38. A STUDY OF THE DIFFUSION EQUATION WITH INCREASE IN THE AMOUNT OF SUBSTANCE, AND ITS APPLICATION TO A BIOLOGICAL PROBLEM *

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Introduction

For the sake of simplicity we consider the two-dimensional diffusion equation

$$\frac{\partial v}{dt} = k\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),\ k > 0, \tag{1}$$

where x and y are the coordinates of a point in the plane, t is time and v is the density of substance at the point (x, y) at time t. We now assume that diffusion is accompanied by increase in the amount of substance at a rate which depends on the density at the given point and time. We then obtain the equation

$$\frac{\partial v}{\partial t} = k\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + F(v). \tag{2}$$

It is natural that we are interested only in the values of $F(v)$ for $v \geq 0$. Assume that $F(v)$ is a function of v which is continuous, differentiable the required number of times and satisfies the conditions

$$F(0) = F(1) = 1; \tag{3}$$

$$F(v) > 0,\ 0 < v < 1; \tag{4}$$

$$F'(0) = \alpha > 0,\ F'(v) < \alpha,\ 0 < v \leq 1. \tag{5}$$

We thus assume that for very small v the rate $F(v)$ of increase in v is proportional to v (with proportionality factor α), and as v approaches 1, there arises a state of "saturation" when v no longer increases. Accordingly we will consider only solutions of equation (2) satisfying the condition

$$0 \leq v \leq 1. \tag{6}$$

Arbitrary initial values of v for $t = 0$ satisfying condition (6) determine one and only one solution [1] of equation (2) for $t > 0$ subject to the same condition (6).

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* *Bull. Moscow Univ., Math. Mech.* **1**:6 (1937), 1–26.

[1] This fact will be proved in §3.

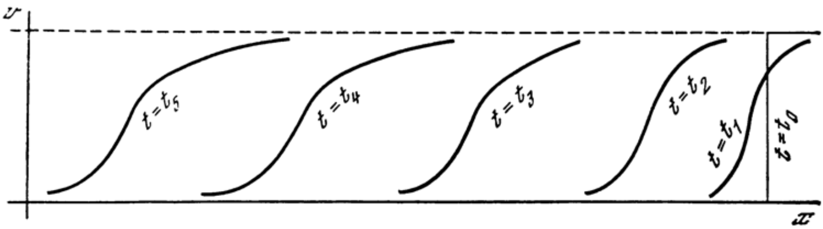

Fig. 1

Further, we will assume that the density v does not depend on the coordinate y. In this case the basic equation (2) is written as

$$\frac{dv}{dx} = k\frac{d^2v}{dx^2} + F(v). \tag{7}$$

We now assume that at the initial time $t = 0$ we have $v = 0$ for $x < a$ and $v = 1$ (that is, the density takes its maximum value) for $x > b \geq a$. Naturally, the region of densities close to 1 will expand, with increasing t, from right to left, displacing thereby the region of small densities to the left. In the special case $a = b$ the pattern looks approximately as shown in Figure 1. The segment of the density curve (as a function of x) on which the major part of density drop from 1 to 0 occurs, moves to the left with increasing time. In its shape the density curve approaches a definite limiting curve for $t \to \infty$. The problem is to find the limiting shape of the density curve and the limiting rate of its motion from right to left. It turns out that the desired limiting rate is

$$\lambda_0 = 2\sqrt{k\alpha}, \tag{8}$$

and the limiting shape of the density curve is determined by the solution of the equation

$$\lambda_0\frac{dv}{dx} = k\frac{d^2v}{dx^2} + F(v) \tag{9}$$

that vanishes when $x = -\infty$ and is equal to one for $x = +\infty$. Such a solution always exists and is unique, to within a transformation $x' = x + c$, which does not change the shape of the curve.

Note that the equation (9) can be obtained in the following manner. We seek a solution of equation (7) such that the curve representing the dependence of v on x does not change its shape with varying t and the curve itself moves from right to left at a rate λ. This solution has the form

$$v(x,t) = v(x + \lambda t). \tag{10}$$

Regarding v as a function of one variable $z = x+\lambda t$, we obtain the equation

$$\lambda\frac{dv}{dz} = k\frac{d^2v}{dz^2} + F(v).$$

This equation turns out to have solutions satisfying the above conditions for equation (9) with any $\lambda \geq \lambda_0$. However, only for $\lambda = \lambda_0$ do we obtain the desired limiting shape of the density curve under the above initial conditions. In order to understand the fact that there are solutions of the form (10) to equation (7) for $\lambda > \lambda_0$, that is, such that the region of large (close to 1) densities moves at a rate greater than λ_0, which at first glance seems astonishing, we examine the limiting case $k = 0$. In this case there is no diffusion and equation (7) can easily be integrated. Under our initial conditions, at points $x < a$ where the density was initially equal to zero it remains zero for any $t > 0$. However, it can easily be shown by calculation that for any $\lambda > \lambda_0$ there exist solutions (10) to equation (7) satisfying all the above conditions. Here the apparent motion of substance from right to left is due to the increase in density at each point occurring independently of the course of the process at other points.

In §1 the facts presented in the Introduction will be applied to the study of biological problems; in §§2, 3 these facts will be proved.

§1

Consider an area populated by a species. We first assume that a dominant gene A is distributed over the area with constant concentration p $(0 \leq p \leq 1)$. Further, we assume that in the struggle for existence, individuals with the character A (that is, those belonging to the genotypes AA and Aa) have an advantage over individuals not possessing this character (that is, those belonging to the genotype aa); more precisely, we assume that the ratio of the probability that an individual with the character A survives to the corresponding probability for an individual without the character is equal to

$$1+\alpha$$

where α is a small positive number. Then, up to terms of order α^2, we obtain for the increment of the concentration p per one generation the formula (see [1])

$$\Delta p = \alpha p(1-p)^2. \tag{11}$$

Now let the concentration p be different at different points in the area occupied by the species, that is, let p depend on the coordinates of the point in the plane (x, y). If the individuals of the species under consideration were fixed at the points of the area, (11) would still be valid. Assume, however, that during the time between birth and reproduction, each individual moves at random (all the directions of motion being equiprobable) and travels some distance. Let $f(r)dr$ be the probability that an individual passes a distance lying between r and $r + dr$; then

$$\rho = \sqrt{\int_0^\infty r^2 f(r)dr}$$

is the root-mean-square path. Therefore, instead of (11) we obtain

$$\Delta p(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\xi, \eta)\frac{f(r)}{2\pi r}d\xi\, d\eta - p(x, y) + \\ + \alpha p(x, y)\{1 - p(x, y)\}^2, \qquad (12)$$

where

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2}.$$

We now assume that p is differentiable with respect to x, y and time t (measured in generations), α and ρ are small, and the third moment

$$\rho^3 = \int_0^\infty |r^3| f(r)dr$$

is small as compared to ρ^2. In this case, expanding $p(\xi, \eta)$ in (12) into a Taylor series in $(\xi - x)$ and $(\eta - y)$ and confining ourselves to the terms of second order (the terms of first order disappear), we obtain [2] an approximate differential equation for p:

$$\frac{\partial p}{\partial t} = \frac{\rho^2}{4}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) + \alpha p(1-p)^2. \qquad (13)$$

All considerations concerning the general equation (2) are applicable to (13).

