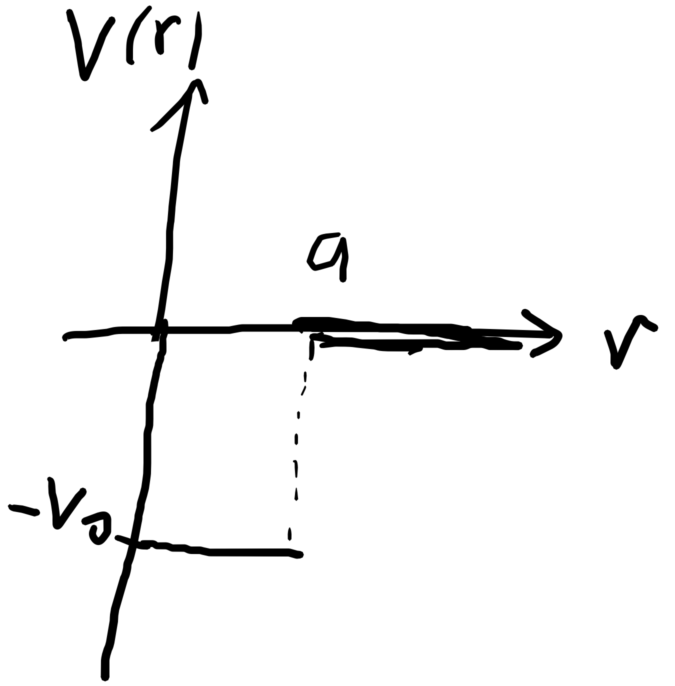
$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \frac{p_f}{p_i} \left|\tilde{V}^{B}(\vec{q})\right|^2$$

$$\tilde{V}^{Born}(\vec{q}) \equiv \int e^{i\vec{q}\cdot\vec{r}}\, V(\vec{r})\, d^3\vec{r} \quad , \quad \vec{q} \equiv (\vec{p}_f - \vec{p}_i)/\hbar$$

κεντρικό δυναμικό $V(\vec{r}) = V(r)$

$$\tilde{V}^{B}(\vec{q}) = \frac{4\pi}{q} \int_0^\infty dr\; r\, V(r)\, \sin(qr)$$

(1) $$V(r) = \begin{cases} -V_0 \;, & r \leq a \\ 0 \;, & r > a \end{cases}$$



$\leftarrow qa \ll 1$

$\Rightarrow q \ll \frac{1}{a}$

$$\tilde{V}^B(\vec{q}) = -\frac{4\pi V_0}{q}\int_0^a dr\, r \sin(\underbrace{qr}_{x})$$

$$= -\frac{4\pi V_0}{q^3}\int_0^{aq} dx\; x\sin x = -\frac{4\pi V_0}{q^3}\left[\sin(qa) - qa\cos(qa)\right]$$

Στο όριο χαμηλών ενεργειών $qa \ll 1$

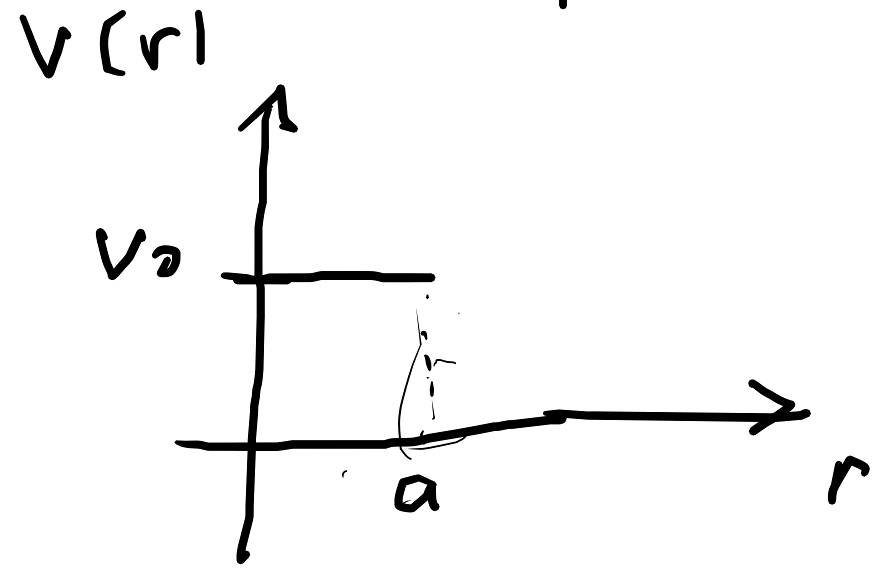
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots$$

