Av $f(x) = F'(x)$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\left[f(x)\right]_a^b = f(b) - f(a)$$

$$f(x)\Big|_a^b =$$

14 (i) $\int_0^1 x^2 e^x\,dx = \int_0^1 x^2 (e^x)'\,dx = \ldots$

$$\int_a^b f(x) g'(x)\,dx = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x) g(x)\,dx$$

$$= f(b)\cdot g(b) - f(a)g(a)$$

$$\dots = \left[x^2 e^x\right]_0^1 - \int_0^1 (x^2)' e^x \, dx = \left[x^2 e^x\right]_0^1 - 2\int_0^1 x \cdot e^x \, dx$$

$$= \left[x^2 e^x\right]_0^1 - 2\int_0^1 x (e^x)' \, dx =$$

$$= \left[x^2 e^x\right]_0^1 - 2\left(\left[x e^x\right]_0^1 - \int_0^1 (x)' e^x \, dx\right)$$

$$= \left[x^2 e^x\right]_0^1 - 2\left[x e^x\right]_0^1 + 2\int_0^1 e^x \, dx =$$

$$= \underbrace{1^2 e^1}_{e} - \cancel{0^2 e^0} - 2\left(\underbrace{1 \cdot e^1}_{e} - \cancel{0 \cdot e^0}\right) + 2\left[e^x\right]_0^1 = -e + 2\left(e^1 - e^0\right)$$

$$= -e + 2e - 2 = e - 2$$

(ii) $\int_0^{\pi/2} x^2 \sin x \, dx = \int_0^{\pi/2} x^2 (-\cos x)' \, dx$

$$= \left[-x^2 \cos x\right]_0^{\pi/2} + \int_0^{\pi/2} 2x \cdot \cos x \, dx \qquad = (\sin x)'$$

$$= -x^2 \cos x\Big]_0^{\pi/2} + 2\left(\left[x \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx\right)$$

$$= -\left[x^2 \cos x\right]_0^{\pi/2} + 2\left[x \sin x\right]_0^{\pi/2} + 2\left[\cos x\right]_0^{\pi/2}$$

$$= \underbrace{-\left(\frac{\pi}{2}\right)^2 \cos\frac{\pi}{2}^{=0} + 0^2 \cos 0}_{=0} + 2\frac{\pi}{2} \sin\frac{\pi}{2}^{=1} - \cancel{2\,0 \sin 0} + 2\left(\cos\frac{\pi}{2}^{0} - \cos 0^{1}\right)$$

$$= \pi - 2$$

$$\circledast \quad I = \int_0^{\pi/2} \sin x \cdot e^x \, dx = \int_0^{\pi/2} \sin x \,(e^x)' \, dx$$

$$= \left(e^x \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} \cos x \cdot (e^x)' \, dx$$

$$= \left[e^x \sin x\right]_0^{\pi/2} - \left(\left[e^x \cos x\right]_0^{\pi/2} + \underbrace{\int_0^{\pi/2} \sin x \, e^x \, dx}_{= I} \right)$$

$$= \left[e^x \sin x\right]_0^{\pi/2} - \left[e^x \cos x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, e^x \, dx$$

$$I = \left[e^x \sin x\right]_0^{\pi/2} - \left[e^x \cos x\right]_0^{\pi/2} - I \Rightarrow 2I =$$

(iii) $$\int x \ln x \, dx = \int \ln x \cdot \left(\frac{x^2}{2}\right)' dx$$

$$\boxed{(\ln x)' = \frac{1}{x}}$$

$$= \frac{x^2}{2}\ln x - \int (\ln x)' \frac{x^2}{2} \, dx$$

$$= \frac{x^2}{2}\ln x - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx$$

$$\int x \, dx = \frac{x^2}{2}$$

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1}$$

$$\alpha \neq -1$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x \, dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\frac{x^2}{2} = \frac{x^2}{2}\ln x - \frac{x^2}{4}.$$

(iv) $\int \cos(\ln x)\, dx \overset{*}{=}$

$\boxed{u = \ln x}$ (αλλαγή μεταβλητής)