[2] As to the passage from (12) to (13), see, for instance, similar considerations in A. Ya. Khinchin [2].

Let us stress again the stipulated assumptions. We have assumed that the concentration p changes smoothly, depending on place and time (differentiability with respect to x, y, t), the changes result from the selection in favour of the dominant character A in the ratio $(1+\alpha):1$ and from the random motion of individuals such that the root-mean-square path of an individual during the time between birth and reproduction is ρ, and finally, α and ρ are small (ρ is small as compared to the paths on which substantial changes in the concentration p occur). In this case, taking one generation as a time unit we obtain equation (13).

We now consider the case when a large area has already been occupied by the gene A with concentration p close to 1. It is natural to assume that there is an intermediate concentration region along the boundary of the area. Beyond this region p is assumed to be close to zero. In view of the positive selection, the area occupied by the gene A expands, in other words, its boundary moves towards places that have not yet been occupied by the gene A, and along this boundary there always remains the intermediate concentration region. Our first problem is to find the *rate of advance of the gene* A, that is, the rate at which the boundary of the area occupied by the gene A moves along the normal to this boundary. Formula (8) readily yields an answer to this question: since in this case k is equal to $\rho^2/4$, it follows that the desired rate is

$$\lambda = \rho\sqrt{\alpha}. \tag{14}$$

The second problem which naturally arises is to find the width of the intermediate region. By formula (9), the concentration p along the normal to the boundary satisfies the equation

$$\lambda\frac{dp}{dn} = \frac{\rho^2}{4}\frac{d^2p}{dn^2} + \alpha p(1-p)^2,$$

whence, on dividing by α and substituting λ from (14), we obtain

$$\frac{\rho}{\sqrt{\alpha}}\frac{dp}{dn} = \tfrac{1}{4}\frac{\rho^2 d^2p}{\alpha dn^2} + p(1-p)^2.$$

Introducing the new variable ν by means of the formula

$$n = (\rho/\sqrt{\alpha})\nu \tag{15}$$

we obtain the equation

$$\frac{dp}{d\nu} = \tfrac{1}{4}\frac{d^2p}{d\nu^2} + p(1-p)^2, \tag{16}$$

which contains neither α nor ρ. The boundary conditions for this equation are the same as in the case of (9):

$$p(-\infty) = 0, \quad p(+\infty) = 1.$$

It follows from (15) that the width of the intermediate region is proportional to

$$L = \rho/\sqrt{\alpha}. \tag{17}$$

For example, let $\rho = 1$, $\alpha = 0.0001$; then $\lambda = 0.01$ and $L = 100$.

§2

In this section we consider the equation

$$\lambda\frac{dv}{dx} = k\frac{d^2v}{dx^2} + F(v), \tag{18}$$

where λ and k are assumed to be positive and the function $F(v)$ satisfies the conditions in the introduction.

We are going to establish the relations between λ, k and $\alpha = F'(0)$ for which this equation has a solution satisfying the conditions

$$0 \le v(x) \le 1,$$
$$v(x) \to 1 \text{ as } x \to +\infty \text{ and } v(x) \to 0 \text{ as } x \to -\infty.$$

Let $dv/dx = p$. Then

$$\frac{d^2v}{dx^2} = \frac{dp}{dv}\frac{dv}{dx} = \frac{dp}{dv}p.$$

On substituting this into equation (18) we obtain

$$\frac{dp}{dv} = \frac{\lambda p - F(v)}{kp}. \tag{19}$$

We are interested in those integral curves of this equation that pass between the straight lines $v = 0$ and $v = 1$ in the plane (p, v). Generally, these can include curves of the following types:

1. integral curves that are separated from the lines $v = 0$ and $v = 1$ by at least a distance $\epsilon > 0$;

2. integral curves that go infinitely far away from the axis v and asymptotically approach one of the lines $v = 0$ or $v = 1$;

3. integral curves that intersect one of these lines at a finite point lying on the v-axis;

4. integral curves that approach the points $v = 0,\ p = 0$ and $v = 1,\ p = 0$ and do not belong to any of the former types.

However, it can be easily seen that integral curves of the first type cannot correspond to solutions of equation (18) satisfying the stated conditions, since for such solutions v cannot be arbitrarily close to 0 or 1.

Integral curves of the second type do not exist at all, since curves of this type must have points such that for very large $|p|$ the value of $|dp/dv|$ is very large. However, the ratio $(\lambda p - F(v))/kp$ is approximately equal to λ/k for large $|p|$ (in view of the boundedness of $F(v)$ on the interval $(0,1)$).

Corresponding to the integral curves of the third type are solutions of equation (18) that do not necessarily remain between 0 and 1. Indeed, assume, for example, that a curve of this type approaches a point $v = 1,\ p = p_1 \neq 0$. In the vicinity of the line $v = 1$ we have

$$\frac{dp}{dv} \approx \frac{\lambda}{k} \neq 0,$$

therefore p can be regarded here as a function of v. Let $p = \phi(v)$. Since $\phi(1) = p_1 \neq 0$, it follows that $|\phi(v)|$ is greater than a positive constant C on a small interval $(1-\epsilon, 1+\epsilon)$. We denote by x_0 the value of x for which $v = 1-\epsilon$. Then, integrating the equation $dv/dx = \phi(x)$, we find

$$\int_{x_0}^{x} dx = x - x_0 = \int_{1-\epsilon}^{v} \frac{dv}{\phi(v)}.$$

It follows that x does not exceed $2\epsilon/C$ in absolute value when v varies from $1-\epsilon$ to $1+\epsilon$. Therefore, when x changes from x_0 to $x_0 + 2\epsilon/C$, v necessarily passes through 1.

It remains to consider integral curves of the fourth type. Each of the points $v = 0,\ p = 0$ and $v = 1,\ p = 0$ is a singular point of the differential equation (19). An integral curve of the fourth type must approach each of these points without intersecting the lines $v = 0$ and $v = 1$ and therefore it does not twist.

Thus, in order that such curves can exist, the characteristic equation for each of these points must have real roots. We write $F(v)$ in the form

$$F(v) = \alpha v + \phi_1(v).$$

Then, clearly, $\phi_1(v) = o(v)$. Therefore the characteristic equation at the point $v = 0,\ p = 0$ is given by

$$\begin{vmatrix} \lambda - \rho & -\alpha \\ k & -\rho \end{vmatrix} = 0,$$

whence

$$\rho^2 - \lambda\rho + \alpha k = 0. \tag{20}$$

This equation has real roots if

$$\lambda^2 \geq 4\alpha k.$$

To obtain the characteristic equation at the point $v = 1,\ p = 0$ we make a change of variables, putting $v = 1 - u$. This results in

$$\frac{dp}{du} = \frac{-\lambda p + \Phi(u)}{kp},$$

where $\Phi(u) = F(1 - u)$.

