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Group Velocity

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This paper gives a survey of the theory of group velocity for one-dimensional and three-dimensional, isotropic and anisotropic, homogeneous and inhomogeneous, conservative and dissipative, linear and non-linear, classical and relativistic systems exhibiting wave propagation under free and forced-motion conditions.

1. Introduction

IN choosing a subject for this lecture, I had in mind the aim which permeates the policy of the I.M.A. journals and Programme Committee, namely, to choose subject matter that will be of interest to as many as possible of our members, and in particular, to mathematicians working in more than just one branch or field of application of mathematics. It occurred to me that these conditions would be met by a general survey of what is known today about group velocity. The theory of group velocity is an essentially mathematical theory that has been developed over the years with an eye on a great variety of spheres of application. I am going to give an account of what seem to me the most important parts of our knowledge of this subject, largely ignoring history and concentrating rather on mathematics (and, of course, on its applications!).

I should like to apologize to those who attended Professor Synge's three very erudite lectures here at King's College recently on the same subject. I recognized that the subject was indeed the same too late to be able to change significantly the material of this lecture, but I hope that even those who attended both may still find useful an elementary and relatively brief account of the subject, that proceeds by degrees from simpler cases to more and more general ones.

I am sure that the properties of group velocity always surprise somebody who meets them for the first time. The most striking feature of waves is, without doubt, their capability of carrying energy over long distances; sometimes, of course, as well as the energy, they carry also, more or less imperfectly, information. With many waves, furthermore, the velocity of crests and troughs and their regular progress is extremely evident. It is natural to imagine, then, that this "phase velocity" is also the velocity with which energy is propagated by the wave, particularly because in some simple cases, including sound waves and waves on a flexible string, the two velocities are indeed the same. For the vast bulk of wave motions occurring in nature, however, the phase velocity, with which the crests and troughs are propagated, and the group velocity, with which the energy is propagated, have quite different magnitudes; and

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where the system possesses any degree of anisotropy they are normally also quite different in direction.

Waves of interest include one-dimensional waves, such as may propagate along a string, or a transmission line; two-dimensional waves, such as may propagate over a water surface, or other surfaces of separation between different phases of matter; and three-dimensional waves, propagating freely in space. One-dimensional systems, and isotropic two- and three-dimensional systems, that is systems without quantitative differences between waves travelling in different directions, have at least the consoling property that energy is propagated in the same direction as that in which the crests move. The magnitudes of the phase and group velocities are, however, not equal for any waves whose phase velocity takes different values for waves of different length. This state of affairs is usually described as dispersion, because it means that if we imagine any general disturbance split up into components of different wavelength, all these components will progress at different speeds, and therefore will tend to get separated out, that is "dispersed", into a long wave train with the wavelength varying rather gradually along it.

In this process of dispersion, the energy associated with waves of a given length is propagated at the group velocity, say u, of those waves. Hence, after a time t has elapsed, waves of that length will be found a distance ut farther on. Anyone who imagined that, because the crests travel with the phase velocity v, those waves should be observed at the quite different distance vt, would fail to find them. There he would find waves, if any, of quite different length, namely those whose group velocity has the value v. If he were mesmerized, in fact, into trying to follow individual crests, he would find that these evolve into crests of waves of continually changing length, or even disappear altogether. Only if he rigorously fixed his gaze on a point moving with the group velocity u would he find that he was continually observing waves of the same length. Thus, although it is variations in the phase velocity which cause the phase and group velocities to be different, it is variations in the group velocity which produce the dispersion.

Waves, as I said earlier, are often used as carriers, not merely of energy, but also of information. Needless to say, the transmission of information is not assisted by dispersion, and the dispersed waveform is hard to unscramble into the original waveform. This is why every effort is made in transmission lines to match constants so that dispersion becomes negligible. When this is successfully done, then the phase and group velocities once more coincide. However, there are relatively few systems where this is possible.

Many different approaches to the mathematics of group velocity have been used. They can be divided into, first, kinematic approaches, which consider a general linear combination of waves of varying length with frequency functionally related to wavelength, and work out how the waves of different lengths become dispersed from one another, and, secondly, dynamic approaches, which are basically quadratic rather than linear. They prove, in particular, that energy flow takes place at the group velocity and so make it possible to predict not only where waves of a given length will be, but also what their amplitudes will be. I propose to describe both kinematic and dynamic methods in this lecture, for one-dimensional and multi-dimensional systems, with and without isotropy, with and without homogeneity and with and without

dissipation. The work will be confined to systems satisfying linear equations, except for a brief final description of extensions to non-linear problems, in which Prof. Whitham (1965a,b) and others have been active in recent years.

2. The Kinematics of Wave Crests (One-dimensional)

I shall begin, then, with the simple case of one-dimensional wave motions, to which it is relatively easy to reduce the case of isotropic multi-dimensional wave motions. The simplest method of deriving the formula for group velocity in this case, as indeed in more general cases, is to consider the kinematics of wave crests (Lighthill & Whitham, 1955; see also Havelock, 1914). This procedure is suitable at any rate when dispersion has already caused the lengths of adjoining waves to differ by only a small fraction. Under these circumstances, it is so rare for a crest to cease to be a crest, or to divide into two crests, that we can to a good approximation apply a law of conservation of numbers of crests. This law takes the form of a hydrodynamical equation of continuity:

$$\frac{\partial k}{\partial t}+\frac{\partial \omega}{\partial x}=0, \tag{1}$$

where ω is the number of wave crests per second, i.e. the frequency, and k is the number of wave crests per cm, i.e. the wavenumber (the reciprocal of the wavelength). Equation (1) means that the change in number of crests in a fixed length equals the difference between inflow of crests at one end and outflow at the other. If in equation (1) we write $\omega = kv$, where v is the phase velocity, the analogy with the equation of continuity becomes particularly apparent.

In a wave motion in which the frequency ω is a function of the wavenumber k, equation (1) becomes

$$\frac{\partial k}{\partial t}+u\frac{\partial k}{\partial x}=0, \tag{2}$$

where $u = d\omega/dk$ will be called the group velocity. Equation (2) says that, in the (x,t)-plane, along paths $x-ut$ = constant (travelling with speed u), the wavenumber k remains unchanged (Fig. 1). Waves of a given length are found, in fact, at points

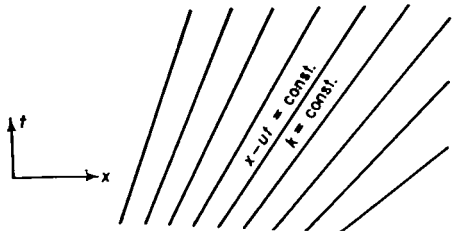

FIG. 1. Homogeneous one-dimensional system. Geometrical significance of equation (2) which governs wave dispersion.

along such paths, and there is a different path for each wavelength. Thus, the basic fact about group velocity for such a system has already been derived.

Here we have taken the system as homogeneous, but an inhomogeneous system is not much more complicated to treat. This is a system where the relationship between

frequency and wavenumber changes gradually with position, so that $\omega = \omega(k,x)$. We cannot then deduce equation (2) for k from equation (1). We can, however, multiply equation (1) by

$$u = \left(\frac{\partial\omega}{\partial k}\right)_x \tag{3}$$

(where the subscript x signifies a derivative keeping x constant) and deduce

$$\frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial x} = 0, \tag{4}$$

because

$$\left(\frac{\partial k}{\partial t}\right)_x \left(\frac{\partial\omega}{\partial k}\right)_x = \left(\frac{\partial\omega}{\partial t}\right)_x. \tag{5}$$

Equation (4) says that the frequency ω remains constant along paths (Fig. 2) satisfying

$$\frac{dx}{dt} = u. \tag{6}$$

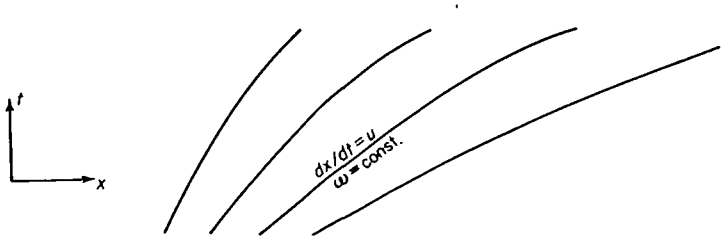

FIG. 2. Inhomogeneous one-dimensional system. Geometrical significance of equation (4) which governs wave dispersion.

Here, u will vary with position along such a path, but in a known manner because it can be expressed as a function of x and of the frequency ω which is constant along the path. Wave packets of a given frequency propagate, then, with velocity u although their crests travel with the quite different velocity $v = \omega/k$.

All this argument can be put into strictly physical language, in terms of an observer who travels (say) with velocity U, and who therefore passes $Uk-\omega$ crests per second. If another observer follows on behind him, passing every point a time T later, and the two observers follow a path on which the frequency ω remains constant, then the number of wavecrests between the observers is ωT which is constant, so that the value of $Uk-\omega$, the number of wavecrests per second passed by each, must be the same. It follows that

$$U = \frac{\Delta\omega}{\Delta k}, \tag{7}$$

which is the change in frequency during time T at a fixed point divided by the change in wavenumber during the same time interval at the same point—in full agreement with expression (3).

It is, of course, quite possible for waves of two more different frequencies to have the same value of the group velocity u; in this case, all of them may be found travelling together along the same path, where an apparently irregular motion would be observed

and Fourier analysis would be needed to resolve it into a small number of sine wave components. The argument I have given could nevertheless be justified by applying it separately to these different components. Another case where apparent local chaos is intelligible only in the light of Fourier analysis is when one wave packet passes through another wave packet which has lower group velocity.

The solution of equation (6) can be written in various forms, but the most convenient is one in which we begin by writing the wavenumber k as a function of the frequency ω rather than the other way round. This is because ω remains constant along the path. Then the equation of such paths in an (x,t) diagram is

$$\begin{aligned} t &= \int \frac{dx}{u} + \text{constant} \\ &= \int \left(\frac{\partial k}{\partial \omega}\right)_x dx + \text{constant} \\ &= \frac{\partial}{\partial \omega} \int k(\omega,x)dx + \text{constant}, \end{aligned} \tag{8}$$

a form requiring us merely to integrate a known function with respect to one of its variables and differentiate it with respect to the other.

