Παρασκευή 12-1-24. Διάλεξη ....

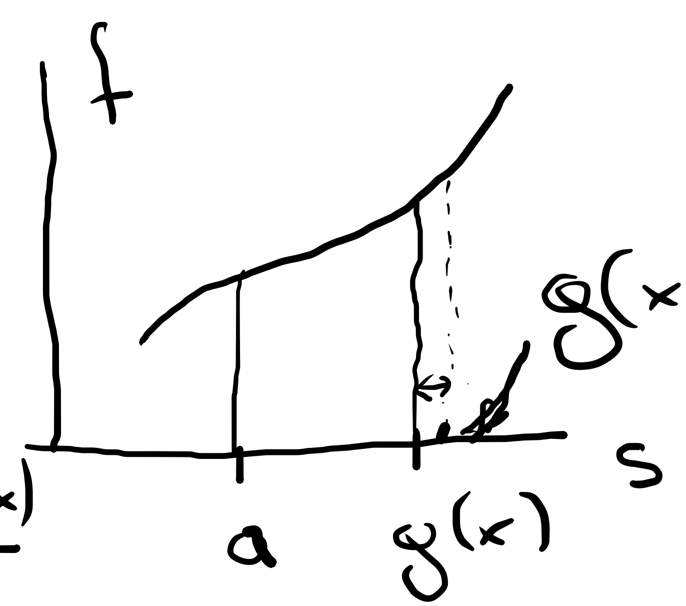
Τεχνικές ολοκλήρωσης

άσκ. 7. $f:[0,\infty)\to\mathbb{R}$ συνεχής

$$F(x)=\int_0^x x\, f(t)\,dt \quad \leftarrow$$

• $$F(x)=\int_a^{g(x)} f(s)\,ds$$

$$F'(x)=\lim_{\epsilon\to 0}\frac{F(x+\epsilon)-F(x)}{\epsilon}$$



$$g(x+\epsilon)\simeq g(x)+\epsilon g'(x)+O(\epsilon^2)$$

$$F'(x) = \lim_{\epsilon \to 0} \frac{\displaystyle\int_{g(x)}^{g(x)+\varepsilon g'(x) + \mathcal{O}(\varepsilon^2)} f(s)\,ds}{\varepsilon}$$

$$\simeq \lim_{\epsilon \to 0} \frac{\dfrac{f(\text{avw}) + f(\text{raiw})}{2}\left(\cancel{\varepsilon} g'(x) \cancel{+ \cdots}\right)}{\cancel{\varepsilon}}$$

$$= \lim_{\epsilon \to 0} \frac{f(g(x+\cancel{\varepsilon})) + f(g(x))}{2}\, g'(x) =$$

$$= f(g(x))\, g'(x)$$

$$\left(x \int_0^x f(t)\,dt \right)' = 1 \cdot \int_0^x f(t)\,dt$$

$$+ x \; f(x) \; x'$$

$$= \int_0^x f(t)\,dt + x f(x)$$

- $F(x) = \displaystyle\int_{g_1(x)}^{g_2(x)} f(x,s)\,ds$

$$F'(x) = f(x, g_2(x))\, g_2'(x) - f(x, g_1(x))\, g_1'(x) + \int_{g_1}^{g_2} \frac{\partial f}{\partial x}\,ds$$

αςκ. 8

$$F(x) = \int_1^x f\left(\frac{x}{t}\right) dt = \int_1^x x f\left(\frac{x}{t}\right) d\left(\frac{t}{x}\right)$$

$$\boxed{f'(1/y) \neq (f(1/y))'}$$

$$= \int_{"1"}^{"x"} x f\left(\frac{1}{y}\right) dy \qquad y = t/x$$

$$= x \int_{"1"}^{"x"} f(\xi)\left(-\frac{d\xi}{\xi^2}\right) \qquad \xi = 1/y \qquad y = 1/\xi$$