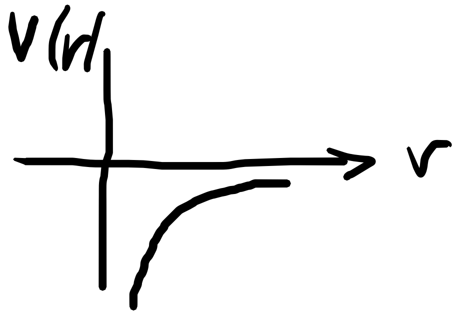
$$\tilde{V}^B(\vec{q}) \xrightarrow{qa \ll 1} -\frac{4\pi V_0}{q^3}\left[qa - \frac{1}{3!}(qa)^3 - qa + \frac{1}{2!}(qa)^3\right] = -\frac{4\pi}{3}V_0 a^3$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{9}\left(\frac{2mV_0 a^2}{\hbar^2}\right)^2 a^2 \qquad (\text{αν } qa \ll 1)$$

Το ίδιο αποτέλεσμα παίρνουμε και για το δυναμικό



② Δυναμικό Coulomb



$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0}\,\frac{1}{r} \longleftrightarrow V(r) = -\frac{Ze^2}{4\pi\epsilon_0}\,\frac{e^{-\mu r}}{r} \;\cdots\; \tilde{V}^B(q)\ (\mu=0)$$

$$\tilde{V}^B(\vec{q}) = -\frac{Ze^2}{4\pi\epsilon_0}\,\frac{4\pi}{q}\int_0^\infty dr\,\sin(qr) \quad \text{αποκλίνει}$$

αλλά το $\tilde{V}^B(\vec{q}) = -\frac{Ze^2}{4\pi\epsilon_0}\,\frac{4\pi}{q}\int_0^\infty dr\,\sin(qr)\,e^{-\mu r}$ αν $(\mu \to 0)$

$$\sin(qr) = \frac{1}{2i}\left(e^{iqr} - e^{-iqr}\right)$$

$$\tilde{V}^B(\vec{q}) = -\frac{Ze^2}{\epsilon_0} \underbrace{\frac{1}{q}\,\frac{1}{2i}\int_0^\infty dr \left\{ e^{i(q+i\mu)r} - e^{-i(q-i\mu)r} \right\}}_{I}$$

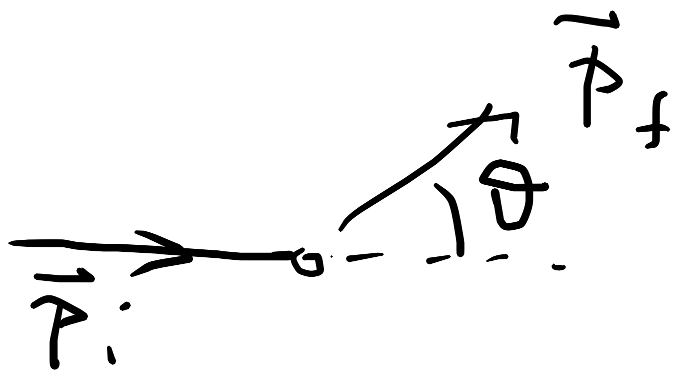
$$I = \frac{1}{2iq}\left[\frac{1}{i(q+i\mu)}\, e^{i(q+i\mu)r}\Big|_0^\infty + \frac{1}{i(q-i\mu)}\, e^{-i(q-i\mu)r}\Big|_0^\infty \right]$$

