, a fluid known to be more viscous than either of its constituents. The effect of varying the viscosity was found to be very distinct. A jet which would not bear a pressure of more than ¼ inch (·63 cm.) of water without flaring when the liquid was water at a temperature under the boiling-point required about 25 inches (63 cm.) pressure to make it flare when the alcoholic mixture was substituted. The importance of viscosity in these phenomena was thus abundantly established.

The manner in which viscosity operates is probably as follows. At the root of the jet, just after it issues from the nozzle, there is a near approach to discontinuous motion, and a high degree of instability. If a disturbance of sufficient intensity and of

[1] *Math. Soc. Proc.* Feb. 12, 1880. See § 366.

[2] *Camb. Phil. Trans.* 1850, "On the Effect of Internal Friction of Fluids on the Motion of Pendulums," § 5. See also Helmholtz, *Wied. Ann.* Bd. vii. p. 337 (1879) or Reprint, vol. i. p. 891.

suitable period have access, the regular motion is lost and cannot afterwards be recovered. But the instability has a very short time in which to produce its effect. Under the influence of viscosity the changes of velocity become more gradual, and the instability decreases rapidly if it does not disappear altogether. Thus if the disturbance be insufficient to cause disintegration during the brief period of instability, the jet may behave very much as though it had not been disturbed at all, and may reach the full developement observed in long flames and smoke-jets. This temporary character of the instability is a second feature differentiating strongly these jets from those of Savart, in which capillarity has an unlimited time of action.

When a flame is lighted at the burner, there are further complications of which it is difficult to give an adequate explanation. The high temperature leads indeed to increased viscosity, and this tends to explain the higher pressure then admissible, and the graver notes which then become operative. But it is probable that the change due to ignition is of a still more fundamental nature.

An ingenious method of observation, due to Mr C. Bell[1], may be applied so as to give valuable information with regard to the disintegration of jets; but the results obtained by the author are not in harmony with the views of Mr Bell, who favours the symmetrical theory. In this method a second similar nozzle faces directly the nozzle from which the air issues, and is connected with the ear of the observer by means of rubber tubing. Suitable means are provided whereby the position of the hearing nozzle may be adjusted with accuracy, both longitudinally and laterally. When the distance is properly chosen, small disturbances acting upon the jet are perceived upon a magnified scale. Thus a fork vibrating feebly and presented to the jet is loudly heard; and that the effect is due to the peculiar properties of the jet is proved at once by cutting off the supply of air, when the sound becomes feeble, if not inaudible. Mr Bell proved that the efficacy of the arrangement requires a *small* area in the hearing nozzle; if the latter be large enough to receive the whole stream of air accompanying the jet, comparatively little is heard.

In the following experiments an air-jet from a well-regulated bellows issued from a glass nozzle and impinged upon a similar

[1] *Phil. Trans.* vol. CLXXVII. p. 383, 1886.

hearing nozzle. It was excited by forks (c' or c'') held in the neighbourhood.

If the position of the fork was such that the plane of its prongs was perpendicular to the jet, and that the prolongation of the axis of the stalk intersected the delivery end of the nozzle, the sound perceived was much less than when the fork was displaced laterally in its own plane so as to bring the nozzle nearer to one prong. This appears to prove that here again the effect is due, not to variation of pressure, but to transverse motion, causing the jet to become sinuous.

Confirmatory evidence may be drawn from observations upon the effect of slight movements of the hearing nozzle. When this is adjusted axially, but little is perceived of the fundamental tone of a fork presented laterally to the jet nozzle, but the octave tone is heard and often very strongly. When, however, the hearing nozzle is displaced laterally, the fundamental tone of the fork comes in loudly.

371. In that very convenient source of sounds of high pitch, the "bird-call," a stream of air issuing from a circular hole in a thin plate impinges centrically upon a similar hole in a parallel plate held at a little distance. The circumstances upon which the pitch depends have been investigated by Sondhauss[1], but much remains obscure as regards the manner in which the vibrations are excited.

According to Sondhauss the pitch is comparatively independent of the size and shape of the plates, varying directly as the velocity (v) of the jet and inversely as the distance (d) between the plates. If we assume independence of other elements, and that the frequency (n) is a function only of v, d, and b the diameter of the jet, it follows from dynamical similarity that

$$n = v/d \,.\, f(b/d) \quad\text{.......................(1)},$$

where f is an arbitrary function. Thus, if b/d be constant, Sondhauss' law must hold. From the very small dimensions employed it might fairly be argued that the action must be nearly independent of the velocity of sound, and therefore (v being given) of the density of the gas; but the question arises whether viscosity may not be an element of importance. If we suppose

[1] *Pogg. Ann.* vol. XCI. p. 126, 1854.

that geometrical similarity is maintained (b proportional to d), the theoretical form, when viscosity is retained, is

$$n = v/d \,.\, F(\nu/vd) \qquad (2),$$

ν being the kinematic coefficient of viscosity, of dimensions 2 in space and − 1 in time. But when we take a numerical example, it appears improbable that the degree of viscosity can play much part in determining the pitch. In C.G.S. measure $\nu = \cdot 16$ for air; and if the pressure propelling the jet be 1 cm. of mercury, $v =$ 4000 (cm./sec.). Thus, if we take $d = \cdot 1$ cm., we have $\nu/vd = \cdot 0004$, so that $F(\nu/vd)$ could hardly differ much from $F(0)$.

Bird-calls are very easily made. The first plate, of 1 or 2 cm. in diameter, is cemented, or soldered, to the end of a short supply tube. The second plate may conveniently be made triangular, the turned down corners being soldered to the first plate. For calls of medium pitch the holes may be made in tin plate, but when it is desired to attain a very high pitch thin brass, or sheet silver, is more suitable. The holes may then be as small as ½ mm. in diameter, and the distance between them as little as 1 mm. In any case the edges of the holes should be sharp and clean[1].

In order to test a bird-call it should be connected with a well-regulated supply of wind and with a manometer by which the operative pressure can be measured with precision. When it is found to speak well, the pressure and corresponding wave-length should be recorded. If the tones are high or inaudible, a high-pressure sensitive flame is required, the wave-length being deduced from the interval between the positions in which a reflector must be held in order that the flame may shew the least disturbance (§ 370). There is no difficulty in obtaining wave-lengths (complete) as low as 1 cm., and with care wave-lengths of ·6 cm. may be reached, corresponding to about 50,000 vibrations per second. In experimenting upon minimum wave-lengths, the distance between the call and the flame should not exceed 50 cm., and the flame should be adjusted to the verge of flaring.

In many cases a bird-call, which otherwise will not speak, may be made to do so by a reflecting plate held at a short distance in front. In practice the reflector is with advantage reduced to a

[1] Prof. A. M. Mayer has constructed beautifully finished bird calls in which the distance between the plates is adjustable by a screw motion.

strip of metal, e.g. 1 cm. wide; and, when this assistance is required, the right distance is an (even or odd) multiple of the half wave-length. In some cases the necessary position of the strip is very sharply defined.

On the question whether the disturbance of the jet accompanying the production of the sound is varicose or sinuous, some evidence may be derived from observations upon the manner in which the sound radiates. Upon the latter view we might expect that the sound would fall off, or even disappear altogether, in the axial direction, as happens, for example, in the case of the sound radiated from a bell (§ 282). But, so far as I have been able to observe, the sound emitted from a bird-call, speaking without the aid of a reflecting strip, is uniform through a wide angle; and this fact may be regarded as telling strongly in favour of the view that the disturbance is here symmetrical, or varicose, in character. Other evidence tending in the same direction is afforded by the behaviour of resonating pipes made to speak with the aid of bird-calls. The pair of perforated plates is mounted symmetrically at one end of a pipe 40 or 50 cm. long. The other end of the pipe is acoustically open, and a gentle stream of air is made to pass the bird-call, most easily with the aid of a very narrow tube inserted into the open end and supplied from the mouth. By careful regulation of the force of the blast, the pipe may be made to speak in various harmonics, and the fact that it speaks at all seems to shew that the issue of air through the bird-call is variable.

The manner of action is perhaps somewhat as follows. When a symmetrical excrescence reaches the second plate, it is unable to pass the hole with freedom, and the disturbance is thrown back, probably with the velocity of sound, to the first plate, where it gives rise to a further disturbance, to grow in its turn during the progress of the jet. But the elucidation of this and many kindred phenomena remains still to be effected.

372. Æolian tones, as in the æolian harp, are generated when wind plays upon a stretched wire capable of vibration at various speeds, and their production also is doubtless connected with the instability of vortex sheets. It is not essential, however, that the wire should partake in the vibration, and the general phenomenon has been investigated by Strouhal[1], under the name of *reibungstöne*.

[1] *Wied. Ann.* vol. v. p. 216, 1878. See also W. Kohlrausch, *Wied. Ann.* vol. XIII. p. 545, 1881.

In Strouhal's experiments a vertical wire attached to a suitable frame was caused to revolve with uniform velocity about a parallel axis. The pitch of the æolian tone generated by the relative motion of the wire and of the air was found to be independent of the length and of the tension of the wire, but to vary with the diameter (d) and with the speed (v) of the relative motion. Within certain limits the relation between the frequency (n) and these data was expressible by

$$n = \cdot 185\ v/d \qquad (1),$$

the centimetre and second being units.

When the speed is such that the æolian tone coincides with one of the proper tones of the wire, supported so as to be capable of free independent vibration, the sound is greatly reinforced, and with this advantage Strouhal found it possible to extend the range of the observations. Under the more extreme conditions then practicable the observed pitch deviated sensibly from the value given by (1). He shewed further that with a given diameter and a given speed a rise of temperature was attended by a fall in pitch.

Observations[1] upon a string, vibrating after the manner of the æolian harp under the stimulus of a chimney draught, have shewn that, contrary to the opinion generally expressed, the vibrations are effected in a plane perpendicular to the direction of the wind. According to (1) the distance travelled over by the wind during one complete vibration is about 6 times the diameter of the wire.

If, as appears probable, the compressibility of the fluid may be left out of account, we may regard n as a function of v, d, and ν the kinematic coefficient of viscosity. In this case n is necessarily of the form

$$n = v/d \,.\, f(\nu/vd) \qquad (2),$$

where f represents an arbitrary function; and there is dynamical similarity, if $\nu \propto v\,d$. In observations upon air at one temperature ν is constant; and, if d vary inversely as v, nd/v should be constant, a result fairly in harmony with the observations of Strouhal. Again, if the temperature rises, ν increases, and in order to accord with observation, we must suppose that the function f diminishes with increasing argument.

[1] *Phil. Mag.* vol. VII. p. 161, 1879.

An examination of the actual values in Strouhal's experiments shew that ν/vd was always small; and we are thus led to represent f by a few terms of Mac Laurin's series. If we take

$$f(x) = a + bx + cx^2,$$

we get

$$n = a\frac{v}{d} + b\frac{\nu}{d^2} + c\frac{\nu^2}{vd^3} \quad\text{.....................}(3).$$

If the third term in (3) may be neglected, the relation between n and v is linear. This law was formulated by Strouhal, and his diagrams shew that the coefficient b is negative, as is also required to express the observed effect of a rise of temperature. Further

$$d\,.\frac{dn}{dv} = a - \frac{c\nu^2}{v^2d^2} \quad\text{........................}(4),$$

so that $d\,.\,dn/dv$ is very nearly constant, a result also given by Strouhal on the basis of his measurements.

On the whole it would appear that the phenomena are satisfactorily represented by (2) or (3), but a dynamical theory has yet to be given. It would also be of interest to extend the experiments to liquids.

CHAPTER XXII.

VIBRATIONS OF SOLID BODIES.

373. IT is impossible in the present work to attempt anything approaching to a full consideration of the problems suggested by vibrating solid bodies: and yet the simpler parts of the theory seem to demand our notice. We shall limit ourselves entirely to the case of *isotropic* matter.

The general equations of equilibrium have already been stated in § 345. If ρ be the density, and

$$a^2 = (\kappa + \tfrac{4}{3} n)/\rho, \qquad b^2 = n/\rho \text{ .... } (1),$$

we have
$$(a^2 - b^2)\frac{d\delta}{dx} + b^2 \nabla^2 \alpha + X' = 0, \text{ etc. .... } (2),$$

where X', Y', Z' are the impressed forces reckoned per unit of *mass*.

If from these we separate the reactions against acceleration, we obtain by D'Alembert's principle

$$\frac{d^2\alpha}{dt^2} = (a^2 - b^2)\frac{d\delta}{dx} + b^2\nabla^2\alpha + X' \text{ .... } (3),$$

and two similar equations. In (3) δ is the *dilatation*, related to α, β, γ according to

$$\delta = d\alpha/dx + d\beta/dy + d\gamma/dz \text{ .... } (4).$$

If α, β, γ, etc. be proportional to e^{ipt}, $d^2\alpha/dt^2 = -p^2\alpha$, and (3) becomes

$$(a^2 - b^2)\frac{d\delta}{dx} + b^2\nabla^2\alpha + p^2\alpha + X' = 0 \text{ .... } (5).$$

Differentiating equation (3) and its companions with respect to x, y, z, and adding, we obtain by (4)

$$\frac{d^2\delta}{dt^2} = a^2\nabla^2\delta + \frac{dX'}{dx} + \frac{dY'}{dy} + \frac{dZ'}{dz} \text{..............}(6).$$

Similar equations may be obtained for the *rotations* (compare § 239), defined by

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 2\varpi', \quad \frac{d\alpha}{dz} - \frac{d\gamma}{dx} = 2\varpi'', \quad \frac{d\beta}{dx} - \frac{d\alpha}{dy} = 2\varpi''' \text{.....}(7).$$

Thus, if we differentiate the third of equations (3) with respect to y, the second with respect to z, and subtract,

$$\frac{d^2\varpi'}{dt^2} = b^2\nabla^2\varpi' + \tfrac{1}{2}\frac{dZ'}{dy} - \tfrac{1}{2}\frac{dY'}{dz} \text{................}(8);$$

and there are two similar equations relative to ϖ'', ϖ'''. It is to be observed that ϖ', ϖ'', ϖ''' are subject to the relation

$$d\varpi'/dx + d\varpi''/dy + d\varpi'''/dz = 0 \text{..............}(9).$$

We will now consider briefly certain cases of the propagation of plane waves in the absence of impressed forces. In (6), if X', Y', Z' vanish, and δ be a function of x only,

$$d^2\delta/dt^2 = a^2 d^2\delta/dx^2 \text{.....................}(10),$$

of which the solution is, as in § 245,

$$\delta = f(x - at) + F(x + at) \text{..................}(11).$$

In this wave $\delta = d\alpha/dx$, while β and γ vanish; so that the case is similar to that of the propagation of waves in a compressible fluid. It should be observed, however, that by (1) the velocity depends upon the constant of rigidity (n) as well as upon that of compressibility (κ).

In the dilatational wave (11) the rotations ϖ', ϖ'', ϖ''' vanish, as appears at once from their expressions in (7). We have now to consider a wave of transverse vibration for which δ vanishes. If, for example, we suppose that α and β vanish and that γ is a function of x (and t) only, we have

$$\delta = 0, \qquad \varpi' = \varpi''' = 0, \qquad 2\varpi'' = -d\gamma/dx.$$

The equation for ϖ'' is

$$d^2\varpi''/dt^2 = b^2 d^2\varpi''/dx^2 \text{......................}(12),$$

of the same form as (10); and the same equation obtains for γ. The transverse vibration is thus propagated in plane waves with velocity b, a velocity *less* than that (a) of the dilatational waves.

The formation of stationary waves by superposition of positive and negative progressive waves of like wave-length need not be dwelt upon. If $k = 2\pi/\lambda$, where λ is the wave-length, the superposition of the positive wave $\gamma = \Gamma \cos k(bt - x)$ upon the negative wave $\gamma = \Gamma \cos k(bt + x)$ gives

$$\gamma = 2\Gamma \cos kbt \,.\, \cos kx \quad\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

The second progressive wave may be the *reflection* of the first at a bounding surface impenetrable to energy. This may be either a free surface, or one at which γ is prevented from varying.

374. The problem of the propagation in three dimensions of a disturbance initially limited to a finite region of the solid was first successfully considered by Poisson, and the whole subject has been exhaustively treated by Stokes[1]. By (6), (8) § 373 the dilatation and the rotations satisfy the equations

$$d^2\delta/dt^2 = a^2\nabla^2\delta, \qquad d^2\varpi/dt^2 = b^2\nabla^2\varpi \ldots\ldots\ldots\ldots (1),$$

the solutions of which, applicable to the present purpose, have already been fully discussed in §§ 273, 274. It appears that distinct waves of dilatation and distortion are propagated outwards with different velocities, so that at a sufficient distance from the source they become separated. If we consider what occurs at a distant point, we see that at first there is neither dilatation nor distortion. When the wave of dilatation arrives this effect commences, but there is no distortion. After a while the wave of dilatation passes, and there is an interval of no dilatation and no distortion. Then the wave of distortion arrives and for a time produces its effect, after which there is never again either dilatation or distortion.

The complete discussion requires the expressions for the displacements in terms of δ, ϖ_1, ϖ_2, ϖ_3, for the derivation of which we have not space. From these it may be proved that before the arrival of the wave of dilatation and subsequently to the passage of the wave of distortion, the medium remains at rest. Between the two waves the medium is not absolutely undisturbed, although there is neither dilatation nor distortion.

If the initial disturbances be of such a character that there is no wave of distortion, the whole disturbance is confined to the wave of dilatation.

[1] "Dynamical Theory of Diffraction," *Camb. Phil. Trans.* Vol. IX. p. 1, 1849.

375. The subject of § 374 was the free propagation of waves resulting from a disturbance initially given. A problem at least equally important is that of divergent waves maintained by harmonic forces operative in the neighbourhood of a given centre.

We may take first the case of a harmonic force of such a character as to generate waves of dilatation. By equation (6) § 373 we may suppose that at all points except the origin of coordinates

$$d^2\delta/dt^2 = a^2\nabla^2\delta \qquad (1);$$

or, if δ as a function of x, y, z depend upon r, or $\sqrt{(x^2+y^2+z^2)}$, only, and as a function of the time be proportional to e^{ipt}, § 241,

$$\frac{d^2\delta}{dr^2} + \frac{2}{r}\frac{d\delta}{dr} + h^2\delta = 0 \qquad (2),$$

where $h = p/a$. The solution of (2) is, as in § 277,

$$\delta = \frac{e^{-ihr}}{r} \qquad (3).$$

In terms of real quantities

$$\delta = \frac{A\cos(pt - hr + \epsilon)}{r} \qquad (4),$$

in which A and ϵ are arbitrary.

By transformation of (4) § 373, the relation between δ and the radial displacement w may be shewn to be

$$\delta = r^{-2}d(r^2w)/dr \qquad (5),$$

or at a great distance from the origin simply

$$\delta = dw/dr \qquad (6).$$

Thus, when r is great, corresponding to (4)

$$w = -\frac{A}{hr}\sin(pt - hr + \epsilon) \qquad (7).$$

In these purely dilatational waves the motion is radial, that is, parallel to the direction of propagation, and the distribution is symmetrical with respect to the origin.

The theory of forced waves of distortion proceeding outwards from a centre is of still greater interest. The simplest case is when the waves are due to a periodic force, say Z', acting through a space T at the origin. If we suppose in (8) § 373 that X', Y' vanish, and that all the quantities are proportional to e^{ipt}, we find

$$\nabla^2\varpi' + k^2\varpi' + \tfrac{1}{2}b^{-2}dZ'/dy = 0 \qquad (8),$$

$$\nabla^2\varpi'' + k^2\varpi'' - \tfrac{1}{2}b^{-2}dZ'/dx = 0 \qquad (9),$$

$$\nabla^2\varpi''' + k^2\varpi''' = 0 \qquad (10),$$

k being written for p/b.

These equations are solved as in § 277. We get $\varpi''' = 0$, and

$$\varpi' = \frac{1}{8\pi b^2}\iiint \frac{dZ'}{dy}\cdot\frac{e^{-ikr}}{r}\,dx\,dy\,dz,$$

r denoting the distance between the element at x, y, z near the origin (O) and the point (P) under consideration. If we integrate partially with respect to y, we find

$$\varpi' = -\frac{1}{8\pi b^2}\iiint Z'\frac{d}{dy}\left(\frac{e^{-ikr}}{r}\right)dx\,dy\,dz \qquad (11),$$

the integrated term vanishing in virtue of the condition that Z' is finite only within the space T. Moreover, since the dimensions of T are supposed to be very small in comparison with the wave-length, $d(r^{-1}e^{-ikr})/dy$ may be removed from under the integral sign. It will be convenient also to change the meaning of x, y, z, so that they shall now represent the coordinates of P relatively to O. Thus, if Z' now stand for the mean value of Z' throughout the space T,

$$\varpi' = +\frac{TZ'}{8\pi b^2}\frac{d}{dr}\left(\frac{e^{-ikr}}{r}\right)\cdot\frac{y}{r} \qquad (12).$$

In like manner

$$\varpi'' = -\frac{TZ'}{8\pi b^2}\frac{d}{dr}\left(\frac{e^{-ikr}}{r}\right)\cdot\frac{x}{r} \qquad (13);$$

and

$$\varpi''' = 0 \qquad (14).$$

In virtue of the symmetry round the axis of z it suffices to consider points which lie in the plane ZX. Then ϖ' vanishes, so that the rotation takes place about an axis perpendicular both to the direction of propagation (r) and to that of the force (z). If θ denote the angle between these directions, the resultant rotation, coincident with ϖ'', is

$$\varpi = -\frac{TZ'\sin\theta}{8\pi b^2}\frac{d}{dr}\left(\frac{e^{-ikr}}{r}\right) \qquad (15).$$

If we confine our attention to points at a great distance, this becomes simply

$$\varpi = \frac{ikTZ' \sin\theta}{8\pi b^2} \frac{e^{-ikr}}{r} \text{..................} (16).$$

The displacement, corresponding to (16), is perpendicular to r and in the plane zr. Its value is given by

$$-2\int \varpi \, dr = \frac{TZ' \sin\theta}{4\pi b^2} \frac{e^{-ikr}}{r};$$

or, if we restore the factor e^{ikbt}, and reject the imaginary part of the solution,

$$-2\int \varpi \, dr = \frac{TZ' \sin\theta}{4\pi b^2} \frac{\cos k(bt-r)}{r} \text{............} (17).$$

If $Z_1 \cos kbt$ denote the whole force applied at the origin,

$$Z_1 = TZ' . \rho \text{........................} (18),$$

so that (17) may be written

$$-2\int \varpi \, dr = \frac{Z_1 \sin\theta}{4\pi\rho b^2} \frac{\cos k(bt-r)}{r} \text{..............} (19).$$

The amplitude of the vibration radiated outwards is thus inversely as the distance, and directly as the sine of the angle between the ray and the direction in which the force acts. In the latter direction itself there is no transverse vibration propagated.