$$= -x \int_x^1 f(\xi)\, d\xi/\xi^2 = \qquad x = 1 \quad y = 1/x,\ \xi = x$$

$$x = x \quad y = 1,\ \xi = 1$$

$$= +x \int_1^x f(\xi)\, d\xi/\xi^2$$

$$F(x)' = 1 \int_1^x f(\xi)\, d\xi / \xi^2 + x \frac{f(x)}{x^2} x'$$

$$= \int_1^x f(\xi) \frac{d\xi}{\xi^2} + \frac{f(x)}{x} .$$

$f(s) = s$

$$F(x) = \int_1^x \frac{x}{t}\, dt = x \int_1^x \frac{dt}{t} = x \log x$$

$$F(x)' = \log x + 1$$

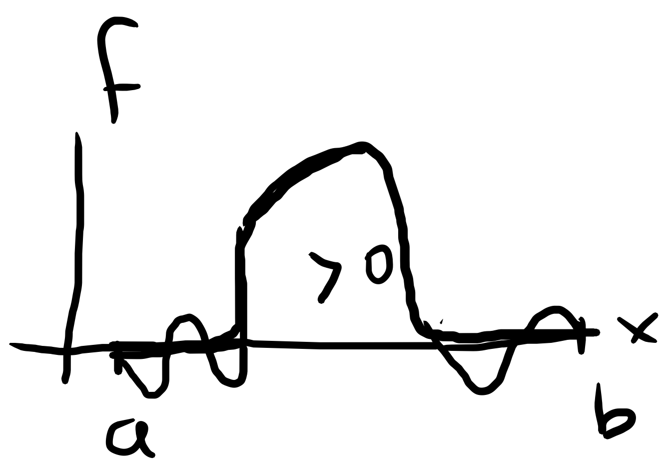
$$F(x)' = \int_1^x \xi \frac{d\xi}{\xi^2} + \frac{x}{x} = \log x + 1$$

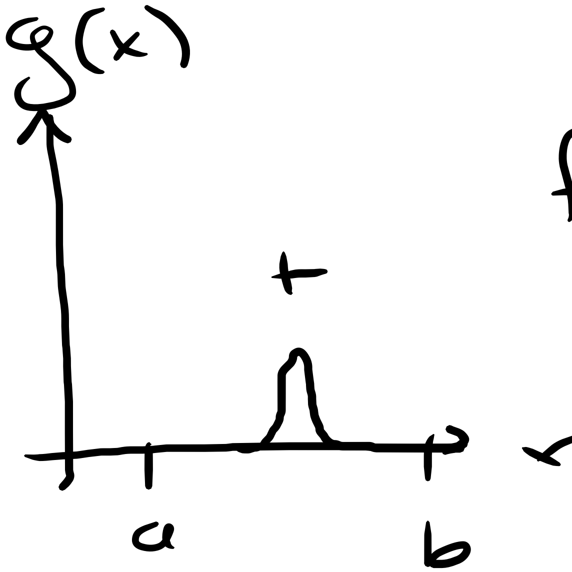
ασκ. 13

αν f: συνεχής στο $[a,b]$

$\forall g$: συνεχή στο $[a,b]$

ισχύει $\displaystyle\int_a^b f(x)g(x)\,dx = 0 \Rightarrow$





$f(x)=0$

$\longrightarrow \displaystyle\int_a^b f g\,dx \neq 0$

Διαλέγω $g(x)=f(x)$ $\qquad \displaystyle\int_a^b f^2(x)\,dx > 0$

$$\int_a^b f(x)g(x)\,dx \cong (x_2-x_1)f(x_1)g(x_1) + f(x_2)g(x_2)(x_3-x_2) + \cdots = \vec{f}\cdot\vec{g}$$

ασκ. 14.