στο $r \to \infty$ $\left\{ e^{i\infty} \right.$ αρμονικώς φραγμένοι $\left. e^{-r} \right\} \to 0$

$$= \frac{1}{2iq}\left(-\frac{1}{i(q+i\mu)} - \frac{1}{i(q-i\mu)} \right) = \frac{1}{q^2+\mu^2} \xrightarrow[\mu \to 0]{} \frac{1}{q^2}$$

$$\tilde{V}^{R}(\vec{q}) = -\frac{Ze^2}{\epsilon_0}\,\frac{1}{q^2}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left(\frac{Ze^2}{\epsilon_0}\right)^2 \frac{1}{q^4}$$



$$\hbar^2 q^2 = |\vec{p}_f - \vec{p}_i|^2 = p_f^2 + p_i^2 - 2p_f p_i \cos\theta$$

Ελαστική σκέδαση $\quad p_i = p_f \Rightarrow \hbar^2 q^2 = 2p^2(1-\cos\theta)$

$$= 4p^2 \sin^2(\theta/2)$$

Άρα η Rutherford x-section με $p = mv$

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z e^2}{8\pi \epsilon_0 m v} \right)^2 \frac{1}{\sin^4 \theta/2}$$

Φορμαλισμός Θεωρίας σκέδασης

Εξίσωση Lippman-Schwinger

Έστω $H = H_0 + V$

$$H|\psi\rangle = E|\psi\rangle$$

$$H_0|\psi_0\rangle = E|\psi_0\rangle$$

$$|\psi_0\rangle = |\psi_{0\vec{k}}\rangle \quad , \quad E = \frac{\hbar^2 k^2}{2m}$$

$$\langle \vec{r}|\psi_0\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}} = \frac{1}{(2\pi)^{3/2}} e^{i\vec{p}\cdot\vec{r}/\hbar}$$

$$|\psi_0\rangle \xrightarrow{?} |\psi\rangle$$

$$H|\phi\rangle = E|\phi\rangle$$

$$\rightarrow (E - H_0)|\phi\rangle = V|\phi\rangle$$

ή

$$|\phi\rangle = \frac{1}{E-H_0} V|\phi\rangle \quad , \quad G_0 \equiv \frac{1}{E-H_0} \Rightarrow G_0(E-H_0) = 1$$

$$|\phi\rangle = |\phi_0\rangle + G_0 V|\phi\rangle \quad (1)$$

αν πολλαπλ. $(E-H_0)$ ταυτότητα.

Η (1) δέχεται διατ. λύσεις.

$$|\phi\rangle^{(0)} = |\phi_0\rangle \quad , \quad |\phi\rangle^{(1)} = |\phi_0\rangle + G_0 V|\phi_0\rangle$$