Obviously, $F'(1) \leq 0$ and $\Phi'(0) = -F'(1) = A \geq 0$. Consequently,

$$\Phi(u) = Au + o(u),$$

and the characteristic equation at the point $v = 1,\ p = 0$ takes the form

$$\begin{vmatrix} -\lambda - \rho & A \\ k & -\rho \end{vmatrix} = 0,$$

whence

$$\rho^2 + \lambda\rho - Ak = 0. \tag{21}$$

This equation has real roots when

$$\lambda^2 \geq -4Ak.$$

Since $\alpha > 0$, it follows that equation (20) has real roots of the same sign. Therefore the point (0, 0) is a node. All integral curves that fall in a sufficiently small neighbourhood of this point pass through it. As to equation (21), it has roots of different signs when $A > 0$. Therefore if $A > 0$, then only two integral curves pass through the point (1, 0), along well-defined directions. Let these directions be specified by the equations

$$m_1 u + n_1 p = 0, \quad m_2 u + n_2 p = 0. \tag{22}$$

The coefficients m_1, n_1, m_2, n_2 are known [3] to be determined from the equations

$$k m_1 - \rho_1 n_1 = 0, \quad k m_2 - \rho_2 n_2 = 0, \tag{23}$$

where ρ_1 and ρ_2 are the roots of the characteristic equation (21). Since the roots are of different sign, it follows that the slopes of the lines (22) have different signs as well. [4] Therefore each of the angles formed at the intersection of the lines $v = 1$ and $p = 0$ contains only one integral curve of equation (19) passing through the point $v = 1, p = 0$. Figure 2 shows an approximate configuration of these curves. The curve II intersects the p-axis below the origin since equation (19) implies that $dp/dv > 0$ in the part of the strip between $v = 0$ and $v = 1$ that lies below the v-axis. Therefore the curve II may be excluded from further consideration. It remains to examine the curve I. [5]

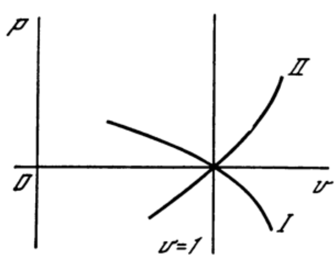

Fig. 2

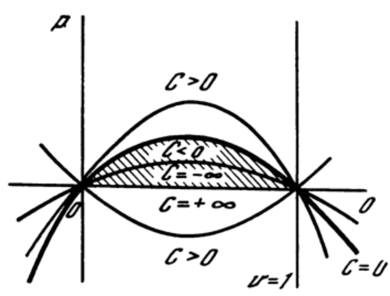

Fig. 3

We are going to prove that each curve of type I intersects the p-axis at the origin. We first of all prove that this curve cannot intersect the p-axis below

[3] See, for example, Bendixson [3] or Stepanov [4].

[4] Using this technique one can show in just the same way that both slopes of the tangents to the integral curves of (19) at the origin are positive.

[5] If $A = 0$, then one can only assert that there exists at least one integral curve of the type I approaching the point (1, 0) along a definite direction whose slope is negative (see [5]).

the origin. To this end we consider the isoclines of equation (19). The equation of the family of these lines has the form

$$\frac{\lambda p = F(v)}{kp} = C. \tag{24}$$

Here C is the value of dp/dv at the point v, p. Hence

$$p = \frac{F(v)}{\lambda - Ck}. \tag{24$'$}$$

Equation (24) specifies a family of curves passing through the points (0, 0) and (1, 0). Figure 3 schematically demonstrates this family. Each curve is supplied with the corresponding value of C. The heavy line represents the curve corresponding to $C = 0$. As the vertices of the curves go up, C increases accordingly and approaches the value λ/k which corresponds to the lines $v = 0$ and $v = 1$. In the region lying between the curve $C = 0$ and the v-axis (shaded in Figure 3) we have $C < 0$, and C becomes very large in absolute value at points near the v-axis. Below the v-axis we have $C > 0$, and C decreases from $+\infty$ to λ/k as the vertex of the curve goes down from the v-axis to $-\infty$.

It is now easily seen that an integral curve I (see Figure 2) cannot intersect the axis Op below the origin. Indeed, in this case the curve I would intersect the v-axis. Since $dp/dv = -\infty$ on the upper side of this axis and $dp/dv = +\infty$ on the lower side, the integral curve I is convex toward the line $v = 1$ at the point where it intersects the v-axis. Therefore in order that this curve pass through the point (1, 0) it is necessary that dp/dv become infinite above the v-axis, which is impossible. For the same reason, an integral curve I cannot intersect the line $v = 1$ above the v-axis.

We now prove that an integral curve I cannot intersect the p-axis above the origin. To this end it suffices to prove that there exists a ray passing through the origin and lying in the first quadrant that does not intersect any integral curve intersecting the positive semi-axis p. From equation (24) we obtain [6]

$$\left(\frac{\overline{dp}}{dv}\right)_{v=0} = \frac{\alpha}{\lambda - Ck}.$$

We now find C for which $(\overline{dp}/dv)_{v=0} = C$. To this end we have to solve the equation

$$\frac{\alpha}{\lambda - Ck} = C,$$

[6] $\frac{\overline{dp}}{dv}$ denotes the derivative of the function $p = p(v)$ determined by equation (24$'$).

that is,

$$kC^2 - C\lambda + \alpha = 0,$$

whence

$$C = \frac{\lambda \pm \sqrt{\lambda^2 - 4\alpha k}}{2k}. \tag{25}$$

Since we are assuming that

$$\lambda^2 \geq 4\alpha k,$$

it follows that the two values of C in (25) are real and positive. Denote by C_0 one of these values and draw the line

$$p = C_0 v. \tag{26}$$

It is easily seen that for all points of the strip between $v = 0$ and $v = 1$ that lie above the line (26), or even on the line itself (except for the origin),[7] we have

$$\frac{dp}{dv} > C_0.$$

Therefore none of the integral curves passing through a point on the p-axis above the origin can cross the part of the line (26) located above the v-axis. We have thus proved that each integral curve of type I (see Figure 2) passes through the origin.

We now prove that there exists only one integral curve of type I. (Of course, we have to prove this only for $A = 0$.) Indeed, we have proved that all curves of type I pass through the origin. On the other hand, it follows from (19) that for $p > 0$ and fixed v the derivative dp/dv increases with p. It follows that two integral curves issuing from the origin cannot pass through the point (1, 0).

We now prove that the curve I corresponds to the solution of equation (18) satisfying the conditions stated at the beginning. First we note that any perpendicular to the v-axis intersects the integral curve I of equation (19) only at one point, since otherwise dp/dv would take the value ∞ above the v-axis. Therefore p is a function of v, that is, $p = \phi(v)$, along this curve. Also recall that the curve I intersects the v-axis at the point (1, 0) at an angle whose tangent is negative and at the origin at an angle with positive tangent. Therefore for small values of v we have

$$p = k_1 v + o(v), \tag{27}$$

[7] Here p is the function of v defined by equation (18).

while for small values of $1 - v$ we obtain

$$p = k_2(1 - v) + o(1 - v), \tag{28}$$

where k_1 and k_2 are positive.

