— GREEN'S Theorem

For two functions ϕ and ψ consider

$$\int_V d^3x\ \vec{\nabla}\cdot(\phi\vec{\nabla}\psi) = \oint_S \phi\vec{\nabla}\psi\cdot d\vec{S}$$

$$\int_V d^3x\ \vec{\nabla}\cdot(\psi\vec{\nabla}\phi) = \oint_S \psi\vec{\nabla}\phi\cdot d\vec{S}$$

$$\therefore\ \boxed{\int_V d^3x\left(\phi\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\phi\right) = \oint_S\left(\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi\right)\cdot d\vec{S}}$$

Green's theorem

Consider the Green's function $G(x,x')$ satisfying:

$$\vec{\nabla}'^2 G(x,x') = -4\pi\,\delta^{(3)}(x-x').$$

In general $G(x,x') = \dfrac{1}{|x-x'|} + F(x,x')$

(Also $G(x,x') = G(x',x)$)

where $\vec{\nabla}^2 F(x,x') = 0$ inside the volume V

Hence $F(x,x')$ can be due to charges outside the volume V and is chosen so that to satisfy for $G(x,x')$ certain boundary conditions.

Choosing $\phi = \Phi$, $\psi = G(x,x')$ we have: (9)

$$\int_V d^3x' \Big[\Phi(x') \underbrace{\vec{\nabla}'^2 G(x,x')}_{= -4\pi\delta^{(3)}(x-x')} - G(x,x') \underbrace{\vec{\nabla}'^2 \Phi(x')}_{= -4\pi\rho(x')} \Big]$$

$$= \oint_S \Big[\Phi(x') \vec{\nabla}' G(x,x') - G(x,x') \vec{\nabla}' \phi(x') \Big] \cdot d\vec{S}'$$

$$\Rightarrow \boxed{\begin{aligned} \phi(x) &= \int_V dx' \, \rho(x') G(x,x') \\ &+ \frac{1}{4\pi} \oint_S \Big[G(x,x') \vec{\nabla}' \phi(x') - \phi(x') \vec{\nabla}' G(x,x') \Big] \cdot d\vec{S}' \\ &\text{If } x \in \text{Volume.} \end{aligned}}$$

If $x \notin$ volume r.h.s. $= 0$ and result represents a constraint between b.c. and Green's function.

Dirichlet b.c. $\boxed{G_D(x,x') = 0 \quad \text{for } x' \in S}$

$$\therefore \quad \phi(x) = \int_V dx' \, \rho(x') G_D(x,x') - \frac{1}{4\pi} \oint_S \phi(x') \vec{\nabla}' G_D(x,x') \cdot d\vec{S}'$$

This b.c. is particularly favolable is we know the potential in the boundary.

Neumann b.c.

From $\vec{\nabla}'^2 G = -4\pi\delta^{(3)}(x-x') \rightarrow \oint_{S'} \vec{\nabla}' G(x,x') \cdot d\vec{S}' = -4\pi$

the easiest b.c. is

$$\boxed{\begin{gathered}\vec{\nabla}' G_N(x,x') = -4\pi/S\,\hat{n}' \quad \text{for } x' \in S \\ S = \text{total boundary surface}\end{gathered}}$$

$$\phi(x) = \int_V dx' \rho(x') G_N(x,x') + \frac{1}{4\pi}\oint_S \vec{\nabla}'\phi \cdot G_N(x,x')\, d\vec{S}'$$

$$+ \langle\phi\rangle_S, \qquad \langle\phi\rangle_S \equiv \frac{1}{S}\oint_S \phi\, dS = \text{averge value}$$

- If the boundary consists of a surface that has infinite area, then $\underline{\underline{S \to \infty}}$.