These expressions may be applied to find the secondary vibration dispersed in various directions when plane waves impinge upon a small obstacle of density different from that of the rest of the solid. We may suppose that the plane waves are expressed by

$$\gamma = \Gamma \cos k(bt - x) \text{...................} (20),$$

and that they impinge at the origin upon an obstacle of volume T and density ρ'. The additional inertia of the solid at this place would be compensated by a force $(\rho' - \rho)\ddot{\gamma}$, or $-(\rho' - \rho) k^2 b^2 \Gamma \cos kbt$, acting throughout T; and, if this force be actually applied, the primary waves would proceed without interruption. The secondary waves may thus be regarded as due to a force equal to the opposite of this, acting at O parallel to Z. The whole amount of the force is given by

$$Z_1 \cos kbt = (\rho' - \rho) k^2 b^2 T \Gamma \cos kbt \text{...........} (21);$$

so that by (19) the secondary displacement at a distant point (r, θ) is

$$\frac{(\rho' - \rho) k^2 T \Gamma \sin\theta}{4\pi\rho} \cdot \frac{\cos k(bt-r)}{r} \text{............} (22).$$

The intensity of the scattered vibration is thus inversely as the fourth power of the wave-length (Γ being given), and as the square of the sine of the angle between the scattered ray and the direction of vibration in the primary waves. Thus, if the primary ray be along x and the secondary ray along z, there are no secondary vibrations if (as above supposed) the primary vibrations are parallel to z; but if the primary vibrations are parallel to y, there are secondary vibrations of full amplitude ($\sin\theta = 1$), and these vibrations are themselves executed in a direction parallel to y.[1]

376. In § 375 we have examined the effect of a periodic force $Z_1 \cos kbt$, localized at the origin. We now proceed to consider the case of a force uniformly distributed along an infinite line.

Of this there are two principal sub-cases: the first where the force, itself always parallel to z, is distributed along the axis of z, the second where the distribution is along the axis of y. In the first, with which we commence, the entire motion is in two dimensions, symmetrical with respect to OZ, and further is such that α and β vanish, while γ is a function of (x^2+y^2) only. If, as suffices, we limit ourselves to points situated along OX, ϖ', ϖ''' vanish, and we have only to find ϖ''.

The simplest course to this end is by integration of the result given in (16) § 375. $\rho TZ'$ will be replaced by $Z_{11}dz$, the amount of the force distributed on dz; r denotes the distance between P on OX and dz on OZ; θ the angle between r and z. The rotation ϖ'' about an axis parallel to y and due to this element of the force is thus

$$\frac{ikZ_{11}dz}{8\pi b^2\rho}\,\frac{xe^{-ikr}}{r^2} \qquad\qquad (1).$$

In the integration x is constant, and $r^2 = x^2 + z^2$, so that we have to consider

$$\int \frac{i\,e^{-ikr}dr}{r\sqrt{(r^2-x^2)}} \quad \text{or} \quad \int \frac{i\,e^{-ikx}e^{-ikh}dh}{(x+h)\,.\,\sqrt{(2x+h)}\,.\,\sqrt{h}} \qquad (2),$$

if we write $r - x = h$.

[1] "On the Light from the Sky, its Polarization and Colour." *Phil. Mag.* Vol. XLI. pp. 107, 274, 1871; see also *Phil. Mag.* Vol. XLI. p. 447, 1871, for an investigation of the case where the obstacle differs in elastic quality, as well as in density, from the remainder of the medium.

From this integral a rigorous solution may be developed, but, as in § 342, we may content ourselves with the limiting form assumed when kx is very great. Thus, as the equivalent of (2), we get

$$\frac{2e^{-ikx}}{x \, . \, \sqrt{(2x)}} \int_0^\infty \frac{ie^{-ikh}dh}{\sqrt{h}} = \frac{2\sqrt{\pi} \, . \, e^{-i(kx-\frac{1}{4}\pi)}}{x \, . \, \sqrt{(2kx)}} \quad \text{..........} \quad (3);$$

so that as the integral of (1)

$$\varpi'' = \frac{kZ_{11}}{4\pi b^2 \rho} \frac{\sqrt{\pi}}{\sqrt{(2kx)}} e^{-i(kx-\frac{1}{4}\pi)} \quad \text{................} \quad (4).$$

From this γ may be at once deduced. We have

$$\gamma = -2\int\varpi'' dx = \frac{Z_{11}}{2\pi b^2 \rho} \frac{\sqrt{\pi}}{\sqrt{(2kx)}} e^{-i(kx+\frac{1}{4}\pi)} \quad \text{..........} \quad (5);$$

or, if we restore the time-factor, and omit the imaginary part of the solution,

$$\gamma = \frac{Z_{11}}{2\pi b^2 \rho} \frac{\sqrt{\pi}}{\sqrt{(2kx)}} \cos k(bt - x - \tfrac{1}{8}\lambda) \quad \text{...........} \quad (6).$$

This corresponds to the force $Z_{11} \cos kbt$ per unit of length of the axis of z. In virtue of the symmetry we may apply (6) to points not situated upon the axis of x, if we replace x by $\sqrt{(x^2+y^2)}$. That the value of γ would be inversely as the square root of the distance from the axis of z might have been anticipated from the principle of energy.

The solution might also be investigated directly in terms of γ without the aid of the rotations ϖ.

It now remains to consider the case in which the applied force, still parallel to z, is distributed along OY, instead of along OZ. The point P, at which the effect is required, may be supposed to be situated in the plane ZX at a great distance R from O and in such a direction that the angle ZOP is θ.

In virtue of the two-dimensional character of the force, $\beta = 0$, while α, γ are independent of y. Hence ϖ', ϖ''' vanish. But, although these component rotations vanish as regards the resultant effect, the action of a single element of the force $Z_{11}dy$, situated at y, would be more complicated. Into this, however, we need not enter, because, as before, the effect in reality depends only upon the elements in the neighbourhood of O. Thus, in place of (1), we may take

$$\frac{ikZ_{11}dy \, . \sin\theta}{8\pi b^2 \rho} \frac{e^{-ikr}}{r} \quad \text{......................} \quad (7),$$

r being the distance between dy and P, so that

$$dy/r = dr/y = dr/\sqrt{(r^2 - R^2)}.$$

Writing $r - R = h$, we get, as in (2), (3), (4),

$$\varpi'' = \frac{kZ_{11}\sin\theta}{4\pi b^2\rho}\frac{\sqrt{\pi}}{\sqrt{(2kR)}}e^{-i(kR-\frac{1}{4}\pi)} \quad \text{.............. (8)};$$

and for the displacement, perpendicular to R,

$$-2\int\varpi'' dR = \frac{Z_{11}\sin\theta}{2\pi b^2\rho}\frac{\sqrt{\pi}}{\sqrt{(2kR)}}e^{-i(kR+\frac{1}{4}\pi)} \quad \text{........... (9)}.$$

Hence, corresponding to the force $Z_{11}\cos kbt$ per unit of length of the axis of y, we have the displacement perpendicular to R at the point (R, θ)

$$\frac{Z_{11}\sin\theta}{2\pi b^2\rho}\frac{\sqrt{\pi}}{\sqrt{(2kR)}}\cos k(bt - R - \tfrac{1}{8}\lambda) \quad \text{........... (10)}.$$

377. As in § 375, we may employ the results of § 376 to form expressions for the secondary waves dispersed from a small cylindrical obstacle, coincident with OZ and of density ρ', upon which primary parallel waves impinge. If the expression for the primary waves be (20) § 375, we have

$$Z_{11} = (\rho' - \rho)\,k^2b^2\,.\,\pi c^2\,.\,\Gamma \quad \text{..................... (1)},$$

πc^2 being the area of the cross section of the obstacle. Thus, if we denote $\sqrt{(x^2 + y^2)}$ by r, we have from (6) § 376 as the expression of the secondary waves,

$$\gamma = \frac{(\rho' - \rho)\,k^2\,.\,\pi c^2\,.\,\Gamma}{2\pi\rho}\frac{\sqrt{\pi}}{\sqrt{(2kr)}}\cos k(bt - r - \tfrac{1}{8}\lambda)$$

$$= \frac{\pi(\rho' - \rho)\,.\,\pi c^2\,.\,\Gamma}{\rho\lambda^{\frac{3}{2}}r^{\frac{1}{2}}}\cos\frac{2\pi}{\lambda}(bt - r - \tfrac{1}{8}\lambda) \quad \text{......... (2)},$$

k being replaced by its equivalent $(2\pi/\lambda)$. In this case the secondary waves are symmetrical, and their intensity varies inversely as the distance and as the *cube* of the wave-length.

The solution expressed by (10) § 376 shews that if primary waves

$$\beta = B\cos k(bt - x) \quad \text{...................... (3)}$$

impinge upon the same small cylindrical obstacle, the displacement perpendicular to the secondary ray, viz. r, will be

$$\frac{\pi(\rho' - \rho)\,.\,\pi c^2\,.\,B\,.\cos\theta}{\rho\lambda^{\frac{3}{2}}r^{\frac{1}{2}}}\cos\frac{2\pi}{\lambda}(bt - r - \tfrac{1}{8}\lambda) \quad \text{......... (4)},$$

θ denoting the angle between the direction of the primary ray (x) and the secondary ray (r). In this case the secondary disturbance vanishes in one direction, that is along a ray parallel to the primary vibration.

Returning to the first case, in which α and β vanish throughout, while γ is a function of x and y only, let us suppose that the material composing the cylindrical obstacle differs from its surroundings in rigidity (n') as well as in density (ρ'). The conditions to be satisfied at the cylindrical surface are

$$\gamma \text{ (inside)} = \gamma \text{ (outside)},$$

$$n' d\gamma/dr \text{ (inside)} = n\, d\gamma/dr \text{ (outside)}.$$

In the exterior space γ satisfies the equation (§ 373)

$$d^2\gamma/dx^2 + d^2\gamma/dy^2 + k^2\gamma = 0,$$

where $k = p/b$; and in the space interior to the cylinder γ satisfies

$$d^2\gamma/dx^2 + d^2\gamma/dy^2 + k'^2\gamma = 0,$$

where $k' = p/b'$ and b' denotes the velocity of transverse vibrations in the material composing the cylinder. The investigation of the secondary waves thrown off by the obstacle when primary plane waves impinge upon it is then analogous to that of § 343, and the conclusion is that, corresponding to primary waves

$$\gamma = \Gamma \cos \frac{2\pi}{\lambda}(bt - x) \dots\dots\dots\dots\dots\dots\dots (5),$$

the secondary waves thrown off by a small cylinder in a direction making an angle θ with x are given by

$$\gamma = \frac{2\pi \,.\, \pi c^2 \,.\, \Gamma}{\lambda^{\frac{3}{2}} r^{\frac{1}{2}}} \left\{ \frac{\rho' - \rho}{2\rho} - \frac{n' - n}{n' + n} \cos\theta \right\} \cos \frac{2\pi}{\lambda}(bt - r - \tfrac{1}{8}\lambda) \dots (6),$$

which includes (2) as a particular case.

378. We now return to the fundamental problem, already partially treated in § 375, of the vibrations in an unlimited solid due to the application of a periodic force at the origin of coordinates. Equations (12), (13), (14) § 375 give the solution so far as to specify the values of the component rotations. If, as we shall ultimately suppose, the solid be incompressible, we have in addition $\delta = 0$. On this basis the solution might be completed, but it may be more instructive to give an independent investigation.

Since in the notation of § 373 $X' = Y' = 0$, we have by (5)

$$(a^2 - b^2)\, d\delta/dx + b^2\nabla^2\alpha + p^2\alpha = 0 \qquad (1),$$

$$(a^2 - b^2)\, d\delta/dy + b^2\nabla^2\beta + p^2\beta = 0 \qquad (2),$$

$$(a^2 - b^2)\, d\delta/dz + b^2\nabla^2\gamma + p^2\gamma = -Z' \qquad (3).$$

Let us assume

$$\alpha = d^2\chi/dx\,dz, \quad \beta = d^2\chi/dy\,dz, \quad \gamma = d^2\chi/dz^2 + w \qquad (4),$$

and accordingly

$$\delta = d\,(\nabla^2\chi)/dz + dw/dz \qquad (5).$$

The substitution of these values in (1) gives

$$\frac{d^2}{dx\,dz}\{a^2\nabla^2\chi + p^2\chi + (a^2 - b^2)\, w\} = 0\,;$$

so that (1) and (2) are satisfied if

$$a^2\nabla^2\chi + p^2\chi + (a^2 - b^2)\, w = 0 \qquad (6).$$

The same substitutions in (3) give

$$\frac{d^2}{dz^2}\{a^2\nabla^2\chi + p^2\chi + (a^2 - b^2)\, w\} + b^2\nabla^2 w + p^2 w + Z' = 0,$$

or in virtue of (6)

$$b^2\nabla^2 w + p^2 w + Z' = 0 \qquad (7).$$

By this equation w is determined, and thence χ by (6).

In the notation of § 375, $k = p/b$, $h = p/a$. Since $Z' = 0$ at all points other than the origin, (7) becomes

$$(\nabla^2 + k^2)\, w = 0 \qquad (8),$$

whence by (6) $$(\nabla^2 + h^2)(\nabla^2 + k^2)\,\chi = 0 \qquad (9)$$

is to be satisfied everywhere except at the origin. The solution of (9) is

$$\chi = A\,\frac{e^{-ikr}}{r} + B\,\frac{e^{-ihr}}{r} \qquad (10),$$

where A and B are constants. The corresponding values of w and δ are by (6) and (5)

$$w = k^2 A\,\frac{e^{-ikr}}{r}, \qquad \delta = -h^2 B\,\frac{d}{dz}\left(\frac{e^{-ihr}}{r}\right) \qquad (11).$$

To connect A and B with Z', we have from (7), as in § 375,

$$w = \frac{1}{4\pi b^2}\iiint Z'\,\frac{e^{-ikr}}{r}\,dx\,dy\,dz = \frac{Z_1}{4\pi b^2\rho}\,\frac{e^{-ikr}}{r},$$

so that $$A = \frac{Z_1}{4\pi k^2 b^2\rho} \qquad (12).$$

Again, by (6) § 373,

$$\nabla^2\delta + h^2\delta + a^{-2}\,dZ'/dz = 0\,;$$

so that, as in § 375,

$$\delta = \frac{1}{4\pi a^2}\iiint \frac{dZ'}{dz}\frac{e^{-ihr}}{r}\,dx\,dy\,dz = \frac{Z_1}{4\pi a^2\rho}\frac{d}{dz}\left(\frac{e^{-ihr}}{r}\right).$$

Thus, by comparison with (11),

$$B = -\frac{Z_1}{4\pi h^2 a^2\rho} = -A \;\ldots\ldots\ldots\ldots\ldots\ldots (13);$$

and

$$\chi = \frac{Z_1}{4\pi k^2 b^2\rho}\,\frac{e^{-ikr} - e^{-ihr}}{r} \;\ldots\ldots\ldots\ldots\ldots\ldots (14).$$

From the values of χ and w thus fully determined α, β, γ are found by simple differentiations, as indicated in (4). We have

$$\frac{d^2}{dx\,dz}\left(\frac{e^{-ikr}}{r}\right) = \frac{xze^{-ikr}}{r^2}\left(-\frac{k^2}{r} + \frac{3ik}{r^2} + \frac{3}{r^3}\right) \ldots\ldots\ldots (15),$$

$$\frac{d^2}{dz^2}\left(\frac{e^{-ikr}}{r}\right) = e^{-ikr}\left[\frac{z^2}{r^2}\left(-\frac{k^2}{r} + \frac{3ik}{r^2} + \frac{3}{r^3}\right) - \frac{ik}{r^2} - \frac{1}{r^3}\right] \ldots (16).$$

As the complete expressions are rather long, we will limit ourselves to the case of incompressibility ($h = 0$). Thus, if we restore the time-factor (e^{ipt}) and throw away the imaginary part of the solution, we get

$$\alpha = \frac{Ak^2xz}{r^3}\left[\left(-1 + \frac{3}{k^2r^2}\right)\cos(pt - kr) - \frac{3}{kr}\sin(pt - kr) - \frac{3}{k^2r^2}\cos pt\right] \ldots\ldots\ldots (17),$$

$$\gamma = \frac{Ak^2}{r}\left[\left(1 - \frac{z^2}{r^2} + \frac{3z^2}{k^2r^4} - \frac{1}{k^2r^2}\right)\cos(pt - kr) + \left(\frac{1}{kr} - \frac{3z^2}{kr^3}\right)\sin(pt - kr) - \left(\frac{3z^2}{k^2r^4} - \frac{1}{k^2r^2}\right)\cos pt\right] \ldots (18),$$

the value of β differing from that of α merely by the substitution of y for x. The value of A is given in (12), and $Z_1 \cos pt$ is the whole force operative at the origin at time t.

At a great distance from the origin (17), (18) reduce to

$$\alpha = -\frac{Z_1}{4\pi b^2\rho}\,\frac{xz}{r^2}\,\frac{\cos(pt - kr)}{r} \;\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

$$\gamma = \frac{Z_1}{4\pi b^2\rho}\left(1 - \frac{z^2}{r^2}\right)\frac{\cos(pt - kr)}{r} \;\ldots\ldots\ldots\ldots (20),$$

in agreement with (19) § 375.

W. König[1] has remarked upon the non-agreement of the complete solution (17), (18), first given in a different form by Stokes[2], with the results of a somewhat similar investigation by Hertz[3], in which the terms involving $\cos pt$, $\sin pt$ do not occur, and he seems disposed to regard Stokes' results as affected by error. But the fact is that the problems treated are essentially different, that of Hertz having no relation to elastic solids. The source of the discrepancy is in the first terms of (1) &c., which are omitted by Hertz in his theory of the ether. But assuredly in a theory of elastic solids these terms must be retained. Even when the material is supposed to be incompressible, so that δ vanishes, the retention is still necessary, because, as was fully explained by Stokes in the memoir referred to, the factor $(a^2 - b^2)$ is infinite at the same time.

If we suppose in (17), (18) that p and k are very small, and trace the limiting form, we obtain the solution of the statical problem of the deformation of an incompressible solid by a force localized at a point in its interior.

379. In § 373 we saw that in a uniform medium plane waves of transverse vibration

$$\alpha = 0, \quad \beta = 0, \quad \gamma = \Gamma \cos (pt - kx) \quad \text{...........(1)}$$

may be propagated without limit. We will now suppose that on the positive side of the plane $x = 0$ the medium changes, so that the density becomes ρ_1 instead of ρ, while the rigidity becomes n_1 instead of n. In the transmitted wave p remains the same, but k is changed to k_1, where

$$k_1^2/k^2 = n\rho_1/n_1\rho \quad \text{.........................(2).}$$

Assuming, as will be verified presently, that no change of phase need be allowed for, we may take as the expressions for the transmitted and reflected waves

$$\gamma_1 = \Gamma_1 \cos (pt - k_1 x), \qquad \gamma = \Gamma' \cos (pt + kx) \quad \text{......(3),}$$

so that altogether the value of γ in the first medium is

$$\gamma = \Gamma \cos (pt - kx) + \Gamma' \cos (pt + kx) \quad \text{.........(4),}$$

and in the second

$$\gamma_1 = \Gamma_1 \cos (pt - k_1 x) \quad \text{....................(5).}$$

[1] *Wied. Ann.* vol. XXXVII. p. 651, 1889.

[2] *Camb. Phil. Trans.* vol. IX. p. 1, 1849; *Collected Works*, vol. II. p. 243.

[3] *Wied. Ann.* vol. XXXVI. p. 1, 1889.