Recall now that $p = dv/dx$. Therefore $dv/dx = \phi(v)$, whence $dx = dv/\phi(v)$. Integrating the latter equation we obtain

$$x - x_0 = \int_{v_0}^{v} \frac{dv}{(v)}, \quad 0 < v_0 < 1.$$

By virtue of (27) and (28), it follows that $x \to -\infty$ as $v \to 0$ and $x \to \infty$ for $v \to 1$, as required.

§3

Instead of equation (7), which was discussed in the Introduction, we here consider the equation

$$\frac{dv}{dt} - \frac{d^2v}{dx^2} = F(v), \tag{29}$$

where the function $F(v)$ satisfies the following conditions:

$$F(0) = F(1) = 0; \tag{30}$$

$$F(v) > 0,\ 0 < v < 1; \tag{31}$$

$$F'(0) = 1; \tag{32}$$

$$F'(v) < 1,\ 0 < v \leq 1; \tag{33}$$

$F'(v)$ is bounded and continuous on (0, 1). Moreover, we assume that $F(v)$ is differentiable the required number of times. The general equation (7) presented in the Introduction can always be reduced to the form (29) by means of the change of variables

$$x = \sqrt{k/\alpha}\bar{x} \text{ and } t = \bar{t}/\alpha.$$

In this section our primary aim is to prove that as $t \to \infty$ the part of the density curve $v(t, x)$ (as a function of x) corresponding to the major portion of density drop from 1 to 0 moves to the left with increasing time at a rate approaching 2 (from below) and the shape of the curve itself approaches that of the graph of the solution $u(x)$ of the equation

$$\frac{d^2u}{dx^2} - 2\frac{du}{dx} + F(v) = 0 \tag{34}$$

that vanishes as $x \to -\infty$ and tends to 1 as $x \to \infty$. The existence of this solution has been proved in §2.

Before proving the basic assertions in this section we consider the equation

$$\frac{dv}{dt} - \frac{d^2v}{dx^2} = F(x,t,v),$$

a special case of which is equation (29). We will prove the existence of a solution taking prescribed values at $t = 0$, and study some properties of the solution.

Theorem 1. *Consider the equation*

$$\frac{dv}{dt} - \frac{d^2v}{dx^2} = F(x,t,v), \tag{35}$$

where the continuous bounded function $F(x,t,v)$ satisfies the Lipschitz condition with respect to v and x, that is,

$$|F(x_2,t,v_2) - F(x_1,t,v_1)| < k|v_2 - v_1| + k|x_2 - x_1| \tag{36}$$

(where k is a constant not depending on x,t,v). Let $f(x)$ be a bounded function defined for all values of x. For simplicity, we assume that $f(x)$ has only a finite number of points of discontinuity. Then there exists a unique function $v(x,t)$, bounded for bounded values of t, which for $t > 0$ satisfies equation (35) *and for $t = 0$ is equal to $f(x)$ at each point at which this function is continuous. In what follows, when saying that $v(x,t)$ is equal to $f(x)$ for $t = 0$ we will always mean only the points of continuity of $f(x)$.*

Proof. Let $v_0(x,t)$ be a bounded function satisfying for $t > 0$ the equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 \tag{37}$$

and equal to $f(x)$ for $t = 0$. Substituting this function for v into the right-hand side of (35) we obtain the solution of this equation that vanishes on the x-axis (see [6]), using the formula

$$\tilde{v}_1(x,t) = \frac{1}{2\sqrt{\pi}}\int_0^t d\eta \int_{-\infty}^{\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}} F(\xi,\eta,v_0(\xi,\eta))d\xi. \tag{38}$$

The function

$$v_1(x,t) = v_0(x,t) + \tilde{v}_1(x,t)$$

is equal to $f(x)$ for $t = 1$, and for $t > 0$ it satisfies the equation

$$\frac{\partial v_1}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} = F[x, t, v_0(x,t)].$$

More generally, using the formula

$$v_{i+1}(x,t) = v_0(x,t) + \frac{1}{2\sqrt{\pi}} \int_0^t d\eta \int_{-\infty}^{+\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}} F(\xi, \eta, v_i) d\xi \tag{39}$$

we find the function $v_{i+1}(x,t)$ satisfying the equation

$$\frac{\partial v_{i+1}}{\partial t} - \frac{\partial^2 v_{i+1}}{\partial x^2} = F(x, t, v_i) \tag{40}$$

for $t > 0$ and equal to $f(x)$ for $t = 0$.

We prove that the sequence of functions $v_i(x,t)$ is uniformly convergent. Indeed, taking into account (36) we find from (39) that

$$M_{i+1}(t) = \sup_{\eta \le t} |v_{i+1}(x,\eta) - v_i(x,\eta)| \le$$

$$\le \frac{1}{2\sqrt{\pi}} \int_0^t d\eta \int_{-\infty}^{+\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}} |F(\xi, \eta, v_i(\xi,\eta)) -$$

$$- F(\xi, \eta, v_{i-1}(\xi,\eta))| d\xi \le \int_0^t k M_i(\eta) d\eta, \tag{41}$$

since

$$\int_{-\infty}^{\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}} d\xi = 2\sqrt{\pi}.$$

However, denoting by M_0 an upper bound of the values of $|f(x)|$ and $F(x,t,0)$ we obtain

$$|v_0(x,t)| \le M_0,$$

and, by virtue of (38), it follows that

$$M_1 \le \int_0^t (k+1) M_0 dt = (k+1) M_0 t = Mt.$$

Hence, using inequality (41) we easily obtain

$$M_i \le \frac{M k^{i-1} t^i}{i!},$$

which readily implies the uniform convergence of the sequence v_i.

We set

$$v(x,t) = \lim_{i\to\infty} v_i(x,t).$$

When $t = 0$, the function $v(x,t)$ is equal to $f(x)$. Moreover, as is easily seen, this function satisfies the equation

$$v(x,t) = v_0(x,t) + \frac{1}{2\sqrt{\pi}} \int_0^t d\eta \int_{-\infty}^{+\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}} F(\xi,\eta,v(\xi,\eta))d\xi. \tag{42}$$

Hence, it is clear that $v(x,t)$ is a continuous function of x and t for $t > 0$. It is proved in the above-cited memoir of Gevrey [6], pp. 343–344, that for any bounded function F the second term on the right-hand side of (42) has a bounded derivative with respect to x. By condition (36), it follows that for $t > 0$ the function $F(x,t,v(x,t))$ has bounded derivatives of any order with respect to x, and therefore $v(x,t)$ satisfies equation (35) (see [6], p. 351).

The uniqueness of the bounded solution is proved in the following way. Assume that there exist two bounded functions $v_1(x,t)$ and $v_2(x,t)$ taking the same value at $t = 0$. Then they satisfy the equation

$$v_2(x,t) - v_1(x,t) =$$

$$= \frac{1}{2\sqrt{\pi}} \int_0^t d\eta \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}} [F(\xi,\eta,v_2) - F(\xi,\eta,v_1)]d\xi. \tag{43}$$

We set

$$M(t) = \sup_{t\geq\eta} |v_2(x,\eta) - v_1(x,\eta)|.$$

Using (36) we obtain from (43) the inequality

$$M(t) \leq k \int_0^t M(\eta)d\eta,$$

which is impossible.