If, furthermore, we argue that the total energy in each part of the frequency spectrum remains constant, then it follows that the energy between two adjacent paths must remain constant. Obviously this can be used to infer changes in amplitude if the relationship of energy density to amplitude is known. This method can be used just as reliably for two- and three-dimensional propagation in isotropic systems, where the normal law of dependence of energy density on amplitude is modified by an additional factor x^{n-1}, where n is the number of dimensions. However, in the neighbourhood of a caustic, i.e. an envelope of paths, the method gives a locally infinite value of the energy density, which in reality is not found; the true peak in energy density near the caustic cannot be calculated by this simple approximate method, which as we shall see has to be replaced by a more refined one within a few wavelengths of the caustic.

3. The Method of Stationary Phase (One-dimensional)

The method so far described gives results quickly, but is not demonstrably firm on its foundations, which therefore need bolstering up with some rather more rigorous analysis of the result of linearly combining waves of different length when the frequency is a function of the wavelength. The classical argument of Stokes (1876) infers a surprising amount from the simple special case when waves with just two values of wavelength are combined:

$$\begin{aligned} &a\cos 2\pi(k_1x-\omega_1t)+a\cos 2\pi(k_2x-\omega_2t) \\ &\quad = \{2a\cos 2\pi[(k_2-k_1)x-(\omega_2-\omega_1)t]\}\cos 2\pi[(k_1+k_2)x-(\omega_1+\omega_2)t]. \end{aligned} \tag{9}$$

The first term on the right is a slowly varying amplitude for the rapidly varying second term, so that the formula can be interpreted (Fig. 3) as a series of packets travelling with the velocity

$$U = \frac{\omega_2-\omega_1}{k_2-k_1} \tag{10}$$

and incapable of exchanging energy with one another through the nodal points where this amplitude is zero. It might be expected from this that a general wave packet, for which neighbouring portions of wave have only slightly different wavenumbers, would

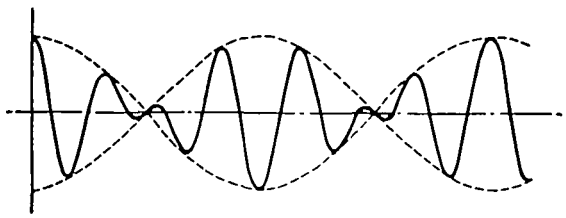

FIG. 3. The linear combination of two sine-waves interpreted as a series of wave packets (equation (9)).

propagate at a speed equal to the limit of (10) for low frequency difference, namely,

$$u = \frac{d\omega}{dk}. \tag{11}$$

Obviously, the argument is still far too special, but it contains one idea that is applicable to much more general cases. A point where the amplitude of (9) takes the full value $2a$, equal to the sum of the amplitudes of the component waves, travels along at the velocity (10) for one very good reason: this point passes wavecrests of the two wave systems at rates, $Uk_1-\omega_1$ and $Uk_2-\omega_2$, which are *equal*. It is possible, therefore, for these wavecrests to remain in phase, which for any other value of U could not happen.

We now consider a much more general combination of waves of different lengths in a homogenous system where ω is a function of k. Such a general linear combination is

$$\phi = \int_{-\infty}^{\infty} f(k) \exp\{2\pi i[kx-\omega(k)t]\}\, dk. \tag{12}$$

We consider the problem of estimating this integral when t is large and x may also be large. We assume that ω is an analytic function of k. We also assume that that the form taken initially, when $t = 0$, by the dependent variable ϕ in which we are interested, namely

$$\phi = \int_{-\infty}^{\infty} f(k)\, \mathrm{e}^{2\pi ikx} dk, \tag{13}$$

represents a disturbance confined to a limited region, that is, to a limited interval of values of x. This requires that $f(k)$ also is an analytic function of k.

We shall see that the main contribution to the integral (12) from a given small interval of wavenumber is found when x and t have values such that the phase $kx-\omega(k)t$ is stationary, i.e. practically constant throughout the interval, so that once more it is possible for the crests of different wave components in this interval to reinforce one another, instead of tending to cancel out by interference. We can prove this most easily (Jeffreys & Jeffreys, 1950, p. 505) by writing the phase in equation (12) as $t\psi(k)$, so that

$$\psi(k) = k\frac{x}{t} - \omega(k). \tag{14}$$

We then transform the integral into one with respect to ψ, namely,

$$\phi = \int f \frac{dk}{d\psi} e^{2\pi i t\psi} d\psi, \tag{15}$$

the integral being over that set of values of ψ that corresponds to the interval $-\infty < k < \infty$.

If $\psi(k)$, which is an analytic function of k, is also one-valued, with $\psi'(k)$ everywhere positive, or everywhere negative, then $f\,dk/d\psi$ is an analytic function of ψ itself, and the theory of Fourier integrals then tells us that, for large t, ϕ is very small, of smaller order than any inverse power of t. The situation is different, however, if $\psi'(k)$ becomes zero anywhere. This usually implies that more than one value of k corresponds to each value of ψ, but, more important for our purpose, it means that $dk/d\psi$ is singular. Where $\psi'(k)$ has a simple zero, $dk/d\psi$ has an inverse-square-root singularity; where $\psi'(k)$ has a double zero, $dk/d\psi$ has an inverse-two-thirds-power singularity, and so on. We can then use the theory of the asymptotic behaviour of Fourier integrals (Lighthill, 1958), which tells us that the asymptotic value of equation (15) for large t is determined by the behaviour of $f\,dk/d\psi$ at its *worst singularities*. These are the zeros of $\psi'(k)$, in other words, the points of stationary phase. In the commonest case, when only simple zeros are present, there is a contribution from the inverse-square-root singularity corresponding to each. If the zeros of $\psi'(k)$ are $k_1, k_2, \ldots, k_m$, the asymptotic form of the integral (15) is

$$\begin{aligned}\phi &\sim \sum_{r=1}^{m} \frac{f(k_r)\exp\left[2\pi i t\psi(k_r) + \tfrac{1}{4}\pi i \operatorname{sgn}\psi''(k_r)\right]}{\left[t\,|\,\psi''(k_r)\,|\,\right]^{\frac{1}{2}}} \\ &= \sum_{r=1}^{m} \frac{f(k_r)\exp\left\{2\pi i[k_r x - \omega(k_r)t] - \tfrac{1}{4}\pi i \operatorname{sgn}\omega''(k_r)\right\}}{\left[t\,|\,\omega''(k_r)\,|\,\right]^{\frac{1}{2}}}.\end{aligned} \tag{16}$$

Here $\operatorname{sgn}\psi''(k_r)$ is $+1$ where $\psi''(k_r) > 0$ so that ψ has a minimum and is -1 where $\psi''(k_r) < 0$ so that ψ has a maximum.

Physically, the zeros of $\psi'(k)$ are the points where

$$x = t\omega'(k), \tag{17}$$

that is, the values of the wavenumber k for which the speed of propagation $\omega'(k)$ of the energy associated with that wavenumber carries it a distance x during the (large) time t. There is only one term in the sum (16) when the group velocity $\omega'(k)$ takes each value only once, but in general, as mentioned before, two or more wave packets may be superimposed on one another and travel along together because they happen to have the same value of the group velocity.

The amplitude falls off like the inverse square root of the time because after time t the energy is spread over a distance proportional to t, so that the energy density varies as t^{-1} and the amplitude, therefore, as $t^{-\frac{1}{2}}$. Specifically, the energy between wavenumbers k_r and $k_r + dk$ is initially $Ef^2(k_r)\,dk$, where E is some constant, and later spreads out to fill a distance

$$\left|\, t\omega'(k_r) - t\omega'(k_r + dk)\,\right| = t\,|\,\omega''(k_r)\,|\,dk. \tag{18}$$

Then, therefore, the energy density is

$$\frac{Ef^2(k_r)}{t\,|\,\omega''(k_r)\,|}, \tag{19}$$

consistently with the value

$$\frac{f(k_r)}{[t\,|\,\omega''(k_r)\,|\,]^{\frac{1}{2}}} \tag{20}$$

for the amplitude. Thus, in the sum (16) the only extra piece of information that we would not have been able to predict from the kinematic analysis is the phase factor

$$\exp\left[-\tfrac{1}{4}\pi i \operatorname{sgn}\omega''(k_r)\right] \tag{21}$$

which simply decreases the phase by 45° where ω'' is positive and increases it by the same amount where ω'' is negative.

The special case, when $\omega''(k_r) = 0$ for a particular solution $k = k_r$ of the equation $x/t = \omega'(k)$, needs individual treatment, and corresponds to the caustic that I mentioned earlier (Fig. 4). The simple form (16) of the solution is then not applicable.

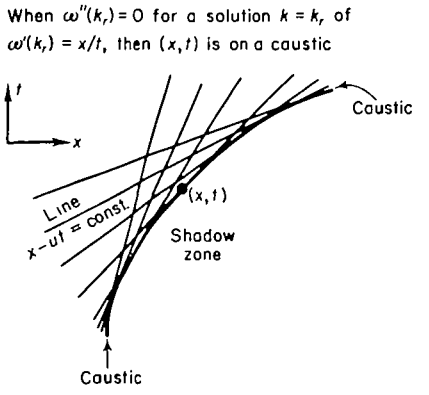

FIG. 4. Homogeneous one-dimensional systems; geometry of a caustic.

The integral (15) with respect to ψ has an inverse-two-thirds-power singularity, and the asympotic contribution from waves of wavenumber k_r is

$$\frac{f(k_r)\exp\{2\pi i[k_r x - \omega(k_r)t]\}(\frac{1}{3})!\sqrt{3}}{[\frac{1}{3}\pi t\omega'''(k_r)]^{\frac{1}{3}}}, \tag{22}$$

falling off only as the inverse cube root of the time. Sometimes we need to approximate ϕ not actually on the caustic, as here, but rather near it. Then it is best to evaluate the contribution from $k = k_r$, where $\omega''(k_r)$ is nearly but not exactly zero, by replacing $\omega(k)$ in the original Fourier integral (13) by the first four terms of its Taylor expansion about $k = k_r$ and evaluating the remaining integral in terms of the Airy function (Jeffreys & Jeffreys, 1950, p. 508). This function decays exponentially on the shadow side of the caustic, and is oscillatory on the other side. In between it rises to a peak close to the value (22).