The conditions to be satisfied at the interface ($x=0$), upon which no external force acts, are

$$\gamma_1 = \gamma, \qquad n_1 d\gamma_1/dx = n\, d\gamma/dx \quad\text{..............(6)};$$

so that $$\Gamma + \Gamma' = \Gamma_1, \qquad nk(\Gamma - \Gamma') = n_1 k_1 \Gamma_1 \quad\text{..............(7)}.$$

If, as can plainly be done, Γ', Γ_1 be determined in accordance with (7), the conditions are all satisfied. We have

$$\frac{\Gamma'}{\Gamma} = \frac{nk - n_1 k_1}{nk + n_1 k_1} = \frac{\sqrt{(n\rho)} - \sqrt{(n_1\rho_1)}}{\sqrt{(n\rho)} + \sqrt{(n_1\rho_1)}} \quad\text{..............(8)},$$

$$\frac{\Gamma_1}{\Gamma} = \frac{\Gamma + \Gamma'}{\Gamma} = \frac{2\sqrt{(n\rho)}}{\sqrt{(n\rho)} + \sqrt{(n_1\rho_1)}} \quad\text{..................(9)},$$

by which the reflected and transmitted waves are determined. The particular cases in which $\rho_1 = \rho$, or $n_1 = n$, may be specially noted.

When the incidence upon the plane separating the two bodies is oblique, the problem becomes more complicated, and divides itself into two parts according as the vibrations (always perpendicular to the incident ray) are executed in the plane of incidence, or in the perpendicular plane. Into these matters, which have been much discussed from an optical point of view, we shall not enter. The method of investigation, due mainly to Green, is similar to that of § 270. A full account with the necessary references is given in Basset's *Treatise on Physical Optics*, Ch. XII.

380. The vibrations of solid bodies bounded by free surfaces which are plane, cylindrical, or spherical, can be investigated without great difficulty, but the subject belongs rather to the Theory of Elasticity. For an infinite plate of constant thickness the functions of the coordinates required are merely circular and exponential[1]. The solution of the problem for an infinite cylinder[2] depends upon Bessel's functions, and is of interest as giving a more complete view of the longitudinal and flexural vibrations of a thin rod.

The case of the sphere is important as of a body limited in all directions. The symmetrical radial vibrations, purely dilatational in their character, were first investigated by Poisson and

[1] *Proc. Lond. Math. Soc.* vol. XVII. p. 4, 1885; vol. XX. p. 225, 1889.

[2] Pochhammer, *Crelle*, vol. LXXXI. 1876; Chree, *Quart. Journ.* 1886. See also Love's *Theory of Elasticity*, ch. XVII.

Clebsch[1]. The complete theory is due to Jaerisch[2] and especially to Lamb[3]. An exposition of it will be found in Love's treatise already cited.

The calculations of frequency are complicated by the existence of *two* elastic constants κ and n § 373, or q and μ § 214. From the principle of § 88 we may infer, as Lamb has remarked, that the frequency increases with any rise either of κ or of n, for as appears from (1) § 345 either change increases the potential energy of a given deformation.

381[4]. In the course of this work we have had frequent occasion to notice the importance of the conclusions that may be arrived at by the method of dimensions. Now that we are in a position to draw illustrations from a greater variety of acoustical phenomena relating to the vibrations of both solids and fluids, it will be convenient to resume the subject, and to develope somewhat in detail the principles upon which the method rests.

In the case of systems, such as bells or tuning-forks, formed of uniform isotropic material, and vibrating in virtue of elasticity, the acoustical elements are the shape, the linear dimension c, the constants of elasticity q and μ (§ 149), and the density ρ. Hence, by the method of dimensions, the periodic time varies *cæteris paribus* as the linear dimension, at least if the amplitude of vibration be in the same proportion; and, if the law of isochronism be assumed, the last-named restriction may be dispensed with. In fact, since the dimensions of q and ρ are respectively $[ML^{-1}T^{-2}]$ and $[ML^{-3}]$, while μ is a mere number, the only combination capable of representing a time is $q^{-\frac{1}{2}}.\rho^{\frac{1}{2}}.c$.

The argument which underlies this mathematical shorthand is of the following nature. Conceive two geometrically similar bodies, whose mechanical constitution at corresponding points is the same, to execute similar movements in such a manner that the corresponding changes occupy times[5] which are proportional to the

[1] *Theorie der Elasticität Fester Körper*, Leipzig, 1862.

[2] *Crelle*, vol. LXXXVIII. 1879.

[3] *Proc. Lond. Math. Soc.* vol. XIII. p. 189, 1882.

[4] This section appeared in the First Edition as § 348.

[5] The conception of an alteration of scale in space has been made familiar by the universal use of maps and models, but the corresponding conception for time is often less distinct. Reference to the case of a musical composition performed at different speeds may assist the imagination of the student.

linear dimensions—in the ratio, say, of $1 : n$. Then, if the one movement be possible as a consequence of the elastic forces, the other will be also. For the masses to be moved are as $1 : n^3$, the accelerations as $1 : n^{-1}$, and therefore the necessary forces are as $1 : n^2$; and, since the strains are the same, this is in fact the ratio of the elastic forces due to them when referred to corresponding areas. If the elastic forces are competent to produce the supposed motion in the first case, they are also competent to produce the supposed motion in the second case.

The dynamical similarity is disturbed by the operation of a force like gravity, proportional to the cubes, and not to the squares, of corresponding lines; but in cases where gravity is the sole motive power, dynamical similarity may be secured by a different relation between corresponding spaces and corresponding times. Thus if the ratio of corresponding spaces be $1 : n$, and that of corresponding times be $1 : n^{\frac{1}{2}}$, the accelerations are in both cases the same, and may be the effects of forces in the ratio $1 : n^3$ acting on masses which are in the same ratio. As examples coming under this head may be mentioned the common pendulum, sea-waves, whose velocity varies as the square root of the wave-length, and the whole theory of the comparison of ships and their models by which Froude predicted the behaviour of ships from experiments made on models of moderate dimensions.

The same comparison that we have employed above for elastic solids applies also to aerial vibrations. The pressures in the cases to be compared are the same, and therefore when acting over areas in the ratio $1 : n^2$, give forces in the same ratio. These forces operate on masses in the ratio $1 : n^3$, and therefore produce accelerations in the ratio $1 : n^{-1}$, which is the ratio of the actual accelerations when both spaces and times are as $1 : n$. Accordingly the periodic times of similar resonant cavities, filled with the same gas, are directly as the linear dimension—a very important law first formulated by Savart.

Since the same method of comparison applies both to elastic solids and to elastic fluids, an extension may be made to systems into which both kinds of vibration enter. For example, the scale of a system compounded of a tuning-fork and of an air resonator may be supposed to be altered without change in the motion other than that involved in taking the times in the same ratio as the linear dimensions.

Hitherto the alteration of scale has been supposed to be uniform in all dimensions, but there are cases, not coming under this head, to which the principle of dynamical similarity may be most usefully applied. Let us consider, for example, the flexural vibrations of a system composed of a thin elastic lamina, plane or curved. By §§ 214, 215 we see that the thickness of the lamina b, and the mechanical constants q and ρ, will occur only in the combinations qb^3 and $b\rho$, and thus a comparison may be made even although the alteration of thickness be not in the same proportion as for the other dimensions. If c be the linear dimension when the thickness is disregarded, the times must vary *cœteris paribus* as $q^{-\frac{1}{2}} . \rho^{\frac{1}{2}} . c^2 . b^{-1}$. For a given material, thickness, and shape, the times are therefore as the *squares* of the linear dimension. It must not be forgotten, however, that results such as these, which involve a law whose truth is only approximate, stand on a different level from the more immediate consequences of the principle of similarity.

CHAPTER XXIII.

FACTS AND THEORIES OF AUDITION.

382. THE subject of the present chapter has especial relation to the ear as the organ of hearing, but it can be considered only from the physical side. The discussion of anatomical or physiological questions would accord neither with the scope of this book nor with the qualifications of the author. Constant reference to the great work of Helmholtz is indispensable[1]. Although, as we shall see, some of the positions taken by the author have been relinquished, perhaps too hastily, by subsequent writers, the importance of the observations and reasonings contained in it, as well as the charm with which they are expounded, ensure its long remaining the starting point of all discussions relating to sound sensations.

383. The range of pitch over which the ear is capable of perceiving sounds is very wide. Naturally neither limit is well defined. From his experiments Helmholtz concluded that the sensation of musical tone begins at about 30 vibrations per second, but that a determinate musical pitch is not perceived till about 40 vibrations are performed in a second. Preyer[2] believes that he heard pure tones as low as 15 per second, but it seems doubtful whether the octave was absolutely excluded. On a recent review of the evidence and in the light of some fresh experiments, Van Schaik[3] sees no reason for departing greatly from Helmholtz's estimate, and fixes the limit at about 24 vibrations per second.

[1] *Tonempfindungen*, 4th edition, 1877; *Sensations of Tone*, 2nd English edition translated from the 4th German edition by A. J. Ellis. Citations will be made from this English edition, which is further furnished by the translator with many valuable notes.

[2] *Physiologische Abhandlungen*, Jena, 1876.

[3] *Arch. Néerl.* vol. XXIX. p. 87, 1895.

On the upper side the discrepancies are still greater. Much no doubt depends upon the intensity of the vibrations. In experiments with bird-calls (§ 371) nothing is heard above 10,000, although sensitive flames respond up to 50,000. But forks carefully bowed, or metal bars struck with a hammer, appear to give rise to audible sounds of still higher frequencies. Preyer gives 20,000 as near the limit for normal ears.

In the case of very high sounds there is little or no appreciation of pitch, so that for musical purposes nothing over 4000 need be considered.

The next question is how accurately can we estimate pitch by the ear only? The sounds are here supposed to be heard in succession, for (§ 59) when two uniformly sustained notes are sounded together there is no limit to the accuracy of comparison attainable by the method of beats. From a series of elaborate experiments Preyer[1] concludes that at no part of the scale can ·20 vibration per second be distinguished with certainty. The sensitiveness varies with pitch. In the neighbourhood of 120, ·4 vibration per second can be just distinguished; at 500 about ·3 vibration; and at 1000 about ·5 vibration per second. In some cases where a difference of pitch was recognised, the observer could not decide which of the two sounds was the graver.

384. In determinations of the limits of pitch, or of the perceptible differences of pitch, the sounds are to be chosen of convenient intensity. But a further question remains behind as to the degree of intensity at given pitch necessary for audibility. The earliest estimate of the amplitude of but just audible sounds appears to be that of Toepler and Boltzmann[2]. It depends upon an ingenious application of v. Helmholtz's theory of the open organ-pipe (§ 313) to data relating to the maximum condensation within the pipe, as obtained by the authors experimentally (§ 322 *d*). They conclude that plane waves, of pitch 181, in which the maximum condensation (s) is $6{\cdot}5 \times 10^{-8}$, are just audible.

It is evident that a superior limit to the amplitude of waves giving an audible sound may be derived from a knowledge of the energy which must be expended in a given time in order to

[1] An account of Preyer's work was given by A. J. Ellis in the *Proceedings of the Musical Association*, 3rd session, p. 1, 1877.

[2] *Pogg. Ann.* vol. CXLI. p. 321, 1870.

generate them and of the extent of surface over which the waves so generated are spread at the time of hearing. An estimate founded on these data will necessarily be too high, both because sound-waves must suffer some dissipation in their progress and also because a part, and in some cases a large part, of the energy expended never takes the form of sound-waves at all.

In the first application of the method[1], the source of sound was a whistle, mounted upon a Wolfe's bottle, in connection with which was a siphon manometer for the purpose of measuring the pressure of the wind.. The apparatus was inflated from the lungs, and with a little practice there was no difficulty in maintaining a sufficiently constant blast of the requisite duration. The most suitable pressure was determined by preliminary trials, and was measured by a column of water $9\frac{1}{2}$ cm. high.

The first point to be determined was the distance from the source to which the sound remained clearly audible. The experiment was tried upon a still winter's day and it was ascertained that the whistle could be heard without effort (in both directions) to a distance of 820 metres.

The only remaining datum necessary for the calculation is the quantity of air which passes through the whistle in a given time. This was determined by a laboratory experiment from which it appeared that the consumption was 196 cub. cents. per second.

In working out the result it is most convenient to use consistently the C. G. S. system. On this system of measurement the pressure employed was $9\frac{1}{2} \times 981$ dynes per sq. cent., and therefore the work expended per second in generating the waves was $196 \times 9\frac{1}{2} \times 981$ ergs[2].

Now (§ 245) the mechanical value of a series of progressive waves is the same as the kinetic energy of the whole mass of air concerned, supposed to be moving with the maximum velocity (v) of vibration; so that, if S denote the area of the wave-front considered, a the velocity of sound, ρ the density of air, the mechanical value of the waves passing in a unit of time is expressed by $S . a . \rho . \frac{1}{2}v^2$, in which the numerical value of a is about 34100, and that of ρ about ·0013. In the present application S is the area of the surface of a hemisphere whose radius is

[1] *Proc. Roy. Soc.* vol. XXVI. p. 248, 1877.

[2] Nearly 2×10^6 ergs.

82000 centimetres; and thus, if the whole energy of the escaping air were converted into sound and there were no dissipation on the way, the value of v at a distance of 82000 centimetres would be given by the equation

$$v^2 = \frac{2 \times 196 \times 9\tfrac{1}{2} \times 981}{2\pi (82000)^2 \times 34100 \times \cdot 0013},$$

whence $v = \cdot 0014 \frac{\text{cm.}}{\text{sec.}}, \quad s = \frac{v}{a} = 4\cdot 1 \times 10^{-8}.$

This result does not require a knowledge of the pitch of the sound. If the period be τ, the relation between the maximum excursion x and the maximum velocity v is $x = v\tau/2\pi$. In the experiment under discussion the note was f^{IV}, with a frequency of about 2730. Hence

$$x = \frac{\cdot 0014}{2\pi \times 2730} = 8\cdot 1 \times 10^{-8}\,\text{cm.},$$

or the amplitude of the aerial particles was less than a ten-millionth of a centimetre. It was estimated that under favourable conditions an amplitude of 10^{-8} cm. would still have been audible.

It is an objection to the above method that when such large distances are concerned it is difficult to feel sure that the disturbing influence of atmospheric refraction is sufficiently excluded Subsequently experiments were attempted with pipes of lower pitch which should be audible to a less distance, but these were not successful, and ultimately recourse was had to tuning-forks.

"A fork of known dimensions, vibrating with a known amplitude, may be regarded as a store of energy of which the amount may readily be calculated. This energy is gradually consumed by internal friction and by generation of sound. When a resonator is employed the latter element is the more important, and in some cases we may regard the dying down of the amplitude as sufficiently accounted for by the emission of sound. Adopting this view for the present, we may deduce the rate of emission of sonorous energy from the observed amplitude of the fork at the moment in question and from the rate at which the amplitude decreases. Thus if the law of decrease be $e^{-\frac{1}{2}kt}$ for the amplitude of the fork, or e^{-kt} for the energy, and if E be the total energy at time t, the rate at which energy is emitted at that time is $-dE/dt$, or kE. The value of k is deducible from observations of the rate of decay, *e.g.* of the time during which the amplitude is halved. With these arrangements there is no difficulty in converting energy into sound upon a small scale, and thus in reducing the distance of audibility to such a figure as 30 metres. Under these circumstances the observations are much more manageable than when the operators are separated by half a mile, and there is no reason to fear disturbance from atmospheric refraction.

The fork is mounted upon a stand to which is also firmly attached the observing-microscope. Suitable points of light are obtained from starch grains, and the line of light into which each point is extended by the vibration is determined with the aid of an eyepiece-micrometer. Each division of the micrometer-scale represents ·001 centim. The resonator, when in use, is situated in the position of maximum effect, with its mouth under the free ends of the vibrating prongs.

The course of an experiment was as follows:—In the first place the rates of dying down were observed, with and without the resonator, the stand being situated upon the ground in the middle of a lawn. The fork was set in vibration with a bow, and the time required for the double amplitude to fall to half its original value was determined. Thus in the case of a fork of frequency 256, the time during which the vibration fell from 20 micrometer-divisions to 10 micrometer-divisions was 16^s without the resonator, and 9^s when the resonator was in position. These times of halving were, as far as could be observed, independent of the initial amplitude. To determine the minimum audible, one observer (myself) took up a position 30 yards (27·4 metres) from the fork, and a second (Mr. Gordon) communicated a large vibration to the fork. At the moment when the double amplitude measured 20 micrometer-divisions the second observer gave a signal, and immediately afterwards withdrew to a distance. The business of the first observer was to estimate for how many seconds after the signal the sound still remained audible. In the case referred to the time was 12^s. When the distance was reduced to 15 yards (13·7 metres), an initial double amplitude of 10 micrometer-divisions was audible for almost exactly the same time.

These estimates of audibility are not made without some difficulty. There are usually 2 or 3 seconds during which the observer is in doubt whether he hears or only imagines, and different individuals decide the question in opposite ways. There is also of course room for a real difference of hearing, but this has not obtruded itself much. A given observer on a given day will often agree with himself surprisingly well, but the accuracy thus suggested is, I think, illusory. Much depends upon freedom from disturbing noises. The wind in the trees or the twittering of birds embarrasses the observer, and interferes more or less with the accuracy of results.

The equality of emission of sound in various horizontal directions was tested, but no difference could be found. The sound issues almost entirely from the resonator, and this may be expected to act as a simple source.

When the time of audibility is regarded as known, it is easy to deduce the amplitude of the vibration of the fork at the moment when the sound ceases to impress the observer. From this the rate of emission of sonorous energy and the amplitude of the aerial vibration as it reaches the observer are to be calculated.

The first step in the calculation is the expression of the total energy of the fork as a function of the amplitude of vibration measured at the extremity of one of the prongs. This problem is considered in § 164. If l be the length, ρ the density, and ω the sectional area of a rod damped at one end and free at the other, the kinetic energy T is connected with the displacement η at the free end by the equation (10)

$$T = \tfrac{1}{8}\rho l\omega(d\eta/dt)^2.$$

At the moment of passage through the position of equilibrium $\eta = 0$ and $d\eta/dt$ has its maximum value, the whole energy being then kinetic. The maximum value of $d\eta/dt$ is connected with the maximum value of η by the equation

$$(d\eta/dt)_{\text{max.}} = 2\pi/\tau\,.\,(\eta)_{\text{max.}};$$

so that if we now denote the double amplitude by 2η, the whole energy of the vibrating bar is $\frac{1}{8}\rho\omega l\pi^2/\tau^2.(2\eta)^2$,

or for the two bars composing the fork

$$E = \tfrac{1}{4}\rho\omega l\pi^2/\tau^2.(2\eta)^2, \qquad\qquad \text{(A)}$$

where $\rho\omega l$ is the mass of each prong.

The application of (A) to the 256-fork, vibrating with a double amplitude of 20 micrometer-divisions, is as follows. We have

$$l = 14{\cdot}0 \text{ cm.}, \qquad \omega = {\cdot}6\times 1{\cdot}1 = {\cdot}66 \text{ sq. cm.},$$

$$1/\tau = 256, \qquad \rho = 7{\cdot}8, \qquad 2\eta = {\cdot}050 \text{ cm.};$$

and thus $$E = 4{\cdot}06\times 10^3 \text{ ergs.}$$

This is the whole energy of the fork when the actual double amplitude at the ends of the prongs is ·050 centim.

As has already been shewn, the energy lost per second is kE, if the amplitude vary as $e^{-\frac{1}{2}kt}$. For the present purpose k must be regarded as made up of two parts, one k_1 representing the dissipation which occurs in the absence of the resonator, the other k_2 due to the resonator. It is the latter part only which is effective towards the production of sound. For when the resonator is out of use the fork is practically silent; and, indeed, even if it were worth while to make a correction on account of the residual sound, its phase would only accidentally agree with that of the sound issuing from the resonator.

The values of k_1 and k are conveniently derived from the times, t_1 and t, during which the amplitude falls to *one half*. Thus

$$k = 2\log_e 2\,.\,/t, \quad k_1 = 2\log_e 2\,.\,/t_1;$$

so that

$$k_2 = 2\log_e 2\,.\,(1/t - 1/t_1) = 1{\cdot}386\,(1/t - 1/t_1).$$

And the energy converted into sound per second is k_2E.

We may now apply these formulæ to the case, already quoted, of the 256-fork, for which $t = 9$, $t_1 = 16$. Thus t_2, the time which would be occupied in halving the amplitude were the dissipation due entirely to the resonator, is 20·6; and $k_2 = {\cdot}0674$. Accordingly,

$$k_2E = 267 \text{ ergs per second,}$$

corresponding to a double amplitude represented by 20 micrometer-divisions. In the experiment quoted the duration of audibility was 12 seconds, during which the amplitude would fall in the ratio $2^{12/9} : 1$, and the energy in the ratio $4^{12/9} : 1$. Hence at the moment when the sound was just becoming inaudible the energy emitted as sound was 42·1 ergs per second[1].

[1] It is of interest to compare with the energy-emission of a source of light. An incandescent electric lamp of 200 candles absorbs about a horse-power, or say 10^{10} ergs per second. Of the total radiation only about $\frac{1}{100}$ part acts effectively upon the eye; so that radiation of suitable quality consuming 5×10^5 ergs per second corresponds to a candle-power. This is about 10^4 times that emitted as sound by the fork in the experiment described above. At a distance of $10^2 \times 30$, or 3000 metres, the stream of energy from the ideal candle would be about equal to the stream of energy just audible to the ear. It appears that the streams of energy required to influence the eye and the ear are of the same order of magnitude, a conclusion already drawn by Toepler and Boltzmann.

The question now remains, What is the corresponding amplitude or condensation in the progressive aerial waves at 27·4 metres from the source? If we suppose, as in my former calculations, that the ground reflects well, we are to treat the waves as hemispherical. On the whole this seems to be the best supposition to make, although the reflexion is doubtless imperfect. The area S covered at the distance of the observer is thus $2\pi \times 2740^2$ sq. centim., and since (§ 245)

$$S \, . \, \tfrac{1}{2} a \rho v^2 = S \, . \, \tfrac{1}{2} \rho a^3 s^2 = 42{\cdot}1,$$

we find
$$s^2 = \frac{42{\cdot}1}{\pi \times 2740^2 \times {\cdot}00125 \times 34100^3},$$

and
$$s = 6{\cdot}0 \times 10^{-9}.$$

The condensation s is here reckoned in atmospheres; and the result shews that the ear is able to recognize the addition and subtraction of densities far less than those to be found in our highest vacua.