(i) $\int_0^1 x^2 e^x \,dx = \int_0^1 x^2 (e^x)' \,dx = x^2 e^x \Big|_0^1 - \int_0^1 (x^2)' e^x \,dx$

$= e - 2\int_0^1 x (e^x)' \,dx = e - 2\Big[\underbrace{x e^x}_{e}\Big]_0^1 + 2\int_0^1 x' e^x \,dx$

$= e - 2e + 2\int_0^1 e^x \,dx = -e + 2e\Big|_0^1$

$= e - 2$

(ii) $\int_0^{\pi/2} x^2 \sin x\,dx = \int_0^{\pi/2} x^2 (-\cos x)'\,dx =$

$$= \underbrace{-x^2\cos x\Big|_0^{\pi/2}}_{0} + \int_0^{\pi/2} 2x\cos x\,dx = 2\int_0^{\pi/2} x\,(\sin x)'\,dx$$

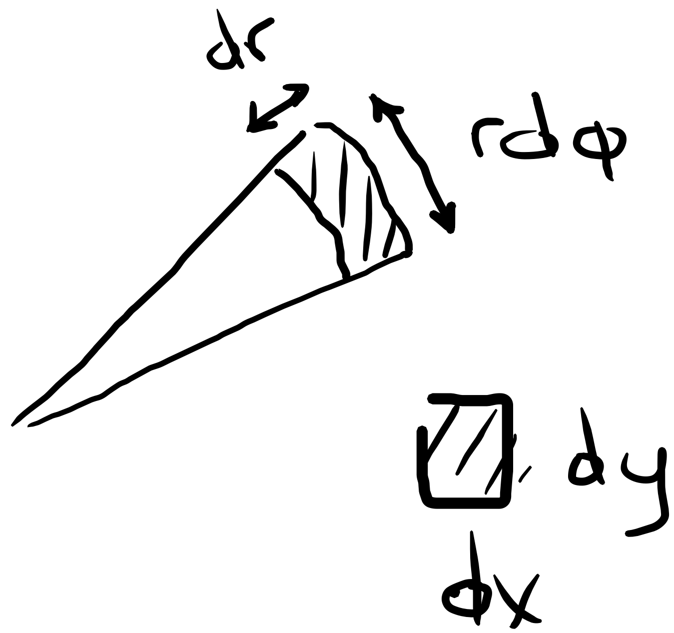
$$= \underbrace{2x\sin x\Big|_0^{\pi/2}}_{\pi} - 2\int_0^{\pi/2} 1\cdot\sin x\,dx$$

$$= \pi - 2\,(-\cos x)\Big|_0^{\pi/2} = \pi + 2(0-1) =$$

$$= \pi - 2$$

$$I = \int_{-\infty}^{+\infty} e^{-x^2}\,dx$$

$$I^2 = \int_{-\infty}^{+\infty} e^{-x^2}\,dx \int_{-\infty}^{+\infty} e^{-y^2}\,dy$$



$$= \iint_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$J\left(\frac{\partial(x,y)}{\partial(r,\varphi)}\right)$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\, r\,d\varphi\,dr$$

$$= 2\pi \int_0^{\infty} e^{-r^2} \underbrace{r\,dr}_{d\frac{r^2}{2}}$$

$$= 2\pi \frac{1}{2}\int_0^{\infty} e^{-\xi}\,d\xi = \pi \quad !$$

$$I = \sqrt{\pi} \quad !$$

αsk. 14iv

$$\int \cos(\underbrace{\ln x}_{y})dx$$

$\ln x = y$

$x = e^y$

$dx = e^y dy$

$$I = \int \cos y\, e^y dy = \cos y e^y - \int(-\sin y) e^y dy$$

$$= \cos y\, e^y + \sin y e^y - \underbrace{\int \cos y e^y dy}_{I}$$

$2I = \ldots\ldots \qquad (y = \ln x)$

ask. 15 (i)

$$\int_0^1 e^x \sin e^x \, dx = \int \sin e^x \, \underbrace{d(e^x)}_{(e^x)' dx}$$

$$= \int_1^e \sin y \, dy \qquad y = e^x$$

$$= -\cos y \Big|_1^e = \cos 1 - \cos e$$

15 (iii)

$$\int \frac{e^x}{e^{2x} + 2e^x + 1} dx = \int \frac{dy}{(y+1)^2} \qquad y = e^x$$