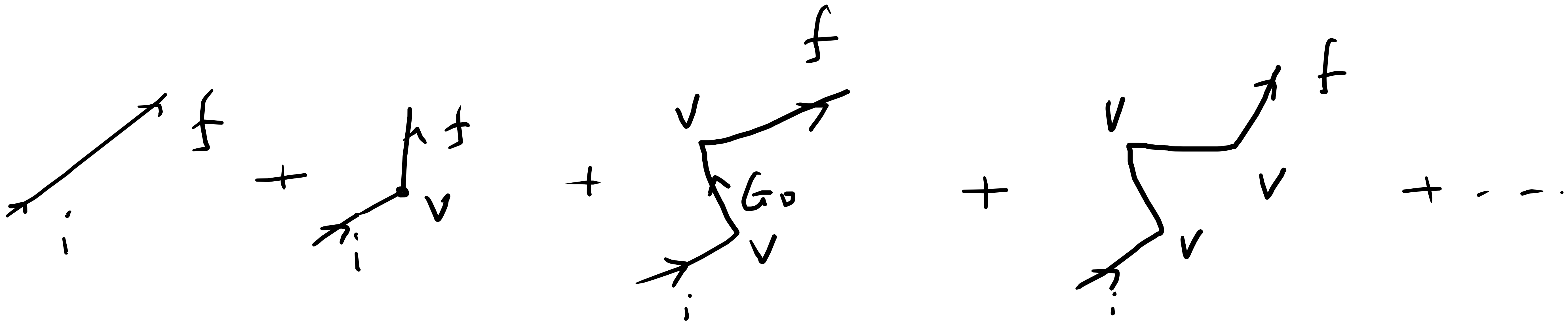
Born

$$|\phi\rangle^{(2)} = |\phi_0\rangle + G_0 V|\phi\rangle^{(1)} = |\phi_0\rangle + G_0 V|\phi_0\rangle + G_0 V G_0 V|\phi_0\rangle$$

$$\cdots$$

Γενικά

$$|\psi\rangle = |\psi_0\rangle + G_0 V|\psi\rangle + G_0 V G_0 V|\psi\rangle + G_0 V G_0 V G_0 V|\psi_0\rangle + \dots \quad (2)$$



$$G_0 = \frac{1}{E - H_0}\,, \qquad G = \frac{1}{E - H} = \frac{1}{E - H_0 - V}$$

Θα δείξουμε ότι $|\psi\rangle = |\phi_0\rangle + GV|\phi_0\rangle$

$$H|\psi\rangle = E|\psi\rangle \to (H_0+V)|\psi\rangle = E|\psi\rangle \to (E-H_0-V)|\psi\rangle = 0$$

$$(E-H_0-V)|\psi\rangle = -V|\phi_0\rangle + V|\phi_0\rangle$$

$$= (E-H_0)|\phi_0\rangle - V|\phi_0\rangle + V|\phi_0\rangle \to$$

$$(E-H_0-V)|\psi\rangle = (E-H_0-V)|\phi_0\rangle + V|\phi_0\rangle \qquad G = \frac{1}{E-H_0-V}$$

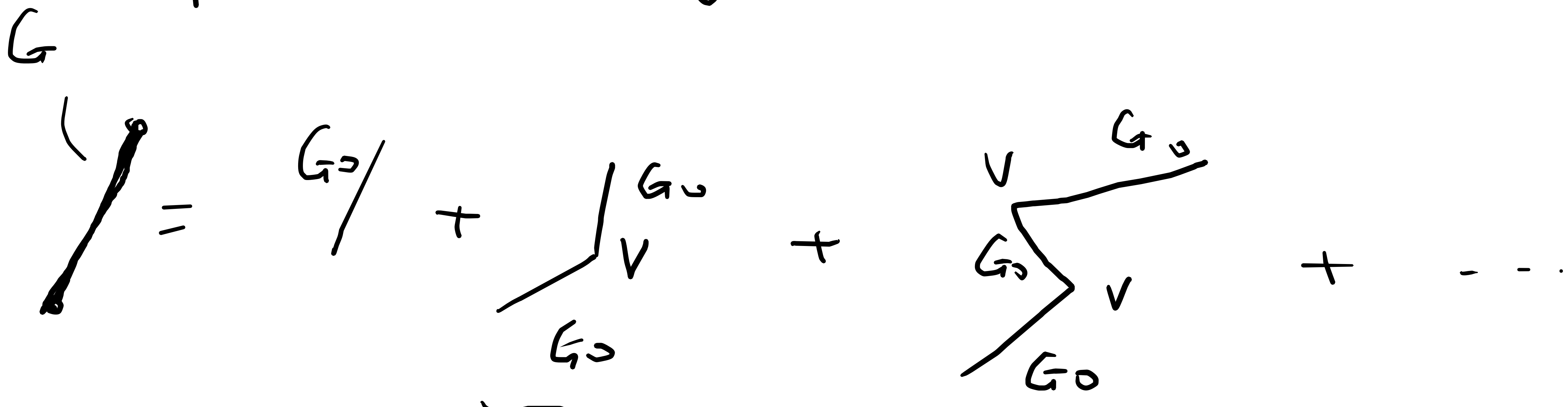
$$\Rightarrow \boxed{|\psi\rangle = |\phi_0\rangle + GV|\phi_0\rangle} \quad (3)$$