The amplitude of aerial vibration is given by $as\tau/2\pi$, where $1/\tau = 256$, and is thus equal to $1{\cdot}27 \times 10^{-7}$ cm.

It is to be observed that the numbers thus obtained are still somewhat of the nature of superior limits, for they depend upon the assumption that all the dissipation *due to the resonator* represents production of sound. This may not be strictly the case even with the moderate amplitudes here in question, but the uncertainty under this head is far less than in the case of resonators or organ-pipes caused to speak by wind. From the nature of the calculation by which the amplitude or condensation in the aerial waves is deduced, a considerable loss of energy does not largely influence the final numbers.

Similar experiments have been tried at various times with forks of pitch 384 and 512. The results were not quite so accordant as was at first hoped might be the case, but they suffice to fix with some approximation the condensation necessary for audibility. The mean results are as follows:—

c', frequency = 256, $s = 6{\cdot}0 \times 10^{-9}$,
g', „ = 384, $s = 4{\cdot}6 \times 10^{-9}$,
c'', „ = 512, $s = 4{\cdot}6 \times 10^{-9}$,

no reliable distinction appearing between the two last numbers. Even the distinction between 6·0 and 4·6 should be accepted with reserve; so that the comparison must not be taken to prove much more than that the condensation necessary for audibility varies but slowly in the singly dashed octave[1]."

Results of the same order of magnitude have been obtained also by Wien[2], who used an entirely different method.

385. For most purposes of experiment and for many of ordinary life it makes but little difference whether we employ one ear only, or both; and yet there can be no doubt that we can derive most important information from the simultaneous use of the two ears. How this is effected still remains very obscure.

Although the utmost precautions be taken to ensure separate action, it is certain that a sound led into one ear is capable of giving beats with a second sound of slightly different pitch led into the other ear. There is, of course, no approximation to such silence as would occur at the moment of antagonism were the two sounds conveyed to the same ear; but the beats are perfectly distinct, and remain so as the sounds die away so as to become singly all but inaudible[3]. It is found, however, that combination tones (§ 391) are not produced under these conditions[4]. Some curious observations with the telephone are thus described by Prof. S. P. Thompson[5]. "Almost all persons who have experimented with the Bell telephone, when using a pair of instruments to receive the sound, one applied to each ear, have at some time or other noticed the apparent localization of the sounds of the telephone at the back of the head. Few, however, seemed to be aware that this was the result of either reversed order in the connection of the terminals of the instruments with the circuit, or reversed order in the polarity of the magnet of one of the receiving instruments. When the two vibrating discs execute similar vibrations, both advancing or both receding at once, the sound is heard as usual in the ears; but if the action of one instrument be reversed, so that when one disc advances the other recedes, and the vibrations have opposite phases, the sound apparently changes its place from the interior of the ear, and is heard as if proceeding from the back of the head, or, as I would rather say, from the top

[1] *Phil. Mag.* vol. xxxviii. p. 366, 1894.
[2] *Wied. Ann.* vol. xxxvi. p. 834, 1889.
[3] S. P. Thompson, *Phil. Mag.* vol. iv. p. 274, 1877.
[4] See also Dove, *Pogg. Ann.* vol. cvii. p. 652, 1859.
[5] *Phil. Mag.* vol. vi. p. 385, 1878.

of the cerebellum."....."I arranged a Hughes's microphone with two cells of a Fuller's battery and two Bell telephones, one of them having a commutator under my control. Placing the telephones to my ears, I requested my assistant to tap on the wooden support of the microphone. The result was deafening. I felt as if simultaneous blows had been given to the tympana of my ears. But on reversing the current through one telephone, I experienced a sensation only to be described as of some one tapping with a hammer *on the back of the skull from the inside.*"

In our estimation of the direction in which a sound comes to us we are largely dependent upon the evidence afforded by binaural audition. This is one of those familiar and instinctive operations which often present peculiar difficulties to scientific analysis. A blindfold observer in the open air is usually able to indicate within a few degrees the direction of a sound, even though it be of short duration, such as a single vowel or a clap of the hands. The decision is made with confidence and does not require a movement of the head.

To obtain further evidence experiments were made with the approximately pure tones emitted from forks in association with resonators; but in order to meet the objection that the first sound of the fork, especially when struck, might give a clue, and so vitiate the experiment, *two* similar forks and resonators, of pitch 256, were provided. These were held by two assistants, between whom the observer stood midway. In each trial both forks were struck, and afterwards *one* only was held to its resonator. The results were perfectly clear. When the forks were to the right and to the left, the observer could distinguish them instinctively and without fail. But when he turned through a right angle, so as to bring the forks to positions in front and behind him, no discrimination was possible, and an attempt to pronounce was felt to be only guessing.

That it should be impossible to distinguish whether a pure tone comes from in front or from behind is intelligible enough. On account of the symmetry the two ears would be affected alike in both cases, and any difference of intensity due to the position could not avail in the absence of information as to the original intensity. The difficulty is rather to understand how the discrimination between front and rear is effected in other cases, *e.g.* of the voice, where it is found to be easy. It can only be conjectured that the quality of a compound sound is liable to modification by the external ear, which is differently presented in the two cases.

The ready discrimination between right and left, even when pure tones are concerned, is naturally attributed to the different intensities with which the sound would be perceived by the two ears. But this explanation is not so complete as might be supposed. It is true that very high sounds, such as a hiss, are ill heard with the averted ear: but when the pitch is moderate, *e.g.* 256 per second, the difference of intensity on the two sides does not seem very great. The experiment may easily be tried roughly by stopping one ear with the finger and turning round backwards and forwards while listening to a sound held steadily. Calculation (§ 328) shews, moreover, that the human head, considered as an obstacle to the waves of sound, is scarcely big enough in relation to the wave length to throw a distinct shadow. As an illustration I have calculated the intensity of sound due to a distant source at various points on the surface of a fixed spherical obstacle. The result depends upon the ratio (kc) between the circumference of the sphere and the length of the wave. If we call the point upon the spherical surface nearest to the source the anterior pole, and the opposite point (where the shadow might be expected to be most intense) the posterior pole, the results on three suppositions as to the relative magnitudes of the sphere and wave length are as follows:—

	$kc=2$	$kc=1$	$kc=\frac{1}{2}$
Anterior pole	·69	·50	·29
Posterior pole	·32	·28	·26
Equator	·36	·24	·23

When for example the circumference of the sphere is but half the wave length, the intensity at the posterior pole is only about a tenth part less than at the anterior pole, while the intensity is least of all in a lateral direction. When kc is less than $\frac{1}{2}$, the difference of the intensities at the two poles is still less important, amounting to about 1 per cent. when $kc = \frac{1}{4}$.

The case of the head and a pitch c' would correspond to $kc = \cdot 4$ about, so that the differences of intensity indicated by theory are decidedly small. The explanation of the power of discrimination actually observed would be easier, if it were possible to suppose account taken of the different phases of the vibrations by which the two ears are attacked[1].

386. Passing on to another branch of our subject, we have now to consider more closely the impression produced upon the ear by an arbitrary sequence of aerial pressures fluctuating about a certain mean value. According to the literal statement of Ohm's law (§ 27) the ear is capable of hearing as separate tones all the simple vibrations into which the sequence of pressures may be analysed by Fourier's theorem, provided that the pitch of these components lies between certain limits. Components whose pitch lies outside the limits would be ignored. Moreover, within the limits of audibility the relative phases of the various components would be a matter of indifference.

To the law stated in this extreme form there must obviously be exceptions. It is impossible to suppose that the ear would hear as separate tones simple components of extremely nearly the same frequency. Such components, it is well known, give rise to beats, and their relative phase is a material element in the question. Again, it will be evident that the corresponding tone will not be heard unless a vibration reaches a certain intensity. A finite intensity would be demanded, even if the vibration stood by itself; and we should expect that the intensity necessary for audibility would be greater in the presence of other vibrations, especially perhaps when these correspond to harmonic undertones. It will be advisable to consider these necessary exceptions to the universality of Ohm's law a little more in detail.

The course of events, when the interval between two simple vibrations is gradually increased, has been specially studied by Bosanquet[2]. As in §§ 30, 65*a*, if the components be $\cos 2\pi n_1 t$, $\cos 2\pi n_2 t$, we have for the resultant,

$$u = \cos 2\pi n_1 t + \cos 2\pi n_2 t$$
$$= 2\cos \pi(n_2 - n_1)t \,.\, \cos \pi(n_2 + n_1)t \quad\ldots\ldots\ldots (1);$$

shewing that the resultant of two simple vibrations of equal amplitudes and of frequencies n_1, n_2 can be represented mathematically by a single vibration whose frequency is the mean, viz.

[1] *Nature*, vol. XIV. p. 32, 1876. *Phil. Mag.* vol. III. p. 456, 1877; vol. VII. p. 149, 1879.

[2] *Phil. Mag.* vol. XI. p. 420, 1881.

$\frac{1}{2}(n_1+n_2)$, and whose amplitude varies according to the cosine law, involving a change of sign (§ 65*a*), with a frequency $\frac{1}{2}(n_2-n_1)$. This single vibration is not *simple*. The question now arises under which of the two forms in (1) will the ear perceive the sound. According to the strict reading of Ohm's law the two tones n_1 and n_2 would be perceived separately. We know that when n_1 and n_2 are nearly enough equal this does not and could not happen. The second form then represents the phenomenon; and it indicates *beats*, the tone $\frac{1}{2}(n_1+n_2)$ having an *intensity* which varies between 0 and 4 with a frequency (n_2-n_1) equal to the difference of frequencies of the original tones. Mr Bosanquet found that "(α) the critical interval at which two notes begin to be heard beside their beats, or resultant displacements, is about two commas, throughout that medium portion of the scale which is used in practical music; (β) this critical interval appears to be not exactly the same for all ears." But in both the cases examined the beats alone were heard with an interval of one comma, and the two notes were quite clear beside the beats with an interval of three commas. "As the interval increases, the separate notes become more and more prominent, and the beats diminish in loudness and distinctness, till, by the time that a certain interval is reached, which is about a minor third in the middle of the scale, the beats practically disappear and the two notes alone survive."

On the second question as to the strength in which a component simple vibration, of sufficiently distinct pitch, must be present in order to assert itself as a separate tone there is but little evidence, and that not very accordant. According to the experiments of Brandt and Helmholtz (§ 130) Young's law as to the absence in certain cases of particular components from the sound of a plucked string is verified. Observations of this kind are easily made with resonators; but for the present purpose the use of resonators is inadmissible, the question relating to the behaviour of the unassisted ear.

On the other hand A. M. Mayer[1] found that sounds of considerable intensity when heard by themselves were liable to be completely obliterated by *graver* sounds of sufficient force. In some experiments the graver note was from an open organ-pipe which sounded steadily, while the higher was that of a fork, excited vigorously and then allowed to die down. The action of the fork could be

[1] *Phil. Mag.* vol. II. p. 500, 1876.

made intermittent by moving the hand to and fro over the mouth of its resonance box. The results are thus described. "At first every time that the mouth of the box is open the sound of the fork is distinctly heard and changes the quality of the note of the open pipe. But as the vibrations of the fork run down in amplitude the sensations of its effect become less and less till they soon entirely vanish, and not the slightest change can be observed in the quality or intensity of the note of the organ-pipe, whether the resonance box of the fork be open or closed. Indeed at this stage of the experiment the vibrations of the fork may be suddenly and totally stopped without the ear being able to detect the fact. But if instead of stopping the fork when it becomes inaudible we stop the sound of the organ-pipe, it is impossible not to feel surprised at the strong sound of the fork which the open pipe had smothered and had rendered powerless to affect the ear."

But "no sound, even when very intense, can diminish or obliterate the sensation of a concurrent sound which is lower in pitch. This was proved by experiments similar to the last, but differing in having the more intense sound higher (instead of lower) in pitch. In this case, when the ear decides that the sound of the (lower and feebler) tuning-fork is just extinguished, it is generally discovered on stopping the higher sound that the *fork*, which should produce the lower sound, *has ceased to vibrate.* This surprising experiment must be made in order to be appreciated. I will only remark that very many similar experiments, ranging through four octaves, have been made, with consonant and dissonant intervals, and that scores of different hearers have confirmed this discovery."

These results, which are not difficult to verify[1], involve a serious deduction from the universality of Ohm's law, and must have an important bearing upon other unsettled questions relating to audition. It is to be observed that in Mayer's experiments the question is not merely whether a particular tone can be heard as such. The higher sound of feebler intensity is not heard *at all.*

The audibility of a sound, even when isolated, is influenced by the state of the ear as regards fatigue. The effect is especially

[1] Instead of a box screwed to the fork, I found it better to use an independent resonator, to the mouth of which the fork is made to approach and recede in a definite manner.

apparent with the very high notes of bird-calls (§ 371). "A bird-call was mounted in connection with a loaded gas-bag and a water-manometer, by which means the pressure could be kept constant for a considerable time. When the ear is placed at a moderate distance from the instrument, a disagreeable sound is heard at first, but after a short interval, usually not more than three or four seconds, fades away and disappears altogether. A very short intermission suffices for at any rate a partial recovery of the power of hearing. A pretty rapid passage of the hand, screening the ear for a fraction of a second, allows the sound to be heard again[1]."

But although Ohm's law is subject to important limitations, it can hardly be disputed that the ear is capable of making a rough analysis of a compound vibration into its simple parts. The nature of the difficulty commonly met with has already been referred to (§§ 25, 26), but a few further remarks may here be made.

In resolving compound notes a certain control over the attention is the principal requisite, and Helmholtz found that the advantage does not always lie with musically trained ears. Before a particular tone is listened for, it ought to be sounded so as to become fixed in the memory, but not too loudly, lest the sensitiveness of the ear be unduly impaired. As a rule the uneven component tones, twelfth, higher third, &c., are more easily recognised than the octaves.

On the pianoforte, for example, let g' be first gently given, and as soon as the key is released, let c be sounded strongly. The tone g' on which the attention should be kept rivetted throughout, may now be heard as part of the compound note c. A similar experiment may be made with the higher third e'', and an acute ear may detect a slight fall in pitch. This is a consequence of the equal temperament tuning (§ 19), and shews clearly that the apparent prolongation of the tone is not due to imagination. In modern pianos the seventh and ninth component tones are often weak or altogether absent, but on the harmonium these tones may usually be heard.

It is still better when the tone to be listened for is first obtained as a harmonic from the string c itself. In the case of

[1] *Phil. Mag.* vol. XIII. p. 344, 1882.

the twelfth, for example, strike the key gently while the string is lightly touched at one-third of its length, and then after removal of the finger more strongly. The proper point may be conveniently found by sliding the finger slowly along the string, while the key is continually struck. When a point of aliquot division is reached, the corresponding harmonic rings out clearly; otherwise the sound is feeble and muffled. In this way Helmholtz succeeded in hearing the overtones of thin strings as far as the sixteenth. From this point they lie too close together to be easily distinguished.

A further slight modification of this method is especially recommended by Helmholtz. Instead of using the finger, the nodal point is touched with a small camel's hair brush. This allows the degree of damping to be varied at pleasure, and a gradual transition to be made from the pure harmonic, free from all admixture of components which have not a node at the point touched, to the natural note of the string.

But it is with the assistance of resonators that overtones are most easily heard in the first instance. For this purpose a resonator is chosen, tuned, say, to g', and the ear is placed in communication with its cavity. When c is sounded, either on the piano or harmonium, or with the human voice, the tone g' may usually be heard very loud and distinct. Indeed on many pianofortes a tone g' may be heard as loudly from its harmonic undertones g or c as from the string g' itself. When an overtone has once been heard, the assistance of the resonator should be gradually withdrawn, which may be done either by removing it from the ear, or putting it out of tune by an obstacle (such as the finger) held near its mouth.

387. If it be admitted that the ear is capable of analysing a musical note into components, or partials, it follows almost of necessity that these more elementary sensations correspond to simple vibrations. So long as we keep within the range of the principle of superposition, this is the kind of analysis effected by mechanical appliances, such as resonators, and all the more patent facts go to prove that the ear resolves according to the same laws. Moreover, the *à priori* probabilities of the case seem to tend in the same direction. It is difficult to suppose that physiological effects—electrical, chemical, or of some unknown character,—are produced directly by the impact of sonorous waves involving merely a variable fluid pressure. Helmholtz's theory of audition is based upon the more natural supposition that the immediate effect of the waves is to set into ordinary mechanical vibration certain internal vibrators[1], and that nervous excitation follows as a secondary consequence.

The *modus operandi* is conceived to be as follows. When a simple tone finds access to the ear, all the parts capable of motion vibrate in synchronism with the source. If there be any part, approximately isolated, whose natural period nearly agrees with that of the sound, then the vibration of that part is far more intense than it would otherwise be. Practically this part of the system may be said to respond only to tones whose pitch lies within somewhat narrow limits. Now it is supposed that the auditory nerves are in communication with vibrating parts of the kind described, whose natural pitch ranges at small intervals between the limits of hearing in such a manner that when any part vibrates the corresponding nerve is excited and conveys the impression to the brain. In the case of a simple tone, one (or at most a relatively small number) of the whole series of nerves is excited, the excitation of the nerve being the proximate cause of the hearing of the tone.

At this point the question presents itself whether more than one simple vibration may not have the power of exciting the same nerve? *À priori*, this might well be the case; for the vibrating parts might be susceptible of more than one mode of vibration, and therefore of more than one natural period. If we were to suppose that the natural periods of any vibrating part formed a harmonic scale, so that the same auditory nerve was excited by a tone and its octave, the supposition would certainly give a very ready explanation of the remarkable resemblance of octaves, and would tend to mitigate some of the difficulties which at present stand in the way of accepting Helmholtz's theory as a complete account of the facts of audition[2]. As we shall see presently,

[1] The drum-skin and its attachments are here regarded as external to the true auditory mechanism. However important may be the part they play, it is analogous rather to that of a hearing tube or of the disc of a mechanical telephone.

[2] A curious question suggests itself as to what would happen in case the vibrations capable of exciting the same nerve deviated sensibly or considerably from the harmonic scale. In this way ears naturally confused in their appreciation of musical relations may easily be imagined.

Helmholtz would admit, or rather assert, that *when the sounds are strong* two originally simple vibrations, such as c and c', would excite to some extent the same nerve, but he regards this as depending upon a failure in the law of superposition, due to excessive vibration.

388. It is evident that Helmholtz's theory gives a very natural account of Ohm's law, as well as of the limitation to which it is subject when two simple vibrations are in operation of nearly the same pitch. Some of the internal vibrators are then within the influence of both disturbing causes, and are accordingly excited in an intermittent manner, giving rise to beats, when the period is long, and to a sensation of roughness as the beats become too quick to be easily perceived separately. But when the interval between the two vibrations is increased, a point is soon reached after which no internal vibrator is sensibly affected by both disturbing causes, so that from this point onwards the resulting sensation is free from beats or roughnesses, or at least should be so according to the strict interpretation of the law. To this point we shall return later.

The magnitude of the interval, over which a single internal vibrator will respond sensibly, is an element of considerable importance in the theory. It has already been shewn (§ 49) that there is a relation between this interval and the number of free vibrations which can be executed by the vibrating body. Thus, if the interval between the natural and the forced vibration required to reduce the resonance to $\frac{1}{10}$ of the maximum be a semitone, this implies that after 9·5 free vibrations the intensity would be reduced to $\frac{1}{10}$ of its original value, and so on for other intervals. From a consideration of the effect of trills in music, Helmholtz concludes that the case of the ear corresponds somewhat to that above specified, and he gives the accompanying table shewing the

Difference of pitch	Intensity of resonance	Difference of pitch	Intensity of resonance
0·0	100	0·6	7·2
0·1	74	0·7	5·4
0·2	41	0·8	4·2
0·3	24	0·9	3·3
0·4	15	Whole tone	2·7
Semitone	10		

relation obtaining in this case between the difference of free and forced pitch and the intensity of resonance, measured by the square of the amplitude of vibration.

Although according to Helmholtz's theory the sensation of dissonance is caused by intermittent excitation of those vibrating parts which are within the range of two or more elements of the sound, it is not to be inferred that the number of beats is a sufficient measure of the dissonance. On the contrary it is found that if the number of beats be retained constant (e.g. 33 per second), the effect is more and more free from roughness as the sounds are made deeper, the *intervals* being correspondingly increased.

The experiments of A. M. Mayer[1] extend over a considerable range of pitch, and have been made by two methods. In the first method a sound, which would otherwise be a pure tone, is rendered intermittent, and the rate of intermittence is gradually raised to the point at which the effect upon the ear again becomes smooth. The results are shewn in the accompanying table, in which the first column gives the pitch of the sound and the second the minimum number of intermittences per second required to eliminate the roughness.

Pitch (n)	Frequency of Intermittence (m)
64	23·1
128	36
256	62
320	73
384	88
512	108
640	126
768	143
1023	170

The theory of intermittent vibrations has already been given § 65 *a*. It is to be remembered that by the nature of the case an intermittent vibration cannot be simple. To a first approximation it may be supposed to be equivalent to *three* simple vibrations of frequencies, $n-m$, n, $n+m$, and the roughness experienced by

[1] *Phil. Mag.* vol. XLIX. p. 352, 1875; vol. XXXVII. p. 259, 1894.

the ear may be looked upon as due to the beats of these three tones.