$du = (\ln x)'\, dx = \frac{1}{x} dx$

$x = e^u \rightsquigarrow (dx = x\,du = e^u du)$

$\overset{*}{=} \int \cos u \cdot e^u du = I$

$= \frac{1}{2} e^u (\cos u + \sin u) = \frac{x}{2} (\cos \ln x + \sin \ln x)$

$u = \ln x \quad x = e^u$

$$I = \int \cos u \,(e^u)' \,du =$$

$$= \cos u \cdot e^u + \int \sin u \cdot (e^u)' \,du$$

$$= \cos u \cdot e^u + \left(\sin u \cdot e^u - \underbrace{\int \cos u \cdot e^u \,du}_{} \right)$$

$$\Rightarrow I = e^u \cos u + e^u \sin u - I$$

$$\Rightarrow \boxed{I = \frac{1}{2} e^u (\cos u + \sin u)}$$

15 (i)

$$\int_0^1 e^x \sin(e^x)\,dx = \left|\begin{array}{lll} x & 0 \rightsquigarrow & 1 \\ u & e^0 \rightsquigarrow & e^1 \\ & \text{''}\ 1 & \text{''}\ e \end{array}\right.$$

$u = e^x,$

$du = (e^x)' dx = e^x dx$

$$= \int_1^e \sin(u)\,du = \left[-\cos u\right]_1^e$$

$$= -\cos e + \cos 1.$$

$f(x) = e^x \quad f'(x) = e^x$

$g(x) = -\cos x \rightsquigarrow g'(x) = \sin x$

$g'(f(x)) \cdot f'(x)$

$= (g \circ f)'(x) = \left(\sin(e^x)\right)'$

16 (i) $\int \frac{2x}{x^2+2x+2}\,dx = \int \frac{2x+2-2}{x^2+2x+2}\,dx$

$= \int \frac{2x+2}{x^2+2x+2}\,dx - 2\int \frac{dx}{x^2+2x+2}$

$\int \frac{2x+2}{x^2+2x+2}\,dx = \int \frac{du}{u} = \ln|u|$

$u = x^2+2x+2$
$du = (2x+2)\,dx$

$= \ln|x^2+2x+2|.$

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1} = \int \frac{du}{1+u^2} =$$

$u = x+1$

$du = dx$

$$\frac{1}{x^2+2x+2} = \frac{1}{\underbrace{x^2+2x+1}+1} = \frac{1}{(x+1)^2+1}$$

όχι

$$= \arctan u$$

$$= \arctan(x+1)$$

αν $\dfrac{1}{x^2+bx+c} = \dfrac{1}{(x-c_1)(x-c_2)} = \dfrac{A}{x-c_1} + \dfrac{B}{x-c_2}$

Βρίσκουμε A, B ώστε να ισχύει

$$\int \frac{dx}{1+x^2} = \arctan x$$

16 (iii) $\int \ln(\cos x)\tan x\,dx$

$= \int \ln(\cos x)\frac{\sin x}{\cos x}\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$= -\int \frac{\ln u}{u}\,du = -\int v\,dv = -\frac{v^2}{2} = -\frac{(\ln u)^2}{2}$

$2\ln u \cdot (\ln u)'$

$= [(\ln u)^2]'$

$v = \ln u$

$dv = \frac{1}{u}\,du$

$= -\frac{(\ln\cos x)^2}{2}$

$$\Rightarrow \int \frac{2x\,dx}{x^2+2x+2} = \ln|x^2+2x+2| - 2\arctan(x+1)$$

17 (i) $\int \sin^4 x \, dx =$

$= \int \sin^2 x \sin^2 x \, dx$

$\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$= \int \sin^2 x (1 - \cos^2 x) \, dx$

$= \int \sin^2 x \, dx - \int \sin^2 x \cos^2 x \, dx$

$\left(\frac{1}{2} \sin 2x \right)^2$

$$1.\int \sin^4(x)\,dx = \int \sin^3(x)\cdot \sin(x)\,dx \qquad \sin(x) = (-\cos x)'$$

$$= -\sin^3(x)\cos x + \int 3\sin^2(x)\cdot \underbrace{\cos^2 x}_{1-\sin^2(x)}\,dx$$

$$= -\sin^3(x)\cos x + 3\int \sin^2 x\,dx - 3\int \sin^4(x)\,dx$$

$$\Rightarrow 4\int \sin^4(x)\,dx = -\sin^3(x)\cos(x) + 3\int \sin^2 x\,dx$$

$$= -\sin^3 x\cos x + 3\int \frac{1-\cos 2x}{2}\,dx = \ldots$$

Γενικό:

$$\int \sin^{2k}(x)\,dx =$$

$$= \frac{-\sin^{2k-1}(x)\cos^{2}x + (2k-1)\int \sin^{2k-2}(x)\,dx}{2k}$$

⊛ $\int \sin^5 x \cdot \cos^3 x \, dx =$ περιττός

$$= \int \sin^5 x \cdot \underbrace{\cos^2 x}_{1-\sin^2 x} \cdot \underline{\cos x \, dx}$$

$$= \int \underline{\sin^5 x \cdot (1-\sin^2 x)} \cos x \, dx$$

$u = \sin x \qquad du = \cos x \, dx$

$$= \int u^5 (1-u^2) \, du = \frac{u^6}{6} - \frac{u^8}{8} = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8}.$$