Remark. For regions bounded by lines $x = \phi_1(t)$ and $x = \phi_2(t)$ on the left and on the right and by straight lines $t = t_0$ and $t = t_1 > t_0$ from above and below, respectively, the above-mentioned memoir by Gevrey [6] contains a proof of the existence and uniqueness of a bounded function satisfying equation (35) in the interior of the region and taking prescribed bounded continuous values on the lines $x = \phi_1(t)$, $x = \phi_2(t)$, and $t = t_0$. It can similarly be proved that there exists a unique bounded function satisfying equation (35) in the interior of a region G bounded only on one side by a line $x = \phi(t)$ and by straight lines $t = t_0$ and $t = t_1 > t_0$ from above and below and taking prescribed bounded continuous values on the lines $x = \phi(t)$ and $t = t_0$.

Theorem 2. *If the function $F(x,t,v)$ is replaced by another function $F_1(x,t,v)$ such that always*

$$F_1(x,t,v) \geq F(x,t,v),$$

then the function $v(x,t)$ does not decrease, provided that the initial conditions remain unchanged.

Remark. If (35) is interpreted as the heat equation, then the function $F(x,t,v)$ characterizes the heat source intensity, and physically Theorem 2 becomes quite clear.

Proof. Let the functions $v(x,t)$ and $v_1(x,t)$ satisfy equation (35) and the equation

$$\frac{\partial v_1}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} = F_1(x,t,v_1),$$

respectively.

After term-by-term subtraction we find that the function

$$w(x,t) = v_1(x,t) - v(x,t)$$

satisfies the equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = F_1(x,t,v_1) - F(x,t,v).$$

We set

$$w(x,t) = \bar{w}(x,t)\ \exp(-kt),$$

where k is the same as in inequality (36). Then

$$\frac{\partial \bar{w}}{\partial t} - \frac{\partial^2 \bar{w}}{\partial x^2} = k\bar{w} + \exp(kt)[F_1(x,t,v_1) - F(x,t,v)].$$

Hence,

$$\bar{w}(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^t d\eta \int_{-\infty}^{\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}} \{k\bar{w} + \exp(k\eta)[F_1(\xi,\eta,v_1) -$$

$$-F(\xi,\eta,v)]\}d\xi = \frac{1}{2\sqrt{\pi}}\int_0^t d\eta \int_{-\infty}^{\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}}\{k\bar{w}+$$

$$+\exp(k\eta)[F_1(\xi,\eta,v_1) - F(\xi,\eta,v_1) + F(\xi,\eta,v_1)-$$

$$-F(\xi,\eta,v)]d\xi \geq \frac{1}{2\sqrt{\pi}}\int_0^t d\eta \int_{-\infty}^{\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}}\{k\bar{w}+$$

$$+\exp(k\eta)[F(\xi,\eta,v_1) - F(\xi,\eta,v)]\}d\xi \geq$$

$$\geq \frac{1}{2\sqrt{\pi}}\int_0^t d\eta \int_{-\infty}^{\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}}(k\bar{w} - k|\bar{w}|)d\xi. \tag{44}$$

The expression in the square brackets is equal to zero if $\bar{w} \geq 0$ and to $-2k\bar{w}$ if $\bar{w} \leq 0$. We denote by $-m(t)$ the infimum of $\bar{w}(\xi,\eta) - |\bar{w}(\xi,\eta)|$ for $\eta \leq t$. To prove the theorem it obviously suffices to show that $m(t) \equiv 0$. To this end we note that (44) implies the inequality

$$\bar{w}(x,t) \geq -k\int_0^t m(\eta)d\eta,$$

and therefore

$$m(t) \leq k\int_0^t m(\eta)d\eta,$$

which is only possible if $m(t) \equiv 0$.

Theorem 3. *When $f(x)$ increases, the value of $v(x,t)$ does not decrease.*

The physical meaning of this theorem is as clear as that of the foregoing theorem, provided that (35) is interpreted as the heat equation for a bar. The function $f(x)$ represents the initial temperature of the bar. When this temperature increases, the subsequent temperature also increases.

Proof of Theorem 3. Let $v_1(x,t)$ and $v_2(x,t)$ satisfy equation (35), let these functions be equal to $f_1(x)$ and $f_2(x)$ respectively, for $t=0$, and let $f_2(x) \geq f_1(x)$. We prove that $v_2 \geq v_1$.

The function $w = v_2 - v_1$ satisfies the equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = F(x,t,v_2) - F(x,t,v_1).$$

By condition (36),

$$F(x,t,v_2) - F(x,t,v_1) \geq -k|w|.$$

Therefore by Theorem 2, the function $w(x,t)$ is not less than the function $v^*(x,t)$, that is equal to $f_2(x) - f_1(x)$ when $t = 0$, while for $t > 0$ it satisfies the equation

$$\frac{\partial v^*}{\partial t} - \frac{\partial^2 v^*}{\partial x^2} = -k|v^*|.$$

If t is bounded, then $\exp(-kt)v^{**}$ is a (unique, by Theorem 1) bounded solution of this equation, where $v^{**}(x,t)$ satisfies equation (37) with the initial condition $v^{**}(x,0) = f_2(x) - f_1(x)$ (this function is clearly non-negative). Consequently,

$$w = v_2 - v_1 \geq 0,$$

as required.

Theorem 4. *If everywhere $f(x) \geq 0$ and $F(x,t,0) = 0$, then we also have*

$$v(x,t) \geq 0.$$

Proof. By Theorem 3, when $f(x)$ decreases, the function $v(x,t)$ does not increase. When $f(x) \equiv 0$, we have $v(x,t) \equiv 0$. Consequently, $f(x) \geq 0$ implies that $v(x,t) \geq 0$, as required.

Theorem 5. *If, in addition to the hypothesis of Theorem 4, $f(x) > 0$ on an interval of positive length, then for $t > 0$ we have*

$$v(x,t) > 0.$$

Proof. The proof follows from that of Theorem 3 by setting $v_2 = v$ and $v_1 = 0$ and taking into account the fact that the function $v^{**}(x,t)$ can be represented by a Poisson integral and is therefore necessarily positive for $t > 0$.

Theorem 6. *If $F(x,t,1) \equiv 0$ and $f(x) \leq 1$, then $v(x,t) \leq 1$.*

Proof. By Theorem 3, when $f(x)$ increases, the function $v(x,t)$ does not decrease. When $f(x) \equiv 1$, we have $v(x,t) \equiv 1$. The theorem now follows.

Theorem 7. *If for $t = 0$ the function $v(x,t)$ is equal to a monotone increasing differentiable function $f(x)$ and, for $t > 0$, satisfies the equation*

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = F(t,v), \tag{45}$$

then $v(x,t)$ is a non-decreasing function of x for any $t > 0$.