Mayer has experimented also upon the "smallest consonant intervals among simple tones," i.e. upon the intervals at which the roughness due to beats just disappears, the plural being used since it is found that the necessary interval varies at different parts of the scale.

Pitch (n_1)	Additional vibrations required ($n_2 - n_1$)	Smallest consonant intervals in semitones
48	......	none
64	......	none
96	41	6·15
128	38	4·50
192	48	3·86
256	58	3·53
316	68	3·34
432	85	3·12
575	107	2·95
766	130	2·70
1707	210	2·00
2304	245	1·76
2560	256	1·64
2806	266	1·54

Different observers agreed very closely as to the point at which roughness disappeared.

According to the theory of intermittent sounds it is to be expected that for a given pitch m in the first set of experiments should be nearly the same as $(n_2 - n_1)$ in the second, and this is pretty well verified by Mayer's numbers, at least over the middle region of the scale.

389. From the degree of damping above determined it follows that the natural pitch of the internal vibrators, which respond sensibly to a given simple sound, ranges over about a whole tone, and it may excite surprise that we are able to compare with such accuracy the pitch of musical sounds heard in succession. The explanation probably depends a good deal upon the symmetry of the effects on the two sides of the maximum. A comparison with the capabilities of the eye in a similar case may be instructive. In setting the cross wires of a telescope upon the centre of a symmetrical luminous band, e.g. an interference band, it is found that the error need not exceed $\frac{1}{100}$ of the width. A similarly accurate judgment as to the centre of the region excited by a given musical note would lead to an estimation of pitch accurate to about $\frac{1}{1000}$, agreeing well enough with the facts to be explained.

In the light of the same principle we may consider how far the perception of pitch would be prejudiced by a limitation of the number of vibrations executed during the continuance of a sound. According to the estimate of Helmholtz already employed (§ 388) the internal vibrations, excited and then left to themselves, would remain sensible over about 10 periods. The number of impulses required to produce nearly the full effect is of this order of magnitude. If the number were increased beyond 20 or 30, there would be little further concentration of effect in the neighbourhood of the maximum, and therefore little foundation for greater accuracy in the estimation of pitch.

Experiments upon this subject have been made by Seebeck[1], Pfaundler[2], S. Exner[3], Auerbach[4], and W. Kohlrausch[5], those of the last being the most extensive. An arc of a circle carrying a limited number of teeth was attached to a pendulum, which could be let go under known conditions. In their passage the teeth struck against a card suitably held; and the sound thus generated was compared with that of a monochord. By varying the length in the usual manner the chord was tuned until the pitch was just perceptibly higher, or just perceptibly lower, than that proceeding from the card, and the interval between the two, called the characteristic interval, determined the precision with which the pitch could be estimated in the case of a given total number of vibrations. The best results were obtained only after considerable practice and in the entire absence of extraneous sounds.

Sixteen teeth appeared to define the pitch with all the precision attainable, the characteristic interval (on the mean of a number of experiments) being in this case ·9922. Even with 9

[1] *Pogg. Ann.* vol. LIII. p. 417, 1841.
[2] *Wien. Ber.* vol. LXXVI. p. 561, 1877.
[3] *Pflüger's Archiv*, vol. XIII. p. 228, 1876.
[4] *Wied. Ann.* vol. VI. p. 591, 1879.
[5] *Wien. Ann.* vol. X. p. 1, 1880.

teeth the characteristic interval was 9903, shewing that this small number of vibrations was capable of defining the pitch to within one part in 200. But the most surprising results were those obtained with a very low number of teeth. For 3 teeth the characteristic interval was ·9790, and for 2 teeth ·9714.

The fact that pitch can be defined with considerable accuracy by so small a sequence of vibrations has sometimes been regarded as an objection to Helmholtz's theory of audition. I do not think that there is any ground for this opinion. So far as there is a difficulty, it is one that would tell equally against any other theory that could be proposed.

It would seem that the delimitation of pitch in Kohlrausch's experiments may have been greatly favoured by the approximate discontinuity of the impulses. For it is to be remembered that the internal vibrators concerned are not those only whose period ranges round the interval between the taps, but also those whose periods are approximately submultiples of this quantity. As regards the vibrators in the octave, the number of impulses is practically doubled, for the twelfth trebled, and so on, just as in optics the resolving power of a grating with a limited number of lines is increased in the spectra of the second and higher orders.

Vibrations limited to a moderate number of periods are sometimes generated by reflection of short sounds from railings or steps. At Terling there is a flight of about 20 steps which returns an echo of a clap of the hands as a note resembling the chirp of a sparrow. In all such cases the action is exactly analogous to that of a grating in optics.

390. When two sounds nearly in unison are compound, we have to consider not only the beats of their first partials, or primes, but also the beats of the overtones. The beats of the octave components are twice, and those of the twelfth three times, as quick as the simultaneous beats of the primes. In some cases, especially where the pitch is very low, mistakes may easily be made by overlooking the prime beats, which affect the ear but feebly. If the octave beats be reckoned as though they were the beats of the primes, the difference of pitch will be taken to be the double of its true value.

But it is in the case of disturbed consonances other than the unison that the importance of upper partials, or overtones, makes itself specially felt. For example, take the Fifth *c*—*g*. The third partial of *c* and the second partial of *g* coincide at *g*′. If the interval be true, there are no beats; but if it be slightly disturbed from the true ratio 3 : 2, the two previously coincident tones separate from one another and give rise to beats. The frequency of the beats follows at once from the manner of their genesis. Thus if the lower note be disturbed from its original frequency by one vibration per second, its third partial is changed by 3 vibrations per second, and 3 per second is accordingly the frequency of the beats. But if the upper note undergoes a disturbance of one vibration per second, while the lower remains unaltered, the frequency of the beats is 2. This rule is evidently general. If the consonance be such that the hth partial of the lower note coincides with the kth partial of the upper note, then when the lower note is altered by one vibration per second, the frequency of the beats is h, and when the upper note is altered by the same quantity, the frequency of the beats is k.

"We have stated that the beats heard are the beats of those partial tones of both compounds which nearly coincide. Now it is not always very easy on hearing a Fifth or an Octave which is slightly out of tune, to recognise clearly with the unassisted ear which part of the whole sound is beating. On listening we are apt to feel that the whole sound is alternately reinforced and weakened. Yet an ear accustomed to distinguish upper partial tones, after directing its attention upon the common upper partials concerned, will easily hear the strong beats of these particular tones, and recognise the continued and undisturbed sound of the primes. Strike the note (*c*), attend to its upper partial (*g*′), and then strike a tempered Fifth (*g*); the beats of (*g*′) will be clearly heard. To an unpractised ear the resonators already described will be of great assistance. Apply the resonator for (*g*′), and the above beats will be heard with great distinctness. If, on the other hand, a resonator, tuned to one of the primes (*c*) or (*g*), be employed, the beats are heard much less distinctly, because the continuous part of the tone is then reinforced[1]."

Experiments of this kind are conveniently made on the harmonium. Small changes of pitch may be obtained by only partially (instead of fully) depressing the key, the effect of which is to flatten the note. The beats of the common overtone are

[1] *Sensations of Tone*, 2nd ed. p. 181.

easily heard when a (tempered) Fifth is sounded; those of the equal temperament Third are somewhat rapid.

The harmonium is also a suitable instrument for experiments illustrative of just intonation. A reed may be flattened by loading the free end of the tongue with a fragment of wax, and sharpened by a slight filing at the same place. It is easy, especially with the aid of resonators, to tune truly the chords $c'—e'—g'$, $f'—a'—c''$, whose consonance will then contrast favourably with the unaltered tempered chord $g'—b'—d''$. It is not consistent with the plan of this work to enter at length into questions of temperament and just intonation. Full particulars will be found in the English edition of Helmholtz (with Ellis's notes) and in Mr Bosanquet's treatise.

According to Helmholtz's theory it is mainly the beats of the upper partials which determine the ordinary consonant intervals, any departure from which is made evident by the beats of the previously coincident overtones. But even when the notes are truly tuned, the various consonances differ in degree, on account of the disturbances which may arise from overtones which approach one another too nearly.

The unison, octave, twelfth, double octave, etc., may be regarded as absolute consonances, the second component introducing no new element but merely reinforcing a part of the other.

The remaining consonant intervals, such as the Fifth and the Major Third, are in a manner disturbed by their neighbourhood to other consonant intervals. In the case of the truly tuned Fifth, for example, with frequencies represented by 3 and 2, there is indeed coincidence between the second partial of the higher note and the third partial of the graver note, but the partials which define the Fourth, of pitch $3 \times 3 = 9$ and $4 \times 2 = 8$, are within a whole-Tone of one another and accordingly near enough to produce disturbance. In like manner the Major Third may be regarded as disturbed by its neighbourhood to the Fourth, and so on in the case of other intervals.

The importance of these disturbances, and consequently the order in which the various intervals stand in respect to their degree of consonance, varies with the quality of the sounds. As an example where overtones are present in considerable strength, Helmholtz has estimated the degree of consonance of various intervals on the violin, and has exhibited the results in the form of a curve[1].

391. The principle of superposition (§ 83), assumed in ordinary acoustical discussions, depends for its validity upon the assumption that the vibrations concerned are infinitely small, or at any rate similar in their character to infinitely small vibrations, and it is only upon this supposition that Ohm's law finds immediate application. One apparent exception to the law has long been known. This is the combination-tone discovered by Sorge and Tartini in the last century. If two notes, at the interval for example of a Major Third, be sounded together strongly, there is heard a grave sound in addition to the two others. In the case specified, where the primary sounds, or generators, as they may conveniently be called, are represented by the numbers 4 and 5, the combination-tone is represented by 1, being thus two octaves below the graver generator.

In the above example the new tone has the period of the cycle of the generating tones; but Helmholtz found that this rule fails in many cases. The following table[2] exhibits his results as obtained by means of tuning-forks:

Generators		Combination tone	Relative Frequency	
			Generators	Combination tone
b	f′	B	2 : 3	1
f′	b′	B	3 : 4	1
b	d′	B_{-1}	4 : 5	1
d′	f′	B_{-1}	5 : 6	1
f′	as′	B_{-1}	6 : 7	1
b	g′	es	3 : 5	2
d′	as′	B	5 : 7	2
d′	b′	f	5 : 8	3

In the last three cases the tones heard were not those in the period of the complete cycle, but their frequencies are the differences of the frequencies of the generators. In virtue of this rule, which was found to apply in all cases[3], the combination-tones in question are called difference-tones.

[1] *Sensations of Tone*, p. 193.
[2] *Berlin Monatsber.*, 1856.
[3] It is, however, disputed by other writers.

According to Helmholtz it is necessary to the distinct audibility of combination-tones that the generators be *strong*. We shall see presently that this statement has been contested. "They are most easily heard when the two generating tones are less than an octave apart, because in that case the differential is deeper than either of the two generating tones. To hear it at first, choose two tones which can be held with great force for some time, and form a justly intoned harmonic interval. First sound the low tone and then the high one. On properly directing attention, a weaker low tone will be heard at the moment that the higher note is struck; this is the required combinational tone. For particular instruments, as the harmonium, the combinational tones can be made more audible by properly tuned resonators. In this case the tones are generated in the air contained within the instrument. But in other cases where they are generated solely within the ear, the resonators are of little or no use[1]."

On the strength of some observations by Bosanquet and Preyer, doubts have been expressed as to the correctness of Helmholtz's statement that combination-tones may exist outside the ear, and strangely enough they have been adopted by Ellis. The question has an important bearing upon the theory of combination-tones; and it has recently been examined by Rücker and Edser[2], who used apparatus entirely independent of the ear. They conclude that "Helmholtz was correct in stating that the siren produces two objective notes the frequencies of which are respectively equal to the sum and difference of the frequencies of the fundamentals." My own observations have been made upon the harmonium, and leave me at a loss to understand how two opinions are possible. The resonator is held with its mouth as near as may be to the reeds which sound the generating notes, and is put in and out of tune to the difference-tone by slight movements of the finger. When the tuning is good, the difference-tone swells out with considerable strength, but a slight mistuning (probably of the order of a semitone) reduces it almost to silence. In some cases, e.g. when he interval between the generators is a (tempered) Fifth, the ifference-tone is heard to *beat*.

The last observation proves that in some cases there exist two difference-tones of nearly the same pitch. Helmholtz finds the

[1] *Sensations of Tone*, p. 153.

[2] *Phil. Mag.* vol. XXXIX. p. 357, 1895.

explanation of this in the compound nature of the sounds. Thus in the case of the Fifth, represented by the numbers 2 and 3, we have not only the primes to consider, but the overtones 2×2, 3×2, etc., 2×3, 3×3, etc. Accordingly the difference-tone 1 may be derived from $2\times 2=4$ and 3, as well as from 3 and 2, and since the octave partial is usually strong, the one source may be as important as the other. But if we substitute the Major Third (5 : 4) for the Fifth, we do not get a second difference-tone 1 until we come to the fourth partial (16) of the graver note and the third (15) of the higher, and these would usually be too feeble to produce much effect.

As regards the frequency of the beats, let us return to the case of the Fifth, supposing it to be so disturbed that the frequencies are 200 and 301. The difference tone due to the primes is $301-200=101$, and that due to the octave partial is

$$2\times 200-301=99\,;$$

and these difference-tones sounding together will give beats with frequency 2. This, it will be observed, is the same number of beats as is due to the common overtone, viz. $2\times 301-3\times 200$; but while the latter beats are those of the tone 600, the beats of the combination-tone are at pitch 100.

392. According to the views of the older theorists Chladni, Lagrange, Young, etc., the explanation of the difference-tone presented no particular difficulty. As the generators separate in pitch, the beats quicken and at last become too rapid for appreciation as such, passing into a difference-tone, whose frequency is continuous with the frequency of the beats. This view of the matter, which has commended itself to many writers, was rejected by Helmholtz, as inconsistent with Ohm's law; and that physicist has elaborated an alternative theory, according to which the failure is not in Ohm's law, but in the principle of superposition.

Helmholtz's calculation of the effect of a want of symmetry in the forces of restitution, when the vibrations of a system cannot be regarded as infinitely small, has already been given (§ 68). It appears that in addition to the terms in pt, qt, corresponding to the generating forces, there must be added other terms of the second order in $2pt$, $2qt$, $(p+q)\,t$, $(p-q)\,t$, the last of which represents the difference-tone. This explanation depends, as Hermann[1]

[1] *Pflüger's Archiv*, vol. XLIX. p. 507, 1891.

has remarked, upon the assumed failure of symmetry. If, as in § 67, we suppose a force of restitution proportional partly to the first power and partly to the cube of the displacement, we do *not* obtain a term in $(p-q)t$, but in place of it terms of the *third* order involving $(2p-q)t$, $(2q-p)t$, etc. This objection, however, is of little practical importance, because the failure of symmetry almost always occurs. It may suffice to instance the all important case of aerial vibrations. Whether we are considering progressive waves advancing from a source, or the stationary vibrations of a resonator, there is an essential want of symmetry between condensation and rarefaction, and the formation in some degree of octaves and combination-tones is a mathematical necessity.

The production of external, or objective, combination-tones demands the coexistence of the generators at a place where they are strong[1]. This will usually occur only when the generating sounds are closely associated, as in the polyphonic siren and in the harmonium. In these cases the conditions are especially favourable, because the limited mass of air included within the instrument is necessarily strongly affected by both tones. When the generating sources are two organ-pipes, even though they stand pretty near together, the difference-tone is not appreciably strengthened by a resonator, from which we may infer that but little of it exists externally to the ear.

We have as yet said nothing about the summation-tone, corresponding to the term in $(p+q)t$. The existence of this tone was deduced by Helmholtz theoretically; and he afterwards succeeded in hearing it, not only from the siren and harmonium, where it exists objectively and is reinforced by resonators, but also from tuning-forks and organ-pipes. Helmholtz narrates also an experiment in which he caused a membrane to vibrate in response to the summation-tone, and similar experiments have recently been carried out with success by Rücker and Edser (l. c.).

Nevertheless, it must be admitted that summation-tones are extremely difficult to hear. Hermann (l. c.) asserts that he can neither hear them himself nor find any one able to do so; and he regards this difficulty as a serious objection to Helmholtz's theory, according to which the summation and the difference tone should be about equally strong.

[1] The estimates of condensation (§ 384) for sounds just audible make it highly improbable that the principle of superposition could fail to apply to sounds of that order of magnitude.

An objection of another kind has been raised by König[1]. He remarks that even if a tone exist of the pitch of the summation-tone, it may in reality be a difference-tone, derived from the upper partials of the generators. As a matter of arithmetic this argument cannot be disputed; for if p and q be commensurable, it will always be possible to find integers h and k, such that

$$p+q=hp-kq.$$

But this explanation is plausible only when h and k are *small* integers.

It seems to me that the comparative difficulty with which summation-tones are heard is in great measure, if not altogether, explained by the observations of Mayer (§ 386). These tones are of necessity higher in pitch than their generators, and are accordingly liable to be overwhelmed and rendered inaudible. On the other hand the difference-tone, being usually graver, and often much graver, than either of its generators, is able to make itself felt in spite of them. And even as regards difference-tones, it had already been remarked by Helmholtz that they become more difficult to hear when they cease to constitute the gravest element of the sound by reason of the interval between the generators exceeding an octave.

393. In the numerous cases where differential tones are audible which are not reinforced by resonators, it is necessary in order to carry out Helmholtz's theory to suppose that they have their origin in the vibrating parts of the outer ear, such as the drum-skin and its attachments. Helmholtz considers that the structure of these parts is so unsymmetrical that there is nothing forced in such a supposition. But it is evident that this explanation is admissible only when the generating sounds are loud, *i.e.* powerful as they reach the ear. Now, the opponents of Helmholtz's views, represented by Hermann, maintain that this condition is not at all necessary to the perception of difference-tones. Here we have an issue as to facts, the satisfactory resolution of which demands better experiments, preferably of a quantitative nature, than any yet executed. My own experience tends rather to support the view of Helmholtz that loud generators are necessary. On several occasions stopped organ-pipes d''', e''', were blown with

[1] *Pogg. Ann.* vol. 157, p. 177, 1876.

a steady wind, and were so tuned that the difference-tone gave slow beats with an electrically maintained fork, of pitch 128, mounted in association with a resonator of the same pitch. When the ear was brought up close to the mouths of the pipes, the difference-tone was so loud as to require all the force of the fork in order to get the most distinct beats. These beats could be made so slow as to allow the momentary disappearance of the grave sound, when the intensities were rightly adjusted, to be observed with some precision. In this state of things the two tones of pitch 128, one the difference-tone and the other derived from the fork, were of equal strength as they reached the observer; but as the ear was withdrawn so as to enfeeble both sounds by distance, it seemed that the combination-tone fell off more quickly than the ordinary tone from the fork. It might be possible to execute an experiment of this kind which should prove decisively whether the combination-tone is really an effect of the second order, or not.

In default of decisive experiments we must endeavour to balance the *a priori* probabilities of the case. According to the views of the older theorists, adopted by König, Hermann, and other critics of Helmholtz, the beats of the generators, with their alternations of swellings and pauses, pass into the differential tone of like frequency, without any such failure of superposition as is invoked by Helmholtz. The critics go further, and maintain that the ear is capable of recognising as a tone any periodicity within certain limits of frequency[1].

Plausible as this doctrine is from certain points of view, a closer examination will, I think, shew that it is encumbered with difficulties. Among these is the ambiguity, referred to in § 12, as to what exactly is meant by period. A periodicity with frequency 128 is also periodicity with frequency 64. Is the latter tone to be heard as well as the former? So far as theory is concerned, such questions are satisfactorily answered by Ohm's law. Experiment may compel us to abandon this law, but it is well to remember that there is nothing to take its place. Again, by consideration of particular cases it is not difficult to prove that the general doctrine above formulated cannot be true. Take the example above mentioned in which two organ-pipes gave a difference-tone of pitch 128. There is periodicity with frequency 128, and the

[1] Hermann, *loc. cit.* p. 514.

corresponding tone is heard[1]. So far, so good. But experiment proves also that it is only necessary to superpose upon this another tone of frequency 128, obtained from a fork, in order to neutralize the combination-tone and reduce it to silence. The periodicity of 128 remains, if anything in a more marked manner than before, but the corresponding tone is *not* heard.

I think it is often overlooked in discussions upon this subject that a difference-tone is not a mere sensation, but involves a *vibration* of definite amplitude and phase. The question at once arises, how is the phase determined? It would seem natural to suppose that the maximum swell of the beats corresponds to one or other extreme elongation in the difference-tone, but upon the principles under discussion there seems to be no ground for a selection between the alternatives. Again, how is the amplitude determined? The tone certainly vanishes with either of the generators. From this it would seem to follow that its amplitude must be proportional to the product of the amplitudes of the generators, exactly as in Helmholtz's theory. If so, we come back to difference-tones of the second order, and their asserted easy audibility from feeble generators is no more an objection to one theory than to another.

An observation, of great interest in itself, and with a possible bearing upon our present subject, has been made by König and Mayer[2]. Experimenting both with forks and bird-calls, they have found that audible difference-tones may arise from generators whose pitch is so high that they are separately inaudible. Perhaps an interpretation might be given in more than one way, but the passage of an inaudible beat into an audible difference-tone seems to be more easily explicable upon the basis of Helmholtz's theory.