17(ii) ĉ fuıla

$$\int \cos^5 x\,dx = \int \cos^4 x \cdot \cos x\,dx$$

$$= \int (1-\sin^2 x)^2 \cos x\,dx$$

$$= \int \underbrace{(1-u^2)^2}_{\textcolor{red}{1+u^4-2u^2}} du = u + \frac{u^5}{5} - \frac{2u^3}{3}.$$

$u = \sin x$

$du = \cos x\,dx$

$$= \sin x + \frac{\sin^5 x}{5} - \frac{2\sin^3 x}{3}$$

8 (iv)

$$\int \frac{1}{\sqrt{\underbrace{2x - x^2}}}\,dx = \int \frac{dx}{\sqrt{1 \underbrace{-1 + 2x - x^2}}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$$

$$\left[\underbrace{-1 + 2x - x^2} = -(x^2 - 2x + 1) = -(x-1)^2 \right]$$

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u$$

Θέτουμε $u = x-1$

$du = dx$

$$= \int \frac{du}{\sqrt{1-u^2}} = \arcsin u = \arcsin(x-1)$$

19 (ii) $\int \tan^2 x \, dx =$

$$= \int \frac{\sin^2 x}{\cos^2 x} \, dx = \int \frac{1-\cos^2 x}{\cos^2 x} \, dx$$

$$= \int \frac{1}{\cos^2 x} \, dx - \int dx = \tan x - x$$

$= \tan x.$

19 (iii)

Προσοχή $\int \frac{1}{x^2+1} \neq \ln(x^2+1)$

$= \arctan x$

$$\int \frac{x+2}{x(x+1)}\,dx = \int \frac{2}{x}\,dx - \int \frac{1}{x+1}\,dx$$

$u \quad du = dx$

$$= 2\ln|x| - \ln|x+1|$$

A, B :

$$\frac{x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)}$$

$x+2$

$(A+B)x + A = x+2$

$A = 2$

$A+B = 1 \rightarrow B = 1-2 = -1$

3)(i) $$\int \frac{1}{\sin x}\,dx = \int \frac{\sin x}{\sin^2 x}\,dx$$

$$= \int \frac{\sin x\,dx}{1-\cos^2 x} \underset{\substack{u=\cos x \\ du=-\sin x\,dx}}{=} -\int \frac{du}{1-u^2}$$

$$\frac{1}{1-u^2} = \frac{\overset{\frac{1}{2}}{A}}{1-u} + \frac{\overset{\frac{1}{2}}{B}}{1+u} = \frac{A(1+u)+B(1-u)}{\qquad} =$$

$$1-u^2 = (1-u)(1+u)$$

$$= \underbrace{(A-B)u}_{0=} + \underbrace{A+B}_{=1}$$

$$= -\left(\frac{1}{2}\int \frac{du}{1-u} + \frac{1}{2}\int \frac{du}{1+u} \right)$$

$= v$

$dv = -du$

$$= \frac{1}{2}\log|1-u| - \frac{1}{2}\log|1+u|$$

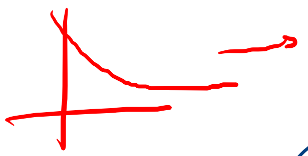
$$= \frac{1}{2}\log|1-\cos x| - \frac{1}{2}\log|1+\cos x|$$

$$\cos x = 2\cos^2\frac{x}{2} - 1 \Rightarrow \underbrace{\cos x + 1}_{} = 2\cos^2\frac{x}{2}$$

$$\cos x = 1 - 2\sin^2\frac{x}{2} \Rightarrow \underbrace{1 - \cos x}_{} = 2\sin^2\frac{x}{2}$$

$$= \ln\left|\sin\frac{x}{2}\right| - \ln\left|\cos\frac{x}{2}\right|$$

21) $\int_0^{+\infty} x \, e^{-x^2} \, dx$ $\quad x > 0, \quad e^{-x^2} \to 0$



$$= \lim_{m \to +\infty} \int_0^m x \, e^{-x^2} \, dx = \lim_{m \to \infty} -\frac{1}{2}\left(e^{-m^2} - 1\right) = \frac{1}{2}$$