Proof. By Theorem 1,

$$v(x,t) = v_0(x,t) + \frac{1}{2\sqrt{\pi}} \int_0^t d\eta \int_{-\infty}^{\infty} \frac{F(\eta, v(\xi,\eta)) \exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}} d\xi, \tag{46}$$

where for $t > 0$ $v_0(x,t)$ satisfies the equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 \tag{47}$$

and equals $f(x)$ when $t = 0$. If $f(x)$ is differentiable, then $\partial v_0(x,t)/\partial x$ tends to $f'(x)$ as (x,t) tends to $(x,0)$ (see [6], pp. 330–331). On the other hand, the partial derivative with respect to x of the second term on the right-hand side of (46) does not exceed $(4/\sqrt{\pi})Mt^{1/2}$ in absolute value, provided that $|F| \leq M$ (see [6], p. 344). Therefore $\partial v(x,t)/\partial x$ tends to $f'(x)$ as $t \to 0$. If we also assume that the function $v(x,t)$ has derivatives $\partial^2 v/\partial t \partial x$ and $\partial^3 v/\partial x^3$, which is true when $F(t,v)$ is three times differentiable with respect to v, then the function $w(x,t) = \partial v(x,t)/\partial x$ satisfies the equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = \frac{\partial F}{\partial v} w. \tag{48}$$

Whence, applying Theorem 4, we find that $w(x,t) \geq 0$, which is the desired result.

Theorem 8. *If*

$$f^{(\epsilon)}(x) \to f^{(0)}(x) \text{ as } \epsilon \to 0,$$

so that

$$\int_{-\infty}^{+\infty} |f^{(\epsilon)} - f^{(0)}| dx \to 0,$$

then for any $t > 0$ the function $v^{(\epsilon)}(x,t)$ satisfying (35) when $t > 0$ and equal to $f^{(\epsilon)}(x)$ when $t = 0$, tends to the function $v^{(0)}(x,t)$ (also) satisfying equation (35) when $t > 0$ and equal to $f^{(0)}(x)$ when $t = 0$.

Proof. We will look for $v^{(\epsilon)}(x,t)$ and $v^{(0)}(x,t)$ by the technique of successive approximation, as in the proof of Theorem 1. The functions $v_0^{(\epsilon)}$ and $v_0^{(0)}$ are determined by the formulas

$$v_0^{(\epsilon)}(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} f^{(\epsilon)}(\xi) \frac{\exp(-(x-\xi)^2/4t)}{\sqrt{t}} d\xi,$$

$$v_0(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} f^{(0)}(\xi) \frac{\exp(-(x-\xi)^2/4t)}{\sqrt{t}} d\xi.$$

It follows immediately that

$$v_0^{(\epsilon)}(x,t) \to v_0(x,t) \text{ as } \epsilon \to 0$$

when $t > 0$.

The difference $\tilde{v}_1^{(\epsilon)}(x,t) - \tilde{v}_1^{(0)}(x,t)$ (we use the same notation as in the proof of Theorem 1) is given by the formula

$$\frac{1}{2\sqrt{\pi}}\int_0^t d\eta \int_{-\infty}^{+\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}}[F(\xi,\eta,v_0^{(\epsilon)}) - F(\xi,\eta,v_0^{(0)})]d\xi,$$

whence

$$v_1^*(x,t) = |\tilde{v}_1^{(\epsilon)}(x,t) - \tilde{v}_1^{(0)}(x,t)| \le$$

$$\le \frac{k}{2\sqrt{\pi}}\int_0^t d\eta \int_{-\infty}^{+\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}}|v_0^{(\epsilon)}(\xi,\eta) - v_0^{(0)}(\xi,\eta)|d\xi.$$

We set

$$v_0^*(x,t) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} |f^{(\epsilon)}(\xi) - f^{(0)}(\xi)|\frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t}}d\xi.$$

Obviously,

$$v_0^*(x,t) \ge |v_0^{(\epsilon)}(x,t) - v^{(0)}(x,t)|,$$

and therefore

$$v_1^*(x,t) \le \frac{k}{2\sqrt{\pi}}\int_0^t d\eta \int_{-\infty}^{+\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}} v_0^*(\xi,\eta)d\xi =$$

$$= k\int_0^t v_0^*(x,t)d\eta = ktv_0^*(x,t).$$

The latter inequality follows from the fact that $v_0^*(x,t)$ satisfies equation (47). Thus,

$$v_1^*(x,t) \le ktv_0^*(x,t).$$

Using the same technique we find that

$$v_i^*(x,t) = |v_i^{(\epsilon)}(x,t) - v_i^{(0)}(x,t)| \le \frac{(kt)^i}{i!}v_0^*(x,t).$$

It follows that, by selecting a sufficiently small ϵ, we can make the sum $\sum_{i=0}^{\infty} v_i^*(x,t)$ and hence the expression $|v^{(\epsilon)}(x,t) - v^{(0)}(x,t)|$ arbitrarily small, as required.

Theorem 9. *The function $v(x,t)$ that satisfies equation (45) for $t > 0$ is zero for $t = 0$ and $x < 0$ and is equal to 1 for $t = 0$ and $x < 0$, is a non-decreasing function of x for any $t > 0$, and $\partial v(x,t)/\partial x > 0$ for $t > 0$.*

Proof. According to the above lemma, the function $v(x,t)$ can be regarded as the limit of the functions $v^{(\epsilon)}(x,t)$ as $\epsilon \to 0$ which assume the same values as $v(x,t)$ on the x-axis for $|x| \geq \epsilon$, are continuous together with their derivatives with respect to x throughout the x-axis, and are monotone. However, it has just been proved (Theorem 7) that for $t > 0$ the function $v^{(\epsilon)}(x,t)$ is monotone increasing in x. Therefore this is also true for $v(x,t)$.

We now prove that for $t > 0$ the derivative $\partial v(x,t)/\partial x$ is positive. To this end we have only to show that for $t > 0$ the relation $\partial v(x,t)/\partial x = 0$ is impossible. This can be done in the following way. For $t > 0$ the derivative $\partial v(x,t)/\partial x$ satisfies equation (48). Therefore the expression $\bar{w}(x,t) = \exp(Mt)\partial v(x,t)/\partial x$ satisfies the equation

$$\frac{\partial \bar{w}}{\partial t} - \frac{\partial^2 \bar{w}}{\partial x^2} = \left[\frac{\partial F}{\partial v} + M\right]\bar{w},$$

where M is an upper bound of $|\partial F/\partial v|$.

Since

$$\partial F/\partial v + M \geq 0,$$

it follows from Theorem 2 that for $t > t_0 > 0$ the function $\bar{w}(x,t)$ is not less than the function $\bar{\bar{w}}(x,t)$ equal to $\bar{w}(x,t)$ for $t = t_0$ and satisfying the equation

$$\frac{\partial \bar{\bar{w}}}{\partial t} - \frac{\partial^2 \bar{\bar{w}}}{\partial x^2} = 0$$

for $t > t_0$.

The function $\bar{\bar{w}}(x,t)$ is positive for all $t > t_0$ since for $t = t_0$ it is not identically equal to zero if t_0 is sufficiently small.

In what follows we will denote by $v(x,t)$ the function satisfying equation (29) for $t > 0$, equal to 0 for $t = 0$ and $x < 0$, and equal to 1 for $t = 0$ and $x > 0$.