Upon the whole this theory seems to afford the best explanation of the facts thus far considered, but it presupposes a more ready departure from superposition of vibrations within the ear than would have been expected.

394. In § 390 we saw that in the case of ordinary compound sounds, containing upper partials fairly developed, the recognised consonant intervals are distinguished from neighbouring intervals

[1] In strictness, the periodicity is incomplete, unless p and q are multiples of $(p-q)$.

[2] Mayer, *Rep. Brit. Ass.* p. 573, 1894.

by well marked phenomena, of which there was no difficulty in rendering a satisfactory account. We have now to consider the more difficult subject of consonance among pure tones; and we shall have to encounter considerable differences of opinion, not only as to theoretical explanations, but as to matters of observation. Here, as elsewhere, it will be convenient to begin with a statement of Helmholtz's views[1] according to which, in a word, the beats of such mistuned consonances are due to combination-tones.

"If combinational tones were not taken into account, two simple tones, as those of tuning-forks, or stopped organ-pipes, could not produce beats unless they were very nearly of the same pitch, and such beats are strong when their interval is a minor or major second, but weak for a Third, and then only recognisable in the lower parts of the scale, and they gradually diminish in distinctness as the interval increases, without shewing any special differences for the harmonic intervals themselves. For any larger interval between two simple tones there would be absolutely no beats at all, if there were no upper partial or combinational tones, and hence the consonant intervals...would be in no way distinguished from adjacent intervals; there would in fact be no distinction at all between wide consonant intervals and absolutely dissonant intervals.

Now such wider intervals between simple tones are known to produce beats, although very much weaker than those hitherto considered, so that even for such tones there is a difference between consonances and dissonances, although it is very much more imperfect than for compound tones[2]."

Experiments upon this subject are difficult to execute satisfactorily. In the first place it is not easy to secure simple tones. As sources recourse is usually had to stopped organ-pipes or to tuning-forks, but much precaution is required. From the free ends of the vibrating prongs of a fork many overtones may usually be heard[3]. Again, if a fork be employed after the manner of musicians with its stalk pressed against a resonating board, the octave is loud and often predominant[4]. The best way is to hold

[1] Ascribed by him to Hällström and Scheibler.

[2] *Sensations of Tone*, p. 199.

[3] König's experiments shew that this is especially the case when the prongs are thin. *Wied. Ann.* vol. xiv. p. 373, 1881.

[4] The prime tone may even disappear altogether. If in their natural position the prongs of a fork are closest below, an outward movement during the vibration

the free ends of the prongs over a suitably tuned resonator. But even then we cannot be sure that a loud sound thus obtained is absolutely free from the octave partial.

In the case of the octave the differential tone already considered suffices. "If the lower note makes 100 vibrations per second, while the imperfect octave makes 201, the first differential tone makes $201 - 100 = 101$, and hence nearly coincides with the lower note of 100 vibrations, producing one beat for each 100 vibrations. There is no difficulty in hearing these beats, and hence it is easily possible to distinguish imperfect octaves from perfect ones, even for simple tones, by the beats produced by the former."

The frequency of the beats is the same as if it were due to overtones; but there is one important difference between the two cases noted by Ellis though scarcely, if at all, referred to by Helmholtz. In the latter the beats would affect the octave tone, whereas according to the above theory the beats will belong to the lower tone. Bosanquet, König and others are agreed that in this respect the theory is verified.

Again, if the beats were due to combination-tones, they must tend to disappear as the sounds die away. The experiment is very easily tried with forks, and according to my experience the facts are in harmony. When the sounds are much reduced, the mistuning fails to make itself apparent.

"For the Fifth, the first order of differential tones no longer suffices. Take an imperfect Fifth with the ratio 200 : 301; then the differential tone of the first order is 101, which is too far from either primary to generate beats. But it forms an imperfect Octave with the tone 200, and, as just seen, in such a case beats ensue. Here they are produced by the differential tone 99 arising from the tone 101 and the tone 200, and this tone 99 makes two beats in a second with the tone 101. These beats then serve to distinguish the imperfect from the justly intoned Fifth, even in the case of two simple tones. The number of these beats is also exactly the same as if they were the beats due to

will depress the centre of inertia, the stalk being immovable, but if the prongs are closest above, the contrary result may ensue. There must be some intermediate construction for which the centre of inertia will remain at rest during the vibration. In this case the sound from a resonance board is of the second order, and is destitute of the prime tone.

the upper partial tones. But to observe these beats the two primary tones must be loud, and the ear must not be distracted by any extraneous noise. Under favourable circumstances, however, they are not difficult to hear."

It is important to be clear as to the order of magnitude of the various differential tones concerned. If the primary tones, with frequencies represented by p and q, have amplitudes e and f respectively, quantities of the first order, then (§ 68) the first difference and summation tones have frequencies corresponding to

$$2p, \quad 2q, \quad p+q, \quad p-q,$$

and are of the second order in e and f. A complete treatment of the second differential tones requires the retention of another term βu^3 (§ 67) in the expression of the force of restitution. From this will arise terms of the third order in e and f with frequencies corresponding to

$$3p, \quad 2p \pm q, \quad p \pm 2q, \quad 3q;$$ [1]

and there are in addition other terms of the same frequencies and order of magnitude, independent of β, arising from the full development to the third order of αu^2. In the case of the disturbed Fifth above taken, the beats are between the tone $2q-p=99$, which is of the third order of magnitude, and $p-q=101$ of the second order. The exposition, quoted from Helmholtz, refers to the terms last mentioned, which are independent of β.

The beats of a disturbed Fourth or major Third depend upon difference-tones of a still higher order of magnitude, and according to Helmholtz's observations they are scarcely, if at all, audible, even when the primary tones are strong. This is no more than would have been expected; the difficulty is rather to understand how the beats of the disturbed Fifth are perceptible and those of the disturbed Octave so easy to hear.

When more than two simple tones are sounded together, fresh conditions arise. "We have seen that Octaves are precisely limited even for simple tones by the beats of the first differential tone with the lower primary. Now suppose that an Octave has been tuned perfectly, and that then a third tone is interposed to act as a Fifth. Then if the Fifth is not perfect, beats will ensue from the first differential tone.

[1] Bosanquet, *Phil. Mag.* vol. XI. p. 497, 1881.

Let the tones forming the perfect Octave have the pitch numbers 200 and 400, and let that of the imperfect Fifth be 301. The differential tones are

$$\begin{array}{r} 400-301= \;\;99 \\ 301-200= \underline{101} \\ \text{Number of beats} \;\;\; 2. \end{array}$$

These beats of the Fifth which lies between two Octaves are much more audible than those of the Fifth alone without its Octave. The latter depend on the weak differential tones of the second order, the former on those of the first order. Hence Scheibler some time ago laid down the rule for tuning tuning-forks, first to tune two of them as a perfect Octave, and then to sound them both at once with the Fifth, in order to tune the latter. If Fifth and Octave are both perfect, they also give together the perfect Fourth.

The case is similar, when two simple tones have been tuned to a perfect Fifth, and we interpose a new tone between them to act as a major Third. Let the perfect Fifth have the pitch numbers 400 and 600. On intercalating the impure major Third with the pitch number 501 in lieu of 500, the differential tones are

$$\begin{array}{r} 600-501= \;\;99 \\ 500-400= \underline{101} \\ \text{Number of beats} \;\;\; 2.\text{''} \end{array}$$

395. In Helmholtz's theory of imperfect consonances the cycles heard are regarded as risings and fallings of intensity of one or more of the constituents of the sound, whether these be present from the first, or be generated by transformation, to use Bosanquet's phrase, in the transmitting mechanism of the ear. According to Ohm's law, such changes of intensity are the only thing that could be heard, for the relative phases of the constituents (supposed to be sufficiently removed from one another in pitch) are asserted to be matters of indifference.

This question of independence of phase-relation was examined by Helmholtz in connection with his researches upon vowel sounds (§ 397). Various forks, electrically driven from one interrupter (§ 64), could be made to sound the prime tone, octave, twelfth etc., of a compound note, and the intensities and phases of the constituents could be controlled by slight modifications in the (natural) pitch of the forks and associated resonators. According to Helmholtz's observations changes of phase were without distinct effect upon the quality of the compound sound.

It is evident, however, that the question of the effect, if any, upon the ear of a change in the phase relationship of the various components of a sound can be more advantageously examined by the method of slightly mistuned consonances. If, for example, an Octave interval between two pure tones be a very little imperfect, the effect upon the ear at any particular moment will be that of a true interval with a certain relation of phases, but after a short time, the phase relationship will change, and will pass in turn through every possible value. The audibility of the cycle is accordingly a criterion for the question whether or not the ear appreciates phase relationship; and the results recorded by Helmholtz himself, and easily to be repeated, shew that in a certain sense the answer must be in the affirmative. Otherwise slow beats of an imperfect Octave would not be heard. The explanation by means of combination-tones does not alter the fact that the ear appreciates the phase relationship of two originally simple tones, at any rate when they are moderately loud[1].

According to the observations of Lord Kelvin[2] the "beats of imperfect harmonies," other than the Octave and Fifth, are not so difficult to hear as Helmholtz supposed. The tuning-forks employed were mounted upon box resonators, and it might indeed be argued that the sounds conveyed down the stalks were not thoroughly purged from Octave partials. But this consideration would hardly affect the result in some of the cases mentioned. It appeared that the beats on approximations to each of the harmonies 2 : 3, 3 : 4, 4 : 5, 5 : 6, 6 : 7, 7 : 8, 1 : 3, 3 : 5 could be distinctly heard, and that they all "fulfil the condition of having the whole period of the imperfection, and not any sub-multiple of it, for their period," the same rule as would apply were the beats due to nearly coincident overtones. As regards the necessity for loud notes, Kelvin found that the beats of an imperfectly tuned chord 3 : 4 : 5 were sometimes the very last sound heard, as the vibrations of the forks died down, when the intensities of the three notes chanced at the end to be suitably proportioned.

[1] König, *Wied. Ann.* vol. XIV. p. 375, 1881.

[2] *Proc. Roy. Soc. Edin.* vol. IX. p. 602, 1878.

The last observation is certainly difficult to reconcile with a theory which ascribes the beats to combination-tones. But on the other side it may be remarked that the relatively easy audibility of the beats from a disturbed Octave and from a disturbed chord of three notes (3 : 4 : 5), which would depend upon the first differential tone, is in good accord with that theory, and (so far as appears) is not explained by any other.

396. But the observations most difficult of reconciliation with the theory of Helmholtz are those recorded by König[1], who finds tones, described as beat-tones, not included among the combination-tones; and these observations, coming from so skilful and so well equipped an investigator, must carry great weight. The principal conclusions are thus summarised by Ellis[2]. "If two simple tones of either very slightly or greatly different pitches, called generators, be sounded together, then the upper pitch number necessarily lies between two multiples of the lower pitch number, one smaller and the other greater, and the differences between these multiples of the pitch number of the lower generator and the pitch number of the upper generator give two numbers which either determine the frequency of the two sets of beats which may be heard or the pitch of the beat-notes which may be heard in their place.

The frequency arising from the lower multiple of the lower generator is called the frequency of the *lower* beat or lower beat-note, that arising from the higher multiple is called the frequency of the *higher* beat or beat-note, without at all implying that one set of beats should be greater or less than the other, or that one beat-note should be sharper or flatter than the other. They are in reality sometimes one way and sometimes the other.

Both sets of beats, or both beat-notes, are not usually heard at the same time. If we divide the intervals examined into groups (1) from 1 : 1 to 1 : 2, (2) from 1 : 2 to 1 : 3, (3) from 1 : 3 to 1 : 4, (4) from 1 : 4 to 1 : 5, and so on, the lower beats and beat-tones extend over little more than the lower half of each group, and the upper beats and beat-tones over little more than the upper half. For a short distance in the middle of each period both sets of beats, or both beat-notes are audible, and these beat-notes beat with each

[1] *Pogg. Ann.* vol. CLVII. p. 177, 1876.
[2] *Sensations of Tone*, p. 529.

other, forming secondary beats, or are replaced by new or secondary beat-notes."

In certain cases the beat-notes coincide with the differential tone, but König considers that the existence of combinational tones has not been proved with certainty. It is to be observed that in these experiments the generating tones were as simple as König could make them; but the possibility remains that overtones, not audible except through their beats, may have arisen within the ear by transformation. This is the view favoured by Bosanquet, who has also made independent observations with results less difficult of accommodation to Helmholtz's views.

It will be seen that König adopts in its entirety the opinion that beats, when quick enough, pass into tones. Some objections to this idea have already been pointed out; and the question must be regarded as still an open one. Experiments upon these subjects have hitherto been of a merely qualitative character. The difficulties of going further are doubtless considerable; but I am disposed to think that what is most wanted at the present time is a better reckoning of the intensities of the various tones dealt with and observed. If, for example, it could be shewn that the intensity of a beat-tone is proportional to that of the generators, it would become clear that something more than combination-tones is necessary to explain the effects.

König has also examined the question of the dependence of quality upon phase relation, using a special siren of his own construction[1]. His conclusion is that while quality is mainly determined by the number and relative intensity of the harmonic tones, still the influence of phase is not to be neglected. A variation of phase produces such differences as are met with in different instruments of the same class, or in various voices singing the same vowel. A ready appreciation of such minor differences requires a series of notes, upon which a melody can be executed, and they may escape observation when only a single note is available. To me it appears that these results are in harmony with the view that would ascribe the departure from Ohm's law, involved in any recognition of phase relations, to secondary causes.

397. The dependence of the quality of musical sounds of given pitch upon the proportions in which the various partial tones are

[1] *Wied. Ann.* vol. XIV. p. 392, 1881.

present has been investigated by Helmholtz in the case of several musical instruments. Further observations upon wind instruments will be found in a paper by Blaikley[1]. But the most interesting, and the most disputed, application of the theory is to the vowel sounds of human speech.

The acoustical treatment of this subject may be considered to date from a remarkable memoir by Willis[2]. His experiments were conducted by means of the free reed, invented by Kratzenstein (1780) and subsequently by Grenié, which imitates with fair accuracy the operation of the larynx. Having first repeated successfully Kempelen's experiment of the production of vowel sounds by shading in various degrees the mouth of a funnel-shaped cavity in association with the reed, he passed on to examine the effect of various lengths of cylindrical tube, the mounting being similar to that adopted in organ-pipes. The results shewed that the vowel quality depended upon the length of the tube. From these and other experiments he concluded that cavities yielding (when sounded independently) an identical note " will impart the same vowel quality to a given reed, or indeed to any reed, provided the note of the reed be flatter than that of the cavity." Willis proceeds (p. 243): " A few theoretical considerations will shew that some such effects as we have seen, might perhaps have been expected. According to Euler, if a single pulsation be excited at the bottom of a tube closed at one end, it will travel to the mouth of this tube with the velocity of sound. Here an echo of the pulsation will be formed which will run back again, be reflected from the bottom of the tube, and again present itself at the mouth where a new echo will be produced, and so on in succession till the motion is destroyed by friction and imperfect reflection....The effect therefore will be a propagation from the mouth of the tube of a succession of equidistant pulsations alternately condensed and rarefied, at intervals corresponding to the time required for the pulse to travel down the tube and back again; that is to say, a short burst of the musical note corresponding to a stopped pipe of the length in question, will be produced.

Let us now endeavour to apply this result of Euler's to the case before us, of a vibrating reed, applied to a pipe of any length,

[1] *Phil. Mag.* vol. VI. p. 119, 1878.

[2] On the Vowel Sounds, and on Reed Organ-pipes. *Camb. Phil. Trans.* vol. III. p. 231, 1829.

and examine the nature of the series of pulsations that ought to be produced by such a system upon this theory.

The vibrating tongue of the reed will generate a series of pulsations of equal force, at equal intervals of time, but alternately condensed and rarefied, which we may call the primary pulsations; on the other hand each of these will be followed by a series of secondary pulsations of decreasing strength, but also at equal intervals from their respective primaries, the interval between them being, as we have seen, regulated by the length of the attached pipe."

And further on (p. 247): "Experiment shews us that the series of effects produced are characterized and distinguished from each other by that quality we call the vowel, and it shews us more, it shews us not only that the pitch of the sound produced is always that of the reed or primary pulse, but that the vowel produced is always identical for the same value of s [the period of the secondary pulses]. Thus, in the example just adduced, g'' is peculiar to the vowel A° [as in *Paw, Nought*]; when this is repeated 512 times in a second, the pitch of the sound is c', and the vowel is A°: if by means of another reed applied to the same pipe it were repeated 340 times in a second, the pitch would be f, but the vowel still A°. Hence it would appear that the ear in losing the consciousness of the pitch of s, is yet able to identify it by this *vowel* quality."

From the importance of his results and from the fact that the early volumes of the Cambridge Transactions are not everywhere accessible, I have thought it desirable to let Willis speak for himself. It will be seen that so far as general principles are concerned, he left little to be effected by his successors. Somewhat later in the same memoir (p. 249) he gives an account of a special experiment undertaken as a test of his theory. "Having shewn the probability that a given vowel is merely the rapid repetition of its peculiar note, it should follow that if we can produce this rapid repetition in any other way, we may expect to hear vowels. Robison and others had shewn that a quill held against a revolving toothed wheel, would produce a musical note by the rapid equidistant repetition of the snaps of the quill upon the teeth. For the quill I substituted a piece of watch-spring pressed lightly against the teeth of the wheel, so that each snap became the musical note of the spring. The spring being at the same time grasped in a pair of pincers, so as to admit of any alteration in length of the vibrating portion. This system evidently produces a compound sound similar to that of the pipe and reed, and an alteration in the length of the spring ought therefore to produce the same effect as that of the pipe. In effect the sound produced retains the same pitch as long as the wheel revolves uniformly, but puts on in succession all the vowel qualities, as the effective length of the spring is altered, and that with considerable distinctness, when due allowance is made for the harsh and disagreeable quality of the sound itself."

In his presentation of vowel theory Helmholtz, following Wheatstone[1], puts the matter a little differently. The aerial vibrations constituting natural or artificial vowels are, when a uniform regime has been attained (§§ 48, 66, 322 *k*), truly periodic, and the period is that of the reed. According to Fourier's theorem they are susceptible of analysis into simple vibrations, whose periods are accurately submultiples of the reed period. The effect of an associated resonator can only be to modify the intensity and phase of the several components, whose periods are already prescribed. If the note of the resonating cavity—the mouth-tone—coincide with one of the partial tones of the voice- or larynx-note, the effect must be to exalt in a special degree the intensity of that tone; and whether there be coincidence or not, those partial tones whose pitch approximates to that of the mouth-tone will be favoured.

This view of the action of a resonator is of course perfectly correct; but at first sight it may appear essentially different from, or even inconsistent with, the account of the matter given by Willis. For example, according to the latter the mouth-tone may be, and generally will be, inharmonic as regards the larynx-tone. In order to understand this matter we must bear in mind two things which are often imperfectly appreciated. The first is the distinction between forced and free vibrations. Although the *natural* vibrations of the oral cavity may be inharmonic, the *forced* vibrations can include only harmonic partials of the larynx note. And again, it is important to remember the definition of simple vibrations, according to which no vibrations can be simple that are not permanently maintained without variation of amplitude or phase. The secondary vibrations of Willis, which

[1] *London and Westminster Review*, Oct. 1837; *Wheatstone's Scientific Papers*, London, 1879, p. 348.

die down after a few periods, are not simple. When the complete succession of them is resolved by Fourier's theorem, it is represented, not by one simple vibration, but by a large or infinite number of such.

From these considerations it will be seen that both ways of regarding the subject are legitimate and not inconsistent with one another. When the relative pitch of the mouth-tone is low, so that, for example, the partial of the larynx note most reinforced is the second or the third, the analysis by Fourier's series is the proper treatment. But when the pitch of the mouth-tone is high, and each succession of vibrations occupies only a small fraction of the complete period, we may agree with Hermann that the resolution by Fourier's series is unnatural, and that we may do better to concentrate our attention upon the actual form of the curve by which the complete vibration is expressed. More especially shall we be inclined to take this course if we entertain doubts as to the applicability of Ohm's law to partials of high order.

Since the publication of Helmholtz's treatise the question has been much discussed whether a given vowel is characterized by the prominence of partials of given *order* (the relative pitch theory), or by the prominence of partials of given *pitch* (the fixed pitch theory), and every possible conclusion has been advocated. We have seen that Willis decided the question, without even expressly formulating it, in favour of the fixed pitch theory. Helmholtz himself, if not very explicitly, appeared to hold the same opinion, perhaps more on *a priori* grounds than as the result of experiment. If indeed, as has usually been assumed by writers on phonetics, a particular vowel quality is associated with a given oral configuration, the question is scarcely an open one. Subsequently under Helmholtz's superintendence the matter was further examined by Auerbach[1], who along with other methods employed a direct analysis of the various vowels by means of resonators associated with the ear. His conclusion on the question under discussion was the intermediate one that *both* characteristics were concerned. The analysis shewed also that in all cases the first, or fundamental tone, was the strongest element in the sound.

A few years later Edison's beautiful invention of the phono-

[1] *Pogg. Ann.* Ergänzung-band VIII. p. 177, 1876.

graph stimulated anew inquiry upon this subject by apparently affording easy means of making an *experimentum crucis*. If vowels were characterized by fixed pitch, they should undergo alteration with the speed of the machine; but if on the other hand the relative pitch theory were the true one, the vowel quality should be preserved and only the pitch of the note be altered. But, owing probably to the imperfection of the earlier instruments, the results arrived at by various observers were still discrepant. The balance of evidence inclined perhaps in favour of the fixed pitch theory[1]. Jenkin and Ewing[2] analysed the impressions actually made upon the recording cylinder, and their results led them to take an intermediate view, similar to that of Auerbach. It is clear, they say, "that the quality of a vowel sound does not depend either on the absolute pitch of reinforcement of the constituent tones alone, or on the simple grouping of relative partials independently of pitch. Before the constituents of a vowel can be assigned, the pitch of the prime must be given; and, on the other hand, the pitch of the most strongly reinforced partial is not alone sufficient to allow us to name the vowel."