$e^{-m^2} \to 0$

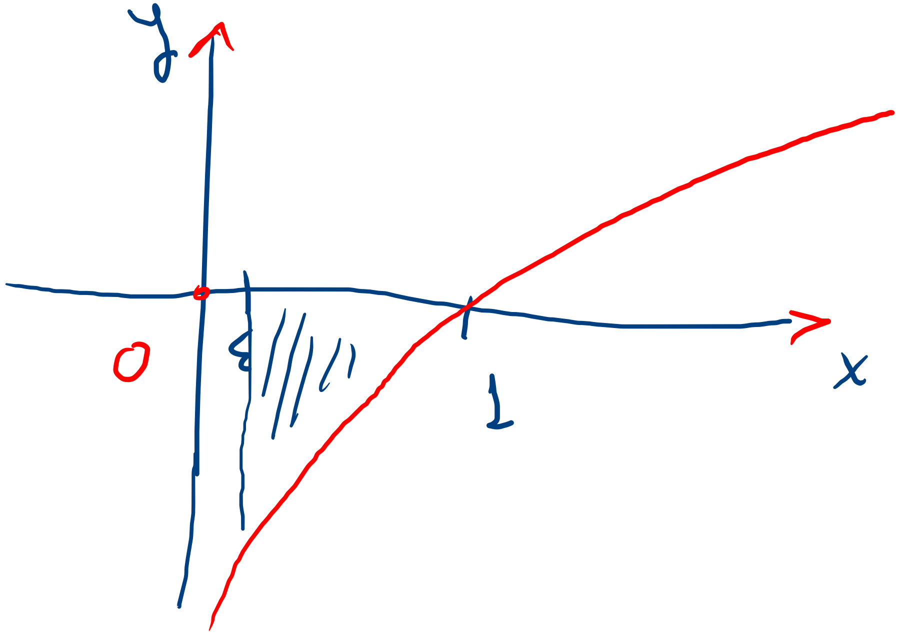
$$\int_0^m x \, e^{-x^2} \, dx = \frac{1}{2} \int_0^{m^2} e^{-u} \, du = \frac{1}{2} \int_0^{m^2} e^{-u} \, du$$

$u = x^2$

$du = 2x \, dx$

$$= \dots = -\frac{1}{2}\left(e^{-m^2} - 1\right)$$

(ii) $$\int_0^1 \ln x \, dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \ln x \, dx = -1$$



$$\int_{\varepsilon}^{1} \underline{\ln x}\, dx = \int_{\varepsilon}^{1} \ln x \cdot (x)'\, dx =$$

$$= \Big[x \ln x\Big]_{\varepsilon}^{1} - \int_{\varepsilon}^{1} \overbrace{\frac{1}{x} \cdot x}^{=1}\, dx$$

$$= \Big[x \ln x\Big]_{\varepsilon}^{1} - \Big[x\Big]_{\varepsilon}^{1} = \overset{0}{\cancel{1 \ln 1}} - \varepsilon \ln \varepsilon - (1 - \varepsilon)$$

$$= -\overset{0}{\varepsilon}\, \overset{-\infty}{\ln \varepsilon} - 1 \overset{0}{+\varepsilon} \xrightarrow[\varepsilon \to 0]{} -1$$

$$\lim_{\varepsilon\to 0^+} \varepsilon \ln \varepsilon = \lim_{\varepsilon\to 0^+} \frac{\overset{\to -\infty}{\ln \varepsilon}}{\underset{\to +\infty}{1/\varepsilon}}$$

$$= \lim_{\varepsilon\to 0^+} \frac{1/\varepsilon}{-1/\varepsilon^2} = -\lim_{\varepsilon\to 0^+} \underbrace{\frac{\varepsilon^2}{\varepsilon}}_{=\varepsilon} \Longrightarrow 0.$$

7

f [illegible]

$$F(x) = x\int_0^x f(t)\,dt$$

$$F'(x) = \int_0^x f(t)\,dt + x\,f(x)$$

6

$$\int_0^x t\,g(t)\,dt = x + x^2$$

$g : [0,b] \to \mathbb{R}$

[illegible] on $[0,b]$

(i) C, $x\,g(x) = 1 + 2x \rightarrow g(x) = \frac{1+2x}{x}$