ισοδ. $|\psi\rangle = |\phi_0\rangle + G_0V|\psi\rangle$

$$\to G \leftrightarrow G_0$$

(2) & (3) $$\boxed{G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \cdots}$$

Σειρά Born για το G



T-matrix

$$G = G_0 + G_0 V \left(\underbrace{G_0 + G_0 V G_0 + \cdots}_{G} \right)$$

$\rightarrow \boxed{G = G_0 + G_0 V G}$ ή $\boxed{G = G_0 + G V G_0}$

Αγνή $\quad GVG_0 = G_0 V G$

διοτί $\quad A^{-1} - B^{-1} = B^{-1}(B-A)A^{-1}$

για $\quad A = G \;,\; B = G_0 \longrightarrow \quad G^{-1} - G_0^{-1} = G_0^{-1}(G_0 - G)G^{-1}$

και $\quad A = G_0 \;,\; B = G \longrightarrow \quad G_0^{-1} - G^{-1} = G^{-1}(G - G_0)G_0^{-1}$

$$G^{-1} = E - H \;,\quad G_0^{-1} = E - H_0 \;,\quad H = H_0 + V$$

$$\therefore \quad V = G^{-1}(G - G_0)G_0^{-1} = G_0^{-1}(G - G_0)G^{-1}$$

$$\therefore \quad \boxed{GVG_0 = G_0 V G}$$

▷ **Πίνακας T (T-matrix)**

ή Πίνακας σκέδασης (Scattering matrix)

$$|\psi\rangle = \underbrace{|\psi_0\rangle}_{\text{μέρος χωρίς σκέδαση}} + G_0 T|\psi_0\rangle \quad \text{(ορισμός του T-matrix)} \quad (4)$$

$$|\psi\rangle = |\psi_0\rangle + G_0 V|\psi\rangle$$

Από την (1) & (4) $\Rightarrow$ $\boxed{T|\psi_0\rangle = V|\psi\rangle}$ (5)

Χρησιμοποιώντας $|\psi\rangle = |\psi_0\rangle + G_0 V|\psi_0\rangle + G_0 V G_0 V|\psi_0\rangle + \dots$

(5) $\rightarrow$ $$\boxed{T = V + V G_0 V + V G_0 V G_0 V + \dots}$$

σειρά Born για τον T-matrix

$$T = V + V\Big(\underbrace{G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \dots}_{G}\Big)V$$