With the improved phonographs of recent years the question can be attacked with greater advantage, and observations have been made by McKendrick and others, but still with variable results. Especially to be noted are the extensive researches of Hermann published in *Pflüger's Archiv*. Hermann pronounces unequivocally in favour of the fixed pitch characteristic as at any rate by far the more important, and his experiments apparently justify this conclusion. He finds that the vowels sounded by the phonograph are markedly altered when the speed is varied.

Hermann's general view, to which he was led independently, is identical with that of Willis. "The vowel character consists in a mouth-tone of amplitude variable in the period of the larynx tone[3]." The propriety of this point of view may perhaps be considered to be established, but Hermann somewhat exaggerates the difference between it and that of Helmholtz.

His examination of the automatically recorded curves was effected in more than one way. In the case of the vowel *A*[4] the

[1] Graham Bell, *Ann. Journ. of Otology*, vol. I. July, 1879.

[2] *Edin. Trans.* vol. XXVIII. p. 745, 1878.

[3] *Pflüg. Arch.* vol. XLVII. p. 351, 1890.

[4] The vowel signs refer of course to the continental pronunciation.

amplitudes of the various partials, as given by the Fourier analysis, are set forth in the annexed table, from which it appears that the favoured partial lies throughout between e^2 and g^2.

VOWEL A.

Note	Ordinal number of partial. 1	2	3	4	5	6	7	8	9	10
G						·12 d^2	·37 $<f^2$	·42 g^2	·11 a^2	·12 h^2
A					·13 cis^2	·30 e^2	·33 $<g^2$	·10 a^2	·09 h^2	·08 cis^3
H	·05 H		·09 fis′	·22 h′	·37 dis^2	·45 fis^2	·10 $<a^2$	·15 h^2		
c	·11 c			·19 c^2	·54 e^2	·38 g^2	·16 $<ais^2$	·09 c^3	·10 d^3	
d				·29 d^2	·52 fis^2	·08 a^2	·18 $<c^3$		·06 e^3	
e			·13 h′	·55 e^2	·28 gis^2	·24 h^2	·07 $<d^2$			
fis			·30 cis^2	·61 fis^2	·07 ais^2	·11 cis^2	·11 $<e^2$			
g	·11 g		·39 d^2	·55 g^2	·21 h^2	·11 d^3	·08 $<f^3$			
a			·71 e^2	·18 a^2	·18 cis^3	·09 e^3				
h			·74 fis^2	·17 h^2	·13 dis^3					
c′		·41 c^3	·54 g^2	·40 c^3	·11 e^3					
d′		·71 d^2	·31 a^2	·26 d^3						

The analysis of the curves into their Fourier components involves a great deal of computation, and Hermann is of opinion that the principal result, the pitch of the vowel characteristic, can be obtained as accurately and far more simply by direct measurement on the diagram of the wave-lengths of the intermittent vibrations. The application of this method to the curves for *A* before used gave

VOWEL A.

Note		L mm.	l mm.	Characteristic tone	
				Frequency	Note
G	98	18·5	2·4	756	> fis² (740)
A	110	16·3	2·5	717	> f² (698·5)
H	123·5	14·9	2·6	708	> f² (698·5)
c	130·8	13·6	2·55	698	f²
d	146·8	11·6	2·4	710	> f² (698·5)
e	164·8	10·9	2·3	781	< g² (784)
fis	185	9·8	2·5	725	< fis² (744)
g	196	9·1	2·5	714	> f² (698·5)
a	220	8·2	2·5	714	> f² (698·5)
h	246·9	7·3	2·6	693	< f² (698·5)
c′	261·7	6·8	?	?	
d′	293·7	6·2	?	?	

Here L is the double period of the complete vibration and l the double period of the vowel characteristic. It appears plainly that l preserves a nearly constant value when L varies over a considerable range.

A general comparison of his results with those obtained by other methods has been given by Hermann, from which it will be seen that much remains to be done before the perplexities involving this subject can be removed. Some of the discrepancies

Vowel	Characteristic tone from graphical records Hermann	Mouth-tones according to			
		Donders	Helmholtz	König	Auerbach
A	e²—gis²	b′	b²	b²	f²
E	h³—c⁴	cis³	f₁, b³	b³	a′ (—g′)
I	d⁴—g⁴	f³	f, d⁴	b⁴	f′
O	d²—e²	d′	b′	b′	a′
U	c²—d²	f′	f	b	f′

that have been encountered may probably have their origin in real differences of pronunciation to which only experts in phonetics are sufficiently alive[1]. Again, the question of double resonance has to be considered, for the known shape of the cavities concerned

[1] Lloyd, *Phonetische Studien*, vol. III. part J.

renders it not unlikely that the complete characterization of a vowel is of a multiple nature (§ 310). It should be mentioned that in Lloyd's view the double characteristic is essential, and that the identity of a vowel depends not upon the absolute pitch of one or more resonances, but upon the relative pitch of two or more. In this way he explains the difficulty arising from the fact that the articulation for a given vowel appears to be the same for an infant and for a grown man, although on account of the great difference in the size of the resonating cavities the absolute pitch must vary widely.

It would not be consistent with the plan of this work to go further into details with regard to particular vowels; but one remarkable discrepancy between the results of Hermann and Auerbach must be alluded to. The measurements by the former of graphical records shew in all cases a nearly complete absence of the first, or fundamental, tone from the general sound, which Auerbach on the contrary, using resonators, found this tone the most prominent of all. Hermann, while admitting that the tone is heard, regards it as developed within the ear after the manner of combination-tones (§ 393). I have endeavoured to repeat some of Auerbach's observations, and I find that for all the principal vowels (except perhaps *A*) the fundamental tone is loudly reinforced, the contrast being very marked as the resonator is put in and out of tune by a movement of the finger over its mouth. This must be taken to prove that the tone in question does exist externally to the ear, as indeed from the manner in which the sound is produced could hardly fail to be the case; and the contrary evidence from the records must be explained in some other way.

An important branch of the subject is the artificial imitation of vowel sounds. The actual synthesis by putting together in suitable strengths the various partials was effected by Helmholtz[1]. For this purpose he used tuning-forks and resonators, the forks being all driven from a single interrupter (§§ 63, 64). These experiments are difficult, and do not appear to have been repeated. Helmholtz was satisfied with the reproduction in some cases, although in others the imitation was incomplete. Less satisfactory results were attained when organ-pipes were substituted for the forks.

[1] *Sensations of Sound*, ch. VI.

Vowel sounds have been successfully imitated by Preece and Stroh[1], who employed an apparatus upon the principle of the phonograph, in which the motion of the membrane was controlled by specially shaped teeth, cut upon the circumference of a revolving wheel. They found that the vowel quality underwent important changes as the speed of rotation was altered.

For artificial vowels, illustrative of his special views, Hermann recommends the polyphonic siren (§ 11). If when the series of 12 holes is in operation and a suitable velocity has been attained, the series of 18 holes be put alternately into and out of action, the difference-tone (6) is heard with great loudness and it assumes distinctly the character of an *O*. At a greater speed the vowel is *Ao*, and at a still higher speed an unmistakable *A*.

With the use of double resonators, suitably proportioned, Lloyd has successfully imitated some of the *whispered* vowels.

In the account here given of the vowel question it has only been possible to touch upon a few of the more general aspects of it. The reader who wishes to form a judgment upon controverted points and to pursue the subject into detail must consult the original writings of recent workers, among whom may be specially mentioned Hermann, Pipping, and Lloyd. The field is an attractive one; but those who would work in it need to be well equipped, both on the physical and on the phonetic side.

[1] *Proc. Roy. Soc.* vol. XXVIII. p. 358, 1879.

NOTE TO § 86[1].

It may be observed that the motion of any point belonging to a system of n degrees of freedom, which executes a harmonic motion, is in general linear. For, if x, y, z be the space coordinates of the point, we have

$$x = X \cos nt, \qquad y = Y \cos nt, \qquad z = Z \cos nt,$$

where X, Y, Z are certain constants; so that at all times

$$x : y : z = X : Y : Z.$$

If there be more than one mode of the frequency in question, the coordinates are not necessarily in the same phase. The most general values of x, y, z, subject to the given periodicity, are then

$$x = X_1 \cos nt + X_2 \sin nt,$$

$$y = Y_1 \cos nt + Y_2 \sin nt,$$

$$z = Z_1 \cos nt + Z_2 \sin nt,$$

equations which indicate elliptic motion in the plane

$$x (Y_1 Z_2 - Z_1 Y_2) + y (Z_1 X_2 - X_1 Z_2) + z (X_1 Y_2 - Y_1 X_2) = 0.$$

[1] This note appears now for the first time.

APPENDIX TO CHAPTER V[1].

ON THE VIBRATIONS OF COMPOUND SYSTEMS WHEN THE AMPLITUDES ARE NOT INFINITELY SMALL.

In §§ 67, 68 we have found second approximations for the vibrations of systems of one degree of freedom, both in the case where the vibrations are free and where they are due to the imposition of given forces acting from without. It is now proposed to extend the investigation to cases where there is more than one degree of freedom.

In the absence of dissipative and of impressed forces, everything may be expressed (§ 80) by means of the functions T and V. In the case of infinitely small motion in the neighbourhood of the configuration of equilibrium, T and V reduce themselves to quadratic functions of the velocities and displacements with constant coefficients, and by a suitable choice of coordinates the terms involving *products* of the several coordinates may be made to disappear (§ 87). Even though we intend to include terms of higher order, we may still avail ourselves of this simplification, choosing as coordinates those which have the property of reducing the terms of the second order to sums of squares. Thus we may write

$$T = \tfrac{1}{2}A_{11}\dot{\phi}_1^2 + \tfrac{1}{2}A_{22}\dot{\phi}_2^2 + \ldots + A_{12}\dot{\phi}_1\dot{\phi}_2 + A_{13}\dot{\phi}_1\dot{\phi}_3 + \ldots\ldots\ldots (1),$$

in which A_{11}, A_{22},... are functions of ϕ_1, ϕ_2,...including constant terms a_1, a_2, ..., while A_{12}, A_{13}, ... are functions of the same variables *without constant terms*:

$$V = \tfrac{1}{2}c_1\phi_1^2 + \tfrac{1}{2}c_2\phi_2^2 + \ldots + V_3 + V_4 + \ldots\ldots\ldots\ldots (2),$$

where V_3, V_4, ... denote the parts of V which are of degree 3, 4, ... in ϕ_1, ϕ_2, ...

For the first approximation, applicable to infinitely small vibrations, we have

$$A_{11} = a_1, \quad A_{22} = a_2, \ \ldots \ A_{12} = 0, \quad A_{13} = 0 \ \ldots, \quad V_3 = 0, \quad V_4 = 0, \ \ldots;$$

[1] This appendix appears now for the first time.

so that (§ 87) Lagrange's equations are

$$a_1\ddot{\phi}_1 + c_1\phi_1 = 0, \quad a_2\ddot{\phi}_2 + c_2\phi_2 = 0, \text{ \&c.} \qquad \text{.........(3)},$$

in which the coordinates are separated. The solution relative to ϕ_1 may be taken to be

$$\phi_1 = H_1 \cos nt, \qquad \phi_2 = 0, \qquad \phi_3 = 0, \ldots \text{ \&c.} \qquad \text{.........(4)},$$

where

$$c_1 - n^2 a_1 = 0 \qquad \text{.........(5)}.$$

Similar solutions exist relative to the other coordinates.

The second approximation, to which we now proceed, is to be founded on (4), (5); and thus $\phi_2, \phi_3, \ldots$ are to be regarded as small quantities relatively to ϕ_1.

For the coefficients in (1) we write

$$A_{11} = a_1 + \alpha_{11}\phi_1 + \alpha_{12}\phi_2 + \ldots, \quad A_{12} = \alpha_2\phi_1 + \ldots, \quad A_{13} = \alpha_3\phi_1 + \ldots \qquad \text{.........(6)},$$

and in (2)

$$V_3 = \gamma_1\phi_1^3 + \gamma_2\phi_1^2\phi_2 + \ldots \qquad \text{.........(7)};$$

so that for a further approximation

$$dT/d\dot{\phi}_1 = (a_1 + \alpha_{11}\phi_1)\,\dot{\phi}_1 + \alpha_2\phi_1\dot{\phi}_2 + \alpha_3\phi_1\dot{\phi}_3 + \ldots,$$

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\phi}_1}\right) = (a_1 + \alpha_{11}\phi_1)\,\ddot{\phi}_1 + \alpha_{11}\dot{\phi}_1^2 + \alpha_2\phi_1\ddot{\phi}_2 + \alpha_2\dot{\phi}_1\dot{\phi}_2 + \alpha_3\phi_1\ddot{\phi}_3 + \alpha_3\dot{\phi}_1\dot{\phi}_3 + \ldots,$$

$$dT/d\phi_1 = \tfrac{1}{2}\alpha_{11}\dot{\phi}_1^2 + \alpha_2\dot{\phi}_1\dot{\phi}_2 + \alpha_3\dot{\phi}_1\dot{\phi}_3 + \ldots$$

Thus as the equation (§ 80) for ϕ_1, terms of the order ϕ_1^2 being retained, we get

$$(a_1 + \alpha_{11}\phi_1)\,\ddot{\phi}_1 + \tfrac{1}{2}\alpha_{11}\dot{\phi}_1^2 + c_1\phi_1 + 3\gamma_1\phi_1^2 = 0 \qquad \text{.........(8)}.$$

To this order of approximation the coordinate ϕ_1 is separated from the others, and the solution proceeds as in the case of but one degree of freedom (§ 67). We have from (4)

$$\phi_1\ddot{\phi}_1 = -n^2H_1^2\cos^2 nt = -\tfrac{1}{2}n^2H_1^2(1 + \cos 2nt),$$
$$\dot{\phi}_1^2 = n^2H_1^2\sin^2 nt = \tfrac{1}{2}n^2H_1^2(1 - \cos 2nt),$$
$$\phi_1^2 = H_1^2\cos^2 nt = \tfrac{1}{2}H_1^2(1 + \cos 2nt);$$

so that (8) becomes

$$a_1\ddot{\phi}_1 + c_1\phi_1 + (-\tfrac{1}{4}n^2\alpha_{11} + \tfrac{3}{2}\gamma_1)\,H_1^2 + (-\tfrac{3}{4}n^2\alpha_{11} + \tfrac{3}{2}\gamma_1)\,H_1^2\cos 2nt = 0 \qquad \text{.........(9)}.$$

The solution of (9) may be expressed in the form

$$\phi_1 = H_0 + H_1\cos nt + H_2\cos 2nt + \ldots \qquad \text{.........(10)},$$

and a comparison gives

$$c_1H_0 = (\tfrac{1}{4}n^2\alpha_{11} - \tfrac{3}{2}\gamma_1)\,H_1^2,$$
$$(c_1 - n^2a_1)\,H_1 = 0,$$
$$(c_1 - 4n^2a_1)\,H_2 = (\tfrac{3}{4}n^2\alpha_{11} - \tfrac{3}{2}\gamma_1)\,H_1^2.$$

Thus to a second approximation

$$\phi_1 = \frac{(\frac{1}{4}n^2 a_{11} - \frac{3}{2}\gamma_1) H_1^2}{c_1} + H_1 \cos nt + \frac{(\frac{3}{4}n^2 a_{11} - \frac{3}{2}\gamma_1) H_1^2}{c_1 - 4n^2 a_1} \cos 2nt \ldots (11);$$

and the value of n is the same, i.e. $\sqrt{(c_1/a_1)}$, as in the first approximation.

We have now to express the corresponding values of ϕ_2, ϕ_3.... From (6)

$$dT/d\dot{\phi}_2 = a_2 \phi_1 \dot{\phi}_1 + a_2 \dot{\phi}_2 + \ldots,$$
$$dT/d\phi_2 = \tfrac{1}{2} a_{12} \dot{\phi}_1^2 + \ldots,$$

and Lagrange's equation becomes, terms of order ϕ_1^2 being retained,

$$a_2 \ddot{\phi}_2 + c_2 \phi_2 + a_2 \phi_1 \ddot{\phi}_1 + (a_2 - \tfrac{1}{2} a_{12}) \dot{\phi}_1^2 + \gamma_2 \phi_1^2 = 0,$$

or on substitution from (4) in the small terms

$$a_2 \ddot{\phi}_2 + c_2 \phi_2 + (-\tfrac{1}{4} n^2 a_{12} + \tfrac{1}{2}\gamma_2) H_1^2 + (-n^2 a_2 + \tfrac{1}{4} n^2 a_{12} + \tfrac{1}{2}\gamma_2) H_1^2 \cos 2nt = 0 \quad \ldots\ldots\ldots (12).$$

Accordingly, if

$$\phi_2 = K_0 + K_1 \cos nt + K_2 \cos 2nt + \ldots \quad \ldots\ldots\ldots\ldots (13),$$

we find on comparison with (12)

$$c_2 K_0 = (\tfrac{1}{4} n^2 a_{12} - \tfrac{1}{2}\gamma_2) H_1^2 \ldots\ldots\ldots\ldots\ldots\ldots (14),$$
$$(c_2 - n^2 a_2) K_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15),$$
$$(c_2 - 4n^2 a_2) K_2 = (n^2 a_2 - \tfrac{1}{4} n^2 a_{12} - \tfrac{1}{2}\gamma_2) H_1^2 \ldots\ldots\ldots (16).$$

Thus $K_1 = 0$, and the introduction of the values of K_0 and K_2 from (14), (16) into (13) gives the complete value of ϕ_2 to the second approximation.

The values of ϕ_3, ϕ_4, &c. are obtained in a similar manner, and thus we find to a second approximation the complete expression for those vibrations of a system of any number of degrees of freedom which to a first approximation are expressed by (4).

The principal results of the second approximation are (i) that the motion remains periodic with frequency unaltered, (ii) that terms, constant and proportional to $\cos 2nt$, are added to the value of that coordinate which is finite in the first approximation, as well as to those which in the first approximation are zero.

We now proceed to a third approximation; but for brevity we will confine ourselves to the case (α) where there are but two degrees of freedom, and (β) where the kinetic energy is completely expressed as a sum of squares of the velocities with constant coefficients. This will include the vibrations of a particle moving in two dimensions in the neighbourhood of a place of equilibrium.

We have

$$T = \tfrac{1}{2}a_1\dot{\phi}_1^2 + \tfrac{1}{2}a_2\dot{\phi}_2^2, \qquad V = \tfrac{1}{2}c_1\phi_1^2 + \tfrac{1}{2}c_2\phi_2^2 + V_3 + V_4,$$

where

$$V_3 = \gamma_1\phi_1^3 + \gamma_2\phi_1^2\phi_2 + \gamma'\phi_1\phi_2^2 + \dots \quad\dots\dots(17),$$

$$V_4 = \delta_1\phi_1^4 + \delta_2\phi_1^3\phi_2 + \dots \quad\dots\dots(18);$$

so that Lagrange's equations are

$$a_1\ddot{\phi}_1 + c_1\phi_1 + 3\gamma_1\phi_1^2 + 2\gamma_2\phi_1\phi_2 + 4\delta_1\phi_1^3 = 0 \quad\dots\dots(19),$$

$$a_2\ddot{\phi}_2 + c_2\phi_2 + \gamma_2\phi_1^2 + 2\gamma'\phi_1\phi_2 + \delta_2\phi_1^3 = 0 \quad\dots\dots(20).$$

As before, we are to take for the first approximation

$$\phi_1 = H_1\cos nt, \qquad \phi_2 = 0 \quad\dots\dots(21).$$

For the solution of (19), (20) we may write

$$\phi_1 = H_0 + H_1\cos nt + H_2\cos 2nt + H_3\cos 3nt + \dots \quad\dots\dots(22),$$

$$\phi_2 = K_0 + K_1\cos nt + K_2\cos 2nt + K_3\cos 3nt + \dots \quad\dots\dots(23).$$

In (22), (23) H_0, H_2, K_0, K_2 are quantities of the second order in H_1, whose values have already been given, while K_1, H_3, K_3 are of the third order. Retaining terms of the third order, we have

$$\phi_1^2 = \tfrac{1}{2}H_1^2 + (2H_0H_1 + H_1H_2)\cos nt + \tfrac{1}{2}H_1^2\cos 2nt + H_1H_2\cos 3nt,$$

$$\phi_1\phi_2 = (H_1K_0 + \tfrac{1}{2}H_1K_2)\cos nt + \tfrac{1}{2}H_1K_2\cos 3nt,$$

$$\phi_1^3 = \tfrac{3}{4}H_1^3\cos nt + \tfrac{1}{4}H_1^3\cos 3nt.$$

Substituting these values in the small terms of (19), (20), and from (22), (23) in the two first terms, we get the following 8 equations, correct to the third order,

$$c_1H_0 + \tfrac{3}{2}\gamma_1H_1^2 = 0 \quad\dots\dots(24),$$

$$c_1 - n^2a_1 + 3\gamma_1(2H_0 + H_2) + 2\gamma_2(K_0 + \tfrac{1}{2}K_2) + 3\delta_1H_1^2 = 0 \quad\dots(25),$$

$$(c_1 - 4n^2a_1)H_2 + \tfrac{3}{2}\gamma_1H_1^2 = 0 \quad\dots\dots(26),$$

$$(c_1 - 9n^2a_1)H_3 + 3\gamma_1H_1H_2 + \gamma_2H_1K_2 + \delta_1H_1^3 = 0 \quad\dots(27);$$

$$c_2K_0 + \tfrac{1}{2}\gamma_2H_1^2 = 0 \quad\dots\dots(28),$$

$$(c_2 - n^2a_2)K_1 + \gamma_2H_1(2H_0 + H_2) + \gamma'H_1(2K_0 + K_2) + \tfrac{3}{4}\delta_2H_1^3 = 0 \quad\dots(29),$$

$$(c_2 - 4n^2a_2)K_2 + \tfrac{1}{2}\gamma_2H_1^2 = 0 \quad\dots\dots(30),$$

$$(c_2 - 9n^2a_2)K_3 + \gamma_2H_1H_2 + \gamma'H_1K_2 + \tfrac{1}{4}\delta_2H_1^3 = 0 \quad\dots(31).$$

Of these (24), (26), (28), (30) give immediately the values of H_0, H_2, K_0, K_2, which are the same as to the second order of approximation, and the substitution of these values in (27), (29), (31) determines H_3, K_1, K_3 as quantities of the third order. The remaining equation (25) serves to determine n. We find as correct to this order

$$\frac{n^2a_1 - c_1}{H_1^2} = -\tfrac{3}{2}\gamma_1^2\left(\frac{6}{c_1} + \frac{3}{c_1 - 4n^2a_1}\right) - \tfrac{1}{2}\gamma_2^2\left(\frac{2}{c_2} + \frac{1}{c_2 - 4n^2a_2}\right) + 3\delta_1 \quad\dots(32).$$

If $\gamma_2 = 0$, this result will be found to harmonize with (9) § 67, when the differences of notation are allowed for, and the first approximation to n is substituted in the small terms.