The analysis I have just given in the particular case of systems homogenous in x has confirmed the value of the approximation derived from the kinematics of wave-crests. Now, instead of duplicating the extension to the non-homogenous case, which is cumbersome by this method, I shall give rather a brief analysis of the extension to dissipative systems, considering only such small dissipation as reduces the wave amplitude by only a small fraction in one wave period. I suppose in fact that for

waves of wavenumber k the amplitude is reduced by a factor $e^{-2\pi\sigma(k)t}$ in time t, where $\sigma(k) \ll \omega(k)$. (We assume further in what follows that $\sigma'(k) \ll \omega'(k)$.) Then $\omega(k)$ must be replaced by

$$\omega(k) - i\sigma(k) \tag{23}$$

in the integral (12). The analysis then goes through as before, but the k_r must satisfy

$$\omega'(k_r) - i\sigma'(k_r) = \frac{x}{t}. \tag{24}$$

An approximate solution for small σ is

$$k_r = k_r^{(0)} + \frac{i\sigma'(k_r^{(0)})}{\omega''(k_r^{(0)})}, \tag{25}$$

where $k_r^{(0)}$ is the solution with no dissipation, satisfying

$$\omega'(k_r^{(0)}) = \frac{x}{t}. \tag{26}$$

When we use the earlier expression (16), with $\omega(k)$ replaced by (23), it becomes

$$\phi \sim \sum_{r=1}^{m} \frac{f(k_r^{(0)}) \exp\{2\pi i[k_r^{(0)}x - \omega(k_r^{(0)})t] - 2\pi\sigma(k_r^{(0)})t - \frac{1}{4}\pi i \operatorname{sgn} \omega''(k_r^{(0)})\}}{\{t \mid \omega''(k_r^{(0)}) \mid\}^{\frac{1}{2}}} \tag{27}$$

to a first approximation, when σ is small. The replacement here of k_r by $k_r^{(0)}$ can be shown to introduce very small errors, in particular because $k_r^{(0)}$ is a point where the phase term in square brackets is stationary.

The physical meaning of the asymptotic form (27) is simply that every wave packet exhibits a time rate of decay of amplitude to a close approximation the same as for periodic waves of the same wavelength, and that in other respects the theory of group velocity is not affected by the presence of a small amount of dissipation. By contrast, when the dissipation rate $\sigma(k)$ is comparable with the total variation in $\omega(k)$, as is the case in anomalous dispersion of an electromagnetic wave, the theory of group velocity is affected very greatly indeed (Stratton, 1941).

4. The Kinematics of Wave Crests (Homogeneous Anisotropic Systems)

But it is time to consider the propagation of waves in anisotropic systems. These are important, of course, in relation to waves such as electromagnetic or elastic waves in anisotropic media such as crystals; they are also important in relation to waves in the presence of an external field; for example, rotation about an axis makes many types of mechanical waves anisotropic; so does a gravitational field when the medium has non-uniform density, and a magnetic field when it is conductive. Now, it is reasonable, encouraged by the success of the method which used the kinematics of wave crests in homogeneous one-dimensional systems, where the conclusions were fully substantiated by the more rigorous analysis, to apply the same method to anisotropic three-dimensional systems, first homogeneous and then non-homogeneous (Whitham, 1960; see also Hamilton, 1837). In making this jump we are leaving out the quite important two-dimensional systems, but that gap is, obviously, rather easy to fill in.

In this three-dimensional case, the geometry and kinematics of the wavecrests can be specified (Fig. 5) by means of a phase function $U(x_1,x_2,x_3,t)$, which is continuous and changes by unity between one crest and the next; more generally, the phase

function U changes by 1 in any distance of one wavelength perpendicular to the surfaces of constant U. Thus, we suppose that each of the local physical quantities takes the form

$$\mathscr{R}\{a(x_1,x_2,x_3,t)\exp[2\pi iU(x_1,x_2,x_3,t)]\}, \tag{28}$$

in which a complex amplitude, in general different for each quantity, is multiplied by a phase factor the same for each quantity. Obviously, different physical quantities may

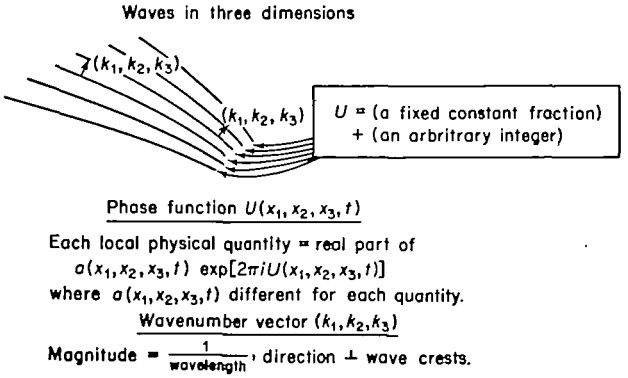

FIG. 5. Phase function and wavenumber vector for waves in three dimensions.

have their crest or peak values at different places, but the function U describes all of them; the crests for every physical quantity lie on curves of the form

$$U = (\text{a fixed constant fraction}) + (\text{an arbitrary integer}).$$

We suppose, furthermore, that a functional relationship exists between frequency and wavenumber. The wavenumber in three dimensions has direction as well as magnitude (Fig. 5); its magnitude, as usual, is the reciprocal of the wavelength, and its direction is perpendicular to the wavecrests. If the components of the wavenumber are (k_1,k_2,k_3), then locally the phase factor varies in proportion to

$$\exp[2\pi i(k_1x_1+k_2x_2+k_3x_3-\omega t)]. \tag{29}$$

This means that

$$k_1 = \frac{\partial U}{\partial x_1},\ k_2 = \frac{\partial U}{\partial x_2},\ k_3 = \frac{\partial U}{\partial x_3},\ \omega = -\frac{\partial U}{\partial t}. \tag{30}$$

I have said that, in a homogeneous anisotropic system, we assume that the frequency is a function of the wavenumber vector:

$$\omega = f(k_1,k_2,k_3). \tag{31}$$

In other words,

$$-\frac{\partial U}{\partial t} = f\left(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \frac{\partial U}{\partial x_3}\right). \tag{32}$$

We may then ask how wave packets of given wavenumber vector (k_1,k_2,k_3), and hence also given frequency, propagate. We obtain the answer by differentiating equation (32) with respect to x_α (where α may be 1, 2 or 3). This differentiation gives

$$-\frac{\partial^2 U}{\partial x_\alpha \partial t} = \frac{\partial f}{\partial k_1}\frac{\partial^2 U}{\partial x_\alpha \partial x_1} + \frac{\partial f}{\partial k_2}\frac{\partial^2 U}{\partial x_\alpha \partial x_2} + \frac{\partial f}{\partial k_3}\frac{\partial^2 U}{\partial x_\alpha \partial x_3}, \tag{33}$$

which can also be written as

$$\frac{\partial k_\alpha}{\partial t}+\frac{\partial f}{\partial k_1}\frac{\partial k_\alpha}{\partial x_1}+\frac{\partial f}{\partial k_2}\frac{\partial k_\alpha}{\partial x_2}+\frac{\partial f}{\partial k_3}\frac{\partial k_\alpha}{\partial x_3}=0. \tag{34}$$

This states simply that the wavenumber vector k_α remains constant under changes in the time and the three position co-ordinates which are in these ratios:

$$\frac{dx_1}{dt}=\frac{\partial f}{\partial k_1},\quad \frac{dx_2}{dt}=\frac{\partial f}{\partial k_2},\quad \frac{dx_3}{dt}=\frac{\partial f}{\partial k_3}. \tag{35}$$

Thus, waves of given wavenumber (and hence also of given frequency) propagate at a velocity

$$\mathbf{u}=\left(\frac{\partial\omega}{\partial k_1},\frac{\partial\omega}{\partial k_2},\frac{\partial\omega}{\partial k_3}\right), \tag{36}$$

which is the gradient of the function $f(k_1,k_2,k_3)$ with respect to the wavenumber vector. This formula for group velocity is seen to be a natural extension of the one-dimensional result.

The direction in which these waves propagate is normal to the so-called wavenumber surface, that is, the surface of constant frequency $f(k_1, k_2, k_3)$ = constant. Furthermore, since they are waves with fixed values of k_1, k_2 and k_3, the direction and the speed of propagation remain constant for the wave packet, which therefore travels in a straight line. On the other hand, both the speed and direction of movement of the wavecrests are quite different, being represented actually by the vector

$$\mathbf{v}=\left(\frac{\omega k_1}{k_1^2+k_2^2+k_3^2},\frac{\omega k_2}{k_1^2+k_2^2+k_3^2},\frac{\omega k_3}{k_1^2+k_2^2+k_3^2}\right). \tag{37}$$

To sum up, energy travels in straight lines in a homogeneous anistropic system, with a direction and speed given by the group velocity vector (36), and both are in general different from the phase velocity's direction and magnitude.

We can illustrate this result by considering the particular case when the waves originate from a small region during a small interval of time (Lighthill, 1960). In this case, waves can spread out from this region in straight lines in all directions in which normals to the wavenumber surface (f = constant) lie. Some of these directions may, actually, be normal to that surface at several points. In such a direction, waves with wavenumber corresponding to each one of these points may be found, and will be found if waves with the wavenumber in question were among those originally generated. However, the magnitude of the group velocity will in general be different for each, so that the wave packets with different wavenumbers will at a given time have reached different points along that direction.

Figure 6 illustrates this for waves generated at an approximately constant frequency during a short time interval in a homogeneous plasma in a uniform magnetic field, when the magnetic pressure is small compared with the plasma pressure. If the frequency, divided by the gyrofrequency of the ions, is negligibly small, then the wavenumber surface is a plane (marked "0" in the figure). This means that energy can be propagated only in the direction normal to the plane.