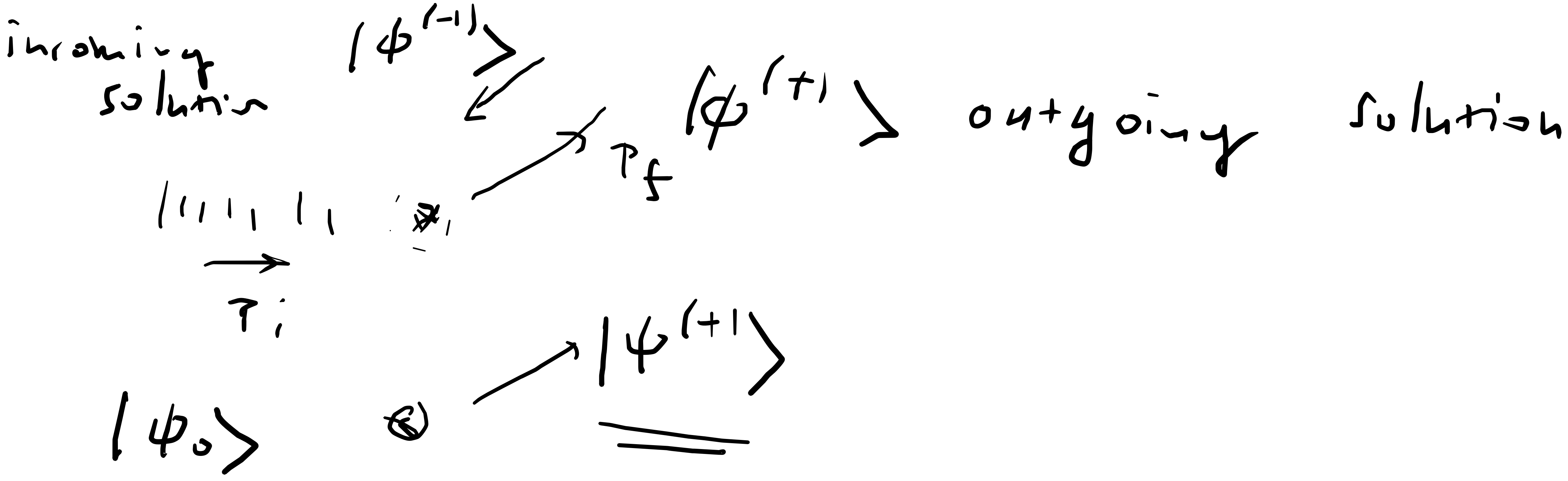
Άρα $\quad T = V + V G V \longrightarrow \boxed{T = V + V G_0 T}$

$GV = G_0 T$

Διατάξ. για να βρούμε το T

Υπολογισμός του $G_0(\vec{r},\vec{r}\,')$

Lippman - Schwinger $\quad |\phi\rangle = |\phi_0\rangle + \dfrac{1}{E - H_0} V|\phi\rangle$



Αναζητούνται εξερχόμενες λύσεις από την

$$|\phi^{(\pm)}\rangle = |\phi_0\rangle + \frac{1}{E - H_0 \pm i\epsilon} V|\phi^{(\pm)}\rangle$$

$$|\psi^{(+)}\rangle = |\phi_0\rangle + \frac{1}{E-H_0+i\epsilon} V|\psi^{(+)}\rangle$$

γενικά $G_0(E) \equiv \dfrac{1}{E-H_0}$

$$\downarrow \quad G_0(E+i\epsilon) \equiv G_0^{(+)}$$

$|\vec{r}''\rangle\langle\vec{r}''|$

$$|\psi^{(+)}\rangle = |\phi_0\rangle + G_0^{(+)} V |\psi^{(+)}\rangle \quad \text{(Εξ. Lippmann-Schwinger)}$$

$|\vec{r}'\rangle\langle\vec{r}'|$

Στην αναπαράσταση θέσης: $\langle\vec{r}|$

$$\langle\vec{r}|\psi^{(+)}\rangle = \langle\vec{r}|\phi_0\rangle + \int d^3\vec{r}' \int d^3\vec{r}'' \langle\vec{r}|G_0^{(+)}|\vec{r}'\rangle\langle\vec{r}'|V|\vec{r}''\rangle$$

$$\langle\vec{r}''|\psi^{(+)}\rangle$$

Υποθέτοντας ότι το $V(r)$ είναι τοπικό

$$\langle \vec{r}\,' | V | \vec{r}\,'' \rangle = V(r')\,\delta^{(3)}(\vec{r}\,' - \vec{r}\,'')$$

$$\langle \vec{r} | \phi^{(+)} \rangle = \phi^{(+)}(\vec{r})$$

$$\langle \vec{r} | \phi_0 \rangle = \phi_0(\vec{r}) \quad \& \quad \langle \vec{r} | G_0^{(+)} | \vec{r}\,' \rangle = G_0^{(+)}(\vec{r}, \vec{r}\,')$$

$$\text{L-S} \Rightarrow \boxed{\phi^{(+)}(\vec{r}) = \phi_0(\vec{r}) + \int d^3\vec{r}\,' \, G_0^{(+)}(\vec{r}, \vec{r}\,')\, V(r')\, \phi^{(+)}(\vec{r}\,')}$$