αςκ. 16 (iii)

$$\int \ln(\underbrace{\cos x}_{y})\tan x\,dx =$$

$y = \cos x$ $\quad \tan x\,dx = \dfrac{\overbrace{\sin x\,dx}}{\cos x} = \dfrac{d(-\cos x)}{\cos x}$

$$\int = -\int \ln y \frac{dy}{y} = -\int \ln y\, d(\ln y) = -(\ln y)^2/2$$

$$= -\frac{(\ln \cos x)^2}{2}$$

$\underbrace{f'(x)dx} = \underbrace{d(f(x))}$

$$f'(x) = \frac{d(f(x))}{dx}$$

ασκ. 16 (iv)

$$\int \frac{\ln(\ln x)}{x \ln x}\,dx = \int \frac{\ln y}{\cancel{e^y}\, y}\,\cancel{e^y}\,dy$$

$\ln x = y \quad x = e^y \quad dx = e^y dy$

$$= \int \ln y \,\frac{dy}{y}$$

$$= \int \ln y \; d(\ln y)$$

$$= (\ln y)^2/2$$

$$d(\ln x) = \underbrace{\frac{d(\ln x)}{d\cancel{x}}}\, d\cancel{x} = \frac{1}{x}\, dx$$

$$f'(x)\, dx = d(f(x))$$

AGK 17(i) $I = \int \sin^4 x \, dx = \int \sin^3 x \, (-\cos x)' \, dx$

$= -\sin^3 x \cos x + \int 3 \sin^2 x \cos x \cos x \, dx$

$= -\sin^3 x \cos x + 3 \int \sin^2 x \cos^2 x \, dx$

$+ 3 \int \sin^2 x \, (1 - \sin^2 x) \, dx$

$4I = -\sin^3 x \cos x + 3 \int \sin^2 x \, dx$

$$I = \int \sin^4 x \, dx = \int \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^4 dx$$

$$= \int \frac{e^{4ix} - 4e^{2ix} + 6 - 4e^{-2ix} + e^{-4ix}}{16 \cdot 1} dx$$

$$= \frac{1}{8} \int \cos 4x \, dx + \frac{1}{8}(-4) \int \cos 2x \, dx + \frac{6}{16} \int dx$$

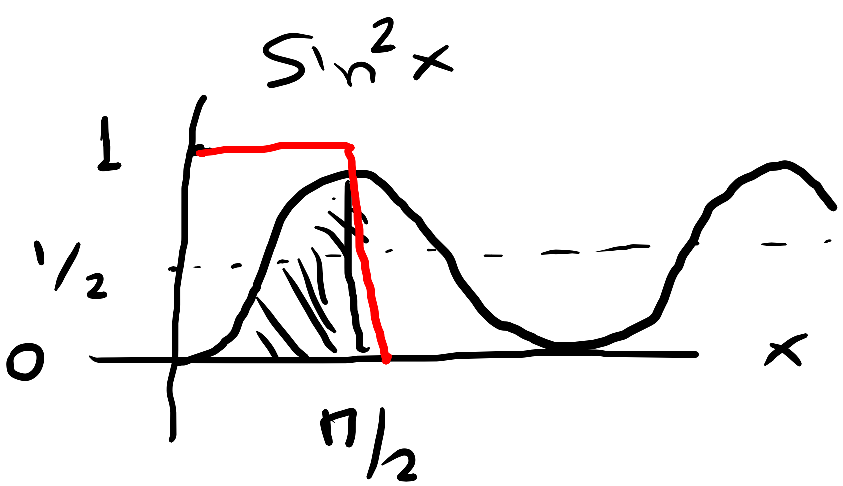
$$\int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} \left(\frac{1 - \cos 2x}{2} \right) dx$$