Magnetohydrodynamicists are, in fact, familiar with the proposition that disturbances are, in such a case, transmitted only along magnetic lines of force. However, when the ratio of the frequency to the ion gyrofrequency is not negligibly small, the

wavenumber surface splits into two. When, for example, the ratio is one-quarter, the two sheets are those marked "$\frac{1}{4}$" in the figure. The energy of waves whose wavenumber corresponds to a given point on the surface propagates in the direction of the normal to the surface at that point, and in this particular case the directions normal to the left-hand sheet all lie within a certain cone N_1; while those normal to the right-hand sheet lie within a somewhat larger cone N_2. All the energy that is created remains, therefore, within this larger cone N_2, while the speed with which energy of a given wavenumber is propagated is the gradient of frequency along the normal to the wavenumber surface at the corresponding point. The upper diagram on the right shows the waves, that were created in a certain short time interval, at a given later

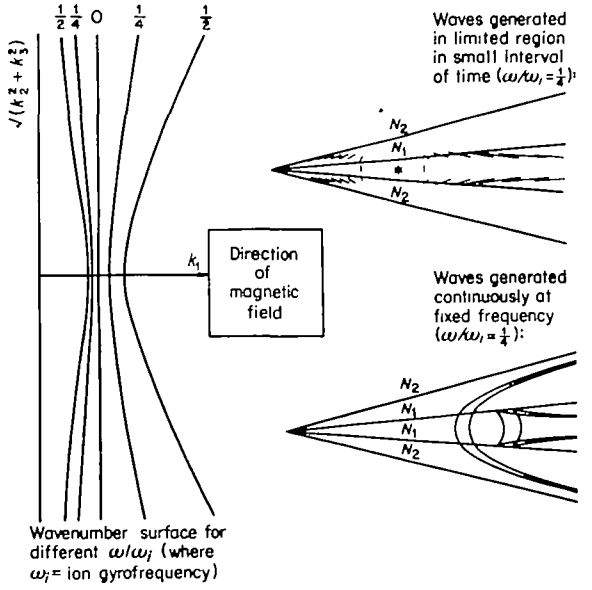

FIG. 6. Diagrams illustrating waves in a homogeneous plasma in a uniform magnetic field, when the magnetic pressure is small compared with the plasma pressure.

time t. Those associated with the left-hand sheet all lie within the cone N_1, and have travelled considerably further than those associated with the right-hand sheet (which lie within N_2). The spacing and angle of the crests is representative of the direction and magnitude of the local wavenumber vector. From a physical point of view the interest lies in the wide dispersion of the waves exhibited already when the ratio of frequency to ion gyrofrequency is one-quarter, a situation completely different from that in which the ratio is very small, when all the energy would be concentrated at the point marked with a star.

This upper diagram illustrates, then, the problems arising when the waves are all generated during a small interval. Quite a different appearance is assumed by the waves when they are, instead, generated continuously, at a fixed frequency, within a small region. The different magnitude of the group velocity in different directions is not then particularly important; time is unlimited, and so waves are found at all distances from the region in every direction in which they propagate. The angle of the crests for waves propagated in a given direction is, however, the same as in Fig. 5. It is not

difficult, in fact, to calculate the shape of one of the crests from this condition. Such a surface of constant phase for the waves generated turns out to be what the geometers have for a long time called the reciprocal polar of the wavenumber surface, that is the locus of the poles of the tangent planes to that surface. This follows from the fact that the point x_α on a surface of constant phase, corresponding to a given point k_α on the wavenumber surface, must lie in the direction perpendicular to the tangent plane at that point, and is furthermore at a distance inversely proportional to the distance from the origin to that plane, because the scalar product of x_α and k_α^* must be constant on a surface of constant phase. For the problem we have just been discussing some surfaces of constant phase for continuously excited waves are shown in the lower diagram on the right of Fig. 6. The somewhat complicated look of these waves is in only apparent contradiction with the very simple law governing their propagation: namely, that the energy travels in straight lines.

5. The Kinematics of Wavecrests (Inhomogeneous Anisotropic Systems)

It is desirable to turn now to the inhomogeneous case, when the relation between frequency and wavenumber varies with position, although at any fixed point it does not change with time. Thus we assume that

$$\omega = f(k_\alpha, x_\alpha). \tag{38}$$

Naturally, we expect that energy will continue to be propagated locally at the same velocity

$$u_\alpha = \frac{\partial \omega}{\partial k_\alpha}, \tag{39}$$

although this will now vary from point to point. Also, because the equations of motion do not involve the time explicitly, energy in waves of a given frequency should remain always in waves of the same frequency. However, the equations do involve the space co-ordinates explicitly, so that, as these waves propagate into different regions of space, their wavenumber vector k_α can be expected to change. We may hope, therefore, first, to prove that, when changes in x_α and t are in a proportion given by the velocity (39), that is, when

$$\frac{dx_\alpha}{dt} = \frac{\partial f}{\partial k_\alpha}, \tag{40}$$

the frequency ω remains constant, and, secondly, we may hope to find how the wavenumber vector k_α varies.

To do both these things, we write equation (38) as

$$-\frac{\partial U}{\partial t} = f\left(\frac{\partial U}{\partial x_\alpha}, x_\alpha\right), \tag{41}$$

and differentiate with respect to x_α, giving

$$-\frac{\partial^2 U}{\partial x_\alpha \partial t} = \frac{\partial f}{\partial k_1}\frac{\partial^2 U}{\partial x_\alpha \partial x_1} + \frac{\partial f}{\partial k_2}\frac{\partial^2 U}{\partial x_\alpha \partial x_2} + \frac{\partial f}{\partial k_3}\frac{\partial^2 U}{\partial x_\alpha \partial x_3} + \frac{\partial f}{\partial x_\alpha}.$$

This can be rewritten as

$$\frac{\partial k_\alpha}{\partial t} + \frac{\partial f}{\partial k_1}\frac{\partial k_\alpha}{\partial x_1} + \frac{\partial f}{\partial k_2}\frac{\partial k_\alpha}{\partial x_2} + \frac{\partial f}{\partial k_3}\frac{\partial k_\alpha}{\partial x_3} = -\frac{\partial f}{\partial x_\alpha}, \tag{42}$$

and we can interpret this by saying that when x_1, x_2, x_3 and t change according to equation (40), k_α changes according to the equation

$$\frac{dk_\alpha}{dt} = -\frac{\partial f}{\partial x_\alpha}. \tag{43}$$

This very simple rule, governing how the wavenumber associated with a given packet of energy varies as the packet is propagated through space, is very strongly reminiscent of the Hamiltonian form of the equations of motion of a dynamical system, and it comes as no surprise to learn that this rule also was discovered by Hamilton; see for example equation (T^2) on p. 182 of Hamilton (1837). Equations (40) and (43) have to be solved simultaneously to find out how the wave packet is propagated through x_α space and k_α space. With modern computational aids this is no great problem, however, whereas the direct computation of the waves themselves, involving a complicated partial differential equation with four independent variables, is expected to remain out of reach of such aids for some time to come.

The check, that, when f is a function of k_α and x_α without explicit dependence on the time, the two equations (40) and (43) imply that the frequency, i.e. the value of f itself, remains constant for the wave packet, is now very straightforward, being identical with the proof that, for a dynamical system whose Hamiltonian is without explicit dependence on the time, every motion of the system carries a constant value of the Hamiltonian, that is, of the total energy. The fact that frequency behaves like energy in this way and wavenumber like momentum is directly related to the fact that quantum mechanics associates with every particle a wave of frequency proportional to its energy and of wavenumber proportional to its momentum.

As one simple illustration of the practical application of these results, I should like to refer to the propagation of sound through an atmosphere in non-uniform motion. The effects of non-uniform winds influence greatly the propagation of sound over long distances and needed to be studied very seriously when accurate calculations of the possible intensity on the ground of "booms" from supersonic aircraft were being made (Warren, 1964). Now, the relation between frequency and wavenumber at a point where the sound speed is a and the wind speed is (V_1, V_2, V_3) is

$$\omega = V_1k_1 + V_2k_2 + V_3k_3 + a\sqrt{(k_1^2 + k_2^2 + k_3^2)}. \tag{44}$$

This gives, first, that the group velocity, with which energy is propagated, is

$$u_\alpha = \frac{\partial \omega}{\partial k_\alpha} = V_\alpha + a\frac{k_\alpha}{\sqrt{(k_1^2 + k_2^2 + k_3^2)}}, \tag{45}$$

namely, the vector sum of the wind velocity and of a vector whose magnitude is a and whose direction is normal to the wavecrests. This result may appear rather obvious, but one of the very few mistakes in Rayleigh's *Theory of Sound* occurs in Section 289, where the energy is assumed in this problem to be propagated exactly at right angles to the wavecrests. Secondly, it gives that, as energy travels along one of these rays of sound, at the group velocity, the wavenumber vector changes according to the law

$$\frac{dk_\alpha}{dt} = -\frac{\partial V_1}{\partial x_\alpha}k_1 - \frac{\partial V_2}{\partial x_\alpha}k_2 - \frac{\partial V_3}{\partial x_\alpha}k_3 - \frac{\partial a}{\partial x_\alpha}\sqrt{(k_1^2 + k_2^2 + k_3^2)}. \tag{46}$$

In the most general case, the rays must be computed by numerically solving this, with the equation for dx_α/dt, as a system of ordinary differential equations, which is relatively straightforward. One particular type of inhomogeneity, however, is of great practical importance and allows us to reduce the problem to a simple integration. When the relation between ω and k_α depends on only one space co-ordinate, say x_1, which in the atmospheric problem might be the altitude, then the wavenumber components k_2 and k_3 remain constant, and the differential equation for the component k_1 has the simple integral ω = constant. This enables the rays to be calculated directly, by integrating equation (40) with k_1 given by the condition ω = constant.

For example, in the atmospheric case with only the V_2 component of wind non-zero, equation (44) for the frequency ω gives

$$k_1 = \sqrt{[(\omega - V_2 k_2)^2 a^{-2} - (k_2^2 + k_3^2)]}. \tag{47}$$

The equation (40) for the rays can then be written

$$\frac{dx_2}{dx_1} = \frac{V_2\omega + k_2(a^2 - V_2^2)}{a\sqrt{[(\omega - V_2 k_2)^2 - a^2(k_2^2 + k_3^2)]}},$$
$$\frac{dx_3}{dx_1} = \frac{k_3 a}{\sqrt{[(\omega - V_2 k_2)^2 - a^2(k_2^2 + k_3^2)]}}, \tag{48}$$

which can be numerically integrated with exceptional ease since the right-hand sides are just known functions of x_1. The acoustic amplitudes are then determined from the condition that energy propagates along ray tubes in inverse proportion to the cross-sectional area of a ray tube. A convenient form of this condition is that

$$u_1 \mathscr{E} \frac{\partial(x_2, x_3)}{\partial(k_2, k_3)} = \text{constant along a ray}, \tag{49}$$

where $\mathscr{E}$ is the energy density.

Hitherto I have discussed inhomogeneous systems in which the relation between ω and k_α varies with the space co-ordinates only. In the still more general case, which however is less often of practical importance, when it depends explicitly also on the time, the results can be put in a similar form most neatly if we use the idea from relativity of regarding the time simply as a fourth co-ordinate, x_4, and the frequency as minus a wavenumber component,

$$\omega = -\frac{\partial U}{\partial x_4} = -k_4. \tag{50}$$

The most general relation between frequency, wavenumber, time and the space co-ordinates can then be written

$$F(k_\lambda, x_\lambda) = 0, \tag{51}$$

where λ goes from 1 to 4 and k_λ is the derivative $\partial U/\partial x_\lambda$ of the phase function with respect to x_λ.