αντίστοιχα για τα εισερχόμενα $\phi^{(-)}(\vec{r})$

• Υπολογισμός του $G_0^{(+)}(\vec{r},\vec{r}\,')$

$$G_0^{(+)}(\vec{r},\vec{r}\,') = \langle \vec{r}\,| \frac{1}{E-H_0+i\epsilon} |\vec{r}\,'\rangle$$

$$|\phi_0(\vec{k}\,')\rangle\langle\phi_0(\vec{k}\,')| \qquad |\phi_0(\vec{k}\,'')\rangle\langle\phi_0(\vec{k}\,'')|$$

$$E = \frac{\hbar^2 k^2}{2m} \qquad H_0|\phi_0(\vec{k})\rangle = \frac{\hbar^2 k^2}{2m}|\phi_0(\vec{k})\rangle$$

$$\langle \vec{r}\,|\phi_0^{\vec{k}}\rangle = \phi_{0\vec{k}}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}$$

$$|\phi_0(\vec{k})\rangle$$

$$G_0^{(+)}(\vec{r},\vec{r}\,') = \int d^3\vec{k}\,' \int d^3\vec{k}\,'' \langle r|\phi_0(\vec{k}\,')\rangle\langle\phi_0(\vec{k}\,')| \frac{1}{E-H_0+i\epsilon} |\phi_0(\vec{k}\,'')\rangle\langle\phi_0(\vec{k}\,'')|\vec{r}\,'\rangle$$

$$= \int d^3\vec{k}\,' \int d^3\vec{k}\,'' \frac{e^{i\vec{k}\,'\cdot r}}{(2\pi)^{3/2}} \frac{\delta^3(\vec{k}\,'-\vec{k}\,'')}{\frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 k''^2}{2m} + i\epsilon} \frac{e^{-i\vec{k}\,''\cdot\vec{r}\,'}}{(2\pi)^{3/2}}$$

$$\Rightarrow G_0^{(+)}(\vec{r},\vec{r}\,') = \frac{1}{(2\pi)^3}\int d^3\vec{k}\,' \frac{e^{i\vec{k}'\cdot(\vec{r}-\vec{r}\,')}}{\frac{\hbar^2}{2m}(k^2-k'^2)+i\epsilon}$$

$$n' \quad \boxed{G_0^{(+)}(\vec{r},\vec{r}\,') = \frac{1}{(2\pi)^3}\frac{2m}{\hbar^2}\underbrace{\int d^3\vec{k}\,' \frac{e^{i\vec{k}'\cdot(\vec{r}-\vec{r}\,')}}{k^2-k'^2+i\epsilon'}}_{I}}$$

$$\epsilon' = \epsilon\frac{2m}{\hbar^2}$$

Σε σφαιρικές συντ.