Theorem 10. *For any fixed $x < 0$ we have*

$$v(x - 2t, t) \to 0 \text{ as } t \to +\infty.$$

Proof. The function $\bar{v}(x,t) = v(x-2t,t)$ satisfies the equation

$$\frac{\partial \bar{v}}{\partial t} - \frac{\partial^2 \bar{v}}{\partial x^2} = -2\frac{\partial \bar{v}}{\partial x} + F(\bar{v}).$$

On the other hand, the function $v^*(x,t) = v(x-2t,t)\exp(-x)$ satisfies the equation[8]

$$\frac{\partial v^*}{\partial t} - \frac{\partial^2 v^*}{\partial x^2} = [F(\bar{v}) - \bar{v}]\exp(-x).$$

By the conditions (32) and (33) imposed on $F(v)$, the function $F(v) - v$ is non-positive. Therefore $v^*(x,t)$ is less than the function satisfying equation (37) for $t > 0$ and, for $t = 0$, equal to zero for $x < 0$ and to $\exp(-x)$ for $x > 0$. The latter function tends to zero uniformly with respect to x as $t \to \infty$.

Theorem 11. *For a fixed t we regard the derivative $\partial v(x,t)/\partial x$ as a function of v. This is possible in view of Theorem 9. Let*

$$\partial v(x,t)/\partial x = \psi(v,t). \tag{49}$$

Then as t increases and v is fixed, the function ψ does not increase.

Proof. Consider the functions $v(x,t)$ and $v(x+c,t+t_0) = v_{t_0}(x,t)$, where c is a constant and $t_0 > 0$. We put

$$w(x,t) = v(x,t) - v_{t_0}(x,t).$$

Let $\mathfrak{M}$ be the set of points (x,t) such that $w(x,t) > 0$. First we prove that this set is bounded only on the left by a line emanating from the origin and along which the coordinate t is nowhere decreasing. To prove this we note that $w(x,t)$ satisfies the equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = k(x,t), \tag{50}$$

where $k(x,t)$ is a bounded function, namely

$$k(x,t) = F'(\bar{u}(x,t)),$$

[8] It is easily seen that the function $v^*(x,t)$ remains bounded when $t > 0$ is bounded.

and $\bar{u}(x,t)$ is a number lying between $v(x,t)$ and $v_{t_0}(x,t)$. Therefore the set $\mathfrak{M}$ cannot contain isolated portions[9] not adjoining the x-axis. Thus, the set consists of a single portion adjoining the positive semi-axis x, as is clear. In order to prove that the set $\mathfrak{M}$ is bounded on the left by a line along which t is non-decreasing we assume that, on the contrary, this line contains a portion of the form shown in Figure 4. For example, suppose that, starting from the point A, this line goes downward. Then the function $w(x,t)$ assumes negative values to the right of the line OA whereas on the line OA it is equal to 0, while for $x > 0$ it takes on positive values on the x-axis. However, using the same methods as in the proof of Theorem 4 we can show that this is impossible.

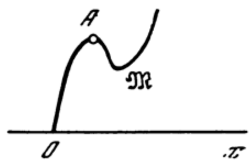

Fig. 4

In just the same way it can be proved that the set $\mathfrak{M}$ is unbounded from the right.

The above remarks allow us to prove the stated theorem quite simply. Indeed, in view of the arbitrariness in the choice of c, $v(x_0,t)$ and $v_{t_0}(x_0,t)$ can be made equal for any pre-assigned t and some $x = x_0$. Then, by the above argument,

$$v(x,t) \geq v_{t_0}(x,t)$$

for all $x > x_0$, and consequently

$$\frac{\partial v}{\partial x}(x_0,t) \geq \frac{\partial v_{t_0}}{\partial x}(x_0,t),$$

which is the desired result.

Theorem 12. *For any t we have*

$$\frac{\partial v(x,t)}{\partial x} \geq u'(x)$$

[9] The proof of a similar assertion for the case of finite portions can be found in [7], pp. 386–387. It can be shown that the same is true for infinite portions as well. Cf. the Remark to Theorem 1.

provided that $v(x,t) = u(x)$. *Here* $u(x)$ *is the solution to equation* (34) *which was discussed at the beginning of this section.*

The proof is completely similar to that of the foregoing theorem. It is only necessary to replace the function $v_{t_0}(x,t)$ by $u(x+c)$ and the function $w(x,t)$ by the difference $v(x,t) - u(x+c)$.

Theorem 13. *Let*

$$v^*(x,t) = v(x+\phi(t),t),$$

where the function $\phi(t)$ *is chosen in such a way that*

$$v^*(0,t) = c = \text{const}.$$

Then

$$v^*(x,t) \to v^*(x) \text{ uniformly with respect to } x \text{ as } t \to \infty.$$

Proof. It follows from (49) that

$$x = \int_c^{v^*} \frac{dv}{\psi(v,t)}. \tag{51}$$

By Theorem 11, the integrand increases monotonically as $t \to \infty$. Moreover, by Theorem 12, the integral $\int_c^{v^*} \frac{\partial v}{\psi(v,t)}$ cannot increase indefinitely. Therefore we can pass to the limit under the integral sign in (51). Let

$$\psi(v,t) \to \psi(v) \text{ as } t \to \infty.$$

Then the limit of (51) is

$$x = \int_c^{v^*} \frac{dv}{\psi(v)}.$$

Since, by Theorem 12,

$$\psi(v) > 0,$$

it follows that this condition determines a function v^* of x. It remains to show that $v^*(x,t)$ converges to $v^*(x)$ uniformly. To this end we note that (51) implies that $x(v^*,t)$ converges uniformly to $x(v)$ on any interval $\epsilon < v^* < 1-\epsilon$. If we now take into account the fact that, by Theorem 11, the function $\psi(v^*,t)$ is bounded on each such interval, it follows that the function $v^*(x,t)$ converges uniformly to $v^*(x)$ for the values of x such that $v(x)$ is contained between ϵ and $1-\epsilon$ (where ϵ is arbitrarily small). But outside this interval of values of x, $v^*(x,t) \to v^*(x)$ uniformly since for sufficiently large t the function $v^*(x,t)$ assumes values that differ slightly from 0 or 1.

Theorem 14. *As $t_0 \to +\infty$ the sequence of functions*

$$v_{t_0}(x,t) = v[x + \phi(t_0),\ t + t_0]$$

converges uniformly to a solution $\bar{v}(x,t)$ of equation (29) *in the region $t \leq T =$* const. *The function $\phi(t_0)$ is defined so that*

$$v_{t_0}(0,0) = c = \text{const}$$

for all t_0.

Proof. The function

$$w(x,t) = v_{t_0}(x,t) - v_{t_0+T}(x,t)$$

satisfies the equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = F'(\bar{v})w, \tag{52}$$

where $\bar{v}(x,t)$ is contained between $v_{t_0}(x,t)$ and $v_{t_0+T}(x,t)$. By Theorem 13, for sufficiently large t_0 we have

$$|w(x,0)| < \epsilon,$$

where $\epsilon > 0$ is arbitrarily small. By Theorems 2 and 3, the function $w(x,t)$ is less than the function $\bar{w}(x,t) = \epsilon \exp(kt)$, where k is an upper bound for the values of $|F'(u)|$, since for $t = 0$ the latter function assumes a value that is not less than $w(x,0)$ and for $t > 0$ satisfies the equation

$$\frac{\partial \bar{w}}{\partial t} - \frac{\partial^2 w}{\partial x^2} = k|\bar{w}|,$$

the right-hand side of which is not less than the right-hand side of (52) for $w = \bar{w}$. It can similarly be proved that

$$\bar{w}(x,t) > -\epsilon \exp(kt).$$

Thus, we have shown that the sequence of functions $v_{t_0}(x,t)$ converges uniformly in a region $t < T$ to a function $\bar{v}(x,t)$ as $t_0 \to +\infty$. Let us show that $\bar{v}(x,t)$ satisfies equation (29).