If now we differentiate (51) with respect to x_λ, we obtain

$$\sum_{\mu=1}^{4} \frac{\partial F}{\partial k_\mu} \frac{\partial^2 U}{\partial x_\lambda \partial x_\mu} + \frac{\partial F}{\partial x_\lambda} = 0, \tag{52}$$

which says that if a ray in space–time is defined by the equations

$$\frac{dx_\mu}{d\tau} = \frac{\partial F}{\partial k_\mu}, \tag{53}$$

where τ is a parameter, then along such a ray

$$\frac{dk_\lambda}{d\tau} = -\frac{\partial F}{\partial x_\lambda}. \tag{54}$$

Thus, the Hamiltonian form of the equations is preserved in this still more general case. Professor Synge in his lectures pointed out that the same equations would describe an even more general situation, namely, the situation defined by specifying that a ray is a path in space–time between two points such that the integral

$$\int (k_1\,dx_1 + k_2\,dx_2 + k_3\,dx_3 + k_4\,dx_4) \tag{55}$$

along the path is stationary, where the k_λ are constrained to satisfy $F(k_\lambda, x_\lambda) = 0$. This integral represents the phase change between the two points, so that stationary phase is being applied even though, in this greatly generalized theory, the quantity in brackets is no longer assumed equal to the exact derivative of a phase function U.

6. The Method of Stationary Phase (Decay of an Initial Disturbance in Three Dimensions)

I shall not pursue this any farther, however, but rather go back to the case of the homogeneous system and check the results inferred from the kinematics of wave-crests against an asymptotic analysis of the result of linearly combining waves of different wavenumber vector when the frequency is a function of the wavenumber vector. It is interesting to sketch briefly the methods involved in this, even though there is not time to describe them in detail. I shall consider separately two types of problem; the first is that of how a disturbance, initially confined to a small region, spreads out in space.

We shall suppose that it can be Fourier-analysed as

$$\phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_\alpha) \exp\{2\pi i[k_1x_1 + k_2x_2 + k_3x_3 - \omega(k_\alpha)t]\}\,dk_1\,dk_2\,dk_3, \tag{56}$$

so that initially (when $t = 0$) we have

$$\phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_\alpha) \exp[2\pi i(k_1x_1 + k_2x_2 + k_3x_3)]\,dk_1\,dk_2\,dk_3. \tag{57}$$

The same methods as in the one-dimensional case can be used to show (see, for example, Chako, 1965) that the asymptotic behaviour of the integral (56) for large t is dominated by terms associated with the points, if any, where the phase term in square brackets is stationary, that is, where

$$x_\alpha = \frac{\partial \omega}{\partial k_\alpha} t = u_\alpha t. \tag{58}$$

This says that waves of a given wavenumber travel a distance t times the group velocity in time t.

In order to derive the contribution from one such point, say $k_\alpha^{(r)}$, where the phase is stationary, we might hope to reduce the problem to a one-dimensional one, by expanding the phase in a Taylor series up to terms quadratic in $k_\alpha - k_\alpha^{(r)}$ and then

estimating the integral as a product of terms involving separate integrations with respect to k_1, k_2 and k_3. This is impeded in practice by the fact that the quadratic terms include not only the squares of $k_\alpha - k_\alpha^{(r)}$ but also products of two of these. The quickest way to get an answer is temporarily to rotate the axes in (k_1,k_2,k_3) space in such a way that in the new axes

$$\frac{\partial^2\omega}{\partial k_\alpha \partial k_\beta} = 0 (\alpha \neq \beta) \quad \text{at} \quad k_\alpha = k_\alpha^{(r)}. \tag{59}$$

(See Lighthill (1960, Appendix B), where the analysis given is more complicated, however, because the relationship between ω and k is supposed given in an implicit rather than explicit form.) Then the contribution to the integral (56) from $k_\alpha = k_\alpha^{(r)}$ can be approximated as a product of three simple error integrals

$$f(k_\alpha^{(r)}) \exp\{2\pi i[k_1^{(r)}x_1 + k_2^{(r)}x_2 + k_3^{(r)}x_3 - \omega(k_\alpha^{(r)})t]\} \int_{-\infty}^{\infty} \exp\left[-\pi i\left(\frac{\partial^2\omega}{\partial k_1^2}\right)^{(r)}(k_1 - k_1^{(r)})^2 t\right] dk_1$$
$$\int_{-\infty}^{\infty} \exp\left[-\pi i\left(\frac{\partial^2\omega}{\partial k_2^2}\right)^{(r)}(k_2 - k_2^{(r)})^2 t\right] dk_2 \int_{-\infty}^{\infty} \exp\left[-\pi i\left(\frac{\partial^2\omega}{\partial k_3^2}\right)^{(r)}(k_3 - k_3^{(r)})^2 t\right] dk_3$$
$$= \frac{f(k_\alpha^{(r)}) \exp\left\{2\pi i\left[\sum_{\alpha=1}^{3} k_\alpha^{(r)}x_\alpha - \omega(k_\alpha^{(r)})t\right] - \frac{1}{4}\pi i \sum_{\alpha=1}^{3} \operatorname{sgn} \frac{\partial^2\omega}{\partial k_\alpha^2}\right\}}{t^{\frac{3}{2}} \left|\left(\frac{\partial^2\omega}{\partial k_1^2}\frac{\partial^2\omega}{\partial k_2^2}\frac{\partial^2\omega}{\partial k_3^2}\right)^{(r)}\right|^{\frac{1}{2}}}. \tag{60}$$

Having found the answer in these special axes we can then throw it into a form invariant under rotation of axes by replacing the product of second derivatives in the denominator by the determinant

$$\Delta^{(r)} = \left| \begin{array}{ccc} \frac{\partial^2\omega}{\partial k_1^2} & \frac{\partial^2\omega}{\partial k_1 \partial k_2} & \frac{\partial^2\omega}{\partial k_1 \partial k_3} \\ \frac{\partial^2\omega}{\partial k_2 \partial k_1} & \frac{\partial^2\omega}{\partial k_2^2} & \frac{\partial^2\omega}{\partial k_2 \partial k_3} \\ \frac{\partial^2\omega}{\partial k_3 \partial k_1} & \frac{\partial^2\omega}{\partial k_3 \partial k_2} & \frac{\partial^2\omega}{\partial k_3^2} \end{array} \right|_{(k_\alpha = k_\alpha^{(r)})} \tag{61}$$

which is invariant under rotation of axes and in the special axes where the cross-derivatives vanish takes that value. In other words, if we write the asymptotic form of ϕ as the sum

$$\phi \sim \sum_r \frac{f(k_\alpha^{(r)}) \exp\left\{2\pi i\left[\sum_{\alpha=1}^{3} k_\alpha^{(r)}x_\alpha - \omega(k_\alpha^{(r)})t\right] + i\theta^{(r)}\right\}}{t^{\frac{3}{2}} \left|\Delta^{(r)}\right|^{\frac{1}{2}}} \tag{62}$$

over all $k_\alpha^{(r)}$ where $\sum_{\alpha=1}^{3} k_\alpha x_\alpha - \omega(k_\alpha)t$ is stationary, the phase addition term $\theta^{(r)}$ being $\frac{3}{4}\pi$ at a minimum (where the signs $\partial^2\omega/\partial x_\alpha^2$ in (60) are all -1), $-\frac{3}{4}\pi$ at a maximum, and $\frac{1}{4}\pi$ or $-\frac{1}{4}\pi$ at a saddle-point according as the function increases along two or along only one of the three principal directions, then each term is invariant under change of axes and therefore is valid in any system of axes, not just the one in which it was derived.

We can interpret this result physically, by using (58) to note that the volume of space occupied at time t by energy whose wavenumber vector lies in an elementary volume $dk_1\,dk_2\,dk_3$, centred on the value $k_\alpha^{(r)}$, is

$$dx_1\,dx_2\,dx_3 = \left[\left|\frac{\partial(x_1,x_2,x_3)}{\partial(k_1,k_2,k_3)}\right|\right]_{k_\alpha=k_\alpha^{(r)}} dk_1\,dk_2\,dk_3$$

$$= t^3\left[\left|\frac{\partial(u_1,u_2,u_3)}{\partial(k_1,k_2,k_3)}\right|\right]_{k_\alpha=k_\alpha^{(r)}} dk_1\,dk_2\,dk_3 = t^3\,|\,\Delta^{(r)}\,|\,dk_1\,dk_2\,dk_3. \qquad (63)$$

Accordingly, as this energy spreads out to fill greater and greater volume, the energy density must vary inversely as $t^3|\,\Delta^{(r)}\,|$ and therefore the amplitude would be expected to vary as the inverse square root of this as we found above.

7. The Method of Stationary Phase (Forced Motions in Three Dimensions)

After this brief treatment of the propagation and decay of an initially limited disturbance, I shall describe, though again only briefly, a second problem treated by the method of stationary phase in three dimensions, but intrinsically different in that the disturbance is of fixed frequency and is maintained by means of a steady source operating at that frequency within a limited region. This, then, is a problem of forced motion. Specifically, I suppose that the quantity to be determined, ϕ, satisfies the equation

$$P\left(\frac{1}{2\pi i}\frac{\partial}{\partial x_1}, \frac{1}{2\pi i}\frac{\partial}{\partial x_2}, \frac{1}{2\pi i}\frac{\partial}{\partial x_3}, -\frac{1}{2\pi i}\frac{\partial}{\partial t}\right)\phi = \mathrm{e}^{-2\pi i\omega_0 t} f(x_1,x_2,x_3), \qquad (64)$$

where P is some polynomial in the partial differential operators shown, and the source term of fixed frequency ω_0 on the right vanishes outside a limited region. The problem is to determine the form of ϕ at distances from that region large compared with its size.

In cases relevant to this lecture, that is, when the homogeneous equation (without a forcing term) $P\phi = 0$ possesses solutions in the form of waves, and the problem in the forced motion is to determine the wavenumbers and amplitudes of the waves found at large distances along each direction out from the source region, there is a well-known mathematical difficulty, namely, that equation (64) does not have a unique solution tending to zero at infinity. On the contrary, it has a large multiplicity of such solutions. Out of all these, however, only one is of any physical interest. This is the one commonly described as satisfying the "radiation condition". I should like to say something about how to obtain this unique solution of physical interest in rather general wave problems.