$$\cos 2x = 2\cos^2 x - 1$$

$$= \cos^2 x - \sin^2 x$$

$$= 1 - 2\sin^2 x$$

$$= \ldots\ldots$$



$$\int_0^{\pi/2} \sin^2 x \, dx = \frac{1}{2} \int_0^{\pi/2} dx = \frac{\pi}{4}$$

$$\underbrace{\langle \sin^2 x \rangle}_{1/2} \int_0^{\pi/2} dx = \pi/4$$

$$I = \int \frac{dx}{P_2(x)} = \int \frac{dx}{a(x-\rho_1)(x-\rho_2)} =$$

$$\frac{1}{(x-\rho_1)(x-\rho_2)} = \frac{A}{x-\rho_1} + \frac{B}{x-\rho_2} \qquad A, B$$

$$= \frac{(A+B)x - (\rho_2 A + \rho_1 B)}{(x-\rho_1)(x-\rho_2)}$$

$$A + B = 0$$

$$\rho_2 A + \rho_1 B = -1$$

$$B = -A \qquad (\rho_2 - \rho_1)A = 1$$

$$A = 1/\rho_2 - \rho_1$$

$$I = \frac{1}{a}\left[\frac{1}{\rho_2-\rho_1}\int \frac{dx}{x-\rho_1} - \frac{1}{\rho_2-\rho_1}\int \frac{dx}{x-\rho_2}\right]$$

$$= \frac{1}{a(\rho_2-\rho_1)} \log \frac{x-\rho_1}{x-\rho_2}$$

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{\underbrace{x^2+x+\frac{1}{4}}_{(x+1/2)^2} + \frac{3}{4}} =$$

$$\xi^2 = \frac{3}{4}\psi^2$$

$$\int \frac{dx}{(x+1/2)^2+3/4} \overset{\xi = x+1/2}{=} \int \frac{d\xi}{\xi^2+3/4} \overset{\psi = \sqrt{\frac{4}{3}}\,\xi}{=}$$

$$= \int \frac{\sqrt{\frac{3}{4}}\,d\psi}{\frac{3}{4}(\psi^2+1)} = \sqrt{\frac{4}{3}} \underbrace{\int \frac{d\psi}{\psi^2+1}}_{\tan^{-1}\psi}$$

$$\int \frac{dx}{(x-a)(x^2+x+1)} = \int dx \left[\frac{A}{x-a} + \frac{Bx+\Gamma}{x^2+x+1} \right]$$

$$[\ ] = \frac{A(x^2+x+1)+(Bx+\Gamma)(x-a)}{(x-a)(x^2+x+1)} =$$

$$\frac{x^2(A+B)+x(A-Ba+\Gamma)+A-\Gamma a}{(x-a)(x^2+x+1)}$$

$B=-A \qquad \Gamma=-A+Ba=-A(1+a) \qquad A-\Gamma a=A\left(1+a(1+a)\right)=1$

$B=-\frac{1}{1+a+a^2} \qquad \Gamma=\frac{-(1+a)}{1+a+a^2} \qquad A=\frac{1}{1+a+a^2}$

$$\underbrace{\frac{1}{1+a+a^2}}\int dx \left[\frac{1}{x-a} + \frac{-x-(1+a)}{x^2+x+1}\right]$$

$$\frac{-\left(x+1/2\right) - 1/2 - a}{\left(x+1/2\right)^2 + 3/4}$$

$$-\frac{\left(x+1/2\right)dx}{\left(x+1/2\right)^2+3/4} + \frac{\left(-1/2-a\right)dx}{\left(x+1/2\right)^2+3/4}$$

$$\int_{\alpha}^{\beta} \frac{dx}{\sqrt{x^2 + \lambda x + \gamma}} = \pi \; !$$

α, β pijes