The vibration above determined is that founded upon (21) as first approximation. The other mode, in which approximately $\phi_1 = 0$, can be investigated in like manner.

If V be an even function both of ϕ_1 and ϕ_2, γ_1, γ_2, γ', δ_2 vanish, and the third approximation is expressed by

$$H_0 = 0, \qquad H_2 = 0, \qquad H_3 = -\delta_1 H_1^3/(c_1 - 9n^2 a_1);$$
$$K_0 = 0, \qquad K_1 = 0, \qquad K_2 = 0, \qquad K_3 = 0;$$
$$n^2 a_1 - c_1 = 3\delta_1 H_1^2.$$

Indeed under this condition ϕ_2 vanishes to any order of approximation.

These examples may suffice to elucidate the process of approximation. An examination of its nature in the general case shews that the following conclusions hold good however far the approximation may be carried.

(*a*) The solution obtained by this process is periodic, and the frequency is an *even* function of the amplitude of the principal term (H_1).

(*b*) The Fourier series expressive of each coordinate contains cosines only, without sines, of the multiples of nt. Thus the whole system comes to rest at the same moment of time, e.g. $t = 0$, and then retraces its course.

(*c*) The coefficient of $\cos rnt$ in the series for any coordinate is of the rth order (at least) in the amplitude (H_1) of the principal term. For example, the series of the third approximation, in which higher powers of H_1 than H_1^3 are neglected, stop at $\cos 3nt$.

(*d*) There are as many types of solution as degrees of freedom; but, it need hardly be said, the various solutions are not superposable.

One important reservation has yet to be made. It has been assumed that all the factors, such as $(c_2 - 4n^2 a_2)$ in (30), are finite, that is, that no coincidence occurs between a harmonic of the actual frequency and the natural frequency of some other mode of infinitesimal vibration. Otherwise, some of the coefficients, originally assumed to be subordinate, e.g. K_2 in (30), become infinite, and the approximation breaks down. We are thus precluded from obtaining a solution in some of the cases where we should most desire to do so.

As an example of this failure we may briefly notice the gravest vibrations in one dimension of a gas, obeying Boyle's law, and contained in a cylindrical tube with stopped ends. The equation to be satisfied throughout, (4) § 249, is of the form

$$\left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dt^2} = \frac{d^2y}{dx^2},$$

and the procedure suggested by the general theory is to assume

$$y = x + y_0 + y_1 \cos nt + y_2 \cos 2nt + \ldots,$$

where

$$y_0 = H_{01} \sin x + H_{02} \sin 2x + H_{03} \sin 3x + \ldots,$$

$$y_1 = H_{11} \sin x + H_{12} \sin 2x + H_{13} \sin 3x + \ldots,$$

$$y_2 = H_{21} \sin x + H_{22} \sin 2x + H_{23} \sin 3x + \ldots,$$

and so on. In the first approximation

$$y = x + H_{11} \sin x \cos nt,$$

with $n = 1$. But when we proceed to a second approximation, we find

$$(4n^2 - 4) H_{22} = -\tfrac{1}{2} n^2 H_{11}{}^2,$$

still with n equal to 1, so that the method breaks down. The term $H_{22} \sin 2x \cos 2nt$ in the value of y, originally supposed to be subordinate, enters with an infinite coefficient.

It is possible that we have here an explanation of the difficulty of causing long narrow pipes to speak in their gravest mode.

The behaviour of a system vibrating under the action of an impressed force may be treated in a very similar manner. Taking, for example, the case of two degrees of freedom already considered in respect of its free vibrations, let us suppose that the impressed forces are

$$\Phi_1 = E_1 \cos pt, \qquad \Phi_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(33),$$

so that the solution to a first approximation is

$$\phi_1 = \frac{E_1 \cos pt}{c_1 - p^2 a_1}, \qquad \phi_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(34).$$

With substitution of p for n equations (22), (23) are still applicable, and also the resulting equations (24) to (31), except that in (25) the left-hand member is to be multiplied by H_1 and that on the right E_1 is to be substituted for zero. This equation now serves to determine H_1, instead of, as before, to determine n.

It is evident that in this way a truly periodic solution can always be built up. The period is that of the force, and the phases are such that the entire system comes to rest at the moment when the force is at a maximum (positive or negative). After this the previous course is retraced, as in the case of free vibrations, each series of cosines remaining unchanged when the sign of t is reversed.

NOTE TO § 273[1].

A METHOD of obtaining Poisson's solution (8) given by Liouville[2] is worthy of notice.

If r be the polar radius vector measured from any point O, and the general differential equation be integrated over the volume included between spherical surfaces of radii r and $r + dr$, we find on transformation of the second integral by Green's theorem

$$\frac{d^2(r\lambda)}{dt^2} = a^2 \frac{d^2(r\lambda)}{dr^2} \quad\text{.........................}(\alpha),$$

in which $\lambda = \iint \phi\, d\sigma$, that is to say is proportional to the mean value of ϕ reckoned over the spherical surface of radius r. Equation (α) may be regarded as an extension of (1) § 279; it may also be proved from the expression (5) § 241 for $\nabla^2\phi$ in terms of the ordinary polar co-ordinates r, θ, ω.

The general solution of (α) is

$$r\lambda = \chi(at + r) + \theta(at - r) \quad\text{........................}(\beta),$$

where χ and θ are arbitrary functions; but, as in § 279, if the pole be not a source, $\chi(at) + \theta(at) = 0$, so that

$$r\lambda = \chi(at + r) - \chi(at - r) \text{.......................}(\gamma).$$

It appears from (γ) that at O, when $r = 0$, $\lambda = 2\chi'(at)$, which is therefore also the value of $4\pi\phi$ at O at time t. Again from (γ)

$$2\chi'(at + r) = \frac{d(r\lambda)}{d(at)} + \frac{d(r\lambda)}{dr},$$

so that

$$2\chi'(r) = \left[\frac{d(r\lambda)}{d(at)} + \frac{d(r\lambda)}{dr}\right]_{(t=0)},$$

or in the notation of § 273

$$2\chi'(r) = \frac{r}{a}\iint F(r)\, d\sigma + \frac{d}{dr}\left[r\iint f(r)\, d\sigma\right] \text{...........}(\delta).$$

By writing at in place of r in (δ) we obtain the value of $2\chi'(at)$, or $4\pi\phi$, which agrees with (8) § 273.

[1] This note appeared in the first edition.

[2] Liouville, tom. I. p. 1, 1856.

APPENDIX A. (§ 307[1].)

CORRECTION FOR OPEN END.

THE problem of determining the correction for the open end of a tube is one of considerable difficulty, even when there is an infinite flange. It is proved in the text (§ 307) that the correction α is greater than $\frac{1}{4}\pi R$, and less than $(8/3\pi) R$. The latter value is obtained by calculating the energy of the motion on the supposition that the velocity parallel to the axis is constant over the plane of the mouth, and comparing this energy with the square of the total current. The actual velocity, no doubt, increases from the centre outwards, becoming infinite at the sharp edge; and the assumption of a constant value is a somewhat violent one. Nevertheless the value of α so calculated turns out to be not greatly in excess of the truth. It is evident that we should be justified in expecting a very good result, if we assume an axial velocity of the form

$$1 + \mu r^2/R^2 + \mu' r^4/R^4,$$

r denoting the distance of the point considered from the centre of the mouth, and then determine μ and μ' so as to make the whole energy a minimum. The energy so calculated, though necessarily in excess, must be a very good approximation to the truth.

In carrying out this plan we have two distinct problems to deal with, the determination of the motion (1) outside, and (2) inside the cylinder. The former, being the easier, we will take first.

The conditions are that ϕ vanish at infinity, and that when $x = 0$, $d\phi/dx$ vanish, except over the area of the circle $r = R$, where

$$d\phi/dx = 1 + \mu r^2/R^2 + \mu' r^4/R^4 \quad \text{..................(1).}$$

Under these circumstances we know (§ 278) that

$$\phi = -\frac{1}{2\pi}\iint \frac{d\phi}{dx}\frac{d\sigma}{\rho} \quad \text{..........................(2),}$$

where ρ denotes the distance of the point where ϕ is to be estimated from the element of area $d\sigma$. Now

$$2\,(\text{kinetic energy})^2 = -\tfrac{1}{2}\iint \phi\,\frac{d\phi}{dx}\,d\sigma = \frac{1}{2\pi}\iint \frac{d\phi}{dx}\,.\iint \frac{d\phi}{dx}\frac{d\sigma}{\rho}\,.\,d\sigma = \frac{P}{\pi},$$

[1] This appendix appeared in the first edition.

[2] The density of the fluid is supposed to be unity.

if P represent the potential on itself of a disc of radius R, whose

$$\text{density} = 1 + \mu r^2/R^2 + \mu' r^4/R^4.$$

The value of P is to be calculated by the method employed in the text (§ 307) for a uniform density. At the edge of the disc, when cut down to radius a, we have the potential

$$V = 4a + \frac{20}{9}\frac{\mu a^3}{R^2} + \frac{356}{225}\frac{\mu' a^5}{R^4} \ldots \qquad (3),$$

and thus

$$P = \int_0^R 2\pi a\, da\, V \left\{1 + \mu \frac{a^2}{R^2} + \mu' \frac{a^4}{R^4}\right\}$$

$$= \frac{8\pi R^3}{3}\left\{1 + \frac{14}{15}\mu + \frac{5}{21}\mu^2 + \frac{314}{525}\mu' + \frac{214}{675}\mu\mu' + \frac{89}{825}\mu'^2\right\} \qquad (4),$$

on effecting the integration. This quantity divided by π gives twice the kinetic energy of the motion defined by (1).

The total current

$$= \int_0^R 2\pi r\, dr \left(1 + \mu \frac{r^2}{R^2} + \mu' \frac{r^4}{R^4}\right) = \pi R^2 (1 + \tfrac{1}{2}\mu + \tfrac{1}{3}\mu') \qquad (5).$$

We have next to consider the problem of determining the motion of an incompressible fluid within a rigid cylinder under the conditions that the axial velocity shall be uniform when $x = -\infty$, and when $x = 0$ shall be of the form

$$d\phi/dx = 1 + \mu r^2/R^2 + \mu' r^4/R^4.$$

It will conduce to clearness if we separate from ϕ, that part of it which corresponds to a uniform flow. Thus, if we take

$$d\phi/dx = 1 + \tfrac{1}{2}\mu + \tfrac{1}{3}\mu' + d\psi/dx,$$

ψ will correspond to a motion which vanishes when x is numerically great. When $x = 0$,

$$d\psi/dx = \mu (r^2 - \tfrac{1}{2}) + \mu' (r^4 - \tfrac{1}{3}) \qquad (6),$$

if for the sake of brevity we put $R = 1$.

Now ψ may be expanded in the series

$$\psi = \Sigma a_p e^{px} J_0 (pr) \qquad (7),$$

where p denotes a root of the equation

$$J_0'(p) = 0 \qquad (8)\,[1].$$

Each term of this series satisfies the condition of giving no radial

[1] The numerical values of the roots are approximately

$p_1 = 3{\cdot}831705$, $p_2 = 7{\cdot}015$, $p_3 = 10{\cdot}174$,
$p_4 = 13{\cdot}324$, $p_5 = 16{\cdot}471$, $p_6 = 19{\cdot}616$.

velocity, when $r=1$; and no motion of any kind, when $x=-\infty$. It remains to determine the coefficients a_p so as to satisfy (6), when $x=0$. From $r=0$ to $r=1$, we must have

$$\Sigma p a_p J_0(pr) = \mu(r^2-\tfrac{1}{2}) + \mu'(r^4-\tfrac{1}{3}),$$

whence multiplying by $J_0(pr)\,r\,dr$ and integrating from 0 to 1,

$$p a_p [J_0(p)]^2 = 2\int_0^1 r\,dr\,J_0(pr)\,\{\mu(r^2-\tfrac{1}{2}) + \mu'(r^4-\tfrac{1}{3})\},$$

every term on the left, except one, vanishing by the property of the functions. For the right-hand side we have

$$\int_0^1 r\,dr\,J_0(pr) = 0,$$

$$\int_0^1 r^3\,dr\,J_0(pr) = \frac{2}{p^2}J_0(p),$$

$$\int_0^1 r^5\,dr\,J_0(pr) = \left(\frac{4}{p^2}-\frac{32}{p^4}\right)J_0(p);$$

so that

$$a_p = \frac{4}{p^3 J_0(p)}\left\{\mu + 2\mu'\left(1-\frac{8}{p^2}\right)\right\} \quad \text{.....................(9).}$$

The velocity-potential ϕ of the whole motion is thus

$$\phi = (1+\tfrac{1}{2}\mu+\tfrac{1}{3}\mu')\,x + 4\Sigma\frac{\mu+2\mu'(1-8p^{-2})}{p^3 J_0(p)}\,e^{px} J_0(pr)\text{....(10)},$$

the summation extending to all the admissible values of p. We have now to find the energy of motion of so much of the fluid as is included between $x=0$, and $x=-l$, where l is so great that the velocity is there sensibly constant.

By Green's theorem

$$2(\text{kinetic energy}) = \int_0^1 \phi\frac{d\phi}{dx}2\pi r\,dr \quad (x=0) - \int_0^1 \phi\frac{d\phi}{dx}2\pi r\,dr \quad (x=-l).$$

Now, when $x=-l$,

$$\phi = -(1+\tfrac{1}{2}\mu+\tfrac{1}{3}\mu')\,l,$$

$$d\phi/dx = 1+\tfrac{1}{2}\mu+\tfrac{1}{3}\mu';$$

so that the second term is $\pi l(1+\tfrac{1}{2}\mu+\tfrac{1}{3}\mu')^2$.

In calculating the first term, we must remember that if p_1 and p_2 be two different values of p,

$$\int_0^1 2\pi r\,dr\,J_0(p_1 r)\,J_0(p_2 r) = 0.$$

Thus

$$\int_0^1 2\pi r\,dr\,\phi\frac{d\phi}{dx}(x=0)=16\Sigma\frac{\{\mu+2\mu'(1-8p^{-2})\}^2}{p^5[J_0(p)]^2}\int_0^1 2\pi r\,dr\,[J_0(pr)]^2$$

$$=16\pi\Sigma\left\{\mu+2\mu'\left(1-\frac{8}{p^2}\right)\right\}^2 p^{-5}.$$

Accordingly, on restoring R,

$$2\,(\text{kinetic energy})=\pi R^2 l\,(1+\tfrac{1}{2}\mu+\tfrac{1}{3}\mu')^2$$
$$+16\pi R^3\Sigma\left\{\mu+2\mu'\left(1-\frac{8}{p^2}\right)\right\}^2 p^{-5}.$$

To this must be added the energy of the motion on the positive side of $x=0$. On the whole

$$\frac{2\text{ kinetic energy}}{(\text{current})^2}=\frac{l}{\pi R^2}+\frac{16}{\pi R(1+\frac{1}{2}\mu+\frac{1}{3}\mu')^2}\Sigma\left\{\mu+2\mu'\left(1-\frac{8}{p^2}\right)\right\}^2 p^{-5}$$
$$+\frac{8}{3\pi^2 R}\,\frac{1+\frac{14}{15}\mu+\frac{5}{21}\mu^2+\frac{314}{525}\mu'+\frac{214}{675}\mu\mu'+\frac{89}{825}\mu'^2}{(1+\frac{1}{2}\mu+\frac{1}{3}\mu')^2}.$$

Hence, if α be the correction to the length,

$$3\pi\alpha/8R=[1+\tfrac{14}{15}\mu+\tfrac{314}{525}\mu'+(6\pi\Sigma p^{-5}+\tfrac{5}{21})\mu^2$$
$$+\{24\pi(\Sigma p^{-5}-8\Sigma p^{-7})+\tfrac{214}{675}\}\mu\mu'$$
$$+\{24\pi(\Sigma p^{-5}-16\Sigma p^{-7}+64\Sigma p^{-9})+\tfrac{89}{825}\}\mu'^2]\div(1+\tfrac{1}{2}\mu+\tfrac{1}{3}\mu')^2.$$

By numerical calculation from the values of p

$$\Sigma p^{-5}=\cdot 00128266;\quad \Sigma p^{-5}-8\Sigma p^{-7}=\cdot 00061255,$$
$$\Sigma p^{-5}-16\Sigma p^{-7}+64\Sigma p^{-9}=\cdot 00030351,$$

and thus
$$3\pi\alpha/8R=[1+\cdot 9333333\mu+\cdot 5980951\mu'$$
$$+\cdot 2622728\mu^2+\cdot 363223\mu\mu'+\cdot 1307634\mu'^2]\div(1+\tfrac{1}{2}\mu+\tfrac{1}{3}\mu')^2$$
$$=1-\frac{\cdot 0666667\mu+\cdot 0685716\mu'-\cdot 0122728\mu^2-\cdot 029890\mu\mu'-\cdot 0196523\mu'^2}{(1+\frac{1}{2}\mu+\frac{1}{3}\mu')^2}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

The fraction on the right is the ratio of two quadratic functions of μ, μ', and our object is to determine its maximum value. In general if S and S' be two quadratic functions, the maximum and minimum values of $z=S\div S'$ are given by the cubic equation

$$-\Delta z^{-3}+\Theta z^{-2}-\Theta' z^{-1}+\Delta'=0,$$

where
$$S=a\mu^2+b\mu'^2+c+2f\mu'+2g\mu+2h\mu\mu',$$
$$S'=a'\mu^2+b'\mu'^2+c'+2f'\mu'+2g'\mu+2h'\mu\mu',$$
$$\Delta=abc+2fgh-af^2-bg^2-ch^2=\frac{(h^2-ab)(g^2-ac)-(hg-af)^2}{a},$$
$$\Theta=(bc-f^2)a'+(ca-g^2)b'+(ab-h^2)c'$$
$$+2(gh-af)f'+2(hf-bg)g'+2(fg-ch)h',$$

and Θ', Δ', are derived from Θ and Δ by interchanging the accented and unaccented letters.

In the present case, since S' is a product of linear factors, $\Delta' = 0$; and since the two factors are the same, $\Theta' = 0$, so that $z = \Delta \div \Theta$ simply. Substituting the numerical values, and effecting the calculations, we find $z = {\cdot}0289864$, which is the maximum value of the fraction consistent with real values of μ and μ'.

The corresponding value of α is $\cdot 82422\,R$, than which the true correction cannot be greater.

If we assume $\mu' = 0$, the greatest value of z then possible is $\cdot 024363$, which gives

$$\alpha = \cdot 828146\,R\text{.[1]}$$

On the other hand if we put $\mu = 0$, the maximum value of z comes out $\cdot 027653$, whence

$$\alpha = \cdot 825353\,R.$$

It would appear from this result that the variable part of the normal velocity at the mouth is better represented by a term varying as r^4, than by one varying as r^2.

The value $\alpha = \cdot 8242\,R$ is probably pretty close to the truth. If the normal velocity be assumed constant, $\alpha = \cdot 848826\,R$; if of the form $1 + \mu r^2$, $\alpha = \cdot 82815\,R$, when μ is suitably determined; and when the form $1 + \mu r^2 + \mu' r^4$, containing another arbitrary constant, is made the foundation of the calculation, we get $\alpha = \cdot 8242\,R$.

The true value of α is probably about $\cdot 82\,R$.

In the case of $\mu = 0$, the minimum energy corresponds to $\mu' = 1{\cdot}103$, so that

$$d\phi/dx = 1 + 1{\cdot}103\,r^4/R^4.$$

On this supposition the normal velocity of the edge ($r = R$) would be about double of that near the centre.

[1] Notes on Bessel's functions. *Phil. Mag.* Nov. 1872.

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