Out of many ways of deriving it, I am inclined to think that the most convenient and, physically, the most logical is to require that the steady-state wave motion must be "arrivable at by switching on (the source) and waiting" (see Lighthill, 1960, pp. 412–414 and also p. 430). Different methods of switching on can be considered, varying from the most abrupt, in which the right-hand side of equation (64) takes the form shown for $t > 0$ but is zero for $t < 0$, to far more gradual methods. The most gradual method is that in which the right-hand side is replaced by $\exp[2\pi(\varepsilon - i\omega_0)t]\,f(x_1,x_2,x_3)$ where ε is very small, so that the forcing term has grown to its present strength from zero during all the time from $t = -\infty$; we then find a solution ϕ proportional to $\exp[2\pi(\varepsilon - i\omega_0)t]$. This is, evidently, equivalent to allowing the frequency ω to have a

small positive imaginary part ε (which is later allowed to tend to zero). All these methods of switching on produce identical steady-state solutions, except in the not very physically important condition when the system is unstable to disturbances of certain wavenumbers and the forcing term f includes components with those wavenumbers, for which wave solutions increasing in amplitude exponentially with time are possible. In this condition the more abrupt methods of switching on can trigger off the instability while the very gradual method would not do so. In the other cases (which are of far greater practical importance) the different ways of switching on all give identical results. I shall not refer again to any except the very gradual method.

In this, we obtain a solution tending to zero at large distances from the source region by supposing that f and ϕ have three-dimensional Fourier transforms, in terms of which they can be written as

$$f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_1,k_2,k_3)\exp[2\pi i(k_1x_1+k_2x_2+k_3x_3)]\,dk_1\,dk_2\,dk_3 \qquad (65)$$

and as $\phi = \exp[2\pi(\varepsilon - i\omega_0)t]\phi_\varepsilon$, where

$$\phi_\varepsilon = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_\varepsilon(k_1,k_2,k_3)\exp[2\pi i(k_1x_1+k_2x_2+k_3x_3)]\,dk_1\,dk_2\,dk_3. \qquad (66)$$

Then equation (64) tells us

$$P(k_1,k_2,k_3,\omega_0+i\varepsilon)\Phi_\varepsilon = F. \qquad (67)$$

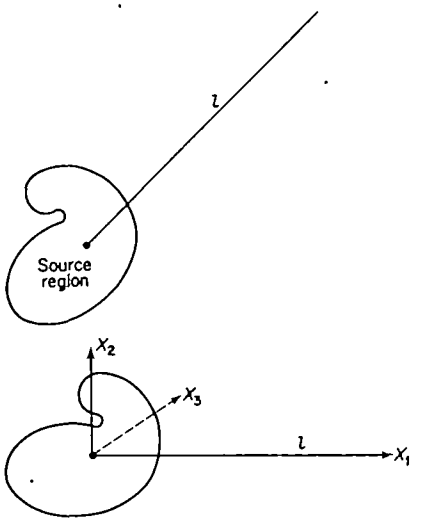

FIG. 7. To make estimates at large distances along a particular straight line l stretching away from the source region, we temporarily use the special axes shown.

The problem is to find ϕ_0, the limit of ϕ_ε as $\varepsilon \to 0$ from above, at large distances from the source region along any straight line l stretching away from it. As in the previous problem, this estimation is carried out most easily if we first effect a rotation of axes. In order to estimate ϕ_0 on a particular line l, we temporarily choose axes (Fig. 7)

such that l is the positive x_1-axis. On l, therefore, x_2 and x_3 are zero, while x_1 is positive, and

$$\phi_\varepsilon = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_2\,dk_3 \int_{-\infty}^{\infty} \frac{F(k_1,k_2,k_3)\,e^{2\pi i k_1 x_1} dk_1}{P(k_1,k_2,k_3,\omega_0+i\varepsilon)}. \tag{68}$$

We estimate the inner integral in equation (68) by moving the path of integration to one (Fig. 8) on which the imaginary part of k_1 takes a suitably chosen positive (constant) value h. The integral over the new path is small of order $e^{-2\pi h x_1}$ for large x_1, and we shall regard quantities of this order of magnitude as negligible. Between the two paths the only singularities of the integrand are poles at zeros of the denominator, since the vanishing of f outside a finite region implies that its Fourier transform F is regular. We must therefore calculate the sum of the residues at those poles.

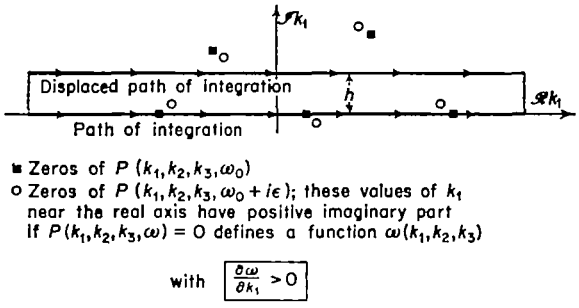

FIG. 8. Illustrating how the radiation condition is derived by displacement of the path of integration in equation (68).

Now, for the systems capable of wave propagation in which we are chiefly interested, there are zeros of the denominator for *real* k_1 when $\varepsilon = 0$ (at least in some ranges of k_2, k_3 and ω; these represent plane-wave solutions). For small positive ε these zeros may be expected to be displaced from the real axis, and will then contribute to the sum of residues only if they are displaced into the region where the imaginary part of k_1 takes *positive* values, between 0 and h. Evidently, giving ω a small positive imaginary part will give the k_1 for which $P = 0$ a positive imaginary part if the equation $P(k_1,k_2,k_3,\omega) = 0$ specifies the frequency ω as a function of the wavenumber (k_1,k_2,k_3) such that

$$\frac{\partial\omega}{\partial k_1} > 0. \tag{69}$$

We have already obtained the result, therefore, that wave energy will be found along the line l only if the component of the group velocity along l is positive. This, then, is the physical significance of the radiation condition. The contribution to the inner integral in equation (68) from each pole will be

$$2\pi i \frac{F(k_1,k_2,k_3)\,e^{2\pi i k_1 x_1}}{\partial P(k_1,k_2,k_3,\omega_0)/\partial k_1} \tag{70}$$

in the limit as $\varepsilon \to 0$ when this inequality is satisfied, and will be zero otherwise. Obviously P may have other zeros, with imaginary parts positive even for $\varepsilon = 0$, but if h is taken less then all those imaginary parts then these singularities do not lie

between the two paths for small enough ε. The asymptotic form of ϕ_0 is then given by substituting for the inner integral in (68) a sum of terms (70) from all real k_1 satisfying $P(k_1,k_2,k_3,\omega_0) = 0$ together with the inequality (69). In other words,

$$\phi_0 \sim 2\pi i \int\int_S \frac{F(k_1,k_2,k_3)\, e^{2\pi i k_1 x_1} dk_2\, dk_3}{\partial P(k_1,k_2,k_3,\omega_0)/\partial k_1}, \tag{71}$$

where the surface S of integration (Fig. 9) is the part of what I earlier called the wave-number surface (defining waves with $\omega = \omega_0$) on which $\partial\omega/\partial k_1 > 0$. This then is the answer for equations which possess plane-wave solutions; for those which do not, the limiting process involving ε does not have to be introduced, and the answer comes out even more simply in terms of those zeros of P which have *smallest* positive imaginary part.

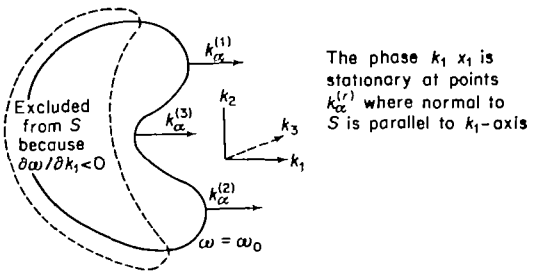

FIG. 9. The surface of integration S, and the points of stationary phase, in the integral (71).

I must return to this answer (71) now and simplify it still further. This is done by the method of stationary phase. The phase k_1x_1 is stationary on the wave-number surface S at those points $k_\alpha^{(r)}$ where the normal to S is parallel to the k_1-axis (Fig. 9). Having already fixed the k_1-axis, along the direction l in which we are estimating ϕ_0, we now find it easiest to calculate the contribution to (71) from each such position of stationary phase by a temporary choice of the k_2- and k_3-axes along the principal directions of curvature of the surface. If we take the associated curvatures κ_2 and κ_3 positive where concave to the positive k_1-direction and negative where convex, then locally the surface S has the approximate equation

$$k_1 = k_1^{(r)} + \tfrac{1}{2}\kappa_2(k_2 - k_2^{(r)})^2 + \tfrac{1}{2}\kappa_3(k_3 - k_3^{(r)})^2, \tag{72}$$

which, substituted in (71), makes it easy to calculate the contribution ϕ_r to the asymptotic form of ϕ_0 from the point $k_\alpha^{(r)}$ where the normal to S is in the k_1-direction as

$$\phi_r = 2\pi i \frac{F(k_\alpha^{(r)}) \exp\left[2\pi i k_1^{(r)} x_1 + \tfrac{1}{4}\pi i(\operatorname{sgn}\kappa_2 + \operatorname{sgn}\kappa_3)\right]}{\left[\partial P(k_1,k_2,k_3,\omega_0)/\partial k_1\right]_{k_\alpha = k_\alpha^{(r)}} x_1 \sqrt{|\kappa_2\kappa_3|}}. \tag{73}$$

These contributions from different points $k_\alpha^{(r)}$, which must be added up, are not in general in identical axes, so that they must first be put into forms invariant under rotation of axes as

$$\phi_r = \frac{2\pi F(k_\alpha^{(r)}) \exp\left[2\pi i(k_1^{(r)}x_1 + k_2^{(r)}x_2 + k_3^{(r)}x_3) + i\theta^{(r)}\right]}{(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}\left[\left(\frac{\partial P}{\partial k_1}\right)^2 + \left(\frac{\partial P}{\partial k_2}\right)^2 + \left(\frac{\partial P}{\partial k_3}\right)^2\right]^{\frac{1}{4}}_{k_\alpha = k_\alpha^{(r)}} |\,\mathrm{K}^{(r)}\,|^{\frac{1}{2}}} \tag{74}$$

where $\mathrm{K}^{(r)} = \kappa_2\kappa_3$ is the Gaussian curvature (product of the two principal curvatures) at $k_\alpha = k_\alpha^{(r)}$, and the phase addition term $\theta^{(r)}$ takes, when $\mathrm{K}^{(r)} > 0$, the value 0 where

the surface is convex to the direction P increasing and π where it is concave to that direction, and, when $\mathrm{K}^{(r)} < 0$, the value $\frac{1}{2}\pi$ or $-\frac{1}{2}\pi$ according as that direction is parallel or antiparallel to l. The asymptotic solution along l,

$$\phi = \phi_0 \, \mathrm{e}^{-2\pi i\omega_0 t} \sim \mathrm{e}^{-2\pi i\omega_0 t} \Sigma \phi_r, \tag{75}$$

where the summation is taken over points $k_\alpha^{(r)}$ of the surface $\omega = \omega_0$ where the normal to the surface is parallel to l and ω increases in the direction of l, is a solution with amplitude decreasing as the inverse first power of the distance $R = \sqrt{(x_1^2 + x_2^2 + x_3^2)}$.