Using (42) we write

$$v_{t_0}(x,t) = v_{t_0,0}(x,t)+ \\ +\frac{1}{2\sqrt{\pi}}\int_0^t d\eta \int_{-\infty}^{+\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}} F(v_{t_0}(\xi,\eta))d\xi. \qquad (53)$$

We can pass to the limit in this relation by substituting $\bar{v}$ for v_{t_0}. However, as was shown in the proof of Theorem 1, a function satisfying equation (53) satisfies equation (29) as well.

Theorem 15. *As $t_0 \to +\infty$ the first partial derivatives of $v_{t_0}(x,t)$ with respect to x and t tend to the corresponding partial derivatives of $\bar{v}(x,t)$, uniformly in a region $\epsilon < t < T$ where ϵ and T are positive constants.*

Proof. The uniform convergence of $\partial v_{t_0}/\partial x$ can be proved by means of relation (53). Indeed, for $t > \epsilon$ the partial derivative of the first term on the right-hand side with respect to x converges uniformly since this term can be represented as a Poisson integral. In order to show that this is also true for the second term for $t < T$ we consider the difference between the two values of it corresponding to $t_0 = t_0'$ and $t_0 = t_0''$. This difference is equal to

$$\frac{1}{2\sqrt{\pi}}\int_0^t d\eta \int_{-\infty}^{+\infty} \frac{\exp(-(x-\xi)^2/4(t-\eta))}{\sqrt{t-\eta}} \left[F(v_{t_0'}) - F(v_{t_0''})\right] d\xi. \qquad (54)$$

By Theorem 14, the difference

$$F(v_{t_0'}(\xi,t)) - F(v_{t_0''}(\xi,t))$$

is arbitrarily small for sufficiently large t_0' and t_0''. By applying to this case the above-mentioned result of Gevrey, we see that for sufficiently large t_0' and t_0'' the partial derivative of (54) with respect to x tends uniformly to zero, provided that $t < T$.

The function $w_{t_0}(x,t) = \partial v_{t_0}(x,t)/\partial x$ satisfies the equation

$$\frac{\partial w_{t_0}}{\partial t} - \frac{\partial^2 w_{t_0}}{\partial x^2} = F'(v_{t_0})w_{t_0}.$$

We have already proved that for $\epsilon < t < T$ the right-hand side of this equation converges uniformly as $t_0 \to +\infty$. Therefore the argument used for proving the uniform convergence of $\partial v_{t_0}/\partial x$ can be applied in the proof of the uniform convergence of $\partial w_{t_0}/\partial x = \partial^2 v_{t_0}/\partial x^2$. And since the function v_{t_0} satisfies equation (29), the uniform convergence of $\partial v_{t_0}/\partial t$ is also proved.

Theorem 16. *Let the function $v_{t_0}(x,t)$ ($\bar{v}(x,t)$) be equal to a constant c on the line $x = \phi_{t_0}(t)$ ($x = \phi(t)$). Then*

$$\phi_{t_0}(t) \to \phi'(t) \text{ as } t_0 \to \infty$$

uniformly with respect to t for $\epsilon < t < T$.

Proof. The value of $\phi'_{t_0}(t)$ ($\phi'(t)$) at the point $(\phi_{t_0}(t), t)$ $(\phi(t), t))$ is equal to

$$-\frac{\partial v_{t_0}/\partial t}{\partial v_{t_0}/\partial x} \quad \left(-\frac{\partial \bar{v}/\partial t}{\partial \bar{v}/\partial x}\right).$$

By Theorems 12 and 14,

$$[\phi_{t_0}(t) - \phi(t)] < \epsilon_1$$

throughout the region $\overline{G}$ ($\epsilon < t < T$), provided that t_0 is sufficiently large and ϵ_1 is arbitrarily small. By Theorem 15, for the same values of the arguments the difference between the numerators and the denominators of the fractions

$$\frac{\partial v_{t_0}/\partial t}{\partial v_{t_0}/\partial x} \text{ and } \frac{\partial \bar{v}/\partial t}{\partial \bar{v}/\partial x} \tag{55}$$

is arbitrarily small, uniformly in the region $\overline{G}$. Moreover, the derivative $\partial \bar{v}/\partial x$ does not exceed a positive constant in the strip $\phi(t) - \epsilon_2 < x < \phi(t) + \epsilon_2$. Consequently, the fractions in (55), for the same values of the arguments and sufficiently large t_0, differ by less than ϵ_3 in the strip

$$\epsilon < t < T, \quad \phi(t) - \epsilon_2 < x < \phi(t) + \epsilon_2.$$

Also, taking into account the fact that the expression $\frac{\partial \bar{v}/\partial t}{\partial \bar{v}/\partial x}$ is uniformly continuous in this strip and therefore the difference between its values at points in this strip with the same t is arbitrarily small for sufficiently small ϵ_3, we complete the proof of the theorem.

Theorem 17. *For any t we have*

$$\bar{v}(x,t) = u(x+2t) \quad \textit{and} \quad d\phi/dt \to -2 \ \textit{as} \ t \to \infty$$

(*the notation is the same as in Theorem* 14).

Proof. Consider the function

$$v^*(x,t) = \bar{v}(x + c_1(t), t),$$

where the function $c_1(t)$ is chosen so that

$$v^*(0,t) \equiv c = \text{const}.$$

Then

$$\frac{\partial v^*}{\partial t} = \frac{\partial^2 v^*}{\partial x^2} + c_1(t)\frac{\partial v^*}{\partial x} + F(v^*).$$

On the other hand, by the definition of $\bar{v}(x,t)$ the value of $v^*(x,t)$ does not depend on t for any x. Therefore

$$\partial v^*/\partial t = 0 \text{ and } c_1'(t) = \text{const}.$$

It follows from §2 that this constant cannot be greater than -2 and, by Theorem 10, the constant cannot be less than -2. Consequently , it is equal to -2, and, by Theorem 16, we have

$$d\phi/dt \to -2 \text{ as } t \to \infty,$$

as required.

Remark. Assume that the initial values of $v(x,t)$ differ from those considered up to now; namely, let

1) $v(x,0) = 1$ for $x \geq c_1$;
2) $v(x,0) = 0$ for $x \leq c_2 < c_1$;
3) $v(x,0)$ assumes arbitrary values between 0 and 1 for $c_2 < x < c_1$.

Then it is easily seen that in this case the rate at which the region in which the major part of the drop of v from 1 to 0 occurs moves to the left, nevertheless tends to 2 since

$$v(x - c_1, t) \leq \bar{v}(x,t) \leq v(x - c_2, t),$$

where $\bar{v}(x,t)$ denotes the solution of equation (29) satisfying the new initial conditions.

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