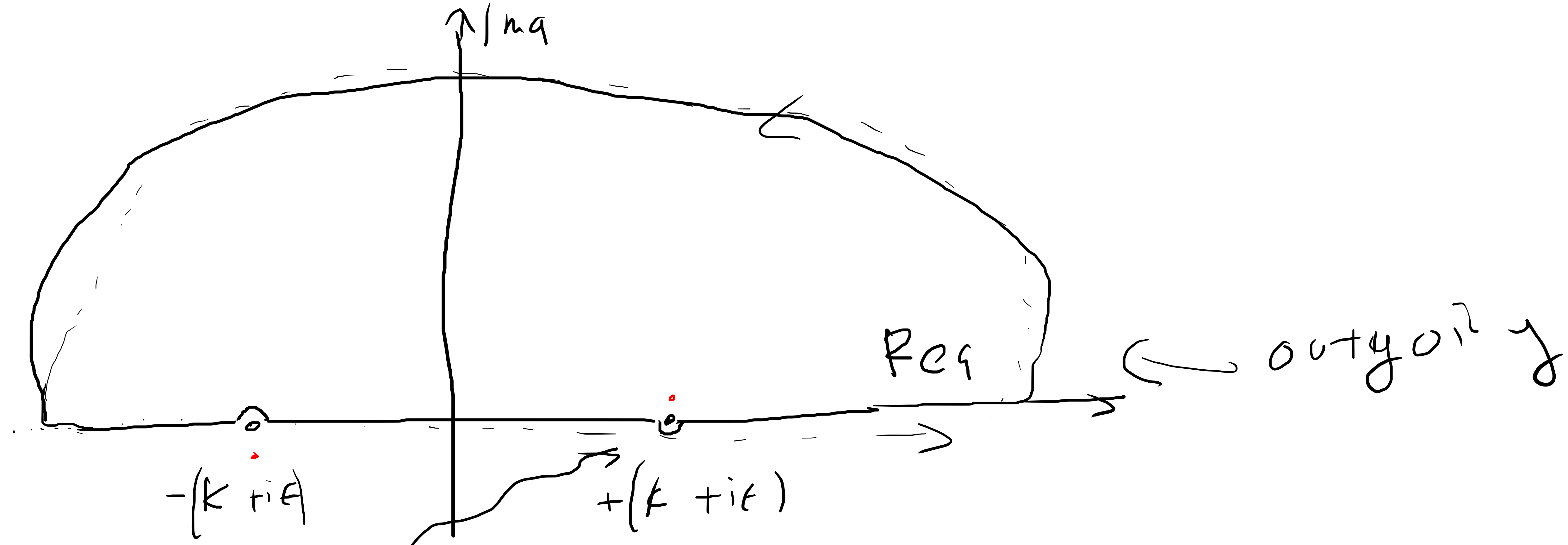
$$I = \int d^3\vec{q}\, \frac{e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')}}{k^2-q^2+i\epsilon} \quad , \quad d^3\vec{q} = q^2 dq \, \sin\vartheta\, d\vartheta\, d\varphi$$

$$= \int_0^{\infty} \frac{q^2\, dq}{k^2-q^2+i\epsilon} \underbrace{\int_0^{\pi} e^{iq|\vec{r}-\vec{r}\,'|\cos\vartheta} \sin\vartheta\, d\vartheta}_{\frac{1}{iq|\vec{r}-\vec{r}\,'|}\left(e^{iq|\vec{r}-\vec{r}\,'|} - e^{-iq|\vec{r}-\vec{r}\,'|}\right)} \underbrace{\int_0^{2\pi} d\varphi}_{\to 2\pi}$$

$$\downarrow \; q \to -q$$

$$I = \frac{2\pi}{i|\vec{r}-\vec{r}\,'|} \int_{-\infty}^{+\infty} \frac{q}{k^2-q^2+i\epsilon}\, e^{iq|\vec{r}-\vec{r}\,'|} = -\frac{2\pi}{i|\vec{r}-\vec{r}\,'|} \int_{-\infty}^{+\infty} \frac{q}{q^2-k^2-i\epsilon}\, e^{iq|\vec{r}-\vec{r}\,'|}$$

$$q^2 - k^2 - i\epsilon \underset{\epsilon^2 \approx 0}{\simeq} q^2 - (k+i\epsilon)^2 = [q-(k+i\epsilon)][q+(k+i\epsilon)]$$



$$I = 2\pi i \, \mathrm{Re}(q=k) = -2\pi^2 \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

$$\Rightarrow \boxed{G_0^{(+)}(\vec{r},\vec{r}') = -\frac{m}{2\pi\hbar^2} \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}}$$