The proportionality to the inverse square root of the modulus of the Gaussian curvature $\mathrm{K}^{(r)}$ at the point $k_\alpha^{(r)}$ can be understood physically as follows. The normals from a small area dS around the point $k_\alpha^{(r)}$ fill a cone (Fig. 10) whose cross-sectional

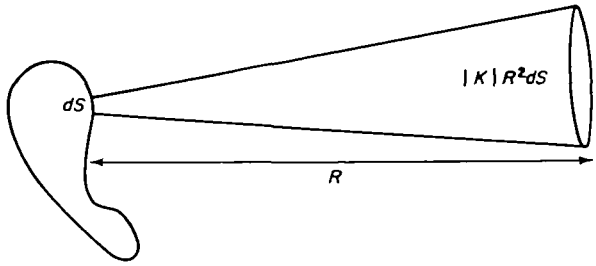

FIG. 10. Physical intepretation of the expression for waves produced in forced motion (equations (74) and (75)). The energy is diminished by a factor $|\mathrm{K}|^{-1}R^{-2}$ in distance R, and the amplitude by $|\mathrm{K}|^{-\frac{1}{2}}R^{-1}$.

area increases with distance R like $|\mathrm{K}^{(r)}| R^2 dS$. The energy created in the source region with wavenumbers lying in this elementary area dS is therefore diminished by a factor $|\mathrm{K}^{(r)}|^{-1}R^{-2}$ at distance R, which makes the factor on amplitude $|\mathrm{K}^{(r)}|^{-\frac{1}{2}}R^{-1}$.

There is not time to discuss further properties of the asymptotic solution (75). As in the one-dimensional case, the method is easily modified to give the proper asymptotic form near a caustic (i.e. a locus of cusps of wave crests) and in other singular cases (Lighthill, 1960, pp. 408–411).

8. Energy Propagation Velocity

I began this lecture by saying that I would describe both the kinematic and the dynamic approaches to the mathematics of group velocity. Hitherto I have described basically kinematic approaches, which, for a general linear combination of waves with frequency and wavenumber functionally related, evaluate how the waves of different wavenumbers become dispersed from one another. From time to time I have shown the results on amplitude variation to be consistent with the assumption that the energy in waves of each wavenumber is propagated at the group velocity, but I have still not proved explicitly that this is so.

This result is in many ways the most surprising of all, for whereas in a motion with varying wavenumber a derivative, like $\partial\omega/\partial k_\alpha$, may be expected to be important, the fact that, *in a perfectly periodic motion of fixed wavenumber*, energy is propagated at a velocity which can be expressed as a ratio of changes of frequency and wavenumber in going to a neighbouring wave solution, appears distinctly odd. To prove this result

for a general periodic wave motion, how are we to introduce the concept of the changes in frequency and wavenumber on passing to a neighbouring solution?

One very amusing answer to this question was supplied by Rayleigh (1877) in the one-dimensional case, and can easily be extended to the three-dimensional case (Lighthill, 1960, Appendix A). This answer supposes that, in a general dynamical system, a pure imaginary change in wavenumber is made, and shows that the corresponding imaginary change in the frequency with which the system oscillates in small disturbances would be replaced by zero frequency change if the motion of every particle in the system were resisted by an additional small force proportional to its momentum. The energy flow across a plane in this steady state is then calculated by balancing it against the dissipative action of those forces throughout the region beyond that plane.

Instead of elaborating further on that, I prefer to give an argument applicable to still more general systems than classical dynamical ones, an argument that utilizes real rather than imaginary changes in frequency and wavenumber, and that calculates energy flow by more localized considerations. I shall, furthermore, give the argument in a generalized form due to G. B. Whitham (unpublished, but see the forthcoming paper Whitham (1965b) for a still more general treatment) which obtains the velocity of energy propagation in any plane periodic wave, even one of finite amplitude, in a homogeneous system in which energy is conserved. (This result, then, unlike those in the rest of the paper, is valid even for non-linear equations of motion.)

Any such system homogeneous in space can be specified (see, for example, Goldstein (1950), p. 350) by a single function: a Lagrangian density $\mathscr{L}$ (or Lagrangian per unit volume), which is a function of (say) n local variables (such as displacements, field strengths, variables of state, etc.) $\eta_1, \eta_2, \ldots, \eta_n$, together with their first derivatives

$$\dot{\eta}_l = \frac{\partial \eta_l}{\partial t} \quad \text{and} \quad \eta_{l\alpha} = \frac{\partial \eta_l}{\partial x_\alpha}. \tag{76}$$

There is no direct dependence of $\mathscr{L}$ on x_1, x_2, x_3 and t, only an indirect dependence, due to the dependence of η_l, $\dot{\eta}_l$ and $\eta_{l\alpha}$ on those variables.

Possible developments of the system in time are specified by Hamilton's principle, that the time integral of the Lagrangian is stationary. The usual form of the principle states that

$$\delta \int_{t_1}^{t_2} dt \int_V \mathscr{L}\, d\tau = 0 \tag{77}$$

for any changes $\delta\eta_l$ in the function $\eta_l(x_1,x_2,x_3,t)$ which vanish at the beginning and end $t = t_1$ and $t = t_2$ of the arbitrary time interval of integration and on the boundary of the arbitrary volume of integration V. This implies the equations of motion

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \dot{\eta}_l}\right) + \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathscr{L}}{\partial \eta_{l\alpha}}\right) - \frac{\partial \mathscr{L}}{\partial \eta_l} = 0. \tag{78}$$

This formulation of the equations of a homogeneous system is of great generality. For a classical dynamical system $\mathscr{L}$ is the kinetic energy minus the potential energy, per unit volume. For a relativistic dynamical system the Lagrangian for each particle can be written in terms of its rest mass m, potential energy V and velocity $c\beta$ (where c is the speed of light) as $-mc^2\sqrt{(1-\beta^2)} - V$. From this Lagrangian for each particle a

Lagrangian density can be calculated. Electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ contribute (in rationalized m.k.s. units) $\frac{1}{2}\varepsilon\mathbf{E}^2-\frac{1}{2}\mu\mathbf{H}^2$ to the Lagrangian density $\mathscr{L}$, and, when charge and current densities ρ and $\mathbf{j}$ are also present, additional terms $-\rho\phi+\mathbf{j}.\mathbf{A}$, where ϕ and $\mathbf{A}$ are the scalar and vector potentials. Other types of field can be taken into account similarly in the Lagrangian formulation. In all cases the density of total energy is

$$\mathscr{E} = \sum_{i=1}^{n} \dot{\eta}_i \frac{\partial \mathscr{L}}{\partial \dot{\eta}_i} - \mathscr{L}, \tag{79}$$

and it follows from the equation of motion (78) that the rate of change of the energy density $\mathscr{E}$ with time can be written as

$$\frac{\partial \mathscr{E}}{\partial t} = -\sum_{\alpha=1}^{3} \frac{\partial I_\alpha}{\partial x_\alpha} \tag{80}$$

where I_α is an "energy flux vector"

$$I_\alpha = \sum_{i=1}^{n} \dot{\eta}_i \frac{\partial \mathscr{L}}{\partial \dot{\eta}_{i,\alpha}}. \tag{81}$$

Equation (80) says that the energy in a rectangular element changes at a rate equal to the differences between the energy flux across opposite faces of the element. For periodic plane waves, we may define an energy propagation velocity u_α as

$$u_\alpha = \frac{\langle I_\alpha \rangle}{\langle \mathscr{E} \rangle} = \frac{\left\langle \sum_{i=1}^{n} \tilde{\eta}_i \partial \mathscr{L}/\partial \dot{\eta}_{i,\alpha} \right\rangle}{\left\langle \sum_{i=1}^{n} \dot{\eta}_i \partial \mathscr{L}/\partial \dot{\eta}_i - \mathscr{L} \right\rangle} \tag{82}$$

where for example $\langle \mathscr{E} \rangle$ signifies the mean energy density $\mathscr{E}$, that is, $\mathscr{E}$ averaged over an integral number of wavelengths or periods. In terms of u_α, the averaged form of equation (80) can be written

$$\frac{\partial \langle \mathscr{E} \rangle}{\partial t} + \sum_{\alpha=1}^{3} \frac{\partial (\langle \mathscr{E} \rangle u_\alpha)}{\partial x_\alpha} = 0, \tag{83}$$

showing that mean energy is convected by the group velocity vector field u_α in exactly the same way as the mass of a compressible fluid is by the hydrodynamic velocity field.

For periodic plane waves, Hamilton's principle can be used in a slightly special form to obtain an expression for the energy propagation velocity u_α. This special form states that such a wave satisfies (77) for all changes $\delta\eta_i$ that are periodic with the same frequency and wavenumber as the η_i themselves, provided that t_2-t_1 is an integral multiple of the (time) period, and V is a rectangular box with four of its sides perpendicular to the wave fronts, and the other two parallel to wave fronts and an integral number of wavelengths apart. For then we have

$$\begin{aligned}
\delta \int_{t_1}^{t_2} dt \int_V \mathscr{L}\, d\tau &= \int_{t_1}^{t_2} dt \int_V \sum_{i=1}^{n} \left(\frac{\partial \mathscr{L}}{\partial \eta_i} \delta\eta_i + \frac{\partial \mathscr{L}}{\partial \dot{\eta}_i} \delta\dot{\eta}_i + \sum_{\alpha=1}^{3} \frac{\partial \mathscr{L}}{\partial \dot{\eta}_{i,\alpha}} \delta\dot{\eta}_{i,\alpha} \right) d\tau \\
&= \int_{t_1}^{t_2} dt \int_V \sum_{i=1}^{n} \left[\frac{\partial \mathscr{L}}{\partial \eta_i} - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \dot{\eta}_i} \right) - \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathscr{L}}{\partial \dot{\eta}_{i,\alpha}} \right) \right] \delta\eta_i\, d\tau + \\
&\int_V d\tau \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\sum_{i=1}^{n} \frac{\partial \mathscr{L}}{\partial \dot{\eta}_i} \delta\eta_i \right) dt + \int_{t_1}^{t_2} dt \int_V \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha}\left(\sum_{i=1}^{n} \frac{\partial \mathscr{L}}{\partial \dot{\eta}_{i,\alpha}} \delta\eta_i \right) d\tau = 0,
\end{aligned} \tag{84}$$

because the first integral vanishes by (78) while, in the second, the inner integral vanishes because t_2-t_1 is an integral multiple of a period, and, in the third, the inner integral vanishes because it equals a surface integral with the contributions from each pair of faces of the rectangular box V cancelling out.

Before using this result to calculate u_α, I will remark that, in the denominator of the expression (82) for u_α, the mean Lagrangian density $\langle\mathscr{L}\rangle$ vanishes in the special case of waves of infinitesimal amplitude. For classical dynamical systems, this is the familiar result (Rayleigh, 1877) that mean kinetic energy equals mean potential energy in waves of infinitesimal amplitude. For more general systems subject to Hamilton's principle, it follows most easily by changing each η_l to $(1+\varepsilon)\eta_l$. This changes $\mathscr{L}$ to $(1+\varepsilon)^2\mathscr{L}$, because the hypothesis of infinitesimal amplitude is equivalent to $\mathscr{L}$ being homogeneous of the second degree in all its variables. The variational result (84) can therefore be true only if

$$\int_{t_1}^{t_2} dt \int_V \mathscr{L}\, d\tau \tag{85}$$

is itself zero (otherwise, multiplying it by $(1+\varepsilon)^2$ would not give it zero variation), and this means that $\langle\mathscr{L}\rangle = 0$.

However, for waves of finite amplitude, this argument fails since $\mathscr{L}$ is not necessarily a homogeneous function of its variables. Simple examples show, in fact, that $\langle\mathscr{L}\rangle$ is, in general, not zero for periodic plane waves of finite amplitude. The corresponding term in the denominator of equation (82) for u_α cannot then be omitted.

One inclines, naturally, to ask whether

$$u_\alpha = \frac{\partial\omega}{\partial k_\alpha} \tag{86}$$

for plane waves of finite amplitude. However, a serious obstacle to the possible truth of such an equation is that for these the frequency ω is in general a function not only of k_1, k_2 and k_3 but also of some quantity representing the amplitude of the wave. (It can in general vary also with other parameters, called "pseudo-frequencies" by Whitham (1965b); but this possibility is not allowed for below.) The derivative in (86) can therefore have meaning only if it is understood in the sense "keeping constant both the wavenumber components other than k_α and also some measure of the amplitude". Now, we have already noted that the mean Lagrangian density $\langle\mathscr{L}\rangle$, although zero for infinitesimal amplitude, has in general different values for finite values of the amplitude. Whitham's result is that equation (86) remains true if the measure of amplitude which is kept constant in calculating the derivative is $\langle\mathscr{L}\rangle/\omega$. That is,

$$u_\alpha = \left(\frac{\partial\omega}{\partial k_\alpha}\right)_{\langle\mathscr{L}\rangle/\omega}. \tag{87}$$

The classical result for infinitesimal amplitude is evidently a special case of this, since $\langle\mathscr{L}\rangle/\omega$ remains constant and equal to zero for all infinitesimal-amplitude waves.

To prove (87), let plane periodic waves have the form

$$\eta_l = f_l\left(\omega t - \sum_{\alpha=1}^{3} k_\alpha x_\alpha\right), \tag{88}$$

where the functions $f_1(z), \ldots, f_n(z)$ are all periodic functions of z with period 1. Then equation (84) holds for any infinitesimal changes whatever in the f_i which leave them still with period 1.

Consider now a more drastic perturbation, in which also the frequency and wavenumber change. If

$$\eta_i + \delta\eta_i = F_i\left(\Omega t - \sum_{\alpha=1}^{3} K_\alpha x_\alpha\right), \tag{89}$$

where

$$\Omega = \omega + \delta\omega, \quad K_\alpha = k_\alpha + \delta k_\alpha, \quad F_i(z) = f_i(z) + \delta f_i(z), \tag{90}$$

and the $\delta f_i(z)$ like the $f_i(z)$ have period 1, then

$$\begin{aligned}\delta\langle\mathscr{L}\rangle &= \delta\int_0^1 \mathscr{L}[f_i(z), \omega f_i'(z), -k_\alpha f_i'(z)]\, dz \\ &= \int_0^1 \sum_{i=1}^{n}\left(\frac{\partial\mathscr{L}}{\partial\eta_i}\delta f_i + \frac{\partial\mathscr{L}}{\partial\dot{\eta}_i}\omega\delta f_i' - \sum_{\alpha=1}^{3}\frac{\partial\mathscr{L}}{\partial\dot{\eta}_i}k_\alpha\delta f_i'\right)dz + \\ &\qquad \left(\int_0^1 \sum_{i=1}^{n}\frac{\partial\mathscr{L}}{\partial\dot{\eta}_i}f_i'dz\right)\delta\omega - \sum_{\alpha=1}^{3}\left(\int_0^1 \sum_{i=1}^{n}\frac{\partial\mathscr{L}}{\partial\dot{\eta}_i}f_i'dz\right)\delta k_\alpha.\end{aligned} \tag{91}$$

Of the three terms on the right-hand side, the first represents the changes resulting from changes in the f_i without changes in ω and k_α and must vanish by equation (84) (Hamilton's principle for changes in the η_i which maintain frequency and wavenumber), while the second and third parts, on multiplication by ω, can be expressed in terms of the mean values that appear in the expression for u_α; that is,

$$\omega\delta\langle\mathscr{L}\rangle = \left\langle\sum_{i=1}^{n}\dot{\eta}_i\frac{\partial\mathscr{L}}{\partial\dot{\eta}_i}\right\rangle\delta\omega - \sum_{\alpha=1}^{3}\left\langle\sum_{i=1}^{n}\dot{\eta}_i\frac{\partial\mathscr{L}}{\partial\dot{\eta}_i}\right\rangle\delta k_\alpha. \tag{92}$$

We can now obtain the relationship between $\delta\omega$ and δk_α, which makes $\langle\mathscr{L}\rangle/\omega$ remain constant, by substituting

$$\omega\delta\langle\mathscr{L}\rangle = \langle\mathscr{L}\rangle\delta\omega \tag{93}$$

in equation (92). If we use expression (82), this gives

$$\delta\omega = \sum_{\alpha=1}^{3} u_\alpha\delta k_\alpha, \tag{94}$$

which finally proves (87).

The quantity $\langle\mathscr{L}\rangle/\omega$ which is kept constant in (87) is the integral of the Lagrangian density with respect to time over a single period. It is the quantity which remains stationary, when we go from a periodic solution η_i to neighbouring values $\eta_i + \delta\eta_i$ which are periodic with the same frequency and wavenumber but are not, in general, solutions. This explains what may appear surprising in the proof I have given, namely, that I did not use the fact that the functions (89), with their perturbed frequency and wavenumber, are solutions of the equations of motion. It was not necessary, because the value of $\langle\mathscr{L}\rangle/\omega$ would be the same for them as for neighbouring functions which are not solutions!

It seems likely that this result of Whitham's, that the velocity of energy propagation in plane periodic waves of finite amplitude, in a general three-dimensional homogeneous conservative system, is equal to the gradient of the frequency with respect

to the wavenumber vector in changes, from one plane periodic wave to another of neighbouring wavenumber vector, in which the measure of amplitude represented by the integral of the Lagrangian with respect to time over a single period remains constant, is the first step in a major process of extension of group velocity and ray theory to non-linear dispersive systems just as far-reaching as was Whitham's work (1956) on non-dispersive systems of intermediate amplitude, based on the principle that the ray geometry will be approximately as for an infinitesimal-amplitude system but that the law of propagation of waveforms along ray tubes takes a new form.

9. Concluding Remarks

I believe I have now said as much as could well be said at one stretch about my subject. Two points about its relationship with modern physics still, perhaps, need to be made. First, that relativity forbids energy to be propagated at a speed exceeding the velocity of light c. It follows that no wave motion of a conservative system can have a group velocity greater than c, although phase velocities greater than c are rather common. (The restriction of this conclusion to conservative systems must be emphasized; as noted at the end of section 3, in the presence of large dissipation the theory of group velocity needs major modifications.)

For example, the relationship between frequency and wavenumber in an electromagnetic wave propagating in an ionized gas is

$$\omega = \left(\frac{4\pi n_e e^2}{m_e} + c^2k^2\right)^{\frac{1}{2}}$$

where n_e is the number of electrons per cm^3 and m_e and e are the mass and charge (in e.s.u.) of the electron. It follows that the phase velocity $v = \omega/k$ exceeds the speed of light c; however, the group velocity is

$$u = \frac{d\omega}{dk} = \frac{c^2k}{\left(\frac{4\pi n_e e^2}{m_e} + c^2k^2\right)^{\frac{1}{2}}} < c.$$

The second point, a related one, concerns the waves that Schrödinger's wave mechanics associates with a particle. They have frequency and wavenumber proportional to the particle's energy and momentum, as I mentioned earlier. In relativistic mechanics, this gives

$$\omega = \frac{E}{h} = \frac{mc^2}{h\sqrt{(1-\beta^2)}}, \quad k = \frac{p}{h} = \frac{mc\beta}{h\sqrt{(1-\beta^2)}}$$

for a particle of rest mass m and velocity $c\beta$. The phase velocity of the associated waves is therefore

$$\frac{\omega}{k} = \frac{c}{\beta} > c,$$

but the group velocity, with which they carry energy, is

$$\frac{d\omega}{dk} = \frac{d\omega/d\beta}{dk/d\beta} = c\beta < c.$$

The waves carry energy, in fact, simply at the velocity of the particle itself, and indeed are equivalent to the particle as energy carriers.

To conclude, then, the mathematical theory I have described in this lecture touches upon several interesting and important parts of physical and engineering science as well as upon more than one major branch of mathematics. Much of the theory is old, and during its long life has found very many applications. This part will find even more applications now that computational aids make the problem of ray tracing by Hamilton's equations (40) and (43) so simple a matter. This can be done easily, now, in the case of complicated internal wave motions in stratified media, rotating media or plasma in magnetic fields, and will be done increasingly. At the same time there are signs that further extensions to this powerful theory are being, and will continue to be made.

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